

The relativistic KMS condition for the thermal n -point functions of the $P(\phi)_2$ model

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Abstract

Thermal quantum field theories are expected to obey a *relativistic KMS condition*, which replaces both the relativistic spectrum condition of Wightman quantum field theory and the KMS condition characterising equilibrium states in quantum statistical mechanics.

In a previous work it has been shown that the two-point function of the thermal $\mathcal{P}(\varphi)_2$ model satisfies the relativistic KMS condition. Here we extend this result to general n -point functions. In addition, we verify that the thermal Wightman distributions are tempered.

1 Introduction

For many practical purposes it may be sufficient to study thermal field theory in finite spatial volume, where the Hamiltonian H_{box} has discrete spectrum and is bounded from below. The thermal expectation value of an observable O at temperature β^{-1} is then given by the Gibbs state

$$\langle O \rangle_\beta := \frac{\text{Tr} e^{-\beta H_{box}} O}{\text{Tr} e^{-\beta H_{box}}} . \quad (1)$$

However, if one wants to investigate the structural properties resulting from Poincaré invariance of the underlying equations of motion, thermal equilibrium states in infinite volume have to be considered. Fortunately, the appropriate generalisation of (1) to infinite volume is well-known: in finite volume the Gibbs states are characterised by their analyticity properties. The latter are summarised in the KMS condition (see, *e.g.*, [3]). Haag, Hugenholtz and Winnink have shown that this characterisation remains valid in the thermodynamic limit [20]. In fact, in infinite volume one can *derive* the KMS condition from

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first principles, which characterise thermal equilibrium states phenomenologically. For example, one can derive the KMS condition from *passivity* [39] or *stability* under small adiabatic perturbations of the dynamics [21][36].

Analyticity properties of the correlation functions were previously used by Wightman to characterise the vacuum state of a relativistic quantum field theory [47]. The intention was to guarantee *stability* of the vacuum, and indeed the requested analyticity properties of the correlation functions ensure that the energy is bounded from below. But soon it turned out that also most peculiar *structural properties* (e.g., the Reeh-Schlieder property [41]) follow from the analyticity properties of the correlation functions. These properties, first rejected as mathematical artifacts resulting from over-idealisation, are nowadays considered to be characteristic for any proper quantum field theory. Experiments testing these properties have yet to be devised, but similar phenomena related to entanglement in quantum mechanics are intensely investigated.

Although vacuum states and thermal equilibrium states are both characterised by analyticity properties of the correlation functions, there is a pronounced difference between them: a state in thermal equilibrium cannot be invariant under Lorentz boosts [35][38], even if the equations of motion are invariant under Poincaré transformations and the propagation speed of signals is finite. Hitherto the correlation functions of a thermal state were required to be analytic only with respect to the time-direction distinguished by the rest-frame of the equilibrium state. Structural results, which are similar to the ones derived from the spectrum condition of Wightman field theory, cannot be derived from the traditional KMS condition alone.

The picture changed fundamentally, when Bros and Buchholz [4] (see also [5][32][33]) recognised that the passivity properties of an equilibrium state should be visible even to an observer, who is moving with respect to the rest frame distinguished by the KMS state. Carefully evaluating the consequences, Bros and Buchholz suggested that the thermal correlation functions of a relativistic system can be continued analytically into the tube domain $\mathbb{R}^d + i(V^+ \cap (\beta e - V^+))$, where β plays the role of the reciprocal temperature of the system, e is the unit vector in the time direction distinguished by the rest-frame, $d = 2, 3, \dots$ is the dimension of space-time and $V^+ := \{(t, \vec{x}) \in \mathbb{R}^d \mid |\vec{x}| < t\}$ denotes the (open) forward light-cone. The consequences of the new *relativistic KMS condition* are profound, aligning thermal field theory and vacuum field theory w.r.t. basic, structural aspects [24][25].

The relativistic KMS condition has been established (see [4]) for a large class of KMS states constructed by Buchholz and Junglas [8]. Moreover, C. Gérard and the first author have shown that the relativistic KMS condition holds for the two-point function of the $\mathcal{P}(\varphi)_2$ model [14]. The present work extends the latter result to general n -point functions.

Content In Section 2 we recall the Euclidean field theory on the cylinder. Using Minlos' theorem, we define Gaussian measures on a space of distributions, supported on a cylinder. Following Glimm and Jaffe, we renormalise the

interaction (Theorem 2.1) by normal ordering the random variables. This allows us to define the Euclidean $\mathcal{P}(\phi)_2$ model on the cylinder with a spatial cut-off $l \in \mathbb{R}^+$ (see also [12]). The corresponding probability measure $d\mu_l$ is absolutely continuous with respect to the Gaussian measure. *Nelson symmetry* (see Theorem 2.3 *ii.*) can be used to remove the spatial cut-off: the Schwinger functions of the thermal $\mathcal{P}(\varphi)_2$ -model on the two-dimensional Minkowski space at temperature β^{-1} exist and are equal to the Schwinger functions of the vacuum $\mathcal{P}(\varphi)_2$ -model on the Einstein universe of spatial circumference β (up to interchanging the interpretation of the Euclidean variables $(\alpha, x) \in S_\beta \times \mathbb{R}$).

The Osterwalder-Schrader reconstructions, presented in Subsection 2.3, provide

- i.) a thermal field theory on Minkowski space \mathbb{R}^{1+1} , consisting of*
 - a Hilbert space \mathcal{H}_β , together with a distinguished vector $\Omega_\beta \in \mathcal{H}_\beta$;
 - a von Neumann algebra $\mathcal{R}_\beta \subset \mathcal{B}(\mathcal{H}_\beta)$, together with an abelian sub-algebra, generated by the bounded functions of the time-zero fields;
 - a one-parameter group of time-translation automorphisms $\{\tau_{t,0} \mid t \in \mathbb{R}\}$ induced by unitary operators e^{itL} , with spectrum of L equal to \mathbb{R} ;
- ii.) a vacuum theory on the Einstein universe (a cylinder, with the position variable taking values on the circle S_β and the time variable real valued), consisting of*
 - a Hilbert space \mathcal{H}_C , together with a distinguished vector $\Omega_C \in \mathcal{H}_C$;
 - a von Neumann algebra $\mathcal{R}_C \subset \mathcal{B}(\mathcal{H}_C)$, together with an abelian sub-algebra, generated by the bounded functions of the time-zero fields;
 - a one-parameter group of time-translation automorphisms $\{\tau'_{0,s} \mid s \in \mathbb{R}\}$ induced by unitary operators e^{isH_C} , with $H_C \geq 0$.

In Subsection 2.4 Wightman field theory on the circle is discussed in some more detail. We identify \mathcal{H}_C with the Fock space $\Gamma(H^{-\frac{1}{2}}(S_\beta))$ over the Sobolev space $H^{-\frac{1}{2}}(S_\beta)$ on the circle S_β , and recall the Glimm-Jaffe ϕ -bounds. According to Theorem 2.6, due to Heifets and Osipov, the joint spectrum of P_C and H_C is contained in the forward light cone $\tilde{V}^+ := \{(p, E) \mid |p| < E\}$. Consequently (Theorem 2.8) the Fourier transform of the Wightman n -point function (expressed in relative variables) has support in $(V^+)^{n-1}$ and the Wightman n -point-distribution $\mathfrak{W}_C^{(n-1)}$ itself is the boundary value of a polynomially bounded function $\mathcal{W}_C^{(n-1)}$, which is analytic in the forward tube $(S_\beta \times \mathbb{R} - iV^+)^{n-1}$.

Lemma 2.9 characterises a set of space-time points in $S_\beta \times \mathbb{R}$, which are mutually space-like to each other. Locality implies that the Wightman n -point functions are real valued, if evaluated at these points. Thus, using Schwarz's reflection principle, we can extend the Wightman n -point functions on the circle to functions, which (expressed in relative variables) are holomorphic in

$$\mathcal{D}^{(n-1)} := (\lambda_1 V_\beta \times \dots \times \lambda_{n-1} V_\beta) + i(V^+ \cup V^-) \times \dots \times (V^+ \cup V^-), \quad (2)$$

where $V_\beta := \{(\alpha, s) \mid |s| < \alpha < \beta - |s|\}$, $\lambda_i > 0$ and $\sum_{i=1}^{n-1} \lambda_i = 1$. The Edge-of-the-Wedge theorem [47, Theorem 2-16][6] implies that the tempered distributions $\mathfrak{W}_C^{(n-1)}$ are the boundary values of functions defined and holomorphic in

$$\mathcal{C}^{(n-1)} := \mathcal{N} \cup \mathcal{D}^{(n-1)}, \quad (3)$$

where \mathcal{N} is a complex neighbourhood of $\lambda_1 V_\beta \times \dots \times \lambda_{n-1} V_\beta$ (Theorem 2.10).

In Section 3 we return to the thermal $\mathcal{P}(\varphi)_2$ model on the two dimensional Minkowski space. Invoking (2), Nelson symmetry implies that analytic continuations of the thermal Wightman distributions $\mathfrak{W}_\beta^{(n-1)}$ can a priori be defined (as analytic functions) in the domain

$$(Q^- \cup Q^+)^{n-1} - i(\lambda_1 V_\beta \times \dots \times \lambda_{n-1} V_\beta),$$

where the right and left space-like wedges are $Q^\pm = \{(t, x) \in \mathbb{R}^2 \mid \pm x > |t|\}$, and, as before, $\lambda_i > 0$ and $\sum_{i=1}^{n-1} \lambda_i = 1$.

The existence of products of thermal sharp-time fields is shown in Lemma 3.1. Taking advantage of their Euclidean heritage, their domain properties are summarised in Proposition 3.2. The spectral theorem is used to extend the functions $\mathcal{W}_\beta^{(n-1)}$ to functions holomorphic in the product of domains

$$(\lambda_1 \mathcal{T}_\beta) \times \dots \times (\lambda_{n-1} \mathcal{T}_\beta), \quad \mathcal{T}_\beta := \mathbb{R}^2 - iV_\beta, \quad \sum_{j=1}^{n-1} \lambda_j = 1,$$

and $\lambda_j > 0$, $j = 1, \dots, n-1$ (Theorem 3.4). The final subsection deals with the boundary values of these functions. A generalisation of Ruelle's Hölder inequality for Gibbs states, suggested by J. Fröhlich in [10], is presented in Theorem 3.5. A fractional ϕ -bound, established in Proposition 3.6, provides a key ingredient in the proof of the final result (Theorem 3.9), which establishes that the thermal Wightman n -point-distributions $\mathfrak{W}_\beta^{(n-1)}$ of the $\mathcal{P}(\varphi)_2$ model on the real line are *tempered distributions* which satisfy the *relativistic KMS condition*.

2 Euclidean fields on the cylinder

In 1974 Høegh-Krohn [23] discovered that the Euclidean field theory on the cylinder allows to reconstruct *two* distinct quantum field theories. In this section we recall the main steps of the two reconstructions [12, 13], leading to a vacuum theory on the Einstein universe (a cylinder, with the position variable taking values on a circle and the time variable real valued) and a thermal theory on $1+1$ -dimensional Minkowski space.

2.1 Probability measures on the cylinder

Consider a cylinder $S_\beta \times \mathbb{R}$, with S_β the circle of circumference β . The coordinates $(\alpha, x) \in S_\beta \times \mathbb{R}$ of a point in the cylinder will refer to either one of the charts $[-\beta/2, \beta/2) \times \mathbb{R}$ or $[0, \beta) \times \mathbb{R}$.

Let $\mathcal{S}(\mathbb{R})$ denote the set of Schwartz functions on the real line. For consistency we denote the set of C^∞ -functions on the circle S_β by $\mathcal{S}(S_\beta)$. The Fréchet space $\mathcal{S}(S_\beta \times \mathbb{R})$ is the space of smooth functions f on the cylinder, which are β -periodic in α and fulfil

$$\left| (1 + |x|)^k \partial_\alpha^p \partial_x^k f(\alpha, x) \right| \leq C_{p,k}, \quad p \in \mathbb{N}, \quad k \in \mathbb{N}.$$

$\mathcal{S}'(\mathbb{R})$, $\mathcal{S}'(S_\beta)$ and $\mathcal{S}'(S_\beta \times \mathbb{R})$ denote the dual spaces of $\mathcal{S}(\mathbb{R})$, $\mathcal{S}(S_\beta)$ and $\mathcal{S}(S_\beta \times \mathbb{R})$. The real-linear subspaces of real valued distributions are indicated by $\mathcal{S}'_{\mathbb{R}}(\mathbb{R})$, $\mathcal{S}'_{\mathbb{R}}(S_\beta)$ and $Q := \mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R})$. The Borel σ -algebra Σ on Q is the minimal σ -algebra containing all open sets in the $\sigma(\mathcal{S}', \mathcal{S})$ -topology. The evaluation map $\phi(f)$, $f \in \mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R})$,

$$\phi(f): Q \rightarrow \mathbb{R}, \quad q \mapsto \langle q, f \rangle,$$

is defined in terms of the duality bracket $\langle \cdot, \cdot \rangle$. In the present context ϕ is called the *Euclidean quantum field*.

If $d\mu$ is a probability measure on the space (Q, Σ) , then its Fourier transform

$$E(f) = \int_Q e^{i\phi(f)} d\mu, \quad f \in \mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}),$$

satisfies

- i.) $E(0) = 1$;
- ii.) $\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}) \ni f \mapsto E(f) \in \mathbb{C}$ is continuous;
- iii.) for all $f_i, f_j \in \mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R})$ and $z_i, z_j \in \mathbb{C}$, $i, j = 1, \dots, n$,

$$\sum_{i,j=1}^n z_i \bar{z}_j E(f_i - f_j) \geq 0.$$

On the converse, *Minlos' theorem* [7][15][34][46] states that any function E on $\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R})$ satisfying the properties i.)–iii.) is the Fourier transform of a probability measure $d\mu$ on Q .

Generating functionals of the form

$$E_0(f) = e^{-C(f,f)/2}, \quad f \in \mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}), \quad (4)$$

with $C(\cdot, \cdot)$ a weakly continuous positive semi-definite quadratic form, clearly satisfy the conditions i.)–iii.) of Minlos' theorem and thus give rise to probability measures on (Q, Σ) . These measures are called *Gaussian* measures. The Gaussian measure on Q , with *covariance*

$$C(f_1, f_2) := (f_1, (D_\alpha^2 + D_x^2 + m^2)^{-1} f_2), \quad f_1, f_2 \in \mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}), \quad (5)$$

is denoted by $d\phi_C$. The scalar product (\cdot, \cdot) in (5) refers to $L^2(S_\beta \times \mathbb{R})$ and $D_\alpha = -i\partial_\alpha$, $D_x = -i\partial_x$. The Euclidean quantum field ϕ on the cylinder is

called *free*, if $\phi(f)$ is viewed as a measurable function on the probability space $(Q, \Sigma, d\phi_C)$.

In this work we are interested in non-Gaussian measures (see Theorem 2.3 below). They formally result from adding a polynomial of the form $\mathcal{P}(\varphi)$, where $\mathcal{P}(\lambda)$, $\lambda \in \mathbb{R}$, is a polynomial which is bounded from below, to the Hamiltonian of the free massive boson field.

In two dimensions, the singularities, which arise from taking powers of the Euclidean field ϕ at a point $(\alpha, x) \in S_\beta \times \mathbb{R}$, can be removed by first normal ordering $:\cdot:_c$ (see [15, 46]) the monomials $\phi(f)^n$, $n \in \mathbb{N}$,

$$:\phi(f)^n:_c := \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{n!}{m!(n-2m)!} \phi(f)^{n-2m} \left(-\frac{1}{2}c(f, f) \right)^m \quad (6)$$

with respect to a covariance c , and then taking appropriate limits. $[\cdot]$ denotes taking the integer part. We will normal order with respect to different covariances c , some of them being limiting cases of the covariance C defined in (5).

Normal-ordering of point-like fields is ill-defined (*i.e.*, one cannot replace the test function $f \in \mathcal{S}_\mathbb{R}(S_\beta \times \mathbb{R})$ in (6) by a two dimensional Dirac δ -function), but integrals over normal-ordered point-like fields can be defined rigorously: set, for $k \in \mathbb{N}$ and $\kappa \in \mathbb{R}^+$,

$$\delta_k(\alpha) := \beta^{-1} \sum_{|n| \leq k} e^{i\nu_n \alpha} \quad \text{and} \quad \delta_\kappa(x) := \kappa \chi(\kappa x),$$

where $\nu_n = 2\pi n/\beta$, $n \in \mathbb{N}$, and χ is an arbitrary, absolutely integrable function in $C_0^\infty(\mathbb{R})$ with $\int \chi(x) dx = 1$. With these notations we have the following result due to Glimm and Jaffe:

Theorem 2.1 (Ultraviolet renormalisation [13][15]). *For $f \in L^1(S_\beta \times \mathbb{R}) \cap L^2(S_\beta \times \mathbb{R})$, the following limit exists in $\bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, d\phi_C)$:*

$$\lim_{k, \kappa \rightarrow \infty} \int_{S_\beta \times \mathbb{R}} f(\alpha, x) : \phi(\delta_k(\cdot - \alpha) \otimes \delta_\kappa(\cdot - x))^n :_C d\alpha dx. \quad (7)$$

We denote it by $\int_{S_\beta \times \mathbb{R}} f(\alpha, x) : \phi(\alpha, x)^n :_C d\alpha dx$.

Remark This key theorem, which follows from exactly the same arguments as in the vacuum case analyzed by Glimm and Jaffe [15], establishes a crucial step forward in the construction of the $\mathcal{P}(\varphi)_2$ model in finite volume, as it takes care (see Eq. (8) below) of the *ultraviolet renormalisation*.

Let $\mathcal{P}(\lambda) = \sum_j c_j \lambda^j$ be a real valued polynomial, which is bounded from below. Replacing the function f in (7) by the characteristic function of the set $S_\beta \times [-l, l]$, $l \in \mathbb{R}^+$, and applying [46, Lemma V.5], we deduce that

$$e^{-\int_{-\beta/2}^{\beta/2} \int_{-l}^l : \mathcal{P}(\phi(\alpha, x)) :_C d\alpha dx} \in L^1(Q, \Sigma, d\phi_C) \quad \text{if} \quad 0 < l < \infty. \quad (8)$$

The *Euclidean* $\mathcal{P}(\phi)_2$ model on the cylinder with a spatial cut-off $l \in \mathbb{R}^+$ is specified by setting

$$d\mu_l := \frac{1}{Z_l} e^{-\int_{-\beta/2}^{\beta/2} \int_{-l}^l : \mathcal{P}(\phi(\alpha, x)) :_C d\alpha dx} d\phi_C. \quad (9)$$

The partition function Z_l is chosen such that $\int_Q d\mu_l = 1$. If $l < \infty$, then the measure $d\mu_l$ is absolutely continuous with respect to the Gaussian measure $d\phi_C$, with Radon-Nikodym derivative $d\mu_l/d\phi_C$ given by (8). However, the limit of the functions in (8) fails to be in $L^1(Q, \Sigma, d\phi_C)$ as $l \rightarrow \infty$, and therefore the formal limiting measure can not be absolutely continuous with respect to the Gaussian measure. In fact, in order to show that a countably additive Borel measure exists in the limit $l \rightarrow \infty$, it is sufficient to show (see Theorem 2.3 below) that

$$\lim_{l \rightarrow +\infty} \int_Q e^{i\phi(f)} d\mu_l =: E_{\mathcal{P}}(f), \quad f \in \mathcal{S}(S_\beta \times \mathbb{R}), \quad (10)$$

defines a generating functional on $\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R})$ satisfying the properties *i.)–iii.)* of Minlos' theorem.

2.2 Sharp-time fields, Existence of the Euclidean measure in the thermodynamic limit, and Nelson symmetry

Cluster expansions (see *e.g.* [15]) certainly allow one to control the limit in (10). But for the thermal $\mathcal{P}(\varphi)_2$ model, in which we are interested, there is another option, which was first explored in this context by Høegh-Krohn [23]: *Nelson symmetry*. It results from replacing the product measure $d\alpha dx$ in the exponent in (9) by iterated integrals with respect to the two measures $d\alpha$ and dx , in different orders. The delicate point, which will now be addressed in some more detail, is that *one* of the limits in (7) can be interchanged with *one* of the integrations.

In [13] it has been shown that

i.) for $h_1, h_2 \in \mathcal{S}'_{\mathbb{R}}(\mathbb{R})$ and $0 \leq \alpha_1, \alpha_2 \leq \beta$,

$$\begin{aligned} & \lim_{k, k' \rightarrow \infty} C(\delta_k(\cdot - \alpha_1) \otimes h_1, \delta_{k'}(\cdot - \alpha_2) \otimes h_2) \\ &= \left(h_1, \frac{e^{-|\alpha_1 - \alpha_2|\epsilon} + e^{-(\beta - |\alpha_1 - \alpha_2|)\epsilon}}{2\epsilon(1 - e^{-\beta\epsilon})} h_2 \right)_{L^2(\mathbb{R}, dx)}, \end{aligned} \quad (11)$$

with $\epsilon := (D_x^2 + m^2)^{\frac{1}{2}}$;

ii.) for $g_1, g_2 \in \mathcal{S}'_{\mathbb{R}}(S_\beta)$ and $x_1, x_2 \in \mathbb{R}$,

$$\lim_{\kappa, \kappa' \rightarrow \infty} C(g_1 \otimes \delta_\kappa(\cdot - x_1), g_2 \otimes \delta_{\kappa'}(\cdot - x_2)) = \left(g_1, \frac{e^{-|x_1 - x_2|\nu}}{2\nu} g_2 \right)_{L^2(S_\beta, d\alpha)}, \quad (12)$$

with $\nu := (D_\alpha^2 + m^2)^{\frac{1}{2}}$.

Thus, for $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$, $g \in \mathcal{S}_{\mathbb{R}}(S_{\beta})$ and $\alpha \in S_{\beta}$, $x \in \mathbb{R}$ fixed, the sequences of functions

$$\{\phi(\delta_k(\cdot - \alpha) \otimes h)\}_{k \in \mathbb{N}} \quad \text{and} \quad \{\phi(g \otimes \delta_{\kappa}(\cdot - x))\}_{\kappa \in \mathbb{N}}$$

are Cauchy sequences in $\bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, d\phi_C)$. This can be derived from the definition of the generating Gaussian functional, as (4) implies

$$\int_Q \phi(f)^p d\phi_C = \begin{cases} 0, & p \text{ odd}, \\ (p-1)!! C(f, f)^{p/2}, & p \text{ even}, \end{cases} \quad (13)$$

with $n!! = n(n-2)(n-4) \cdots 1$. We can therefore define sharp-time fields

$$\phi(\alpha, h) := \lim_{k \rightarrow \infty} \phi(\delta_k(\cdot - \alpha) \otimes h), \quad \phi(g, x) := \lim_{\kappa \rightarrow \infty} \phi(g \otimes \delta_{\kappa}(\cdot - x)). \quad (14)$$

We note that both $\phi(\alpha, h)$ and $\phi(g, x)$ belong to $\bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, d\phi_C)$.

Lemma 2.2 (Integrals over sharp-time fields [13]).

i.) For $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\alpha \in [0, 2\pi)$ the limit

$$\lim_{\kappa \rightarrow \infty} \int_{\mathbb{R}} h(x) : \phi(\alpha, \delta_{\kappa}(\cdot - x))^n :_{C_0} dx \quad (15)$$

exists in $\bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, d\phi_C)$. Denote it by $\int_{\mathbb{R}} h(x) : \phi(\alpha, x)^n :_{C_0} dx$. Normal ordering in (15) is with respect to the temperature β^{-1} covariance on \mathbb{R} : for $h_1, h_2 \in \mathcal{S}(\mathbb{R})$

$$C_0(h_1, h_2) := \left(h_1, \frac{(1 + e^{-\beta\epsilon})}{2\epsilon(1 - e^{-\beta\epsilon})} h_2 \right)_{L^2(\mathbb{R}, dx)}. \quad (16)$$

ii.) For $g \in L^1(S_{\beta}) \cap L^2(S_{\beta})$ and $x \in \mathbb{R}$ the limit

$$\lim_{\kappa \rightarrow \infty} \int_{S_{\beta}} g(\alpha) : \phi(\delta_k(\cdot - \alpha), x)^n :_{C_{\beta}} d\alpha \quad (17)$$

exists in $\bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, d\phi_C)$. Denote it by $\int_{S_{\beta}} g(\alpha) : \phi(\alpha, x)^n :_{C_{\beta}} d\alpha$. Normal ordering in (17) is w.r.t. the covariance

$$C_{\beta}(g_1, g_2) := \left(g_1, \frac{1}{2\nu} g_2 \right)_{L^2(S_{\beta}, d\alpha)}, \quad g_1, g_2 \in \mathcal{S}(S_{\beta}). \quad (18)$$

Returning to the integral in (7), we let f be the characteristic function on $S_{\beta} \times [-l, l]$. This enables us to rewrite (7) as $\lim_{k, \kappa \rightarrow \infty} F(k, \kappa)$, where

$$F(k, \kappa) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{n! \left(-\frac{1}{2} C(\delta_{k, \kappa}^{(2)}, \delta_{k, \kappa}^{(2)}) \right)^m}{m!(n-2m)!} \int_{S_{\beta} \times [-l, l]} d\alpha dx \phi(\delta_k(\cdot - \alpha) \otimes \delta_{\kappa}(\cdot - x))^{n-2m},$$

and $\delta_{k,\kappa}^{(2)}(\alpha, x) := \delta_k(\alpha) \otimes \delta_\kappa(x)$. Interchanging integrals and limits is permitted by the existence of the limits in (7), (15) and (17). Performing the two limits in different orders results in

$$\lim_{k,\kappa \rightarrow \infty} F(k, \kappa) = \lim_{\kappa \rightarrow \infty} \sum_{m=0}^{[n/2]} \frac{n! \left(-\frac{1}{2} C_0(\delta_\kappa, \delta_\kappa)\right)^m}{m!(n-2m)!} \int_{S_\beta} d\alpha \int_{[-l,l]} dx \phi(\alpha, \delta_\kappa(\cdot - x))^{n-2m}$$

and

$$\lim_{k,\kappa \rightarrow \infty} F(k, \kappa) = \lim_{k \rightarrow \infty} \sum_{m=0}^{[n/2]} \frac{n! \left(-\frac{1}{2} C_\beta(\delta_k, \delta_k)\right)^m}{m!(n-2m)!} \int_{[-l,l]} dx \int_{S_\beta} d\alpha \phi(\delta_k(\cdot - \alpha), x)^{n-2m}.$$

Note that in the latter expression normal ordering is done w.r.t. the covariance C_β , whilst in the former normal ordering is done with respect to the temperature β^{-1} covariance C_0 on \mathbb{R} .

Now let $U(\alpha, x)$, with $\alpha \in [0, 2\pi)$ and $x \in \mathbb{R}$, denote the unitary operators implementing the rotations and translations on the cylinder in $L^2(Q, \Sigma, d\mu)$ (for further details see next section). It follows that the L^1 -function (8) equals

$$e^{-\int_{-l}^l U(0,x) \left(\int_{-\beta/2}^{\beta/2} : \mathcal{P}(\phi(\alpha,0)) :_{C_\beta} d\alpha\right) dx} = e^{-\int_{-\beta/2}^{\beta/2} U(\alpha,0) \left(\int_{-l}^l : \mathcal{P}(\phi(0,x)) :_{C_0} dx\right) d\alpha}. \quad (19)$$

A proof of this identity can be found in [13, Lemma 5.3]. The analog of (19) in the case $\beta = \infty$ is known as *Nelson symmetry* (see *e.g.* [46]). Interpreting x in (19) as the imaginary time one notices that $d\mu = \lim_{l \rightarrow \infty} d\mu_l$ is the Euclidean measure of the vacuum $\mathcal{P}(\varphi)_2$ model on the circle. This argument can be made rigorous (see [13, Theorem 7.2], [23]) by exploiting various properties of a time dependent heat equation (see [13, Appendix A]).

Theorem 2.3. *Consider sharp-time fields as defined in (14), and integrals over normal ordered products as defined in (15) and (17).*

i.) (Thermodynamic limit of Euclidean measures). For $f \in C_{0\mathbb{R}}^\infty(S_\beta \times \mathbb{R})$

$$E_{\mathcal{P}}(f) = \lim_{l \rightarrow +\infty} \frac{1}{Z_l} \int_Q e^{i\phi(f)} e^{-\int_{-l}^l U(0,x) \left(\int_{-\beta/2}^{\beta/2} : P(\phi(\alpha,0)) :_{C_\beta} d\alpha\right) dx} d\phi_C. \quad (20)$$

ii.) (Nelson symmetry). For $f \in C_{0\mathbb{R}}^\infty(S_\beta \times \mathbb{R})$

$$E_{\mathcal{P}}(f) = \lim_{l \rightarrow +\infty} \frac{1}{Z_l} \int_Q e^{i\phi(f)} e^{-\int_{-\beta/2}^{\beta/2} U(\alpha,0) \left(\int_{-l}^l : P(\phi(0,x)) :_{C_0} dx\right) d\alpha} d\phi_C. \quad (21)$$

The map $f \mapsto E_{\mathcal{P}}(f)$ is continuous in some Schwartz semi-norm and thus extends to $\mathcal{S}(S_\beta \times \mathbb{R})$ [13, Theorem 7.2 ii.]. It satisfies the conditions of Minlos' theorem and thus defines a probability measure $d\mu$.

Remark This result solves the *infrared problem* for the thermal field theory under consideration. As mentioned before, we could have used cluster expansions to resolve this problem. However, Nelson symmetry will play a key role in the sequel, enabling us to transfer results between the two models it connects.

Before we continue, we recall two results, which refer to the L^p -spaces for the interacting measure $d\mu$:

Lemma 2.4. [13, Propositions 7.3 and 7.5]

- i.) (Sharp-time fields are in $L^p(Q, \Sigma, d\mu)$). Let $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ and $\alpha \in S_\beta$. Then the sequence $\phi(\delta_k(\cdot - \alpha) \otimes h)$ is Cauchy in $\bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, d\mu)$ and hence

$$\phi(\alpha, h) := \lim_{k \rightarrow \infty} \phi(\delta_k(\cdot - \alpha) \otimes h) \in \bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, d\mu).$$

Moreover, the map

$$\begin{array}{ccc} S_\beta & \rightarrow & \bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, d\mu) \\ \alpha & \mapsto & \phi(\alpha, h) \end{array}$$

is continuous for $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ fixed.

- ii.) (Convergence of sharp-time Schwinger functions, Part I). Let $h_i \in C_0^\infty(\mathbb{R})$ and $\alpha_i \in S_\beta$, $1 \leq i \leq n$. Then

$$\lim_{l \rightarrow \infty} \int_Q \left(\prod_{j=1}^n e^{i\phi(\alpha_j, h_j)} \right) d\mu_l = \int_Q \left(\prod_{j=1}^n e^{i\phi(\alpha_j, h_j)} \right) d\mu.$$

In Section 3.1 we will show that products of Euclidean sharp-time fields are as well elements of $\bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, d\mu)$. This will allow us to extend results of Fröhlich [10], Fröhlich and Birke [2], and Klein and Landau [30] concerning the reconstruction of thermal Green's functions.

2.3 The Osterwalder-Schrader Reconstruction

The cylinder $S_\beta \times \mathbb{R}$ is invariant under rotations and translations

$$\mathbf{t}_{(\alpha', x')} : (\alpha, x) \mapsto (\alpha + \alpha', x + x'), \quad \alpha' \in [0, 2\pi), \quad x' \in \mathbb{R},$$

as well as the reflections $\mathbf{r} : (\alpha, x) \mapsto (-\alpha, x)$ and $\mathbf{r}' : (\alpha, x) \mapsto (\alpha, -x)$. The pull-backs

$$(\mathbf{t}_*^{(\alpha', x')} f)(\alpha, x) := f(\mathbf{t}_{(\alpha', x')}^{-1}(\alpha, x)) = f(\alpha - \alpha', x - x')$$

acting on the testfunctions $f \in \mathcal{S}(S_\beta \times \mathbb{R})$, induce actions on the tempered distributions $q \in Q$:

$$(t_{(\alpha', x')} q)(f) := \langle q, \mathbf{t}_*^{(-\alpha', -x')} f \rangle, \quad (r q)(f) := \langle q, \mathbf{r}_* f \rangle, \quad \text{and} \quad (r' q)(f) := \langle q, \mathbf{r}'_* f \rangle.$$

Lifting these maps to measurable functions of distribution one finds that

- i.) the map $U(\alpha, x)F(q) := F(t_{(\alpha, x)}^{-1}q)$, $q \in Q$, defines a two-parameter group of measure-preserving $*$ -automorphisms of $L^\infty(Q, \Sigma, d\mu)$, strongly continuous in measure, and strongly continuous two-parameter groups of isometries of $L^p(Q, \Sigma, d\mu)$ for $1 \leq p < \infty$;
- ii.) the maps $RF(q) := F(rq)$ and $R'F(q) := F(r'q)$ extend to two measure preserving $*$ -automorphisms of $L^\infty(Q, \Sigma, d\mu)$ and to isometries of $L^p(Q, \Sigma, d\mu)$ for $1 \leq p < \infty$.

Since $d\mu$ is translation and rotation invariant, $U(\gamma, y)$ is unitary on the Hilbert space $L^2(Q, \Sigma, d\mu)$ for $\gamma \in [0, \beta)$ and $y \in \mathbb{R}$.

Notation. For $0 \leq \gamma \leq \beta$ (resp. $0 \leq y \leq \infty$) we denote by $\Sigma_{[0, \gamma]}$ (resp. $\Sigma^{[0, y]}$) the sub σ -algebra of the Borel σ -algebra Σ generated by the functions $e^{i\phi(f)}$ with $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$ and $\text{supp } f \subset [0, \gamma] \times \mathbb{R}$ (resp. $\text{supp } f \subset S_\beta \times [0, y]$).

Next define two scalar products:

$$\forall F, G \in L^2(Q, \Sigma_{[0, \beta/2]}, d\mu) : (F, G) := \int_Q R(\overline{F})G \, d\mu,$$

and

$$\forall F, G \in L^2(Q, \Sigma^{[0, \infty)}, d\mu) : (F, G)' := \int_Q R'(\overline{F})G \, d\mu.$$

The measure $d\mu$ is *Osterwalder-Schrader positive* with respect to *both* reflections R and R' :

$$\forall F \in L^2(Q, \Sigma_{[0, \beta/2]}, d\mu) : (F, F) \geq 0$$

and

$$\forall G \in L^2(Q, \Sigma^{[0, \infty)}, d\mu) : (G, G)' \geq 0.$$

Let $\mathcal{N} \subset L^2(Q, \Sigma_{[0, \beta/2]}, d\mu)$ be the kernel of the positive quadratic form (\cdot, \cdot) and $\mathcal{N}' \subset L^2(Q, \Sigma^{[0, \infty)}, d\mu)$ the kernel of the positive quadratic form $(\cdot, \cdot)'$. Set

$$\mathcal{H}_\beta := \overline{L^2(Q, \Sigma_{[0, \beta/2]}, d\mu) / \mathcal{N}} \quad \text{and} \quad \mathcal{H}_C := \overline{L^2(Q, \Sigma^{[0, \infty)}, d\mu) / \mathcal{N}'}$$

The completions of the pre-Hilbert spaces are taken w.r.t. the norms $(\cdot, \cdot)^{\frac{1}{2}}$ and $(\cdot, \cdot)'^{\frac{1}{2}}$, respectively. The canonical projection from $L^2(Q, \Sigma_{[0, \beta/2]}, d\mu)$ to \mathcal{H}_β and from $L^2(Q, \Sigma^{[0, \infty)}, d\mu)$ to \mathcal{H}_C are denoted by \mathcal{V} and \mathcal{V}' , respectively. The distinguished vectors

$$\Omega_\beta := \mathcal{V}(1), \quad \Omega_C := \mathcal{V}'(1),$$

arise as the image of 1, the constant function equal to 1 on Q .

The abelian algebra

- i.) $L^\infty(Q, \Sigma_{\{0\}}, d\mu)$ preserves $L^2(Q, \Sigma_{[0, \beta/2]}, d\mu)$ and \mathcal{N} . Thus a representation π_β of $L^\infty(Q, \Sigma_{\{0\}}, d\mu)$ on the Hilbert spaces \mathcal{H}_β is given by

$$\pi_\beta(A)\mathcal{V}(F) := \mathcal{V}(AF), \quad F \in L^2(Q, \Sigma_{[0, \beta/2]}, d\mu), \quad A \in L^\infty(Q, \Sigma_{\{0\}}, d\mu);$$

- ii.) $L^\infty(Q, \Sigma^{\{0\}}, d\mu)$ preserves $L^2(Q, \Sigma^{[0, \infty)}, d\mu)$ and \mathcal{N}' . Thus one obtains a representation π_C of $L^\infty(Q, \Sigma^{\{0\}}, d\mu)$ on \mathcal{H}_C , specified by

$$\pi_C(B)\mathcal{V}'(G) := \mathcal{V}'(BG), \quad G \in L^2(Q, \Sigma^{[0, \infty)}, d\mu), \quad B \in L^\infty(Q, \Sigma^{\{0\}}, d\mu).$$

The corresponding von Neumann algebras can be interpreted as the algebras generated by bounded functions of the thermal time-zero fields on the real line and the vacuum time-zero fields on the circle, respectively.

The reconstruction of the dynamics requires a more pronounced distinction of the two cases under consideration, which in the thermal case relies on a remarkable result on local symmetric semi-groups by Fröhlich [11] and, independently, Klein and Landau [31]:

- i.) The semigroup $\{U(\alpha, 0)\}_{\alpha>0}$ does *not* preserve $L^2(Q, \Sigma_{[0, \beta/2]}, d\mu)$. But setting, for $0 \leq \gamma \leq \beta/2$,

$$\mathcal{D}_\gamma := \mathcal{V}\mathcal{M}_\gamma, \quad \text{with } \mathcal{M}_\gamma := L^2(Q, \Sigma_{[0, \beta/2-\gamma]}, d\mu),$$

one can define, for $0 \leq \alpha \leq \gamma$, a linear operator $P(\alpha): \mathcal{D}_\gamma \rightarrow \mathcal{H}_\beta$ with domain \mathcal{D}_γ by setting

$$P(\alpha)\mathcal{V}\psi := \mathcal{V}U(\alpha, 0)\psi, \quad \psi \in \mathcal{M}_\gamma.$$

The triple $(P(\alpha), \mathcal{D}_\alpha, \beta/2)$ forms a *local symmetric semigroup* (see [11][31]):

- a.) for each α , $0 \leq \alpha \leq \beta/2$, \mathcal{D}_α is a linear subset of \mathcal{H}_β such that $\mathcal{D}_\alpha \supset \mathcal{D}_\gamma$ if $0 \leq \alpha \leq \gamma \leq \beta/2$, and

$$\mathcal{D} := \bigcup_{0 < \alpha \leq \beta/2} \mathcal{D}_\alpha$$

is dense in \mathcal{H}_β ;

- b.) for each α , $0 \leq \alpha \leq \beta/2$, $P(\alpha)$ is a linear operator on \mathcal{H}_β with domain \mathcal{D}_α ;

- c.) $P(0) = 1$, $P(\alpha)\mathcal{D}_\gamma \subset \mathcal{D}_{\gamma-\alpha}$ for $0 \leq \alpha \leq \gamma \leq \beta/2$, and

$$P(\alpha)P(\gamma) = P(\alpha + \gamma)$$

on $\mathcal{D}_{\alpha+\gamma}$ for $\alpha, \gamma, \alpha + \gamma \in [0, \beta/2]$;

d.) $P(\alpha)$ is symmetric, i.e.,

$$(\Psi, P(\alpha)\Psi') = (P(\alpha)\Psi', \Psi), \quad 0 \leq \alpha \leq \beta/2,$$

for all $\Psi, \Psi' \in \mathcal{D}_\alpha$ and $0 \leq \alpha \leq \beta/2$;

e.) $P(\alpha)$ is weakly continuous, i.e., if $\Psi \in \mathcal{D}_\gamma$, $0 \leq \gamma \leq \beta/2$, then

$$\alpha \mapsto (\Psi, P(\alpha)\Psi)$$

is a continuous function of α for $0 \leq \alpha \leq \gamma$.

By the results cited [11][31] there exists a selfadjoint operator L on \mathcal{H}_β such that for $0 \leq \alpha \leq \gamma$

$$\mathcal{V}(U(\alpha, 0)F) = e^{-\alpha L}\mathcal{V}(F), \quad F \in L^2(Q, \Sigma_{[0, \beta/2 - \gamma]}, d\mu).$$

The selfadjoint operator L is said to be *associated* to the local symmetric semigroup $(P(\alpha), \mathcal{D}_\alpha, \beta/2)$. Since $1 \in \mathcal{M}_\gamma$ and $L^\infty(Q, \Sigma_{\{0\}}, d\mu)\mathcal{M}_\gamma \subset \mathcal{M}_\gamma$ for all $0 \leq \gamma \leq \beta/2$, it follows that $e^{i\phi_\beta(h)}\Omega_\beta \in \mathcal{D}(e^{-\frac{\beta}{2}L})$, where $e^{i\phi_\beta(h)} \doteq \pi_\beta(e^{i\phi(0, h)})$ with $h \in C_{0\mathbb{R}}^\infty(\mathbb{R})$.

Lemma 2.5. \mathcal{D}_γ is dense in \mathcal{H}_β for $0 < \gamma < \beta/2$.

Proof. Assume that

$$(\Psi, \Phi) = 0 \quad \forall \Phi \in \mathcal{D}_\gamma. \quad (22)$$

Now consider, for $h_1, h_2 \in C_{0\mathbb{R}}^\infty(\mathbb{R})$ fixed, the analytic function

$$z \mapsto (\Psi, e^{i\phi_\beta(h_1)}e^{-zL}e^{i\phi_\beta(h_2)}\Omega_\beta), \quad \{z \in \mathbb{C} \mid 0 < \Re z < \beta/2\}. \quad (23)$$

Clearly, $e^{i\phi_\beta(h_1)}e^{-\Re z L}e^{i\phi_\beta(h_2)}\Omega_\beta \in \mathcal{D}_\gamma$ for $0 < \Re z < \gamma$ and consequently, because of (22), the analytic function (23) vanishes on an open line segment in the interior of its domain, and is therefore identically zero. It follows that

$$(\Psi, e^{i\phi_\beta(h_1)}e^{-\frac{\beta}{2}L}e^{i\phi_\beta(h_2)}\Omega_\beta) = 0 \quad \forall h_1, h_2 \in C_{0\mathbb{R}}^\infty(\mathbb{R}). \quad (24)$$

The set $\{e^{i\phi_\beta(h_1)}e^{-\frac{\beta}{2}L}e^{i\phi_\beta(h_2)}\Omega_\beta \mid h_1, h_2 \in C_{0\mathbb{R}}^\infty(\mathbb{R})\}$ is dense in \mathcal{H}_β [30, Theorem 11.2], and therefore (24) implies $\Psi = 0$. In other words, \mathcal{D}_γ is dense in \mathcal{H}_β . \square

ii.) The semi-group $U(0, x)$, $x \geq 0$, preserves the half-space $L^2(Q, \Sigma^{[0, \infty)}, d\mu)$ as $U(0, x)$ maps $L^2(Q, \Sigma_{[0, \infty)}, d\mu)$ into itself. Following [29] one can therefore define a self-adjoint positive operator H_C on \mathcal{H}_C such that for $G \in L^2(Q, \Sigma_{[0, \infty)}, d\mu)$

$$\mathcal{V}'(U(0, x)G) = e^{-xH_C}\mathcal{V}'(G), \quad x > 0. \quad (25)$$

The operators e^{-xH_C} , $x > 0$, form a strongly continuous semigroup of contractions on \mathcal{H}_C .

The next step in the reconstruction program is to define non-abelian von Neumann algebras $\mathcal{R}_\beta \subset \mathcal{B}(\mathcal{H}_\beta)$ and $\mathcal{R}_C \subset \mathcal{B}(\mathcal{H}_C)$, generated by the operators

$$\tau_{t,0}(\pi_\beta(A)) := e^{itL} \pi_\beta(A) e^{-itL}, \quad t \in \mathbb{R}, \quad A \in L^\infty(Q, \Sigma_{\{0\}}, d\mu),$$

and

$$\tau'_{0,\sigma}(\pi_C(A)) := e^{i\sigma H_C} \pi_C(A) e^{-i\sigma H_C}, \quad \sigma \in \mathbb{R}, \quad A \in L^\infty(Q, \Sigma^{\{0\}}, d\mu),$$

respectively. Clearly $\tau_{t,0}$ and $\tau'_{0,\sigma}$ extend to *-automorphisms of \mathcal{R}_β and \mathcal{R}_C , respectively.

The algebra $\mathcal{R}_\beta \subset \mathcal{B}(\mathcal{H}_\beta)$ has a cyclic and separating vector, namely Ω_β . The time-translation invariant state ω_β (a normalised positive linear functional) on \mathcal{R}_β defined by

$$\omega_\beta(a) := (\Omega_\beta, a \Omega_\beta), \quad a \in \mathcal{R}_\beta,$$

is invariant under the spatial translations induced by $\mathfrak{t}_{(0,y)}$, $y \in \mathbb{R}$. Furthermore, it satisfies the *KMS condition* [30]: the functions

$$F_{h_1, \dots, h_n}(t_1 - t_2, \dots, t_{n-1} - t_n) := (\Omega_\beta, \tau_{t_1}(e^{i\phi_\beta(h_1)}) \dots \tau_{t_n}(e^{i\phi_\beta(h_n)}) \Omega_\beta)$$

extend to analytic functions in the domain

$$\left\{ (z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1} \mid \Im z_k < 0, \quad -\beta < \sum_{k=1}^{n-1} \Im z_k \right\}$$

and satisfy the KMS boundary condition: for each $1 \leq k < n$

$$\begin{aligned} & F_{h_1, \dots, h_n}(s_1, \dots, s_{k-2}, s_{k-1} - i\beta, s_k, \dots, s_{n-1}) \\ &= F_{h_k, \dots, h_n, h_1, \dots, h_{k-1}}(s_k, \dots, s_{n-1}, s_n, s_1, \dots, s_{k-2}) \end{aligned} \quad (26)$$

with $s_n = t_n - t_1$ and $s_k = t_k - t_{k+1}$, $k = 1, \dots, n-1$, and $h_1, \dots, h_n \in C_{0\mathbb{R}}^\infty$.

The algebra $\mathcal{R}_C \subset \mathcal{B}(\mathcal{H}_C)$ has a cyclic vector, namely Ω_C . The state ω_C on \mathcal{R}_C ,

$$\omega_C(a) := (\Omega_C, a \Omega_C), \quad a \in \mathcal{R}_C,$$

is invariant under the rotations induced by $\mathfrak{t}_{(\gamma,0)}$, $\gamma \in [0, 2\pi)$, and satisfies the *spectrum condition* (see Theorem 2.6 below), which characterises *vacuum states*. Since ω_C is the unique vacuum state (see below), the commutant \mathcal{R}'_C of \mathcal{R}_C equals $\mathbb{C} \cdot 1$ and therefore $\mathcal{R}_C = \mathcal{B}(\mathcal{H}_C)$.

2.4 The Wightman functions on the Einstein universe

The Hilbert space \mathcal{H}_C reconstructed in the previous section is unitarily equivalent to the Fock space $\Gamma(H^{-\frac{1}{2}}(S_\beta))$ over the Sobolev space $H^{-\frac{1}{2}}(S_\beta)$ of order $-\frac{1}{2}$ on S_β , equipped with the norm

$$\|g\|^2 = (g, (2\nu)^{-1}g)_{L^2(S_\beta, d\alpha)}, \quad \nu = (D_\alpha^2 + m^2)^{\frac{1}{2}}.$$

To ease the notation we simply identify corresponding operators and vectors. For $g \in \mathcal{S}_{\mathbb{R}}(S_{\beta})$ the Segal field operator on \mathcal{H}_C , given by

$$\phi_C(g) := -i \frac{d}{d\lambda} \mathcal{V}'(e^{i\phi(0,\lambda g)}) \Big|_{\lambda=0},$$

is thereby identified with the Fock space field operator

$$\phi_C(g) = \frac{1}{\sqrt{2}} (a^*(\nu^{-1/2}g) + a(\nu^{-1/2}g))^{-}, \quad (27)$$

built up from bosonic creation and annihilation operators $a^*(f)$ and $a(f)$ (see, *e.g.*, [40]). Note that the map $f \mapsto a^*(f)$ is linear, while the map $f \mapsto a(f)$ is anti-linear.

The (angular) momentum operator $P_C := d\Gamma(D_{\alpha})$ on the circle S_{β} has discrete spectrum. Define

$$V := \int_{S_{\beta}} : \mathcal{P}(\phi_C(\alpha)) :_{C_{\beta}} d\alpha.$$

The operator sum

$$d\Gamma(\nu) + V - E_C$$

is essentially selfadjoint on its natural domain $\mathcal{D}(d\Gamma(\nu)) \cap \mathcal{D}(V)$ and bounded from below. Its closure equals the Hamiltonian H_C of the $\mathcal{P}(\phi_C)_2$ model on the circle S_{β} , which has been (re-)constructed in the previous section (see (25)). The additive constant E_C is chosen such that zero is the lowest eigenvalue, *i.e.*, $\inf \text{Spec}(H_C) = 0$. This eigenvalue is non-degenerated¹, and the corresponding eigenvector Ω_C can be chosen such that $(\Omega_C, \Omega^{\circ}) > 0$. Here Ω° denotes the Fock vacuum vector in $\Gamma(H^{-\frac{1}{2}}(S_{\beta}))$.

Moreover, the Glimm-Jaffe ϕ -bounds (see *e.g.* [9][18][19], the exact variant we use can be found in [13, Proposition 5.4]) hold: for $c \gg 1$ and some $\mathfrak{C} \in \mathbb{R}^+$,

$$\pm \phi_C(g) \leq \mathfrak{C} \|g\|_{H^{-\frac{1}{2}}(S_{\beta})} (H_C + c)^{1/2} \quad \forall g \in H^{-\frac{1}{2}}(S_{\beta}), \quad (28)$$

and

$$\pm \phi_C(g) \leq \mathfrak{C} \|g\|_{H^{-1}(S_{\beta})} (H_C + c) \quad \forall g \in H^{-1}(S_{\beta}). \quad (29)$$

The following remarkable result is due to Heifets & Osipov [22]; see also [27].

Theorem 2.6 (Spectrum Condition [22]). *The joint spectrum of P_C and H_C is purely discrete and contained in the forward light cone $\tilde{V}^+ := \{(p, E) \mid |p| < E\}$.*

¹Glimm and Jaffe have shown in [17] that the Hamiltonian H with a spatial cutoff, rather than on a spatial circle, *i.e.*, with periodic boundary conditions, satisfies the properties stated in this paragraph. Similar arguments apply to H_C , see the proof of Proposition 5.4 in [13].

The unitary operators $U_C(\alpha, \sigma) \in \mathcal{B}(\mathcal{H}_C)$ given by

$$U_C(\alpha, \sigma) := e^{i(\sigma H_C - \alpha P_C)}, \quad \alpha \in [0, 2\pi), \sigma \in \mathbb{R}, \quad (30)$$

implement the two parameter group of automorphisms $\tau'_{\alpha, \sigma}$ of \mathcal{R}_C on the Hilbert space \mathcal{H}_C . Let $g_i \in \mathcal{S}(S_\beta)$ and set

$$\phi_C(g_i, \sigma_i) := e^{i\sigma_i H_C} \phi_C(g_i) e^{-i\sigma_i H_C}, \quad i = 1, \dots, n.$$

By Stone's theorem, the map $\sigma \mapsto U_C(0, \sigma)$ is strongly continuous. Together with the bound (28) this implies that

$$\mathcal{W}_C^{(n)}(g_1, \sigma_1, \dots, g_n, \sigma_n) := (\Omega_C, \phi_C(g_1, \sigma_1) \cdots \phi_C(g_n, \sigma_n) \Omega_C)$$

exists and is a separately continuous multi-linear functional of the arguments (g_i, σ_i) , $i = 1, \dots, n$, as they vary over $\mathcal{S}(S_\beta) \times \mathbb{R}$. It follows from the nuclear theorem [47] that this functional can be uniquely represented as a tempered distribution of the n vectors $(\alpha_i, \sigma_i) \in S_\beta \times \mathbb{R}$. Denote the corresponding distribution by

$$\mathcal{W}_C^{(n)}(\alpha_1, \sigma_1, \dots, \alpha_n, \sigma_n) \equiv (\Omega_C, \phi_C(\alpha_1, \sigma_1) \cdots \phi_C(\alpha_n, \sigma_n) \Omega_C). \quad (31)$$

Translation invariance implies that $\mathcal{W}_C^{(n)}$ depends only on the relative coordinates

$$\xi_i = (\alpha_i - \alpha_{i+1}, \sigma_i - \sigma_{i+1}), \quad i = 1, \dots, n-1,$$

or more precisely, that there exists a tempered distribution $\mathfrak{W}_C^{(n-1)}$ such that

$$\mathfrak{W}_C^{(n-1)}(\xi_1, \xi_2, \dots, \xi_{n-1}) = \mathcal{W}_C^{(n)}(\alpha_1, \sigma_1, \alpha_2, \sigma_2, \dots, \alpha_n, \sigma_n). \quad (32)$$

We interpret $\mathfrak{W}_C^{(n-1)}$ as a periodic generalised function, and so its continuous Fourier transform is a tempered distribution, which can be identified with its discrete Fourier transform.

Lemma 2.7. *Let $\widetilde{\mathfrak{W}}_C^{(n-1)}$ denote the Fourier transform of $\mathfrak{W}_C^{(n-1)}$. Then the distributional support of $\widetilde{\mathfrak{W}}_C^{(n-1)}$ is contained in the joint spectrum of P_C and H_C .*

Proof. The Fourier transform of $\mathfrak{W}_C^{(n-1)}$ is

$$\begin{aligned} \widetilde{\mathfrak{W}}_C^{(n-1)}((p_1, E_1), (p_2, E_2), \dots, (p_{n-1}, E_{n-1})) &= \\ &= (2\pi\beta)^{-(n-1)} \int d\xi_1 \cdots d\xi_{n-1} e^{i \sum_{j=1}^{n-1} (p_j, E_j) \cdot \xi_j} \mathfrak{W}_C^{(n-1)}(\xi_1, \dots, \xi_{n-1}), \end{aligned}$$

where

$$\begin{aligned} \mathfrak{W}_C^{(n-1)}(\alpha_1 - \alpha_2, \sigma_1 - \sigma_2, \dots, \alpha_{n-1} - \alpha_n, \sigma_{n-1} - \sigma_n) &= \\ &= (\Omega_C, \phi_C(\alpha_1) e^{i(\sigma_2 - \sigma_1) H_C} \cdots \phi_C(\alpha_{n-1}) e^{i(\sigma_n - \sigma_{n-1}) H_C} \phi_C(\alpha_n) \Omega_C). \end{aligned}$$

Next insert, as suggested in [47], a basis of common eigenfunctions $\Psi_{\epsilon,k}$ of the operators P_C, H_C : for all $\Phi \in \mathcal{H}_C$ the unitary operators $U_C(\alpha, s)$ defined in (30) can be expressed as

$$U_C(\alpha, \sigma)\Phi = \sum_{(k,\epsilon) \in \text{Sp}(P_C, H_C)} e^{i(\sigma\epsilon - \alpha k)} (\Psi_{k,\epsilon}, \Phi) \Psi_{k,\epsilon}.$$

Now consider, for $\Phi, \Phi' \in \mathcal{H}_C$ fixed, the map

$$\begin{aligned} (E, p) &\mapsto \int_{S_\beta} d\alpha \int_{\mathbb{R}} d\sigma e^{-i(E\sigma - p\alpha)} (\Phi', U(\alpha, \sigma)\Phi) \\ &= \sum_{(k,\epsilon) \in \text{Sp}(P_C, H_C)} \int_{\mathbb{R}} d\sigma e^{-i(E-\epsilon)\sigma} \times \\ &\quad \times \int_{S_\beta} d\alpha e^{i(p-k)\alpha} (\Psi_{k,\epsilon}, \Phi) (\Phi', \Psi_{k,\epsilon}) \\ &= \sum_{(k,\epsilon) \in \text{Sp}(P_C, H_C)} 2\pi\delta(E-\epsilon)\delta_{k,p}(\Psi_{k,\epsilon}, \Phi) (\Phi', \Psi_{k,\epsilon}). \end{aligned}$$

The sum on the r.h.s. vanishes, if $(p, E) \notin \text{Sp}(P_C, H_C)$. This implies that the distributional support of $\widetilde{\mathfrak{W}}_C^{(n-1)}$ is contained in the joint spectrum of P_C and H_C . \square

Theorem 2.8. *For each $n \geq 1$, $\widetilde{\mathfrak{W}}_C^{(n-1)}$ has support in $(\widetilde{V}^+)^{n-1}$ and $\mathfrak{W}_C^{(n-1)}$ is the boundary value of a polynomially bounded function $\mathcal{W}_+^{(n-1)}$ analytic in the forward tube $(S_\beta \times \mathbb{R} - iV^+)^{n-1}$, where $V^+ := \{(t, x) \in \mathbb{R}^2 \mid |x| < t\}$.*

Proof. The support property of $\widetilde{\mathfrak{W}}_C^{(n-1)}$ was established in Lemma 2.7. By the Bros-Epstein-Glaser Lemma [40, Theorem IX.15] there exists a polynomial P and a polynomially bounded function $G^{(n-1)}: \mathbb{R}^{2(n-1)} \rightarrow \mathbb{C}$ obeying

$$\text{supp } G^{(n-1)} \subseteq \overline{(\widetilde{V}^+)^{(n-1)}},$$

such that $\widetilde{\mathfrak{W}}_C^{(n-1)} = P(D)G^{(n-1)}$, with

$$P(D) = \frac{\partial^{k_1 + \dots + k_{n-1} + l_1 + \dots + l_{n-1}}}{\partial E_1^{k_1} \partial p_1^{l_1} \dots \partial E_{n-1}^{k_{n-1}} \partial p_{n-1}^{l_{n-1}}}, \quad k_i, l_i \in \mathbb{N}.$$

Consequently an analytic continuation $\mathcal{W}_+^{(n-1)}$ of $\mathfrak{W}_C^{(n-1)}$ to $(S_\beta \times \mathbb{R} - iV^+)^{n-1}$ can be defined:

$$\begin{aligned} \mathcal{W}_+^{(n-1)}(\xi_1 - i\eta_1, \dots, \xi_{n-1} - i\eta_{n-1}) &= \\ &= (2\pi\beta)^{-(n-1)} P(-i(\xi_1 - i\eta_1, \dots, \xi_{n-1} - i\eta_{n-1})) \times \\ &\quad \times \int_{(S_\beta \times \mathbb{R})^{n-1}} \prod_{j=1}^{n-1} dp_j dE_j e^{-i(\xi_j - i\eta_j) \cdot (p_j, E_j)} \times \\ &\quad \times G^{(n-1)}((p_1, E_1), \dots, (p_{n-1}, E_{n-1})). \end{aligned}$$

If $\eta_j \in V^+$ for all $j \in \{1, \dots, n-1\}$, this integral exists. Furthermore, its boundary value for $(\eta_1, \dots, \eta_{n-1}) \searrow 0$ is $\mathfrak{W}_C^{(n-1)}$. Polynomial boundedness of the analytic function $\mathcal{W}_+^{(n-1)}$ results from the following inequality [40, Theorem IX.16]:

$$\begin{aligned} & \left| \mathcal{W}_+^{(n-1)}(\xi_1 - i\eta_1, \dots, \xi_{n-1} - i\eta_{n-1}) \right| \\ & \leq C \left| P(-i(\xi_1 - i\eta_1, \dots, \xi_{n-1} - i\eta_{n-1})) \right| (1 + d((\eta_1, \dots, \eta_{n-1})))^{-N}. \end{aligned}$$

C is a constant, $d((\eta_1, \dots, \eta_{n-1}))$ is the distance of $(\eta_1, \dots, \eta_{n-1})$ to $\partial(V^+)^{n-1}$ and N is a positive integer. \square

Next we investigate the consequences of locality on the circle $S_\beta \equiv [0, \beta)$.

Lemma 2.9. *The tempered distributions $\mathscr{W}_C^{(n)}(\alpha_1, \sigma_1, \dots, \alpha_n, \sigma_n)$ defined in (31) are real valued for $(\alpha_1, \sigma_1, \dots, \alpha_n, \sigma_n) \in J^{(n)}$, where*

$$(\alpha_1, \sigma_1, \dots, \alpha_n, \sigma_n) \in J^{(n)} \Leftrightarrow \begin{cases} (\alpha_i, \sigma_i) \in S_\beta \times \mathbb{R}, \\ (\alpha_{i+1} - \alpha_i, \sigma_{i+1} - \sigma_i) \in \lambda_i V_\beta, \\ \sum_{i=1}^{n-1} \lambda_i = 1, \quad \lambda_i > 0, \end{cases} \quad (33)$$

with $V_\beta := \{(\alpha, \sigma) \mid |\sigma| < \alpha < \beta - |\sigma|\} \subseteq S_\beta \times \mathbb{R}$ and $i = 1, \dots, n-1$.

Proof. Assume that the space-time points (α_i, σ_i) and (α_j, σ_j) are space-like to each other for all choices of $i \neq j$ and $i, j \in \{1, \dots, n\}$. Then, as a consequence of locality, all the field operators $\phi_C(\alpha_i, \sigma_i)$ commute (as quadratic forms) with each other and therefore $\mathscr{W}_C^{(n)}(\alpha_1, \sigma_1, \dots, \alpha_n, \sigma_n)$ equals

$$\begin{aligned} (\Omega_C, \phi_C(\alpha_1, \sigma_1) \cdots \phi_C(\alpha_n, \sigma_n) \Omega_C) &= (\Omega_C, \phi_C(\alpha_n, \sigma_n) \cdots \phi_C(\alpha_1, \sigma_1) \Omega_C) \\ &= \overline{\mathscr{W}_C^{(n)}(\alpha_1, \sigma_1, \dots, \alpha_n, \sigma_n)}. \end{aligned}$$

In other words, the tempered distributions $\mathscr{W}_C^{(n)}(\alpha_1, \sigma_1, \dots, \alpha_n, \sigma_n)$ are real valued. Thus the lemma follows, once we have shown that the set $J^{(n)}$ consists of points, which are pairwise space-like to each other.

A point (α, σ) on the cylinder is space-like to the origin $(0, 0)$ iff $(\alpha, \sigma) \in V_\beta$. Space-likeness is a symmetric relation and therefore it suffices to prove that (α_i, σ_i) is space-like to (α_j, σ_j) for $i > j$, i.e.,

$$(\alpha_j, \sigma_j) - (\alpha_i, \sigma_i) \in V_\beta \quad \text{for } i > j. \quad (34)$$

Moreover, for $0 < \lambda \leq 1$,

$$V_{\lambda\beta} = \{(\alpha, \sigma) \in W \mid |\alpha| + |\sigma| < \lambda\beta\},$$

with W the wedge $\{(\alpha, \sigma) \in [0, \beta) \times \mathbb{R} \mid \alpha > |\sigma|\}$. The map $\mathbf{n}: [0, 2\pi) \times \mathbb{R} \rightarrow \mathbb{R}^+$,

$$(\alpha, \sigma) \mapsto |\alpha| + |\sigma|,$$

defines a norm. Denote its restriction to the wedge W by $\mathfrak{n}|_W$. Equ. (34) now follows from the triangle inequality:

$$\begin{aligned} \mathfrak{n}|_W((\alpha_j, \sigma_j) - (\alpha_i, \sigma_i)) &= \mathfrak{n}|_W((\alpha_j - \alpha_{j-1}, \sigma_j - \sigma_{j-1}) + \dots \\ &\quad \dots + (\alpha_{i+1} - \alpha_i, \sigma_{i+1} - \sigma_i)) \\ &\leq \mathfrak{n}|_W((\alpha_j - \alpha_{j-1}, \sigma_j - \sigma_{j-1})) + \dots \\ &\quad \dots + \mathfrak{n}|_W((\alpha_{i+1} - \alpha_i, \sigma_{i+1} - \sigma_i)) \\ &< \lambda_{j-1}\beta + \lambda_{j-2}\beta + \dots + \lambda_i\beta \leq \beta \sum_{k=1}^{n-1} \lambda_k = \beta, \end{aligned}$$

and therefore (33) implies (34). We note that the set of n points on the cylinder, which are space-like to each other, is actually larger than $J^{(n)}$. \square

Because the tempered distributions $\mathscr{W}_C^{(n)}(\alpha_1, \sigma_1, \dots, \alpha_n, \sigma_n)$ defined in (31) are real valued for $(\alpha_1, \sigma_1, \dots, \alpha_n, \sigma_n) \in J^{(n)}$, we can apply the Schwarz reflection principle. The function

$$\begin{aligned} \mathcal{W}_-^{(n-1)}(\xi_1 + i\eta_1, \dots, \xi_{n-1} + i\eta_{n-1}) &= \overline{\mathcal{W}_+^{(n-1)}(\xi_1 - i\eta_1, \dots, \xi_{n-1} - i\eta_{n-1})} \\ &= \int_{(S_\beta \times \mathbb{R})^{n-1}} \frac{\prod_{j=1}^{n-1} dp_j dE_j}{(2\pi\beta)^{n-1}} e^{i(\xi_j + i\eta_j) \cdot (p_j, E_j)} \overline{\mathfrak{W}_C^{(n-1)}((p_1, E_1), \dots, (p_{n-1}, E_{n-1}))}. \end{aligned}$$

is analytic on $(S_\beta \times \mathbb{R} + iV^+) \times \dots \times (S_\beta \times \mathbb{R} + iV^+)$ and polynomially bounded as $\eta_i \searrow 0$. Since V^+ is a cone, $V^+ \times \dots \times V^+$ is a cone (by definition). Applying the Edge-of-the-Wedge theorem [47, Theorem 2-16], we conclude that there exists a complex neighbourhood \mathcal{N} of $\lambda_1 V_\beta \times \dots \times \lambda_{n-1} V_\beta$ and a function $\mathcal{W}_C^{(n-1)}$ defined and holomorphic in $\mathcal{N} \cup (S_\beta \times \mathbb{R} - iV^+)^{n-1} \cup (S_\beta \times \mathbb{R} + iV^+)^{n-1}$, which coincides with the restriction of the distributions $\mathfrak{W}_C^{(n-1)}(\xi_1, \xi_2, \dots, \xi_{n-1})$ defined in (32) to $\lambda_1 V_\beta \times \dots \times \lambda_{n-1} V_\beta$. In fact, by only partially reordering the fields (see the proof of Lemma 2.9) and using the support properties of the Fourier transform stated in Theorem 2.8, we can extend $\mathcal{W}_C^{(n-1)}$ into the regions $(S_\beta \times \mathbb{R} \mp iV^+) \times \dots \times (S_\beta \times \mathbb{R} \mp iV^+)$ (the \mp all being independent). Note that relative coordinates are used in $\mathcal{W}_C^{(n-1)}$ and therefore reordering of the arguments results in iV^+ being replaced by $-iV^+$. Thus we arrive at the following result:

Theorem 2.10. *There exists of a function $\mathcal{W}_C^{(n-1)}$ holomorphic in*

$$\mathcal{C}^{(n-1)} := \mathcal{N} \cup \mathcal{D}^{(n-1)}, \quad (35)$$

which coincides with the restriction of the distributions $\mathfrak{W}_C^{(n-1)}(\xi_1, \xi_2, \dots, \xi_{n-1})$ defined in (32) to $\lambda_1 V_\beta \times \dots \times \lambda_{n-1} V_\beta$. Here \mathcal{N} is a complex neighbourhood of $\lambda_1 V_\beta \times \dots \times \lambda_{n-1} V_\beta$ and

$$\mathcal{D}^{(n-1)} := (\lambda_1 V_\beta \times \dots \times \lambda_{n-1} V_\beta) + i(V^+ \cup V^-) \times \dots \times (V^+ \cup V^-)$$

with $\lambda_i > 0$ and $\sum_{i=1}^{n-1} \lambda_i = 1$. (In fact, one can take the union over these λ_j 's, $j = 1, \dots, n-1$).

3 The relativistic KMS condition for the $P(\phi)_2$ model

In the previous section we have seen that the Wightman functions on the circle are the boundary values of a function $\mathcal{W}_C^{(n-1)}$ holomorphic in the region $\mathcal{C}^{(n-1)}$ (see (35)). Now define a new function

$$\mathcal{W}_\beta^{(n-1)} := \mathcal{W}_C^{(n-1)} \circ \Xi^{-1}, \quad (36)$$

where Ξ is the coordinate transformation

$$(z_1, w_1, \dots, z_{n-1}, w_{n-1}) \mapsto (iz_1, -iw_1, \dots, iz_{n-1}, -iw_{n-1})$$

on $\mathbb{C}^{2(n-1)}$. Then $\mathcal{W}_\beta^{(n-1)}$ is analytic in the domain

$$((Q^- \cup Q^+)^{n-1} - i\lambda_1 V_\beta \times \dots \times \lambda_{n-1} V_\beta) \cup \Xi\mathcal{N}, \quad (37)$$

with $\lambda_i > 0$ and $\sum_{i=1}^{n-1} \lambda_i = 1$, where the right and left wedges are

$$Q^\pm = \{(\tau, y) \in \mathbb{R}^2 \mid \pm y > |\tau|\}.$$

Our aim is to show that

- i.*) the thermal Wightman functions $\mathcal{W}_\beta^{(n-1)}$ introduced in (36) extend to functions analytic in the product of domains

$$(\lambda_1 \mathcal{T}_\beta) \times \dots \times (\lambda_{n-1} \mathcal{T}_\beta), \quad \mathcal{T}_\beta := \mathbb{R}^2 - iV_\beta, \quad \sum_{j=1}^{n-1} \lambda_j = 1, \quad (38)$$

and $\lambda_j > 0, j = 1, \dots, n-1$. In fact, one can take the union over these λ_j 's;

- ii.*) the boundary values of the analytic functions $\mathcal{W}_\beta^{(n-1)}$ as $\Im z_j \searrow 0$ yield tempered distributions.

We will also ensure that these tempered distributions are indeed the Wightman distributions of the thermal field theory on the real line. We proceed in several steps.

3.1 Products of sharp-time fields and their domains

The representation π_β defined in Section 2.3 is a regular CCR representation (see [13]), and therefore one can define for $h \in C_{0\mathbb{R}}^\infty(\mathbb{R})$ the Segal field operators

$$\phi_\beta(h) := -i \frac{d}{ds} \pi_\beta \left(e^{i\phi(0, sh)} \right) \Big|_{s=0}. \quad (39)$$

While Stone's theorem is convenient to show that $\phi_\beta(h)$ exists as a self-adjoint unbounded operator, it provides little control on the domain of $\phi_\beta(h)$. In fact, a priori it is not even clear whether Ω_β is an element of $\mathcal{D}(\phi_\beta(h))$. We will need several steps to resolve these domain problems.

Lemma 3.1.

i.) (Products of sharp-time fields). Let $h_i \in \mathcal{S}_{\mathbb{R}}^{\infty}(\mathbb{R})$ for $i = 1, \dots, j$, $j \in \mathbb{N}$, and $0 \leq \alpha_1 \leq \dots \leq \alpha_j < \beta$. Then

$$\phi(\alpha_j, h_j) \cdots \phi(\alpha_1, h_1) \in \bigcap_{1 \leq p < \infty} L^p(Q, \Sigma_{[0, \alpha_j]}, d\mu), \quad j \in \mathbb{N}. \quad (40)$$

ii.) (Convergence of sharp-time Schwinger functions, Part II). Let $h_i \in C_0^{\infty}(\mathbb{R})$ and $\alpha_i \in S_{\beta}$, $1 \leq i \leq n$. Then

$$\lim_{l \rightarrow \infty} \int_Q \phi(\alpha_n, h_n) \cdots \phi(\alpha_1, h_1) d\mu_l = \int_Q \phi(\alpha_n, h_n) \cdots \phi(\alpha_1, h_1) d\mu.$$

Proof. i.) Consider an approximation of the Dirac δ -function: $\delta_{\kappa}(x) := \kappa \chi(\kappa x)$, with χ a function in $C_0^{\infty}(\mathbb{R})$ and $\int \chi(x) dx = 1$. It has been shown in [13, Proposition 7.3] that

$$\lim_{k \rightarrow \infty} \phi(\delta_k(\cdot - \alpha_i) \otimes h_i) \in \bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, d\mu), \quad h_i \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}).$$

For later purpose, we briefly recall the proof:

$$\begin{aligned} & \int_Q (\phi(\delta_k(\cdot - \alpha_i) \otimes h_i))^p d\mu \\ &= (-i)^p \frac{d^p}{d\lambda^p} (\Omega_C, W_{[-\infty, +\infty]}(\lambda(\delta_k(\cdot - \alpha_i) \otimes h_i)) \Omega_C) \Big|_{\lambda=0}, \end{aligned}$$

where $W_{[a,b]}(f)$ is a solution of the heat equation

$$\frac{d}{db} W_{[a,b]}(f) = W_{[a,b]}(f)(-H_C + i\phi_C(f_b)), \quad a \leq b,$$

with the boundary condition $W_{[a,a]}(f) = \mathbb{1}$ and with $f_b(\cdot) := f(\cdot, b) \in \mathcal{S}_{\mathbb{R}}(S_{\beta})$ for $f \in \mathcal{S}_{\mathbb{R}}(S_{\beta} \times \mathbb{R})$. Now, if $f = \delta_k(\cdot - \alpha_i) \otimes h_i$, then the function $f_x \in \mathcal{S}_{\mathbb{R}}(S_{\beta})$ is equal to $\delta_k(\cdot - \alpha_i)h_i(x)$. It follows from (29), *i.e.*, estimate (5.9) in [13, Proposition 5.4], that $h_i \in C_0^{\infty}(\mathbb{R})$ implies

$$\begin{aligned} \pm \phi_C(\delta_k(\cdot - \alpha_i)h_i(x)) &\leq c \|\delta_k(\cdot - \alpha_i)h_i(x)\|_{H^{-1}(S_{\beta})} (H_C + 1) \\ &\leq c \|h_i(x)\| \|\delta_k\|_{H^{-1}(S_{\beta})} (H_C + 1). \end{aligned}$$

Set $r_k(x) := c \|h_i(x)\| \|\delta_k\|_{H^{-1}(S_{\beta})}$ and apply [13, Lemma A.8] to obtain

$$\left\| \frac{d^p}{d\lambda^p} W_{[-\infty, +\infty]}(\lambda(\delta_k(\cdot - \alpha_i) \otimes h_i)) \right\| \leq p! \|r_k\|_{\infty}^p e^{\|r_k\|_1 \|r_k\|_{\infty}^{-1}}. \quad (41)$$

Since $\delta_k(\cdot - \alpha_i)$ converges to $\delta(\cdot - \alpha_i)$ in $H^{-1}(S_{\beta})$ and $h_i \in C_0^{\infty}(\mathbb{R})$ for $i = 1, \dots, j$, we see that $\lim_{k \rightarrow \infty} \|r_k\|_1 < \infty$ and $\lim_{k \rightarrow \infty} \|r_k\|_{\infty} < \infty$. Thus

$$\int_Q |\phi(\alpha_i, h_i)|^j d\mu < \infty. \quad (42)$$

The estimate (40) with $p = 1$ then follows from the Hölder inequality

$$\int_Q |\phi(\alpha_j, h_j) \cdots \phi(\alpha_1, h_1)| \, d\mu \leq \prod_{i=1}^j \left(\int_Q |\phi(\alpha_i, h_i)|^j \, d\mu \right)^{1/j}.$$

The higher L^p -estimates follow as well, as (α_k, h_k) may equal (α_l, h_l) , $k, l = 1, \dots, j$. $\Sigma_{[0, \alpha]}$ -measurability follows from the fact that a.) for all k there is an ϵ_k (the δ_k were chosen to have compact support) such that

$$\phi(\delta_k(\cdot - \alpha_i) \otimes h_i) \in \bigcap_{1 \leq p < \infty} L^p(Q, \Sigma_{[0, \alpha_i + \epsilon_k]}, d\mu)$$

and b.) the upper continuity of μ .

ii.) Now let $h_i \in C_0^\infty(\mathbb{R})$ and $\alpha_i \in S_\beta$, $1 \leq i \leq n$. Part ii.) follows from

$$\begin{aligned} & \lim_{l \rightarrow \infty} \int_Q \phi(\alpha_n, h_n) \cdots \phi(\alpha_1, h_1) \, d\mu_l \\ &= \lim_{k \rightarrow \infty} \frac{d^n}{d\lambda_1 \cdots d\lambda_n} (\Omega_C, W_{[-a, a]}(\sum_{i=1}^n \lambda_i (\delta_k(\cdot - \alpha_i) \otimes h_i)) \Omega_C) \Big|_{\lambda_i=0} \\ &= \int_Q \phi(\alpha_n, h_n) \cdots \phi(\alpha_1, h_1) \, d\mu, \end{aligned}$$

for $\text{supp } \delta_k(\cdot - \alpha_i) \otimes h_i \subset S_\beta \times [-a, a]$, $i = 1, \dots, n$, as for $s \leq -a \leq a \leq t$ the map $(s, t) \mapsto (\Omega_C, W_{[s, t]}(f) \Omega_C)$ is constant. \square

The existence of products of sharp time fields in $L^2(Q, \Sigma, d\mu)$ allows us to investigate their domains, taking advantage of their Euclidean heritage:

Proposition 3.2. *Let $h_i \in C_0^\infty(\mathbb{R})$, $1 \leq i \leq n$. Then*

i.) $\Omega_\beta \in \mathcal{D}(L)$ and $L\Omega_\beta = 0$;

ii.) If $\alpha_1, \dots, \alpha_n \geq 0$ and $\sum_{j=1}^n \alpha_j \leq \beta/2$, then

$$e^{-\alpha_{n-1}L} \phi_\beta(h_{n-1}) \cdots e^{-\alpha_1L} \phi_\beta(h_1) \Omega_\beta \in \mathcal{D}(\phi_\beta(h_n)) \quad (43)$$

and

$$\phi_\beta(h_n) e^{-\alpha_{n-1}L} \phi_\beta(h_{n-1}) \cdots e^{-\alpha_1L} \phi_\beta(h_1) \Omega_\beta \in \mathcal{D}(e^{-\alpha_nL}). \quad (44)$$

Moreover, the linear span of such vectors is dense in \mathcal{H}_β and

$$\begin{aligned} & e^{-\alpha_nL} \phi_\beta(h_n) e^{-\alpha_{n-1}L} \phi_\beta(h_{n-1}) \cdots e^{-\alpha_1L} \phi_\beta(h_1) \Omega_\beta \\ &= \mathcal{V} \left(U(\alpha_n, 0) \phi(0, h_n) U(\alpha_{n-1}, 0) \phi(0, h_{n-1}) \cdots U(\alpha_1, 0) \phi(0, h_1) \right); \end{aligned}$$

iii.) If $0 \leq \alpha_1 \leq \dots \leq \alpha_k \leq \beta/2$ and $\beta/2 \leq \alpha_{k+1} \leq \dots \leq \alpha_n \leq \beta$, then

$$\begin{aligned} & \int_Q \left(\prod_{j=1}^n \phi(\alpha_j, h_j) \right) d\mu \\ &= (e^{(\alpha_n - \beta)L} \phi_\beta(h_n) e^{(\alpha_{n-1} - \alpha_n)L} \phi_\beta(h_{n-1}) \dots e^{(\alpha_{k+1} - \alpha_{k+2})L} \phi_\beta(h_{k+1}) \Omega_\beta, \\ & \quad e^{-\alpha_1 L} \phi_\beta(h_1) e^{(\alpha_1 - \alpha_2)L} \phi_\beta(h_2) \dots e^{(\alpha_{k-1} - \alpha_k)L} \phi_\beta(h_k) \Omega_\beta). \end{aligned} \quad (45)$$

$$iv.) \quad \|e^{-(\beta/2)L} \phi_\beta(h_n) \dots \phi_\beta(h_1) \Omega_\beta\| = \|\phi_\beta(h_n) \dots \phi_\beta(h_1) \Omega_\beta\|.$$

Proof. Let us first note that (45) formally results from differentiating the following identity, which is a consequence of $e^{i\phi(0, h_j)} \in L^\infty(Q, \Sigma_{\{0\}}, d\mu)$ for $h_i \in C_{0, \mathbb{R}}^\infty(\mathbb{R})$ and the Osterwalder-Schrader reconstruction outlined in Section 2.3:

$$\begin{aligned} & \int_Q \left(\prod_{j=1}^n e^{i\phi(\alpha_j, h_j)} \right) d\mu \\ &= \int_Q R \left(\overline{U(\beta, 0) \prod_{j=k+1}^n e^{-i\phi(-\alpha_j, h_j)}} \right) \prod_{j=1}^k e^{i\phi(\alpha_j, h_j)} d\mu \\ &= \left(\mathcal{V}(U(\beta, 0) e^{-i\phi(-\alpha_n, h_n)} \dots e^{-i\phi(-\alpha_{k+1}, h_{k+1})}), \mathcal{V}(e^{i\phi(\alpha_k, h_k)} \dots e^{i\phi(\alpha_1, h_1)}) \right) \\ &= (e^{(\alpha_n - \beta)L} e^{-i\phi_\beta(h_n)} e^{(\alpha_{n-1} - \alpha_n)L} e^{-i\phi_\beta(h_{n-1})} \dots e^{(\alpha_{k+1} - \alpha_{k+2})L} e^{-i\phi_\beta(h_{k+1})} \Omega_\beta, \\ & \quad e^{-\alpha_1 L} e^{i\phi_\beta(h_1)} e^{(\alpha_1 - \alpha_2)L} e^{i\phi_\beta(h_2)} \dots e^{(\alpha_{k-1} - \alpha_k)L} e^{i\phi_\beta(h_k)} \Omega_\beta), \end{aligned}$$

for $1 \leq i \leq n$, and $0 \leq \alpha_1 \leq \dots \leq \alpha_k \leq \beta/2$ and $\beta/2 \leq \alpha_{k+1} \leq \dots \leq \alpha_n \leq \beta$. Note that inserting the identity $U(\beta, 0) = \mathbb{1}$ ensures that $(\beta - \alpha_i) \in [0, \beta/2]$ for $i = k+1, \dots, n$. However, we have to ensure that (45) is well-defined.

i.) See [31, Lemma 8.4]: $1 \in \mathcal{M}_\alpha$, thus $\Omega_\beta \in \mathcal{D}_\alpha$ and $e^{-\alpha L} \Omega_\beta = P(\alpha) \Omega_\beta = \Omega_\beta$ as $U(\alpha, 0)1 = 1$ for $0 \leq \alpha \leq \beta$;

ii.) The case $n = 1$, namely $\Omega_\beta \in \mathcal{D}(\phi_\beta(h_1))$ and

$$e^{-\alpha_1 L} \phi_\beta(h_1) \Omega_\beta \in \mathcal{H}_\beta \quad \text{for } 0 \leq \alpha_1 \leq \beta/2$$

was proven in [14]. In fact,

$$e^{-\alpha_1 L} \phi_\beta(h_1) \Omega_\beta \in \mathcal{D}(\phi_\beta(h_2)),$$

as $\phi(0, h_2)$ acts as a multiplication operator on $\phi(\alpha_1, h_1)$ and

$$\phi(0, h_2) \phi(\alpha_1, h_1) \in \mathcal{M}_{\beta/2 - \alpha_1}$$

by Lemma 3.1 i.). As $P(\alpha) \mathcal{D}_\gamma \subset \mathcal{D}_{\gamma - \alpha}$, it follows that

$$e^{-\alpha_2 L} \phi_\beta(h_2) e^{-\alpha_1 L} \phi_\beta(h_1) \Omega_\beta \in \mathcal{D}_{\beta/2 - \alpha_1 - \alpha_2}$$

and $\phi(0, h_3)\phi(\alpha_2, h_2)\phi(\alpha_1 + \alpha_2, h_1) \in \mathcal{M}_{\beta/2 - \alpha_1 - \alpha_2}$ implies

$$e^{-\alpha_2 L} \phi_\beta(h_2) e^{-\alpha_1 L} \phi_\beta(h_1) \Omega_\beta \in \mathcal{D}(\phi_\beta(h_3)).$$

Iterating this argument it follows that

$$\mathcal{V}(\phi(\alpha_k, h_k) \cdots \phi(\alpha_1 + \dots + \alpha_k, h_1)) \in \mathcal{D}_{\beta/2 - \gamma},$$

if $\sum_{i=1}^k \alpha_i \leq \gamma \leq \beta/2$. Thus (43) and (44) follow.

Next we prove that

$$e^{-\alpha_n L} \phi_\beta(h_n) e^{-\alpha_{n-1} L} \phi_\beta(h_{n-1}) \dots e^{-\alpha_1 L} \phi_\beta(h_1) \Omega_\beta$$

is dense in \mathcal{H}_β for $\alpha_1, \dots, \alpha_n \geq 0$ and $\sum_{j=1}^n \alpha_j \leq \beta/2$. Assume that, for $\Psi \in \mathcal{H}_\beta$ and all $f, g \in C_0^\infty(\mathbb{R})$,

$$\forall m, n \in \mathbb{N} : \quad (\Psi, \phi_\beta(f)^n e^{-\beta L/2} \phi_\beta(g)^m \Omega_\beta) = 0. \quad (46)$$

(Note that (46) is well-defined as a consequence of (44).) Then²

$$(\Psi, e^{i\phi_\beta(f)} e^{-\beta L/2} e^{i\phi_\beta(g)} \Omega_\beta) = 0. \quad (47)$$

But vectors of the form $e^{i\phi_\beta(f)} e^{-\beta L/2} e^{i\phi_\beta(g)} \Omega_\beta$, $f, g \in C_0^\infty(\mathbb{R})$, are dense [30, Theorem 11.2] in \mathcal{H}_β , and therefore (47) implies $\Psi = 0$, establishing the claim.

iii.) If $0 \leq \alpha_1 \leq \dots \leq \alpha_k \leq \beta/2$ and $\beta/2 \leq \alpha_{k+1} \leq \dots \leq \alpha_n \leq \beta$, then according to ii.)

$$\begin{aligned} & (e^{(\alpha_n - \beta)L} \phi_\beta(h_n) e^{(\alpha_{n-1} - \alpha_n)L} \phi_\beta(h_{n-1}) \dots e^{(\alpha_{k+1} - \alpha_{k+2})L} \phi_\beta(h_{k+1}) \Omega_\beta, \\ & e^{-\alpha_1 L} \phi_\beta(h_1) e^{-(\alpha_2 - \alpha_1)L} \phi_\beta(h_2) \dots e^{-(\alpha_k - \alpha_{k-1})L} \phi_\beta(h_k) \Omega_\beta) \end{aligned}$$

is well-defined and equals

$$\begin{aligned} & \left(\mathcal{V}(\phi(\beta - \alpha_n, h_n) \dots \phi(\beta - \alpha_{k+1}, h_{k+1})), \mathcal{V}(\phi(\alpha_k, h_k) \cdots \phi(\alpha_1, h_1)) \right) = \\ & = \int_Q R \left(\overline{\prod_{j=k+1}^n \phi(\beta - \alpha_j, h_j)} \right) \prod_{j=1}^k \phi(\alpha_j, h_j) \, d\mu \\ & = \int_Q R \left(\overline{U(\beta, 0) \prod_{j=k+1}^n \phi(-\alpha_j, h_j)} \right) \prod_{j=1}^k \phi(\alpha_j, h_j) \, d\mu \\ & = \int_Q R \left(\prod_{j=k+1}^n \phi(-\alpha_j, h_j) \right) \prod_{j=1}^k \phi(\alpha_j, h_j) \, d\mu \\ & = \int_Q \left(\prod_{j=k+1}^n \phi(\alpha_j, h_j) \right) \prod_{j=1}^k \phi(\alpha_j, h_j) \, d\mu \\ & = \int_Q \prod_{j=1}^n \phi(\alpha_j, h_j) \, d\mu. \end{aligned}$$

We made again use of $U(\beta, 0) = \mathbb{1}$, which holds by periodicity.

²From Prop. A6 i.) and Theorem 7.2 i) in [13] it follows that the vector valued function $s, t \mapsto \mathcal{V}(e^{is\phi(0,f)} e^{it\phi(\beta/2,g)})$ is entire for $f, g \in C_0^\infty(\mathbb{R})$.

iv.) By ii.) we have $\phi_\beta(h_n)\phi_\beta(h_{n-1})\cdots\phi_\beta(h_1)\Omega_\beta \in \mathcal{D}(e^{-\beta L/2})$. Now

$$\begin{aligned}
& \|e^{-\beta L/2}\phi_\beta(h_n)\phi_\beta(h_{n-1})\cdots\phi_\beta(h_1)\Omega_\beta\|^2 = \\
& = \|\mathcal{V}(U(\beta/2, 0)\phi(0, h_n)\cdots\phi(0, h_1))\|^2 \\
& = \int_Q \overline{U(\beta/2, 0)\phi(0, h_n)\cdots\phi(0, h_1)} RU(\beta/2, 0)\phi(0, h_n)\cdots\phi(0, h_1) d\mu \\
& = \int_Q \overline{\phi(0, h_n)\cdots\phi(0, h_1)} U(-\beta/2, 0) RU(\beta/2, 0)\phi(0, h_n)\cdots\phi(0, h_1) d\mu \\
& = \int_Q \phi(0, h_n)\cdots\phi(0, h_1) RU(\beta, 0)\phi(0, h_n)\cdots\phi(0, h_1) d\mu \\
& = \|\mathcal{V}(\phi(0, h_n)\cdots\phi(0, h_1))\|^2 = \|\phi_\beta(h_n)\phi_\beta(h_{n-1})\cdots\phi_\beta(h_1)\Omega_\beta\|^2,
\end{aligned}$$

again using $U(\beta, 0) = \mathbb{1}$.

□

The extension of these results to real times is our next objective. Given the self-adjoint operator $\phi_\beta(h)$, $h \in C_{0\mathbb{R}}^\infty(\mathbb{R})$, set

$$\phi_\beta(t, h) := e^{itL}\phi_\beta(h)e^{-itL}, \quad t \in \mathbb{R}.$$

The domain of the self-adjoint operator $\phi_\beta(t, h)$ is $e^{itL}\mathcal{D}(\phi_\beta(h))$. That products of field operators smeared out in time can be applied to the distinguished vector Ω_β will be shown in the final subsection.

3.2 Analyticity properties of the thermal Wightman distributions

We can now proceed by using the following remarkable consequence of the KMS condition established by Araki (see Equ. (1.27) in Lemma A, [1]).

Lemma 3.3 (Araki). *Let ω_β be a (τ, β) -KMS state over a von Neumann algebra \mathcal{R} . Let $(z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$ with $\Im z_j \geq 0$ for $j = 1, \dots, n-1$, and*

$$\begin{aligned}
& \Im z_1 + \dots + \Im z_{k-1} + \Im z'_k \leq \beta/2, & \Im z'_k & \geq 0, \\
& \Im z_{n-1} + \dots + \Im z_{k+1} + \Im z''_k \leq \beta/2, & \Im z''_k & \geq 0, \quad z'_k + z''_k = z_k.
\end{aligned}$$

Moreover, let $z_n = i - \sum_{i=1}^{n-1} z_i$. It follows that there exists some $j = 1, \dots, k$ such that for $A_0, \dots, A_n \in \mathcal{R}$ one has

$$\begin{aligned}
& (e^{iz'_k L} A_k^* e^{\overline{iz_{k-1} L}} A_{k-1}^* \cdots e^{\overline{iz_1 L}} A_1^* \Omega_\beta, e^{iz''_k L} A_{k+1} e^{iz_{k+1} L} A_{k+2} \cdots e^{iz_{n-1} L} A_n e^{iz_n L} \Omega_\beta) \\
& = (e^{\overline{iz'_{k+1} L}} A_{k+1}^* e^{\overline{iz_k L}} A_k^* \cdots e^{\overline{iz_{j+1} L}} A_{j+1}^* \Omega_\beta, \\
& \quad e^{iz''_{k+1} L} A_{k+2} e^{iz_{k+2} L} A_{k+3} \cdots e^{iz_{n-1} L} A_n e^{iz_n L} A_1 \cdots e^{iz_{j-1} L} A_j e^{iz_j L} \Omega_\beta),
\end{aligned}$$

with $\Im z'_{k+1}, \Im z''_{k+1} \geq 0$,

$$\begin{aligned} \Im z_j + \Im z_{j+1} + \dots + \Im z_k + \Im z''_{k+1} &\leq \beta/2, \\ \Im z_{j-1} + \Im z_{j-2} + \dots + \Im z_1 + \Im z_n + \dots + \Im z_{k+2} + \Im z'_{k+1} &\leq \beta/2, \end{aligned}$$

and $z'_{k+1} + z''_{k+1} = z_{k+1}$ for some $k = 0, 1, \dots, n-1$.

The identity stated applies to bounded operators of the fields, but the fields themselves may be approximated by bounded operators (see, *e.g.*, Equ. (55) below). After removing these approximations, one finds

$$\begin{aligned} &(e^{\overline{iz''_k}L} \phi(h_k) e^{\overline{iz_{k-1}}L} \phi(h_{k-1}) \dots e^{\overline{iz_1}L} \phi(h_1) \Omega_\beta, \\ &\quad e^{iz'_k L} \phi(h_{k+1}) e^{iz_{k+1}L} \phi(h_{k+2}) \dots e^{iz_{n-1}L} \phi(h_n) e^{iz_n L} \Omega_\beta) \\ &= \int_{\mathbb{R}^2} dx_1 \dots dx_n h_1(x_1) \dots h_n(x_n) \mathcal{W}_\beta^{(n-1)}(z_1, x_1 - x_2, \dots, z_{n-1}, x_{n-1} - x_n). \end{aligned}$$

Note that for $\Re z_1 = \dots = \Re z_n = 0$ the existence of the l.h.s. follows from Proposition 3.2 *ii.*). The extension to non-vanishing real parts will be discussed below. But before we do so, we choose sequences of absolutely integrable functions $h_i^{(k)} \in C_0^\infty(S_\beta)$, $i = 1, \dots, n$, tending to the Dirac distributions $\delta(\cdot - x_k)$ as $k \rightarrow \infty$. For $\Re z_i = 0$ and $\Im z_i > 0$, $i = 1, \dots, n-1$, the limit $k \rightarrow \infty$ exists and yields

$$\begin{aligned} &\mathcal{W}_\beta^{(n-1)}(z_1, x_1 - x_2, \dots, z_{n-1}, x_{n-1} - x_n) \\ &= (e^{\overline{iz''_k}L + i(x_k - x_{k+1})P} \phi(\delta) e^{\overline{iz_{k-1}}L + i(x_{k-1} - x_k)P} \phi(\delta) \dots e^{\overline{iz_1}L + i(x_1 - x_2)P} \phi(\delta) \Omega_\beta, \\ &\quad e^{iz'_k L} \phi(\delta) e^{iz_{k+1}L - i(x_{k+1} - x_{k+2})P} \phi(\delta) \dots e^{iz_{n-1}L - i(x_{n-1} - x_n)P} \phi(\delta) \Omega_\beta). \end{aligned}$$

Setting $x'_1 = x_1 - x_2$, $x'_2 = x_2 - x_3$, etc., this identity takes the following form

$$\begin{aligned} &\mathcal{W}_\beta^{(n-1)}(z_1, x'_1, \dots, z_{n-1}, x'_{n-1}) \\ &= (e^{\overline{iz''_k}L + ix'_k P} \phi(\delta) e^{\overline{iz_{k-1}}L + ix'_{k-1} P} \phi(\delta) \dots e^{\overline{iz_1}L + ix'_1 P} \phi(\delta) \Omega_\beta, \\ &\quad e^{iz'_k L - ix'_k P} \phi(\delta) e^{iz_{k+1}L - ix'_{k+1} P} \phi(\delta) \dots e^{iz_{n-1}L - ix'_{n-1} P} \phi(\delta) \Omega_\beta). \end{aligned}$$

We have set $x_k = x'_k + x''_k$, using the same ratio of the absolute values as in the splitting of $z_k = z'_k + z''_k$.

In particular,

$$\begin{aligned} &\|e^{iz'_k L - ix'_k P} \phi(\delta) e^{iz_{k+1}L - ix_{k+1}P} \phi(\delta) \dots e^{iz_{2k-1}L - ix_{2k-1}P} \phi(\delta) \Omega_\beta\|^2 \\ &= \mathcal{W}_\beta^{(2k-1)}(z_{2k-1}, x_{2k-1}, \dots, z_{k+1}, x_{k+1}, z_k, x_k, z_{k+1}, x_{k+1} \dots, z_{2k-1}, x_{2k-1}) \end{aligned}$$

with $z_k = 2z'_k$ and $x_k = 2x'_k$. It follows that the vector valued function

$$\begin{aligned} &(t_k, x_k, \dots, t_{n-1}, x_{n-1}) \mapsto \\ &\quad e^{it_k L - ix_k P} \phi(\delta) e^{it_{k+1}L - ix_{k+1}P} \phi(\delta) \dots e^{it_{2k-1}L - ix_{2k-1}P} \phi(\delta) \Omega_\beta \end{aligned}$$

can be analytically continued into the region

$$(Q^- \cup Q^+)^k - i\lambda_k V_\beta \times \dots \times \lambda_{2k-1} V_\beta, \quad (48)$$

with $\lambda_i > 0$ and $\sum_{i=k}^{2k-1} \lambda_i = 1/2$.

On the other hand, by first applying Lemma 3.3 and then removing the approximations in a similar manner as above, one can establish the following identity:

$$\begin{aligned} & \mathcal{W}_\beta^{(n-1)}(z_1, x_1, \dots, z_{n-1}, x_{n-1}) \\ &= \left(e^{\overline{iz'_{k+1}L + ix'_{k+1}P}} \phi(\delta) e^{\overline{iz_kL + ix'_kP}} \phi(\delta) \dots e^{\overline{iz_{j+1}L + ix'_{j+1}P}} \phi(\delta) \Omega_\beta, \right. \\ & \left. e^{iz''_{k+1}L} \phi(\delta) e^{iz_{k+2}L - ix_{k+2}P} \dots e^{iz_nL - ix_nP} \phi(\delta) e^{iz_1L - ix_1P} \dots e^{iz_{j-1}L - ix_{j-1}P} \phi(\delta) \Omega_\beta \right). \end{aligned} \quad (49)$$

where $x_n = -x_{n-1} - x_{n-2} - \dots - x_1$.

Clearly, there are $n-1$ different expressions for $\mathcal{W}_\beta^{(n-1)}$ which can be gained by repeated application of Lemma 3.3.

Theorem 3.4. *The thermal Wightman functions $\mathcal{W}_\beta^{(n-1)}$ introduced in (36) are analytic in the product of domains*

$$(\lambda_1 \mathcal{T}_\beta) \times \dots \times (\lambda_{n-1} \mathcal{T}_\beta), \quad \mathcal{T}_\beta := \mathbb{R}^2 - iV_\beta, \quad \sum_{j=1}^{n-1} \lambda_j = 1, \quad (50)$$

and $\lambda_j > 0$, $j = 1, \dots, n-1$. In fact, one can take the union over these λ_j 's.

Proof. We recall that $\mathcal{W}_\beta^{(n-1)}$ is an analytic function in the domain (37). Within the domain (37), the Cauchy Schwarz inequality yields

$$\begin{aligned} & \left| \mathcal{W}_\beta^{(n-1)}(z_1, x_1 + iy_1, \dots, z_{n-1}, x_{n-1} + iy_{n-1}) \right| \\ & \leq \left\| e^{\overline{iz'_kL + y'_kP}} \phi_\beta(\delta_0) e^{\overline{iz_{k-1}L + i(x_{k-1} - iy_{k-1})P}} \dots e^{\overline{iz_1L + i(x_1 - iy_1)P}} \phi_\beta(\delta_0) \Omega_\beta \right\| \\ & \quad \times \left\| e^{iz''_kL - i(x_k + iy''_k)P} \phi_\beta(\delta_0) \dots e^{iz_{n-1}L - i(x_{n-1} + iy_{n-1})P} \phi_\beta(\delta_0) \Omega_\beta \right\|. \end{aligned}$$

Here $y_k = y'_k + y''_k$ is split according to the same ratio as $z_k = z'_k + z''_k$.

As L and P are self-adjoint operators, the spectral theorem implies that the vector valued function

$$(z''_k, w''_k) \mapsto e^{iz''_kL - iw''_kP} \phi_\beta(\delta_0) \dots e^{iz_{n-1}L - i(x_{n-1} + iy_{n-1})P} \phi_\beta(\delta_0) \Omega_\beta$$

is analytic in the domain $(z''_k, w''_k) \in \mathbb{R}^2 + i\frac{\lambda_k}{2} V_\beta$ (as the norm of the vector is preserved by applying the unitary $e^{i\Re z''_kL - i\Im w''_kP}$). And consequently, the function $\mathcal{W}_\beta^{(n-1)}$ extends to an analytic function in the domain

$$\begin{aligned} & \left((Q^- \cup Q^+) - i\lambda_1 V_\beta \right) \times \dots \times \left((Q^- \cup Q^+) - i\lambda_{k-1} V_\beta \right) \\ & \times \left(\mathbb{R}^2 - i\lambda_k V_\beta \right) \times \left((Q^- \cup Q^+) - i\lambda_{k+1} V_\beta \right) \times \dots \times \left((Q^- \cup Q^+) - i\lambda_{n-1} V_\beta \right), \end{aligned} \quad (51)$$

with $\lambda_i > 0$ and $\sum_{i=1}^{n-1} \lambda_i = 1$. Using the expression (49) one can replace k by $k + 1$ in (51). Iterating this procedure (eventually renaming the variables) and applying Hartogs' theorem [49, p. 30] one concludes that $\mathcal{W}_\beta^{(n-1)}$ is analytic in the domain (50). \square

3.3 Temperedness of the thermal Wightman distributions

For a quantum system confined to a box, Ruelle [43][44] has used a Hölder inequality, which applies to the trace in the Gibbs state. It was pointed out by Fröhlich [10] that this Hölder inequality is crucial in the present context.

Theorem 3.5 (Hölder inequality). *Let ω_β be a (τ, β) -KMS state over a von Neumann algebra \mathcal{R} . Define for, $p \in \mathbb{N}$ and $A \in \mathcal{R}^+$,*

$$\|A\|_p := \omega_\beta \left(\underbrace{e^{itL/p} A \cdots e^{itL/p} A}_{p \text{ times}} \right) \Big|_{t=i\beta}^{1/p}.$$

Let $(z_1, \dots, z_n) \in \mathbb{C}^n$ with $0 \leq \Re z_j$, $\sum_{j=1}^m \Re z_j \leq 1/2$ and $\sum_{j=m+1}^n \Re z_j \leq 1/2$, and let p_j be the smallest, positive integer such that $\frac{1}{p_j} \leq \min\{\Re z_{j+1}, \Re z_j\}$, with $z_{n+1} = z_n$ and $z_0 = z_1$. Then

$$\left| \omega_\beta (A_n e^{it_n \beta L} \cdots A_1 e^{it_1 \beta L} A_0) \Big|_{t_j = iz_j} \right| \leq \|A_0\|_{p_0} \cdots \|A_n\|_{p_n} \quad (52)$$

for all $A_0, \dots, A_n \in \mathcal{R}^+$. (The subscript $\Big|_{t_j = iz_j}$ indicates the analytic continuation from t_j to iz_j , $j = 1, \dots, n$.)

Proof. The proof of this results relies on the theory of non-commutative L^p -spaces and is given in [26]. \square

Note that because of the time-invariance of the KMS state, the r.h.s. in (52) does not depend on $\Im z_i$, $i = 1, \dots, n$.

Proposition 3.6. *For $0 \leq \epsilon \leq 1$ fixed there exist constants $c_1, c_2 > 0$ such that*

$$\pm \phi_C(g) \leq c_1 \|g\|_{H^{-\frac{1}{2}-\frac{\epsilon}{2}}(S_\beta)} (H_C + c_2)^{\frac{1}{2}+\epsilon} \quad (53)$$

for all $g \in H^{-\frac{1}{2}-\frac{\epsilon}{2}}(S_\beta)$.

Proof. Set $H_0 = d\Gamma(\nu)$. It is sufficient to prove that

$$A(g) \equiv (H_0 + 1)^{-\frac{1}{4}-\frac{\epsilon}{2}} \phi_C(\nu^{\frac{1}{2}+\frac{\epsilon}{2}} g) (H_0 + 1)^{-\frac{1}{4}-\frac{\epsilon}{2}}$$

is a bounded operator on Fock space, uniformly bounded for $\|g\|_2 \leq 1$. The first order estimate (see, e.g., [42, Equ. (2.21)])

$$(H_0 + 1) \leq c_3 (H_C + c_2) \quad \text{for } c_2, c_3 \gg 1$$

and operator monotonicity of the map $\lambda \mapsto \lambda^\alpha$ for $0 \leq \alpha \leq 1$ (see, e.g., [28, Example 4.6.46]) then ensure the fractional ϕ -bound (53).

We now follow ideas of Rosen (see, *e.g.*, [42, Proof of Lemma 6.2]). We show that $A(g)$ is a bounded bilinear form in Fock space. The desired operator extension then follows from the Riesz representation theorem. It is sufficient to show that, for $\|g\|_2 \leq 1$,

$$|(\Phi, A(g)\Psi)| \leq c_4 \|\Phi\| \cdot \|\Psi\|,$$

for Φ, Ψ arbitrary vectors on Fock space and $c_4 > 0$ a constant. Now (see (27))

$$\begin{aligned} |(\Phi, A(g)\Psi)| &\leq \frac{1}{\sqrt{2}} \left(|((H_0 + 1)^{-\frac{1}{4} - \frac{\epsilon}{2}} \Phi, a^*(\nu^{\frac{\epsilon}{2}} g)(H_0 + 1)^{-\frac{1}{4} - \frac{\epsilon}{2}} \Psi)| \right. \\ &\quad \left. + |(a^*(\nu^{\frac{\epsilon}{2}} \bar{g})(H_0 + 1)^{-\frac{1}{4} - \frac{\epsilon}{2}} \Phi, (H_0 + 1)^{-\frac{1}{4} - \frac{\epsilon}{2}} \Psi)| \right). \end{aligned}$$

Since $H_C^{(0)}$ commutes with the number operator, and both terms are of the same structure, it is sufficient to prove that for $\Phi_n \in \mathcal{H}_C^{(n)}$ and $\Psi_{n-1} \in \mathcal{H}_C^{(n-1)}$ with $\|\Phi_n\|, \|\Psi_{n-1}\| \leq 1$

$$\begin{aligned} &|((H_C^{(0)} + 1)^{-\frac{1}{4} - \frac{\epsilon}{2}} \Phi_n, a^*(\nu^{\frac{\epsilon}{2}} g)(H_C^{(0)} + 1)^{-\frac{1}{4} - \frac{\epsilon}{2}} \Psi_{n-1})| \\ &\leq \|(n+1)^{-\frac{1}{4}} \Phi_n\| \cdot \|a^*(\nu^{\frac{\epsilon}{2}} g)(H_C^{(0)} + 1)^{-\frac{1}{4} - \frac{\epsilon}{2}} \Psi_{n-1}\| \end{aligned}$$

is uniformly bounded in n . For simplicity it is assumed (in the second inequality below) that the mass $m \geq 1$, so that $\sum_{i=1}^{n-1} \nu(k_i) + 1 \geq n$; otherwise one is left with yet another n -independent constant. Now

$$\begin{aligned} &(n+1)^{-1/2} \|\Phi_n\|^2 \|a^*(\nu^{\frac{\epsilon}{2}} g)(H_C^{(0)} + 1)^{-\frac{1}{4} - \frac{\epsilon}{2}} \Psi_{n-1}\|^2 \\ &\leq \frac{n}{(n+1)^{1/2}} \|\Phi_n\|^2 \\ &\quad \times \frac{1}{\beta^n} \int \prod_{j=1}^n \frac{dk_j}{\nu(k_j)} \left| \frac{\nu(k_n)^{\frac{\epsilon}{2}} \tilde{g}(k_n)}{\left(\sum_{i=1}^{n-1} \nu(k_i) + 1\right)^{1/4 + \epsilon/2}} \tilde{\Psi}_{n-1}(k_1, k_2, \dots, k_{n-1}) \right|^2 \\ &\leq \frac{n}{(n+1)^{1/2} n^{1/2}} \|\Phi_n\|^2 \\ &\quad \times \frac{1}{\beta^n} \int \prod_{j=1}^n \frac{dk_j}{\nu(k_j)} \left| \left(\frac{\nu(k_n)}{\sum_{i=1}^{n-1} \nu(k_i) + 1} \right)^{\frac{\epsilon}{2}} \tilde{g}(k_n) \tilde{\Psi}_{n-1}(k_1, k_2, \dots, k_{n-1}) \right|^2 \\ &\leq \|g\|_2^2 \|\Phi_n\|^2 \|\Psi_{n-1}\|^2, \end{aligned}$$

which establishes the claim. \square

The Euclidean time zero field $\phi(0, h) \in L^p(Q, \Sigma, d\mu)$, $1 \leq p < \infty$, can be approximated by a sequence of functions in $L^\infty(Q, \Sigma, d\mu)$. The latter can be decomposed in their positive and negative part. Define, for $h \in \mathcal{S}_\mathbb{R}(\mathbb{R})$ and $\alpha \in [0, \beta)$,

$$\phi_\pm^{(\ell)}(\alpha, h) = \begin{cases} \pm \phi(0, h) & \text{if } 0 \leq \pm \phi(\alpha, h) \leq \ell \\ 0 & \text{otherwise.} \end{cases} \quad (54)$$

It follows from [45, Lemma 3.5] that $\phi_+^{(\ell)}(0, h) - \phi_-^{(\ell)}(0, h)$ converges to $\phi(0, h)$ as $\ell \rightarrow \infty$ in any L_q -norm with $q < p$. We can use this result to define approximations for the thermal time-zero field $\phi_\beta(h)$: set

$$\phi_\ell^\pm(h) := \pi_\beta(\phi_\pm^{(\ell)}(0, h)) \quad (55)$$

and $\phi_\ell^\pm(t, h) := e^{itL}\phi_\ell^\pm(h)e^{-itL}$, $t \in \mathbb{R}$.

Lemma 3.7. *For $h \in \mathcal{S}_\mathbb{R}(\mathbb{R})$ and $p \in \mathbb{N}$ even, the expressions*

$$\|h\|_p := \max \left\{ \lim_{\ell \rightarrow \infty} \|\phi_\ell^+(t, h)\|_p, \lim_{\ell \rightarrow \infty} \|\phi_\ell^-(t, h)\|_p \right\} \quad (56)$$

are bounded from above by $\sqrt[p]{p!} \cdot |h|_{\mathcal{S}}$, for some Schwartz norm $|\cdot|_{\mathcal{S}}$.

Proof. Set $\phi_\pm(\alpha, h) = \lim_{\ell \rightarrow \infty} \phi_\pm^{(\ell)}(\alpha, h)$, $\alpha \in [0, \beta]$. We have

$$\begin{aligned} \|h\|_p^p &= \max \left\{ \lim_{\ell \rightarrow \infty} \int_Q \prod_{k=1}^p \phi_+^{(\ell)}\left(\frac{k\beta}{p}, h\right) d\mu, \lim_{\ell \rightarrow \infty} \int_Q \prod_{k=1}^p \phi_-^{(\ell)}\left(\frac{k\beta}{p}, h\right) d\mu \right\} \\ &= \max \left\{ \int_Q \prod_{k=1}^p \phi_+\left(\frac{k\beta}{p}, h\right) d\mu, \int_Q \prod_{k=1}^p \phi_-\left(\frac{k\beta}{p}, h\right) d\mu \right\} \\ &\leq \max \left\{ \int_Q \phi_+(0, h)^p d\mu, \int_Q \phi_-(0, h)^p d\mu \right\}. \end{aligned} \quad (57)$$

In the first line we have used the definition of the norm $\|\cdot\|_p$, in the third line we have used the Hölder inequality for $L^p(Q, \Sigma, d\mu)$ and translation invariance of $d\mu$.

Since $p \in \mathbb{N}$ is even and the supports of $\phi_+(0, h)$ and $\phi_-(0, h)$ are disjoint,

$$\begin{aligned} \int_Q \phi_\pm(0, h)^p d\mu &\leq \int_{\text{supp } \phi_+(0, h)} \phi_+(0, h)^p d\mu + \int_{\text{supp } \phi_-(0, h)} \phi_-(0, h)^p d\mu \\ &= \int_Q \phi(0, h)^p d\mu. \end{aligned} \quad (58)$$

Taking advantage of the fractional ϕ -bound (53) we can apply [13, Lemma A.7, p. 167], which states that there is a constant $c > 0$ such that

$$\left| \int d\mu e^{i\lambda\phi(\delta_k \otimes h)} \right| \leq e^{c|\Im\lambda|^\varkappa \|r_k\|_\varkappa^\varkappa} \quad (59)$$

for $\lambda \in \mathbb{C}$, $\varkappa = (\frac{1}{2} - \epsilon)^{-1}$, $0 < \epsilon < 1/2$ and

$$r_k(x) := |h(x)| \|\delta_k\|_{H^{-\frac{1}{2} - \frac{\epsilon}{2}}(S_\beta)}.$$

The limit $k \rightarrow \infty$ exists, as the Dirac δ -function is in all Sobolev spaces H^q for $q < -1/2$. Denote $r(x) := \lim_{k \rightarrow \infty} r_k(x)$. Applying Cauchy's formula³ on the

³As mentioned before, Prop. A6 i.) and Theorem 7.2 i) in [13] imply that the map $\lambda \mapsto \int d\mu e^{i\lambda\phi(\delta_k \otimes h)}$ is entire.

circle of radius R centred around $\lambda = 0$ yields

$$\int_Q \phi(0, h)^p \, d\mu \leq p! R^{-p} e^{cR^\varkappa} \|r\|_\varkappa^p .$$

Optimizing this bound w.r.t. R yields

$$\int_Q \phi(0, h)^p \, d\mu \leq p! \left(\frac{c\varkappa e}{p} \right)^{p/\varkappa} \|r\|_\varkappa^p \leq p! |h|_{\mathcal{S}}^p . \quad (60)$$

Here we have used $\sup_{p \in \mathbb{N}} \left(\frac{c\varkappa e}{p} \right)^{p/\varkappa} < \infty$ and the fact that

$$\|r\|_\varkappa = \|\delta\|_{H^{-\frac{1}{2}-\frac{\varepsilon}{2}}(S_\beta)} \left(\int |h(x)|^\varkappa \, dx \right)^{1/\varkappa}$$

can be estimated by a Schwartz semi-norm if $h \in \mathcal{S}(\mathbb{R})$ and $\varkappa > 2$. Combining (58) and (60) we arrive at

$$\|h\|_p^p \leq p! |h|_{\mathcal{S}}^p ,$$

which establishes the lemma. \square

Remark 3.8. *Using the ϕ -bound (28) one can use the equation preceding [13, Equ. (A.9), p. 165] to arrive at Fröhlich's bound*

$$\int e^{\pm\phi(g \otimes h)} \, d\mu \leq e^{c \int_{\mathbb{R}} |h(x)|^2 C_\beta(g, g)} , \quad g \in H^{-1/2}(S_\beta) , \quad h \in L_{\mathbb{R}}^2(\mathbb{R}) ,$$

stated (for the special case $g = \mathbb{1}_{[0,1]}$ a characteristic function) in [10, Equ. (7)]. However, using only this bound, we were unable to establish the existence of the products estimated in Lemma 3.7.

Theorem 3.9. *The thermal Wightman functions*

$$\mathfrak{W}_\beta^{(n-1)}(t_1 - t_2, x_1 - x_2, \dots, t_{n-1} - t_n, x_{n-1} - x_n)$$

are tempered distributions, which satisfy the relativistic KMS condition, i.e., they

- i.) are the boundary values of functions $\mathcal{W}_\beta^{(n-1)}$ analytic in the interior of the product of domains*

$$(\lambda_1 \mathcal{T}_\beta) \times \dots \times (\lambda_{n-1} \mathcal{T}_\beta), \quad \mathcal{T}_\beta := \mathbb{R}^2 - iV_\beta, \quad (61)$$

where $\lambda_i > 0$, $i = 1, \dots, n-1$ and $\sum_{i=1}^{n-1} \lambda_i = 1$;

ii.) satisfy the following boundary condition: for any time-like vector $e = (e_0, e_1) \in V^+$

$$\begin{aligned} & \lim_{e \rightarrow 0} \mathcal{W}_\beta^{(n-1)}(s_1, y_1, \dots, s_{k-1}, y_{k-1}, s_k - ie_0, y_k - ie_1, \dots, s_{n-1}, y_{n-1}) \\ &= \mathfrak{W}_\beta^{(n-1)}(s_1, y_1, \dots, s_{n-1}, y_{n-1}) \end{aligned} \quad (62)$$

and

$$\begin{aligned} & \lim_{e \rightarrow 0} \mathcal{W}_\beta^{(n-1)}(s_1, y_1, \dots, s_{k-1}, y_{k-1}, s_k - i\beta + ie_0, y_k + ie_1, \dots, s_{n-1}, y_{n-1}) \\ &= \mathfrak{W}_\beta^{(n-1)}(s_k, y_k, \dots, s_n, y_n, s_1, y_1, \dots, s_{k-2}, y_{k-2}) \end{aligned} \quad (63)$$

for all $(s_1, y_1, \dots, s_{n-1}, y_{n-1}) \in \mathbb{R}^{2(n-1)}$. We have set $s_k = t_k - t_{k+1}$ and $y_k = x_k - x_{k+1}$, $1 \leq k < n$, and in addition, $s_n = t_n - t_1$ and $y_n = x_n - x_1$.

Proof. The domain of analyticity of the thermal Wightman functions $\mathcal{W}_\beta^{(n-1)}$ stated in *i.*) was established in Theorem 3.4. Thus it remains to establish *ii.*). We first prove that the boundary values in the distinguished time direction $(1, 0)$ define tempered distributions. In the sequel, we show that the boundary values in a time-like direction $e = (e_0, e_1)$ coincide with them.

Within their domain of analyticity the Wightman functions can be approximated by the expectation values of bounded operators: let $h \in C_{0\mathbb{R}}^\infty(\mathbb{R})$ and set

$$\phi_\ell(t, h) := \phi_\ell^+(t, h) - \phi_\ell^-(t, h), \quad t \in \mathbb{R}.$$

To ease the notation, put

$$\begin{aligned} \underline{s} &= (t_1 - t_2, \dots, t_{n-1} - t_n), \\ \underline{\alpha} &= (\alpha_1 - \alpha_2, \dots, \alpha_{n-1} - \alpha_n), \\ \underline{y} &= (x_1 - x_2, \dots, x_{n-1} - x_n). \end{aligned}$$

Now define, using the nuclear theorem, the kernels $\mathbf{W}_{\ell_1, \dots, \ell_n}(\underline{s} - i\underline{\alpha}, \underline{y})$ with $0 < \alpha_n < \dots < \alpha_1 < \beta$, by requiring that

$$\begin{aligned} & \int dx_1 \cdots dx_n \mathbf{W}_{\ell_1, \dots, \ell_n}(\underline{s} - i\underline{\alpha}, \underline{y}) h_1(x_1) \cdots h_n(x_n) \\ & \doteq \omega_\beta(\phi_{\ell_1}(r_1, h_1) \cdots \phi_{\ell_n}(r_n, h_n)) \upharpoonright_{r_i = t_i + i\alpha_i} \end{aligned}$$

for all $h_1, \dots, h_n \in C_{0\mathbb{R}}^\infty(\mathbb{R})$. As before, the subscript $\upharpoonright_{r_i = t_i + i\alpha_i}$ indicates the analytic continuation from t_i to $t_i + i\alpha_i$, $i = 1, \dots, n$.

Clearly,

$$\begin{aligned} & \lim_{\ell_i \rightarrow \infty} \mathbf{W}_{\ell_1, \dots, \ell_n}(-i\underline{\alpha}, \underline{y}) \\ &= \lim_{\ell_i \rightarrow \infty} \int_Q \phi^{(\ell_1)}(\alpha_1, x_1) \cdots \phi^{(\ell_n)}(\alpha_n, x_n) d\mu \\ &= \mathcal{W}_\beta^{(n-1)}(-i(\alpha_1 - \alpha_2), y_1, \dots, -i(\alpha_{n-1} - \alpha_n), y_{n-1}). \end{aligned}$$

We have used the notation introduced in paragraph *ii.*), Theorem 3.9. In addition, we have set $\phi^{(\ell)}(\alpha, h) := \phi_+^{(\ell)}(\alpha, h) - \phi_-^{(\ell)}(\alpha, h)$. Since the functions involved are all bounded on compact sets of their domain of analyticity, it follows that for $0 < \alpha_n < \dots < \alpha_1 < \beta$, $i = 1, \dots, n$,

$$\begin{aligned} & \lim_{\ell_i \rightarrow \infty} \mathbf{W}_{\ell_1, \dots, \ell_n}(\underline{s} - i\underline{\alpha}, \underline{y}) \\ &= \mathcal{W}_\beta^{(n-1)}(s_1 - i(\alpha_1 - \alpha_2), y_1, \dots, s_{n-1} - i(\alpha_{n-1} - \alpha_n), y_{n-1}), \end{aligned} \quad (64)$$

uniformly on compact sets in their domain of analyticity. We denote this limit by $\mathbf{W}(\underline{s} - i\underline{\alpha}, \underline{y})$.

We now show that there exist uniform bounds (independent of $\ell_i, i = 1, \dots, n$) as we approach the real boundary of the domain of analyticity: by construction

$$\begin{aligned} & \int dx_1 \cdots dx_n \mathbf{W}(\underline{s} - i\underline{\alpha}, \underline{y}) h_1(x_1) \cdots h_n(x_n) \\ &= \lim_{\ell_i \rightarrow \infty} \omega_\beta(\phi_{\ell_1}(r_1, h_1) \cdots \phi_{\ell_n}(r_n, h_n)) \Big|_{r_i = t_i + i\alpha_i} \end{aligned} \quad (65)$$

for $0 < \alpha_n < \dots < \alpha_1 < \beta$. Now the Hölder inequality (52) implies that each of the 2^n terms arising from the linear polar decomposition can be estimated: for $0 < \alpha_n < \dots < \alpha_1 < \beta$, $i = 1, \dots, n$, we have

$$\begin{aligned} & \lim_{\ell_i \rightarrow \infty} \left| \omega_\beta(\phi_{\ell_1}^\pm(r_1, h_1) \cdots \phi_{\ell_n}^\pm(r_n, h_n)) \Big|_{r_i = t_i + i\alpha_i} \right| \\ & \leq \lim_{\ell_i \rightarrow \infty} \|\phi_{\ell_1}^\pm(t_1, h_1)\|_{p_1} \cdots \|\phi_{\ell_n}^\pm(t_n, h_n)\|_{p_n} \\ & \leq \frac{p_1}{2} \cdots \frac{p_n}{2} \cdot |h_1|_{\mathcal{S}} \cdots |h_n|_{\mathcal{S}}, \quad t_1, \dots, t_n \in \mathbb{R}, \end{aligned} \quad (66)$$

with $p_i \equiv p_i(\underline{\alpha})$ the smallest integer such that

$$\frac{1}{p_i(\underline{\alpha})} < \frac{1}{\beta} \min \{ \alpha_{i+1} - \alpha_i, \alpha_i - \alpha_{i-1} \}, \quad i = 1, \dots, n.$$

(Setting $\alpha_0 = \beta - \alpha_n$ and $\alpha_{n+1} = \beta - \alpha_1$.) In the second inequality in (66) we have used Lemma 3.7 to conclude that for p sufficiently large⁴

$$\lim_{\ell \rightarrow \infty} \|\phi_\ell^\pm(t, h)\|_p \leq \|h\|_p \leq \sqrt[p]{p!} \cdot |h|_{\mathcal{S}} < \frac{p}{2} \cdot |h|_{\mathcal{S}}.$$

Thus, for $0 < \alpha_n < \dots < \alpha_1 < \beta$,

$$\begin{aligned} & \int dx_1 \cdots dx_n \mathbf{W}(\underline{s} - i\underline{\alpha}, \underline{y}) h_1(x_1) \cdots h_n(x_n) \\ & \leq \frac{p_1(\underline{\alpha})}{2} \cdots \frac{p_n(\underline{\alpha})}{2} \cdot |h_1|_{\mathcal{S}} \cdots |h_n|_{\mathcal{S}}, \quad t_1, \dots, t_n \in \mathbb{R}. \end{aligned}$$

Note that $p_i(\lambda\underline{\alpha}) \sim \lambda^{-1} p_i(\underline{\alpha})$ for $\lambda \searrow 0$.

⁴Recall that $p! < (p/2)^p$ for $p \geq 6$.

We will now show, following ideas in [40, p. 24], that this bound ensures that the boundary values exist as tempered distributions as $\alpha_i \searrow 0$: define, for $\lambda \in (0, 1]$ fixed, a tempered distribution $T_{\underline{\alpha}}(\lambda) \in \mathcal{S}'(\mathbb{R}^{n-1})$ by setting, for $g \in \mathcal{S}(\mathbb{R}^{n-1})$,

$$T_{\underline{\alpha}}(\lambda)(g) := \int_{\mathbb{R}^{n-1}} d\underline{s} g(\underline{s}) \int dx_1 \cdots dx_n \mathbf{W}(\underline{s} - i\lambda\underline{\alpha}, \underline{y}) h(x_1) \cdots h(x_n).$$

Let $T_{\underline{\alpha}}^{(k)}(\lambda)$, $k = 1, 2, \dots$, denote the k -th distributional derivative, specified by setting

$$\begin{aligned} T_{\underline{\alpha}}^{(k)}(\lambda)(g) & \\ &= \int_{\mathbb{R}^{n-1}} d\underline{s} \int dx_1 \cdots dx_n \mathbf{W}(\underline{s} - i\lambda\underline{\alpha}, \underline{y}) h_1(x_1) \cdots h_n(x_n) \left(i\underline{\alpha} \cdot \frac{\partial}{\partial \underline{s}} \right)^k g(\underline{s}). \end{aligned} \quad (67)$$

Thus, by the fundamental theorem of calculus,

$$\begin{aligned} T_{\underline{\alpha}}(\lambda) &= T_{\underline{\alpha}}(1) + \sum_{j=1}^{k-1} Q_j(\lambda) T_{\underline{\alpha}}^{(j)}(1) \\ &\quad - \int_{\lambda}^1 d\lambda_k \int_{\lambda_k}^1 d\lambda_{k-1} \cdots \int_{\lambda_2}^1 d\lambda_1 T_{\underline{\alpha}}^{(k)}(\lambda_1). \end{aligned} \quad (68)$$

The Q_j 's in (68) are suitable polynomials. The limit $\lambda \downarrow 0$ in (68) can be taken, provided that there exists a k such that

$$\lim_{\lambda \downarrow 0} \left| \int_{\lambda}^1 d\lambda_k \int_{\lambda_k}^1 d\lambda_{k-1} \cdots \int_{\lambda_2}^1 d\lambda_1 T_{\underline{\alpha}}^{(k)}(\lambda_1)(g) \right| < c \cdot \|g\|_{\mathcal{S}}, \quad (69)$$

with $c > 0$ a constant and $\|g\|_{\mathcal{S}}$ a Schwartz semi-norm. This is done by estimating $T_{\underline{\alpha}}^{(k)}(\lambda)(g)$ as given in (67) for $\lambda \in (0, 1]$: choose some $m \in \mathbb{N}$ large enough so that $\int_{\mathbb{R}^{n-1}} d\underline{s} (1 + |\underline{s}|)^{-m} < \infty$. Then, for $\lambda \in (0, 1]$,

$$\begin{aligned} \left| T_{\underline{\alpha}}^j(\lambda)(g) \right| &= C \sup_{\underline{t} \in \mathbb{R}^{n-1}} |(1 + |\underline{s}|)^m| \left| \left(i\underline{\alpha} \cdot \frac{\partial}{\partial \underline{s}} \right)^j g(\underline{s}) \right| \\ &\quad \times p_1(\lambda\underline{\alpha}) \cdots p_n(\lambda\underline{\alpha}) \cdot |h_1|_{\mathcal{S}} \cdots |h_n|_{\mathcal{S}} \\ &\leq C' \cdot \lambda^{-n}, \quad C, C' > 0. \end{aligned} \quad (70)$$

Note that

$$\lim_{\lambda \downarrow 0} \left| \int_{\lambda}^1 d\lambda_k \int_{\lambda_k}^1 d\lambda_{k-1} \cdots \int_{\lambda_2}^1 d\lambda_1 \lambda_1^{-n} \right| < \infty \quad (71)$$

for k sufficiently large, *i.e.*, $k \geq n + 1$. Combining (69), (70), and (71) one concludes that the limit of $T_{\underline{\alpha}}(\lambda)$ exists as $\lambda \downarrow 0$ and that each term in the limit is less than or equal to a constant times an $\mathcal{S}(\mathbb{R}^{n-1})$ -seminorm of g . Thus

$$\mathbf{W}(\underline{s} - i\underline{\alpha}, \underline{x}) = \mathcal{W}_{\beta}^{(n-1)}(s_1 - i(\alpha_1 - \alpha_2), y_1, \dots, s_{n-1} - i(\alpha_{n-1} - \alpha_n), y_{n-1}).$$

converges in $\mathcal{S}'(\mathbb{R}^{n-1})$ as $\underline{\alpha} \downarrow 0$ to a tempered distribution. The latter is denoted by $\mathfrak{W}_\beta^{(n-1)}(t_1 - t_2, x_1 - x_2, \dots, t_{n-1} - t_n, x_{n-1} - x_n)$.

Now suppose that $e = (\tau_1, z_1, \dots, \tau_{n-1}, z_{n-1}) \in \mathbb{R}^{2(n-1)}$ is an $(n-1)$ -tuple of time-like unit vectors $(\tau_i, z_i) \in V^+$, and that $\tilde{h}_i \in C_{0\mathbb{R}}^\infty(\mathbb{R})$, $i = 1, \dots, n$. To ease the notation we set

$$\begin{aligned} \underline{h}(\underline{x}) &= h_1(x_1) \cdots h_n(x_n) , \\ \underline{\alpha}_\tau &= (\tau_1(\alpha_1 - \alpha_2), \dots, \tau_{n-1}(\alpha_{n-1} - \alpha_n)) . \end{aligned}$$

Then

$$\begin{aligned} T_{\underline{\alpha}e}(\lambda)(g) &\doteq \int_{\mathbb{R}^{n-1}} d\underline{s} g(\underline{s}) \int dx_1 \cdots dx_n h_1(x_1) \cdots h_n(x_n) \\ &\quad \times \mathbf{W}((\underline{s}, \underline{y}) - i\lambda\underline{\alpha}e + i\lambda\underline{\alpha}(e - (1, 0))) \\ &= \int_{\mathbb{R}^{n-1}} d\underline{s} g(\underline{s}) \int dx_1 \cdots dx_n h(x_1 + i\lambda z_1) \cdots h(x_{n-1} + i\lambda z_{n-1}) h(x_n) \\ &\quad \times \mathbf{W}((\underline{s} - i\lambda\underline{\alpha}_\tau, \underline{y})) , \end{aligned}$$

where we have used the fact that the h_i 's, $i = 1, \dots, n$, are entire and the estimates in the Paley-Wiener theorem (Theorem IX.11 [40]) to shift the hyperplane of integration in second equality.

Since $\tilde{h}_i \in C_{0\mathbb{R}}^\infty(\mathbb{R})$, $h(x_i + i\lambda z_i) \rightarrow h(x_i)$ as $\lambda \searrow 0$. Since such h_i 's, are dense in $\mathcal{S}(\mathbb{R})$,

$$\lim_{\lambda \downarrow 0} T_{\underline{\alpha}, e}(\lambda)(g) = T_{\underline{\alpha}_\tau}(0)(g) = T_{\underline{\alpha}}(0)(g) .$$

Thus, the limit of $\mathbf{W}((\underline{s}, \underline{x}) - i\underline{\alpha}e)$ coincides with the tempered distribution

$$\mathfrak{W}_\beta^{(n-1)}(t_1 - t_2, x_1 - x_2, \dots, t_{n-1} - t_n, x_{n-1} - x_n)$$

encountered before.

The KMS boundary condition follows by differentiating (see (39)) the boundary condition of the corresponding Weyl operators given in (26). \square

Remark 3.10. *We note that the thermal Wightman distributions*

$$\mathfrak{W}_\beta^{(n-1)}(t_1 - t_2, x_1 - x_2, \dots, t_{n-1} - t_n, x_{n-1} - x_n)$$

are analytic functions as long as the (t_i, x_i) , $i = 1, \dots, n$, are mutually space-like points. This can be shown by an argument similar to the one outlined in the discussion preceding Theorem 2.10.

4 Summary and Outlook

For quite some time the pioneering work of Høegh-Krohn [23] did not find the recognition it deserves. However, the thermal $\mathcal{P}(\varphi)_2$ model should be seen as

a binding link between statistical mechanics and quantum field theory. The authors believe that by providing more detail on the construction of this model (see [12][13]) and verifying that it satisfies key axioms, other scientists might get motivated to look at this model in more detail. In fact, the physical properties of this model have hardly been explored so far. It would be interesting to know how, for instance, the specific heat behaves as a function of the temperature and the coupling constants. A more challenging question is to investigate the particle content of this model. Eventually, one may want to set up scattering theory at positive temperature or prove the uniqueness of the KMS state for all temperatures and all allowed values of the coupling constant. There are strong indications that the correlation functions decay exponentially in space-like directions, and thus it seems to the authors that all of these questions can be resolved with reasonable amount of work.

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