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Bullwhip in a Multi-Product Production Setting

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Abstract

We consider the bullwhip problem in a multi-product scenario, specifically looking at the impact of product demands on production and inventory cost performance in a factory setting. A Vector Auto-Regressive process of the first order (VAR(1)) is used to represent two different demand processes. The Order-Up-To (OUT) policy is used to manage inventory levels and generate production orders. The two demand processes could represent two different products, or the same product being sold to two different retailers. In this manner we can quantity the impact of "inventory pooling" on performance. The two demand streams could be satisfied by production on a single production line (with or without a changeover), or production on two separate production lines. In this way we can investigate the issue of "capacity pooling". We provide a real-life case study to motivate our model and support our technical findings. We discuss the implications of our research against the background of the current proliferation of "SKU customization" in the fast moving consumer goods industry.

Keywords: Bullwhip Effect, Vector Auto-Regressive Demand, Safety Stocks, SKU customization.

Economic Literature Codes: C02, M11

Word Count: 6610

1. Introduction

Over the last fifteen years the bullwhip, or demand amplification, problem has extensively been studied in the operations management literature. The supply chain bullwhip effect, as described by Lee et al. (1997a), is "the phenomenon where orders to the supplier tend to have larger variance than sales to the buyer (i.e., demand distortion), and the distortion propagates upstream in an amplified form (i.e., variance amplification)." This phenomenon has been observed in many supply chains. To name just a few examples (see Chen and Lee 2011): Hammond (1994) showed large fluctuations of weekly order quantities in Barilla's pasta supply chain. Lee et al. (1997b) observed excess volatility in weekly orders in both Procter & Gamble's diaper supply chain and Hewlett-Packard's printer supply chain. Fransoo and Wouters (2000) reported strong bullwhip effects in daily data from a supply chain of ready-

made meals and salads. Waller et al. (2008) studied the weekly order and sales data obtained from a major U.S. retailer and found strong bullwhip effects in all 115 products.

The bullwhip effect has a number of highly undesirable cost implications. In his seminal paper, Lee (1997b) describes how the bullwhip effect may lead to excessive inventory investment, poor customer service, lost revenues, misguided capacity plans, ineffective transportation, and missed production schedules.

Disney et al. (2007) define bullwhip as the key example of supply chain inefficiency, creating unstable production schedules and increased inventory requirements. To measure these inefficiencies, it is common to define two variance ratios: the ratio of order variance to demand variance (the typical bullwhip measure) and the ratio of the inventory variance to the demand variance, since this may have a significant impact on customer service. A high bullwhip measure implies highly fluctuating orders, meaning that the production level is changed frequently, resulting in higher average production (capacity) costs per period. An increased inventory variance results in higher holding and backlog costs, inflating the average inventory cost per period. The variance amplification in orders and inventory levels has been quantified by Disney et al. (2006) under a generalised order-up-to policy for i.i.d. and the weakly stationary auto regressive (AR), moving average (MA) and auto regressive moving average (ARMA) demand processes.

It is worth noting that the original definition of the bullwhip effect and the variance amplification ratios studied in the literature are typically based on a single product and a single customer. It is common practice, however, to produce a *group* of products on the same production line, in which case it makes more sense to measure bullwhip on a 'product group' level, rather than on an individual product level. Indeed, from a capacity point of view, when different products share the same production facilities there is a pooling effect that can be exploited.

Similarly, branded label manufacturers have the common practice of selling the same SKU to a variety of customers. This means that the branded label manufacturer can aggregate (or pool) the demands over its retail customers with a potential to reduce inventories. In this case the inventory variance ratio should be evaluated on the 'customer group' level, rather than at a individual customer level. Interestingly, this grouping at customer level is often not the case for private label suppliers, who usually sell customer-specific (customized) SKU's. In this situation enormous swings in orders may hurt inventories and service levels as there is no aggregation or inventory pooling effect across customers. So it is also important to define inventory variance at the appropriate level. Farasyn et al. (2011) report on the significant product proliferation at Procter & Gamble: its number of finished goods has doubled over the past 10 years. For some business units, P&G commonly customizes packaging to match a customer's unique requirements: these specialized SKU's pass through multiple tiers of distribution and inventories should be kept accordingly. Depending on the supply chain echelon, demand can be aggregated and its relative uncertainty decreases. Clearly, the need for more customization will impact P&G's inventory performance.

A number of papers have discussed the level of aggregation in data and bullwhip measurements. For example, Fransoo and Wouters (2000) experienced that exactly the same basic data can lead to different measures of the bullwhip effect, dependent on the sequence of aggregation in the analysis. Sucky (2009) argues that pooling cannot be ignored when

studying the bullwhip effect. With moving average forecasts of i.i.d. demand that is correlated between retailers he shows how negative correlation can reduce the bullwhip effect. Also, Chen and Lee (2011) show how aggregating data over longer time periods or aggregating across products can affect the magnitude of the bullwhip effect and its resulting cost implications. Hedenstierna and Disney (2012) investigate the economics of consolidating capacity across products and over time by considering the impact of the re-planning frequency on costs.

In this paper we consider a multi-product setting and analyse its impact on both production and inventory related costs. The remainder of the paper is organised as follows. We introduce a real life example in the next section that will be used throughout the paper to motivate, illustrate and verify our findings. Our model is then described in section 3. Section 4 provides stability conditions for the assumed demand process, and expressions for the order and inventory variances are given in section 5. Section 5 also investigates the economic consequences of consolidating capacity and inventory. Section 6 concludes.

2. Motivating Example: The Chocolate Case

We motivate the managerial impact of our model by means of a real life example. We analysed the sales data of an international food corporation which is, among others, active in the consumer packaged chocolate business. The products are strongly branded and distributed in the retail market, primarily in the Nordic and Baltic countries. The company faces a high proliferation of SKU's, both in terms of different flavours as well as various package offerings, which are demanded by its retail customers. Figure 1 shows the weekly (de-trended) sales data of two of its products.
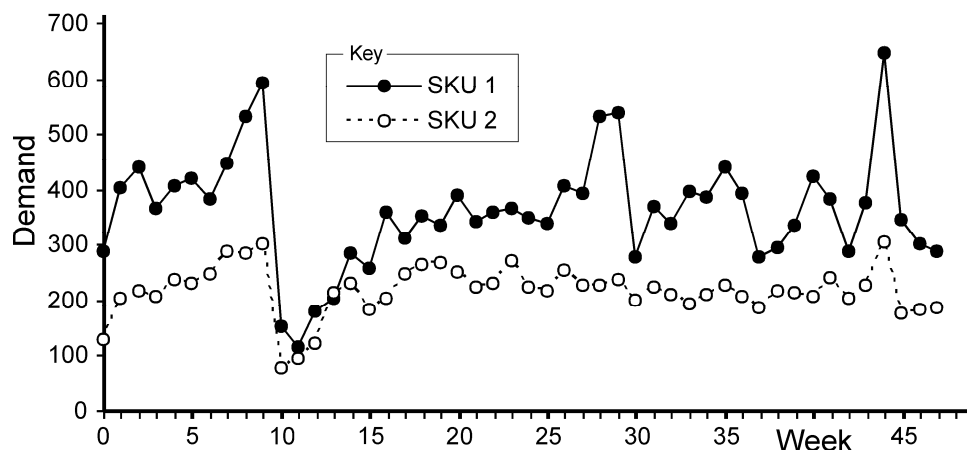
Figure 1. Weekly sales data of two consumer packaged chocolate SKU's

We analyse the production and inventory requirements for these two SKU's. The company can either produce the SKU's each on a separate production line, or alternatively produce them together on the same production line. In the latter case, a changeover may be needed to allow the production line to be adapted to produce the new incoming product. We will ignore this cost as it is context specific. Furthermore in the chocolate case it can be ignored, as when only one product is produced on a line, the line still has to be cleaned periodically to meet hygiene requirements, and the change-over cost will be incurred anyhow. When both SKU's

use the same production capacity, the aggregation of demands may partially cancel out the fluctuations in orders. This pooling effect is strongly present in case of negative correlation between both products, but as we will show later in this paper, also positively correlated demands might provide pooling benefits in capacity.

Analogously, we can keep separate inventories for both SKU's, or consolidate them to hold only one (virtual) inventory. Inventories can be consolidated, when e.g. the same SKU can be distributed in an identical form to various retail customers. This may also be when the customized packaging is postponed until the order is effectively received, or when lateral trans-shipments are used between different production sites that are producing the same product. In this case we may have pooling benefits in inventory since both demands are aggregated to hold one (combined) safety stock.

3. Model Description

We consider a manufacturer who supplies different SKU's to its retail customers (the fast moving consumer goods business is a good setting to have in mind). Without loss of generality, we restrict our discussion to only two SKU's in order to limit the size of the mathematical calculations. The two SKU's are two products that allow inventory pooling (because they are the same product sold to two different markets) or not (because they are two different products). The two SKU's may also allow capacity pooling (they can be produced on the same production line with or without a changeover) or not (they have to be produced on two different production lines). The demands of both SKU's are interdependent over time – to capture these interdependencies, we use the Vector Auto-Regressive (VAR) process. The use of the VAR process in supply chain studies is not new, but it is rather rare. Chaharsooghi and Sadeghi (2008) use a VAR model to study the bullwhip problem in multiple product supply chains. Ratanchote (2011) used the VAR model in a study of a distribution network. It is also interesting to note that Christopher Sims was awarded the 2011 Nobel Prize in Economics for his work in applying VAR models to macroeconomic problems.

The first order Vector Auto-Regressive (VAR(1)) process for the demand of SKU $k \in \{1,2\}$ in period t, $d_{k:t}$, is given by

$$\left.\begin{array}{l} \left(d_{1:t} - \mu_1\right) = \phi_1\left(d_{1:t-1} - \mu_1\right) + \theta_1\left(d_{2:t-1} - \mu_2\right) + \varepsilon_{1:t} \\ \left(d_{2:t} - \mu_2\right) = \phi_2\left(d_{2:t-1} - \mu_2\right) + \theta_2\left(d_{1:t-1} - \mu_1\right) + \varepsilon_{2:t} \end{array}\right\} \tag{1}$$

where $\{\varepsilon_1, \varepsilon_2\} \in N\left(0, \sigma_\varepsilon\right)$ are i.i.d. random variables drawn from a normal distribution with a mean of zero and a standard deviation of σ_ε. We assume for simplicity that ε_1 and ε_2 are independent. μ_k is the mean (average) demand for product k, ϕ_k is a auto-regressive parameter and θ_k is a cross-correlation parameter. Note that $d_{2:t}$ could represent the sum of the other $N-1$ SKU's expect SKU 1. For our sample data set portrayed in Figure 1, assuming beforehand a VAR(1) process exists and minimising the squared one period ahead forecast error of the demand process, we obtained the following VAR(1) parameters: $\phi_1 = 0.2$, $\phi_2 = 0.56$, $\theta_1 = 0.65$ and $\theta_2 = -0.15$. These parameter values are all within the range that intuition would predict as reasonable. SKU 1 has a mean of $\mu_1 = 354$, SKU 2 has a mean of $\mu_2 = 206$. The noise processes have a variance of $\sigma_{\varepsilon_1}^2 = 8309$ and $\sigma_{\varepsilon_2}^2 = 2796$.

The manufacturer controls its inventory with a periodic review Order-Up-To (OUT) policy. The sequence of events is such that previous orders placed $T_p + 1$ periods ago are received at the beginning of the period, demand is satisfied from inventory during the period and inventory levels are observed and replenishment orders placed at the end of the period (see Figure 2).
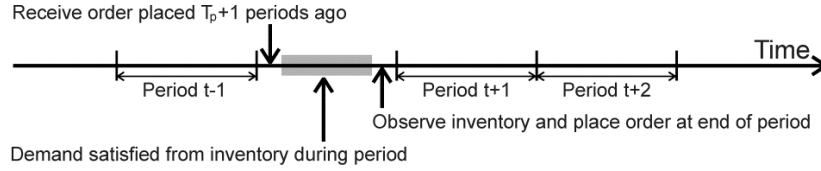

Figure 2. The sequence of events in the OUT policy

Under this sequence of events, the inventory balance equation per SKU is given by

$$i_{k:t} = i_{k:t-1} + o_{k:t-T_p-1} - d_{k:t} \,, \tag{2}$$

with $i_{k:t}$ the end-of-period net inventory of product k in period t. The inventory levels are regulated by the production orders per SKU, $o_{k:t}$, which are generated with the OUT policy,

$$o_{k:t} = \underbrace{\hat{d}_{k:t+T_p+1,t}}_{\text{Forecast}} + \underbrace{\overbrace{tns}^{\text{Safety stock}} - i_{k:t}}_{\text{Inventory feedback}} + \underbrace{\overbrace{\sum_{i=1}^{T_p} \hat{d}_{k:t+i,t}}^{\text{DWIP}} - \overbrace{\sum_{j=1}^{T_p} o_{k:t-j}}^{\text{WIP}}}_{\text{WIP feedback}} \,. \tag{3}$$

Here, $\hat{d}_{k:t+T_p+1,t}$ is a forecast of demand for product k in periods $t+T_p+1$ made at time period t. The target net stock, tns, is a safety stock set to minimise piece-wise linear holding (H) and backlog (B) costs associated with the inventory levels, $i_{k:t}$, satisfying a newsvendor fractile

$$tns = \sigma_{ns} z; \; z = \Phi^{-1}\left[\tfrac{B}{B+H}\right]. \tag{4}$$

Note that we denote the probability density function (pdf) of the standard normal distribution by $\varphi[x]$ and its cumulative distribution function (cdf) by $\Phi[x]$. There is also a forecast that sets the desired work-in-progress (DWIP). DWIP is made up of the sum of forecast in the periods over the lead-time. The actual work-in-progress (WIP) is the sum of the orders that have been placed but not yet received because of the lead-time, T_p. An alternative, and useful in this paper, formulation of the OUT policy is

$$o_{k:t} = \left(\sum_{i=1}^{T_p+1} \hat{d}_{k:t+i,t}\right) - \left(\sum_{i=1}^{T_p+1} \hat{d}_{k:t+i-1,t-1}\right) + d_{k:t} \,, \tag{5}$$

Hosoda and Disney (2006). The replenishment orders are sent to a production line to be processed. We assume from now on that the planning period for capacity in production is one week in duration, because the natural cycle of our daily life is based on the week. The production line is manned by flexible labour that is equally proficient at producing (assembling) the order for any SKU. The production line has a nominal weekly capacity to produce $K = (\mu_O + s)$ units, with μ_O the weekly average order size and s is a *safety* capacity to cope with higher than average orders (s is a decision variable depending on the order

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fluctuations). s can be changed over the long term by altering the amount of labour and machines in a production line. Note: s can be negative, which means that the nominal weekly capacity, $K < \mu_O$; $K > 0$ as long as $s < \mu_O$. The unit capacity cost within a normal working week is U. This means the weekly wage is to be paid, even if the weekly orders are below the capacity of $K = (\mu_O + s)$ units (hence the notation U for *undertime* cost). If the orders are above the available capacity of $K = (\mu_O + s)$ units, then over-time is offered to produce the remaining, as yet incomplete orders. The unit overtime cost per item is charged at a higher price W (the notation *"double U" denotes that it should be higher than U, otherwise it would be never optimal to invest in normal-time labour*). Overtime volume can be (and is always) flexed to meet the required weekly production orders. This means that when just a few hours of overtime is required, just a few hours of overtime is used (rather than a whole day). Denote $C = E[C_t]$ the average weekly cost to be minimized, with the weekly cost C_t given by

$$C_t = \underbrace{U\left(s + \mu_O\right)}_{\text{Guarenteed hours}} + \underbrace{W\left(o_t - s - \mu_O\right)^+}_{\text{Over-time costs}} + \underbrace{H\left(i_t\right)^+ + B\left(-i_t\right)^+}_{\text{Inventory holding \& backlog costs}} , \qquad (6)$$

with $(x)^+ = \max[x, 0]$. By taking the expected values of the cost function, differentiating it w.r.t. s and solving for the first order conditions we may obtain an expression for the optimal amount of slack capacity, above (below) a mean production order (Disney, Gaalman and Hosoda, 2012). It is given by

$$s^* = \sigma_O z; \; z = \Phi^{-1}\left[\tfrac{W-U}{W}\right], \qquad (7)$$

and when the capacity level and the safety stock has been optimised, the minimised expected per period costs are given by

$$C^* = C_O^* + C_{NS}^* = \underbrace{\mu_O U + W\sigma_O \varphi\left[\Phi^{-1}\left[\tfrac{W-U}{W}\right]\right]}_{C_O^*} + \underbrace{\sigma_{NS}\left(B + H\right)\varphi\left[\Phi^{-1}\left[\tfrac{B}{B+H}\right]\right]}_{C_{NS}^*}. \qquad (8)$$

We notice that optimised costs are linear in both the standard deviation of the order fluctuations σ_O (with an offset from the origin due to $\mu_O U$) and the standard deviation of the inventory fluctuations σ_{NS}.

We assume the following cost parameters in our chocolate case example, $\{H = 1, B = 9, U = 4, W = 6\}$. When the Target Net Stock has been set optimally then the critical fraction $\tfrac{B}{B+H} = p$ of periods end with inventory on-hand. Similarly when the capacity level have been optimised then a fraction $1 - \tfrac{U}{W} = \tfrac{W-U}{W} = p$ of periods require over-time to complete the production orders. It is interesting to note that the capacity requirements are function of the standard deviation of the order and the costs and are not solely a percentage of the average demand as is often advocated. Due to its prominent role in (8), Figure 3 illustrates how $\varphi\left[\Phi^{-1}[p]\right]$ behaves.
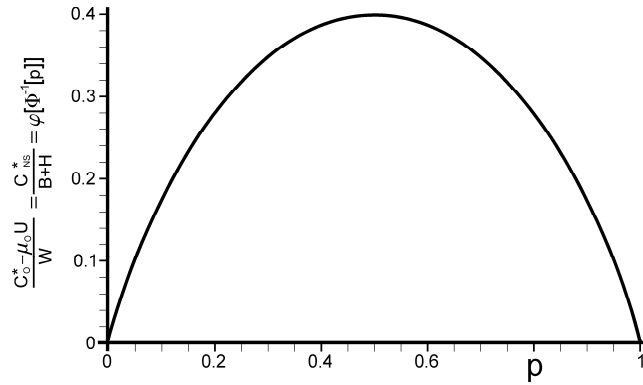
Figure 3. The behaviour of $\varphi\left[\Phi^{-1}\left[p\right]\right]$

4. Stability of the VAR(1) Demand Process

The first step to studying the dynamic behaviour of the production and inventory control system is to consider the question of stability. The OUT policy that we study is a stable policy. However the VAR(1) demand process that we are using can become unstable. By unstable we mean that the demand process can have an infinite variance and no natural mean. This obviously creates problems in a stationary cost analysis because if the demand is unstable so will the forecasts and also the orders. Thus we first consider the question of the stability of the demand process. We will do this using z-transforms. We refer interested readers to a good control engineering text, for example Moudgalya (2007), for more information on the z-transform. The z-transform of (1) which relate each of the demands, d_t, to each of the noises, ε_t,

$$\frac{d_1(z)}{\varepsilon_1(z)} = \frac{z(z-\phi_2)}{A(z)}; \ \frac{d_2(z)}{\varepsilon_1(z)} = \frac{z\theta_2}{A(z)}; \ \frac{d_1(z)}{\varepsilon_2(z)} = \frac{z\theta_1}{A(z)}; \ \frac{d_2(z)}{\varepsilon_2(z)} = \frac{z(z-\phi_1)}{A(z)} \Bigg\} \tag{9}$$

where $A(z) = z^2 + z(-\phi_1 - \phi_2) + \phi_1\phi_2 - \theta_1\theta_2$. $A(z)$ is in the form of $a_n z^n + a_{n-1} z^{n-1} + ... + a_0$ with $a_n > 0$ and $n = 2$. We can see that the denominators of each of the equations in (9) are identical. As the stability criterion only depends on the denominator of the transfer function then this means that the stability criteria of both demands in relation to each of the noises is the same. That is, as we would naturally expect, there is only one set of stability criteria that is valid for the whole demand model.

The stability boundaries can be obtained by using Inners approach of Jury (1974). Jury shows that the necessary and sufficient conditions for stability of a system are that $A(1) > 0$, $(-1)^n A(-1) > 0$ and (because $n = 2$) $a_2 \pm a_0 > 0$. These three conditions reveal that the VAR(1) is stable if

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$$
\left.\begin{array}{l}
\left(\left(\phi_1 = 1 \wedge \left(\left(\theta_2 > 0 \wedge \theta_1 < 0\right) \vee \left(\theta_1 > 0 \wedge \theta_2 < 0\right)\right)\right) \vee \left(\theta_1\theta_2 + \phi_1 + \phi_2 < 1 + \phi_1\phi_2 \wedge \phi_1 \neq 1\right)\right) \wedge \\[2mm]
\left(\left(\phi_1 = -1 \wedge \left(\left(\theta_2 > 0 \wedge \theta_1 < 0\right) \vee \left(\theta_1 > 0 \wedge \theta_2 < 0\right)\right)\right) \vee \left(\theta_1\theta_2 < 1 + \phi_1 + \phi_2\left(1 + \phi_1\right) \wedge 1 + \phi_1 \neq 0\right)\right) \wedge \\[2mm]
\left(\theta_2 = 0 \vee \phi_1 \neq 0 \vee \theta_1\theta_2 < 1\right) \wedge \left(\phi_1 = 0 \vee \theta_1\theta_2 < 1 + \phi_1\phi_2\right) \wedge \\[2mm]
\left(\theta_2 = 0 \vee \phi_1 \neq 0 \vee 1 + \theta_1\theta_2 > 0\right) \wedge \left(\phi_1 = 0 \vee \phi_1\phi_2 < 1 + \theta_1\theta_2\right)
\end{array}\right\}
$$

$$(10)$$

holds true. Here $\wedge$ is the logical "and" and $\vee$ is the logical "or" function. Perhaps it is best to visualise the stability boundary graphically, see Figure 4. Consider first the contour plot where either $\theta_1 = 0$ or $\theta_2 = 0$ (or both $\theta_1 = \theta_2 = 0$), which shows that the VAR(1) demand process is stable when $-1 < \{\phi_1, \phi_2\} < 1$. We would expect this result: when $\theta_1 = 0$ or $\theta_2 = 0$, we effectively have two independent Auto Regressive (AR) processes which are well known to be stable when the AR parameter is within $-1 < \phi < 1$ (Box et al., 1994). When $-1 < \theta_1\theta_2 < 0$, then the stable area is a single area bounded by six curves in the $\{\phi_1, \phi_2\}$ plane. At $\theta_1\theta_2 = -1$, this stable area splits into two distinct areas each bounded by three curves that become smaller and drift apart when $\theta_1\theta_2$ becomes more negative. When $\theta_1\theta_2 > 0 \wedge \theta_2 < \theta_1^{-1}$, then there is only a single stable area in the $\{\phi_1, \phi_2\}$ plane bounded by two curves. This stable area gets smaller as $\theta_2 \to \theta_1^{-1}$ from below. Figure 4 also shows the stability boundary near the parameter setting for our chocolate example.
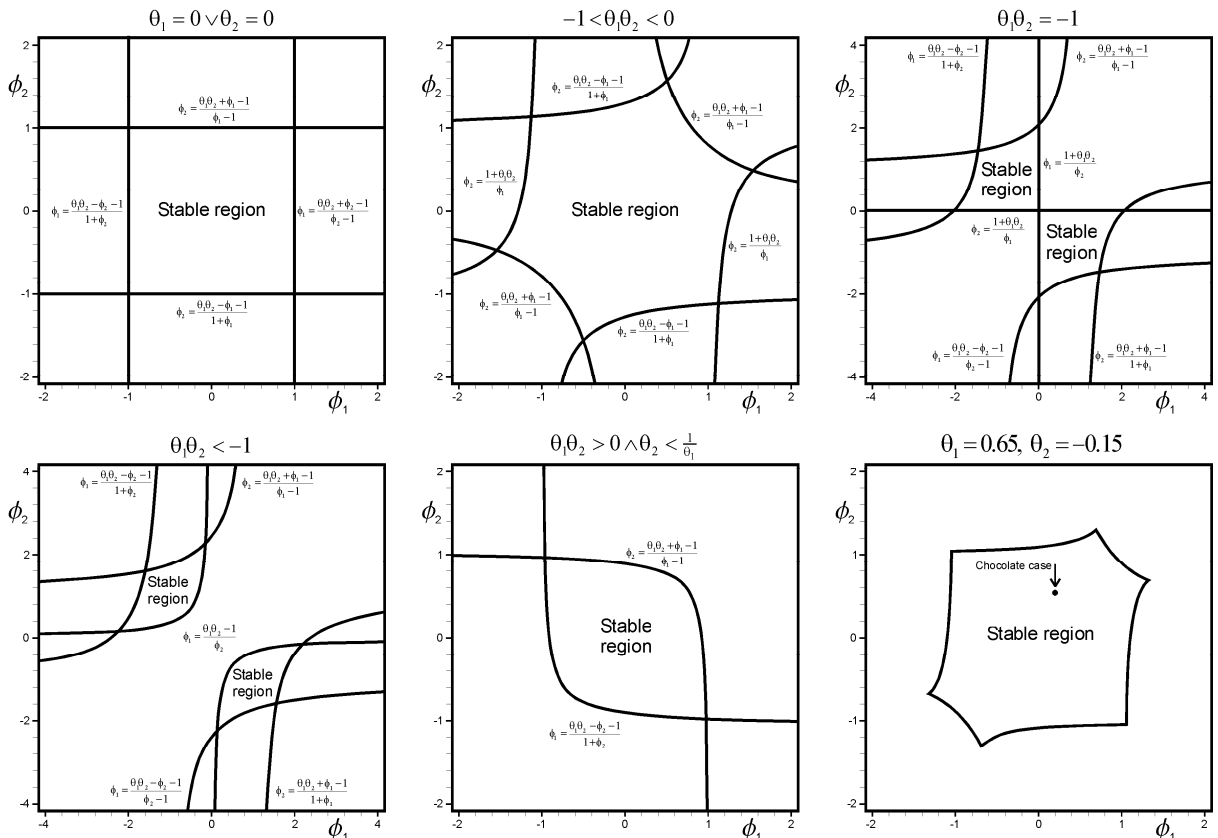

Figure 4. The stability region for VAR(1) demands

5. Determining the OUT Policy Variance Ratios

In this section we highlight how the different variance ratios are obtained. First we consider the demand variance, which illustrates the main steps in the analysis procedure. Then we turn our attention to the variance of the production orders and the variance of the inventory levels. We illustrate our results throughout this section with the chocolate case example, introduced in section 2. This industrially relevant numerical example will also serve as a validation case to verify our theoretical findings.

5.1. Variance of the Demand Process

The z-transforms that link each of the demands to each of the noises were previously given in (9). In order to determine the variance of a particular demand, we need to add the two demand variances from each of the two sources of noise together. This is a consequence of our assumption that the two noises are independent. In general terms, the ratio of the variance of the systems output (the demand, forecast, orders, inventory, etc.) to the variance of the systems input (the noise) when the input is a white noise random process can be obtained using the Inners approach of Jury (1974),

$$\frac{\sigma_{\text{Output}}^2}{\sigma_{\text{White noise input}}^2} = \frac{\left| \mathbf{X_{n+1}} + \mathbf{Y_{n+1}} \right|_b}{a_n \left| \mathbf{X_{n+1}} + \mathbf{Y_{n+1}} \right|}. \tag{11}$$

Using (11) the variance ratio is calculated from the determinants of matrices. Determinants can always be obtained and it is easy to automate this procedure on computers: we used Mathematica in this research. To use Jury's Inners approach we first write the nth order transfer function, $F[z]$, in standard form

$$F[z] = \frac{\sum_{i=1}^{n} b_i z^i}{\sum_{i=1}^{n} a_i z^i} = \frac{b_n z^n + b_{n-1} z^{n-1} + \dots + b_1 z + b_0}{a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0}, \ a_n > 0, \tag{12}$$

from which we identify the coefficients of z in the transfer function's numerator $\{b_0, b_1, \dots, b_n\}$ and denominator $\{a_0, a_1, \dots, a_n\}$. We then construct the following matrices based on the coefficients of the denominator

$$\mathbf{X_{n+1}} = \begin{bmatrix} a_n & a_{n-1} & \cdots & a_0 \\ 0 & a_n & \cdots & a_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}, \ \mathbf{Y_{n+1}} = \begin{bmatrix} 0 & 0 & \cdots & a_0 \\ 0 & 0 & \ddots & a_1 \\ \vdots & a_0 & \cdots & \vdots \\ a_0 & a_1 & \cdots & a_n \end{bmatrix} \tag{13}$$

and use these to obtain the $\left[\mathbf{X_{n+1}} + \mathbf{Y_{n+1}} \right]$ matrix. Another matrix $\left[\mathbf{X_{n+1}} + \mathbf{Y_{n+1}} \right]_b$ is obtained by replacing the last row of $\left[\mathbf{X_{n+1}} + \mathbf{Y_{n+1}} \right]$ with $\left\{ 2b_n b_o, \ 2\sum b_i b_{i+n-1}, \ \cdots, \ 2\sum b_i b_{i+1}, \ 2\sum_{i=1}^{n} b_i^2 \right\}$. Finally the required variance ratio can be obtained from

$$\frac{\sigma_{\text{Output}}^2}{\sigma_{\text{White noise}}^2} = \frac{\left| \mathbf{X_{n+1}} + \mathbf{Y_{n+1}} \right|_b}{a_n \left| \mathbf{X_{n+1}} + \mathbf{Y_{n+1}} \right|}. \tag{14}$$

For illustration let's consider the variance ratio between d_1 and ε_1. The transfer function was given in (9). It has the following coefficients of z,

$$\left\{ b_2 = 1, \; b_1 = -\phi_2, \; b_0 = 0, \; a_2 = 1, \; a_1 = -\phi_1 - \phi_2, \; a_0 = \phi_1\phi_2 - \theta_1\theta_2 \right\}. \tag{15}$$

Collecting together the coefficients of the denominator for the $\mathbf{X}_{n+1}$ and $\mathbf{Y}_{n+1}$ matrix yields

$$\mathbf{X}_{n+1} = \begin{pmatrix} 1 & -\phi_1 - \phi_2 & \phi_1\phi_2 - \theta_1\theta_2 \\ 0 & 1 & -\phi_1 - \phi_2 \\ 0 & 0 & 1 \end{pmatrix}, \; \mathbf{Y}_{n+1} = \begin{pmatrix} 0 & 0 & \phi_1\phi_2 - \theta_1\theta_2 \\ 0 & \phi_1\phi_2 - \theta_1\theta_2 & -\phi_1 - \phi_2 \\ \phi_1\phi_2 - \theta_1\theta_2 & -\phi_1 - \phi_2 & 1 \end{pmatrix} \tag{16}$$

which we sum together to yield

$$\left[\mathbf{X}_{n+1} + \mathbf{Y}_{n+1} \right] = \begin{pmatrix} 1 & -\phi_1 - \phi_2 & 2\left(\phi_1\phi_2 - \theta_1\theta_2\right) \\ 0 & 1 + \phi_1\phi_2 - \theta_1\theta_2 & -2\left(\phi_1 + \phi_2\right) \\ \phi_1\phi_2 - \theta_1\theta_2 & -\phi_1 - \phi_2 & 2 \end{pmatrix}. \tag{17}$$

The $\left[\mathbf{X_{n+1}} + \mathbf{Y_{n+1}} \right]_b$ matrix is

$$\left[\mathbf{X}_{n+1} + \mathbf{Y}_{n+1} \right]_b = \begin{pmatrix} 1 & -\phi_1 - \phi_2 & 2\left(\phi_1\phi_2 - \theta_1\theta_2\right) \\ 0 & 1 + \phi_1\phi_2 - \theta_1\theta_2 & -2\left(\phi_1 + \phi_2\right) \\ 0 & -2\phi_2 & 2\left(1 + \phi_2^2\right) \end{pmatrix}. \tag{18}$$

Finally from (14), (15), (17) and (18) we may obtain the following expression that describes the contribution of $\sigma_{\varepsilon_1}^2$ to $\sigma_{d_1}^2$.

$$\frac{\sigma_{d_1}^2}{\sigma_{\varepsilon_1}^2} = \frac{\left(1 + \theta_1\theta_2 - \phi_1\phi_2\right)\left(\theta_1\theta_2 + \phi_1\left(1 - \phi_2\right) + \phi_2 - 1\right)\left(\theta_1\theta_2 - \left(1 + \phi_1\right)\left(1 + \phi_2\right)\right)}{\left(\phi_1\phi_2 - 1\right)\left(\phi_2^2 - 1\right) - \theta_1\theta_2\left(1 + \phi_2^2\right)}. \tag{19}$$

In a like manner we can obtain the variance ratio expression between $\sigma_{\varepsilon_2}^2$ and $\sigma_{d_1}^2$. We do this by departing from the relevant transfer function in (9), the third one. We notice that only the coefficients of z in the numerator change, $\left\{ b_0 = b_2 = 0, b_1 = \theta_1 \right\}$. This implies that

$$\left[\mathbf{X}_{n+1} + \mathbf{Y}_{n+1} \right]_b = \begin{pmatrix} 1 & -\phi_1 - \phi_2 & 2\left(\phi_1\phi_2 - \theta_1\theta_2\right) \\ 0 & 1 + \phi_1\phi_2 - \theta_1\theta_2 & -2\left(\phi_1 + \phi_2\right) \\ 0 & 0 & 2\theta_1^2 \end{pmatrix}. \tag{20}$$

from which, together with (14), (17) and $a_n = a_2 = 1$ from (15) yields

$$\frac{\sigma_{d_1}^2}{\sigma_{\varepsilon_2}^2} = \frac{\left(\phi_1\phi_2 - 1 - \theta_1\theta_2\right)\left(\theta_1\theta_2 + \phi_1 + \phi_2 - \phi_1\phi_2 - 1\right)\left(\theta_1\theta_2 - \left(1 + \phi_1\right)\left(1 + \phi_2\right)\right)}{\theta_1^2\left(\theta_1\theta_2 - \phi_1\phi_2 - 1\right)}. \tag{21}$$

10

Finally we may obtain the expression for the variance of demand process d_1 by adding together (19) and (21) because the two noise processes are independent:

$$\sigma_{d_1}^2 = \frac{\sigma_{\varepsilon_2}^2 \theta_1^2 \left(1 - \theta_1\theta_2 + \phi_1\phi_2\right) - \sigma_{\varepsilon_1}^2 \left(\theta_1\theta_2\left(1+\phi_2^2\right) + \phi_2\left(\phi_2 + \phi_1\left(1-\phi_2^2\right)\right) - 1\right)}{\left(1+\theta_1\theta_2 - \phi_1\phi_2\right)\left(\theta_1\theta_2 - \left(1-\phi_1\right)\left(1-\phi_2\right)\right)\left(\theta_1\theta_2 - \left(1+\phi_1\right)\left(1+\phi_2\right)\right)}. \tag{22}$$

Following the same procedure for the demand process d_2 leads us to

$$\sigma_{d_2}^2 = \frac{\sigma_{\varepsilon_1}^2 \theta_2^2 \left(1 - \theta_1\theta_2 + \phi_1\phi_2\right) - \sigma_{\varepsilon_2}^2 \left(\theta_1\theta_2(1+\phi_1^2) + \phi_1\left(\phi_1 + \phi_2\left(1-\phi_1^2\right)\right) - 1\right)}{\left(1+\theta_1\theta_2 - \phi_1\phi_2\right)\left(\theta_1\theta_2 - \left(1-\phi_1\right)\left(1-\phi_2\right)\right)\left(\theta_1\theta_2 - \left(1+\phi_1\right)\left(1+\phi_2\right)\right)}. \tag{23}$$

We notice in (22) and (23) that there is symmetry between the two equations. That is, to obtain the variance of demand process 2, we simply make the substitutions $\{\phi_1 \rightleftarrows \phi_2,$ $\theta_1 \rightleftarrows \theta_2$ and $\sigma_{\varepsilon_1}^2 \rightleftarrows \sigma_{\varepsilon_2}^2\}$ in the variance expression for demand process 1. This symmetry always exists in the variance expressions and because of this we will now only present one of the two required variance expressions in order to save space. For our chocolate case example, we find that the demand variance of SKU 1, $\sigma_{d_1}^2 = 10799.2$ and for SKU 2, $\sigma_{d_2}^2 = 4134.5$ from (22) and (23) respectively.

5.2. Transfer Functions for the OUT Policy and its Forecasts

In order to generate the production orders with the OUT replenishment policy we need to specify two forecasts. The first forecast is a forecast of the demand in the next period. This is used as the Desired WIP within the OUT policy, see (3), as we assume the lead-time is one period long. The second forecast that the OUT policy requires is a forecast of the demand two periods ahead – a forecast of the demand in the period after the lead-time. Let's consider the one period ahead forecast first. The conditional expectation of demand one period ahead in the time domain is given by

$$\left. \begin{array}{l} \hat{d}_{1:t+1,t} = \mu_1 + \phi_1\left(d_{1:t} - \mu_1\right) + \theta_1\left(d_{2:t} - \mu_2\right) \\ \hat{d}_{2:t+1,t} = \mu_2 + \phi_2\left(d_{2:t} - \mu_2\right) + \theta_2\left(d_{1:t} - \mu_1\right) \end{array} \right\}. \tag{24}$$

The impulse responses of (24) can be described by the following set of z-transforms,

$$\left. \frac{\hat{d}_{1:t+1}(z)}{\varepsilon_1(z)} = \frac{z\left(\theta_1\theta_2 + \phi_1\left(z-\phi_2\right)\right)}{A(z)}; \ \frac{\hat{d}_{2:t+1}(z)}{\varepsilon_1(z)} = \frac{z^2\theta_2}{A(z)}; \ \frac{\hat{d}_{1:t+1}(z)}{\varepsilon_2(z)} = \frac{z^2\theta_1}{A(z)}; \ \frac{\hat{d}_{2:t+1}(z)}{\varepsilon_2(z)} = \frac{z\left(\theta_1\theta_2 + \phi_2\left(z-\phi_1\right)\right)}{A(z)} \right\} \tag{25}$$

which describe the contribution of each of the noises processes to each of the one period ahead demand forecasts. The forecast of the demand two periods ahead is also required for the OUT policy when $T_p = 1$. The time domain difference equations are

$$
\left.\begin{aligned}
\hat{d}_{1:t+2,t} &= \mu_1 + \phi_1\left(\hat{d}_{1:t+1,t} - \mu_1\right) + \theta_1\left(\hat{d}_{2:t+1,t} - \mu_2\right) \\
\hat{d}_{2:t+2,t} &= \mu_2 + \phi_2\left(\hat{d}_{2:t+1,t} - \mu_2\right) + \theta_2\left(\hat{d}_{1:t+1,t} - \mu_1\right)
\end{aligned}\right\}.
\tag{26}
$$

(26) can be described by the following set of z-transforms,

$$
\left.\begin{aligned}
\frac{\hat{d}_{1:t+2}(z)}{\varepsilon_1(z)} &= \frac{z\left(\theta_1\theta_2(z+\phi_1)+(z-\phi_2)\phi_1^2\right)}{A(z)}; & \frac{\hat{d}_{2:t+2}(z)}{\varepsilon_1(z)} &= z\theta_2\left(\frac{z^2}{A(z)}-1\right); \\
\frac{\hat{d}_{1:t+2}(z)}{\varepsilon_2(z)} &= z\theta_1\left(\frac{z^2}{A(z)}-1\right); & \frac{\hat{d}_{2:t+2}(z)}{\varepsilon_2(z)} &= \frac{z\left(\theta_1\theta_2(z+\phi_2)+(z-\phi_1)\phi_2^2\right)}{A(z)}
\end{aligned}\right\}.
\tag{27}
$$

The transfer function that relates the orders in the OUT policy for product k to each of the noises driving the demand processes, $\varepsilon_j \in \{1,2\}$, can then be obtained from

$$
\frac{o_k(z)}{\varepsilon_j(z)} = \left(1-z^{-1}\right)\left(\frac{\hat{d}_{k:t+1}(z)}{\varepsilon_j(z)} + \frac{\hat{d}_{k:t+2}(z)}{\varepsilon_j(z)}\right) + \frac{d_k(z)}{\varepsilon_j(z)}
\tag{28}
$$

as highlighted by Li and Disney (2012). Finally the transfer function that relates the inventory levels maintained by the OUT policy for each product k to each of the noises driving the demand processes, $\varepsilon_j \in \{1,2\}$, can then be obtained from

$$
\frac{ns_k(z)}{\varepsilon_j(z)} = \frac{z}{z-1}\left(\frac{o_k(z)}{\varepsilon_j(z)}\left(z^{-Tp-1}\right) - \frac{d_k(z)}{\varepsilon_j(z)}\right).
\tag{29}
$$

We will now use these transform functions to derive the variance ratio expressions for the inventory and order variance in four settings. We consider the cases where either production or inventory (or both) can be consolidated, or kept separate. The variance ratio expressions for the inventory and the order variance can be derived using Jury's Inners approach as we have illustrated in Section 5.1. We will not present the detailed mathematics involved in order to save space. It is however a fairly straightforward procedure with no deviations from the already presented method – we just start from different transfer functions.

5.3. The Case of Separate Production Capacity and Separate Inventory

We first consider the case where both products are produced on separate production lines, and inventory is kept accordingly. Note that this is the common setting for bullwhip and inventory variance definitions in many papers. For the order variance we use (25) and (27) in (28), to obtain the transfer functions that relates the production orders for each product to each of the two noise processes. These four transfer functions are

$$
\left.\begin{array}{ll}
\dfrac{o_1(z)}{\varepsilon_1(z)}=1+\phi_1(1-z)-z^2+\dfrac{z^3(z-\phi_2)}{A(z)}; & \dfrac{o_2(z)}{\varepsilon_1(z)}=\theta_2\left(1-z+\dfrac{z^3}{A(z)}\right); \\[4mm]
\dfrac{o_1(z)}{\varepsilon_2(z)}=\theta_1\left(1-z+\dfrac{z^3}{A(z)}\right); & \dfrac{o_2(z)}{\varepsilon_2(z)}=1+\phi_2(1-z)-z^2+\dfrac{z^3(z-\phi_1)}{A(z)}
\end{array}\right\}
\tag{30}
$$

Then from each of these transfer functions, using Jury's Inners approach we may obtain the following variance ratio expression that relates each of the production orders to each of the noises, see (31). This can be obtained by rearranging the two variance ratios $\left\{\dfrac{\sigma_{O_1}^2}{\sigma_{\varepsilon_1}^2},\ \dfrac{\sigma_{O_1}^2}{\sigma_{\varepsilon_2}^2}\right\}$ for $\sigma_{O_1}^2$ and adding them together. $\left\{\dfrac{\sigma_{O_1}^2}{\sigma_{\varepsilon_1}^2},\ \dfrac{\sigma_{O_1}^2}{\sigma_{\varepsilon_2}^2}\right\}$ are acquired from $\left\{\dfrac{o_1(z)}{\varepsilon_1(z)},\ \dfrac{o_1(z)}{\varepsilon_2(z)}\right\}$ using Jury's Inners approach.

$$
\sigma_{O_1}^2=\left(\begin{array}{l}
2\sigma_{\varepsilon_2}^2\theta_1^2\phi_2+\dfrac{2\theta_1\left(\sigma_{\varepsilon_1}^2\theta_2-\sigma_{\varepsilon_2}^2\theta_1\right)\phi_1^2}{\left(\theta_1\theta_2+(\phi_1-1)^2\right)\left(\theta_1\theta_2+(1+\phi_1)^2\right)(1+\theta_1\theta_2-\phi_1\phi_2)}+ \\[6mm]
\sigma_{\varepsilon_1}^2\left(2\phi_1-\dfrac{1}{\phi_1^2-1}+2(1+\phi_1)\left(\theta_1\theta_2+\phi_1^2\right)\right)+\dfrac{\theta_1^2\left(\sigma_{\varepsilon_1}^2\theta_2^2+\sigma_{\varepsilon_2}^2(1+\phi_1)^2\right)}{2(1+\phi_1)\left(\theta_1\theta_2+(1+\phi_1)^2\right)\left((1+\phi_1)(1+\phi_2)-\theta_1\theta_2\right)}+ \\[6mm]
2\sigma_{\varepsilon_2}^2\theta_1^2\phi_1+\dfrac{\theta_1^2\left(\sigma_{\varepsilon_1}^2\theta_2^2+\sigma_{\varepsilon_2}^2(\phi_1-1)^2\right)}{2(\phi_1-1)\left(\theta_1\theta_2-(1-\phi_1)(1-\phi_2)\right)\left(\theta_1\theta_2+(\phi_1-1)^2\right)}
\end{array}\right)
\tag{31}
$$

The behaviour of (31) is illustrated in Figure 5 when $\sigma_{\varepsilon_1}^2=\sigma_{\varepsilon_2}^2=1$. The first graph in Figure 5 shows that when $\theta_1=\theta_2=0$, $\sigma_{O_1}^2$ (and $\sigma_{O_2}^2$ with the proper substitutions) is only influenced by ϕ_1 (ϕ_2). When $\theta_1=0.5$, $\theta_2=-0.5$, $\sigma_{O_1}^2$ has a single minimum at $\phi_1=-0.601785$, $\phi_2=-0.0303873$ of $\sigma_{O_1}^2=0.319776$. However we note (but have not shown) that when $-1<\theta_1\theta_2<0$ there could be two minimums in the stability range as $\theta_1\theta_2\to-1$ from above. When $\theta_1\theta_2>0\wedge\theta_2<\frac{1}{\theta_1}$ then there is only one minimum. We have also illustrated the variance of $\sigma_{O_1}^2$ for the chocolate case in the last graph in Figure 5. For our actual chocolate case example, the order variance of SKU 1 is $\sigma_{O_1}^2=14771.7$ and for SKU 2 it is $\sigma_{O_2}^2=9345$ (remember that in the actual chocolate case $\sigma_{\varepsilon_1}^2=8309$ and $\sigma_{\varepsilon_2}^2=2796$, whereas Figure 5 assumes $\sigma_{\varepsilon_1}^2=\sigma_{\varepsilon_2}^2=1$).

Using (9) and (30) in (29) we may obtain the following transfer function that relates inventory levels $ns_k(z)$; $k\in\{1,2\}$ to each of the noises from the demand process, $\varepsilon_j(z)$; $j\in\{1,2\}$:

$$
\left.\dfrac{ns_1(z)}{\varepsilon_1(z)}=-\dfrac{1+z+\phi_1}{z};\ \dfrac{ns_1(z)}{\varepsilon_2(z)}=-\dfrac{\theta_1}{z};\ \dfrac{ns_2(z)}{\varepsilon_1(z)}=-\dfrac{\theta_2}{z};\ \dfrac{ns_2(z)}{\varepsilon_2(z)}=-\dfrac{1+z+\phi_2}{z}\right\}.
\tag{32}
$$

Then, after acquiring $\left\{ \frac{\sigma_{NS_1}^2}{\sigma_{\varepsilon_1}^2}, \frac{\sigma_{NS_1}^2}{\sigma_{\varepsilon_2}^2} \right\}$ from $\left\{ \frac{ns_1(z)}{\varepsilon_1(z)}, \frac{ns_1(z)}{\varepsilon_2(z)} \right\}$, rearranging the two variance ratios

$\left\{ \frac{\sigma_{NS_1}^2}{\sigma_{\varepsilon_1}^2}, \frac{\sigma_{NS_1}^2}{\sigma_{\varepsilon_2}^2} \right\}$ for $\sigma_{NS_1}^2$ and adding them together we arrive at the variance of the inventory
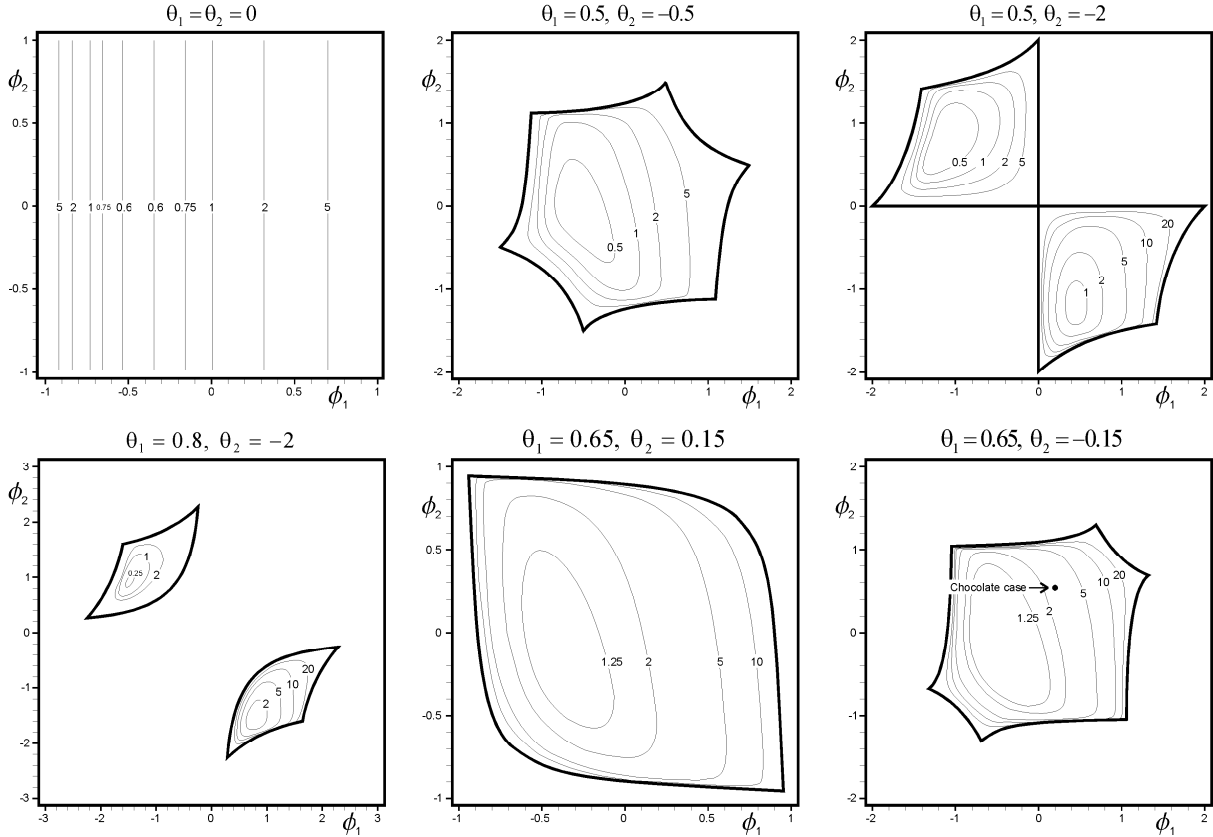
levels. It is

Figure 5. Variance of the orders for product 1 with separated capacity

$$\sigma_{NS_1}^2 = \sigma_{\varepsilon_2}^2 \theta_1^2 + \sigma_{\varepsilon_1}^2 \left(2\left(1+\phi_1\right) + \phi_1^2 \right), \tag{33}$$

which is interesting as the inventory variance of each product is independent of the other products auto-regressive and cross correlation coefficients. We also note that the inventory variance is finite, even in the presence of an unstable demand process. Figure 6 plots the variance of the inventory levels when the demand noise term is $\sigma_{\varepsilon_1}^2 = \sigma_{\varepsilon_2}^2 = 1$. Counter-intuitively we note that the inventory variance also increases for negative cross-correlation. The inventory variance has a minimum at $\theta_1 = 0$, $\left(\theta_2 = 0\right)$ and $\phi_1 = -1$, $\left(\phi_2 = -1\right)$ of $\sigma_{NS_1}^2 = 1$, $\left(\sigma_{NS_2}^2 = 1\right)$. For our chocolate case example, it turns out that the inventory variances are $\sigma_{NS_1}^2 = 146.48$ and $\sigma_{NS_2}^2 = 98.93$.
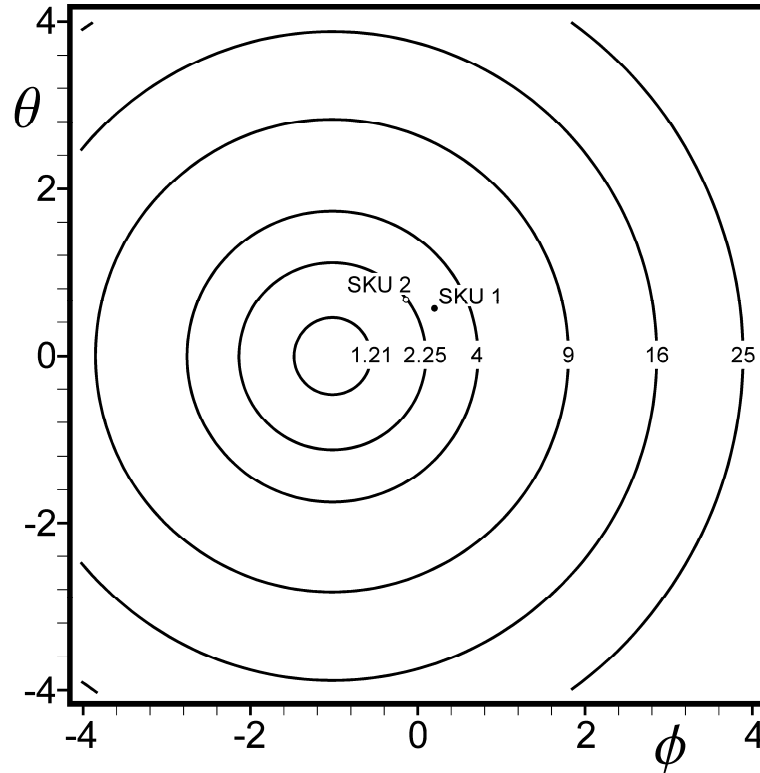
Figure 6. Contour plot of the variance of separate inventory levels

Using (4), (7), (8), (31) and (33) we may verify our theoretical findings with a simulation (of one million time periods) and calculate the cost implications for our chocolate case example, as shown in Table 1. We note that the optimal safety stocks, TNS^*, slack capacity, S^* and the corresponding costs, require the additional assumption that the forecast errors are normally distributed. The variance ratio expressions do not require this assumption (only that the two noises are independent).

Cost	TNS^* / S^*	Cost Formula	Chocolate case cost	
			Theory	Simulation
Inventory cost for product 1	$TNS_1^* = \sigma_{NS_1} \Phi^{-1}\left[\frac{B}{B+H}\right]$ $= 187.717$	$C_{NS_1} = \sigma_{NS_1}(B+H)\varphi\left[\Phi^{-1}\left[\frac{B}{B+H}\right]\right]$	257.06	257.26
Inventory cost for product 2	$TNS_2^* = \sigma_{NS_2} \Phi^{-1}\left[\frac{B}{B+H}\right]$ $= 126.785$	$C_{NS_2} = \sigma_{NS_2}(B+H)\varphi\left[\Phi^{-1}\left[\frac{B}{B+H}\right]\right]$	173.62	173.76
Capacity cost for product 1	$S_1^* = \sigma_{O_1} \Phi^{-1}\left[\frac{W-U}{W}\right]$ $= -52.35$	$C_{O_1} = \mu_{d_1}U + \sigma_{O_1}W\varphi\left[\Phi^{-1}\left[\frac{W-U}{W}\right]\right]$	1681.15	1681.60
Capacity cost for product 2	$S_2^* = \sigma_{O_2} \Phi^{-1}\left[\frac{W-U}{W}\right]$ $= -41.83$	$C_{O_2} = \mu_{d_2}U + \sigma_{O_2}W\varphi\left[\Phi^{-1}\left[\frac{W-U}{W}\right]\right]$	1035.91	1035.74
Total cost of case 1 =			3147.73	3148.36

Table 1. Cost optimisation and validation of the separated inventory, separated capacity case

5.4. Aggregated Production Capacity with Separated Inventory

In this scenario we consider that both products are produced on the same production line, while keeping two separate inventories. It could be that a set-up cost is required to change the production line from producing one product to producing the other product (which we ignore in our analysis, for reasons explained in section 2). It could also be that a single product is produced, but packaging requirements are different and thus two different inventories are maintained. It could also be the case that a single product is produced, but the inventory is sent to two different geographical locations to serve remote markets and lateral trans-shipments are not allowed (or are uneconomic).

To obtain the variance of the order when both SKU's are produced on the same production line then we must add the two transfer functions $\left(\frac{o_1(z)}{\varepsilon_1(z)} + \frac{o_2(z)}{\varepsilon_1(z)}\right)$ together to yield

$$\frac{o_1(z)}{\varepsilon_1(z)} + \frac{o_2(z)}{\varepsilon_1(z)} = \frac{\begin{pmatrix} \left(1+\theta_2+\phi_1\right)\left(\theta_1\theta_2-\phi_1\phi_2\right)-z^2\left(1+\phi_1+\phi_1^2+\theta_2\left(1+\theta_1+\phi_1+\phi_2\right)\right)+ \\ z\left((1+\theta_2)\phi_2+(1+\theta_2)\phi_1(1+\phi_2)+\phi_1^2(1+\phi_2)-\theta_1\theta_2(\theta_2+\phi_1)\right) \end{pmatrix}}{\theta_1\theta_2-\phi_1\phi_2+z(\phi_1+\phi_2)-z^2} \tag{34}$$

from which we obtain

$$\frac{\sigma_{O_1+O_2}^2}{\sigma_{\varepsilon_1}^2} = \begin{pmatrix} 2\begin{pmatrix} \phi_1^3+\theta_2\left(1+\theta_1(1+\theta_2)+\phi_2(1+\theta_2)\right)+\phi_1\left(1+\theta_2\left(1+\theta_1+\theta_2+\phi_2\right)\right)+ \\ (1+2\theta_2)\phi_1^2 - \dfrac{\theta_2\phi_2\left(1+(\theta_2-\phi_2)(\theta_1+\phi_2)\right)}{\left(\theta_1\theta_2+(\phi_2-1)^2\right)\left(1+\theta_1\theta_2-\phi_1\phi_2\right)\left(\theta_1\theta_2+(1+\phi_2)^2\right)} \end{pmatrix} + \\ \dfrac{(1+\theta_2-\phi_2)^2(\phi_2-1)}{2\left(\theta_1\theta_2+(\phi_2-1)^2\right)\left(\theta_1\theta_2+\phi_1+\phi_2(1-\phi_1)-1\right)} + \dfrac{(1+\phi_2)(1-\theta_2+\phi_2)^2}{2\left(1-\theta_1\theta_2+\phi_1+\phi_2(1+\phi_1)\right)\left(\theta_1\theta_2+(1+\phi_2)^2\right)} \end{pmatrix} \tag{35}$$

which together with $\sigma_{O_1+O_2}^2 / \sigma_{\varepsilon_2}^2$ yields the variance of the combined production orders

$$\sigma_{O_T}^2 = \begin{pmatrix} \dfrac{1}{\phi_1^2-1}\begin{pmatrix} 2\sigma_{\varepsilon_2}^2\theta_1\left(1+\theta_2(1+\theta_1)+\phi_1(1+\theta_1)\right)\left(\phi_1^2-1\right)+ \\ \sigma_{\varepsilon_1}^2\left(2\theta_1\theta_2(1+\theta_2+\phi_1)(\phi_1^2-1)+2\phi_1(\phi_1^2-1)(1+\theta_2(1+\theta_2+2\phi_1)+\phi_1(1+\phi_1))-1\right) \end{pmatrix} + \\ 2\phi_2\begin{pmatrix} \sigma_{\varepsilon_1}^2\theta_2(1+\theta_2+\phi_1)+ \\ \sigma_{\varepsilon_2}^2\left(1+\theta_1(1+\theta_1+\theta_2+\phi_1)\right) \end{pmatrix} + \dfrac{\left(\sigma_{\varepsilon_1}^2\theta_2^2+\sigma_{\varepsilon_2}^2(\phi_1-1)^2\right)(1+\theta_1-\phi_1)^2}{2\left(\theta_1\theta_2+(\phi_1-1)^2\right)(\phi_1-1)\left(\theta_1\theta_2+\phi_2+\phi_1(1-\phi_2)-1\right)} + \\ 2\sigma_{\varepsilon_2}^2(1+2\theta_1)\phi_2^2+2\sigma_{\varepsilon_2}^2\phi_2^3 - \dfrac{2\phi_1\left(\sigma_{\varepsilon_2}^2\theta_1-\sigma_{\varepsilon_1}^2\theta_2\right)\left(1+(\theta_1-\phi_1)(\theta_2+\phi_1)\right)}{\left(\theta_1\theta_2+(\phi_1-1)^2\right)\left(\theta_1\theta_2+(1+\phi_1)^2\right)(1+\theta_1\theta_2-\phi_1\phi_2)} + \\ 2\sigma_{\varepsilon_1}^2\theta_2 + \dfrac{(1-\theta_1+\phi_1)^2\left(\sigma_{\varepsilon_1}^2\theta_2^2+\sigma_{\varepsilon_2}^2(1+\phi_1)^2\right)}{2(1+\phi_1)\left(\theta_1\theta_2+(1+\phi_1)^2\right)\left(1-\theta_1\theta_2+\phi_2+\phi_1(1+\phi_2)\right)} \end{pmatrix}. \tag{36}$$

16

In Figure 7 we have plotted (36) when $\sigma^2_{\varepsilon_1} = \sigma^2_{\varepsilon_2} = 1$. For our specific chocolate case example, we obtain a combined order variance of $\sigma^2_{O_T} = 30710.2$.
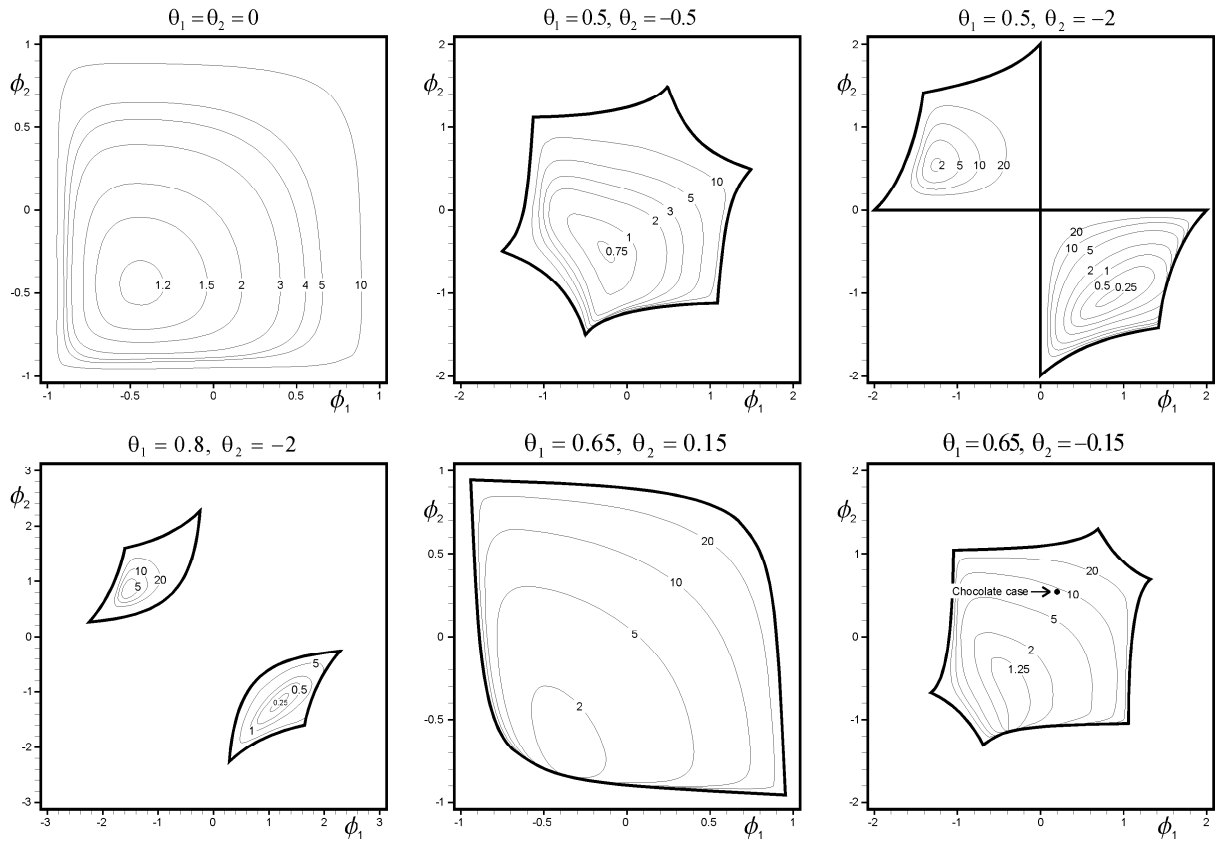

Figure 7. Variance of the combined orders

Table 2 shows the resulting impact on the capacity costs. In this specific example, we find that consolidating the capacity reduces the total costs by 3% when compared to case 1, the separated inventory separate capacity scenario.

Cost	TNS^* / S^*	Cost Formula	Chocolate case cost	
			Theory	Simulation
Inventory cost for product 1	$TNS_1^* = \sigma_{NS_1} \Phi^{-1}\left[\frac{B}{B+H}\right]$ $= 187.717$	$C_{NS_1} = \sigma_{NS_1}(B+H)\varphi\left[\Phi^{-1}\left[\frac{B}{B+H}\right]\right]$	257.06	256.99
Inventory cost for product 2	$TNS_2^* = \sigma_{NS_2} \Phi^{-1}\left[\frac{B}{B+H}\right]$ $= 126.785$	$C_{NS_2} = \sigma_{NS_2}(B+H)\varphi\left[\Phi^{-1}\left[\frac{B}{B+H}\right]\right]$	173.62	173.59
Capacity cost to produce 1 & 2 on same production line	$S_T^* = \sigma_{O_T} \Phi^{-1}\left[\frac{W-U}{W}\right]$ $= -78.48$	$C_{O_T} = \left(\begin{array}{l}\left(\mu_{d_1} + \mu_{d_2}\right)U + \\ \sigma_{O_T}W\varphi\left[\Phi^{-1}\left[\frac{W-U}{W}\right]\right]\end{array}\right)$	2622.31	2621.82
		Total cost of case 2 =	3052.99	3052.40

Table 2. Cost optimisation and validation of the separate inventory, consolidated capacity case

5.5. Consolidated Inventory Levels and Separate Production

In this scenario we consider a consolidated inventory for both SKU's and separate production lines. This may mean that the same product is produced on two different lines, and its customized packaging is postponed until the order is effectively received, or that lateral transhipments are used between the different production sites, to allow the inventory to be considered as a single (or virtual) inventory.

The order variance for each of the two products that are produced on separated production lines was already given in (31). The inventory variance for the combined inventory levels is a little more tedious to obtain but can be found with the following procedure. First we find, for each noise process ε_j, $j \in \{1,2\}$, the transfer function of the inventory levels. It can be obtained from

$$\frac{ns_T(z)}{\varepsilon_j(z)} = \frac{z}{z-1}\left(\left(\frac{o_{d_1}(z)}{\varepsilon_j(z)} + \frac{o_{d_2}(z)}{\varepsilon_j(z)}\right)z^{-Tp-1} - \left(\frac{d_1(z)}{\varepsilon_j(z)} + \frac{d_2(z)}{\varepsilon_j(z)}\right)\right), \tag{37}$$

which lead to the following pair of transfer functions,

$$\left\{\frac{ns_T(z)}{\varepsilon_1(z)} = -\frac{1+z+\theta_1+\phi_2}{z}, \quad \frac{ns_T(z)}{\varepsilon_2(z)} = -\frac{1+z+\theta_2+\phi_1}{z}\right\}. \tag{38}$$

Then, using Jury's Inners Approach we find the contribution of each noise process to the variance of the combined inventory levels. These variance expressions are then added together to yield the following expression describes the variance of the total inventory levels.

$$\sigma^2_{NS_T} = \sigma^2_{\varepsilon_1}\left(2\left(1+\theta_2+\phi_1\left(1+\theta_2\right)\right)+\theta_2^2+\phi_1^2\right) + \sigma^2_{\varepsilon_2}\left(2\left(1+\theta_1+\phi_2\left(1+\theta_1\right)\right)+\theta_1^2+\phi_2^2\right) \tag{39}$$

When $\sigma^2_{\varepsilon_1} = \sigma^2_{\varepsilon_2} = 1$ (39) shows that the Net Stock variance has a circular nature with a minimum of $\sigma^2_{NS} = 2$ at $\{\phi_1 = -1-\theta_2, \phi_2 = -1-\theta_1\}$. In order to visualise the behaviour of the five dimensional space in (39) we have provided Figure 8 where we have plotted the function $2\left(1+\theta_m+\phi_n\left(1+\theta_m\right)\right)+\theta_m^2+\phi_n^2$ (where $\{m,n\} \in \{1,2\}$, $m \neq n$) due to the repeated structure of (39).

Returning to our industrially based numerical example we see from Table 3 that inventory pooling results in a 25% reduction in inventory costs (whilst offering the same 90% service level). In our numerical example, consolidating the inventory has a slightly bigger economic impact than consolidating the capacity, providing a cost reduction of 3.6%. The net stock variance is now $\sigma^2_{NS_T} = 33921.6$.

5.6. Consolidated Inventory and Aggregated Production Capacity

In our last scenario we consider the case when both capacity and inventory can be pooled. The required variance expressions have already been presented in (36) and (39). The resulting costs for our chocolate case are given in Table 4 which show that when we pool both inventory and capacity then the lowest costs from all four scenarios are obtained.
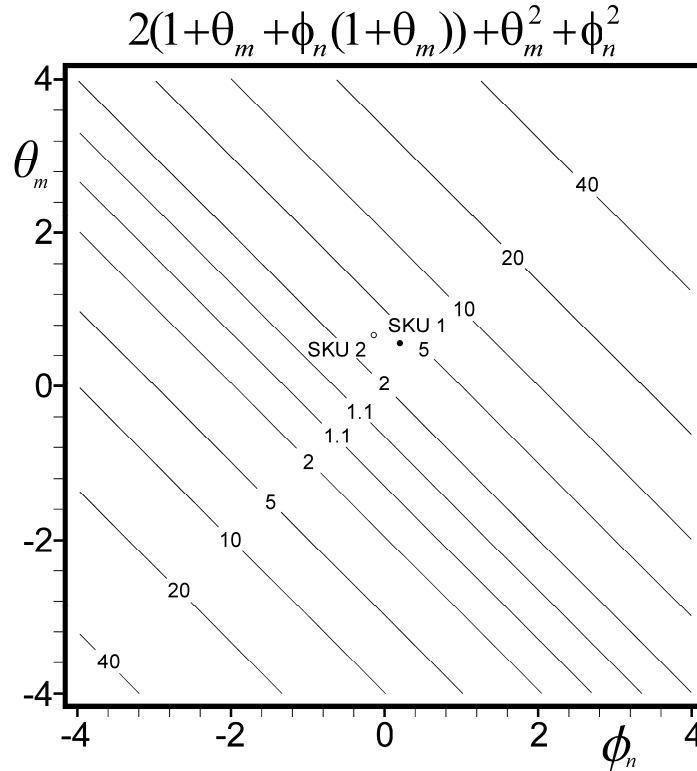
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$$2(1+\theta_m+\phi_n(1+\theta_m))+\theta_m^2+\phi_n^2$$

Figure 8. Components of the combined inventory variance

Cost	TNS^*/S^*	Cost Formula	Chocolate case cost	
			Theory	Simulation
Inventory cost for product 1 & 2 when the inventory is consolidated	$TNS_T^* = \sigma_{NS_T}\Phi^{-1}\left[\frac{B}{B+H}\right]$ $= 236.03$	$C_{NS_T} = \sigma_{NS_T}(B+H)\varphi\left[\Phi^{-1}\left[\frac{B}{B+H}\right]\right]$	323.23	323.18
Capacity cost for product 1	$S_1^* = \sigma_{O_1}\Phi^{-1}\left[\frac{W-U}{W}\right]$ $= -52.35$	$C_{O_1} = \mu_{d_1}U + \sigma_{O_1}W\varphi\left[\Phi^{-1}\left[\frac{W-U}{W}\right]\right]$	1681.15	1680.99
Capacity cost for product 2	$S_2^* = \sigma_{O_2}\Phi^{-1}\left[\frac{W-U}{W}\right]$ $= -41.83$	$C_{O_2} = \mu_{d_2}U + \sigma_{O_2}W\varphi\left[\Phi^{-1}\left[\frac{W-U}{W}\right]\right]$	1035.91	1035.87
		Total cost of case 4 =	3040.29	3040.05

Table 3. Cost optimisation and validation of the consolidated inventory and separate production capacity case

5.7. The question of when to consolidate

Consolidation results in a lower capacity cost if $\sigma_{O_T}-\sigma_{O_1}-\sigma_{O_2}<0$. Using (31), (36) and (8) to model these variances, results in a complex expression but we can numerically evaluate it efficiently, see Figure 9, where we assumed $\sigma_{\varepsilon_1}^2=\sigma_{\varepsilon_2}^2=1$. We see that in virtually all regions, capacity costs can be reduced by consolidating the capacity. Only in the case when $-\theta_2 \gg \theta_1$ are there instances where capacity consolidation is not desirable. Similarly, it is economically beneficial to consolidate inventory if $\sigma_{NS_T}-\sigma_{NS_1}-\sigma_{NS_2}<0$. As the required expressions (33) and (39) are much simpler we can obtain the following expression that describes the $\sigma_{NS_T}-\sigma_{NS_1}-\sigma_{NS_2}=0$ boundary,

Cost	TNS^*/S^*	Cost Formula	Chocolate case cost	
			Theory	Simulation
Inventory cost for product 1 & 2 when the inventory is consolidated	$TNS_T^* = \sigma_{NS_T} \Phi^{-1}\left[\frac{B}{B+H}\right]$ $= 236.03$	$C_{NS_T} = \sigma_{NS_T}(B+H)\varphi\left[\Phi^{-1}\left[\frac{B}{B+H}\right]\right]$	323.23	323.38
Capacity cost to produce 1 & 2 on same production line	$S_T^* = \sigma_{O_T}\Phi^{-1}\left[\frac{W-U}{W}\right]$ $= -75.48$	$C_{O_T} = \left(\begin{array}{c}\left(\mu_{d_1}+\mu_{d_2}\right)U + \\ \sigma_{O_T}W\varphi\left[\Phi^{-1}\left[\frac{W-U}{W}\right]\right]\end{array}\right)$	2622.31	2622.36
		Total cost of case 4 =	2945.54	2945.74

Table 4. Cost optimisation and validation of the consolidated inventory and aggregated capacity case

$$\phi_1 = \frac{\theta_1\theta_2(1+\phi_2)-2-\phi_2(2+\phi_2)\pm\sqrt{-\left(2+\theta_1^2+\phi_2(2+\phi_2)\right)\left(2+\theta_2^2+\phi_2(2+\phi_2)\right)}}{2+\phi_2(2+\phi_2)} . \tag{40}$$

ϕ_1 is complex, meaning that the consolidated inventory is always less than the separated inventory, if

$$-\left(2+\theta_1^2+\phi_2(2+\phi_2)\right)\left(2+\theta_2^2+\phi_2(2+\phi_2)\right) < 0 \tag{41}$$

which is easy to see is always true when $\{\theta_1,\theta_2,\phi_2\}\in\Re$. Numerical investigations also verify that it is always beneficial to consolidate inventory for the settings in Figure 9.

6. Discussion and Concluding Remarks

We have used order and inventory variances as inputs into a cost function to measure and evaluate the performance of a factory setting in a multi-product environment. High bullwhip implies wildly fluctuating production orders, meaning that significant safety capacity should be maintained when planning production. The higher the inventory variance, the more safety stock is required to guarantee a target availability of product. Using a Vector-Autoregressive VAR(1) demand process for two SKU's, we gathered variance expressions on an individual SKU basis and when combined together, and we demonstrated the impact of pooling SKU's in production and / or inventory. Typically, bullwhip is to be defined at the level of a "group of products", since the SKU's may share the same production or distribution facilities. In some cases, inventory variance may also be aggregated, for instance when the SKU's are supplied in identical packaging to different customers. However, in the other case of customer specific packaging, inventory is to be defined at the level of the "individual product".

Using real-life case data from a consumer packaged chocolate supplier, we illustrated the impact of defining the variance amplification ratios at the appropriate level. Clearly, when orders are aggregated for production on the same production capacity, the bullwhip effect must be defined accordingly. Its measure will be different from the traditional bullwhip measure defined at the SKU level, and so will its production capacity requirements. Needless to say that it is not always the case that several products can be aggregated on the same line. In our chocolate example for instance, chocolate with nuts is not produced on the same line

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as chocolate without nuts for safety reasons. In other situations the changeovers may be too time and / or cost intensive. Hence the bullwhip consequences are incurred on an individual SKU basis, which will result in higher safety capacity requirements.
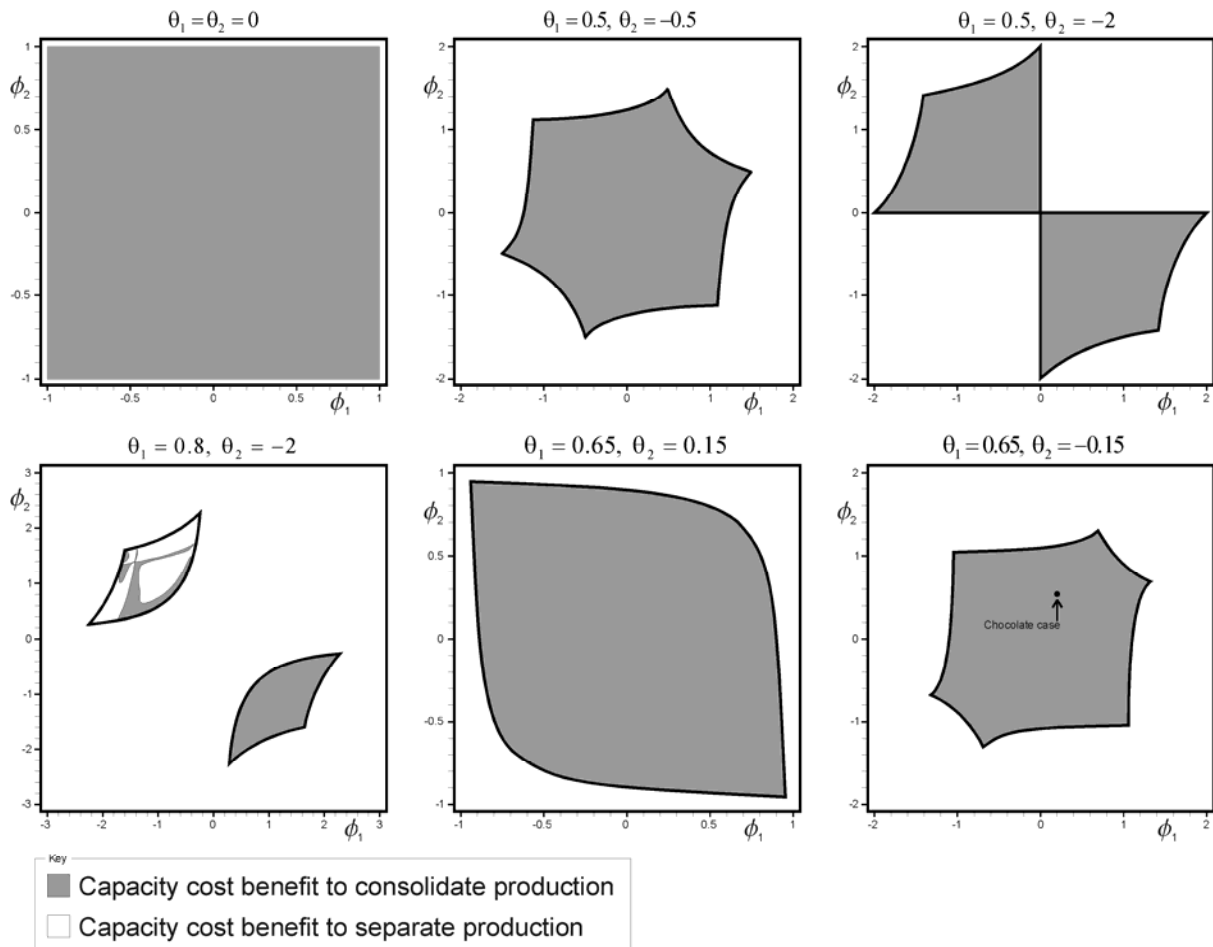

Figure 9. Capacity cost benefit from consolidation

Table 5 summarizes the results of our chocolate case example. Consolidating both products on the same production line reduces the production costs by 3% compared to the dedicated production lines, due to the reduced (aggregated) bullwhip effect over both products. Aggregating inventories to hold one safety stock for both SKU's reduces the inventory requirements by 25% (to guarantee a 90% availability).

Our analysis also sheds light on the differences between private label and branded A-product suppliers. For private labels we usually only have one customer for a given SKU. The enormous swings in orders hurt inventories & service levels. From a capacity point of view, there might be a balancing effect when multiple customer demands can be satisfied on the same production line. Traditionally, branded labels have the benefit to pool across different customers and, depending on the demand parameters, can reap substantial inventory reductions. However, a recent trend in the branded label industry shows that retail customers also demand their own customized SKU's, very similar to the private labels. From the above analysis it is clear to what extent this may increase their inventory requirements. When retailers would additionally impose their individual quality specs in production, then

suppliers would also lose their pooling benefits in production and more safety capacity is required. The increased production and inventory costs introduce a tension between retailers and suppliers, potentially hurting overall supply chain profitability.

Costs		Case 1	Case 2	Case 3	Case 4
		Separate inventory & separate capacity	Separate inventory & shared capacity	Combined inventory & separate capacity	Combined inventory & shared capacity
Absolute cost	Inventory	430.7	430.7	323.2	323.2
	Capacity	2717.1	2622.3	2717.1	2622.3
	Total	3147.7	3053.0	3040.3	2945.5
Cost index	Inventory	100%	100%	75%	75%
	Capacity	100%	97%	100%	97%
	Total	100%	97%	96.6%	93.6%

Table 5. Summary costs in the chocolate case

References

Box, E.P., Jenkins, G.M. and Reinsel, G.C., (1994), *Time series analysis: Forecasting & control*. John Wiley and Sons, New Jersey.

Chen, L. and Lee, H.L., (2011), "Bullwhip effect measurement and its implications". Working paper from the Graduate School of Business, Stanford University.

Chaharsooghi, S. K. and Sadeghi, A., (2008), "On the bullwhip effect measure in supply chains with VAR(1) demand processes". *International Journal of Industrial Engineering & Production Research*, 19 (4): 9-19.

Disney, S.M., Farasyn, I., Lambrecht, M.R., Towill, D.R., Van de Velde, W., (2006), "Taming the bullwhip effect whilst watching customer service in a single supply chain echelon". *European Journal of Operational Research*, 173: 151–172.

Disney, S.M., Gaalman, G. and Hosoda, T., (2012), "Review of stochastic cost functions for production and inventory control". Pre-prints of the 17th International Working Seminar of Production Economics, Innsbruck, Austria, February 20th -24th, 1: 117-128.

Disney, S.M. and Lambrecht, M.R., (2007), "On replenishment rules, forecasting, and the bullwhip effect in supply chains". Foundations and Trends in Technology, *Information and Operations Management*, 2 (1): 1–80.

Farasyn, I., Humair, S., Kahn, J.I., Rosen, O., Ruark, J., Tarlton, W., Van de Velde, W., Wegryn, G. and Willems, S.P., (2011), "Inventory optimisation at Procter & Gamble: Achieving real benefits through user adoption of inventory tools". *Interfaces*, 41 (1): 66-78.

Fransoo, J.C. and Wouters, M.J.F., (2000), "Measuring the bullwhip effect in the supply chain". *Supply Chain Management*, 5 (2): 78-89.

Hammond, J. (1994). Barilla SpA (a). Harvard Business School Case No. 9-694-046.

Hedenstierna, C.P.T. and Disney, S.M., (2012), "Impact of scheduling frequency and shared capacity on production and inventory costs". Pre-prints of the 17th International Working Seminar of Production Economics, Innsbruck, Austria, February 20th -24th, 2: 277-288.

Hosoda, T., Disney, S.M., (2006), "On variance amplification in a three-echelon supply chain with minimum mean squared error forecasting". *OMEGA: The International Journal of Management Science*, 34 (4): 344-58.

Jury, E. I., (1974), *Inners and stability of dynamic systems*. Wiley, London.

Lee, H.L., Padmanabhan, V. and Whang, S., (1997a), "Information distortion in a supply chain: the bullwhip effect". *Management Science*, 43 (4): 546–558.

Lee, H.L., Padmanabhan, V. and Whang, S., (1997b), "The bullwhip effect in supply chains". *Sloan Management Review*, 38 (4): 93–102.

Li, Q. and Disney, S.M., (2012), "Damped trend forecasting and the Order-Up-To replenishment policy". *Pre-prints of the 17th International Working Seminar of Production Economics*, Innsbruck, Austria, February 20th -24th, 3: 205-216.

Moudgalya, K.M., (2007), *Digital Control*, John Wiley, Chichester, UK.

Ratanachote, P., "Distribution network dynamics with correlated demands", PhD Dissertation, Cardiff Business School, Cardiff University, United Kingdom.

Sucky, E., (2009) "The bullwhip effect in supply chains – An overestimated problem?". *International Journal of Production Economics*, 118: 311-322.

Waller, M.A., Williams, B.D. and Eroglu C., (2008), "Hidden effects of variable order review intervals in inventory control". *International Journal of Physical Distribution & Logistics Management*, 38 (3): 244–258.