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STABILITY ANALYSIS OF A CONSTRAINED INVENTORY SYSTEM

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Abstract

The stability of a production and inventory system where negative orders are forbidden is investigated via an eigenvalue analysis of a piecewise linear model and a simulation study. The Automatic Pipeline, Variable Inventory and Order Based Production Control System is adopted. All classes of dynamic behaviour in nonlinear systems can be observed in this stylised model with only one constraint. Exact expressions for the asymptotic stability and Lyapunovian stability boundaries are derived when the replenishment lead-time is both one and two periods long. Asymptotically stable regions in the nonlinear system are identical to the stable regions in its linear counterpart. However, regions of bounded fluctuations that continue forever exist in the parameter plane. Simulation reveals an intriguing and delicate structure within these regions. Our results show that ordering policies have to be both designed properly and use accurate lead-time information to avoid such undesirable behaviour.

Keywords:

Supply chain dynamics; constraint; piecewise linear system; stability; simulation

1 INTRODUCTION

One of the main objectives to design an inventory control system is to maintain its stability and robustness when responding to external disturbances. Since the introduction of control theory and other system dynamic approaches into the field of inventory management, much research has been dedicated to this problem. However, the significance of many results is limited in their ability to cope with uncertainty and complexity in the system structure. For example, capacity and non-negativity constraints are typically omitted.

In earlier works on supply chain stability [1, 2], linear inventory system models are usually adopted. To maintain linearity, order rates are allowed to take negative values. This means that participants are permitted to return product to their supplier without cost and in zero time. If not returned, the excess inventory instead may be considered to be owned by the supplier, until being used as part of a future replenishment [4].

When the return of goods is forbidden (i.e. negative order are not allowed), the behaviour of the replenishment policy becomes much richer, sometimes even chaotic or super-chaotic, see [5, 6, 7]. [8] has attempted to control such inventory systems with H∞ techniques. Border-collision behaviour has been studied in [9]. The mathematical properties of nonlinear systems, such as bifurcations, local and global stability conditions, are "very hard to investigate" and "notoriously challenging" [10].

In an ideal situation, supply chain participants would have perfect knowledge of a constant production lead-time. In practical situations however, this may not be true. Time varying and incorrect lead-times are commonly observed in industry, see Figure 1 derived from [11]. Varying lead-times has been shown to increase supply chain cost [12], exacerbates the bullwhip effect [13] and significantly affects decision making [14]. Incorrect lead-time information also creates an undesirable phenomenon known as 'inventory drift' [15].

This paper investigates the effect of lead-time on constrained supply chain stability. Section 2 models the constrained one echelon supply chain system using piecewise-linear techniques and conducts an eigenvalue analysis; Section 3 investigates the stability of the inventory system with perfect lead-time knowledge when lead-time is both 1 and 2 periods long; Section 4 focuses on the effect of incorrect lead-time information on system stability and dynamic patterns; Section 5 concludes.

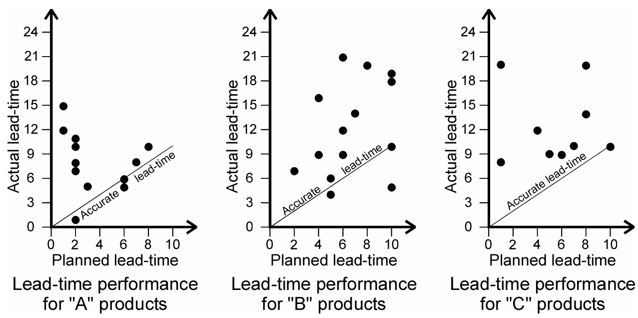

Figure 1: Lead-time performance in an industrial setting

2 MODEL OF CONSTRAINED INVENTORY SYSTEMS

We study a replenishment policy known as the Automatic Pipeline, Variable Inventory and Order Based Production Control System (APVIOPBCS). This policy has been frequently adopted and researched as it is of a very general nature. The popular order-up-to (OUT) policy is a special case of the APVIOBPCS model. We do not define our notation here due to space limitations, but refer interested readers to [15] if they require this information.

The difference equations for the inventory system are

$$AVCON_t = \frac{1}{T_a+1} CONS_t + \frac{T_a}{T_a+1} AVCON_{t-1}\,, \tag{1}$$

$$AINV_t = AINV_{t-1} + COMRATE_t - CONS_t\,, \tag{2}$$

$$WIP_t = WIP_{t-1} + ORATE_{t-1} - COMRATE_t\,, \tag{3}$$

$$TRANS_t = ORATE_{t-1};\; TRANS2_t = TRANS_{t-1} \tag{4}$$

$$COMRATE_t = \begin{cases} TRANS_{t-1} & \text{if } T_p = 1 \\ TRANS2_{t-1} & \text{if } T_p = 2 \end{cases}, \tag{5}$$

$$ORATE_t = \left((1 + a\alpha_S + \alpha_{SL}) AVCON_t - \alpha_S AINV_t - \alpha_{SL} WIP_t\right)^+ \tag{6}$$

Here $\{TRANS, TRANS2\}$ are assistant variables required to describe the transportation delay in matrix form. Forbidden returns (non-negative orders) are enforced with the maximum operator ($(x)^+ = \max(0,x)$) in (6). The above difference equations are easily converted into matrices that describe the system of equations. For example, when $T_p = 1$ we have,

$$\mathbf{A}_1 = \begin{bmatrix} \frac{T_a}{T_a+1} & 0 & 0 & 0 & 0 \\ \frac{T_a(1+a\alpha_S+\alpha_{SL})}{T_a+1} & -\alpha_{SL} & -\alpha_S & -\alpha_{SL} & \alpha_{SL}-\alpha_S \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{b}_1 = \begin{bmatrix} \frac{1}{T_a+1} \\ \alpha_S + \frac{1+a\alpha_S+\alpha_{SL}}{T_a+1} \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} \frac{T_a}{T_a+1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{b}_2 = \begin{bmatrix} \frac{1}{T_a+1} & 0 & -1 & 0 & 0 \end{bmatrix}^T.$$ [1]

The piecewise affine model for this system is given by

$$\mathbf{x}_t = \begin{cases} \mathbf{A}_1\mathbf{x}_{t-1} + \mathbf{b}_1 CONS_t, & ORATE_t > 0 \\ \mathbf{A}_2\mathbf{x}_{t-1} + \mathbf{b}_2 CONS_t, & ORATE_t < 0 \end{cases} \tag{7}$$

Let $\bar{\mathbf{A}}_1 = \begin{bmatrix} \mathbf{A}_1 & \mathbf{b}_1 \\ 0 & 1 \end{bmatrix}$, $\bar{\mathbf{A}}_2 = \begin{bmatrix} \mathbf{A}_2 & \mathbf{b}_2 \\ 0 & 1 \end{bmatrix}$, and

$\bar{\mathbf{x}} = \begin{bmatrix} AVCON & ORATE & AINV & WIP & TRANS & CONS \end{bmatrix}^T$.

Rewriting (7) we may eliminate the affine terms:

$$\bar{\mathbf{x}}_t = \begin{cases} \bar{\mathbf{A}}_1\bar{\mathbf{x}}_{t-1}, & \mathbf{x} \in S_1 \\ \bar{\mathbf{A}}_2\bar{\mathbf{x}}_{t-1}, & \mathbf{x} \in S_2 \end{cases} \tag{8}$$

The bars in $\mathbf{A}$ and $\mathbf{x}$ will be dropped in the following text when no confusion occurs. $S_1 = \{\mathbf{x} \mid ORATE \geq 0\}$ and $S_2 = \{\mathbf{x} \mid ORATE < 0\}$ are both non-degenerate polyhedral partitions of the state space. That is, each region S_i is a (convex) polyhedron with a non-empty interior. The $(n–1)$ dimensional hyper-plane $ORATE = 0$ is the boundary of the partitions. $S_1 \bigcup S_2 = \mathbb{R}^n$, $S_1^\circ \bigcap S_2 = S_1 \bigcap S_2^\circ = \varnothing$. S° is the interior of S and n is the dimension of $\mathbf{x}$. It should be noted that the boundaries are continuous, i.e., $\mathbf{A}_1\mathbf{x}_t = \mathbf{A}_2\mathbf{x}_t$ when $\mathbf{x}_t \in S_1 \bigcap S_2$. Common concepts, such as region and boundary, will be used in either the phase space or the parametrical space.

For $T_p = 1$, the eigenvalues for $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_1 \times \mathbf{A}_2$ are:

$$\boldsymbol{\lambda}_{\mathbf{A}_1} = \left[\frac{T_a}{T_a+1}, 0, \frac{1-\alpha_{SL} \pm \sqrt{(1+\alpha_{SL})^2 - 4\alpha_S}}{2}, 1, 1 \right];$$

$$\boldsymbol{\lambda}_{\mathbf{A}_2} = \left(\frac{T_a}{T_a+1}, 0, 0, 1, 1, 1 \right); \ \boldsymbol{\lambda}_{\mathbf{A}_1 \times \mathbf{A}_2} = \left[\frac{T_a^2}{(T_a+1)^2}, 1-\alpha_S, 0, 0, 1, 1 \right].$$

The eigenvalues associated with the forecasting system can be ignored. However $\mathbf{A}_1$ has 2 and $\mathbf{A}_1 \times \mathbf{A}_2$ has 1 eigenvalue(s) associated with the feedback loops which do need to be investigated. The eigenvalues of these matrices, yield parametrical regions in which the inventory system behaves differently and these will now be derived and investigated.

[1] When lead-time is 2 the size of these matrices increases by one to incorporate the extra assistant variable, *TRANS2*. This change is rather obvious and we omit it for space reasons.

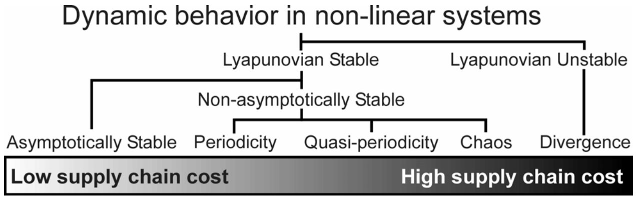

Figure 2: Categorization of dynamic behaviours of a nonlinear system

3 STABILITY OF A CONSTRAINED APVIOBPCS

In linear systems there are only two patterns of dynamic behaviours which are physically realizable: stable (convergence) and unstable (divergence). In nonlinear systems however, the range of dynamic behaviours the system could exhibit is much larger. First, the fixed point of the system could be a saddle-node. A saddle node is a point that attracts trajectories from some directions and repels trajectories from other directions. Second, there are other types of attractors such as limit cycles and strange attractors that may be present. Trajectories of nonlinear systems could be either convergent or divergent and can even oscillate in a bounded fashion. It could oscillate in a regular repeating pattern or in a seemingly random one. The system could also be highly sensitive to initial values. Dynamic behaviours in nonlinear systems are categorized in Figure 2. We have also ranked the different dynamic behaviours for an intuitive supply chain cost viewpoint in Figure 2. Detailed investigations on the stability of the constrained supply chain will now be conducted.

3.1 Lyapunovian stability and divergence

If all solutions of the dynamical system that start out near an equilibrium point $\mathbf{x}_e$ and stay near $\mathbf{x}_e$ forever, then the system is *Lyapunovian stable*. Note that, even when the system fluctuates but never approaches the equilibrium, as long as the fluctuation is bounded, the system is Lyapunovian stable. Lyapunovian instability indicates an unbounded oscillation and a trajectory tends to infinity. That is, it diverges.

The criteria for exponential monotonic divergence is $\mathrm{Im}(\boldsymbol{\lambda}_{\mathbf{A}_1}) = 0$ and $\left|\mathrm{Re}(\boldsymbol{\lambda}_{\mathbf{A}_1})\right| > 1$, where $\mathrm{Im}(z)$ and $\mathrm{Re}(z)$ are imaginary and real parts of complex number z respectively. These relations give the Lyapunovian stability regions. In linear systems, if the eigenvalues of system matrices are complex, systems oscillate with exponential divergence. However, in piecewise linear systems, such trajectories will eventually hit the boundary. In other words, the boundary constrains or limits such trajectories from divergence. The direction of exponential monotonic divergence is always away from the boundary.

We can derive the parametrical boundaries that separate exponential divergence from bounded responses. For T_p = 1, when $\mathbf{A}_1$ has two real eigenvalues greater than 1 exponential divergence can be observed. Thus, the Lyapunovian stability boundary for $T_p = 1$ is

$$\alpha_S = (\alpha_{SL} + 1)^2 / 4. \tag{9}$$

For $T_p = 2$, the Lyapunovian stability boundary is given by

$$27\alpha_S = 4 + 15\alpha_{SL} + 12\alpha_{SL}^2 - 4\alpha_{SL}^3 . \tag{10}$$

3.2 Asymptotic stability

If $\mathbf{x}_e$ is Lyapunovian stable and all solutions that start near $\mathbf{x}_e$ converge to $\mathbf{x}_e$, then more strongly, $\mathbf{x}_e$ is *asymptotically stable*. This means that the trajectory approaches an equilibrium point over time. This concept has similar meaning with stability in classical linear control theory. In the asymptotically stable region, the system will eventually return to equilibrium. The criteria for asymptotic stability is $\left|\boldsymbol{\lambda}_{\mathbf{A}_1}\right| = 1$. When $T_p = 1$ this amounts to

$$\alpha_S = 2\alpha_{SL} - 2; \tag{11}$$

$$\alpha_S = 0; \tag{12}$$

$$\alpha_S = \alpha_{SL} + 1. \tag{13}$$

For $T_p = 2$, the following conditions hold for asymptotic stability:

$$0 < \alpha_S < \frac{2\left(1+\alpha_{SL} - 2\alpha_{SL}^2\right)}{1 - 3\alpha_{SL} + \sqrt{\alpha_{SL}^2 - 2\alpha_{SL} + 5}} \text{ for } -0.5 < \alpha_{SL} < 1; \tag{14}$$

$$\frac{2\left(1+\alpha_{SL} - 2\alpha_{SL}^2\right)}{1 - 3\alpha_{SL} - \sqrt{\alpha_{SL}^2 - 2\alpha_{SL} + 5}} < \alpha_S < 2 \text{ for } 1 < \alpha_{SL} < 2.5. \tag{15}$$

3.3 Periodicity

Periodicity of a function is a point which the system returns to after a certain number of function iterations or a certain amount of time. The boundaries of the periodic region of $T_p = 1$ system can be derived by solving $\left|\boldsymbol{\lambda}_{\mathbf{A}_1 \times \mathbf{A}_2}\right| < 1$, where $S_1S_2S_1S_2\ldots$ is the periodical movement in this region. Here $\mathbf{A}_1$ is unstable but $\mathbf{A}_1 \times \mathbf{A}_2$ is stable. This makes the periodical movement $S_1S_2S_1S_2\ldots$ fall into a repeating pattern. Solving $\left|1-\alpha_S\right| = 1$ to investigate whether the only undetermined eigenvalue of $\mathbf{A}_1 \times \mathbf{A}_2$ is inside the unit circle, we obtain the boundary of periodic region: $0 < \alpha_S < 2$. When $T_p = 2$, the boundary of period-3 is $\alpha_S = 2$, which is identical to $T_p = 1$ case. This boundary can be derived by solving the asymptotic stability of $\mathbf{A}_1\mathbf{A}_2^2$. Generally, if period $S_1^m S_2^n$ is discovered to be stable under certain parameter settings, where $m, n \in \mathbb{Z}^+$ and an exponential form is used to express the trajectory staying in one region, then the matrix $\mathbf{A}_1^m\mathbf{A}_2^n$ is asymptotically stable.

3.4 Quasi-periodicity and chaos

Quasi-periodicity is the property of a system that displays irregular periodicity. Quasi-periodic behaviour is a pattern of recurrence with a component of unpredictability that does not lead itself to precise measurement. Quasi-periodic motion is in rough terms the type of motion executed by a dynamical system containing a finite number (two or more) of incommensurable frequencies. Values of quasi-periodic points are dense everywhere.

Mathematically, *chaos* refers to a very specific kind of unpredictability: deterministic behaviour that is very sensitive to its initial conditions. In other words, infinitesimal variations in initial conditions for a chaotic dynamic system lead to large variations in behaviour.

Quasi-periodicity and chaotic motion are both characterized by the fact that, although bounded in phase space, the trajectory never precisely repeats itself. This research shows that, compared to the four-echelon beer-game model adopted in [7], such complex behaviours can be found even in this simple model of a constrained inventory system.

3.5 Division of the parametrical plane and bifurcation analysis

Stability maps for $T_p = 1$ and 2 are shown as Figures 3 and 4. Asymptotic stable regions are represented by black, periodic darker grey, quasi-periodic and chaos lighter grey, and divergent white. When $T_p = 1$, combining Equations 9 and 11-13, boundaries $\alpha_S = 0$, $\alpha_S = 2$, $\alpha_S = \alpha_{SL} + 1$, $\alpha_S = 2\alpha_{SL} - 2$ and $\alpha_S = (\alpha_{SL} + 1)^2 / 4$ divides the plane into several regions in which the inventory system behaves differently. A positive α_S is essential to maintain Lyapunovian stability. Furthermore, when the absolute values of α_{SL} are small, the system will be asymptotically stable. When α_{SL} is negative, exponential divergence can be observed. Regular periodicity can be discovered when α_S is small and α_{SL} is positive and large. Other areas are filled by quasi-periodic and chaotic movements. Notice that there exist periodic areas shaped as branches. What we can infer from the experiments is that high values of α_S and α_{SL} lead to chaos. When $T_p = 2$, there is a similar stability layout in the parameter plane as $T_p = 1$. Figure 5 shows typical bounded dynamic patterns in the time domain.

4 THE EFFECT OF LEAD-TIMES ON STABILITY

4.1 Known lead-time changes

If we draw boundaries of $T_p = 1$ and $T_p = 2$ together as shown in Figure 6, we are able to better visualise the different dynamic behaviours that exist when the lead-time increases from 1 to 2 and we know of this fact. Table 1 summarizes the dynamic behaviour under each lead-time

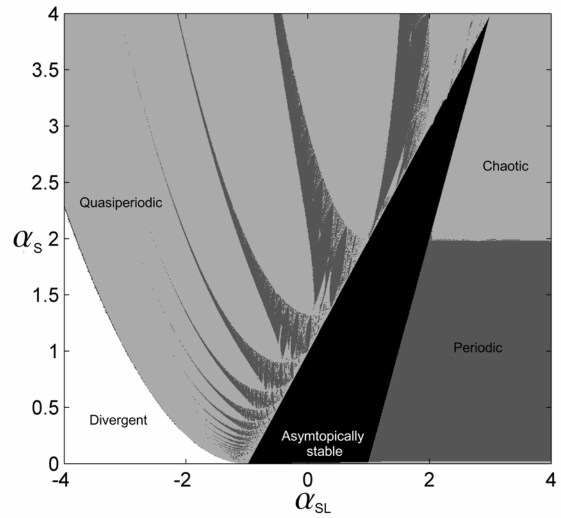

Figure 3: Stability map of $T_p = 1$

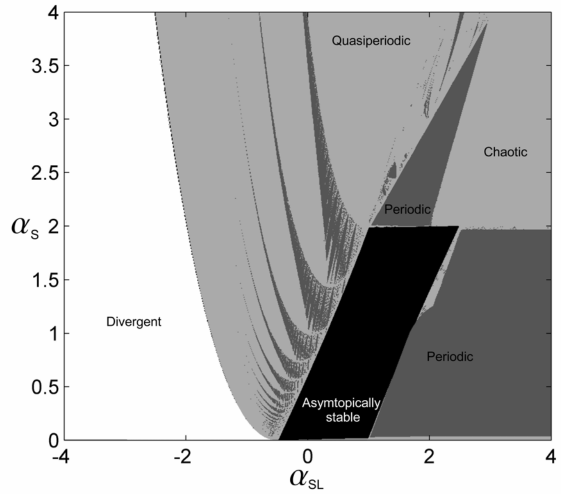

Figure 4: Stability map of $T_p = 2$

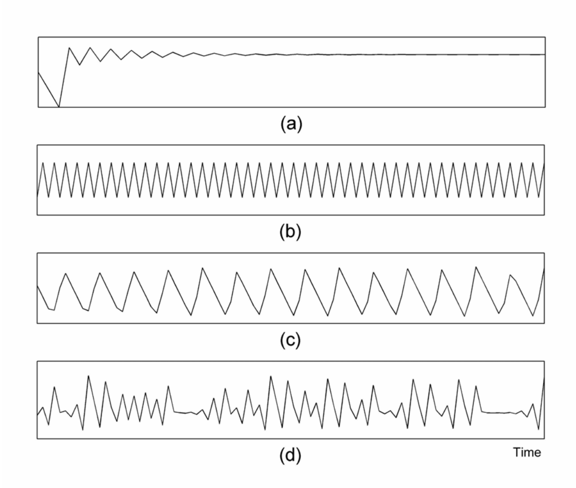

Figure 5: Four typical dynamic patterns (step response) generated by the constrained inventory system. (a) asymptotic stable; (b) periodic; (c) quasi-periodic; (d) chaotic

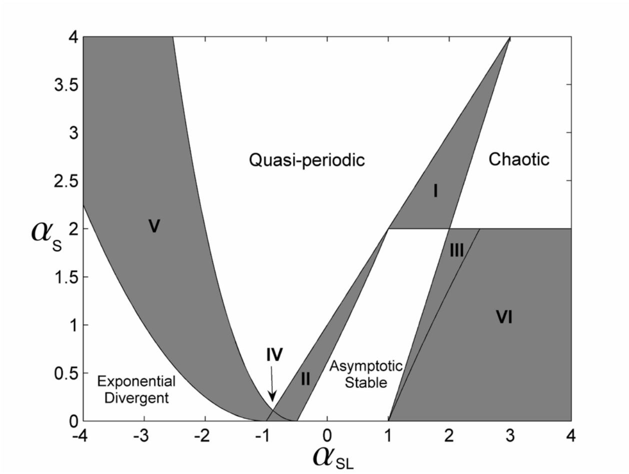

Figure 6: Stability comparison between $T_p = 1$ and $T_p = 2$

Table 1: Change of dynamic behaviour in each region with known T_p changes observed in Figure 6

	$T_p = 1$, $T_{\bar{p}} = 1$ (1-1)	$T_p = 2$, $T_{\bar{p}} = 2$ (2-2)
I	Asymptotically stable	Periodic
II	Asymptotically stable	Quasi-periodic
III	Periodic	Asymptotically stable
IV	Asymptotically stable	Divergent
V	Quasi-periodic	Divergent
VI	Period-2	Period-3

value, and highlights the regions where a structurally different behaviour can be observed.

Although we have assumed the lead-time can be accurately measured and correctly updated the replenishment policy with this lead-time information, a sudden increase of lead-time could still jeopardize system stability. That is, a system could be asymptotically stable with one lead-time but with a different lead-time it could be periodic (region **I** in Figure 6), quasi-periodic (**II**) or even divergent (**IV**). In region **III**, a lead-time increase will actually *stabilize* the system.

4.2 Unknown lead-time changes

In this section, the effect of lead-time on system stability in the Estimated Lead-time Variable Inventory and Order Based Production Control System (EPVIOBPCS) will be analyzed. This policy was introduced in [15] to eliminate a phenomenon known as inventory drift (when the system falls unit a steady state with a difference between the target and actual inventory levels), by calculating work-in-process level using perceived lead-time as

$$WIP_t = WIP_{t-1} + ORATE_{t-1} - ORATE_{t-T_{\bar{p}}-1} . \tag{16}$$

When the perceived lead-time is incorrect, i.e., $T_{\bar{p}} \neq T_p$, this method generates false WIP values, but the WIP values are matched to the DWIP target levels and this allows the inventory to return to target levels. It is already known that when work-in-process is calculated in the conventional way (via Equation 5), the value of $T_{\bar{p}}$ does not affect the stability of the system, since it only appears in a feed-forward path (calculating desired work-in-process). However, when work-in-process is calculated based on the perceived lead-time, $T_{\bar{p}}$, (with Equation 16), then $T_{\bar{p}}$ changes the dimension of the inventory system and thus it has a dramatic effect on stability and the dynamic behaviour.

Let's examine the following cases: Case (1-2), an overestimation of the actual lead-time; Case (2-1), an under estimation of the actual lead-time. Note that we are using two hyphenated numbers in brackets to represent lead-time misspecification scenarios, the first number representing the actual lead-time, T_p, and the second one perceived lead-time, $T_{\bar{p}}$.

4.2.1. Case (1-2)*, an overestimation of lead-time*

Using the same techniques as before, parametrical boundaries for the above two cases can be derived. For the (1-2) case, the asymptotic stability region is:

$$0 < \alpha_S < 1 + \alpha_{SL} - 2\alpha_{SL}^2 \text{ for } -0.5 < \alpha_{SL} < 1 . \tag{17}$$

The Lyapunovian stability boundary is:

$$\begin{aligned} &54\alpha_S\alpha_{SL}^2 - 12\alpha_{SL}^4 + 36\alpha_{SL}^3 - 3\alpha_S^2\alpha_{SL}^2 + 45\alpha_{SL}^2 + 6\alpha_S^2\alpha_{SL} - \\ &54\alpha_S\alpha_{SL} + 12\alpha_{SL} - 3\alpha_S^2 + 12\alpha_S^3 = 0. \end{aligned} \tag{18}$$

The boundary of the periodic region, $\alpha_S = 2$, can be derived by solving the eigenvalue equation $\left|\lambda_{\mathbf{A}_1\mathbf{A}_2^2}\right| = 0$. The system follows a $S_1S_2S_1S_2\ldots$ periodic orbit in this region.

4.2.2. Case (2-1)*, an underestimation of lead-time*

For case (2-1), the asymptotic stability region is:

$$0 < \alpha_S < \frac{\alpha_{SL} - 1 + \sqrt{\alpha_{SL}^2 + 2\alpha_{SL} + 5}}{2} \text{ for } -1 < \alpha_{SL} < 0.5 ; \tag{19}$$

$$0 < \alpha_S < -2\alpha_{SL} + 2 \text{ for } 0.5 < \alpha_{SL} < 1 . \tag{20}$$

The Lyapunovian stability boundary is:

$$\begin{aligned} &12\alpha_S\alpha_{SL}^3 - 3\alpha_{SL}^4 - 6\alpha_{SL}^3 + 18\alpha_S\alpha_{SL}^2 - 3\alpha_{SL}^2 - 18\alpha_S\alpha_{SL} + \\ &81\alpha_S^2 - 12\alpha_S = 0. \end{aligned} \tag{21}$$

The boundaries of the periodic region, $\alpha_S = 1$, can be derived by solving the eigenvalue equation $\left|\lambda_{\mathbf{A}_1\mathbf{A}_2}\right| = 1$. Consequently, the system follows a $S_1S_2S_2S_1S_2S_2\ldots$ periodic orbit in this region.

Similar to the analysis in the previous section, we obtain structurally different behaviours when lead-time mis-specification occurs. First, by overlapping the stability map of case (1-1) (Figure 3) with that of case (1-2) (Figure 7), the effect of incorrectly assuming the lead-time is 2 rather than correctly assuming that it is 1 can be analyzed. It can be seen that the lead-time misspecification leads to a major reduction in the size of asymptotically stable region, and an increase in the size of exponentially divergent region. Moreover, in most regions, the dynamic behaviour of the inventory system deteriorates (Table 2).

Table 2 and Figure 7 highlights a rather worrying situation could exist for the classical OUT policy. Using $\alpha_S = \alpha_{SL} = 1$ when both the actual and the perceived lead-time is 1 results in an asymptotically stable system. However, if the perceived lead-time is 2, but the actual lead-time is 1, then a quasi-periodic / chaotic system is present. This is a rather alarming result given the prevalence of the classical OUT in industrial setting and highlights the need for knowing and using correct lead-time information in replenishment policies.

By overlapping the stability map of case (2-2) with that of case (2-1) (Figure 8), we can analyze the effect of incorrectly using a lead-time of 1 when we should be using 2 in the replenishment system. Dynamic behaviour comparisons are shown in Table 3. This lead-time mis-specification decreases the size of asymptotically stable region and increases the size of the Lyapunovian stable region.

Again, alarmingly, the industrially prevalent setting of $\alpha_S = \alpha_{SL} = 1$ in the OUT policy exhibits periodic behaviour when lead-time mis-specification occurs, although it is asymptotically stable when the replenishment system is set up with correct lead-time information.

Table 4 shows the effect of the actual lead-time and perceived lead-time on the size of asymptotically stable region. It can be seen that when lead-time is perceived correctly (the main diagonal of the matrix), the size of asymptotic stability is much larger than those when lead-time information is incorrect. The importance of accurate lead-time information on asymptotic stable region is

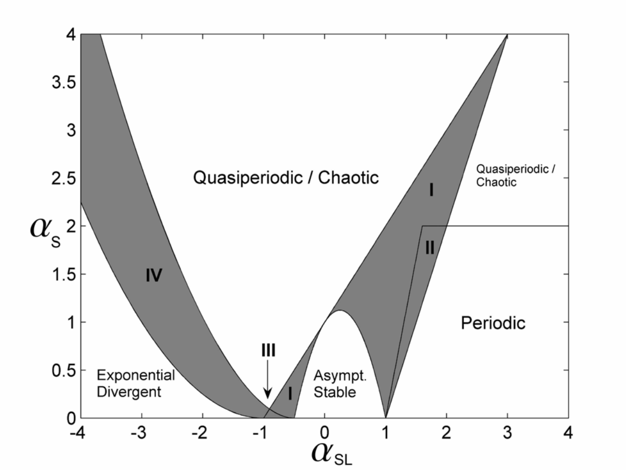

Figure 7: Stability Comparison between (1-1) and (1-2)

Table 2: Change of dynamic behaviour in each region with $T_p = 1$ and $T_{\bar{p}}$ changes observed in Figure 7

	$T_p = 1$, $T_{\bar{p}} = 1$ (1-1)	$T_p = 1$, $T_{\bar{p}} = 2$ (1-2)
I	Asymptotically stable	Quasi periodic / Chaotic
II	Asymptotically stable	Periodic
III	Asymptotically stable	Divergent
IV	Quasi periodic / Chaotic	Divergent

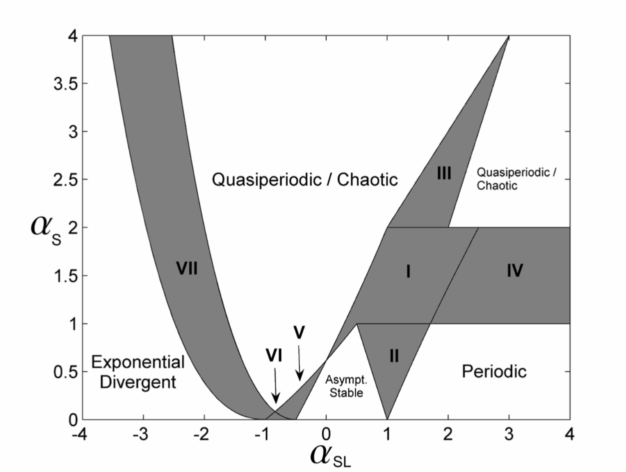

Figure 8: Stability Comparison between (2-2) and (2-1)

Table 3: Change of dynamic behaviour in each region with $T_p = 2$ and $T_{\bar{p}}$ changes observed in Figure 8

	$T_p = 2$, $T_{\bar{p}} = 2$ (2-2)	$T_p = 2$, $T_{\bar{p}} = 1$ (2-1)
I	Asymptotically stable	Quasi-periodic / Chaotic
II	Asymptotically stable	Periodic
III	Periodic	Quasi-periodic / Chaotic
IV	Periodic	Quasi-periodic / Chaotic
V	Quasi-periodic / Chaotic	Asymptotically stable
VI	Divergent	Asymptotically stable
VII	Divergent	Quasi-periodic / Chaotic

Table 4: Size of stability area under different actual and perceived lead-times

		$T_{\bar{p}}$			
		1	2	3	4
T_p	1	4.000	1.125	0.741	0.675
	2	0.943	2.742	0.750	0.496
	3	1.062	0.693	2.398	0.656
	4	0.578	0.541	0.598	2.221

obvious. Moreover, with correct lead-time information, increased lead-time reduces the size of the asymptotic stability region. Furthermore, the degree of lead-time mis-specification seems to have a complex relationship to the size of the stability region.

5 CONCLUSION AND DISCUSSION

This paper highlights the range of dynamic behaviours that are present in a constrained inventory system with only one constraint. It is interesting that even a simple deterministic model with a short lead-time is sufficient to generate such complex phenomena. We wonder what sort of impact stochastic models, longer lead-times and more constraints will have? Compared to the stable regions, the unstable regions are vast and relatively unknown. We have shown that complex and diversified behaviours and patterns exist in the unstable region.

Managerially, in a supply chain or production setting, intuitively we may rank these classes of dynamic behaviours from good to bad as follows; asymptotic stability, periodicity, quasi-periodicity, chaos and divergence. The most surprising result we have revealed herein is the fact that using wrong lead-time information in the EPVIOPBCS can result in a periodic or a chaotic

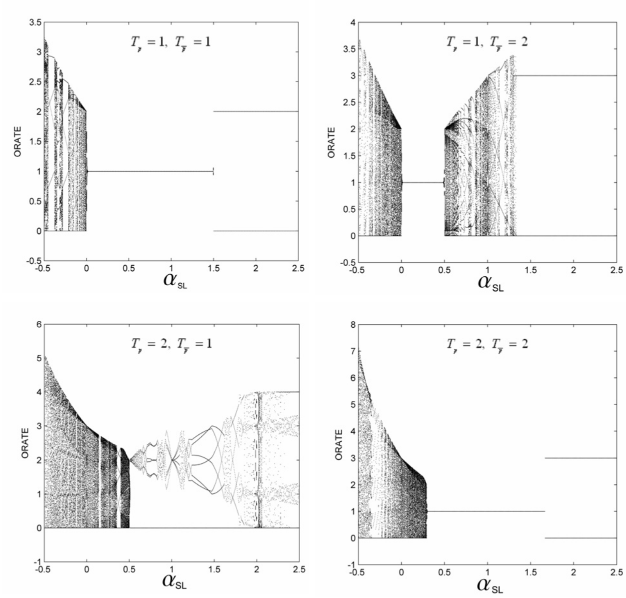

Figure 9: Bifurcation diagrams when $\alpha_S = 1$

system. As the EPVIOPBCS is a general case of the Order-Up-To (OUT) policy this is a worrying result as the OUT policy is probably one of the most popular replenishment algorithms in industry. Figure 9 takes another look at this issue by presenting four bifurcation diagrams, two with correct lead-times, two with mis-specified lead-times. The effect of lead-time and lead-time estimation on stability of the constrained inventory system is complex. It is extremely important to obtain and use accurate lead-time information in a replenishment system. However, as far as Lyapunovian stability is concerned, increasing either lead-time (actual or estimated) will decrease the Lyapunovian stable area. This might suggest that maintaining a low estimation of lead-time seems to be a better choice, regardless of the actual lead-time. However, from a practical supply chain point of view, an asymptotically unstable, but Lyapunovian stable system is a very bad system. We also note that the estimated lead-time will affect system stability only in the EPVIOBPCS version of the OUT policy, which is designed to eliminate inventory drift. Hence, there might be a trade-off between effective inventory control and system stability. This requires more detailed investigation in the future.

In the field of dynamical systems, the problem of high dimensional piecewise linear systems are far from being solved. Hence, to explore the dynamical behaviours of the constrained inventory system, a simulation-based technique has to be incorporated with an eigenvalue analysis and knowledge of dynamical systems. Due to the unique nature of the forbidden returns system, a linear system stability analysis was sufficient to derive the asymptotic stability boundaries of this particular constrained inventory system. In general this may not always be the case.

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