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Solutions in folded geometries, and associated cloaking due to anomalous resonance

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Abstract. Solutions for the fields in a coated cylinder where the core radius is bigger than the shell radius are seemingly unphysical, but can be given a physical meaning if one transforms to an equivalent problem by unfolding the geometry. In particular, the unfolded material can act as an impedance matched hyperlens, and as the loss in the lens goes to zero finite collections of polarizable line dipoles lying within a critical region surrounding the hyperlens are shown to be cloaked having vanishingly small dipole moments. This cloaking, which occurs both in the folded geometry and the equivalent unfolded one, is due to anomalous resonance, where the collection of dipoles generates an anomalously resonant field, which acts back on the dipoles to essentially cancel the external fields acting on them.

Contents

1. Introduction

Analytical solutions have played an important role in understanding the electromagnetic response of inclusions to an applied field. In these analytic solutions nothing prevents one from substituting seemingly unphysical values of the parameters. For example, for a coated spherical inclusion with core radius r_c and shell radius r_s, one may substitute into the analytic solution for the fields parameter values r_c and r_s with $r_c > r_s$. Is there any physical significance to such solutions? Introducing the novel concept (from the viewpoint of classical electromagnetism) of folded geometries and building upon the ideas of Leonhardt and Philbin (2006) let us first show that 'yes there is'.

Specifically, for simplicity, we analyze in the quasistatic limit the transverse magnetic (TM) solution for a coated cylindrical inclusion. In the usual situation, it is filled with an isotropic core material having a homogeneous complex dielectric constant ε_c and radius r_c, embedded in a homogeneous isotropic shell of dielectric constant ε_s having radii r_c and r_s, with $r_s > r_c$, which itself is embedded in a homogeneous isotropic matrix having dielectric constant ε_m. The potential V takes values V_c, V_s and V_m in the core, shell and matrix, respectively. Each of these are harmonic functions (satisfying $\Delta V = 0$) within their respective domains, except at singularities which we assume are confined to a finite set of points in the matrix. At the interfaces they satisfy the boundary conditions

$$\begin{aligned} V_c|_{r=r_c} &= V_s|_{r=r_c}, \qquad V_s|_{r=r_s} = V_m|_{r=r_s}, \\ \varepsilon_c \frac{\partial V_c}{\partial r}\bigg|_{r=r_c} &= \varepsilon_s \frac{\partial V_s}{\partial r}\bigg|_{r=r_c}, \qquad \varepsilon_s \frac{\partial V_s}{\partial r}\bigg|_{r=r_s} = \varepsilon_m \frac{\partial V_m}{\partial r}\bigg|_{r=r_s}. \end{aligned} \tag{1.1}$$

These equations still make mathematical sense if $r_c > r_s$: we look for harmonic potentials V_c, V_s and V_m defined in the respective regions $r \leqslant r_c$, $r_s \leqslant r \leqslant r_c$ and $r > r_s$, and satisfying the boundary conditions (1.1), where now ε_c, ε_s and ε_m are regarded as mathematical parameters entering these boundary conditions. The dielectric tensor field $\boldsymbol{\varepsilon}(\mathbf{x})$ takes values

$$\begin{aligned} \boldsymbol{\varepsilon}(\mathbf{x}) &= \varepsilon_c \mathbf{I}, && \text{in the core,} \\ &= -\varepsilon_s \mathbf{I}, && \text{in the shell,} \\ &= \varepsilon_m \mathbf{I}, && \text{in the matrix,} \end{aligned} \tag{1.2}$$

with the choices of sign here being motivated by the effect of folding of space 'back on itself', which affects the direction of derivatives. Indeed, flux will be conserved only if the radial component of the displacement field $\mathbf{D}(\mathbf{x}) = -\boldsymbol{\varepsilon}(\mathbf{x})\nabla V$ changes sign, but maintains magnitude, at these interfaces: if $\nabla\cdot\mathbf{D} = 0$ then one can draw a flow field for $\mathbf{D}$ with arrows and (by conservation of flux) the arrows must reverse direction at the interface. The interface conditions (1.1) are compatible with this constraint provided $\boldsymbol{\varepsilon}(\mathbf{x})$ is given by (1.2).

To make physical sense of such a solution, we recall the fact that the quasistatic equations (and more generally, the equations of electromagnetism) retain their form under coordinate transformations. Specifically, if $V(\mathbf{x})$ is a solution to

$$\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{x})\nabla V(\mathbf{x}) = 0 \tag{1.3}$$

and $\mathbf{x}'(\mathbf{x})$ is a transformation to a new curvilinear coordinate system, then the potential $V'(\mathbf{x}') \equiv V(\mathbf{x}(\mathbf{x}'))$, where $\mathbf{x}(\mathbf{x}')$ is the inverse transformation, satisfies

$$\nabla'\cdot\boldsymbol{\varepsilon}'(\mathbf{x}')\nabla' V'(\mathbf{x}') = 0, \tag{1.4}$$

where the dielectric tensor, viewed as a contravariant tensor density, has been transformed according to the standard formula

$$\boldsymbol{\varepsilon}'(\mathbf{x}') = |\det\mathbf{A}(\mathbf{x})|^{-1}\mathbf{A}(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{x})\mathbf{A}^T(\mathbf{x}), \tag{1.5}$$

in which $\mathbf{A} = \nabla\mathbf{x}'(\mathbf{x})$ is the Jacobian, and $\mathbf{x} = \mathbf{x}(\mathbf{x}')$. The equation (1.4) can be reinterpreted as a quasistatic equation in a body with dielectric constant $\boldsymbol{\varepsilon}'(\mathbf{x}')$ in which $\mathbf{x}' = (x_1', x_2', x_3')$ are now regarded as Cartesian coordinates. The displacement field and the electric field $\mathbf{E}(\mathbf{x}) = -\nabla V(\mathbf{x})$ transform to

$$\mathbf{D}'(\mathbf{x}') = |\det\mathbf{A}(\mathbf{x})|^{-1}\mathbf{A}(\mathbf{x})\mathbf{D}(\mathbf{x}), \qquad \mathbf{E}'(\mathbf{x}') = [\mathbf{A}^T(\mathbf{x})]^{-1}\mathbf{E}(\mathbf{x}). \tag{1.6}$$

To turn the unphysical solution in the folded geometry, with $r_c > r_s$, into a physical solution we use a coordinate transformation which unfolds the geometry. Consider the standard polar coordinates (r, θ) and (r', θ') in the folded and transformed geometries, respectively. Then the simplest unfolding mapping, as sketched in figure 1, is given by $\theta' = \theta$ and

$$\begin{aligned} r' &= r_c^{-1}[r_s - a(r_c - r_s)]r, && \text{in the core,}\\ &= r_s - a(r - r_s), && \text{in the shell,}\\ &= r, && \text{in the matrix,} \end{aligned} \tag{1.7}$$

where a is a fixed positive constant less than $r_s/(r_c - r_s)$. We emphasize that the pair (r, θ) with $r_c > r > r_s$ and $2\pi > \theta \geqslant 0$ does not suffice to uniquely specify a point in the folded geometry: one has to specify whether the point lies in the core, shell or matrix. In a folded geometry it is as if space overlaps itself but without intersection: as one goes continuously on a straight line trajectory from the origin, first one moves in the core and the radius increases until one encounters the core radius r_c, then one moves into the shell and the radius decreases until one reaches the shell radius $r_s < r_c$, where one moves into the matrix and the radius increases again. With this definition, the unfolding mapping (1.7) is globally a 1-to-1 mapping.

It is clear from (1.7) that $r_s' = r_s > r_c' = r_s - a(r_c - r_s)$. The inverse folding transformation $\mathbf{x}(\mathbf{x}')$ takes the same form as (1.7) with r_c, r_s and a replaced by r_c', r_s' and a^{-1}, respectively, and the roles of r and r' swapped. Using the expression (1.5) and the formula for the unfolding map, which in particular implies that in the shell

$$\mathbf{x}' = -a\mathbf{x} + b\mathbf{x}/\sqrt{\mathbf{x}\cdot\mathbf{x}}, \qquad \mathbf{A} = \nabla\mathbf{x}' = (b/r - a)\mathbf{I} - b\mathbf{x}\otimes\mathbf{x}/r^3, \tag{1.8}$$

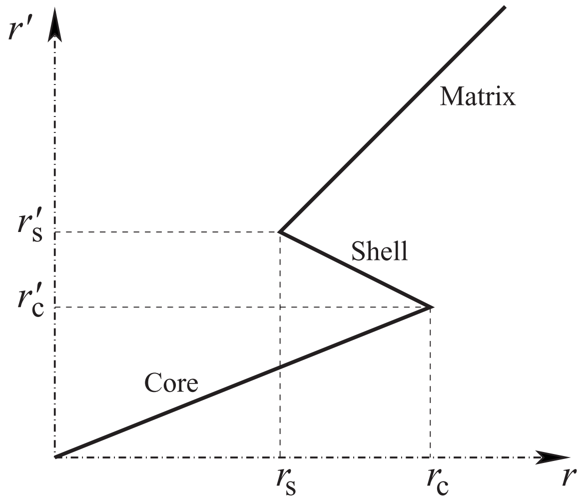

Figure 1. Sketch of the unfolding transformation (1.7), where r and r' are the radial coordinates in the folded and unfolded geometries. Note that $r'_s = r_s$ since the mapping is the identity map in the matrix.

where $b = (1+a)r_s$, we get expressions for the dielectric tensor in the core, shell and matrix in the unfolded geometry

$$\boldsymbol{\varepsilon}'_c = \varepsilon_c \mathbf{I}, \quad \boldsymbol{\varepsilon}'_s = -\frac{(b/r-a)^2\mathbf{I} + (2ab/r^3 - b^2/r^4)\mathbf{x}\otimes\mathbf{x}}{a(b/r-a)}\varepsilon_s, \quad \boldsymbol{\varepsilon}'_m = \varepsilon_m \mathbf{I}. \qquad (1.9)$$

To be physically realizable we require that $\boldsymbol{\varepsilon}'_c$, $\boldsymbol{\varepsilon}'_s$ and $\boldsymbol{\varepsilon}'_m$ have positive semi-definite imaginary parts, which requires that ε_c and ε_m have a non-negative imaginary part, whereas ε_s has a non-positive imaginary part (as can be seen directly from equations (1.2) and (1.5)). In summary, we see that seemingly paradoxical geometries may be transformed into a physically comprehensible form, which may prove an interesting direction for future research.

When $\varepsilon_s = \varepsilon_m$ the response of the coated cylinder in the folded geometry is equivalent to that of a solid cylinder of radius r_c and dielectric constant ε_c. The potential in the shell in the folded region between r_c and r_s is the same as that in the matrix in this region and is the analytic extension of the potential surrounding the solid cylinder provided there are no singularities in this analytic extension—otherwise a solution does not exist. So in the unfolded geometry, the shell with dielectric tensor $\boldsymbol{\varepsilon}'_s(\mathbf{x}')$ acts to magnify the core by a factor of r_c/r'_c so it responds like a solid cylinder of radius r_c and dielectric constant ε_c. We call such a shell an impedance-matched hyperlens in recognition of the pioneering work of Kildishev and Narimanov (2007) who showed that it would magnify fixed sources in core, not just in the quasistatic limit, but also for the full Helmholtz equation (provided the magnetic permeability was also suitably chosen). Such lenses were first considered by Rahm *et al* (2008) as electromagnetic concentrators. Although both groups assumed $r_s > r_c$, their analysis extends directly to the case $r_c > r_s$. Other hyperlenses with magnifying properties were studied by Jacob *et al* (2006) and Salandrino and Engheta (2006).

This equivalence is similar to the result of Nicorovici *et al* (1994) who found that a coated dielectric cylinder with radii $r_s > r_c$ and moduli $\varepsilon_s = -\varepsilon_m$ would have the same quasistatic response as a solid cylinder of radius $r_* = r_s^2/r_c$ and dielectric constant ε_c, i.e. the shell, of dielectric constant $\varepsilon_s = -\varepsilon_m$, now known as a cylindrical superlens, acts to magnify the core by the factor $h = r_s^2/r_c^2$. This equivalence implied that a line source at radius $r_0 > r_s$ in the matrix would generate a potential which appeared like it originated from the line source plus an image line source at the radius r_*^2/r_0 which would *be in the matrix* when $r_*^2/r_0 > r_s$. They found that the actual potential in the matrix converged as $\varepsilon_s \to -\varepsilon_m$ to this singular potential at radii greater than r_*^2/r_0 and numerically found that the actual potential developed large oscillations at smaller radii. (See, in particular, the sentence beginning with 'These fluctuations become less pronounced...' above figure 2 in that paper.) To our knowledge, this was the first discovery of an apparent (ghost) singularity in the field surrounding an inclusion, or in effect the first example of perfect imaging (in quasistatics) of a point or line source. The regions where the field diverges were later called regions of anomalous resonance (Milton *et al* 2005).

In a subsequent development, Pendry (2000) made the bold claim that the Veselago lens (Veselago 1967) consisting of a slab of thickness d with dielectric constant $\varepsilon_s = -1$ and magnetic permittivity $\mu_s = -1$, surrounded by a medium with dielectric constant $\varepsilon_m = 1$ and magnetic permittivity $\mu_m = 1$, would behave as a superlens: a line source at a distance d_0 in front of the slab, would appear to have an image line source at a distance $d - d_0$ behind the slab. When ε_s and μ_s approached $-\varepsilon_m$ and $-\mu_m$ (having a very small imaginary part) the actual fields behind the slab converged to these singular fields behind the image, but diverged between the image and the slab. There was also a seeming paradox (pointed out to GWM by Alexei Efros): if the source was closer than a distance $d/2$ to the lens then the electromagnetic power dissipated in the lens per unit time by a constant amplitude source would approach infinity as the loss went to zero. This paradox was resolved by Milton *et al* (2005) who showed that when $d_0 < d/2$ then the anomalously resonant fields acting on the source act as a sort of 'optical molasses' against which the source has to do a tremendous amount of work to maintain its amplitude. Subsequently, it was found that a polarizable dipolar line source or single constant energy line source becomes 'cloaked' if it is within a distance $d/2$ of the slab lens or within a radius $\sqrt{r_s^3/r_c}$ of a cylindrical superlens (with the core having dielectric constant $\varepsilon_c = \varepsilon_m$). Its dipole moment, and consequently, its effect on the field outside a certain distance from the lens, becomes vanishingly small. The energy generated by a constant energy source, like the energy generated by two opposing sources on opposite sides of a slab lens (Boardman and Marinov 2006, Cui *et al* 2005) is effectively trapped within the cloaking region. This cloaking was proved (Milton and Nicorovici 2006) and numerically verified (Nicorovici *et al* 2007) to extend to collections of finitely many polarizable dipoles. Also arguments were presented (Milton *et al* 2007) which suggested that a line dipole which was 'switched on' at time $t = 0$ in front of a perfect lens with no loss, having $\varepsilon_s = -\varepsilon_m$ and $\mu_s = -\mu_m$, would become cloaked in the limit $t \to \infty$. On the other hand, Bruno and Lintner (2007) showed that a dielectric body such as a solid cylinder of finite radius in the cloaking region would only be partially and not fully cloaked in the limit as the loss goes to zero. One can conclude that a dielectric body *is neither perfectly cloaked nor perfectly imaged by superlenses* (in the limit as the loss goes to zero) if it lies within the cloaking region.

Here we show that anomalous resonance and cloaking extends to folded cylindrical geometries, and therefore also to the equivalent unfolded cylindrical geometries. This is not too surprising. Leonhardt and Philbin (2006) realized that the solution for the electromagnetic fields in the slab superlens can be viewed as the result of an unfolding of space, and we know that anomalous resonance and cloaking are associated with superlenses.

There are important conceptual differences between the work of Leonhardt and Philbin (2006), and our work. In their work, the unfolding transformation is applied to empty space, so that in the appropriate region one point gets mapped to three points, and a field at that point gets mapped to three fields. In this context, it is correct, as they do, to take transformations of the moduli of the form (1.5), but without the absolute value around the Jacobian of the determinant. In our approach, applied to the idealized 'perfect' superlens with the full-Maxwell equations, the unfolding transformation is applied to a folded geometry, and there is globally a one-to-one correspondence between points in the folded geometry and the unfolded geometry. (The value of $\mathbf{x}$ in the folded geometry is not necessarily sufficient to specify a point: one also has to specify the manifold on which the point lies.) Given empty space one first inserts a fold. In the half of the fold that gets mapped to the lens $\nabla\times$ gets replaced by $-\nabla\times$ in the Maxwell equations because of the change in handedness of the space and the moduli are negative to ensure that the Maxwell equations $\nabla\cdot\mathbf{D}=\nabla\cdot\mathbf{B}=0$ remain satisfied in any source free region in the folded geometry. At a given value of $\mathbf{x}$ in the fold the electromagnetic fields take the values $(\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B})$, $(\mathbf{E}, -\mathbf{D}, \mathbf{H}, -\mathbf{B})$ and $(\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B})$ on the three different manifolds, where $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$ and $\mathbf{B}$ are the electromagnetic fields at $\mathbf{x}$ in the original empty space. Thus, the total displacement field density at $\mathbf{x}$ is $\mathbf{D}$ (and not $3\mathbf{D}$). When transforming the moduli absolute values around the Jacobian of the determinant are needed to ensure that Maxwell's equations remain satisfied in the unfolded 'perfect' superlens geometry. Our introduction of folded geometries greatly enlarges the class of geometries to which one can transform to simplify the analysis of a problem. This simplification is analogous to the way one uses conformal transformations to map to a simpler problem.

For simplicity, our analysis (which for the most part only requires minor modifications of the analysis of Milton and Nicorovici (2006)) is for two-dimensional quasistatics. Presumably analogous results hold for the full (time harmonic) Maxwell equations in three-dimensional folded spherical geometries, although we have not explored this. Throughout the paper, we use the symbol $\equiv$ to mean equal by definition, and the symbol $\approx$ to mean approximately equal to.

2. The Green function for a monopole and solutions for a dipole in the matrix

Let us consider the Green function $V(\mathbf{x})$ for a point source (monopole) located in the matrix. Although unphysical (because the net charge associated with the singularity oscillates in time) it is mathematically well defined, and useful for deriving the potential associated with a dipole. This potential, by definition, takes values V_{c}, V_{s} and V_{m} in the core, shell and matrix which satisfy

$$\Delta V_{\mathrm{c}}=0,\quad \Delta V_{\mathrm{s}}=0,\quad \Delta V_{\mathrm{m}}=-\delta(\mathbf{x}-\mathbf{x}_0) \tag{2.1}$$

in their respective domains, together with the boundary conditions (1.1), where $\delta(\mathbf{y})$ is the standard Dirac delta function for a source located at $\mathbf{y}=0$. The problem of finding $V(\mathbf{x})$ can be solved explicitly using power series with respect to the complex coordinate $z=x_1+\mathrm{i}x_2$, as follows. Note that the Green function for the Laplace equation in R^2 is given by the formula

$$V_0 = -\frac{1}{4\pi}\left(\log(z-z_0)+\log(\bar{z}-\overline{z_0})\right) = -\frac{1}{4\pi}\left[2\log|z_0| - \sum_{n=1}^{\infty} n^{-1}\left(\frac{z}{z_0}\right)^n - \sum_{n=1}^{\infty} n^{-1}\left(\frac{\bar{z}}{\overline{z_0}}\right)^n\right]. \tag{2.2}$$

This is the potential of a point monopole in a homogeneous free space.

We are looking for a solution $V_{\mathrm{s,c,m}}$ to the above problem (equations (2.1) and (1.1)) in the form of a power series in each of the three regions:

$$\begin{aligned} V_{\mathrm{c}} &= \sum_{n=0}^{\infty} A_n^{(\mathrm{c})} z^n + \sum_{n=0}^{\infty} B_n^{(\mathrm{c})} \bar{z}^n, \\ V_{\mathrm{s}} &= \sum_{n=-\infty}^{\infty} A_n^{(\mathrm{s})} z^n + \sum_{n=-\infty}^{\infty} B_n^{(\mathrm{s})} \bar{z}^n, \\ V_{\mathrm{m}} &= V_0 + \sum_{n=1}^{\infty} A_n^{(\mathrm{m})} z^{-n} + \sum_{n=1}^{\infty} B_n^{(\mathrm{m})} \bar{z}^{-n}. \end{aligned} \tag{2.3}$$

The substitution of these series in the interface conditions (1.1) yields via the identity $r\partial/(\partial r) = z\partial/(\partial z) + \bar{z}\partial/(\partial\bar{z})$ explicit expressions for the coefficients $A_n^{(\mathrm{c,s,m})}$, $B_n^{(\mathrm{c,s,m})}$. The formulae for $V_{\mathrm{c,s,m}}$ can then be found, and are as follows:

$$\begin{aligned} V_{\mathrm{c}} &= -\frac{1}{2\pi}\log|z_0| + \frac{\varepsilon_{\mathrm{s}}\varepsilon_{\mathrm{m}}}{\pi(\varepsilon_{\mathrm{s}}-\varepsilon_{\mathrm{c}})(\varepsilon_{\mathrm{m}}-\varepsilon_{\mathrm{s}})} \sum_{n=1}^{\infty}\left[\left(\frac{r_{\mathrm{c}}}{r_{\mathrm{s}}}\right)^{2n} + \delta e^{\mathrm{i}\phi}\right]^{-1} \frac{1}{n}\left[\left(\frac{z}{z_0}\right)^n + \left(\frac{\bar{z}}{\overline{z_0}}\right)^n\right], \\ V_{\mathrm{s}} &= -\frac{1}{2\pi}\log|z_0| + \frac{\varepsilon_{\mathrm{m}}}{2\pi\eta_{\mathrm{sc}}(\varepsilon_{\mathrm{m}}-\varepsilon_{\mathrm{s}})} \sum_{n=1}^{\infty}\left[\left(\frac{r_{\mathrm{c}}}{r_{\mathrm{s}}}\right)^{2n} + \delta e^{\mathrm{i}\phi}\right]^{-1} \frac{1}{n}\left[\left(\frac{z}{z_0}\right)^n + \left(\frac{\bar{z}}{\overline{z_0}}\right)^n\right] \\ &\quad + \frac{\varepsilon_{\mathrm{m}}}{2\pi(\varepsilon_{\mathrm{m}}-\varepsilon_{\mathrm{s}})} \sum_{n=1}^{\infty}\left[\left(\frac{r_{\mathrm{c}}}{r_{\mathrm{s}}}\right)^{2n} + \delta e^{\mathrm{i}\phi}\right]^{-1} \frac{1}{n}\left[\left(\frac{z\overline{z_0}}{r_{\mathrm{c}}^2}\right)^{-n} + \left(\frac{\bar{z}z_0}{r_{\mathrm{c}}^2}\right)^{-n}\right], \\ V_{\mathrm{m}} &= V_0 + \frac{1}{4\pi}\sum_{n=1}^{\infty}\left[\frac{1}{\eta_{\mathrm{sc}}} + \delta e^{\mathrm{i}\phi}\eta_{\mathrm{sc}}\left(\frac{r_{\mathrm{c}}}{r_{\mathrm{s}}}\right)^{2n}\right]\left[\left(\frac{r_{\mathrm{c}}}{r_{\mathrm{s}}}\right)^{2n} + \delta e^{\mathrm{i}\phi}\right]^{-1} \frac{1}{n}\left[\left(\frac{z\overline{z_0}}{r_{\mathrm{s}}^2}\right)^{-n} + \left(\frac{\bar{z}z_0}{r_{\mathrm{s}}^2}\right)^{-n}\right], \end{aligned} \tag{2.4}$$

where, in accordance with the definitions in Milton *et al* (2005), we have introduced the real parameters ϕ and δ (not to be confused with the delta function) and the complex parameter η_{sc} defined via

$$\delta e^{\mathrm{i}\phi} = \frac{(\varepsilon_{\mathrm{s}}+\varepsilon_{\mathrm{c}})(\varepsilon_{\mathrm{m}}+\varepsilon_{\mathrm{s}})}{(\varepsilon_{\mathrm{s}}-\varepsilon_{\mathrm{c}})(\varepsilon_{\mathrm{m}}-\varepsilon_{\mathrm{s}})}, \qquad \eta_{\mathrm{sc}} = \frac{\varepsilon_{\mathrm{s}}-\varepsilon_{\mathrm{c}}}{\varepsilon_{\mathrm{s}}+\varepsilon_{\mathrm{c}}}. \tag{2.5}$$

These expressions for V_{c}, V_{s} and V_{m} are valid both for the cases $r_{\mathrm{s}} > r_{\mathrm{c}}$ and $r_{\mathrm{c}} > r_{\mathrm{s}}$.

In figure 2, we show the potential around a monopole when mapped to the unfolded geometry. The contrast is evident between the case of a core of dielectric constant matching that of the matrix, which is non-resonant in this example, and the case when $\varepsilon_{\mathrm{c}} \neq \varepsilon_{\mathrm{m}}$, which exhibits anomalous resonance. Note that in the first case the coated inclusion is almost invisible: the equipotentials outside it are nearly circular.

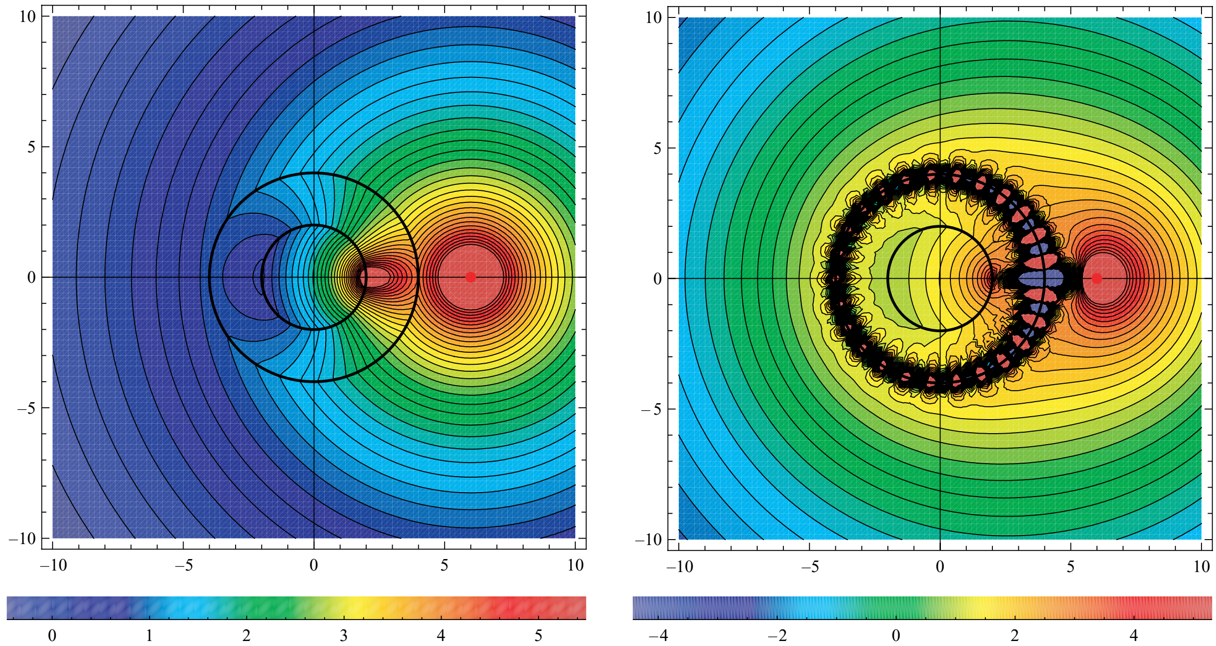

Figure 2. Numerical computations for the potential associated with a monopole at $z_0 = 6$ in the unfolded geometry (unfolding parameter $a = 0.7$) with $\varepsilon_s = -1 + 10^{-9}\mathrm{i}$, (a) $\varepsilon_c = \varepsilon_m = 1$ and (b) $\varepsilon_c = 5$, $\varepsilon_m = 1$. In both cases, $r_c = 5.4$, $r'_c = 2$ and $r_s = r'_s = 4$.

By letting $z_0 = r_0 e^{\mathrm{i}\theta_0}$ and differentiating (2.4) with respect to r_0 and with respect to θ_0 one obtains formula for the potential associated with a dipole at z_0 oriented in the radial direction, and with one oriented in the tangential direction. The potential associated with an arbitrarily oriented dipole is of course a linear combination of these two potentials and is given by the formulae

$$V_c = (k^{(1)} + k^{(2)})/r_0 + k^{(1)} F_c(z, z_0) + k^{(2)} F_c(\bar{z}, \overline{z_0}),$$

$$V_s = (k^{(1)} + k^{(2)})/r_0 + k^{(1)} \underline{F}_{out}(z, z_0) + k^{(2)} \underline{F}_{out}(\bar{z}, \overline{z_0}) + k^{(2)} \underline{F}_{in}(z, z_0) + k^{(1)} \underline{F}_{in}(\bar{z}, \overline{z_0}), \qquad (2.6)$$

$$V_m = \frac{k^{(1)}}{r_0(1 - z/z_0)} + \frac{k^{(2)}}{r_0(1 - \bar{z}/\overline{z_0})} + k^{(2)} F_{in}(z, z_0) + k^{(1)} F_{in}(\bar{z}, \overline{z_0}),$$

where

$$k^{(1)} = (-k^e + \mathrm{i}k^o)/2, \quad k^{(2)} = -(k^e + \mathrm{i}k^o)/2, \qquad (2.7)$$

in which (in accordance with the definition below equation (3.5) in Milton and Nicorovici (2006)) k^e and k^o are the (generally complex) suitably normalized amplitudes of the dipole components which have even and odd symmetry about the line $\theta = \theta_0$, and

$$\begin{aligned} F_c(z, z_0) &= \frac{4\varepsilon_s\varepsilon_m}{r_0(\varepsilon_s - \varepsilon_c)(\varepsilon_m - \varepsilon_s)} S(\delta, hz/z_0), \\ \underline{F}_{out}(z, z_0) &= \frac{2\varepsilon_m}{r_0\eta_{sc}(\varepsilon_m - \varepsilon_s)} S(\delta, hz/z_0), \\ \underline{F}_{in}(z, z_0) &= \frac{2\varepsilon_m}{r_0(\varepsilon_m - \varepsilon_s)} S(\delta, r_s^2/(z\overline{z_0})), \\ F_{in}(z, z_0) &= \frac{S(\delta, r_s^4/(r_c^2 z\overline{z_0}))}{r_0\eta_{sc}} + \frac{\delta e^{i\phi}\eta_{sc} S(\delta, r_s^2/(z\overline{z_0}))}{r_0}, \end{aligned} \tag{2.8}$$

in which

$$h = \frac{r_s^2}{r_c^2}, \qquad S(\delta, w) = \sum_{\ell=1}^{\infty} \frac{w^\ell}{1 + \delta e^{i\phi} h^\ell}, \tag{2.9}$$

and the remaining functions are obtained by replacing z and z_0 with $\bar{z}$ and $\overline{z_0}$ in (2.8). These formulae for the potentials agree with the formulae of Milton *et al* (2005) and (for a dipole not on the x_1-axis) with the formulae in the supporting online material of Nicorovici *et al* (2007) (see http://www.physics.usyd.edu.au/cudos/research/plasmon.html) aside from the (irrelevant) additive constant of $(k^{(1)} + k^{(2)})/r_0$.

It is interesting to see what happens to the potential in the matrix in the limit as ε_s approaches ε_m. Specifically, let us suppose that $k^{(1)}$, $k^{(2)}$, ε_c and ε_m remain fixed with ε_m real and positive, and with ε_c possibly complex (with non-negative imaginary part) but not real and negative, and that ε_s approaches ε_m along a trajectory in the lower half of the complex plane in such a way that $\delta \to \infty$ but ϕ remains fixed. We set

$$\eta = \frac{\varepsilon_m - \varepsilon_c}{\varepsilon_m + \varepsilon_c}. \tag{2.10}$$

When ε_s is close to ε_m (2.5) implies

$$\begin{aligned} \varepsilon_s &\approx [1 - 2e^{-i\phi}/(\delta\eta)]\varepsilon_m, && \text{when } \varepsilon_c \neq \varepsilon_m, \\ &\approx (1 - 2ie^{-i\phi/2}/\sqrt{\delta})\varepsilon_m, && \text{when } \varepsilon_c = \varepsilon_m, \end{aligned} \tag{2.11}$$

and so we have

$$\begin{aligned} \delta &\approx 2\varepsilon_m/(|\varepsilon_s - \varepsilon_m||\eta|), && \text{when } \varepsilon_c \neq \varepsilon_m, \\ &\approx 4\varepsilon_m^2/|\varepsilon_s - \varepsilon_m|^2, && \text{when } \varepsilon_c = \varepsilon_m. \end{aligned} \tag{2.12}$$

Thus for large δ the trajectory approaches ε_m in such a way that the argument of $\varepsilon_s - \varepsilon_m$ is approximately constant. Since the imaginary part of ε_s is strictly negative, whereas the imaginary part of η is negative or zero, we deduce that ϕ is not equal to π or $-\pi$ and this ensures that there are no infinite terms in the series (2.9).

We need an approximation for $S(\delta, w)$ in the limit, where δ is very large. From (2.9), we see that when $|w| < h$ the series expansion for $\delta S(\delta, w)$ converges in the limit $\delta \to \infty$ and as a consequence

$$S(\delta, w) \approx \frac{e^{-i\phi} w}{\delta(h - w)}. \tag{2.13}$$

When $1 > |w| > h$, the terms in the series for $S(\delta, w)$ first increase exponentially until ℓ reaches a transition region, where $\ell \approx n$ in which n is the largest integer such that $\delta h^n \geqslant 1$ and after this transition region the terms in the series decay exponentially. To a good approximation (which becomes better as $\delta \to \infty$) we have

$$S(\delta, w) = w^n \sum_{\ell=1}^{\infty} \frac{w^{\ell-n}}{1+\delta e^{i\phi} h^{\ell}} \approx w^n \sum_{j=-\infty}^{\infty} \frac{w^j}{1+\delta h^n e^{i\phi} h^j}. \tag{2.14}$$

Since $\delta h^n \to 1$ as $\delta \to \infty$, upon solving for n in terms of h and δ, we obtain

$$S(\delta, w) \approx e^{-\log w \log \delta / \log h} T(w), \tag{2.15}$$

where

$$T(w) = \sum_{j=-\infty}^{\infty} \frac{w^j}{1+e^{i\phi} h^j}. \tag{2.16}$$

Assuming z is in the matrix, let us first treat the case when $\varepsilon_c \neq \varepsilon_m$. Then as $\delta \to \infty$, η_{sc} approaches η and for $h > r_s^2/|z\overline{z_0}|$, i.e. for $|z| > r_c^2/r_0$, (2.13) implies

$$\lim_{\delta\to\infty} F_{in}(z, z_0) = \tilde{F}_{in}(z, z_0) \equiv \frac{\eta r_c^2}{r_0(z\overline{z_0} - r_c^2)} \tag{2.17}$$

and as a consequence the potential V_m in the matrix, with $|z| > r_c^2/r_0$ approaches

$$\tilde{V}_m = k^{(1)} \left[\frac{1}{r_0(1-z/z_0)} + \frac{\eta r_c^2}{r_0(\bar{z} z_0 - r_c^2)} \right] + k^{(2)} \left[\frac{1}{r_0(1-\bar{z}/\overline{z_0})} + \frac{\eta r_c^2}{r_0(z\overline{z_0} - r_c^2)} \right], \tag{2.18}$$

which, as might be expected, is exactly the same potential which would be associated with line dipole outside a solid cylinder of dielectric constant ε_c and radius r_c. In the unfolded geometry, it appears as if the shell has the effect of magnifying the core by the factor $r_c/r_c' = r_c/(r_s - a(r_c - r_s))$. When the source is located with $r_s < r_0 < r_c^2/r_s$ it will look like there is a ghost singularity in the matrix positioned at $z = r_c^2/\overline{z_0}$. When $r_s < |z| < r_c^2/r_0$ (2.15) implies $\delta S(\delta, r_s^2/(z\overline{z_0}))$ scales like $\delta^{\log(r_c{}^2/(|z|r_0))/(-\log h)}$ and as a result this is a region of anomalous resonance with the potential V_m diverging inside it, with this same scaling.

When $\varepsilon_c = \varepsilon_m$, the same argument shows that as, $\delta \to \infty$, $F_{in}(z, z_0)$ tends to zero for $|z| > r_c^2/r_0$. In fact, it converges to zero in a larger region. To see this, note that η_{sc} scales as $1/\sqrt{\delta}$, and as a consequence $\delta \eta_{sc} S(\delta, r_s^2/(z\overline{z_0}))$ scales like δ^{τ}, where $\tau = \log(r_c r_s/r_0|z|)/(-\log h)$, in the region $r_s < |z| < r_c^2/r_0$. This converges to zero for $|z| > r_{\#}^2/r_0$, where $r_{\#} = \sqrt{r_c r_s}$, but diverges to infinity (with increasingly rapid spatial oscillations) in the region $r_s < |z| < r_{\#}^2/r_0$. Thus, as $\delta \to \infty$, the potential V_m will converge for $|z| > r_{\#}^2/r_0$ to the potential associated with a line dipole in free space, while diverging to infinity in the anomalously resonant region $r_s < |z| < r_{\#}^2/r_0$.

It is also interesting to consider the limit as ε_s approaches $-\varepsilon_m$ in the folded geometry. The results of Nicorovici *et al* (1994) apply directly to this case, and show that the coated cylinder in the folded geometry is equivalent to a solid cylinder of dielectric constant ε_c of radius r_s^2/r_c, which is less than r_s. In particular, in the unfolded geometry, the inclusion will be invisible when $\varepsilon_c = \varepsilon_m$: presumably such an object acts as a lens to shrink the apparent size of any object inside it. One can check that anomalous resonance and cloaking do not occur for sources outside the inclusion in this circumstance.

3. Cloaking of a single polarizable line dipole

First, we present an example which shows that a polarizable line with polarizability α can be cloaked when immersed in a TM field surrounding a folded coated cylinder with core radius r_c and shell radius $r_s < r_c$ and with cylinder axis $x_1 = x_2 = 0$. The polarizable line is placed along $x_1 = r_0$ and $x_2 = 0$, where $r_0 > r_s$. Suppose $(E_1(x_1, x_2), E_2(x_1, x_2), 0)$ is the field with the polarizable line absent (but with the coated cylinder present) due to fixed sources not varying in the x_3-direction lying outside the radius r_c when $\varepsilon_c \neq \varepsilon_m$, and the radius $r_\# \equiv \sqrt{r_s r_c}$ when $\varepsilon_c = \varepsilon_m$. We assume these sources are not perturbed when the polarizable line is introduced.

Again, let us suppose that ε_c and ε_m remain fixed and that ε_s approaches ε_m along a trajectory in the lower half of the complex plane in such a way that $\delta \to \infty$ but ϕ remains fixed. Let us drop the E_3 field component of the electric field since it is zero for TM fields. The field (E_1^0, E_2^0) acting on the polarizable line has two components:

$$(E_1^0, E_2^0) = (E_1 + E_1^r, E_2 + E_2^r), \tag{3.1}$$

where

$$\begin{aligned} &E_1 \equiv E_1(r_0, 0), \quad E_2 \equiv E_2(r_0, 0), \quad E_1^r \equiv E_1^r(r_0, 0), \quad E_2^r \equiv E_2^r(r_0, 0), \\ &(E_1^r(x, y), E_2^r(x, y)) = (-\partial V_{in}(x_1, x_2)/\partial x_1, -\partial V_{in}(x_1, x_2)/\partial x_2), \end{aligned} \tag{3.2}$$

and $V_{in}(x_1, x_2)$ is the (possibly resonant) response potential in the matrix generated by the coated cylinder responding to the polarizable line itself (not including the field generated by the coated cylinder responding to the other fixed sources). From (2.6), (2.7) and (2.8), or alternatively from (2.5), (3.9) and (3.10) of Milton *et al* (2005), we have

$$V_{in}(x_1, x_2) = [f_{in}^e(z) + f_{in}^e(\bar{z})]/2 + [f_{in}^o(z) - f_{in}^o(\bar{z})]/(2i), \tag{3.3}$$

where $z = x_1 + ix_2$ and for $p = e, o$

$$f_{in}^p(z) = -qk^p F_{in}(z, r_0) = -\frac{qk^p S(\delta, r_*^2/(r_0 z))}{r_0 \eta_{sc}} - \frac{qk^p \delta e^{i\phi} \eta_{sc} S(\delta, r_s^2/(r_0 z))}{r_0}, \tag{3.4}$$

in which k^e and k^o are the (suitably normalized) dipole moments of the polarizable line (k^e gives the amplitude of the dipole component which has even symmetry about the x_1-axis, whereas k^o gives the amplitude of the dipole component which has odd symmetry about the x_1-axis) and in which $q = 1$ for $p = e$ and $q = -1$ for $p = o$. Differentiating (3.3) gives

$$\begin{aligned} &E_1^r(x_1, x_2) = -[f_{in}^{e\prime}(z) + f_{in}^{e\prime}(\bar{z})]/2 - [f_{in}^{o\prime}(z) - f_{in}^{o\prime}(\bar{z})]/(2i), \\ &E_2^r(x, y) = -i[f_{in}^{e\prime}(z) - f_{in}^{e\prime}(\bar{z})]/2 - [f_{in}^{o\prime}(z) + f_{in}^{o\prime}(\bar{z})]/2, \end{aligned} \tag{3.5}$$

where

$$f_{in}^{p\prime}(z) \equiv df_{in}^p(z)/dz = \frac{qk^p r_*^2 S'(\delta, r_*^2/(r_0 z))}{r_0^2 z^2 \eta_{sc}} + \frac{qk^p r_s^2 \delta e^{i\phi} \eta_{sc} S'(\delta, r_s^2/(r_0 z))}{r_0^2 z^2}, \tag{3.6}$$

in which

$$S'(\delta, w) \equiv \frac{dS(\delta, w)}{dw} = \sum_{\ell=1}^{\infty} \frac{\ell w^{\ell-1}}{1 + \delta e^{i\phi} h^\ell}. \tag{3.7}$$

These expressions simplify if z is real, since then $f_{in}^{p'}(z) - f_{in}^{p'}(\bar{z}) = 0$ and $(E_1^r, E_2^r) = (-f_{in}^{e'}(z), -f_{in}^{o'}(z))$. In particular with $z = r_0$, we obtain

$$\begin{pmatrix} E_1^r \\ E_2^r \end{pmatrix} = c(\delta) \begin{pmatrix} k^e \\ -k^o \end{pmatrix}, \tag{3.8}$$

where

$$c(\delta) = -\frac{r_*^2 S'(\delta, r_*^2/r_0^2)}{r_0^4 \eta_{\rm sc}} - \frac{r_{\rm s}^2 \delta {\rm e}^{{\rm i}\phi} \eta_{\rm sc} S'(\delta, r_{\rm s}^2/r_0^2)}{r_0^4}. \tag{3.9}$$

We will see that $|c(\delta)|$ can diverge to infinity as $\delta \to \infty$, and that when this happens the polarizable line becomes cloaked.

Now, if α denotes the polarizability of the line, then we have

$$\begin{pmatrix} k^{\rm e} \\ -k^{\rm o} \end{pmatrix} = \alpha \begin{pmatrix} E_1^0 \\ E_2^0 \end{pmatrix}. \tag{3.10}$$

This implies

$$\begin{pmatrix} k^{\rm e} \\ -k^{\rm o} \end{pmatrix} = \alpha \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} + \alpha c(\delta) \begin{pmatrix} k^{\rm e} \\ -k^{\rm o} \end{pmatrix}, \tag{3.11}$$

which when solved for the dipole moment $(k^{\rm e}, -k^{\rm o})$ gives

$$\begin{pmatrix} k^{\rm e} \\ -k^{\rm o} \end{pmatrix} = \alpha_* \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \tag{3.12}$$

where

$$\alpha_* = [\alpha^{-1} - c(\delta)]^{-1} \tag{3.13}$$

is the 'effective polarizability'. So far no approximation has been made.

Notice that when $|c(\delta)|$ is very large then $\alpha_* \approx -1/c(\delta)$. So, in this limit, the effective polarizability has a very weak dependence on α.

To obtain an asymptotic formula for $c(\delta)$ when δ is very large we use the asymptotic formulae (2.13) and (2.15). Differentiating these gives

$$S'(\delta, w) \approx \frac{{\rm e}^{-{\rm i}\phi} h}{\delta (h-w)^2}, \quad \text{for } |w| < h, \tag{3.14}$$

for $|w| < h$, whereas when $1 > |w| > h$,

$$\begin{aligned} S'(\delta, w) &\approx -[\log\delta/(w\log h)]{\rm e}^{-\log w \log\delta/\log h} T(w) + {\rm e}^{-\log w \log\delta/\log h} T'(w), \\ &\approx -[\log\delta/(w\log h)]{\rm e}^{-\log w \log\delta/\log h} T(w), \end{aligned} \tag{3.15}$$

where $T'(w) = {\rm d}T(w)/{\rm d}w$ and in making the last approximation in (3.15) we have assumed that $|\log\delta|$ is very large. Let us first treat the case where $\varepsilon_{\rm c}$ is fixed and not equal to $\varepsilon_{\rm m}$ and $r_0 < r_{\rm c}$. Then, we have $\eta_{\rm sc} \approx \eta$ and substituting these approximations in (3.4) and (3.9) and keeping only the terms which are dominant because δ is very large gives, for $r_{\rm c}^2/r_0 > |z| > r_{\rm s}$,

$$f_{\rm in}^{\rm p}(z) \approx -qk^{\rm p}\eta{\rm e}^{{\rm i}\phi}{\rm e}^{[\log z - \log(r_{\rm c}^2/r_0)]\log\delta/\log h} r_0^{-1} T(r_{\rm s}^2/(r_0 z)), \tag{3.16}$$

which implies

$$f_{\rm in}^{\rm p\prime}(z) \approx \frac{-qk^{\rm p}\eta{\rm e}^{{\rm i}\phi}\log\delta}{zr_0\log h}{\rm e}^{-\log(r_{\rm c}^2/(zr_0))\log\delta/\log h} T(r_{\rm s}^2/(r_0 z)) \tag{3.17}$$

and

$$c(\delta) \approx \frac{\eta{\rm e}^{{\rm i}\phi}\log\delta}{r_0^2\log h}{\rm e}^{-2\log(r_{\rm c}/r_0)\log\delta/\log h} T(r_{\rm s}^2/r_0^2). \tag{3.18}$$

We see that $|c(\delta)| \to \infty$ as $\delta \to \infty$ when $r_0 < r_c$. Thus for a polarizable line dipole inside the radius r_c the 'effective polarizability' approaches zero in the limit $\delta \to \infty$. When δ is very large from (3.12) and (3.13), we have

$$k^e \approx -E_1/c(\delta), \qquad k^o \approx E_2/c(\delta). \tag{3.19}$$

Thus for z in the annulus $r_c^2/r_0 > |z| > r_s$ the potential associated with the polarizable line has, from (3.16),

$$f_{in}^e \approx E_1 \delta^{\log(z/r_0)/\log h} r_0 T(r_s^2/(r_0 z)) \log h/(T(r_s^2/r_0^2)\log\delta). \tag{3.20}$$

Similarly in this annulus, we have

$$f_{in}^o \approx E_2 \delta^{\log(z/r_0)/\log h} r_0 T(r_s^2/(r_0 z)) \log h/(T(r_s^2/r_0^2)\log\delta). \tag{3.21}$$

For z outside the radius r_c^2/r_0, the potential due to the polarizable line dipole is approximately given by (2.18) and converges to zero because $k^{(1)}$ and $k^{(2)}$ vanish as $\delta \to \infty$. We avoid the technical question of what happens when $|z| = r_c^2/r_0$ but presumably the potential also converges to zero there.

Thus as $\delta \to \infty$ the potential in the matrix due to the polarizable line dipole converges to zero in the region $r > r_0$, but diverges to infinity with increasingly rapid angular oscillations for $r_s \leqslant r < r_0$. (This is to be contrasted with the potential in the matrix associated with a line dipole having fixed k^e and k^o, which as can be seen from (3.16) diverges to infinity in the much larger region $r_s \leqslant r < r_c^2/r_0$.) A simple calculation shows that in the shell the potential associated with the polarizable line similarly converges to zero for $r > r_0$ but diverges to infinity for $r_s < r < r_0$, whereas in the core the potential associated with the polarizable line converges to zero everywhere.

It is instructive to see what happens to the local field (E_1^0, E_2^0) acting on the polarizable line as $\delta \to \infty$. From (3.1), (3.8), (3.12) and (3.13), we see that

$$E_1^0 = E_1 + c(\delta)k^e = E_1 + \frac{c(\delta)E_1}{\alpha^{-1} - c(\delta)} = \frac{E_1}{1 - \alpha c(\delta)} \tag{3.22}$$

goes to zero as $\delta \to \infty$, and similarly so does E_2^0. This explains why the 'effective polarizability' vanishes as $\delta \to \infty$: the effect of the resonant field is to cancel the field (E_1^0, E_2^0) acting on the polarizable line.

Suppose the source outside is a line dipole with a fixed source term $(k_1^e, k_1^o) = (k_1^e, 0)$ located at the point $(r_1, 0)$, where $r_1 > r_c > r_0 > r_s$. When r_1 is chosen with $r_c^2/r_0 > r_1 > r_c$ the polarizable line will be located within the resonant region generated by the line source outside. One might at first think that a polarizable line placed within the resonant region would have a huge response because of the enormous fields there. However, we will see that the opposite is true: the dipole moment of the polarizable line still goes to zero as $\delta \to \infty$. From (3.8), (3.5) and (3.17), with r_0 replaced by r_1, the field at the point $(r_0, 0)$ when the polarizable line is absent will be

$$E_1 = c_1(\delta)k_1^e, \quad E_2 = 0, \tag{3.23}$$

where

$$c_1(\delta) \approx \frac{\eta e^{i\phi}\log\delta}{r_0 r_1 \log h} e^{-\log(r_c^2/(r_0 r_1))\log\delta/\log h} T(r_s^2/(r_0 r_1)). \tag{3.24}$$

This and (3.19) implies the polarizable line has a dipole moment

$$k^{\rm e} \approx -E_1/c(\delta) \approx -c_1(\delta)k_1^{\rm e}/c(\delta) \approx -\frac{r_0 T(r_{\rm s}^2/(r_0 r_1))}{r_1 T(r_{\rm s}^2/r_0^2)}\delta^{\log(r_1/r_0)/\log h}k_1^{\rm e}. \tag{3.25}$$

So $k^{\rm e}$ scales as $\delta^{\log(r_1/r_0)/\log h}$ which goes to zero (since $h<1$) as $\delta\to\infty$ but fairly slowly when r_1 and r_0 are almost equal, i.e. both are close to $r_{\rm c}$.

If the source is outside the critical radius $r_{\rm crit}=r_{\rm c}^2/r_{\rm s}$ then there are no resonant regions associated with it and $k^{\rm e}$ will scale like $1/c(\delta)$, i.e. as $\delta^{2\log(r_{\rm c}/r_0)/\log h}/\log\delta$ which goes to zero at a faster rate as $\delta\to\infty$, but still slowly when r_0 is close to $r_{\rm c}$. On the other hand, when r_0 is close to $r_{\rm s}$ we have $r_{\rm c}/r_0\approx 1/\sqrt{h}$ and this latter scaling is approximately $\delta^{-1}/\log\delta\sim-\varepsilon_{\rm s}''/\log\varepsilon_{\rm s}''$, where $\varepsilon_{\rm s}''$ is the imaginary part of $\varepsilon_{\rm s}$, which is quite fast.

The asymptotic analysis is basically similar when $\varepsilon_{\rm c}=\varepsilon_{\rm m}$ and $r_0<r_\#\equiv\sqrt{r_{\rm s}r_{\rm c}}$. Then $\eta_{\rm sc}\approx-{\rm i}{\rm e}^{-{\rm i}\phi/2}/\sqrt{\delta}$ and from (3.4), (3.9), (2.15) and (3.15) we have for $r_{\rm c}^2/r_0>|z|>r_{\rm s}$ that

$$f_{\rm in}^{\rm p}(z)\approx {\rm i}qk^{\rm p}{\rm e}^{{\rm i}\phi/2}{\rm e}^{[\log z-\log(r_{\rm c}r_{\rm s}/r_0)]\log\delta/\log h}r_0^{-1}T(r_{\rm s}^2/(r_0 z)) \tag{3.26}$$

and

$$c(\delta)\approx\frac{-{\rm i}{\rm e}^{{\rm i}\phi/2}\log\delta}{r_0^2\log h}{\rm e}^{-\log(r_{\rm c}r_{\rm s}/r_0^2)\log\delta/\log h}T(r_{\rm s}^2/r_0^2). \tag{3.27}$$

When all the sources lie outside the critical radius $r_{\rm c}$ so they do not generate any resonant regions in the absence of the polarizable line, both $k^{\rm e}$ and $k^{\rm o}$ will scale as $1/c(\delta)$, i.e. as $\delta^{\log(r_{\rm c}r_{\rm s}/r_0^2)/\log h}/\log\delta$, as $\delta\to\infty$. When r_0 is close to $r_{\rm s}$ we have $r_{\rm c}r_{\rm s}/r_0^2\approx 1/\sqrt{h}$ and this latter scaling is approximately $1/(\sqrt{\delta}\log\delta)\sim-\varepsilon_{\rm s}''/\log\varepsilon_{\rm s}''$ which is the same as when $\varepsilon_{\rm c}\neq\varepsilon_{\rm m}$. By substituting (3.19) in (3.26), we obtain

$$\begin{aligned} f_{\rm in}^{\rm e}(z) &\approx -E_1{\rm i}{\rm e}^{{\rm i}\phi/2}{\rm e}^{[\log z-\log(r_{\rm c}r_{\rm s}/r_0)]\log\delta/\log h}r_0^{-1}T(r_{\rm s}^2/(r_0 z))/c(\delta),\\ &\approx E_1\delta^{\log(z/r_0)/\log h}r_0T(r_{\rm s}^2/(r_0z))\log h/(T(r_{\rm s}^2/r_0^2)\log\delta), \end{aligned} \tag{3.28}$$

which coincides with (3.20). Likewise (3.21) still holds. By similar arguments applied to $V_{\rm c}$ and $V_{\rm s}$ it follows that as $\delta\to\infty$ the potential V diverges with increasingly rapid oscillations in the core in the region $r_{\rm c}>r>r_{\rm c}r_{\rm s}/r_0$, in the shell in the two regions $r_{\rm s}<r<r_0$ and $r_{\rm c}>r>r_{\rm c}r_{\rm s}/r_0$, and in the matrix in the region $r_{\rm s}<r<r_0$. Outside these regions it converges to the potential generated by the fixed sources.

It is possible to get any cloaking radius between $r_{\rm s}$ and $r_{\rm c}$ if we let $\varepsilon_{\rm c}$ depend on δ, so that $\varepsilon_{\rm s}-\varepsilon_{\rm c}$ scales as $\delta^{-\beta}$ and $\varepsilon_{\rm m}-\varepsilon_{\rm s}$ scales as $\delta^{-1+\beta}$, where β is a fixed constant between 0 and 1. Then $\eta_{\rm sc}$ will scale as $\delta^{-\beta}$ and $c(\delta)$ will scale as $\delta^{\tau}\log\delta$ with $\tau=\log(r_{\rm c}^{2-2\beta}r_{\rm s}^{2\beta}/r_0^2)/(-\log h)$ and so the cloaking radius will be $r_{\rm c}^{1-\beta}r_{\rm s}^{\beta}$. Since (based on the results of Milton *et al* (2005) and Bruno and Lintner (2007)) dielectric bodies located in the cloaking region are not perfectly imaged, it is not sufficient that $\varepsilon_{\rm c}$, $\varepsilon_{\rm s}$ and $\varepsilon_{\rm m}$ be arbitrarily close to each other to ensure perfect imaging of a dielectric body which lies inside the radius $r_{\rm c}$. Similarly, for the standard cylindrical quasistatic superlens, it is not sufficient that $\varepsilon_{\rm c}$, $-\varepsilon_{\rm s}$ and $\varepsilon_{\rm m}$ be arbitrarily close to each other to ensure perfect quasistatic imaging of a dielectric body which lies inside the radius $r_*=r_{\rm s}^2/r_{\rm c}$. Also a slab lens of thickness d and permittivity $\varepsilon_{\rm s}$ separating two media with permittivities $\varepsilon_{\rm m}$ and $\varepsilon_{\rm c}$ will not necessarily provide a good quasistatic image of a dielectric body which lies within a distance d of the slab, even when $\varepsilon_{\rm c}$, $-\varepsilon_{\rm s}$ and $\varepsilon_{\rm m}$ are arbitrarily close to each other.

4. A proof of cloaking for an arbitrary number of polarizable line dipoles

The concept of 'effective polarizability' does not have much use when two or more polarizable lines are positioned in the cloaking region since each polarizable line will also interact with the resonant regions generated by the other polarizable lines and if the polarizable lines are not all on a plane containing the coated cylinder axis then these interactions will oscillate as $\delta \to \infty$. However, we will see here that nevertheless the dipole moment of each polarizable line in the cloaking region must go to zero as $\delta \to \infty$ and in such a way that no resonant field extends outside the cloaking region. This is not too surprising. Based on the results for a single dipole line, we expect that a resonant field extending outside the cloaking region would cost infinite energy, and the only way to avoid this is for the dipole moment of each polarizable line in the cloaking region to go to zero as $\delta \to \infty$.

Here, we limit our attention to the cylindrical lens with the core having approximately the same permittivity as the matrix. Also to simplify the analysis, we assume the core (but not the matrix) has some small loss. Specifically, we assume

$$\varepsilon_{\mathrm{m}} = 1, \qquad \varepsilon_{\mathrm{s}} = 1 - \mathrm{i}\kappa, \qquad \varepsilon_{\mathrm{c}} = 1 + \mathrm{i}\gamma\kappa, \tag{4.1}$$

with κ and $\gamma\kappa$ having positive real parts and approaching zero in such a way that γ, which could be complex, remains fixed and ϕ given by (2.5) also remains fixed. In this limit (2.5) implies $(\kappa + \gamma\kappa)\kappa \approx 4/(\delta \mathrm{e}^{\mathrm{i}\phi})$ and since κ and $\gamma\kappa$ have positive real parts we deduce that ϕ is not equal to π or $-\pi$. Solving for κ we see that

$$\kappa \approx 2\mathrm{e}^{-\mathrm{i}\phi/2}/\sqrt{\delta(1+\gamma)}. \tag{4.2}$$

The potential in the core due to a single dipole in the matrix at z_0 is given by (2.6) and (2.8).

If there are m dipoles at $z_1, z_2, \ldots, z_m$ (where $z_i \neq z_j$ for all $i \neq j$) all in the matrix then, by the superposition principle, the potential in the core is

$$V_{\mathrm{c}} = \sum_{\ell=0}^{\infty} (A_\ell^{(\mathrm{c})} z^\ell + B_\ell^{(\mathrm{c})} \bar{z}^\ell), \tag{4.3}$$

where for $\ell \neq 0$

$$A_\ell^{(\mathrm{c})} = \frac{h^\ell \delta \psi(\delta)}{1 + \delta \mathrm{e}^{\mathrm{i}\phi} h^\ell} \sum_{j=1}^{m} (k_j^{(1)}/r_j)(1/z_j)^\ell, \qquad B_\ell^{(\mathrm{c})} = \frac{h^\ell \delta \psi(\delta)}{1 + \delta \mathrm{e}^{\mathrm{i}\phi} h^\ell} \sum_{j=1}^{m} (k_j^{(2)}/r_j)(1/\bar{z}_j)^\ell, \tag{4.4}$$

in which $r_j = |z_j|$ and

$$\psi(\delta) \equiv \frac{4\varepsilon_{\mathrm{s}}}{\delta(\varepsilon_{\mathrm{s}} - \varepsilon_{\mathrm{c}})(1 - \varepsilon_{\mathrm{s}})}, \tag{4.5}$$

depends on δ through the dependence of ε_{s} and ε_{c} on δ but tends to $\mathrm{e}^{\mathrm{i}\phi}$ as $\delta \to \infty$.

Let us suppose the dipoles positioned in the matrix at $z_1, z_2, \ldots, z_g$ with $1 \leqslant g \leqslant m$ are in the cloaking region, while the remainder of the dipoles are outside the cloaking region, i.e.

$$|z_j| \leqslant r_\#, \quad \text{for all} \quad j \leqslant g, \qquad |z_j| > r_\#, \quad \text{for all} \quad j > g, \tag{4.6}$$

where we allow for the special case where some of the dipoles have $|z_j| = r_\#$: as we will see, these are also cloaked. We do not specify how the set of dipole moments $\{k_1, k_2, \ldots, k_m\}$ depends on δ except that:

- We assume that each dipole outside the cloaking region has moments which converge to fixed limits as $\delta \to \infty$;
$$\lim_{\delta\to\infty}(k_j^{(1)}(\delta), k_j^{(2)}(\delta)) = (k_{j0}^{(1)}, k_{j0}^{(2)}), \quad \text{for all } j > g. \tag{4.7}$$
The dipole moments $k_j^{(1)}(\delta)$ and $k_j^{(2)}(\delta)$ inside or outside the cloaking region are assumed to depend linearly on the field acting upon them, since nonlinearities would generate higher order frequency harmonics. Some of them could be energy sinks, although at least one of them should be an energy source.
- We assume that in the unfolded geometry the energy absorbed per unit time per unit length of the coated cylinder remains bounded as $\delta \to \infty$, as, for example, must be the case if the line sources only supply a finite amount of energy per unit time per unit length. We let $W_{\max}$ be the maximum amount of energy available per unit time per unit length. It is supposed that the quasistatic limit is being taken not by letting the frequency ω tend to zero, but instead by fixing the frequency ω and reducing the spatial size of the system and using a coordinate system which is appropriately rescaled.

We need to show that, because the energy absorption in the core remains bounded, the dipole moments in the cloaking region go to zero as $\delta \to \infty$ and the resonant field does not extend outside the cloaking region, $r \leqslant r_\#$. This is certainly true when only one polarizable line is present but as cancellation effects can occur (the energy absorption associated with two line dipoles can be less than the absorption associated with either line dipole acting separately) a proof is needed.

To do this we bound $k_i^{(1)}$ and $k_i^{(2)}$ for any given $i \leqslant g$ using the fact that the energy loss within the lens is bounded by $W_{\max}$. If $W_c = W_c(\delta)$ represents the energy dissipated in the core in the unfolded geometry, then we have the inequality

$$\begin{aligned} W_c &= (\omega/2)\int_0^{r_c'} r'\mathrm{d}r' \int_0^{2\pi} \mathrm{d}\theta'\, \mathbf{E}'(\mathbf{x}')\cdot \mathrm{Imag}(\boldsymbol{\varepsilon}')\overline{\mathbf{E}'(\mathbf{x}')}, \\ &= (\omega/2)\varepsilon_c'' \int_0^{r_c} r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\, \mathbf{E}(\mathbf{x})\cdot\overline{\mathbf{E}(\mathbf{x})} \\ &\geqslant (\omega/2)\varepsilon_c'' \int_0^{r_c} r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\, E_1(z)\overline{E_1(z)}, \end{aligned} \tag{4.8}$$

in which Imag denotes the imaginary part, $\varepsilon_c'' = \mathrm{Imag}(\varepsilon_c)$ and $E_1(z)$ is the x_1 component of the electric field in the core in the folded geometry given by

$$E_1(z) = -\frac{\partial V_c}{\partial x_1} = -\sum_{\ell=1}^{\infty} \ell r^{\ell-1}(A_\ell^{(c)} e^{i(\ell-1)\theta} + B_\ell^{(c)} e^{-i(\ell-1)\theta}), \tag{4.9}$$

where the derivative $\partial V_c/\partial x_1$ is calculated by substituting $z = x_1 + ix_2$ in (4.3).

Substituting this expression for the electric field back in (4.8) and using the orthogonality properties of Fourier modes we then have

$$
\begin{aligned}
2W_{\rm c}/\omega &\geqslant 2\pi\varepsilon_{\rm c}'' \int_0^{r_{\rm c}} {\rm d}r\Big[(A_1^{(\rm c)}+B_1^{(\rm c)})\overline{(A_1^{(\rm c)}+B_1^{(\rm c)})}r+\sum_{\ell=2}^{\infty}\ell^2 r^{2\ell-1}(A_\ell^{(\rm c)}\overline{A_\ell^{(\rm c)}}+B_\ell^{(\rm c)}\overline{B_\ell^{(\rm c)}})\Big],\\
&\geqslant \pi\varepsilon_{\rm c}'' r_{\rm c}^2(A_1^{(\rm c)}+B_1^{(\rm c)})\overline{(A_1^{(\rm c)}+B_1^{(\rm c)})}+\pi\varepsilon_{\rm c}''\sum_{\ell=2}^{\infty}\ell r_{\rm c}^{2\ell}(A_\ell^{(\rm c)}\overline{A_\ell^{(\rm c)}}+B_\ell^{(\rm c)}\overline{B_\ell^{(\rm c)}}),\\
&\geqslant \pi\varepsilon_{\rm c}''\sum_{\ell=n-m+1}^{n}\ell r_{\rm c}^{2\ell}(A_\ell^{(\rm c)}\overline{A_\ell^{(\rm c)}}+B_\ell^{(\rm c)}\overline{B_\ell^{(\rm c)}})=\pi\varepsilon_{\rm c}''\sum_{k=0}^{m-1}b_k(U_k\overline{U_k}+V_k\overline{V_k}),
\end{aligned}
\tag{4.10}
$$

where the last identity is obtained using (4.4) with the definitions

$$
\begin{aligned}
b_k &\equiv (n-k)(r_*/r_i)^{2n-2k}\delta^2|\psi(\delta)/(1+\delta h^{n-k}{\rm e}^{{\rm i}\phi})|^2,\\
U_k &\equiv \sum_{j=1}^{m}u_j(r_i/z_j)^{-k},\quad u_j\equiv(r_i/z_j)^n k_j^{(1)}/r_j,\\
V_k &\equiv \sum_{j=1}^{m}v_j(r_i/\bar z_j)^{-k},\quad v_j\equiv(r_i/\bar z_j)^n k_j^{(2)}/r_j,
\end{aligned}
\tag{4.11}
$$

in which $k=n-\ell$, $n\geqslant m+1$ remains to be chosen, and $i\leqslant g$. From (4.11) it follows that $\mathbf{U}=\mathbf{M}\mathbf{u}$ and $\mathbf{V}=\mathbf{M}\mathbf{v}$, where $\mathbf{M}$ is the Vandermonde matrix

$$
\mathbf{M}=\begin{pmatrix}
1 & 1 & 1 & \dots & 1\\
z_1/r_i & z_2/r_i & z_3/r_i & \dots & z_m/r_i\\
(z_1/r_i)^2 & (z_2/r_i)^2 & (z_3/r_i)^2 & \dots & (z_m/r_i)^2\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
(z_1/r_i)^{m-1} & (z_2/r_i)^{m-1} & (z_3/r_i)^{m-1} & \dots & (z_m/r_i)^{m-1}
\end{pmatrix}.
\tag{4.12}
$$

From the well-known formula for the determinant of a Vandermonde matrix it follows that $\mathbf{M}$ is non-singular. Therefore there exists a constant $c_i>0$ (which is the reciprocal of the norm of $\mathbf{M}^{-1}$ and which only depends on i, m and the z_j) such that $|\mathbf{U}|\geqslant c_i|\mathbf{u}|$ and $|\mathbf{V}|\geqslant c_i|\mathbf{v}|$, implying

$$
\begin{aligned}
|\mathbf{U}|^2+|\mathbf{V}|^2 &\geqslant c_i^2(|\mathbf{u}|^2+|\mathbf{v}|^2)\\
&\geqslant c_i^2(|u_i|^2+|v_i|^2)\\
&= c_i^2(|k_i^{(1)}|^2+|k_i^{(2)}|^2)/r_i^2.
\end{aligned}
\tag{4.13}
$$

Next, we need to select n and find a lower bound on b_k which is independent of k. Let $s=-\log\delta/\log h$ (so $\delta h^s=1$) and take n as the largest integer smaller than or equal to s so $n+1\geqslant s\geqslant n$. Then since $r_*<r_{\rm s}<r_i$ and $r_*=r_\# h^{3/4}$, we have

$$
(r_*/r_i)^{2n}\delta^2\geqslant(r_*/r_i)^{2s}\delta^2=\delta^2\delta^{-2\log(r_*/r_i)/\log h}=\delta^{1/2}\delta^{-2\log(r_\#/r_i)/\log h}.
\tag{4.14}
$$

Also the following inequalities hold for $m-1\geqslant k\geqslant 0$

$$
1=\delta h^s\leqslant\delta h^n\leqslant\delta h^{n-k}\quad\text{and}\quad\delta h^{n-k}\leqslant\delta h^{s-k-1}\leqslant\delta h^{s-m}=h^{-m}.
\tag{4.15}
$$

So it follows that

$$
|1+\delta h^{n-k}{\rm e}^{{\rm i}\phi}|\leqslant a\equiv\max_{1\leqslant t\leqslant h^{-m}}|1+t{\rm e}^{{\rm i}\phi}|,
\tag{4.16}
$$

and a is independent of δ. From the bounds (4.14) and (4.16), we deduce that

$$
\begin{aligned}
b_k &\geqslant (s-m)(r_i/r_*)^{2k}\delta^{1/2}\delta^{-2\log(r_\#/r_i)/\log h}|\psi(\delta)|^2/a^2,\\
&\geqslant -[(\log\delta/\log h)+m]\delta^{1/2}\delta^{-2\log(r_\#/r_i)/\log h}|\psi(\delta)|^2/a^2.
\end{aligned}
\tag{4.17}
$$

Combining inequalities gives

$$2W_c/\omega \geqslant \frac{\pi\varepsilon_c''|\psi(\delta)|^2\sqrt{\delta}}{a^2(-\log h)}(\log\delta + m\log h)\delta^{-2\log(r_\#/r_i)/\log h}(|\mathbf{U}|^2+|\mathbf{V}|^2),$$
$$\geqslant \frac{\pi\varepsilon_c''|\psi(\delta)|^2 c_i^2\sqrt{\delta}}{a^2 r_i^2(-\log h)}(\log\delta + m\log h)\delta^{-2\log(r_\#/r_i)/\log h}(|k_i^{(1)}|^2+|k_i^{(2)}|^2), \tag{4.18}$$

in which the real positive prefactor has the property that

$$\rho_i \equiv \lim_{\delta\to\infty}\frac{\pi\varepsilon_c''|\psi(\delta)|^2 c_i^2\sqrt{\delta}}{a^2 r_i^2(-\log h)} = \frac{2\pi c_i^2}{a^2 r_i^2(-\log h)}\mathrm{Real}(\mathrm{e}^{-\mathrm{i}\phi/2}\gamma/\sqrt{1+\gamma}) \tag{4.19}$$

is strictly positive, where $\mathrm{Real}(w)$ denotes the real part of w. So there exists a δ_0 such that, for all $\delta > \delta_0$ and all $i \leqslant g$,

$$\frac{\pi\varepsilon_c''|\psi(\delta)|^2 c_i^2\sqrt{\delta}}{a^2 r_i^2(-\log h)} \geqslant \rho/2, \quad \text{where } \rho \equiv \min_{i\leqslant g}\rho_i > 0 \tag{4.20}$$

and such that

$$\log\delta + m\log h > \tfrac{1}{2}\log\delta > -2\log h, \tag{4.21}$$

which, in particular, ensures that $n \geqslant m+1$. So we conclude that

$$|k_i^{(1)}|^2+|k_i^{(2)}|^2 \leqslant 2\delta^{\log(r_\#/r_i)/\log h}\sqrt{2W_c/(\omega\rho\log\delta)}, \tag{4.22}$$

which, since $\log(r_\#/r_i)/\log h$ is negative, forces the dipole amplitudes $k_i^{(1)}$ and $k_i^{(2)}$ to go to zero as $\delta\to\infty$ (even when $r_i = r_\#$) because $W_c = W_c(\delta) \leqslant W_{\max}$.

Now, the superposition principle implies that the potential at any point z in the matrix is

$$V(z) = \sum_{j=1}^{m} k_j^{(1)}V_j^{(1)}(z) + k_j^{(2)}V_j^{(2)}(z), \tag{4.23}$$

where $V_j^{(1)}(z)$ (or $V_j^{(2)}(z)$) is the potential in the matrix due to an isolated line dipole in the matrix at the point z_j with $k_j^{(1)}=1, k_j^{(2)}=0$ (respectively with $k_j^{(1)}=0, k_j^{(2)}=1$). Now according to the analysis at the end of section 2 (which is easily extended to the case treated here, where ε_c depends on δ as implied by (4.1) and (4.2)) it follows that for z in the matrix with $|z| > \max\{r_s, r_\#^2/r_j\}$,

$$\begin{aligned}\lim\nolimits_{\delta\to\infty} V_j^{(1)}(z) &= \tilde{V}_j^{(1)}(z) \equiv \tfrac{1}{r_j(1-z/z_j)},\\ \lim\nolimits_{\delta\to\infty} V_j^{(2)}(z) &= \tilde{V}_j^{(2)}(z) \equiv \tfrac{1}{r_j(1-\bar{z}/\bar{z}_j)}.\end{aligned} \tag{4.24}$$

Also, as shown in the analysis at the end of section 2, if $r_\#^2/r_j > |z| > r_s$, then $V_j^{(1)}(z)$ and $V_j^{(2)}(z)$ diverge as δ^τ, where $\tau = \log(r_c r_s/r_j|z|)/(-\log h)$. If z_j is outside the cloaking region (i.e. $j > g$) then $r_\#^2/r_j$ will be less than $r_\#$. So using the well-known fact that

$$\lim_{\delta\to\infty} e(\delta)f(\delta) = e_0 f_0, \quad \text{where } e_0 = \lim_{\delta\to\infty} e(\delta), \quad f_0 = \lim_{\delta\to\infty} f(\delta), \tag{4.25}$$

it follows that for all $|z| > r_\#$ and all $j > g$

$$\begin{aligned}\lim\nolimits_{\delta\to\infty} k_j^{(1)}V_j^{(1)}(z) &= k_{j0}^{(1)}\tilde{V}_j^{(1)}(z),\\ \lim\nolimits_{\delta\to\infty} k_j^{(2)}V_j^{(2)}(z) &= k_{j0}^{(2)}\tilde{V}_j^{(2)}(z).\end{aligned} \tag{4.26}$$

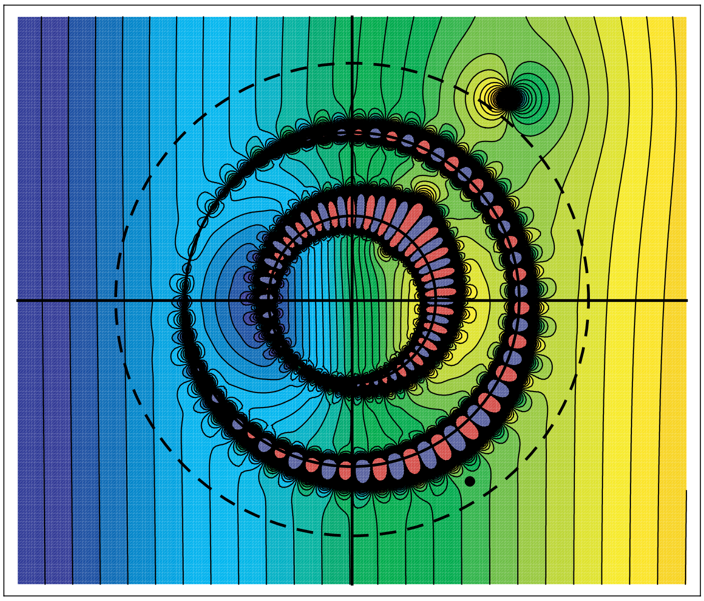

Figure 3. Numerical computations for the potential associated with a pair of polarizable dipoles (polarizability $\alpha = 2$ located at points $z_1 = 2.816 - 4.328\mathrm{i}$ and $z_2 = 3.755 + 4.828\mathrm{i}$) in the unfolded geometry (unfolding parameter $a = 2$) with $\varepsilon_s = -1 + 10^{-9}\mathrm{i}$, $\varepsilon_c = \varepsilon_m = 1$, $r_c = 8$, $r_c' = 2$ and $r_s = r_s' = 4$. The dashed line denotes the cloaking radius, at $r_\# = \sqrt{r_c r_s} \simeq 5.657$. Note that one dipole is outside the cloaking region, while the other is inside.

If z_i is inside the cloaking region (i.e. $i \leqslant g$) and $|z| > r_\#^2/r_i$ then (4.24), (4.25) and the fact that $|k_i^{(1)}|$ and $|k_i^{(2)}|$ tend to zero implies that $k_i^{(1)} V_i^{(1)}(z)$ and $k_i^{(2)} V_i^{(2)}(z)$ will tend to zero. For $r_\#^2/r_i > |z| > r_\#$, we have that $V_i^{(1)}(z)$ and $V_i^{(2)}(z)$ scale as δ^τ with $\tau = \log(r_c r_s/(r_i|z|))/(-\log h)$ while from (4.22) $k_i^{(1)}$ and $k_i^{(2)}$ scale at worst as $\delta^{-t}/(\log\delta)$ with $t = \log(r_\#/r_i)/(-\log h)$. So their product $k_i^{(1)} V_i^{(1)}(z)$ or $k_i^{(2)} V_i^{(2)}(z)$ will scale at worst as $\delta^{\tau-t}/\log\delta$, where $\tau - t = \log(r_\#/|z|)/(-\log h)$. This *goes to zero* as $\delta \to \infty$ when $|z| > r_\#$. By taking the limit $\delta \to \infty$ of both sides of (4.23) we conclude that

$$\lim_{\delta\to\infty} V(z) = \sum_{j=g+1}^{m} [k_{j0}^{(1)} \tilde{V}_j^{(1)}(z) + k_{j0}^{(2)} \tilde{V}_j^{(2)}(z)], \quad \text{for all } |z| > r_\#, \tag{4.27}$$

which proves that the coated cylinder and all the line dipoles inside the cloaking region are invisible outside the cloaking region in this limit.

In this proof, we have assumed that the dipole positions z_j are independent of δ. If they depend on δ and $|z_i(\delta) - z_j(\delta)|$ is not bounded below by a positive constant for all $i \neq j$ then it is an open question as to whether cloaking persists. At least in some cases it may persist since Nicorovici *et al* (2007) show that 'polarizable' quadrupoles are cloaked.

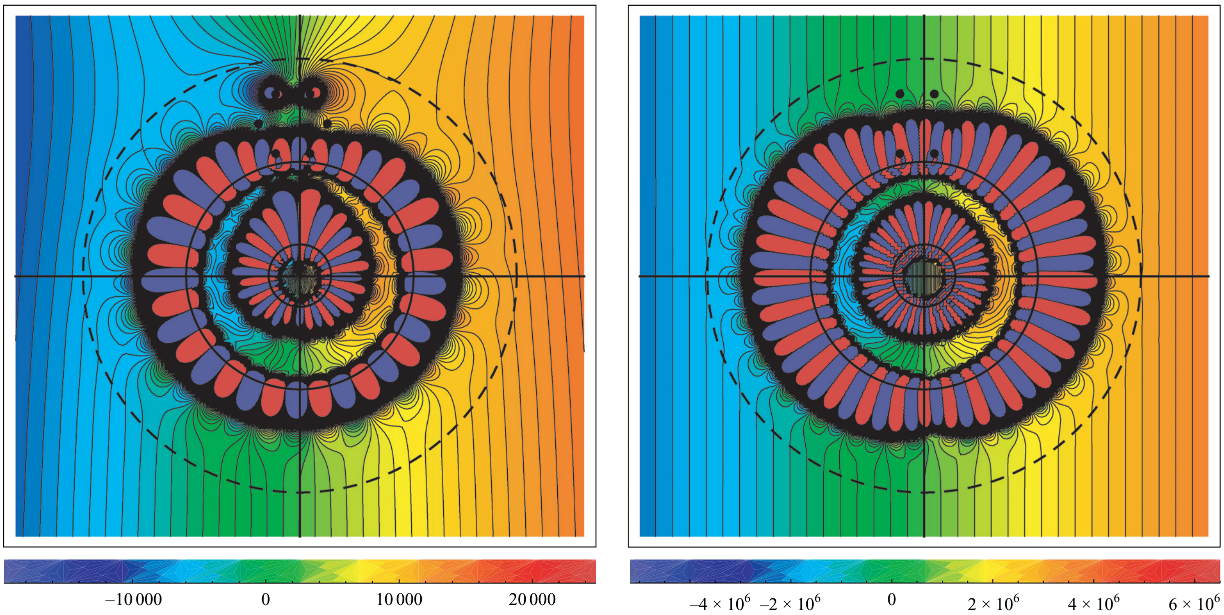

Figure 4. Numerical computations for the potential associated with six polarizable dipoles arranged on the vertices of a hexagon in the unfolded geometry (unfolding parameter $a = r'_s/r'_c$) with (a) $\varepsilon_s = -1 + 10^{-9}$i and (b) $\varepsilon_s = -1 + 10^{-15}$i. In both cases, $\varepsilon_m = \varepsilon_c = 1$, $r_c = 14.5455$, $r'_c = 1.1$, $r_s = r'_s = 4$, while each line dipole has polarizability $\alpha = 2$. The dashed line denotes the cloaking radius, at $r_\# = \sqrt{r_c r_s} \simeq 7.6277$.

5. Numerical examples of cloaking of collections of polarizable line dipoles

Due to the mathematical equivalence between the analysis for the coated cylinder in the cases $r_c > r_s$ and $r_c < r_s$, we can use the same numerical tools here as were employed in the paper (Nicorovici *et al* 2007) to solve for the fields in the folded geometry. Then, we use the unfolding transformation (1.7) to obtain results for the potential in the unfolded geometry, where the permittivity in the shell is anisotropic (with a positive definite imaginary part) and given by (1.9). We have prepared three animations (available from stacks.iop.org/NJP/10/115021/mmedia) illustrating the cloaking action, one for a pair of polarizable dipoles in a uniform external field, and two others for a set of six polarizable dipoles arranged on the vertices of a hexagon. (These animations show snapshots of the potential distribution in space, for a sequence of equilibrium solutions, as discussed by Nicorovici *et al* (2007)). We present here in figures 3 and 4 images from each animation.

Figure 3 shows the potential associated with two polarizable dipoles, of which one is inside the cloaking radius and the other outside it. The resonant region touches the cloaked line dipole, and quenches the field acting on it. As in the previous study (Nicorovici *et al* 2007), the resonance develops first on the shell-core boundary, before developing on the shell-matrix boundary (see movie 1).

Figure 4 shows two frames from the accompanying movies 2 and 3, and compares the cloaking of a set of six polarizable dipoles for two values of the imaginary part of ε_s. As can be seen from the first figure, an imaginary part of 10^{-9} is not sufficient to ensure cloaking of the two dipoles closest to $r_\#$. However, good cloaking of all six dipoles is achieved for an imaginary part of 10^{-15}. As in the papers of Bruno and Lintner (2007) and Nicorovici *et al* (2007), it appears that cloaking becomes more difficult as the number of polarizable particles in the collection increases, and becomes more effective as the particles move more deeply into the cloaking region.

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