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LARGE-TIME BEHAVIOR FOR VISCOUS AND NONVISCOUS HAMILTON–JACOBI EQUATIONS FORCED BY ADDITIVE NOISE*

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Abstract. We study the large-time behavior of the solutions to viscous and nonviscous Hamilton–Jacobi equations with additive noise and periodic spatial dependence. Under general structural conditions on the Hamiltonian, we show the existence of unique up to constants, global-in-time solutions, which attract any other solution.

Key words. Hamilton–Jacobi equations with additive noise, large-time behavior, Lipschitz regularity for Hamilton–Jacobi equations

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1. Introduction.

We are interested in the long-time behavior of solutions to equations of the form

$$\mathrm{d}u - (\mathrm{tr}(A(x)D^2u) - H(Du,x))\mathrm{d}t + \mathrm{d}W(x,t) = 0 \quad \text{in } \mathbb{R}^n \times (t_0,\infty), \tag{1.1}$$

where $t_0 \in \mathbb{R}$ is arbitrary,

$$H \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^n) \quad \text{is } \mathbb{Z}^n\text{-periodic with respect to } \; x, \tag{1.2}$$

and, if S^n and $\mathcal{M}^{n\times m}$ are, respectively, the spaces of $n \times n$ symmetric and $n \times m$ matrices,

$$A \in C^{0,1}(\mathbb{R}^n; S^n) \text{ is } \mathbb{Z}^n\text{-periodic} \tag{1.3}$$

and

$$\text{there exists a } \mathbb{Z}^n\text{-periodic } \sigma \in C^{0,1}(\mathbb{R}^n; \mathcal{M}^{n\times m}) \text{ such that } A = \sigma\sigma^T. \tag{1.4}$$

Here we use the standard notation $C^{0,1}$ and $C^{0,1}_{\mathrm{loc}}$ for the spaces of Lipschitz continuous and locally Lipschitz continuous functions.

We note that (1.4) immediately implies that A is degenerate elliptic, i.e., for all $x, \xi \in \mathbb{R}^n \times \mathbb{R}^n$,

$$(A(x)\xi,\xi) \geq 0.$$

If A is uniformly elliptic, i.e., there exists $\nu > 0$ such that for all $x, \xi \in \mathbb{R}^n \times \mathbb{R}^n$,

$$(A(x)\xi,\xi) \geq \nu|\xi|^2,$$

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then (1.4) holds. As a matter of fact, the latter is true also if A is degenerate elliptic and $A \in C^{1,1}(\mathbb{R}^n; S^n)$.

Let $(\Omega, \mathcal{F}, P)$ be a standard probability space and

$$\Delta = \{(s,t) \in \mathbb{R}^2 : s \leqq t\}.$$

For each $(s,t) \in \Delta$, denote by $W(x,t,s,\omega)$ the increment of the random variable $W(x,\cdot,\omega)$ in the interval $[s,t]$. Then $W(x,t,s,\omega)$ has the form

$$W(x,t,s,\omega) = \sum_{i=1}^{M} F_i(x)(W_i(t,\omega) - W_i(s,\omega)), \tag{1.5}$$

where, for each $i = 1, \ldots, M$,

$$W_i \text{ is a Brownian motion and } F_i \in C^2(\mathbb{R}^n) \text{ is } \mathbb{Z}^n\text{-periodic.} \tag{1.6}$$

In our analysis we do not need to assume that the Brownian motions $W_1, \ldots, W_M$ are mutually independent. Indeed, throughout the paper, we use the fact that $W = (W_1, \ldots, W_M)$ is continuous with respect to t almost surely in ω with increments in time which are independent and identically distributed over disjoint time intervals, and that, for all $\epsilon > 0$ and $\ell \in \mathbb{N}$,

$$\mathbb{P}\left(\sup_{t\in[0,\,l]} |W(t) - W(0)| < \epsilon\right) > 0. \tag{1.7}$$

In view of this, our analysis extends to any random forcing $\zeta(x,t,\omega)$ for which a notion of time integral $Z(x,t,s,\omega) = \int_s^t \zeta(x,\rho)\mathrm{d}\rho$ is defined in such a way that Z has the aforementioned properties. Moreover, using discontinuous viscosity solutions, it is possible to extend our analysis to equations driven by certain jump processes, such as, for example, kicking force (see [IK]). In order to keep the presentation short, we focus here on the Brownian case.

Our results hold for all initial data and initial times and for all realizations of the noise in Ω_C, the set of continuous paths of the Brownian motion, which has full measure ($\mathbb{P}(\Omega_C) = 1$), or a smaller set $\tilde{\Omega}$, also of full measure, to be defined later.

Throughout the paper we write $\mathbb{T} = [0,1]^n$, we denote by $C(\mathbb{T})$ the space of $\mathbb{Z}^n$-periodic continuous real-valued functions, and we use the seminorm $|||\cdot|||$ defined, for each $u \in C(\mathbb{T})$, by

$$|||w||| = \inf_{c\,\in\,\mathbb{R}} \|w - c\|,$$

where $\|\cdot\|$ is the usual *sup*-norm.

The deterministic version of (1.1), i.e., the equation

$$u_t - \operatorname{tr}\left(A(x)D^2u\right) + H(Du,x) = 0 \quad \text{in} \quad \mathbb{R}^n \times (t_0,\infty), \tag{1.8}$$

plays a fundamental role in our analysis.

Indeed our main result says that, under some additional assumptions on A, H and $F = (F_1, \ldots, F_M)$, if (1.8) has a unique up to constants, periodic-in-space, and global-in-time attracting solution, then so does (1.1). In other words, there exists a unique up to constants, periodic with respect to x solution $u_{inv} : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ of (1.1) such that, if u is another solution of (1.1), then

$$\lim_{t\to\infty} |||u(\cdot,t) - u_{inv}(\cdot,t)||| = 0. \tag{1.9}$$

We briefly explain the strategy of the proof. The theory of a fully nonlinear stochastic PDE developed by Lions and one of the authors in [LS1], [LS2], and [LS3], which applies to more general equations, allows us to define pathwise solutions to (1.1). These can be expressed, using a simple transformation, as solutions of a PDE with random coefficients.

The comparison principle for viscosity solutions to (viscous) Hamilton–Jacobi equations implies that the distance between two solutions driven by the same noise cannot increase. Moreover, whenever the excursions of the Brownian motion remain small throughout a time interval, the solutions to (1.1) and (1.8) stay close. In view of (1.9), which holds for solutions of (1.8), the latter converge, as $t \to \infty$ to a unique up to constants attractive solution. It follows that the distance between solutions measured in the seminorm $|||\cdot|||$ decreases throughout such intervals. On the other hand, the independent increments property of W and (1.7) imply that, as $t \to \infty$, there exist enough intervals of small excursions for W. Hence the difference of any two solutions of (1.1) measured in $|||\cdot|||$ tends to 0 as $t \to \infty$. The claim then follows in a standard way.

An important step in showing that the solutions to the deterministic and stochastic equations stay close to each other in intervals of small excursions of the Brownian motion is the fact that, after times of order one, the solutions to (1.1) become Lipschitz continuous with respect to x, with a Lipschitz constant depending on the realization of the noise and not the initial datum. This fact, which is of independent interest, is the main technical result in the paper.

When the equation is of first order, i.e., $A \equiv 0$, the Lipschitz bound follows from the growth conditions on the Hamiltonian, which yield uniform L^∞-bounds on the solutions. For second-order equations, i.e., when $A \not\equiv 0$, there are two distinct cases. When H is superquadratic with respect to the gradient, it is again possible to obtain universal L^∞-bounds on the solutions. The Lipschitz estimate then follows as in the first-order case. When the Hamiltonian is superlinear but not superquadratic, the estimate is more delicate. In this case it is necessary to obtain the Lipschitz bound without using a priori L^∞-bounds for nonnegative solutions, which may not exist. Typically (see, e.g., Barles [B], Crandall, Lions, and Souganidis [CLS], and Lions [L]), the Lipschitz bounds depend on the spatial oscillations of the initial datum, a fact which is not enough for the argument here. We overcome this difficulty by obtaining uniform, after time of order one, estimates on the spacial oscillations of the solutions.

The problem under consideration in this paper is a "toy" example for far more complex models in, for example, phase transitions and growth processes (the so-called KPZ (Kadar–Parisi–Zhang) equation) and fluid mechanics (the stochastically forced Navier–Stokes equation).

The stochastic KPZ equation

$$\mathrm{d}u - \left(\epsilon \Delta u - |Du|^2\right) \mathrm{d}t - \mathrm{d}W = 0$$

is obtained by linearizing the forced mean curvature flow for small gradients and large force. Our results apply directly to this equation with additive forcing and more general operators.

Another concrete example to which our results apply is the stochastic Burgers equation with additive noise. Indeed, if $u \in C(\mathbb{R} \times (0,\infty))$ solves the stochastic Hamilton–Jacobi equation

$$\mathrm{d}u + (u_x)^2 \mathrm{d}t - \mathrm{d}W = 0,$$

then $v = u_x$ solves the Burgers equation

$$\mathrm{d}v + (v^2)_x \mathrm{d}t - \mathrm{d}W_x = 0. \tag{1.10}$$

The unique up to constants random attractor of the Hamilton–Jacobi equation yields a unique invariant measure for the Burgers equation.

Invariant measures for (1.10) and other closely related equations have been the object of extensive study. We refer to E et al. [EKMS], Iturriaga and Khanin [IK], Gomes et al. [GIKP] for the Burgers equation and Mattingly [M1], [M2] for the Navier–Stokes equation with stochastic forcing.

The large-time behavior of solutions of (1.8) depends strongly on whether $A \equiv 0$ or is uniformly elliptic, while very little is known in the degenerate case. When $A \equiv 0$, the problem was studied by Fathi [F], Roquejoffre [R], and Namah and Roquejoffre [NR1], [NR2], the most general results being the ones of Barles and Souganidis [BS2]. The behavior of (1.8) for uniformly elliptic A was studied by Barles and Souganidis [BS3].

When $A = 0$ and H is periodic in time, it was shown by Barles and Souganidis [BS1] (see also Fathi and Mather [FM]) that there are no global attracting solutions. As a matter of fact, phenomena like period doubling can occur. In the uniformly elliptic case, however, it was shown in [BS3] that there exists a unique up to constants attracting solution. Of course, the basic difference between the degenerate and uniformly elliptic settings is that, in the latter case, the equation admits a strong maximum principle.

It follows from our results that even when the equation does not have a strong maximum principle, the stochastic noise is sufficiently irregular for the solutions to lose dependence on the initial data, while this is not true in general for a deterministic time-dependent perturbation.

The proofs in our paper are based on general arguments from the theory of viscosity solutions. This allows us to consider general Hamiltonians H and matrices A. In view of the generality of our assumptions, this paper extends previous works of Iturriaga and Khanin [IK], E et al. [EKMS], and Gomes et al. [GIKP], which consider strictly convex Hamiltonians, and in [GIKP], a space independent uniformly elliptic second-order operator. If the Hamiltonian is strictly convex, the solution of (1.1) can be expressed as the value function of a control problem. The asymptotic behavior of the solutions then reduces to the study of the corresponding controlled stochastic and ordinary differential equations. Here, instead of convexity, we assume some form of asymptotic convexity of the level sets of H. Moreover, in the viscous case, the matrix A can be degenerate elliptic and may depend on space.

We remark that Gomes et al. [GIKP] show that attracting solutions for strictly convex Hamiltonians and $A = \epsilon I$ converge to attracting solutions of the first-order equation. A similar convergence result holds in our case for general A's.

The paper is organized as follows. In section 2 we introduce the notion of solution, we state all the assumptions and the main theorems of the paper, and we prove some preliminary facts. In section 3 we prove the existence of an attracting solution u_{inv} on $\mathbb{R}^n \times (-\infty, \infty)$, assuming that we have the Lipschitz regularization property discussed earlier. Section 4 is devoted to the proof of this property.

2. Assumptions, preliminaries, and results.

We begin with the notion of a solution of (1.1). For this, we need the equation

$$v_t - \mathrm{tr}(A(x)D^2 v) + H(Dv + DW(x,t,t_0), x) = \mathrm{tr}(A(x)D^2 W(x,t,t_0)). \tag{2.1}$$

DEFINITION 2.1. *A function* $u : \mathbb{R}^n \times [a,b] \times \Omega \to \mathbb{R}$ *is a viscosity solution of* (1.1) *if, for all* $[t_0, t_1] \subseteq [a,b]$*, the function*

$$v(x,t,\omega) = u(x,t,\omega) - W(x,t,t_0,\omega)$$

is a viscosity solution of (2.1) *in* $\mathbb{R}^n \times [t_0, t_1]$.

This definition coincides with the more general notion of stochastic viscosity solutions in [LS1], [LS2], [LS3]. Notice that when $A \equiv 0$, for the definition we only need $F \in C^1$. When A is uniformly elliptic and sufficiently smooth—for example, when A has constant coefficients—then it is possible to give an alternative definition requiring less differentiability of the F. Indeed, consider the solution w of the linear stochastic PDE

$$\begin{cases} \mathrm{d}w(x,t,t_0) - \mathrm{tr}(A(x)D^2 w(x,t,t_0))\mathrm{d}t = \mathrm{d}W(x,t), \\ w(x,t_0,t_0) = 0. \end{cases}$$

The basic regularity theory for uniformly parabolic linear equations yields, for some $C > 0$, the estimate

$$\|w(\cdot,t,t_0)\|_{C^2(\mathbb{T})} \le C(\|W\|_{C^{0,\alpha}([t_0,t_1])} + \|F\|_{C^{2,\alpha}(\mathbb{T})}).$$

In this case we say that u is a viscosity solution of (1.1) if $v = u - w(\cdot,\cdot,t_0)$ solves

$$v_t - \mathrm{tr}(A(x)D^2 v) + H\big(Dv + Dw(x,t,t_0), x\big) = 0 \text{ in } \mathbb{R}^n \times [t_0, t_1].$$

Next we state a proposition which asserts the existence and uniqueness of pathwise solutions of (1.1). Since the result is an immediate consequence of the theory of viscosity solutions (see [CIL], [B]) and Definition 2.1, we omit the proof.

PROPOSITION 2.2. *Assume* (1.2)*,* (1.3)*,* (1.4)*,* (1.5)*, and* (1.6)*. For all* $\omega \in \Omega_C$*,* $s \in \mathbb{R}$*, and* $u \in C(T)$*, there exists a unique stochastic viscosity solution* $u(\cdot,\cdot,s,\omega) \in C\big(\mathbb{R}^n \times [s,\infty)\big)$ *of* (1.1) *such that* $u(\cdot,s,s,\omega) = u$.

Throughout the paper we denote by $S^{W,A}(t,s)(u)$ the stochastic viscosity solution of (1.1) starting with initial datum u at s. The solution to (1.8) is denoted by $S^{0,A}(t,s)(u)$. When $A \equiv 0$ and the context allows it, we write $S^W(t,s)$ and $S^0(t,s)$ to denote the solution operators to (1.1) and (1.8), respectively. Finally, whenever it does not create any ambiguity, we write $S^{W,A}(t,s)$ for both $S^{W,A}(t,s)$ and $S^{0,A}(t,s)$.

Since it will be used later, we note here that, as an immediate consequence of Proposition 2.2, both $S^{0,A}(t,s)$ and $S^{W,A}(t,s)$ commute with constants, i.e., for all $c \in \mathbb{R}^n$,

$$S^{W,A}(t,s)(v+c) = S^{W,A}(t,s)(v) + c. \tag{2.2}$$

We proceed with the assumptions on the Hamiltonian H, which we will be using in this paper.

$$\begin{cases} \text{There exist } K > 0 \text{ and } q > 1 \text{ such that for all } (p,x) \in \mathbb{R}^n \times \mathbb{R}^n, \\ \qquad H(p,x) \ge K^{-1}|p|^q - K. \end{cases} \tag{2.3}$$

$$\begin{cases} \text{There exist } R_0 > 0 \text{ and a strictly increasing } \Phi \in C\big([0,\infty),[0,\infty)\big) \\ \text{with } \Phi(0) = 0, \text{ such that for all } (p,x) \in \mathbb{R}^n \times \mathbb{R}^n \text{ with } |p| \ge R_0, \\ \qquad D_pH(p,x)\cdot p - H(p,x) \ge \Phi(|p|). \end{cases} \tag{2.4}$$

$$\left\{\begin{array}{l}\text{There exist } R_0 \text{ and } B>0 \text{ such that for all } (p,x)\in\mathbb{R}^n\times\mathbb{R}^n \\ \text{with } |p|\geq R_0, -D_xH(p,x)\cdot p\leq B|p|^2(D_pH(p,x)\cdot p-H(p,x)).\end{array}\right. \tag{2.5}$$

$$\begin{cases}\text{There exist } R_0>0 \text{ and a strictly increasing } \Phi\in C([0,\infty);[0,\infty)) \text{ with } \Phi(0)=0,\\ \ \text{such that for some } \delta>0, G(r)=\Phi(r)r^{-(1+\delta)} \text{ is increasing,}\\ G(r)\to\infty \ \text{ as } r\to\infty, \text{ and for all } (p,x)\in\mathbb{R}^n\times\mathbb{R}^n \text{ with } |p|\geq R_0,\\ \qquad\qquad D_pH(p,x)\cdot p-H(p,x)\geq\Phi(|p|).\end{cases} \tag{2.6}$$

$$\begin{cases}\text{There exists } C>0 \text{ such that for all } (p,x)\in\mathbb{R}^n\times\mathbb{R}^n \text{ with } |p|\geq R_0,\\ \qquad\qquad -D_xH(p,x)\cdot p\leq C(D_pH(p,x)\cdot p-H(p,x)).\end{cases} \tag{2.7}$$

$$\limsup_{|p|\to\infty}(D_pH(p,x)\cdot p-H(p,x))^{-1}|D_pH(p,x)|=0 \ \text{ uniformly in } x\in\mathbb{R}^n. \tag{2.8}$$

$$\sup_{x\in\mathbb{R}^n}\limsup_{|p|\to\infty}\left(D_pH(p,x)\cdot p-H(p,x)\right)^{-1}|p||D_pH(p,x)|<\infty. \tag{2.9}$$

$$\begin{cases}\text{There exist a unique } \lambda\in\mathbb{R} \text{ and a unique up to constants } U\in C(\mathbb{T}),\\ \text{both depending on } A \text{ and } H, \text{ such that for each } v\in C(\mathbb{T}) \text{ and } t_0\in\mathbb{R},\\ \text{there exists } c\in\mathbb{R} \text{ such that}\\ \lim_{N\to\infty}\sup_{x\in\mathbb{T}}\left|S^{0,A}(t_0+N,t_0)(v)-(U+c)-\lambda N\right|=0.\end{cases} \tag{2.10}$$

Assumptions (2.4) and (2.6) state that the level sets of H as a function of p become convex for large $|p|$. This asymptotic condition is crucial for obtaining Lipschitz bounds which do not depend on the initial data and is much weaker than requiring the Hamiltonian to be convex in p.

The sole purpose of (2.8) and (2.9) is to ensure that the Hamiltonian in (2.1), which arises after incorporating the noise, still satisfies the growth assumptions (2.3), (2.4), (2.5) in the nonviscous case and (2.3), (2.6), (2.7) in the viscous case, with constants which may depend on t_0, t, and ω.

Among all the above, the most important assumption is (2.10). It states that the corresponding deterministic equation has a global attractor, which consists—up to constants—of a single trajectory. We refer to the introduction for a discussion concerning this fact and to [BS1], [BS2], and [BS3] for results yielding (2.10) as well as an extensive list of references.

The main result of this paper is the next theorem. The strategy for the proof of the first part was outlined in the introduction. As we explain later in this section the second part is a simple consequence of the first and the stability properties of the viscosity solutions.

THEOREM 2.3. *Assume* (1.2), (1.5), (1.6), (2.3), *and* (2.10). *There exists* $\widetilde{\Omega}\subseteq\Omega$ *with* $\mathbb{P}(\widetilde{\Omega})=1$ *such that for every* $\omega\in\widetilde{\Omega}$, *the following hold:*

(i) *If* $A\equiv 0$ *and, in addition,* (2.4), (2.5), *and* (2.8) *hold, or if* $A\not\equiv 0$ *satisfies* (1.3), (1.4) *and, in addition,* (2.6), (2.7), (2.9) *hold and* $F_i\in C^3(\mathbb{T})$, *there exists a unique up to constants solution* $u_{inv}(\cdot,\cdot,\omega)\in C(\mathbb{R};C^{0,1}(\mathbb{T}))$ *of* (1.1) *attracting any other solution, i.e., for any* $v\in C(\mathbb{T})$ *and* $s\in\mathbb{R}$,

$$\lim_{t\to\infty}|||u_{inv}(\cdot,t,\omega)-S^W(t,s)(v)(\cdot)|||=0.$$

(ii) *Assume that* $A = \epsilon \tilde{A}$ *is uniformly elliptic and satisfies* (1.3). *If* $u^{\epsilon}_{inv}(\cdot,\cdot,\omega)$ *and* ${u_{inv}}^0(\cdot,\cdot,\omega)$ *are the unique up to constants invariant solutions of* (1.1) *corresponding to* $\epsilon > 0$ *and* $\epsilon = 0$, *respectively, then for any* $[a,b] \subset (-\infty,+\infty)$,

$$\lim_{\epsilon \to 0} \sup_{t \in [a,b]} |||u^{\epsilon}_{inv}(\cdot,t,\omega) - u^0_{inv}(\cdot,t,\omega)||| = 0.$$

As was already mentioned in the introduction, for $A \equiv 0$ and $H(p) = |p|^2$ this result was first proved by [EKMS] in one dimension and by [IK] in all dimensions for general strictly convex H and uniformly elliptic x-independent A. Our assumptions allow, however, to consider nonconvex Hamiltonians and degenerate elliptic A. For example, H can have the form

$$H(p,x) = |p|^2 \widehat{H}(\hat{p},x),$$

where, for $p \in \mathbb{R}^n \setminus \{0\}$, $\hat{p} = |p|^{-1}p$, and $\widehat{H}$ is periodic in x and uniformly bounded away from 0. It is straightforward to check that all structural assumptions on H hold. Moreover, it is proved in [BS2] and [BS3] that for each $v \in C(\mathbb{T})$, $S^0(t)(v)$ has a limit as $t \to \infty$. The up to constants uniqueness of the asymptotic limit of the deterministic equation is here an assumption, which holds, for example, if $\widehat{H}$ is independent of x.

Most of the growth conditions on H are needed for the following lemma, which plays a central role in the paper. In fact, this lemma is of independent interest, as it extends known regularity results for viscous Hamilton–Jacobi equations.

For $(t_1,t_2) \in \Delta$, we write

$$C_W(t_1,t_2,\omega) = \max_i \sup_{t \in [t_1,t_2]} \left| \int_{t_1}^{t} \mathrm{d}W_i(s,\omega) \right|. \tag{2.11}$$

We have the following.

LEMMA 2.4. *Assume* (1.2), (1.3), (1.4), (1.5), (1.6), (2.3) *and either* (2.6), (2.7), (2.9), *and* $F_i \in C^3(\mathbb{T})$ *if* $A \not\equiv 0$ *is degenerate elliptic, or* (2.4), (2.5), *and* (2.8) *if* $A \equiv 0$. *For all* $\omega \in \Omega_C$ *and* $(s,t) \in \Delta$, *there exists* $L(s,t,\omega) > 0$ *such that for all* $v \in C(\mathbb{T})$,

$$\inf_{c \in \mathbb{R}} \|S^{W,A}(t,s)(v) - c\|_{C^{0,1}(\mathbb{T})} \le L(s,t,\omega).$$

Moreover, there exists $\hat{L} : (0,\infty) \times (0,\infty) \to (0,\infty)$ *which is increasing with respect to the second argument, such that*

$$\textit{if } C_W(s,t,\omega) \le K, \qquad \textit{then } L(s,t,\omega) \le \widehat{L}(t-s,K).$$

It follows from Lemma 2.4 that solutions to (1.1) are Lipschitz continuous in space with Lipschitz constant independent of the initial datum. For solutions of the deterministic time-independent equation (1.8), the lemma holds with an L which depends only on $|t-s|$.

The claim about the vanishing viscosity limit asserted in Theorem 2.3 is a simple consequence of our results and standard arguments from the theory of viscosity solutions. Indeed, Lemma 2.4 yields that the family $({u^{\epsilon}}_{inv})_{\epsilon>0}$ is uniformly Lipschitz continuous on any given compact time interval. A simple diagonalization argument yields a subsequence which converges uniformly on compact intervals to a viscosity solution u of (1.1) with $A \equiv 0$. Lemma 3.7 below then asserts that we must have $u(x,t) = u^0_{inv}(x,t) + c(t)$. However, since both u and u^0_{inv} are solutions, the constant c cannot depend on time. Therefore the whole family $({u^{\epsilon}}_{inv})_{\epsilon>0}$ converges up to constants to u^0_{inv}.

3. Proofs. We begin with a number of preliminary lemmas which summarize some of the key properties of the solutions of (1.1). The first lemma is an immediate consequence of the definition of a solution and the comparison principle for viscosity solutions (see [CIL]); hence we omit its proof.

LEMMA 3.1. *For all u, $v \in C(\mathbb{T})$ and $(s,t) \in \Delta$,*

$$\|S^{W,A}(t,s)(u) - S^{W,A}(t,s)(v)\|_{C(\mathbb{T})} \le \|u-v\|_{C(\mathbb{T})}.$$

For $v_0 \in C^{0,1}(\mathbb{T})$ and $(t_1,t_2) \in \Delta$ we denote by

$$L_A(t_1,t_2) = \sup_{s\in[t_1,t_2]} \|DS^{0,A}(s,t_1)(v_0)\|$$

the uniform Lipschitz constant of the solution of the deterministic equation.

We also write C_A and C_0 for the constants

$$C_A = \max_{x\in\mathbb{T},\ |p|\le L_A(t_1,t_2)} \big(|D_pH(p,x)|+1\big)\|F\|_{C^3(\mathbb{T})} \quad \text{if} \quad A \not\equiv 0$$

and

$$C_0 = \max_{x\in\mathbb{T},\ |p|\le L_{A\equiv 0}(t_1,t_2)} (|D_pH(p,x)| \ +1)\|F\|_{C^2(\mathbb{T})} \quad \text{if} \quad A \equiv 0.$$

LEMMA 3.2. *Let $v_0 \in C^{0,1}(\mathbb{T})$ and $(t_1,t_2) \in \Delta$. Then*

$$\big\|S^{0,A}(t_2,t_1)(v_0) - S^{W,A}(t_2,t_1)(v_0)\big\| \le (t_2-t_1)\, C_A\, \|F\|\, C_W(t_1,t_2,\omega).$$

Proof. 1. To simplify the presentation we assume that $t_1 = 0,\ t_2 = T$ and we use the notation $C = C_A\,\|F\|_{L^\infty}\, C_W(t_1,t_2,\omega)$, $u = S^{W,A}(v_0)$, and $v = S^{0,A}(v_0)$.

2. Arguing by contradiction, we assume that there exists $(x_0,t_0) \in \mathbb{T}\times(0,T)$ such that, possibly after exchanging the role of u and v, $u(x_0,t_0) - v(x_0,t_0) - Ct_0 > 0$. Standard arguments from the theory of viscosity solutions (see [CIL]) then yield $\eta > 0$ and $(X_\alpha, p_\alpha, x_\alpha, t_\alpha)$, $(Y_\alpha, p_\alpha, y_\alpha, s_\alpha) \in S^n \times \mathbb{R}^n \times \mathbb{R}^n \times (0,T)$ such that, as $\alpha \to \infty$,

$$\begin{cases} |t_\alpha - s_\alpha| + \alpha|y_\alpha - x_\alpha|^2 \to 0, \quad \mathrm{tr}\,(A(y_\alpha)Y_\alpha) - \mathrm{tr}\,(A(x_\alpha)X_\alpha) \le L\alpha|x_\alpha - y_\alpha|^2, \\ C + \eta(T-t_\alpha)^{-2} + H(p_\alpha,x_\alpha) - \mathrm{tr}(A(x_\alpha)X_\alpha) \\ \le -\eta(T-s_\alpha)^{-2} + H(p_\alpha + DW(y_\alpha,s_\alpha),y_\alpha) - \mathrm{tr}\,(A(y_\alpha)Y_\alpha)\,. \end{cases}$$

The (degenerate) ellipticity of A, the choice of C, and the above inequalities contradict the fact that $\eta > 0$. ∎

Note that the above estimates depend on the Lipschitz constant of the deterministic equation. Hence to use this lemma, it is necessary to have a universal bound on those Lipschitz constants, like the one asserted by Lemma 2.4.

The next claim strengthens the assertion of (2.10), which asserts only pointwise convergence as $t \to \infty$ of the solution operator $S^{0,A}(t,s)$ acting on $C(\mathbb{T})$. It turns out that this convergence is uniform with respect to the initial data.

LEMMA 3.3. *Assume* (2.10) *and the hypotheses of Lemma* 2.4 *hold. There exists a unique up to constants function $U_A^* \in C^{0,1}(\mathbb{T})$ such that, for all $t \in \mathbb{R}$,*

$$\lim_{k\to\infty}\left(\sup_{v\in C^0(\mathbb{T})} |\!|\!| S^{0,A}(t,-k)(v) - U_A^* |\!|\!| \right) = 0.$$

Proof. 1. Since the deterministic equation does not depend on time, we may take $t = 0$. Assume that, for some $\delta > 0$, there exist $(v_k)_{k\in\mathbb{N}} \in C(\mathbb{T})$ such that

$$\||S^{0,A}(0,-k)(v_k) - U_A^*\|| \geq \delta \quad \text{for all } k \in \mathbb{N}, \tag{3.1}$$

where U_A^* is the unique (up to constants) limit which exists in view of (2.10).

2. The Lipschitz continuity asserted in Lemma 2.4 yields constants c_k such that the family $(\hat{v}_k)_{k\in\mathbb{N}}$ defined by

$$\hat{v}_k = S^{0,A}(-k+1,-k)(v_k) - c_k$$

is bounded in $C^{0,1}$ and thus compact in $C(\mathbb{T})$. Hence there exists a subsequence $k_m \to \infty$ such that $\widehat{v}_{k_m} \to \hat{v}$ in C^0.

3. Consider the family of maps $S_k : C(\mathbb{T}) \to C(\mathbb{T})$ given by

$$S_k(v) = S^{0,A}(0,-k+1)(v).$$

The contraction property yields that, as $m \to \infty$,

$$\|S_{k_m}(\hat{v}) - S_{k_m}(\hat{v}_{k_m})\| \to 0.$$

But (2.10) implies that

$$\||S_{k_m}(\hat{v}) - U_A^*\|| \to 0.$$

Hence, $S_{k_m}(\hat{v}_{k_m}) \to U_A^*$, a contradiction to (3.1). ∎

The next result concerns a technical property of the Brownian motion which is a consequence of the fact that the increments are independent and identically distributed. This property plays a fundamental role in our analysis as well as that of [EKMS], [IK], and [GIKP]. To state it, we need the following definition.

DEFINITION 3.4. *Fix $l, m \in \mathbb{N}$ and $k \in \mathbb{Z}$. An interval $[kl,(k+1)l]$ is called an (l,m)-small noise interval if*

$$\sup_{t\in[kl,(k+1)l]} \sup_{1\leq i\leq M} |W_i(t) - W_i(kl)| \leq \frac{1}{m}.$$

We have the following.

LEMMA 3.5. *For almost every path and for any $(l,m) \in \mathbb{N}\times\mathbb{N}$, there are two sequences of integers $(k_i^{l,m,\pm})_{i\in\mathbb{N}}$ such that $k_i^{l,m,\pm} \to \pm\infty$, as $i \to \infty$, and $[k_i^{l,m,\pm}l, (k_i^{l,m,\pm}+1)l]$ are (l,m)-small noise intervals.*

Proof. 1. We present the argument only for positive values of k.

2. Let

$$A_k^{l,m} = \left\{\omega : \sup_{kl\leq t\leq (k+1)l} \sup_{1\leq i\leq M} |W_i(t) - W_i(kl)| \leq \frac{1}{m}\right\}.$$

The increments $W(t) - W(kl)$ of the Brownian motion $W(t) = (W_1(t),\ldots,W_M(t))$ on the interval $[kl,(k+1)l]$ are independent and identically distributed. Hence the events $(A_k^{l,m})_{k\in\mathbb{N}}$ are independent and $\mathbb{P}(A_k^{l,m})$ is strictly positive and independent of k. The second Borel–Cantelli lemma then yields that

$$\mathbb{P}(\{\omega \in A_k^{l,m} \text{ for infinitely many } k\}) = 1. \qquad ∎$$

The subset $\tilde{\Omega}$ of Ω of full measure in which our result holds consists of all of continuous paths which have, for each $(l, m) \in \mathbb{N} \times \mathbb{N}$, infinitely many (l, m)-small noise intervals for both positive and negative times. The precise definition of $\tilde{\Omega}$ is

$$\tilde{\Omega} = \Omega_C \cap_{(l,m)\in\mathbb{N}\times\mathbb{N}} (\cap_{j=1}^{\infty} \cup_{k=j}^{\infty} A_k^{l,m}) \cap (\cap_{j=1}^{\infty} \cup_{k=j}^{\infty} A_{-k}^{l,m}).$$

Next we use Lemmas 2.4, 3.2, 3.3, and 3.5 to establish the following.

LEMMA 3.6. *Fix $\omega \in \widetilde{\Omega}$, t_0 and $\delta > 0$. There exists $k_0 = k_0(\omega) \in \mathbb{N}$ such that for all $k \geq k_0(\omega)$ and $u,\ v \in C(\mathbb{T})$,*

$$|||S^{W,A}(t_0, t_0 - k)(u) - S^{W,A}(t_0, t_0 - k)(v)||| \leq \delta, \ \textit{and}$$

$$|||S^{W,A}(t_0 + k, t_0)(u) - S^{W,A}(t_0 + k, t_0)(v)||| \leq \delta.$$

Proof. 1. Since both estimates are proved similarly, here we establish only the second.

2. Lemma 3.3 yields an $M > 0$ such that for any initial datum $\hat{u}$ and any $m \in \mathbb{N}$,

$$|||S^{0,A}(m + M, m)(\widehat{v}) - U_A^*||| < \delta/4. \tag{3.2}$$

Here we use the fact that, since the deterministic equation is independent of time,

$$\sup_{\hat{v}} |||S^{0,A}(m + M, m)(\hat{v}) - U_A^*||| = \sup_{\hat{v}} |||S^{0,A}(M, 0)(\hat{v}) - U_A^*|||.$$

3. If C_A is the constant in Lemma 3.2 for the Lipschitz constant $L = \widehat{L}(1, 1)$, choose $m \in \mathbb{N}$ such that $4MC_A\|F\| < \delta m$ and recall that Lemma 3.5 yields an $(M + 1, m)$-small noise interval $[j(M + 1), (j + 1)(M + 1)]$ contained in $(t_0, +\infty)$.

Fix $k_0(\omega)$ such that $t_0 + k_0(\omega) > (j + 1)M$. It follows that the small noise interval is contained in $(t_0, t_0 + k_0(\omega)]$.

4. Let $t_M^- = j(M + 1)$ and $t_M^+ = (j + 1)(M + 1)$. Since $[t_M^-, t_M^- + 1]$ is contained in the small noise interval, Lemma 2.4 asserts that

$$u_0 = S^{W,A}(t_M^- + 1, t_0)(u) \quad \text{and} \quad v_0 = S^{W,A}(t_M^- + 1, t_0)(v)$$

are Lipschitz continuous with Lipschitz constant $L = \widehat{L}(1, 1)$.

Applying again Lemma 2.4, we find that the last statement holds on the entire interval $[t_M^- + 1, (t_M^- + 1) + M]$, which has length M.

5. Using (3.2) and Lemma 3.2, we find

$$\begin{aligned} &|||S^{W,A}(t_M^+, t_M^- + 1)(u_0) - S^{W,A}(t_M^+, t_M^- + 1)(v_0)||| \\ &\quad \leq |||S^{W,A}(t_M^+, t_M^- + 1)(u_0) - S^{0,A}(t_M^+, t_M^- + 1)(u_0)||| \\ &\qquad + |||S^{W,A}(t_M^+, t_M^- + 1)(v_0) - S^{0,A}(t_M^+, t_M^- + 1)(v_0)||| \\ &\qquad + |||S^{0,A}(t_M^+, t_M^- + 1)(u_0) - U_A^*||| + |||S^{0,A}(t_M^+, t_M^- + 1)(v_0) - U_A^*||| \\ &\quad \leq 4(\delta/4). \end{aligned}$$

The contraction property guarantees now that the estimate holds for all later times $t > t_M^+$. □

Next we construct the global attracting solution u_{inv}^A.

LEMMA 3.7. *Fix $\omega \in \widetilde{\Omega}$. For all $u_0 \in C(T)$ and all $t \in \mathbb{R}$, the limit*

$$\widetilde{u}(\cdot, t) = \lim_{k\to\infty} S^{W,A}(t, -k)(u_0)(\cdot) \tag{3.3}$$

exists in $C(\mathbb{T})$ and is unique up to constants. Moreover, for any $t_1 < t_2$, there exists $c(t_1, t_2) \in \mathbb{R}$ such that

$$S^{W,A}(t_2, t_1)(\tilde{u}(t_1)) = \widetilde{u}(t_2) + c(t_2, t_1). \tag{3.4}$$

Proof. 1. Lemma 3.6 yields that the family $(u_k(\cdot, t))_{k\in\mathbb{N}}$ defined by

$$u_k(\cdot, k) = S^{W,A}(t, -k)(u_0)(\cdot)$$

is a Cauchy sequence with respect to the seminorm $|||\cdot|||$ for each fixed t. Therefore there exist constants $c_k(t)$ such that the sequence $u_k(\cdot, t) - c_k(t)$ converges in $C(\mathbb{T})$.

2. The identity (3.4) is a consequence of the C^0-continuity of the semigroup. ∎

We are now in a position to present the proof of Theorem 2.3.

Proof. In view of Lemma 3.6 and Lemma 3.7, it remains to show that there exists $c(t)$ such that the function $\tilde{v} = \tilde{u} - c$ satisfies, for all $t_1 < t_2$,

$$S^{W,A}(t_2, t_1)(\widetilde{v}(\cdot, t_1))(\cdot) = \widetilde{v}(\cdot, t_2).$$

Let $t_1 < t_2 < 0$. The semigroup property and (3.4) yield

$$S^{W,A}(0, t_1)(\widetilde{u}(\cdot, t_1)) = S^{W,A}(0, t_2)\left(\widetilde{u}(\cdot, t_2) + c(t_2, t_1)\right).$$

It follows that

$$c(t_2, t_1) = c(0, t_1) - c(0, t_2).$$

Similar expressions for $t_2 < 0 < t_1$ and $t_2,\ t_1 > 0$ yield the existence of a solution on $(-\infty, \infty)$ by setting

$$u_{inv}(x, t) = \widetilde{u}(x, t) + c(t),\ c(t) = c(\max\{t, 0\}, \min\{t, 0\}). \qquad ∎$$

4. The proof of the Lipschitz bounds. The proof of Lemma 2.4 is long and technical. To simplify the presentation, we divide it into a number of lemmas.

We remind the reader that the sole purpose of assumptions (2.8) and (2.9) is to ensure that the Hamiltonian in (2.1), which arises after incorporating the noise, still satisfies the growth assumptions (2.3), (2.4), (2.5) in the nonviscous case and (2.3) and (2.6), (2.7) in the viscous case, with constants depending on the noise only through the expression in (2.11). Therefore, we will usually omit the dependence of the Hamiltonian in (2.1) on t and ω, thus keeping the notation simple.

The first step towards the universal Lipschitz bound is a universal L^∞-bound for nonnegative solutions. This is the object of the following lemma.

LEMMA 4.1. *Fix $\omega \in \Omega_C$, $u_0 \in C(\mathbb{T})$, and $s \in \mathbb{R}$ and assume* (2.3) *and $A \equiv 0$. Let u be the solution of* (2.1) *on $\mathbb{R} \times [s, T]$ with $u(\cdot, s) = u_0$. For all $t \geq s$, there exists a positive constant $C(s, t, \omega)$, which is independent of the initial datum u_0 and depends on ω only through the expression in* (2.11)*, such that*

$$\|u(\cdot, t) - \min u_0\| \leq C(s, t, \omega).$$

Proof. 1. If H satisfies (2.3), a straightforward calculation yields that so does $\bar{H}(p, x, t) = H(p + DW(x, t, s, \omega), x)$ with a constant depending on $\|W\|_{C^\infty(\mathbb{T}\times[0,T])}$.

Without loss of generality, we may assume that $u_0(0) = \min_{\mathbb{T}} u_0 = 0$ and $s = 0$. The extension to the general case is straightforward.

2. For sufficiently large $C = C(K) > 0$, the function

$$g(x,t) = C|x|^{q/(q-1)}t^{-1/(q-1)} + Kt + 1$$

is a supersolution of (2.1).

Indeed, for C large,

$$\begin{aligned} &-C(q-1)^{-1}(|x|t^{-1})^{q/q-1} + K + H(Cq(q-1)^{-1}(|x|t^{-1})^{1/q-1}D|x|, x) \\ &\quad \geq C(q-1)^{-1}(|x|t^{-1})^{q/q-1}(K^{-1}q(qC)^{q-1}(q-1)^{1-q} - 1) - K + K \geq 0. \end{aligned}$$

For t small enough we clearly have $g(\cdot,t) \geq u(\cdot,t)$. Since the infimum of a family of supersolutions is also a supersolution, it follows that

$$\bar{g}(x,t) = \inf_{z\in\mathbb{Z}^n} g(x-z,t)$$

is a periodic supersolution of (2.1). □

When $A \not\equiv 0$, a universal L^∞-bound for nonnegative solutions is available only for Hamiltonians H with superquadratic growth in p. Indeed, we have the following.

LEMMA 4.2. *Fix $\omega \in \Omega_C$ and $u_0 \in C(\mathbb{T})$ and assume that* (2.3) *holds with $q > 2$. Let u solve* (2.1) *on $\mathbb{R}^n \times [s,T]$ with $u(\cdot,s) = u_0$. For $(s,t) \in \Delta$ there exists a constant $C(s,t,\omega)$, independent of the initial datum and depending on ω only via* (2.11)*, such that*

$$\|u(\cdot,t) - \min u_0\| \leq C(s,t,\omega).$$

Before we present the proof, we remark that it is not expected, as follows from the discussion below, to have a universal bound on the L^∞-norm for nonnegative solutions of the viscous Hamilton–Jacobi equations with quadratic or subquadratic growth H. Indeed, for $c > 0$, consider the function $u_c : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ defined by

$$u_c(x,t) = \frac{n}{2}\ln(t+c) + (4(t+c))^{-1}|x|^2 - \frac{n}{2}\ln(c),$$

which is an exact nonnegative solution to

$$u_t - \Delta u + |Du|^2 = 0.$$

It is immediate that, for each $c > 0$, $\min u_c(x,0) = 0$, $u_c(\cdot,t) \geq 0$ for all $t \geq 0$ and $\lim_{c\to 0} u_c(x,1) = +\infty$. However, the oscillation of $u_c(x,1)$ on each bounded subset of $\mathbb{R}^n$ is bounded uniformly in c.

The above solutions were obtained by applying the Hopf–Cole transform to fundamental solutions of the heat equation at time $t+c$. By applying the Hopf–Cole transformation to periodic solutions of the heat equation, it is possible to construct counterexamples in the periodic case in a similar way.

Now we prove Lemma 4.2.

Proof. 1. To simplify the presentation, we assume throughout the proof that $s = 0$ and write u_0 for $u(\cdot,0)$. Finally, as before, we assume that $\min u_0 = 0$.

2. Let

$$\beta = q - 2 > 0, \quad \gamma = (1-\theta)(q-2)(q-1)^{-1}, \quad \text{and} \quad \alpha = \gamma - 1 + 2\theta,$$

where $\theta \in (0, 2^{-1})$ is chosen so that $\alpha > 0$.

3. For $a, b > 0$ consider the function $G_{a,b} : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ given by

$$G_{a,b}(x,t) = Kt + 2b \max_{\mathbb{T}} \operatorname{tr}(A) t^{\gamma} + a t^{\alpha} + b\gamma |x|^2 t^{\gamma - 1}.$$

It is immediate that for any $x \neq 0$, $\lim_{t\to 0} G_{a,b}(x,t) = +\infty$. Hence, for t small,

$$G_{a,b} \geq u_0.$$

4. The constants a, b may be chosen so that $G_{a,b}$ is a supersolution of (2.1). Indeed, since $D^2|x|^2 = 2I$, it remains to show only that

$$R_{a,b}(x,t) = a\alpha t^{\alpha-1} + |x|^2 t^{\gamma-2}\big(K^{-1}(2b\gamma)^q (|x|t^{-\theta})^{q-2} - \gamma(1-\gamma)b\big) > 0.$$

If $|x| \geqq t^{\theta}$, it is possible to find b, depending on q, θ, and K but not on a, so that $R_{a,b} > 0$.

If $|x| \leq t^{\theta}$, it is possible to choose a so that

$$R_{a,b}(x,t) \geq (a\alpha - \gamma(1-\gamma)b)t^{\alpha-1} > 0.$$

5. A periodic supersolution can be constructed as the infimum of supersolutions exactly as in the first-order case. ∎

We remark that since

$$\inf_{\mathbb{T}} u(t,\cdot) \geq -Kt + \inf_{\mathbb{T}} u(0,\cdot)$$

and the equations commute with constants, Lemmas 4.1 and 4.2 yield automatically a bound on the oscillation

$$\operatorname{osc}(u(\cdot,t)) = \sup_{\mathbb{T}} u(\cdot,t) - \inf_{\mathbb{T}} u(\cdot,t).$$

Thus a bound on the oscillation is a weaker statement than the bounds on the L^{∞}-norm of nonnegative solutions asserted by the previous lemmas. We summarize these comments in the following corollary.

COROLLARY 4.3. *Fix $\omega \in \Omega_C$. Under the assumptions of either Lemma* 4.1 *or Lemma* 4.2*, there exists a positive constant $C(s,t,\omega)$, depending on ω only through $C_W(s,t,\omega)$ as in* (2.11)*, such that for all $(s,t) \in \Delta$ and $u_0 \in C(\mathbb{T})$,*

$$\operatorname{osc}(S^{W,A}(t,s)) \leq C(s,t,\omega).$$

The following lemma completes the proof of Lemma 2.4 in the first-order case.

LEMMA 4.4. *If* (2.4), (2.5)*, and* (2.8) *hold and u solves* (1.1) *on $\mathbb{R}^n \times [s,T]$ with $A \equiv 0$, then for all $t \in [s,T]$, $u(\cdot,t)$ is Lipschitz continuous with a Lipschitz constant bounded by $L(s,t,\omega)$, which is nonincreasing for $s < t < s+1$ and depends only on* (2.11)*, H, and* $\sup_{t' \in [s,T]} \|u(\cdot,t')\|$.

Proof. 1. For almost all ω, there exists a $K(t,s,\omega) > 0$ such that if $|p| > K(t,s,\omega)$, there exist $B, R_0 > 0$ such that

$$\widetilde{H}(p,x,t,\omega) = H(p + DW(x,t,s,\omega), x)$$

satisfies (2.4) and (2.5) for fixed ω uniformly in $t \in [s,T]$. Again this is the place where (2.8) is used. In order to simplify notation, next we suppress the dependence of $\tilde{H}$ on t, s, and ω and write simply $\widetilde{H}(p,x)$. Finally, we choose $s = 0$.

2. Following [CLS] (note that (2.4) and (2.5) are (G2) and (3.2) in [CLS]), we consider the solution φ of

$$\varphi'(t) = \varphi(t)\Phi(\varphi(t)^{-1}), \tag{4.1}$$

where Φ is the increasing function in (2.4).

3. For $\lambda > 0$ let

$$z(x,t) = -\varphi(t)e^{-\lambda u(x,t)}.$$

It follows that

$$z_t - G(Dz, z, x) - \varphi'\varphi^{-1}z = 0,$$

where

$$G(p, z, x) = (\lambda z)\widetilde{H}(-(\lambda z)^{-1}p, x).$$

Note that if $q = -(\lambda z)^{-1}p$, then

$$D_z G(p, z, x) = \lambda\big(\widetilde{H}(q, x) - qD_p\widetilde{H}(q, x)\big)$$

and

$$D_x G(p, z, x) = \lambda z D_x\widetilde{H}(q, x) = -|p|\,|q|^{-1}D_x\widetilde{H}(q, x).$$

4. If, for some $C > 0$,

$$w(x, y, t) = z(x, t) - z(y, t) - C|x - y|$$

has a positive maximum M at (x_0, y_0, t_0), then in particular $x_0 \neq y_0$, so $|x - y|$ is smooth in a neighborhood of (x_0, y_0, t_0).

Using the definition of the viscosity solutions with $p = C(x_0 - y_0)|x_0 - y_0|^{-1}$ and noting that $p = C\hat{p}$, we find

$$\begin{aligned}
0 &\leq G(p, z(x_0, t_0), x_0) - G(p, z(y_0, t_0), y_0) + \varphi'(\varphi^{-1})(t_0)(z(x_0, t_0) - z(y_0, t_0)) \\
&= \int_0^1 [|x_0 - y_0|\hat{p}\cdot D_x G(p, z(r), x(r)) + D_z G(p, z(r), x(r))(z(x_0, t_0) - z(y_0, t_0))]\mathrm{d}r \\
&\quad + \varphi'(\varphi^{-1})(t_0)\big(z(x_0, t_0) - z(y_0, t_0)\big),
\end{aligned}$$

where

$$q(r) = -(\lambda z(r))^{-1}p, \quad x(r) = y_0 + r(x_0 - y_0), \quad z(r) = z(y_0, t_0) + r(z(x_0, t_0) - z(y_0, t_0)).$$

Hence

$$\begin{aligned}
0 \leq &\int_0^1 \Big(-C|q(r)|^{-1}|x_0 - y_0|\hat{q}(r)\cdot D_x\widetilde{H}\big(q(r), x(r)\big)\Big)\mathrm{d}r \\
&+ \varphi'(\varphi^{-1})(t_0)\big(z(x_0, t_0) - z(y_0, t_0)\big) \\
&- \lambda\big(z(x_0, t_0) - z(y_0, t_0)\big)\int_0^1 \big(q(r)\cdot D_q\widetilde{H}(q(r), x(r)) - \widetilde{H}(q(r), x(r))\big)\mathrm{d}r.
\end{aligned}$$

Assume next that C is such that

$$C \geq \lambda \sup_{\mathbb{T}} |z| R_0 \geq \lambda \varphi e^{\|u^-\|} R_0,$$

so that $|q| \geq R_0$ and, hence, $D_q \widetilde{H} \cdot q - \widetilde{H} \geqq 0$, and recall that $\varphi'(\varphi)^{-1} \geqq 0$.

Since by assumption

$$z(x_0, t_0) - z(y_0, t_0) \geq C|x_0 - y_0|,$$

there exists, in view of (2.5), a constant $B > 0$ such that

$$0 \leq \left(\int_0^1 (B - \lambda) g(r) \mathrm{d}r + \varphi'(t_0)(\varphi(t_0))^{-1} \right) \left(z(x_0, t_0) - z(y_0, t_0) \right),$$

where

$$g(r) = q(r) \cdot D_q \widetilde{H}(q(r), x(r)) - \widetilde{H}(q(r), x(r)).$$

Choosing $\lambda = B + 1$ and using (2.4) and (4.1), we find

$$0 \leq (z(x_0, t_0) - z(y_0, t_0)) \int_0^1 [\Phi(\varphi^{-1}(t_0)) - \Phi(|q(r)|)] \mathrm{d}r.$$

Recalling that $|q| = Ce^{\lambda u}(\lambda \varphi)^{-1}$ and that Φ is strictly increasing, we obtain, for $C > \lambda e^{\lambda \|u^-\|_\infty}$, the desired contradiction. ☐

We continue with the Lipschitz bound in the second-order case. Here we argue using the classical Bernstein method, which yields a universal Lipschitz bound depending only on the oscillation of the initial datum.

In the subquadratic but superlinear case, we will use this bound iteratively to obtain a bound for the oscillation which is independent of the initial datum (see Lemma 4.7). Of course, for a superquadratic Hamiltonian, the oscillation is easily bounded by Lemma 4.2, so the Lipschitz bound follows directly from Lemma 4.5.

To this end, let $\varphi : [s, T] \to [0, \infty)$ be a solution of the ordinary differential inequality

$$\varphi_t \leq \min(\varphi^{1/2}, 1), \quad \varphi(s) = 0. \tag{4.2}$$

Lemma 4.5. *Let u solve (2.1) on $\mathbb{T} \times [s, T]$ and assume that*

$$\tilde{H}(p, x, t, s, \omega) = H(p + DW(x, t, s, \omega), x) - \operatorname{tr}\left(A(x) D^2 W(x, t, s, \omega)\right)$$

satisfies (2.3), (2.6), and (2.7) on $[s, T]$. There exist $\kappa \in [0, 1)$ and $C_{R_0} > 0$, both independent of the initial datum $u(\cdot, s)$, such that for all $t \in [s, T]$,

$$\|Du(\cdot, t)\| \leq \varphi(t)^{-1/2} C_{R_0} (1 + \operatorname{osc}(u(\cdot, s))^\kappa). \tag{4.3}$$

The fact that $\kappa < 1$ is very critical, since it implies that even if the oscillation is large initially, it will be much smaller at the end of the time interval. It follows from the proof that for δ as in (2.6), $\kappa(\delta) \to 1$ as $\delta \to 0$. Therefore the method does not apply to Hamiltonians with just linear growth.

Further, notice that the constants in (2.3), (2.6), and (2.7) depend on the realization of the noise in a given time interval, but only through (2.11), so they are bounded if the interval is a small noise interval.

Finally, we remark that it is straightforward to check that the particular equation

$$u_t - \epsilon \Delta u + |Du + DW(x,t,s,\omega)|^2 = 0$$

satisfies the conditions of Lemma 4.5.

For the proof of Lemma 4.5 we need a rough a priori bound on the oscillation. To this end, let

$$L(\omega) = \sup_{(x,y,t)\,\in\,\mathbb{R}^n \times \mathbb{R}^n \times [s,T]} |H(DW(x,t,s,\omega),x) - H(DW(y,t,s,\omega),y)|.$$

Note that the dependence on ω is through (2.11).

LEMMA 4.6. *For all $(s,t) \in \Delta$, we have*

$$\operatorname{osc}(u(\cdot,t)) \leq \operatorname{osc}(u(\cdot,s)) + L|t-s|.$$

Proof. The estimate follows directly from the fact that

$$\operatorname{osc}(u(\cdot,t))_t = \left(\sup_{\mathbb{T}} u(\cdot,t) - \inf_{\mathbb{T}} u(\cdot,t)\right)_t \leq L. \qquad \square$$

We continue with the proof of Lemma 4.5, which uses some of the techniques of [CLS].

Proof. 1. To simplify things we assume that $s = 0$. The functions $v(\cdot,t) = u(\cdot,t) + Kt$ and $u(\cdot,t)$ have the same Lipschitz constant and v solves an equation with a nonnegative Hamiltonian. We may therefore assume that the Hamiltonian is nonnegative, i.e., $K = 0$. Moreover, to simplify the presentation, we drop the dependence on ω and write $\tilde{H}(p,x,t)$ instead of $\tilde{H}(p,x,t,0,\omega)$. Finally, we write

$$O_0 = \operatorname{osc}(u(\cdot,0)).$$

2. Let $m(t)$ and $x_m(t)$ denote, respectively, the maximum of the function $u(\cdot,t)$ and the point where the maximum is assumed, i.e., for all $x \in \mathbb{T}$,

$$m(t) = u(x_m(t),t) \geq u(x,t).$$

Then

$$|u(x,t) - m(t)| \leq \operatorname{osc}(u(\cdot,t)) \leq \operatorname{diam}(\mathbb{T})\|Du(\cdot,t)\|.$$

Since $\tilde{H} \geq 0$, we know that

$$m_t(t) = u_t(x_m(t),t) = [\operatorname{tr}(A(x_m(t))D^2u(x_m(t),t)) - \tilde{H}(0,x_m(t),t)] \leq 0.$$

3. For $\lambda > 0$ consider the function

$$z(x,t) = \varphi(t)|Du(x,t)|^2 + \lambda(m(t) - u(x,t)).$$

Let (x_0,t_0) be a point where z achieves its maximum. The goal is to show that there exist $\lambda > 0$ such that either $t_0 = 0$ or $|Du(x_0,t_0)| \leq R_0$.

In order to keep the presentation simple, in what follows we assume that A is the identity matrix. The modifications needed for general A are straightforward, so we omit them.

4. If either $t_0 = 0$ or $|Du(x_0, t_0)| \le R_0$, then

$$z(x,t) \le R_0^2 + \lambda(O_0 + LT).$$

Hence, for all $(x,t) \in \mathbb{T} \times [0,T]$,

$$\varphi(t)|Du(x,t)|^2 \le R_0^2 + \lambda(O_0 + LT) + \lambda(u(x,t) - m(t)) \le R_0^2 + \lambda(O_0 + LT).$$

Assume that

$$O_0 \ge 1 + LT.$$

It then follows that for all $(x,t) \in \mathbb{T} \times [0,T]$,

$$\varphi(t)^{1/2}|Du(x,t)| \le (R_0^2 + \lambda(O_0 + LT))^{1/2} \le (R_0^2 + 2\lambda O_0)^{1/2}.$$

Since R_0 is given, we may assume that $\lambda \ge R_0$. The above estimate then can be simplified to read

$$\|Du(\cdot,t)\| \le C\,\lambda\varphi(t)^{-1/2}(1 + \left(O_0\lambda^{-1}\right)^{1/2}). \tag{4.4}$$

5. Assume that $t_0 > 0$ and $|Du(t_0, x_0)| > R_0$. The classical calculations associated with Bernstein's method then yield the following sequence of inequalities, where C is the constant in (2.7) and where z and $\tilde{H}$ are evaluated at (x_0, t_0) and $(Du(x_0,t_0), x_0, t_0)$:

$$\begin{aligned} 0 \le z_t - \Delta z &= \lambda m_t - \lambda(u_t - \Delta u) \\ &\quad + 2\varphi Du \cdot D(u_t - \Delta u) - 2\varphi|D^2u|^2 + \varphi_t|Du|^2 \\ &\le \lambda\tilde{H} - 2\varphi Du \cdot D\tilde{H} - 2\varphi|D^2u|^2 + \varphi_t|Du|^2 \\ &\le \lambda\tilde{H} - \lambda Du \cdot D_p\tilde{H} - 2\varphi Du \cdot D_x\tilde{H} + \varphi_t|Du|^2 \\ &\le -(\lambda - C)\Phi(|Du|) + \varphi_t|Du|^2. \end{aligned}$$

If $3\lambda \ge 4C$, then

$$0 \le -\lambda\Phi(|\operatorname{grad} u|) + 4\varphi_t|\operatorname{grad} u|^2.$$

Dividing by $|Du|^{1+\delta}$, we obtain, always at (x_0, t_0),

$$0 \le -\lambda G(|Du|) + 4\varphi_t|Du|^{1-\delta}.$$

Consider the set

$$D_{R_0} = \{(x,t) \in \mathbb{T} \times [0,T] \ : \ |\operatorname{grad} u(x,t)| \ge R_0\}$$

and let

$$\lambda_0 = \sup_{(x,t)\in D_{R_0}} 4\varphi_t(t)G(|Du(x,t)|)^{-1}|Du(x,t)|^{1-\delta}. \tag{4.5}$$

If we choose $\lambda > \lambda_0$, then it is impossible for the Bernstein function z to have an interior maximum, unless at the maximum we have $|Du| \le R_0$, in which case (4.4) holds. It remains to show that λ_0 depends only on the data.

6. Let $(\bar{x},\bar{t})$ be such that

$$\lambda_0 = 4\phi_t(\bar{t})(G(|Du(\bar{x},\bar{t})|))^{-1}|Du(\bar{x},\bar{t})|^{1-\delta}.$$

If such $(\bar{x},\bar{t})$ does not exist, we argue using approximate maximizers—we leave the details to the reader. Moreover, since $\phi_t(0)=0$, if $\lambda_0>0$, then $\bar{t}>0$.

Choose $\lambda\in(\lambda_0,2\lambda_0)$. Using (4.4) and (4.5), we find, for some universal constant $C>0$, which is independent of λ and the initial datum, that

$$|Du(\bar{x},\bar{t})| \le C\,\varphi_t(\bar{t})(G(|Du(\bar{x},\bar{t})|)\varphi(\bar{t}))^{-1/2}|Du(\bar{x},\bar{t})|^{1-\delta}(1+(O_0\lambda^{-1})^{1/2}).$$

Note that since $G(|Du(\bar{x},\bar{t})|)\ge G(R_0)$ and $\varphi_t\le\varphi^{1/2}$,

$$|Du(\bar{x},\bar{t})|^{\delta} \le C(1+(O_0\lambda^{-1})^{1/2}).$$

Inserting the above in (4.5) and using (4.2) yield, for a different universal constant C,

$$\lambda_0 \le C(1+(O_0\lambda^{-1})^{1/2})^{(1-\delta)/\delta} \le 2^{(1-\delta)/\delta}C(1+(O_0{\lambda_0}^{-1})^{(1-\delta)/2\delta}).$$

7. We may assume that

$$2^{(1-\delta)/\delta+1}C\le\lambda_0,$$

and hence

$$\lambda_0\le C(O_0{\lambda_0}^{-1})^{(1-\delta)/2\delta},$$

which implies

$$\lambda_0\le CO_0^{(1-\delta)(1+\delta)^{-1}}.$$

It follows that there exists $\rho\in(0,1)$, independent of the initial condition, such that

$$\lambda_0\le CO_0^{1-\rho}.$$

Inserting a λ with $\lambda\in(\lambda_0,2\lambda_0)$ in (4.4) yields (4.3). $\square$

We conclude with a lemma which provides a universal bound on the oscillation via a bootstrap procedure.

LEMMA 4.7. *Assume the hypotheses of Lemma* 4.5. *There exists a universal constant C, which is independent of the initial datum, such that, after time $T=1$, the oscillation of u is bounded by C.*

Proof. 1. Since we may assume that $\varphi(t)\ge t^{\beta}$ for some $\beta>0$, we find that if $\mathrm{osc}(u(\cdot,0))$ is sufficiently large, then Lemma 4.5 asserts the existence of $\hat{\kappa}\in(0,1)$ and $C>1$ such that, after a time interval of length τ,

$$\mathrm{osc}(u(\cdot,t+\tau))\le C\tau^{-\beta}(\mathrm{osc}(u(\cdot,t)))^{\hat{\kappa}}.$$

If the oscillation at some time is already bounded by a power of the universal constant C, there is nothing to prove. Therefore we assume that

$$\text{if } \hat{C}=C^{2(1-\hat{\kappa})^{-1}}, \text{ then } \mathrm{osc}(u)\ge\widehat{C}.$$

If $2\kappa = (1+\hat{\kappa}) < 2$, we obtain the simpler recursion

$$\mathrm{osc}(u(\cdot, t+\tau)) \le \tau^{-\beta}(\mathrm{osc}(u(\cdot,t)))^{\kappa}.$$

2. Choose a sufficiently small $\beta_1 > 0$, let $\bar{\kappa} = \beta\beta_1 + \kappa < 1$, and consider the recursively defined sequences

$$O_l = O_{l-1}^{\bar{\kappa}} \quad \text{and} \quad \tau_l = O_{l-1}^{-\beta_1}.$$

If the numbers O_l are given by $O_l = O_0^{(\bar{\kappa})^l}$, it follows that

$$\mathrm{osc}\left(u\left(\cdot, \sum_{i=0}^{l} \tau_i\right)\right) \le \max(\widehat{C}, O_l).$$

3. Let l_M be the smallest integer such that $O_{l_M} \le 2\widehat{C}$. Then

$$O_{l_M-1} = O_0^{\bar{\kappa}^{(l_M-1)}} \ge 2\widehat{C} \quad \text{and} \quad O_{l_M} \ge (2\widehat{C})^{\bar{\kappa}}.$$

Recall that O_0 and l_M are sufficiently large, β_1 is sufficiently small, $0 \le l \le l_M$, and define

$$s_l = B^{(\bar{\kappa})^{-l}} \quad \text{and} \quad B = (O_{l_M})^{-\beta_1}.$$

We have

$$\sum_{l=0}^{l_M} \tau_l = \sum_{l=0}^{l_M} (O_0^{-\beta_1})^{(\bar{\kappa})^{l-l_M+l_M}} = \sum_{l=0}^{l_M} ((O_0^{-\beta_1})^{(\bar{\kappa}^{l_M})})^{(\bar{\kappa})^{l-l_M}} = \sum_{l=0}^{l_M} s_l,$$

and, since $\bar{\kappa} < 1$,

$$(\bar{\kappa})^{-l}((\bar{\kappa})^{-1} - 1) \ge r(\bar{\kappa}) = (\bar{\kappa})^{-1}((\bar{\kappa})^{-1} - 1) > 0.$$

Moreover

$$B = (O_{l_M})^{-\beta_1} \le (2\widehat{C})^{-\bar{\kappa}\beta_1} < 1.$$

Therefore

$$s_{l+1} {s_l}^{-1} = B^{(\bar{\kappa})^{-(l+1)} - (\bar{\kappa})^{-l}} \le B^{r(\bar{\kappa})} < 1.$$

Thus the series $\sum \tau_l$ converges by comparison with the geometric series. Note that the powers β_1, κ are independent of the length of the a priori chosen time interval. ∎

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