European Journal of Applied Mathematics

http://journals.cambridge.org/EJM

Additional services for ***European Journal of Applied Mathematics:***

Email alerts: Click here
Subscriptions: Click here
Commercial reprints: Click here
Terms of use : Click here

Pulsating wave for mean curvature flow in inhomogeneous medium

N. DIRR, G. KARALI and N. K. YIP

European Journal of Applied Mathematics / Volume 19 / Issue 06 / December 2008, pp 661 - 699
DOI: 10.1017/S095679250800764X, Published online: 07 August 2008

Link to this article: http://journals.cambridge.org/abstract_S095679250800764X

How to cite this article:
N. DIRR, G. KARALI and N. K. YIP (2008). Pulsating wave for mean curvature flow in inhomogeneous medium. European Journal of Applied Mathematics, 19, pp 661-699 doi:10.1017/S095679250800764X

Request Permissions : Click here

Downloaded from http://journals.cambridge.org/EJM, IP address: 131.251.254.13 on 25 Feb 2014

Euro. Jnl of Applied Mathematics (2008), *vol.* 19, *pp.* 661–699. © 2008 Cambridge University Press
doi:10.1017/S095679250800764X Printed in the United Kingdom

Pulsating wave for mean curvature flow in inhomogeneous medium

N. DIRR[1]†, G. KARALI[2,3] and N. K. YIP[4]

[1]*Max-Planck Institute for Mathematics in the Sciences, D-04103 Leipzig, Germany*
email: ndirr@mis.mpg.de
[2]*Department of Applied Mathematics, University of Crete, GR 71409, Heraklion, Greece*
[3]*Institute for Applied and Computational Mathematics, FORTH, Crete, Greece*
email: gkarali@tem.uoc.gr
[4]*Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA*
email: yip@math.purdue.edu

(*Received 28 August 2007; revised 11 June 2008; first published online 7 August 2008*)

We prove the existence and uniqueness of pulsating waves for the motion by mean curvature of an n-dimensional hypersurface in an inhomogeneous medium, represented by a periodic forcing. The main difficulty is caused by the degeneracy of the equation and the fact the forcing is allowed to change sign. Under the assumption of weak inhomogeneity, we obtain uniform oscillation and gradient bounds so that the evolving surface can be written as a graph over a reference hyperplane. The existence of an effective speed of propagation is established for any normal direction. We further prove the Lipschitz continuity of the speed with respect to the normal and various stability properties of the pulsating wave. The results are related to the homogenisation of mean curvature flow with forcing.

1 Introduction

In this paper, we study the mean curvature flow of a hypersurface in a periodic inhomogeneous medium. More precisely, we consider the evolution of an n-dimensional surface $\{\Gamma(t) \subseteq \mathbb{R}^{n+1} : t \geqslant 0\}$ with its motion law given by

$$V_N(p) = H(p) + \delta f(p), \quad p \in \Gamma(t), \tag{1.1}$$

where V_N and H are the normal velocity and mean curvature of $\Gamma(t)$, and δ is a positive number which measures the strength of the spatial inhomogeneity, represented by f. Without loss of generality, we assume $0 < \delta < 1$. The function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfies the following conditions:

$$\mathbf{A}: \begin{cases} \text{(i) } f \text{ is } \mathbb{Z}^{n+1}\text{-periodic, i.e. } f(p+\omega) = f(p) \text{ for all } p \in \mathbb{R}^{n+1} \text{ and } \omega \in \mathbb{Z}^{n+1}. \\ \text{(ii) } f(\cdot) \text{ is twice continuously differentiable and } \|f\|_{C^2(\mathbb{R}^{n+1})} = F < \infty. \end{cases}$$

We emphasize that f is *not* restricted to be either positive or negative.

† Current address: Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY, UK. Email: N.Dirr@maths.bath.ac.uk

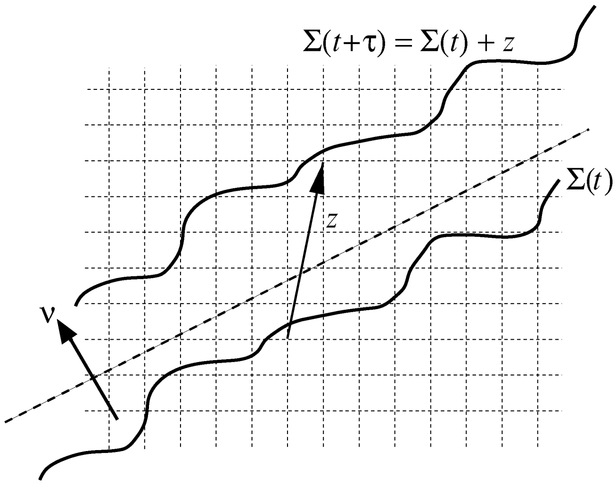

FIGURE 1. The pulsating wave property: time shift corresponds to lattice translation.

The main contribution of the present paper is that under the above rather weak assumption for the forcing, together with δ small enough, we are able to show for any direction v the existence of a unique speed c_v and a number $D < \infty$ such that the solution of (1.1) starting from a plane with normal v stays as a graph over the same plane for all times, and moreover, this graph lies within a distance D from a plane which has normal v and moves with normal velocity c_v. This result is motivated by and extends the geometric arguments of [4] which essentially considers a stationary version of (1.1). Using the language of homogenisation, we have in fact shown the existence of a *homogenised front* – a hyperplane with normal v – which moves with an *effective speed* c_v.

Furthermore, if $c_v \neq 0$, we show that *pulsating waves* exist. A pulsating wave is a special solution defined globally in space and time with the property that a spatial translation that keeps the periodic environment invariant (lattice translation) corresponds to a translation in time. More precisely, $\{\Sigma_v(t) \subseteq \mathbb{R}^{n+1} : t \in \mathbb{R}\}$ is a pulsating hypersurface evolving by (1.1) with normal direction v and velocity $c_v \neq 0$, if it satisfies the following property (see Figure 1):

$$\Sigma(t+\tau) = \Sigma(t) + z, \quad \text{for all } z \in \mathbb{Z}^{n+1} \text{ and } \tau = \frac{v \cdot z}{c_v}. \tag{1.2}$$

The interest in (1.1) stems from models for the motions of material interfaces (such as phase boundaries) in the over-damped limit, i.e. when inertial effects are neglected. Then the time evolution is often the *negative gradient flow* of some underlying *energy functional*. Such models should incorporate heterogeneities, which may arise from the periodic structure of the material or substrate or impurities present in the material on a very fine scale. These heterogeneities create a very *oscillatory* energy landscape and make the analysis of the dynamics very challenging. In particular, the large-scale limit of the energy, obtained for example by means of Γ-convergence [10] and the large-scale limit of the gradient flow dynamics may not commute, i.e. the gradient flow of the limiting energy is *not* the scaling limit of the gradient flows. This is mainly due to the fact that the dynamical state of the gradient flows often gets stuck in the *local minima* created by the heterogeneities. The ultimate limiting behaviour is the result of some non-trivial averaging process between *energetic* and *kinematic* effects. See [11] for some results along these lines. (The work [22] proves some Γ-convergence result in the time-dependent case but the effect of oscillatory energy landscape is not considered.)

The motion law (1.1) is motivated by the evolution of phase boundaries [1] or defects such as dislocation lines in a solid [3, 9, 21]. The 'non-oscillatory' part of the energy for the gradient flow model is chosen to be the interfacial energy (area of a hypersurface). This model thus captures the competition between the tendency to decrease the interfacial energy – flatten the interface – while at the same time adapting to inhomogeneities on a very small spatial scale. The mathematical analysis of this simple 'physical' model is already challenging as the interaction between the non-linearities and heterogeneities can be quite intricate.

One question of interest is the *effective* front and velocity of $\Gamma(t)$ on a large space–time scale. This can be phrased as follows: Given any direction $v \in \mathbb{S}^n$, is there a number c_v such that the solution of (1.1) starting from a plane with normal v stays within *bounded* distance from a plane that has the same normal and moves with normal velocity c_v?

In the framework of homogenisation, the above question can be formulated in the following form. Introduce a small parameter ϵ and rescale (1.1) as

$$V_N^\epsilon = \epsilon H(p) + \delta f(p/\epsilon), \quad p \in \Gamma^\epsilon(t). \tag{1.3}$$

Then questions about the effective behaviour are equivalent to investigating the limits of the solutions $\Gamma^\epsilon(t)$ of (1.3) as $\epsilon \longrightarrow 0$. Note that the highest order (curvature) term is multiplied by the small parameter ϵ which makes the corresponding homogenisation problem *singular*. In such a scaling, the curvature and heterogeneity are coupled together in an elaborate way and hence can lead to interesting phenomena.

The above question, though simply stated, is highly non-trivial. Besides the facts that the motion law (1.1) is extremely non-linear and the equation written in appropriate coordinates is degenerate parabolic, the main technical difficulty in its analysis lies in the fact that the forcing f is allowed to change sign. For a forcing which is *positive* and satisfies some additional technical conditions, the problem on the existence of effective speed is solved in [19] using the machinery of viscosity solutions. This is briefly explained here. Let $U^\epsilon : \mathbb{R}^{n+1} \times \mathbb{R}^+ \to \mathbb{R}$ be a function with the property that each of its level set $\Sigma_\lambda^\epsilon(t) = \{x \in \mathbb{R}^{n+1} : U^\epsilon(x,t) = \lambda\}$ evolves by (1.3), then U^ϵ solves the following non-linear degenerate parabolic equation:

$$U_t^\epsilon = \epsilon \mathrm{tr}[(I - |\nabla U^\epsilon|^{-2}(\nabla U^\epsilon \otimes \nabla U^\epsilon))D^2 U^\epsilon] + \delta f(X/\epsilon)|\nabla U^\epsilon|. \tag{1.4}$$

It is conjectured (and proved in [19] for certain f which remains strictly positive) that the solutions U^ϵ converge to a solution $\overline{U}$ of a homogenised problem which in the level set formulation becomes the following *first*-order equation:

$$\overline{U}_t = c(\nabla\overline{U}/|\nabla\overline{U}|)|\nabla\overline{U}|, \tag{1.5}$$

where $c(\cdot)$ is the speed of the front which depends on its normal direction, given by

$\nabla\overline{U}/|\nabla\overline{U}|$ in the level-set formulation. (See also [2, 3] for results related to the above homogenisation problem. The work [20] studies a semi-linear version of (1.1), but still with positive forcing.)

Another interesting behaviour concerning (1.1) is the *pinning/de-pinning* phenomenon. To explain this, introduce an additional parameter h into (1.1)

$$V_N = H(p) + \delta(f(p) + h) \tag{1.6}$$

which models the presence of some external field imposed on the dynamics. The relevant questions in terms of application and modelling include the *de-pinning threshold* h_c defined as the *smallest* force h required to obtain a *non-zero* velocity c_v, and also the relationship between the effective velocity and the *excess forcing* $h - h_c$. This question is not addressed in the present paper but is studied in detail in [11] for semi-linear PDEs which are approximations of (1.1) when the evolving hypersurface is close to a very 'flat' graph. We expect that for planes with rational normal direction and small δ, the method of [11] can be extended to (1.1), but the estimates will in general not be uniform in the direction. We remark that, unlike the effective velocity c_v, the de-pinning threshold h_c is in general *not* continuous in the direction v (see Section 5 for a simple example and also [5] for some results on a related discrete system).

We now introduce the setting of the present paper. The investigation of effective behaviour is very much tied to the consideration of *plane-like* solutions of (1.1), i.e. there exists a fixed unit vector $v \in \mathbb{S}^n$ such that for all $t \geqslant 0$, the solution $\Gamma(t)$ satisfies

$$D(t) := \sup_{p,q \in \Gamma(t)} (p - q) \cdot v < \infty. \tag{1.7}$$

Furthermore, the existence of effective property relies intimately to the fact that $D(t)$ is *uniformly bounded* in time.

In order to incorporate general v, we introduce two coordinate systems for $\mathbb{R}^{n+1}$. First, we write $\mathbb{R}^{n+1}$ as

$$\mathbb{R}^{n+1} = \left\{ \begin{pmatrix} X \\ X_{n+1} \end{pmatrix} : X \in \mathbb{R}^n,\ X_{n+1} \in \mathbb{R} \right\}.$$

Let $\mathcal{O}_v$ be a positively oriented orthogonal transformation of $\mathbb{R}^{n+1}$ such that $v = \mathcal{O}_v((0, \dots, 0, 1)^T)$. Introduce the new coordinate system: (x, x_{n+1}), $x \in \mathbb{R}^n$ and $x_{n+1} \in \mathbb{R}$ such that

$$\begin{pmatrix} x \\ x_{n+1} \end{pmatrix} = \mathcal{O}_v^T \begin{pmatrix} X \\ X_{n+1} \end{pmatrix}.$$

Observe that the (x, x_{n+1})-coordinate of v is $(0, \dots, 0, 1)$. We call the (X, X_{n+1})- and (x, x_{n+1})-coordinate systems the reference and tilted frames, respectively (see Figure 2).

If $\Gamma(t)$ can be written as a graph over the plane $x_{n+1} = 0$, i.e.

$$\Gamma(t) = \{(x, u(x, t)) : x \in \mathbb{R}^n,\ u \in \mathbb{R}\},$$

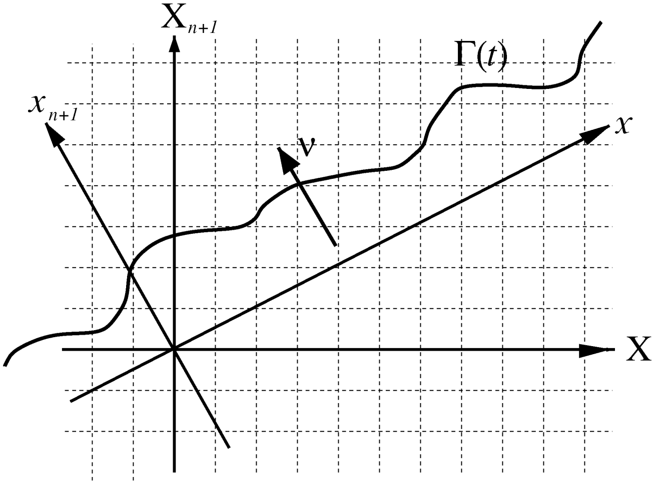

FIGURE 2. The original and the tilted frames. The lattice stands for the period of the forcing.

then u solves the following quasi-linear parabolic differential equation:

$$u_t = A_f(v, x, u) = \sqrt{1 + |\nabla u|^2}\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) + \delta\sqrt{1 + |\nabla u|^2} f(\mathcal{O}_v(x, u)^T). \tag{1.8}$$

The symbol ∇ refers to the gradient operator with respect to the x-variables. Note that (1.8) is *invariant* under the *lattice translation* in the following sense:

$$A_f(v, x + x', u + u') = A_f(v, x, u) \quad \text{for all} \quad \begin{pmatrix} x' \\ u' \end{pmatrix} \in \mathcal{O}_v^T \mathbb{Z}^{n+1}. \tag{1.9}$$

Equation (1.8) plays a fundamental role in this paper. The notation v will sometimes be suppressed, unless needed in the presentation. The main difficulty in the study of (1.8) is that it is not uniformly parabolic and becomes degenerate as the gradient blows up. If the forcing is large, this can indeed happen in finite time even if the initial data is smooth. Furthermore, the graph representation might not be preserved in time (Section 5). However, by a combination of the *periodicity* of the domain and the *smallness* of the inhomogeneity of the medium, we are able to derive several useful uniform estimates for the solution of (1.8) which allow us to employ many techniques for parabolic PDEs to the study of (1.1).

The restriction to small forcing is not just for convenience (so that we only need to deal with classical solutions). In fact, if the forcing is large, it can lead to a quite different phenomenon. First, 'pinch-off' – a portion of the graph becomes detached from the overall surface – can happen. Even though this can still be potentially handled by the level-set formulation [7, 14], it involves a different type of technicality. Second, on a more fundamental level of difficulty, there might not even be an effective front or effective behaviour due to the possibility of fingering. How to define a modified notion of homogenised object and equation is not completely clear. Section 5 gives some explicit examples of these phenomena.

We expect that our results for graph-like pulsating waves can be extended by fairly standard arguments to yield a homogenisation result for the level-set equation (1.4). Indeed, using the fact that our graph-like pulsating wave solution $u(x,t)$ satisfies $u_t > 0$ (Proposition 4.4), one can construct a special solution for (1.4) by setting

$$\{U(x, x_{n+1}, t) = \lambda\} = \{u(x, t+\lambda) = x_{n+1}\}.$$

In a rotated and moving frame

$$\widetilde{U}(x, x_{n+1}, t) = U(x, x_{n+1}, t) - x_{n+1} - c_v t$$

is a *globally bounded* solution of

$$\widetilde{U}_t^\epsilon = \mathrm{tr}[(I - |\nabla\widetilde{U} + P|^{-2}((\nabla\widetilde{U} + P) \otimes (\nabla\widetilde{U} + P)))D^2\widetilde{U}] + \delta f|\nabla\widetilde{U} + P| - c_P,$$

where without loss of generality, $P = v = (0, \ldots, 0, 1)$. This clearly implies a homogenisation result for plane-like initial data. Note that the above equation is a special case of the equation for the so-called 'corrector'. As the limit effective velocity is continuous in the normal (Proposition 3.3), we expect the extension to more general initial data to be straightforward, but in order to keep the present paper focused and of reasonable length, we will not address these issues here.

1.1 Outline of paper

Section 2 proves the key estimates for (1.8) – uniform oscillation and gradient bounds (Theorem 2.4, Corollary 2.5) – to be used for the rest of the paper. The existence of classical solution with Lipschitz initial data (Theorem 2.7) and a gradient decay estimate (Theorem 2.8) are also presented. Section 3 establishes the existence, uniqueness and Lipschitz continuity of the effective speed of propagation for any normal direction v. Section 4 proves the existence, uniqueness and various stability properties of the pulsating wave solutions. Section 5 provides some examples for the formation of singularities if the forcing is large. The Appendix contains the proof of Theorems 2.7 and 2.8 which are somewhat long and technical.

2 Estimates for mean curvature flow in inhomogeneous medium

The following simple geometric lemma is the starting point for the uniform estimates derived later. It essentially shows that starting from a hyperplane, at any fixed time t, if a cube Q is 'above(below)' the interface $\Gamma(t)$, so is any 'tangential' translates $Q + w$. This result is motivated by the work [4].

Lemma 2.1 *Let $\{\Gamma(t) : t \geqslant 0\}$ be a connected hypersurface in $\mathbb{R}^{n+1}$ which is the unique classical solution of (1.1) with initial datum the hyperplane $\Gamma(0) = \{(X, X_{n+1}) : (X, X_{n+1})^T \cdot v = 0\}$, i.e. $x_{n+1} = 0$. Let further $\Sigma^{\pm}(t) \subseteq \mathbb{R}^{n+1}$ be connected open sets such that for all t,*

$\Gamma(t) = \partial\Sigma^+(t) = \partial\Sigma^-(t)$, $\mathbb{R}^{n+1} = \Gamma(t) \cup \Sigma^+(t) \cup \Sigma^-(t)$, and the vector v point into $\Sigma^+(t)$. Let $z \in \mathbb{Z}^{n+1}$ and $Q(z) = Int([0,1]^{n+1} + z)$. Then following statements hold.

If $Q(z) \subseteq \Sigma^+(t)$, then $Q(z+w) \subseteq \Sigma^+(t)$ for all $w \in \mathbb{Z}^{n+1}$ with $w \cdot v \geqslant 0$. Similarly, if $Q(z) \in \Sigma^-(t)$, then $Q(z+w) \subseteq \Sigma^-(t)$ for all $w \in \mathbb{Z}^{n+1}$ with $w \cdot v \leqslant 0$.

Proof Without loss of generality, we will just prove the first statement. Let $\widehat{\Gamma}(t)$ be the solution of (1.1) with initial datum $\Gamma(0) + w$ and $\widehat{\Sigma}^{\pm}(t)$ be the two open sets similarly defined as $\Sigma^{\pm}(t)$ for $\Gamma(t)$. By the periodicity of the inhomogeneity and the assumed uniqueness of classical solution of (1.1) starting from $\Gamma(0)$, we have that $\widehat{\Gamma}(t) = \Gamma(t) + w$ and $\widehat{\Sigma}^+(t) = \Sigma^+(t) + w$. Moreover as $\widehat{\Gamma}(0) \subseteq \Sigma^+(0)$, the comparison principle implies that $\widehat{\Sigma}^+(t) \subseteq \Sigma^+(t)$. Since $Q(z) \subseteq \Sigma^+(t)$, we have

$$Q(z) + w \subseteq \Sigma^+(t) + w = \widehat{\Sigma}^+(t) \subseteq \Sigma^+(t),$$

which proves the claim. □

Remark 2.2 Note that in the above and the rest of the paper, we deal only with classical solutions of equation (1.1), by which we mean smooth hypersurface evolving according to (1.1). Due to the degeneracy of the equation, even in the homogeneous case ($f \equiv 0$), the question of well posedness is already not trivial (see [12, 13]). With forcing ($f \not\equiv 0$), in general the gradient can blow up in finite time. On the other hand, if the forcing is small ($\delta \ll 1$) and the initial data has bounded gradient and oscillation, well posedness can be established. This and related comments are stated in Remark 2.6 (1,2), Theorems 2.7 and 2.8.

The following notation is introduced for convenience:

$$\operatorname{osc}(\Gamma, \mathbf{B}, v) := \sup_{p,q\in\Gamma\cap\mathbf{B}} (p-q)\cdot v \ \ (\text{for } \mathbf{B} \subseteq \mathbb{R}^{n+1}) \quad \text{and} \quad \operatorname{osc}(\Gamma, v) := \operatorname{osc}(\Gamma, \mathbb{R}^{n+1}, v).$$

If $\{(x, u(x)) : x \in \mathbb{R}^n\}$ is the graph representation of Γ over $\mathbb{R}^n$,

$$\operatorname{osc}(u, B) := \sup_{x,y\in\mathbb{R}^n\cap B} u(x) - u(y) \ \ (\text{for } B \subseteq \mathbb{R}^n) \quad \text{and} \quad \operatorname{osc}(u) := \operatorname{osc}(u, \mathbb{R}^n).$$

The previous lemma immediately leads to the following result.

Lemma 2.3 *Let $\{\Gamma(t)\}_{t\geqslant 0}$ be as in Lemma 2.1, in particular $\Gamma(0) = \{(X, X_{n+1}) : (X, X_{n+1})^T \cdot v = 0\}$, i.e. $x_{n+1} = 0$. Let $\mathbf{B} = \{(X, X_{n+1}) \in \mathbb{R}^{n+1} : |X| \leqslant 2\sqrt{n+1}\}$. Then for all $t \geqslant 0$,*

$$\operatorname{osc}(\Gamma(t), \mathcal{O}_v(\mathbf{B}), v) \leqslant \operatorname{osc}(\Gamma(t), v) \leqslant \operatorname{osc}(\Gamma(t), \mathcal{O}_v(\mathbf{B}), v) + 4\sqrt{n+1}. \tag{2.1}$$

In the graph setting, $\Gamma(t) = \{(x, u(x,t)) : x \in \mathbb{R}^n\}$, upon introducing $B = \{x : |x| \leqslant 2\sqrt{n+1}\}$, then it holds similarly that

$$\operatorname{osc}(u(\cdot,t), B) \leqslant \operatorname{osc}(u(\cdot,t)) \leqslant \operatorname{osc}(u(\cdot,t), B) + 4\sqrt{n+1}. \tag{2.2}$$

(*The quantity $\sqrt{n+1}$ comes from the diameter of the unit cube in $\mathbb{R}^{n+1}$.*)

It is crucial for our analysis that $\mathrm{osc}(\Gamma(t), v)$ remains *uniformly bounded for all time*. For the existence and uniqueness of the speed as stated in Theorem 3.1, we could simply make this as a standing assumption, or we can work in the graph setting in which such an assumption can be justified. The next several results show that this assumption is indeed valid provided the forcing is *small compared with the period*. For the clarity of presentation, the proofs are postponed till the results are listed.

In the following, the symbol $C(F)$ denotes some universal constant which depends on the quantity $F = \|f\|_{C^2(\mathbb{R}^{n+1})}$. The constant convention is used: Different constants are denoted by the same symbol $C(F)$, provided they depend only on $\|f\|_{C^2(\mathbb{R}^{n+1})}$. In addition, if $u(x, t)$ is a solution of (1.8), we denote

$$z(x,t) := \sqrt{1 + |\nabla u(x,t)|^2} \quad \text{and} \quad \|z(t)\|_\infty := \sup_{x \in \mathbb{R}^n} z(x,t).$$

Theorem 2.4 (*Bernstein's Method*) *Let* $\{u(x,t) : x \in \mathbb{R}^n, 0 \leqslant t \leqslant T\}$ *be a classical solution of* (1.8) *with uniformly Lipschitz and bounded initial datum* $u_0(x)$*. Further, let* K *be a constant such that* $K > \|z(0)\|_\infty$*. Then*

$$\sup_{t \in [0, T_K]} \|z(t)\|_\infty \leqslant \|z(0)\|_\infty + \lambda(\delta, K, F) \sup_{t \in [0, T_K]} \mathrm{osc}(u(t)), \tag{2.3}$$

where $T_K := T \wedge \inf\{t \geqslant 0 : \|z(t)\|_\infty > K\}$ *and* $\lambda(\delta, K, F) := C(F)\sqrt{\delta}K^2$.

Corollary 2.5 (*Uniform oscillation and gradient bounds*) *Let* $\{u(x,t) : x \in \mathbb{R}^n, 0 \leqslant t \leqslant T\}$ *be as in Theorem 2.4. There is a* $\delta_0(F) > 0$ *such that if* $u_0(x) \equiv 0$*, then for all* $0 \leqslant \delta \leqslant \delta_0$*, the following two estimates hold:*

$$\sup_{t \in [0,T]} \|z(t)\|_\infty \leqslant 1 + C(F)\delta^{\frac{1}{2}} \quad \Big(\textit{or written differently} \ \sup_{t \in [0,T]} \|\nabla u(t)\|_\infty \leqslant C(F)\delta^{\frac{1}{4}}\Big), \tag{2.4}$$

$$\sup_{t \in [0,T]} \mathrm{osc}(u(t)) \leqslant D_0 := C(F)\big(1 + \delta^{\frac{1}{2}}\big). \tag{2.5}$$

For general initial datum $u_0(x)$*, set* $M_0 := \mathrm{osc}(u_0)$*. Then*

$$\sup_{t \in [0,T]} \mathrm{osc}(u(t)) \leqslant D_1(M_0) := D_0 + [M_0]\sqrt{n+1} \tag{2.6}$$

(*where* $[r]$ *denotes the smallest integer bigger or equal to* r). *Furthermore, for all* $K > \|z(0)\|_\infty$ *and* $0 \leqslant \delta \leqslant \delta_1 := C(F)\big[\frac{K - \|z(0)\|_\infty}{K^2(D_0 + [M_0]\sqrt{n+1})}\big]^2$*, then*

$$\sup_{t \in [0,T]} \|z(t)\|_\infty \leqslant \|z(0)\|_\infty + \lambda(\delta, K, F) D_1(M_0). \tag{2.7}$$

Remark 2.6

(1) The above two results show that the solution has uniform gradient bound in space and time as long as δ is small enough. They make equation (1.8) uniformly parabolic and thus allow us to use standard techniques for quasi-linear equations. In addition, note that all the estimates are independent of T. Hence by continuation in the time

variable, we can in fact show that classical solution exists globally in time. This will be stated more precisely in Theorem 2.7.

(2) In contrast to the case of pure mean curvature flow – $f \equiv 0$, due to the degeneracy of the parabolic operator, estimates for solutions of (1.8) of the form $\|z(t)\|_\infty \leqslant \|z(0)\|_\infty$ [13, Corollary 3.1] and $\|z(t)\|_\infty \leqslant G\big(\mathrm{osc}(u(0)), t\big)$ for some function G (see for example [15, Theorem 5.2]) cannot be true. Examples can easily be constructed such that an initial graph will not stay as a graph – the gradient can blow up in *finite time* (see Section 5).

On the other hand, our results show that a global-in-time estimate for the gradient is possible through a combination of small forcing and uniform oscillation bound. In the present paper, the latter is obtained by means of Lemma 2.1.

(3) The dependence of the choice of δ on the size of the period – here assumed to be 1 – of the spatial inhomogeneity can be seen by scaling. Suppose the f in (1.8) is P-periodic in the x- and u-variables. Consider the scaling

$$x = P\tilde{x}, \quad u = P\tilde{u}, \quad t = P^2\tilde{t}.$$

Then equation (1.8) written in the $\tilde{x}, \tilde{u}$ and $\tilde{t}$ variables becomes

$$\tilde{u}_{\tilde{t}} = \sqrt{1+|\tilde{\nabla}\tilde{u}|^2}\,\widetilde{\mathrm{div}}\left(\frac{\tilde{\nabla}\tilde{u}}{\sqrt{1+|\tilde{\nabla}\tilde{u}|^2}}\right) + \delta P\sqrt{1+|\tilde{\nabla}\tilde{u}|^2} f(\mathcal{O}_\nu(P\tilde{x}, P\tilde{u})^T).$$

We need $\tilde{\delta} = \delta P$ to be small. More precisely,

$$\delta P \ll \Delta(\|f(P\cdot, P\cdot)\|_{C^2}) \quad \text{i.e.} \quad \delta \ll \frac{1}{P}\Delta(\|f\|_\infty + P\,\|D_{x,u}f\|_\infty + P^2\|D^2_{x,u}f\|_\infty),$$

where $\Delta(\cdot)$ is some monotonically decreasing function. Qualitatively, small period allows larger δ while large period requires small δ. The results in this paper requires the C^2-norm of f which demands a more stringent condition on the choice of δ. It would be interesting to see if only the dependence on $\|f\|_\infty$ is needed.

Theorem 2.7 (*Existence of classical solution of (1.8)*) *Let $u_0(x)$ be the initial data of (1.8). If $\|\nabla u_0\|_\infty = N_0 < \infty$, then there is a $T = T(\delta, F, N_0) > 0$ such that (1.8) has a unique classical solution for $t \in (0, T)$. Moreover, it holds that*

$$\|D^2u(\cdot,t)\|_{\mathbb{L}^\infty(\mathbb{R}^n)} \leqslant C(N_0, F, T)\frac{1}{\sqrt{t}}. \tag{2.8}$$

If in addition, $\|u_0\|_\infty = M_0 < \infty$, then for all δ smaller than some constant $\delta_2(F, M_0, N_0)$, there exists a unique classical solution of (1.8) for all time. In this case, the following estimate holds:

$$\|D^2u(\cdot,t)\|_{\mathbb{L}^\infty(\mathbb{R}^n)} \leqslant C_1(N_0, F)\frac{1}{\sqrt{t}} + C_2(N_0, F). \tag{2.9}$$

The following statement, though strictly speaking not needed, is interesting in its own right. It indicates the parabolic regularization property of (1.8) and might be useful for other purposes.

Theorem 2.8 (*Gradient decay estimate*) *Let* $\{u(x,t) : x \in \mathbb{R}^n, 0 \leqslant t \leqslant T\}$ *be as in Theorem 2.4. Suppose* $\|z(0)\|_\infty = N_0 < \infty$ *and* $\|u\|_{\mathbb{L}^\infty(\mathbb{R}^n \times [0,T])} \leqslant M < \infty$. *Then there exist constants* $0 < \delta_3(T, N_0, M, F)$ *and* $0 < N_1(\delta, T, M, F) < N_2(\delta, T, M, F)$ *such that for all* $0 < \delta < \delta_3$,

$$\text{if } N_1 \leqslant \|z(0)\|_\infty \leqslant N_2, \text{ then } \|z(T)\|_\infty \leqslant \frac{1}{2}\|z(0)\|_\infty.$$

Furthermore, N_1 *and* N_2 *satisfy* $\lim_{\delta\to 0} N_1(\delta, T, M, F) = N_1^* < \infty$ *and* $\lim_{\delta\to 0} N_2(\delta, T, M, F) = \infty$.

As mentioned earlier, the gradient can blow up in finite time. Hence an upper bound for $\|z(0)\|_\infty$ is necessary for such kind of statement.

We now proceed to prove Theorem 2.4 and Corollary 2.5 which are the core estimates needed for the rest of the paper. The proofs of Theorems 2.7 and 2.8 will be presented in the Appendix.

Proof of Theorem 2.4 Let $\lambda > 0$ be some positive number (to be determined). We define the following function:

$$\Phi(x,t) := z(x,t) + \lambda\left(u^*(t) - u(x,t)\right), \quad u^*(t) := \sup_{x\in\mathbb{R}^n} u(x,t), \quad \Phi^*(t) := \sup_{x\in\mathbb{R}^n} \Phi(x,t).$$

Note that by definition, $0 \leqslant u^*(t) - u(x,t) \leqslant \operatorname{osc}(u(t))$. Furthermore, the function $u^*(t_0) + \delta\|f\|_\infty t$ is a super-solution of (1.1) for all t_0 and $t > 0$. Hence, $\frac{d}{dt}u^*(t) \leqslant \delta\|f\|_\infty$. We will show the existence of a function $\lambda(\delta, K, F)$ such that if $\lambda > \lambda(\delta, K, F)$, then

$$\sup_{t\in[0,T_K]} \Phi^*(t) \leqslant \Phi^*(0) + \lambda \sup_{t\in[0,T_K]} \operatorname{osc}(u(t)). \tag{2.10}$$

First note that for all $t \in [0, T_K]$, there exists a sequence $\{x_j(t)\}_j \subset \mathbb{R}^n$ with the following property

$$\Phi(x_j(t), t) \longrightarrow \Phi^*(t), \;\; \nabla\Phi(x_j(t), t) \longrightarrow 0 \quad \text{and} \quad \lim_j D^2\Phi(x_j(t), t) \leqslant 0. \tag{2.11}$$

The last inequality in (2.11) is understood in the sense that $\lim_j \left\langle [D^2\Phi(x_j(t), t)]v, v\right\rangle \leqslant 0$ for all $v \in \mathbb{R}^n$. (Such a sequence may be constructed by considering the maxima of the functions $\Phi_{\epsilon_j}(x,t) := \Phi(x,t) - \epsilon_j|x|^2$ and upon choosing $\epsilon_j \longrightarrow 0$ appropriately.)

Now consider the above sequence at $t = T^* \in [0, T_K]$ where $\Phi^*(T^*) = \sup_{[0,T_K]} \Phi^*(t)$. We state for later use that $\lim_j \Phi_t(x_j(T^*), T^*) \geqslant 0$. The following two cases can be distinguished:

(i) $\lim_j |\nabla u(x_j(T^*), T^*)| \to 0$.

(ii) There exists a subsequence (still denoted by j) $x_j(t)$'s such that

$$\lim_j |\nabla u(x_j(T^*), T^*)| \text{ exists and is positive.} \tag{2.12}$$

If $T^* = 0$, then we immediately have

$$\sup_{t\in[0,T_K]} \Phi^*(t) \leqslant \Phi^*(0) \leqslant \|z(0)\|_\infty + \lambda \operatorname{osc}(u(0)).$$

If $T^* > 0$ and case (i) above holds, then

$$\sup_{[0,T_K]} \Phi^*(t) \leqslant \Phi^*(T^*) = 1 + \lambda \operatorname{osc}(u(T^*)) \leqslant 1 + \lambda \sup_{[0,T_K]} \operatorname{osc}(u(T^*)).$$

Together, these two cases give (2.10).

We now show that the case with $T^* > 0$ and case (ii) above cannot happen if we choose λ large enough. We first present a claim which will be proved later:

Claim I. *Let V be a vector in $\mathbb{R}^n$ and $\widetilde{G}_V$ be the linear functional on the space of symmetric $n \times n$ matrices defined as*

$$\widetilde{G}_V(S) = tr\left[\left(I - \frac{V \otimes V}{1+|V|^2}\right) S\right] = S_{ii} - \frac{1}{1+|V|^2} V_i V_j S_{ij}.$$

Then $\widetilde{G}_V(S)$ is $\geqslant$ $(\leqslant)$ 0 for any symmetric semi-positive(negative) definite matrix S.

Applying the above claim to $D^2\Phi(x_j(T^*), T^*)$, we have

$$0 \leqslant \lim_j \{\Phi_t(x_j(T^*), T^*) - \widetilde{G}_{\nabla u(x_j(T^*))}(D^2\Phi(x_j(T^*), T^*))\}.$$

Hence

$$\begin{aligned} 0 \leqslant \lim_j \Big\{ & z_t(x_j(T^*), T^*) - \widetilde{G}_{\nabla u(x_j(T^*))}(D^2 z(x_j(T^*), T^*)) \\ & - \lambda\big[u_t(x_j(T^*), T^*) - \widetilde{G}_{\nabla u(x_j(T^*))}(D^2 u(x_j(T^*), T^*))\big] + \lambda \frac{d}{dt} u^*(T^*) \Big\} \end{aligned}$$

which by (A 13) is equivalent to

$$\begin{aligned} 0 \leqslant \lim_j \Big\{ & -\frac{|D^2 u|^2}{z} + \frac{\langle \nabla u, \nabla z\rangle^2}{z^3} + \delta\left(\frac{\langle \nabla u, \nabla z\rangle}{z} f(x,u) + \langle \nabla u, \nabla_x f(x,u)\rangle + |\nabla u|^2 f_u(x,u)\right) \\ & - \lambda\delta f(x,u) + \lambda \frac{d}{dt} u^*(t) \Big\}\Big|_{(x_j(T^*),T^*)}. \end{aligned} \tag{2.13}$$

Note that by (2.11), we have

$$\nabla z(x_j(T^*), T^*) = \lambda \nabla u(x_j(T^*), T^*) + \rho_j \tag{2.14}$$

for some vector ρ_j such that $\lim_j \rho_j = 0$. Now we make another claim which will be shown later.

Claim II. *With case (ii), i.e. (2.12) holds, we have the following statement:*

$$\lim_j \frac{|D^2u|^2(x_j(T^*), T^*)}{z(x_j(T^*), T^*)} \geqslant \lim_j \lambda^2 z(x_j(T^*), T^*). \tag{2.15}$$

With the above, starting from (2.13), we proceed as follows. [The notation $(x_j(T^*), T^*)$ is suppressed.]

$$\begin{aligned}
0 &\leqslant \lim_j \left\{ -\lambda^2 z + \frac{(\lambda|\nabla u|^2 + \langle \rho_j, \nabla u\rangle)^2}{z^3} + \delta C(F) \left(\frac{|\lambda|\nabla u|^2 + \langle \rho, \nabla u\rangle|}{z} + z + z^2 + 2\lambda \right) \right\} \\
&\leqslant \lim_j \left\{ -\lambda^2 z + \frac{\lambda^2|\nabla u|^4}{z^3} + \delta C(F)(\lambda + \lambda z + z^2) \right\} \\
&\leqslant \lim_j \left\{ \frac{-\lambda^2 z^4 + \lambda^2|\nabla u|^4}{z^3} + \delta C(F)(\lambda + \lambda z + z^2) \right\} \\
&\leqslant \lim_j \left\{ \frac{\lambda^2(-1 - 2|\nabla u|^2)}{z^3} + \delta C(F)(\lambda + \lambda z + z^2) \right\} \\
&\leqslant \lim_j \left\{ \frac{-z^2\lambda^2}{z^3} + \delta C(F)(\lambda + \lambda z + z^2) \right\}
\end{aligned}$$

i.e. $\lambda^2 \leqslant \delta C(F) \lim_j (\lambda z + \lambda z^2 + z^3)$.

Using $\delta C(F)\lambda z \leqslant \frac{1}{4}\lambda^2 + \frac{1}{4}\delta^2 C(F)^2 z^2$ and $\delta C(F)\lambda z^2 \leqslant \frac{1}{4}\lambda^2 + \frac{1}{4}\delta^2 C(F)^2 z^4$, we have

$$\lambda^2 \leqslant C(F)(\delta + \delta^2)z^4 \quad \text{or equivalently} \quad \lambda \leqslant C(F)\sqrt{\delta + \delta^2} z^2 \leqslant C(F)\sqrt{\delta} K^2.$$

The above then leads to a contradiction upon choosing $\lambda(\delta, K, F) = 2C(F)\sqrt{\delta}K^2$. □

We now give the proofs of Claims I and II.

Proof of Claim I. Without loss of generality, let S be semi-positive definite. Let also $\tilde{G} = (g^{ij})_{1 \leqslant i,j \leqslant n}$. Then

$$\widetilde{G}_V(S) = \operatorname{tr}(\tilde{G}S^T) = \operatorname{tr}\left(\left[\sqrt{S}\sqrt{\tilde{G}}\right] \left[\sqrt{\tilde{G}}\sqrt{S}\right] \right) = \operatorname{tr}\left(\left[\sqrt{\tilde{G}}\sqrt{S}\right]^T \left[\sqrt{\tilde{G}}\sqrt{S}\right] \right) \geqslant 0,$$

thus proving the claim. (The symbol $\sqrt{\tilde{G}}$ refers to the square root of $\tilde{G}$ and so forth.) □

Proof of Claim II. Note that $z_{x_i} = z^{-1} u_{x_k} u_{x_k x_i}$. We rewrite (2.14) as

$$\frac{1}{z(x_j(T^*), T^*)} [D^2 u](x_j(T^*), T^*) \nabla u(x_j(T^*), T^*) = \lambda \nabla u(x_j(T^*), T^*) + \rho_j.$$

In the following we suppress the notation $(x_j(T^*), T^*)$. Let $\{\mu_l\}_{l=1,\dots,n}$ be the eigenvalues of $D^2 u$. Then

$$\lambda|\nabla u|^2 + \langle \rho_j, \nabla u\rangle = \frac{\langle [D^2u]\nabla u, \nabla u\rangle}{z} \leqslant \frac{\max_l |\mu_l| |\nabla u|^2}{z}$$

so that

$$\lambda^2|\nabla u|^4 + 2\lambda\,|\nabla u|^2\,\langle \rho_j, \nabla u\rangle + \langle \rho_j, \nabla u\rangle^2 \leqslant \frac{(\max_l |\mu_l|)^2|\nabla u|^4}{z^2} \leqslant \frac{|D^2u|^2|\nabla u|^4}{z^2}$$

leading to (2.15). (Recall that $\lim_j |\nabla u(x_j(T^*), T^*)| > 0$.) □

Proof of Corollary 2.5 For the case $u_0(x) \equiv 0$, by (2.2) of Lemma 2.3, we have

$$\operatorname{osc}(u(t)) \leqslant \operatorname{osc}(u(t), \{x \in \mathbb{R}^n : |x| \leqslant \sqrt{n+1}\}) + 4\sqrt{n+1} \leqslant C\,\|z(t)\|_\infty \quad \text{for some } C > 0. \tag{2.16}$$

From (2.3), let $K = 2$, we get $\sup_{t\in[0,T_K]} \|z(t)\|_\infty \leqslant 1 + C\lambda(\delta, 2, F)\sup_{t\in[0,T_K]} \|z(t)\|_\infty$. If δ is chosen small enough such that $C\lambda(\delta, 2, F) \leqslant \frac{1}{2}$, then

$$\sup_{t\in[0,T_K]} \|z(t)\|_\infty \leqslant \frac{1}{1 - C\lambda(\delta, 2, F)} \leqslant 1 + C(F)\delta^{\frac{1}{2}}.$$

Further, if δ is small enough such that $1 + C(F)\delta^{\frac{1}{2}} \leqslant 2$, the above estimate will hold for all t up to time T, giving the desired result (2.4). The estimate (2.5) is a direct consequence of (2.16) and what we have just proved.

For initial data with finite gradient and oscillation bounds, (2.6) follows by using $u_0^+ \equiv \sup_{x\in\mathbb{R}^n} u_0(x)$ and $u_0^- \equiv \inf_{x\in\mathbb{R}^n} u_0(x)$ as comparison data. Statement (2.7) follows from (2.3) and upon choosing δ small enough to ensure that $\|z(t)\| \leqslant K$ for $t \in [0, T]$. □

From now on, we will always assume that δ is taken to be sufficiently small. The smallness depends on the initial quantities $\|\nabla u_0\|_\infty$ and $\operatorname{osc}(u_0)$.

3 Effective speed of front propagation

Theorem 3.1 *Let $u(x,t)$ be the solution of* (1.8) *with initial datum $u(x,0) \equiv 0$, and let*

$$w^c(x,t) := u(x,t) - ct.$$

Then there exists a unique, finite *value c_v, such that*

$$\|w^{c_v}\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}_+)} \leqslant D_2 = D_0 + \sqrt{n+1}, \tag{3.1}$$

where D_0 is the number from equation (2.5)*. Furthermore, $|c_v| \leqslant \delta\,\|f\|_\infty$ and c_v is a* Lipschitz continuous *function of v .*

To facilitate the proof, first define

$$A^c(t) := \sup_{x\in\mathbb{R}^n} w^c(x,t) \quad \text{and} \quad B^c(t) := \inf_{x\in\mathbb{R}^n} w^c(x,t).$$

Note that both quantities are finite for each $t > 0$, as we can compare with constant sub- and super-solutions. Furthermore, by Corollary 2.5(2.5), we have

$$A^c(t) - B^c(t) = \operatorname{osc}(u(t)) \leqslant D_0. \tag{3.2}$$

The proof of Theorem 3.1 is divided into two propositions.

Proposition 3.2 *There exists a unique finite number* c_v ($|c_v| \leqslant \delta \,\|f\|_\infty$) *such that for all* $t \geqslant 0$,

$$\begin{gathered} -\sqrt{n+1} \leqslant A^{c_v}(t) \leqslant D_0 + \sqrt{n+1} \\ (\text{or equivalently: } -D_0 - \sqrt{n+1} \leqslant B^{c_v}(t) \leqslant \sqrt{n+1}) \end{gathered} \tag{3.3}$$

and

$$\lim_{t\to+\infty} A^c(t) \quad (\text{or equivalently: } \lim_{t\to+\infty} B^c(t)) = \begin{cases} +\infty & \text{for } c < c_v \\ -\infty & \text{for } c > c_v \end{cases} \tag{3.4}$$

Proof The uniqueness of c_v and statement (3.4) are immediate consequence of (3.3). The bound $|c_v| \leqslant \delta \|f\|_\infty$ also follows easily by using $A^c(0) + \delta \|f\|_\infty t$ and $B^c(0) - \delta \,\|f\|_\infty t$ as super- and sub-solutions.

Take a value of c. If for this value of c, (3.3) is satisfied, then clearly (3.4) is true by taking $c_v = c$.

We show that either (3.3) is true or A^c and B^c diverge at least *linearly* in time, i.e.

$$\begin{aligned} &\sup_t A^c(t) > D_0 + \sqrt{n+1} \implies \text{there exists } \alpha > 0,\ \beta > 0 \text{ s.t. } A^c(t) \geqslant \alpha t - \beta \\ &\inf_t A^c(t) < -\sqrt{n+1} \qquad\implies \text{there exists } \alpha' > 0,\ \beta' > 0 \text{ s.t. } A^c(t) \leqslant -\alpha' t + \beta'. \end{aligned} \tag{3.5}$$

Consider the first statement. (The second is shown in a similar way.) So suppose there exists t_0 such that $A^c(t_0) > D_0 + \sqrt{n+1}$. By (3.2), $B^c(t_0) > \sqrt{n+1}$.

In this case, there exists a constant h such that $B^c(t_0) > h > \sqrt{n+1}$ and the planar function $u_0^{(1)}(x) \equiv h$ is some *upward lattice translate* of $u_0(x) \equiv 0$ in the sense that

$$\left\{\left(x, u_0^{(1)}(x)\right) : x \in \mathbb{R}^n\right\} = \{(x, u_0(x)) : x \in \mathbb{R}^n\} + (x_n', h)$$

for some $x_n' \in \mathbb{R}^n$ which satisfies $\mathcal{O}_v^T (x_n', h)^T \in \mathbb{Z}^{n+1}$. Let $u^{(1)}(x,t)$ be the solution of (1.8) with initial datum $u_0^{(1)}(x)$. By the invariance of (1.8) under lattice translation and the uniqueness of classical solutions, then up to a delay in time and a translation of the graph in space by (x_n', h), the behaviour of $u^{(1)}(x,t)$ is exactly the same as that of $u(x,t)$. Furthermore, as $u(x,t_0) \geqslant u^{(1)}(x,0)$, by comparison principle, we have

$$u(x, 2t_0) \geqslant u^{(1)}(x, t_0) \geqslant 2h.$$

By induction, we have: $\inf_{x\in\mathbb{R}^n} u(x, it_0) \geqslant ih$.

Let $I_0 := \inf_{t\in[0,t_0]} B^c(t) > -\infty$. By the translational invariance and the comparison principle again, we get $B^c(t) \geqslant ih - I_0$ on $[it_0, (i+1)t_0]$. The first claim of (3.5) then follows with $\alpha = h/t_0$ and $\beta = I_0 + h$. The second claim can be proved similarly.

Now define

$$c_v := \sup\left\{c : \lim_{t\to\infty} A^c(t) = +\infty\right\}. \tag{3.6}$$

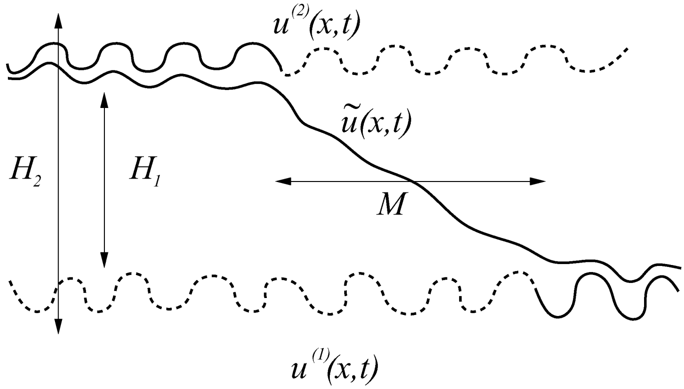

FIGURE 3. Kink-like solution.

(Note that with this definition, it follows that $\lim_{t\to\infty} A^c(t) = +(-)\infty$ for $c < (>) c_v$.) If for this value of c_v, (3.3) is not satisfied, using (3.5), then it holds that either

$$\lim_{t\to\infty} A^{c'}(t) = +\infty \quad \text{for } c' = c_v + \tfrac{1}{2}\alpha \,(\text{if there exists a } t_0 \text{ such that } A^{c_v}(t_0) > D_0 + \sqrt{n+1})$$
$$\text{or } \lim_{t\to\infty} A^{c'}(t) = -\infty \quad \text{for } c' = c_v - \tfrac{1}{2}\alpha' \quad (\text{if there exists a } t_0 \text{ such that } A^{c_v}(t_0) < -\sqrt{n+1}).$$

Both cases contradict the definition (3.6) of c_v and the remark immediately below it. Thus (3.3) must hold and the proposition is proved. □

We now proceed to prove the Lipschitz continuity of c_v.

Proposition 3.3 (*Lipschitz continuity of speed with respect to* v) *The speed* c_v *is a Lipschitz function of* v*, i.e. there exists a* $C(F,\delta) > 0$ *such that for all* $v, \tilde{v} \in \mathbb{S}^n$,

$$|c_v - c_{\tilde{v}}| \leqslant C\,|v - \tilde{v}|. \tag{3.7}$$

Proof Fix $v, \tilde{v} \in \mathbb{S}^n$ with $|v - \tilde{v}| < c_0$ for a small constant $0 < c_0 = c_0(F,\delta) \ll 1$.

Consider (1.8) with v. Recall that in the (x, x_{n+1})-coordinate system, $v = (0,\ldots,0,1)^T$. By choosing an appropriate rotation with respect to the axis v, we can assume $\tilde{v} = (\sin\tilde{\theta}, 0, \ldots, 0, \cos\tilde{\theta})^T$ with $0 < \tilde{\theta} < \frac{\pi}{2}$. The main idea is to construct an approximate solution of (1.8) which is a plane-like surface with effective normal vector $\tilde{v}$. We will show that such a solution cannot have speed much faster then c_v. The construction of the approximating solution and its estimates are carried out in several steps.

Step I. *Kink-like solution* $\tilde{u}$ (*Figure 3*).

Let $u(x,t)$ be a solution of (1.8) with $u(x,0) \equiv 0$ and normal vector v. Let c_v be the speed obtained by Proposition 3.2. By (3.3), then we also have $\|u(\cdot,t) - c_v t\|_{\mathbb{L}^\infty(\mathbb{R}^n)} \leqslant D_2$ for all $t \geqslant 0$.

Let H_1 and H_2 be two fixed positive constants satisfying $H_2 > H_1 > 2\sqrt{n+1}$. Consider two lattice translates $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ of $u(x,t)$ such that

$$H_2 \geqslant u^{(2)}(x,t) - u^{(1)}(x,t) \geqslant H_1 \quad \text{for all } x \in \mathbb{R}^n, t \in \mathbb{R}_+.$$

Further, let M be another fixed and large constant. Consider $\widetilde{u}_0(x)$ which is a smooth function interpolating between $u^{(2)}(x,0)$ and $u^{(1)}(x,0)$ in the following sense:

- $u^{(1)}(x,0) \leqslant \widetilde{u}_0(x) \leqslant u^{(2)}(x,0)$ for all $x \in \mathbb{R}^n$ and $\widetilde{u}_0(x) = u^{(2)}(x,0)$ for $x_1 \leqslant -M$, while $\widetilde{u}_0(x) = u^{(1)}(x,0)$ for $x_1 \geqslant M$;
- $\|\widetilde{u}_0\|_{C^2(\mathbb{R}^n)} \leqslant CH_2M^{-2}\|u\|_{C^2(\mathbb{R}^n)}$, where C is a universal constant which does not depend on f, M or v.

Now define $\widetilde{u}(x,t)$ as the classical solution of (1.8) with initial datum $\widetilde{u}_0(x)$. By Theorem 2.7, $\widetilde{u}(x,t)$ exists globally in time and satisfies $\|\widetilde{u},\widetilde{u}_t,D\widetilde{u},D^2\widetilde{u}\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}_+)} < C(\delta,F,M,H_2)$, with $\lim_{M\to\infty} C(\delta,F,M,H_2) = C(\delta,F)$.

Next we show that $\widetilde{u}(x,t)$ converges to $u^{(i)}(x,t)$ exponentially as $|x_1| \longrightarrow \infty$. Consider $\varphi(x,t) = \tilde{u}(x,t) - u^{(1)}(x,t)$. Then $\varphi(x,t)$ solves a linear, uniformly parabolic equation,

$$\begin{aligned}\varphi_t &= A_f(v,x,\tilde{u}) - A_f\left(v,x,u^{(1)}\right)\\ &= \sum_{ij} a_{ij}(x,t)\varphi_{x_ix_j}(x,t) + \sum_j b_j(x,t)\varphi_{x_j}(x,t) + c(x,t)\varphi(x,t),\end{aligned} \tag{3.8}$$

where $\|a_{ij}\|_{C^0} + \|b_j\|_{C^0} + \|c\|_{C^0} \leqslant C(\delta,F,M,H_2)$. From now on the dependence on δ and F will not be written explicitly.

It is straightforward to verify that if $A(M,H_2)$ and $B(M,H_2)$ are two constants large enough, then $Ae^{-x_1}e^{Bt}$ is a super-solution of (3.8). Hence $0 \leqslant \tilde{u}(x,t) - u^{(1)}(x,t) = \varphi(x,t) \leqslant Ae^{-x_1}e^{Bt}$ for all $x \in \mathbb{R}^n$ and $t \geqslant 0$. Similar argument leads to $0 \leqslant u^{(2)}(x,t) - \tilde{u}(x,t) \leqslant Ae^{x_1}e^{Bt}$. Note that A is of order $e^M H_2$. Combining these estimates gives

$$\max\left\{u^{(2)}(x,t) - Ae^{x_1}e^{Bt}, u^{(1)}(x,t)\right\} \leqslant \tilde{u}(x,t) \leqslant \min\left\{u^{(1)}(x,t) + Ae^{-x_1}e^{Bt}, u^{(2)}(x,t)\right\}. \tag{3.9}$$

The above gives the following statement for $\widetilde{u}$ which justifies it to be called a 'kink-like' solution: Let $D_1 := D_1(H_2)$ be the bound on the oscillation as in (2.6). Then

$$\begin{aligned} u^{(1)}(x,t) \leqslant \widetilde{u}(x,t) \leqslant u^{(1)}(x,t) + \tfrac{D_1}{4} \quad &\text{for} \quad x_1 \geqslant Bt + \ln\tfrac{4A}{D_1},\\ u^{(1)}(x,t) \leqslant \widetilde{u}(x,t) \leqslant u^{(2)}(x,t) \quad &\text{for} \quad -Bt - \ln\tfrac{4A}{D_1} \leqslant x_1 \leqslant Bt + \ln\tfrac{4A}{D_1},\\ u^{(2)}(x,t) - \tfrac{D_1}{4} \leqslant \widetilde{u}(x,t) \leqslant u^{(2)}(x,t) \quad &\text{for} \quad x_1 \leqslant -Bt - \ln\tfrac{4A}{D_1}. \end{aligned} \tag{3.10}$$

Note that the 'width' of the region where $\widetilde{u}$ interpolates between $u^{(1)}$ and $u^{(2)}$ grows most linearly with speed B.

Step II. *Plane-like approximation* (*Figure 4*).

Let $\{u^{(i)}(x,t)\}_{i=-\infty}^{\infty}$ be a sequence of solutions of (1.8) which are lattice translates of each other such that $u^{(i)}(x,0) \equiv -iH$ with some fixed constant $H > 3D_2$, where D_2 is the $\mathbb{L}^\infty$-bound in the moving frame as in (3.1). By Proposition 3.2, we have

$$\left\|u^{(i)}(x,t) + iH - c_v t\right\|_{\mathbb{L}^\infty(\mathbb{R}^n,\mathbb{R}_+)} \leqslant D_2.$$

For the remaining proof, the above H and the M (used in the previous step) will be kept fixed. Let L be a large constant ($\gg M$) which is to be determined.

Define $\tilde{u}^{(i)}(x,t)$ to be the kink-like solution which interpolates between $u^{(i+1)}$ and $u^{(i)}$ as in Step I but now 'centred' at iL, i.e. $\widetilde{u}^{(i)}(x,0) = u^{(i+1)}(x,0)$ for $x_1 \leqslant iL - M$, $\widetilde{u}^{(i)}(x,0) = u^{(i)}(x,0)$

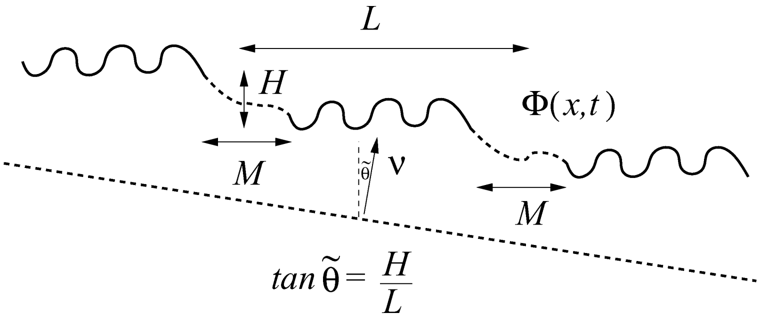

FIGURE 4. Plane-like approximation.

for $x_1 \geqslant iL + M$ and so forth. Now patch the $\{\tilde{u}^{(i)}\}_i$ together by means of a partition of unity: $\Phi(x,t) = \sum_{-\infty}^{\infty} \tilde{u}^{(i)}(x,t)\eta^i(x)$ where the $\{\eta^i\}_i$ is a sequence of smooth functions satisfying

$$\eta^i(x) \geqslant 0,\ \ \eta^i(x) = \begin{cases} 1 & x_1 \in [iL - \frac{L}{4}, iL + \frac{L}{4}] \\ 0 & x_1 \in (-\infty, iL - \frac{3L}{4}] \cup [iL + \frac{3L}{4}, \infty) \end{cases} \quad \text{and} \quad \sum_i \eta^i(x) \equiv 1.$$

The $\Phi(x,t)$ thus constructed has the following properties:

(1) Using (3.10), $\Phi(x,t)$ approximates a tilted plane in the following sense: for all $i \in \mathbb{Z}$

- For $x \in \mathbb{R}^n : (i-1)L + Bt + \ln\frac{4A}{D_1} \leqslant x_1 \leqslant iL - Bt - \ln\frac{4A}{D_1}$:

$$u^{(i-1)}(x,t) + \frac{D_1}{4} \geqslant \tilde{u}(x,t) \geqslant u^{(i-1)}(x,t) - \frac{D_1}{4}; \tag{3.11}$$

- For $x \in \mathbb{R}^n : iL - Bt - \ln\frac{4A}{D_1} \leqslant x_1 \leqslant iL + Bt + \ln\frac{4A}{D_1}$:

$$u^{(i-1)}(x,t) + \frac{D_1}{4} \geqslant \tilde{u}(x,t) \geqslant u^{(i)}(x,t) - \frac{D_1}{4}; \tag{3.12}$$

The above structure is valid if $(i-1)L + Bt + \ln\frac{4A}{D_1} \leqslant iL - Bt - \ln\frac{4A}{D_1}$, i.e.

$$0 < t < T_L := \frac{L}{2B} - \frac{1}{B}\ln\frac{4A}{D_1}. \tag{3.13}$$

Note that as A and B (which are defined through M and H) are fixed, we get $0 < T_L$ if L is sufficiently large.

(2) The upward normal vector of the tilted hyperplane approximated by $\Phi(\cdot,t)$ (for $0 < t < T_L$) is given in the (x, x_{n+1})-coordinate system by

$$\left(\frac{H}{\sqrt{L^2+H^2}}, 0, \ldots, 0, \frac{L}{\sqrt{L^2+H^2}}\right)^T.$$

which can be set to equal $\tilde{\nu} = (\sin\tilde{\theta}, 0, \ldots, 0, \cos\tilde{\theta})^T$ upon choosing

$$L = H\cot\tilde{\theta}. \tag{3.14}$$

i.e. $L \sim \frac{H}{\tilde{\theta}}$ as $\tilde{\theta} \longrightarrow 0$.

(3) Φ solves (1.8) *exactly* for all $t \geqslant 0$ and $x \in \mathbb{R}^n$ such that $x_1 \in \bigcup_i [iL - \frac{L}{4}, iL + \frac{L}{4}]$.

(4) Now statement (3.9) combined with the properties of the η_i and parabolic regularity gives a constant $C = C(M, H)$ such that

$$\sup_i \|\Phi(\cdot, t) - u^{(i)}(\cdot, t)\|_{C^2(\{x: iL + \frac{L}{4} \leqslant x_1 \leqslant iL + \frac{3L}{4}\})} \leqslant Ce^{-\frac{L}{4}} e^{Bt}. \tag{3.15}$$

Step III. *Approximation of speed.*

This step shows that the normal speed of propagation of the tilted plane approximated by $\Phi(x, t)$ cannot be much bigger than c_v.

In fact, by (3.11) and (3.12), there exists a $C_1 > 0$ such that for all $(x, t) \in \mathbb{R}^n \times [0, T_L]$

$$\Phi(x, t) \leqslant -(\tan \tilde{\theta})x_1 + (c_v + B \tan \tilde{\theta})t + C_1.$$

[The extra factor $B \tan \tilde{\theta}$ comes from the linear spread of the width of the kink in the plane-like approximation – see (3.10) and (3.11)–(3.12).] The above shows that $\Phi(x, t)$ can be bounded from *above* by a hyperplane moving with *normal* speed $c_v \cos \tilde{\theta} + B \sin \tilde{\theta}$, at least on the time interval $[0, T_L]$.

Next we show that Φ differs from the actual solution of (1.8) by a very small error. From (3.15), it follows that Φ satisfies the following equation:

$$\Phi_t = A_f(v, x, \Phi) + g(x, t),$$

where $g(x, t)$ is supported on $\bigcup_{i=-\infty}^{\infty} \{x : iL + \frac{L}{4} \leqslant x_1 \leqslant iL + \frac{3L}{4}\}$ and $\|g\|_{C^0} \leqslant Ce^{-\frac{L}{2}} e^{Bt}$. Let $\widetilde{\Phi}(x, t)$ be the solution of (1.8) with initial data $\widetilde{\Phi}(x, 0) = \Phi(x, 0)$. The function $\psi(x, t) = \widetilde{\Phi}(x, t) - \Phi(x, t)$ solves a linear parabolic equation similar to (3.8),

$$\psi_t = \sum_{ij} a_{ij}(x, t)\psi_{x_i x_j}(x, t) + \sum_j b_j(x, t)\psi_{x_j}(x, t) + c(x, t)\psi(x, t) - g(x, t), \quad \psi(x, 0) \equiv 0.$$

Using $\widetilde{\Psi}^{\pm} = \psi \pm \int_0^t \|g(s, \cdot)\|_{C^0}\, ds$ as a comparison function gives $\|\psi(\cdot, t)\|_{\mathbb{L}^\infty(\mathbb{R}^n)} \leqslant Ce^{-\frac{L}{4}} e^{Bt}$. Hence for $0 \leqslant t \leqslant T_L$, we have

$$\widetilde{\Phi}(x, t) \leqslant \Phi(x, t) + Ce^{-\frac{L}{2}} e^{Bt} \leqslant -(\tan \tilde{\theta})x_1 + (c_v + B \tan \tilde{\theta})t + C_1 + Ce^{-\frac{L}{2}} e^{Bt}.$$

Similarly, by definition, $\widetilde{\Phi}$ can be bounded from *below* by some plane-like solution with normal $\tilde{v}$ and speed $c_{\tilde{v}}$. Thus

$$-(\tan \tilde{\theta})x_1 + \frac{c_{\tilde{v}} t}{\cos \tilde{\theta}} - C_2 \leqslant -(\tan \tilde{\theta})x_1 + (c_v + B \tan \tilde{\theta})t + C_1 + Ce^{-\frac{L}{4}} e^{Bt},$$

which gives

$$(c_{\tilde{v}} - c_v \cos \tilde{\theta})t \leqslant B(\sin \tilde{\theta})t + C_3 + C_4 e^{-\frac{L}{4}} e^{Bt}. \tag{3.16}$$

Now choose $t = \frac{T_L}{P}$ for some $P > 1$ which is admissible according to (3.13). Furthermore, by (3.14),

$$t = \frac{1}{P}\left[\frac{L}{2B} - \frac{1}{B}\ln\frac{4A}{D_1}\right] = \frac{1}{P}\left[\frac{H \cot \tilde{\theta}}{2B} - \frac{1}{B}\ln\frac{4A}{D_1}\right].$$

Then (3.16) becomes

$$\frac{c_{\tilde{v}} - c_v \cos\tilde{\theta}}{P}\left[\frac{H\cot\tilde{\theta}}{2B} - \frac{1}{B}\ln\frac{4A}{D_1}\right]$$
$$\leqslant \frac{B\sin\tilde{\theta}}{P}\left[\frac{H\cot\tilde{\theta}}{2B} - \frac{1}{B}\ln\frac{4A}{D_1}\right] + C_3 + C_4\exp\left[-\frac{H\cot\tilde{\theta}}{4} + \frac{H\cot\tilde{\theta}}{2P} - \frac{1}{P}\ln\frac{4A}{D_1}\right].$$

If we choose $P = 3\ (> 2)$ and consider the regime $|\tilde{\theta}| \ll 1$, we obtain $c_{\tilde{v}} - c_v\cos\tilde{\theta} \leqslant C(A,B,H)\tilde{\theta}$, i.e.

$$c_{\tilde{v}} - c_v \leqslant C(A,B,H)\tilde{\theta} + O(\tilde{\theta}^2) \leqslant C(A,B,H)\tilde{\theta}.$$

The lower bound $c_v - c_{\tilde{v}} \geqslant -C\tilde{\theta}$ can be proved similarly. The Lipschitz continuity of c_v is thus established. □

4 Pulsating wave

In this section, we look for a special type of solutions of (1.8) which is invariant under appropriate space–time translation (see equation (1.2) and Figure 1),

$$u(x+x', t+t') = c_v t' + u(x,t) \quad \text{for all } (x',t')^T \text{ such that } \mathcal{O}_v(x', c_v t')^T \in \mathbb{Z}^{n+1}. \tag{4.1}$$

If $c_v \neq 0$, the above condition is equivalent to the following representation of u:

$$u(x,t) = c_v t + U(\mathcal{O}_v(x, c_v t)^T), \tag{4.2}$$

where $U : \omega = (\omega_1, \ldots, \omega_{n+1}) \in \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ is a one-periodic function of its argument, i.e. $U(\omega + p) = U(\omega)$ for all $\omega \in \mathbb{R}^{n+1}$ and $p \in \mathbb{Z}^{n+1}$. We call U the *transformed* function of u, and ω the *transformed* variable. We will show the existence and uniqueness of U and present its various stability properties. The resulting function u and the corresponding U will be called a *pulsating wave* for (1.8). We often identify u with U.

For $c_v \neq 0$, we can relate the gradients of u to those of U. Introducing

$$\omega = \mathcal{O}_v((x, c_v t)^T) \quad \text{and} \quad \mathcal{O}_v = (a_{ij})_{1\leqslant i,j\leqslant n+1},$$

then

$$c_v^{-1}u_t - 1 = \sum_{k=1}^{n+1} a_{k,n+1}\partial_{\omega_k}U \quad \text{and} \quad \partial_{x_i}u = \sum_{k=1}^{n+1} a_{k,i}\partial_{\omega_k}U.$$

Furthermore, U satisfies the following equation:

$$c_v + c_v\sum_{k=1}^{n+1} a_{k,n+1}\partial_{\omega_k}U = \sqrt{1+|\tilde{\nabla}U|^2}\sum_{i=1}^{n}\left\{\sum_{k=1}^{n+1} a_{k,i}\partial_{\omega_k}\left(\frac{\sum_{k=1}^{n+1} a_{k,i}\partial_{\omega_k}U}{\sqrt{1+|\tilde{\nabla}U|^2}}\right)\right\}$$
$$+\delta\sqrt{1+|\tilde{\nabla}U|^2}f(\omega + \mathcal{O}_v((0,\ldots,0,U)^T)), \tag{4.3}$$

where $|\tilde{\nabla}U|^2 = \sum_{i=1}^{n}\left(\sum_{k=1}^{n+1} a_{k,i}\partial_{\omega_k}U\right)^2$.

We first establish the following existence result.

Theorem 4.1 (*Existence of pulsating wave*) *For any $v \in \mathbb{S}^n$, there exists a continuous function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ which solves (1.8) and satisfies (4.1) for the c_v given by Proposition 3.2. Moreover, the transformed function U satisfies*

$$\|U\|_{L^\infty(\mathbb{R}^{n+1})} \leqslant D_3 := 3(D_2 + \sqrt{n+1})$$

(*where D_2 is the constant from Theorem 3.1*) *so that the pulsating wave is bounded in its moving frame.*

There are several methods to establish the existence result. A standard approach is to use Schauder Fixed Point Theorem. This can be accomplished by the gradient decay estimate (Theorem 2.8) which produces a contraction map in an appropriate function space. Here we employ a different, but more elementary method. It uses the comparison principle in its full capacity.

The current proof consists of several steps. First we prove the theorem for rational normal direction v and the case of $c_v \neq 0$. This is accomplished by constructing sub- and super-solutions of (1.8). These objects satisfy uniform Lipschitz bounds in x, t *independent* of v. It turns out that they are in fact *solutions* and hence are actually pulsating waves. The cases of irrational direction and $c_v = 0$ are handled by approximation using the previous case.

4.1 Proof of Theorem 4.1

First, consider a rational normal direction v – the coordinates of v are all rational numbers – and assume $c_v > 0$. Then in the (x, x_{n+1})-coordinate system, the inhomgeneity is periodic with some period $P = P(v)$. In contrast to the one-periodicity of the inhomogeneity, we call this periodicity 'fictitious' as the period depends on the normal direction and it can be extremely large.

Step I. *Construction of 'pulsating' sub- and super-solutions.*

Let $\{u^\pm(x,t)\}_{x\in\mathbb{R}^n, t\in\mathbb{R}_+}$ be a solution of (1.8) starting from $u^\pm(x,0) \equiv \pm 2(D_2 + \sqrt{n+1})$ where D_2 is the number from (3.1). Define

$$U^+(x,t) := \liminf_{|I|\to\infty}\{u^+(x - x_I, t + t_I) - c_v t_I\} \tag{4.4}$$

$$U^-(x,t) := \limsup_{|I|\to\infty}\{u^-(x - x_I, t + t_I) - c_v t_I\} \tag{4.5}$$

where $u^\pm(\cdot, r) = \pm\infty$ if $r < 0$ and $\{I \in \mathbb{Z}^{n+1}\}$ is a fixed sequence which enumerates the set

$$\left\{(x_J, t_J) : \mathcal{O}_v(x_J, c_v t_J) \in \mathbb{Z}^{n+1}, t_J > 0\right\}.$$

Note that $U^\pm(\cdot,\cdot)$ are defined on all of $\mathbb{R}^n \times \mathbb{R}$. Furthermore, they satisfy the following properties:

(i) They both are pulsating functions, i.e. they satisfy (4.1). In particular, they are P-periodic in x.

(ii) $\|U^{\pm}(\cdot,\cdot) - c_v t\|_{L^\infty(\mathbb{R}^n\times\mathbb{R})} < D_2$, and

$$0 < 2(D_2 + \sqrt{n+1}) < \inf_{x\in\mathbb{R}^n, t\in\mathbb{R}} U^+(x,t) - \sup_{x\in\mathbb{R}^n, t\in\mathbb{R}} U^-(x,t) < 6(D_2 + \sqrt{n+1}). \tag{4.6}$$

(iii) They are uniformly Lipschitz on $\mathbb{R}^n \times \mathbb{R}$.

(iv) $U^+(\cdot,\cdot)$ is a super-solution and $U^-(\cdot,\cdot)$ a sub-solution of (1.8).

Proof (i) We will only focus on U^+. For all (x_K, t_K) such that $\mathcal{O}_v(x_K, c_v t_K) \in \mathbb{Z}^{n+1}$,

$$\begin{aligned} U^+(x - x_K, t + t_K) &= \liminf_{|I|\to\infty} u^+(x - x_K - x_I, t + t_K + t_I) \\ &= \liminf_{|I'|\to\infty} u^+(x - x_{I'}, t + t_{I'}) \\ &= U^+(x,t) \end{aligned}$$

since $\mathcal{O}_v(x_K + x_I, c_v(t_K + t_I)) \in \mathbb{Z}^{n+1}$ if both $\mathcal{O}_v(x_K, c_v t_K)$ and $\mathcal{O}_v(x_I, c_v t_I)$ belong to $\mathbb{Z}^{n+1}$. Note that the lim inf and lim sup of a sequence are not changed under finite shifts of the sequence.

(ii) This follows from equation (3.1) which yields

$$|[u^{\pm}(x_1 - x_J, t + t_J) - c_v t_J] - c_v t \mp 2(D_2 + \sqrt{n+1})| \leqslant D_2 + \sqrt{n+1},$$

and hence the estimates as claimed.

(iii) By Corollary 2.5, the $\|\nabla u^{\pm}(x,t)\|_{\mathbb{L}^\infty(\mathbb{R}^n\times\mathbb{R}_+)}$ are bounded. Theorem 2.7 implies that $\|u_t^{\pm}(x,t)\|_{\mathbb{L}^\infty(\mathbb{R}^n\times[1,\infty))}$ is also bounded. Therefore $u^{\pm}$ is uniformly Lipschitz continuous in space and time. As the lim inf and lim sup of uniformly Lipschitz continuous functions are also uniformly Lipschitz (with the same constant), the $U^{\pm}(\cdot,\cdot)$ satisfy the same property.

(iv) The fact that lim inf and lim sup are super- and sub-solutions, respectively, follows from a standard argument (see [8, Lemma 6.1]). Note that we need no monotonicity of $f(x,u)$ with respect to u, because $u^{\pm}$ are uniformly bounded (in the moving frame) and $f(x,u)$ is uniformly Lipschitz. The lemma can be applied instead to $\tilde{u}^{\pm}_{y,\tau}(x,t) = e^{-Mt}[u^{\pm}(x - y, t+\tau) - c_v\tau]$ on a bounded neighbourhood of t_0, for some large constant M. □

Step II. *Existence of pulsating wave for rational slope.*

We show that in fact $U^{\pm}(x,t)$ are classical solutions of (1.8) and thus are pulsating waves.

First define

$$T_* := \sup\left\{\tau > 0 : \inf_{x\in\mathbb{R}^n} \left(U^+(x,0) - U^-(x,\tau)\right) \geqslant 0\right\} \tag{4.7}$$

i.e., the first time $U^-(\cdot,t)$ touches $U^+(\cdot,0)$ from below. By property (ii), the U^- is bounded in a frame moving with velocity $0 < c_v$ so that $T_* < \infty$. By property (iii), the $U^{\pm}$ are uniformly continuous in x and t. The periodicity in x then implies the existence of an $x_0 \in \mathbb{R}^n$ such that $U^-(x_0, T_*) = U^+(x_0, 0)$.

Now consider the classical solutions $V^{\pm}$ of (1.8) with the Lipschitz initial data $V^+(x,0) = U^+(x,0)$ and $V^-(x,0) = U^-(x, T_*)$. These solutions are globally defined (Theorem 2.7) and stay uniformly Lipschitz (Corollary 2.5). By property (iv) and weak comparison principle, we have

$$U^-(x, t + T_*) \leqslant V^-(x, t) \leqslant V^+(x, t) \leqslant U^+(x, t) \quad \text{for } (x, t) \in \mathbb{R}^n \times \mathbb{R}_+. \tag{4.8}$$

On the other hand, by property (i), there exists $T_v > 0$ such that

$$U^-(x_0, T_v + T_*) = U^-(x_0, T_*) = U^+(x_0, 0) = U^+(x_0, T_v + T_*),$$

leading to $V^-(x_0, T_v) = V^+(x_0, T_v)$.

Let $\widetilde{V} := V^+(x, t) - V^-(x, t)$. As $V^\pm$ are $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+)$, the difference $\widetilde{V}$ satisfies a linear parabolic PDE of the form [similar to (3.8)]

$$\partial_t \widetilde{V} = \sum_{i,j} a_{ij}(x, t) \widetilde{V}_{x_i x_j} + \sum_j b_j(x, t) \widetilde{V}_{x_j} + c(x, t) \widetilde{V}$$

with continuous coefficients. As f and $V^\pm$ are uniformly Lipschitz in space–time, the above equation is uniformly parabolic with bounded coefficients. Note that $\widetilde{V} \geqslant 0$ and $\widetilde{V}(x_0, 0) = \widetilde{V}(x_0, T_v) = 0$. Classical strong maximum principle (see for example [16]) implies that $\widetilde{V}(\cdot, t) \equiv 0$ for all $t \in (0, T_v)$. Therefore $V^+ \equiv V^-$. [By the same reasoning as in Step I (iv), we can apply the strong maximum principle without a sign condition on $c(x, t)$.]

As a last step, note that $V^\pm(x, 1/n) \to V^\pm(x, 0)$ (pointwise), we obtain $U^+(\cdot, t) = U^-(\cdot, T_* + t)$ for $t \in [0, T_v]$, and therefore this function is both super- and sub-solution, i.e. a viscosity solution. By the comparison principle for viscosity solutions it must equal $V^\pm$ and thus is a classical solution.

Thus we have established the existence of pulsating waves for rational slopes with $c_v \neq 0$.

Step III. *Existence of pulsating wave for irrational slope.*

The following argument extends the existence result to irrational slopes.

Let v_n (rational slopes) $\to v$. By the continuity of the speed in the normal, we have $c_n \to c_v \neq 0$. Further, let u_n be the corresponding pulsating waves in the frame $\mathcal{O}_{v_n}$. They satisfy uniformly Lipschitz bounds in x, t independent of v.

Using the transformation (4.2), we thus obtain a family of functions $U_n(\omega)$ which are one-periodic in $\mathbb{R}^{n+1}$ and are solutions of (4.3). As $c_n > \frac{c_v}{2} > 0$, the change of variables $\omega = \mathcal{O}_{v_n}(x, c_n t)^T$ are invertible for each n with uniform bounds for the inverse. Therefore, the U_n's also satisfy uniform Lipschitz and (by parabolic regularity of the u_n's) $C^{2,\alpha}$ estimates on $[0, 1]^{n+1}$. Hence we can extract a convergent subsequence leading to a U which solves (4.3) with the limiting normal direction v.

The Theorem is thus proved for the case $c_v \neq 0$.

Step IV. *Existence of 'pulsating wave': Stationary ($c_v = 0$) case.*

Again, we consider separately the case of rational and irrational directions.

For rational direction, the evolution equation described by (1.8) in fact is the *negative gradient flow* of the following energy functional:

$$\mathscr{E}(u) = \int_{[0,P]^n} \left(\sqrt{1+|\nabla u|^2} - \delta F(x,u)\right) dx^n, \quad \text{where } F(x,u) = \int_0^u f(x,s)\,ds \tag{4.9}$$

($u_t = -\sqrt{1+|\nabla u|^2}\frac{\delta \mathscr{E}}{\delta u}(u)$). As $c_v = 0$, we have two solutions of (1.8): $u_*(x,t) < u^*(x,t)$ which are P-periodic in x and are uniformly Lipschitz and bounded in x and t. Hence any solution $u(x,t)$ of (1.8) with $u_*(x,0) \leqslant u(x,0) \leqslant u^*(x,0)$ satisfies $u_*(x,t) \leqslant u(x,t) \leqslant u^*(x,t)$. Furthermore, the following energy identity holds:

$$\mathscr{E}(u(\cdot,t)) + \int_0^t \int_{[0,P]^n} \frac{u_t^2}{\sqrt{1+|\nabla u|^2}}\, dx^n\, dt = \mathscr{E}(u(\cdot,0)).$$

The uniform oscillation and gradient bounds from Corollary 2.5 lead to $\sup_{t\geqslant 0} |\mathscr{E}(u(\cdot,t))| < \infty$. Moreover, the uniform gradient bound implies that $u_t^2(\sqrt{1+|\nabla u|^2})^{-1} \leqslant C u_t^2$. Thus we have

$$\int_0^\infty \int_{[0,P]^n} u_t^2\, dx^n\, dt < \infty.$$

A standard application of parabolic regularity implies that $\partial_t u(\cdot,t)$ is uniformly continuous on $[0,P]^n$, and hence $\partial_t u(\cdot,t_j) \longrightarrow 0$ for some subsequence $t_j \longrightarrow \infty$. A further subsequence gives that the limit $\bar{u}(x) = \lim_{t_{j_k}\to\infty} u(x,t_{j_k})$ exists and it solves the stationary solution for (1.8). Furthermore, the P-periodicity of $\bar{u}(\cdot)$ automatically implies (4.1).

For irrational direction, the same argument can be applied with the modification that the domain $[0,P]^n$ is replaced by a sequence of monotonically increasing balls B_j such that $B_j \longrightarrow \mathbb{R}^n$. The function u is required to satisfy the Dirichlet boundary condition: $u = C$ on ∂B_j (where $u_* \leqslant C \leqslant u^*$). Then for each j, we obtain a stationary solution u^j as before. From the uniform gradient estimates Corollary 2.5, we can extract a subsequence which converges (on compact subsets) to a stationary solution on the whole space. (Note that the result of Corollary 2.5 stated for $\mathbb{R}^n$, can be extended to bounded domains such as balls B_j's by constructing suitable barrier functions with uniformly bounded gradient at the boundary. By the smallness of the forcing and the *a priori* $\mathbb{L}^\infty$ bound, such barriers can be constructed quite easily.)

Finally, for irrational slope, any stationary solution of (1.8) *automatically* satisfies (4.1) as there is *no* $x' \in \mathbb{R}^n$ such that the condition $\mathcal{O}_v(x',0)^T \in \mathbb{Z}^{n+1}$ is fulfilled. Theorem 4.1 is thus proved.

Remark 4.2

(1) Our result for the case $c_v \neq 0$ is related to the result in [4] on the existence of plane-like minimizers: If the forcing is small and sufficiently regular, then stationary solutions of (1.1) not only stay close to a plane, but are even graphs over that plane.

(2) Note that there may be solutions that stay bounded in a frame with $c_v = 0$, but are not stationary, for example a 'travelling kink' or cascades of many kink structures.

4.2 Properties of the pulsating wave

In this section, we present the uniqueness result and some stability properties for the pulsating wave.

Proposition 4.3 (*Uniqueness of pulsating wave*) *For all v, the speed c_v is unique. If $c_v \neq 0$, then the shape U of the pulsating wave is also unique.*

Proof The uniqueness of c_v is already proved in Theorem 3.1, in particular, Proposition 3.2.

When $c_v \neq 0$ and the direction v is rational, the uniqueness of the pulsating wave follows exactly from the same argument as in [11, Proposition 6]. When v is irrational, we proceed similarly, but with the following additional consideration. (Without loss of generality, assume $c_v > 0$.)

Let U and V be two pulsating waves solving (4.3). First, consider $u_0(x) = U(\mathcal{O}_v(x,0)^T)$. Second, let $v(x,t)$ be the solution of (1.8) with initial data $\bar{v}(x,0) = -h + V(\mathcal{O}_v(x,0)^T)$ for some large positive constant h such that $\bar{v}$ is some lattice translation of $V(\mathcal{O}_v(x,0)^T)$. Similar to (4.7), define

$$T_* = \sup\left\{\tau > 0 : \inf_{x\in\mathbb{R}^n} (u(x,0) - \bar{v}(x,\tau)) \geqslant 0\right\}.$$

Note that $T_* < \infty$ as $c_v > 0$. Now let $\tilde{u}(x,t)$ and $\tilde{v}(x,t)$ be the solution of (1.8) with initial data $u_0(x)$ and $\bar{v}(x, T_*)$. As $\tilde{v}(x,0) \leqslant \tilde{u}(x,0)$, weak maximum principle (in the whole space) implies that $\tilde{v}(x,t) \leqslant \tilde{u}(x,t)$ for all $x \in \mathbb{R}^n$ and $t \geqslant 0$. Consider the following two cases:

(1) Suppose there exists an x_* such that $\tilde{u}(x_*,0) = \tilde{v}(x_*,0)$. By the pulsating wave ansatz, $\tilde{u}(x_*+x', c_v t') = \tilde{v}(x_*+x', c_v t')$ for some $(x', c_v t')$ such that $t' > 0$ and $\mathcal{O}_v(x', c_v t') \in \mathbb{Z}^{n+1}$. This would contradict the strong comparison principle (in unbounded domain) unless U is identically equal to V.

(2) Suppose there exists x_i such that $|x_i| \longrightarrow \infty$ and $\tilde{u}(x_i,0) - \tilde{v}(x_i,0) \longrightarrow 0^+$. By the pulsating ansatz again, we have $\tilde{u}(x_i+x_i', c_v t_i') - \tilde{v}(x_i+x_i', c_v t_i') \longrightarrow 0^+$ for some $(x_i', c_v t_i')$ satisfying $\mathcal{O}_v(x_i', c_v t_i')^T \in \mathbb{Z}^{n+1}$. As $c_v \neq 0$, we can always choose the x_i' and t_i''s such that the $(x_i + x_i, t_i)$'s lie in a compact subset of $\mathbb{R}^{n+1}$. Hence, there exists an x_* and t_* such that $\tilde{u}(x_*, t_*) = \tilde{v}(x_*, t_*)$. Thus the situation is the same as in the previous case.

□

For the case $c_v = 0$, we do not expect uniqueness to be true as there could be many stationary solutions corresponding to the local minimizers of the energy functional (4.9). These solutions cannot be related to each other as in the $c_v \neq 0$ case.

The next result leads to a form of stability property of the pulsating waves. It is similar in spirit to the Krein–Rutman type of statement.

Proposition 4.4 (*Monotonicity in time for the pulsating wave*) *Let u be a pulsating wave of (1.8) with $c_v > 0$. Then $u_t > 0$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.*

Proof We first prove the result for rational direction so that the pulsating wave is space–time periodic in a tilted frame with some period $P = P(v)$. The case for irrational direction can be deduced by a limiting procedure together with the strong maximum principle.

Consider $u(x, 0)$ and define

$$T_* = \sup\{t \geqslant 0 : u(x,t) \leqslant u(x,0) \text{ for some } x \in \mathbb{R}^n\}.$$

As $c_v > 0$ and u is bounded in its frame, we have $0 \leqslant T_* < \infty$. By the continuity of $u(x,t)$ in the x- and t-variables and the compactness of the domain (as u is P-periodic), we must have $u(x, T_*) \geqslant u(x, 0)$ *for all* $x \in \mathbb{R}^n$ and $u(x_*, T_*) = u(x_*, 0)$ for some x_*. Now consider the solutions of (1.8) with initial data $u(x, T_*)$ and $u(x, 0)$, respectively. The pulsating wave ansatz implies that $u(x_*, T_* + T_v) = u(x_*, T_v)$ for some $T_v > 0$, contradicting the strong maximum principle unless $u(\cdot, T_*) \equiv u(\cdot, 0)$. As $c_v > 0$, this can only happen if $T_* = 0$. Hence $u(x,t) > u(x,0)$ for all $t > 0$ giving $u_t \geqslant 0$. The fact that $u_t > 0$ follows from strong maximum principle for u_t. [Note that u_t solves a linear parabolic equation (by taking the time derivative of (1.8)) with bounded coefficients.] □

The above result immediately leads to the following corollary.

Corollary 4.5 *Let v be a rational direction and u be the pulsating wave of (1.8) with $c_v \neq 0$. Then there exist $0 < C_1(v, F) < C_2(F) < \infty$ such that for all $x \in \mathbb{R}^n$, $t, s \in \mathbb{R}$, it holds that*

$$C_1 |t - s| \leqslant |u(x,t) - u(x,s)| \leqslant C_2 |t - s|.$$

The next exponential convergence result is a consequence of the above monotonicity property.

Theorem 4.6 (*Stability property of pulsating wave*) *If v is a rational direction and $c_v \neq 0$, then the pulsating wave u satisfies the following stability property:*

Let $\{v(x,t) : x \in \mathbb{R}^n, t \geqslant 0\}$ be a classical solution of (1.8) which is a P-periodic function [where $P = P(v)$]. Then there exists $t_* \in \mathbb{R}$, $\lambda > 0$ and a constant C which might depend on P such that

$$\|v(\cdot,t) - u(\cdot, t^* + t)\|_{\mathbb{L}^\infty(\mathbb{R}^n)} \leqslant Ce^{-\lambda t}.$$

Proof Without loss of generality, we can assume the initial condition $v(x,0)$ is smooth and $v(x, 0) > NP$ for some sufficiently large integer N so that $v(x,0) \geqslant u(x,0)$ for $x \in \mathbb{R}^n$.

Now let $u(x,t)$ be the pulsating wave of (1.8). Define

$$s_0^* = \inf\{t > 0 : u(x,t) = v(x,0) \text{ for some } x \in \mathbb{R}^n\}$$
$$\text{and } t_0^* = \sup\{t > 0 : u(x,t) = v(x,0) \text{ for some } x \in \mathbb{R}^n\}.$$

(Qualitatively, s_0^* is the *first* time $u(x,t)$ touches $v(x,0)$ from *below* and t_0^* is the *last* time $u(x,t)$ touches $v(x,0)$ from *above*. The above definitions make sense as we are working in the compact domain and u and v are periodic functions with uniform Lipschitz bound.)

By Proposition 4.4, we have $s_0^* < t_0^*$ and $u(x,s_0^*) \leqslant v(x,0) \leqslant u(x,t_0^*)$ *for all* $x \in \mathbb{R}^n$ with the equalities valid at some $x_0', x_0'' \in \mathbb{R}^n$.

By comparison principle, we have for all $x \in \mathbb{R}^n$ that $u(x, s_0^*+T) \leqslant v(x,T) \leqslant u(x,t_0^*+T)$ where $c_v T = P$. The pulsating wave ansatz gives

$$u(x,s_0^*) \leqslant v(x,T) - c_v T \leqslant u(x,t_0^*).$$

Now the *strong maximum principle* together with Proposition 4.4 imply the existence of s_1^* and t_1^* such that $s_0^* < s_1^* < t_1^* < t_0^*$ and $u(x,s_1^*) \leqslant v(x,T) - c_v T \leqslant u(x,t_1^*)$ with the equalities valid at some $x_1', x_1'' \in \mathbb{R}^n$. By induction, there exist $s_{n-1}^* < s_n^* < t_n^* < t_{n-1}^*$ such that

$$u(x,s_n^*) \leqslant v(x,nT) - c_v nT \leqslant u(x,t_n^*), \quad x \in \mathbb{R}^n \tag{4.10}$$

and the equalities hold at some $x_n', x_n'' \in \mathbb{R}^n$.

Define: $\tau_n^* = t_n^* - s_n^*$. We claim the existence of a positive number $\rho < 1$ *independent* of n such that

$$\tau_{n+1}^* \leqslant \rho \tau_n^*. \tag{4.11}$$

Granted the above claim, then there exists a $t^* < \infty$ such that $t^* - s_n^*$ and $t_n^* - t^* \leqslant \rho^n$. Furthermore, from (4.10), we have

$$u(x,t^*) + u(x,s_n^*) - u(x,t^*) + c_v nT \leqslant v(x,nT) \leqslant u(x,t^*) + u(x,t_n^*) - u(x,t^*) + c_v nT.$$

Hence, Corollary 4.5 gives

$$\begin{aligned}&\|v(\cdot,nT) - u(x,t^*+nT)\|_{\mathbb{L}^\infty(\mathbb{R}^n)} \\ &\quad \leqslant \|u(\cdot,s_n^*) - u(\cdot,t^*)\|_{\mathbb{L}^\infty(\mathbb{R}^n)} + \|u(\cdot,t_n^*) - u(\cdot,t^*)\|_{\mathbb{L}^\infty(\mathbb{R}^n)} \leqslant 2C_2\rho^n\end{aligned}$$

which will lead to the stated exponential convergence.

Now we proceed to prove (4.11). Consider the time interval: $[nT, nT + \frac{T}{2}]$. Applying the same argument as that leading to (4.10), we obtain the following statement:

$$u\left(x, s_n^* + \frac{T}{2} + \epsilon_1\right) \leqslant v\left(x, nT + \frac{T}{2}\right) - c_v nT \leqslant u\left(x, t_n^* + \frac{T}{2} - \epsilon_2\right), \quad \text{for all } x \in \mathbb{R}^n$$

for some $\epsilon_1, \epsilon_2 > 0$ such that $s_n^* + \frac{T}{2} + \epsilon_1 \leqslant t_n^* + \frac{T}{2} - \epsilon_2$ and the equalities hold at some $x', x'' \in \mathbb{R}^n$.

Let $0 < \mu < 1$ be some fixed number (to be determined later). Consider the following two cases.

Case I. If $\epsilon_1 + \epsilon_2 \geqslant \mu\tau_n^*$, then applying strong comparison principle to (1.8) on the interval $[nT + \frac{T}{2}, (n+1)T]$, we have

$$u(x, s_n^* + T + \epsilon_1) < v(x,(n+1)T) - c_v nT < u(x, t_n^* + T - \epsilon_2) \quad \text{for all } x \in \mathbb{R}^n.$$

Hence $s_n^* + \epsilon_1 \leqslant s_{n+1}^* \leqslant t_{n+1}^* \leqslant t_n^* - \epsilon_2$ which leads to

$$\tau_{n+1}^* = t_{n+1}^* - s_{n+1}^* \leqslant t_n^* - \epsilon_2 - (s_n^* + \epsilon_1) \leqslant (1-\mu)\tau_n^*.$$

Setting $\rho = 1 - \mu$ gives the desired result.

Case II. If $\epsilon_1 + \epsilon_2 \leqslant \mu\tau_n^*$, then either $\epsilon_1 \leqslant \frac{\mu}{2}\tau_n^*$ or $\epsilon_2 \leqslant \frac{\mu}{2}\tau_n^*$. Consider the second case (the first can be treated similarly).

Let $\psi(x,t) = v(x, nT + t) - u(x, s_n^* + t) - c_v nT$. It solves a linear parabolic equation similar to (3.8) with smooth bounded coefficients. Then ψ has the following properties:

(1) $\psi(\cdot, 0) \geqslant 0$ and hence $\psi(\cdot, t) > 0$ for all $t > 0$.

(2) $0 \leqslant \psi(x,0) = v(x, nT) - u(x, s_n^*) - c_v nT \leqslant u(x, t_n^*) - u(x, s_n^*) \leqslant C_2\tau_n^*$. Hence,

$$\|\psi(\cdot,0)\|_{\mathbb{L}^\infty(\mathbb{R}^n)} \leqslant C_2\tau_n^*, \tag{4.12}$$

$$\text{and} \quad \left\|\nabla\psi\left(\cdot, \tfrac{T}{2}\right)\right\|_{\mathbb{L}^\infty(\mathbb{R}^n)} \leqslant C_3(T)\,\|\psi(\cdot,0)\|_{\mathbb{L}^\infty(\mathbb{R}^n)} \leqslant C_3(T)\tau_n^*, \tag{4.13}$$

where the first estimate comes from Corollary 4.5 and the second is a consequence of parabolic regularity – recall that $\psi(\cdot, t)$ is periodic in $x \in \mathbb{R}^n$.

Now the definition and assumption of ϵ_2 implies the existence of some $x'' \in \mathbb{R}^n$ such that

$$\begin{aligned}
\psi\left(x'', \frac{T}{2}\right) &= v\left(x'', nT + \frac{T}{2}\right) - u\left(x'', s_n^* + \frac{T}{2}\right) - c_v nT \\
&= u\left(x'', t_n^* + \frac{T}{2} - \epsilon_2\right) - u\left(x'', s_n^* + \frac{T}{2}\right) \\
&= u\left(x'', t_n^* + \frac{T}{2}\right) - u\left(x'', s_n^* + \frac{T}{2}\right) + u\left(x'', t_n^* + \frac{T}{2} - \epsilon_2\right) - u\left(x'', t_n^* + \frac{T}{2}\right) \\
&\geqslant C_1\tau_n^* - C_2\frac{\mu}{2}\tau_n^* \quad \text{(by Corollary 4.5)} \\
&\geqslant \left(C_1 - C_2\frac{\mu}{2}\right)\tau_n^*.
\end{aligned}$$

Upon choosing μ small enough, we get $\left\|\psi(\cdot, \frac{T}{2})\right\|_{\mathbb{L}^\infty} \geqslant C_3\tau_n^*$. This and the gradient bound in (4.13) implies the existence of a $C_4(T)$ such that *for all* $x \in \mathbb{R}^n$, it holds that $\psi(x, T) \geqslant C_4(T)\tau_n^*$. Without loss of generality, $C_4(T)$ can be chosen to be some small number. This leads to the following sequence of statements:

$$\begin{aligned}
v(x, nT + T) - u(x, s_n^* + T) - c_v nT &\geqslant C_4\tau_n^* \quad (\text{for all } x \in \mathbb{R}^n) \\
v(x, (n+1)T) - u(x, s_n^*) - c_v(n+1)T &\geqslant C_4\tau_n^* \\
v(x, (n+1)T) - c_v(n+1)T &\geqslant C_4\tau_n^* + u(x, s_n^*).
\end{aligned}$$

Now from Corollary 4.5, we deduce that $s_{n+1}^* \geqslant s_n^* + \delta_n^*$ for some $\delta_n^* > \frac{C_4}{C_2}\tau_n^*$. So we have

$$\tau_{n+1}^* = t_{n+1}^* - s_{n+1}^* \leqslant t_n^* - s_n^* - \delta_n^* \leqslant \tau_n^* - \frac{C_4}{C_1}\tau_n^* = \left(1 - \frac{C_4}{C_1}\right)\tau_n^*.$$

(Recall that C_4 can be chosen to be as small as possible.)

Finally, (4.11) follows upon choosing $\mu = \min(\frac{1}{2}, \frac{C_1}{C_2})$ and $\rho = \max(1 - \mu, 1 - \frac{C_4}{C_1})$. (It is clear that the choice of all the constants are independent of n.) □

For general $\mathbb{L}^\infty$ initial data defined on the whole space, the stability issue can be quite complicated. On the other hand, for compactly supported initial perturbation, analogous stability property might still be true. Due to length, we do not pursue to make this statement precise in this paper.

The next result indicates the stability of the pulsating wave with respect to the underlying medium. Due to the availability of the additional equation (4.3), the result is stronger in the case of $c_\nu \neq 0$.

Proposition 4.7 (*Stability of pulsating wave with respect to the inhomogeneity*) *Consider a sequence of inhomogeneous mediums f_i's and f satisfying condition* **A**. *Suppose $\|f_i - f\|_{C^2} \longrightarrow 0$. Let U_i and U be the pulsating waves for f_i and f with speed c_i and c (and the same normal direction ν). Then the following convergence statements hold:*

(i) $c_i \longrightarrow c$.

(ii) *If $c \neq 0$, then $U_i \longrightarrow U$ uniformly in $\mathbb{R}^n \times \mathbb{R}$.*

(iii) *If $c = 0$, then there exists a subsequence $u_{i_j}(x,t) = U_{i_j}(\mathcal{O}_\nu^T(x, c_\nu t))$ converging uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}$ to a solution of (1.8) for f.*

Proof (i) The convergence of the speed follows easily by considering the equation satisfied by $u_i - u$

$$\begin{aligned}\frac{d}{dt}(u_i(x,t) - u(x,t)) &= A_{f_i}(\nu, x, u_i) - A_f(\nu, x, u)\\ &= A_f(\nu, x, u_i) - A_f(\nu, x, u) + A_{f_i}(\nu, x, u_i) - A_f(\nu, x, u_i)\\ &= [D_u A_f](\nu, x, u_*)(u_i - u) + [D_f A_f]_{f_*}(\nu, x, u_i)(f_i - f),\end{aligned}$$

where $D_u A$ and $D_f A$ are the derivatives of A_f with respect to the arguments u and f. (In the above, we have used the mean value theorem for first-order Taylor expansion.) Gronwall's inequality gives $\|u_i(\cdot,t) - u(\cdot,t)\|_{\mathbb{L}^\infty(\mathbb{R}^n)} \leqslant C\,\|f_i - f\|_\infty e^{Ct}$ where the constant C depends on the C^2-norms of the f and f_i's. Hence for any large, but *fixed* T, we have $\|u_i(\cdot,T) - u(\cdot,T)\|_{\mathbb{L}^\infty(\mathbb{R}^n)} \longrightarrow 0$ which implies the convergence of the c_i's.

(ii) If $c \neq 0$, working directly in the transformed equation (4.3) shows that any limit of the U_i's satisfies the same equation as that for U. Uniqueness of U implies the result.

(iii) If $c = 0$, working instead in the original equation (1.8) implies that up to a subsequence, the u_i's converge uniformly in compact subsets in space–time and the limiting function satisfies (1.8) for the inhomogeneity function f. □

5 Examples of fingering and pinching

Here we give some examples in $\mathbb{R}^2$ of the formation of singularities for the mean curvature flow with forcing (1.1) when the forcing is not small.

5.1 Fingering with 'laminate' environment

By a laminate environment we mean a forcing of the form $f(x,u) = g(x)$. Even though simple, it can provide examples amenable to explicit computations which can still capture

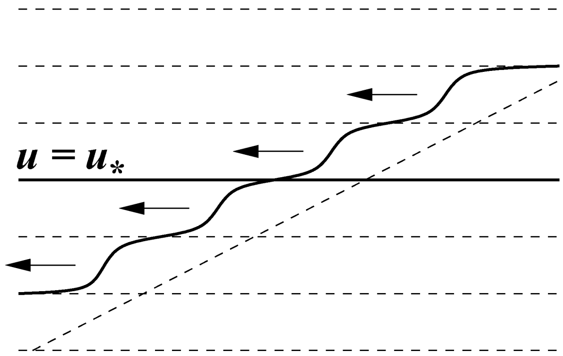

FIGURE 5. Pinned horizontal direction and travelling tilted direction.

some interesting features. Note that after a rotation by $\frac{\pi}{2}$, the forcing in the new frame is described by a function which depends only on the u-variable. This already indicates that questions on effective behaviours can depend crucially on the direction of the front.

Here we give an example that, in contrast to the effective speed c_ν, the *pinning threshold* h_c as mentioned earlier varies *discontinuously* with respect to the normal direction. Consider (after a rotation of the axis by $\frac{\pi}{2}$) $f(x,u) = sin(u) + \lambda + h$ where $0 < \lambda < 1$. If $h = 0$, then any constant function $u = u_*$ where u_* solves $sin(u_*) + \lambda = 0$ with $f_u(\cdot, u_*) = \cos(u_*) < 0$ is a *stable stationary* solution so that h_c must be *strictly positive* for this direction. On the other hand, fronts with any other directions will always have non-zero speed (unless $h = -\lambda$) as they can be approximated by travelling kinks (see Figure 5). (See also [5] for a similar result on a related discrete system.)

Another interesting phenomenon is 'fingering'. A precise analysis of such a situation has been carried out in details in [6], so we will just briefly explain the terminology. If $f(x,u)$ equals some periodic function $g(x)$ such that its amplitude is sufficiently large compared with the period, then the solution $u(x,t)$ starting from $u(x,0) \equiv 0$ remains as a graph, but it can happen that

$$\lim \inf_{[0,1]^n} u(x,t) \to -\infty \text{ as } t \to \infty, \quad \lim \sup_{[0,1]^n} u(x,t) \to +\infty \text{ as } t \to \infty.$$

(See Figure 6.) The solution in a sense can be described by a cascade of a series of translational invariant solitons, or 'grim-reapers'. In this case, it is not *a priori* clear what the 'effective front' should be.

5.2 Pinching with 'hard' obstacles

This section provides an example for the formation of another form of singularities. It can lead to the 'pinch-off' of a portion of the surface, reminiscent to the so-called Orowan loops in dislocation dynamics [21, pp. 624].

We consider strong, almost 'hard' circular obstacles. Consider three positive constants $\rho \ll 1$, and A and $B \gg 1$. Choose a smooth function $f(x,u)$ which satisfies

$$f(x,u) := \begin{cases} -B & \text{for} \quad x^2 + u^2 \leqslant \rho^2, \\ A & \text{for} \quad x^2 + u^2 > (2\rho)^2 \end{cases}$$

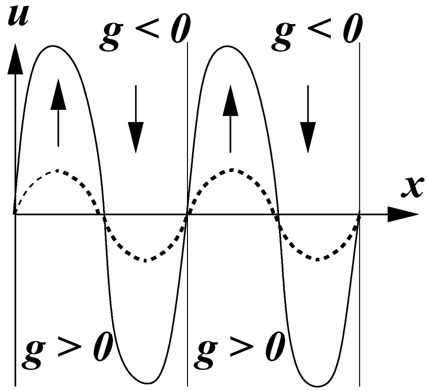

FIGURE 6. Fingering in a laminate: The vertical lines denote the period of $g(x)$.

(and is extended periodically in both the x and u-directions with period two). Let the initial data be $u_0(x) \equiv -1$. Now consider two types of evolving circles (see Figure 7).

Outer barrier. Consider the obstacle $S^- := \{(x,u) : x^2 + u^2 \leqslant \rho^2\}$ and the shrinking ball $S^-(t)$ centred at $(0,0)$ with radius $r_-(t)$ solving

$$\frac{d}{dt} r_-(t) = -\frac{1}{r_-(t)} + B, \quad r_-(0) = \rho.$$

Now $S^-(t)$ acts as barrier for the geometric problem, hence also as a barrier for the graph equation (1.8). The shrinking of $S^-(t)$ can be made arbitrarily slow if B is chosen appropriately ($B \approx \frac{1}{\rho}$).

Inner barrier. Consider a sequence of expanding circles centred at $(1, c_i)$ with $c_i \in [-1, 1]$ and radius denoted by $R_i(t)$. The centres are arranged in such a way that $c_1 < c_2 < \cdots$. These circles are used as inner barriers to the evolving solution $u(x,t)$. Their radii solve

$$\frac{d}{dt} R_i(t) = -\frac{1}{R_i(t)} + A, \quad \text{for } t \in [t_{i-1}, t_i)$$

which are further related by $R_i(t_{i-1}) = R_{i-1}(t_{i-1}) - (c_i - c_{i-1})$ so that only the ith circle is 'active' as a barrier during the time interval $[t_{i-1}, t_i]$. If the constant A and the initial R_i's are large enough, then the circles will expand. The t_i's are chosen such that the ith circle is allowed to continue to expand until it touches $S^-(t)$ at t_i. At this moment, a new circle with parameters c_{i+1} and R_{i+1} fits inside $R_i(t)$ and is used as initial datum for a new active barrier.

Since A is sufficiently large, each of the R_i is growing with speed bounded from below. Thus there exists a certain time T_* such that for some i, the R_i will touch its periodic extension on the vertical line $x \equiv 0$. If the motion of the outer barrier $S^-(t)$ is so slow that $r_-(T_*) > 0$, then the solution cannot remain as a graph, leading to an example of pinching.

The above pinching phenomena can certainly be handled by the level-set formulation as in (1.4). On the other hand, this example is not too much different from the fingering example. If the detached portion around the obstacle $S^-(t)$ persists for a long time, it can be viewed as a part of 'detached' fingers. In order to show a homogenisation result for

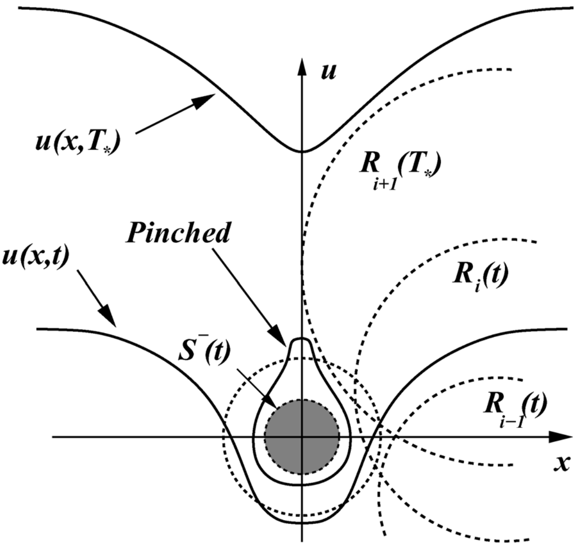

FIGURE 7. The solid lines show the expected behaviour of the solution $u(x, t)$ at different times; the dashed lines denote the outer- ($S^-(t)$) and inner- ($R_i(t)$) barriers.

such kind of situation, in a sense we still need a solution which remains bounded in some appropriate moving frame. This provides work for further investigation.

In summary, we have proved the existence of effective front and studied its property for a model of mean curvature flow in a periodic inhomogeneous medium. This is a difficult problem due to the non-linearity of the geometric motion and the interaction with the background environment. Our results work in the case of weak inhomogeneity. In general, intricate phenomenon such as pinning/de-pinning and fingering pattern formation can occur. Their understanding would require more detailed study.

Acknowledgements

The first author is partially supported by Focus Research Group 718 of the German Science Foundation (DFG), and he would like to thank Pierre Cardaliaguet and Takis Souganidis for helpful discussions on the homogenisation of forced mean curvature flow. The second author is supported by a Marie Curie International Reintegration Grant within the 7th European Community Framework Programme. The third author is partially supported by the Division of Mathematical Sciences of the National Science Foundation. The hospitality of the University of Crete, the Max-Planck Institute for Mathematics in the Sciences in Leipzig and Purdue University is also highly appreciated by the authors.

Appendix A Classical estimates for mean curvature flow with forcing

This appendix proves Theorems 2.7 and 2.8. Since we already have space–time uniform gradient estimates, we could in principle invoke well-known results for quasi-linear parabolic equations, in particular, the interior Schauder estimates to prove the existence of classical solutions starting from Lipschitz initial data. However, in order to take advantage of the structure of the equation and see how the constants are computed in the estimates, we will use a more geometric approach as in [12, 13, 17]. As the overall strategy is already presented quite clearly in the cited references, we only outline here the main steps needed in extending the results to handle equation (1.8).

Let $\{\Gamma(t) \subseteq \mathbb{R}^{n+1} : t \geqslant 0\}$ be parameterised as $\Gamma(p,t)$ so that its motion law is given by (1.1), i.e.

$$\frac{\partial}{\partial t}\Gamma(p,t) = V_N \sigma \quad (\sigma \text{ is the unit normal of } \Gamma(t)).$$

The following notations (with Einstein convention) will be used:

$$g_{ij} = \left\langle \frac{\partial \Gamma}{\partial p_i}, \frac{\partial \Gamma}{\partial p_j} \right\rangle \quad \text{(first fundamental form)}$$

$$\Big(\langle \cdot, \cdot \rangle = \text{standard inner product in } \mathbb{R}^{n+1} \Big)$$

$$g^{ij} = \text{inverse of } (g_{ij}), \text{ i.e. } g^{ik} g_{kj} = \delta^i_j$$

$$h_{ij} = -\left\langle \frac{\partial^2 \Gamma}{\partial p_i \partial p_j}, \sigma \right\rangle = \left\langle \frac{\partial \Gamma}{\partial p_i}, \frac{\partial \sigma}{\partial p_j} \right\rangle \quad \text{(second fundamental form)}$$

$$H = g^{ij} h_{ij} \quad \text{(mean curvature)}$$

$$|A|^2 = g^{ij} g^{kl} h_{ik} h_{jl}$$

$$\nabla_\Gamma = \text{gradient operator on the tangent space of } \Gamma$$

$$(\nabla_\Gamma \varphi = g^{ij} [\partial_{p_i} \varphi] \partial_{p_j} \Gamma)$$

$$\triangle_\Gamma = \text{Laplace Beltrami operator on the tangent space of } \Gamma$$

$$\left(\triangle_\Gamma \varphi = \frac{1}{\sqrt{g}} \partial_{p_i} [\sqrt{g} g^{ij} \partial_{p_j} \varphi] \right)$$

where φ is an arbitrary function defined on Γ and $g = \det(g_{ij})$.

Let $\Gamma_t = \{(x, u(x,t)) : x \in \mathbb{R}^n\}$ so that it has a graphical representation over a hyperplane with normal vector $v = (0, \ldots, 0, 1)$, then we have the following explicit formulas (∇ is the gradient operator with respect to $x \in \mathbb{R}^n$)

$$\sigma = \frac{1}{z}(-\nabla u, 1), \quad g_{ij} = \delta_{ij} + u_{x_i} u_{x_j}, \quad g^{ij} = \delta^{ij} - \eta_i \eta_j, \quad h_{ij} = -\frac{1}{z} u_{x_i x_j} \tag{A 1}$$

where $z = \sqrt{1 + |\nabla u|^2} = \langle \sigma, v \rangle^{-1}$ and $\eta = \frac{u_{x_i}}{z}$. Furthermore,

$$H = -\frac{1}{z}\left(\delta^{ij} - \eta_i \eta_j\right) u_{x_i x_j} \quad \text{and} \quad |A|^2 = \frac{1}{z^2} g^{ij} g^{kl} u_{ik} u_{jl}.$$

We further remark that up to tangential diffeomorphism, the geometric evolution (1.1) is equivalent to the graph equation (1.8) (see [13, pp. 549]) which is written again here in the following form:

$$u_t = g^{ij} u_{x_i x_j} + \delta \sqrt{1 + |\nabla u|^2} f(\mathcal{O}_v(x,u)^T). \tag{A 2}$$

For simplicity, we set $\delta = 1$. As seen in the following derivation, this will not affect the result, as the smallness of δ is only used in deriving the gradient bound in Theorem 2.4. Once this is done or assumed, δ does not play a role in deriving higher regularity.

We now write down the evolution equations for the important geometric quantities relevant for our estimates. In the following, the symbol $C(\cdot)$ denotes some general constant which might depend on its argument(s). Recall that $F = \|f\|_{C^2(\mathbb{R}^{n+1})}$.

$$\begin{aligned}
\frac{\partial\sigma}{\partial t} &= g^{ij}\left\langle \frac{\partial\sigma}{\partial t}, \frac{\partial\Gamma}{\partial p_i}\right\rangle \frac{\partial\Gamma}{\partial p_j} = -g^{ij}\left\langle \sigma, \frac{\partial}{\partial p_i}\left(\frac{\partial\Gamma}{\partial t}\right)\right\rangle \frac{\partial\Gamma}{\partial p_j}\\
&= -g^{ij}\left\langle \sigma, \frac{\partial}{\partial p_i}(V_N\sigma)\right\rangle \frac{\partial\Gamma}{\partial p_j} = -g^{ij}\frac{\partial V_N}{\partial p_i}\frac{\partial\Gamma}{\partial p_j} = -\nabla_\Gamma V_N = \nabla_\Gamma H - \nabla_\Gamma f \qquad \text{(A 3)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= \left\langle \frac{\partial}{\partial p_i}\left(\frac{\partial\Gamma}{\partial t}\right), \frac{\partial\Gamma}{\partial p_j}\right\rangle + \left\langle \frac{\partial\Gamma}{\partial p_i}, \frac{\partial}{\partial p_j}\left(\frac{\partial\Gamma}{\partial t}\right)\right\rangle\\
&= \left\langle \frac{\partial}{\partial p_i}(V_N\sigma), \frac{\partial\Gamma}{\partial p_j}\right\rangle + \left\langle \frac{\partial\Gamma}{\partial p_i}, \frac{\partial}{\partial p_j}(V_N\sigma)\right\rangle = V_N\left\langle \frac{\partial\sigma}{\partial p_i}, \frac{\partial\Gamma}{\partial p_j}\right\rangle + V_N\left\langle \frac{\partial\Gamma}{\partial p_i}, \frac{\partial\sigma}{\partial p_j}\right\rangle\\
&= 2V_N h_{ij} = (-2H + 2f)h_{ij} \qquad \text{(A 4)}
\end{aligned}$$

$$\frac{\partial g^{ij}}{\partial t} = -g^{ik}\frac{\partial g_{kl}}{\partial t}g^{lj} = -2V_N g^{ik}g^{lj}h_{kl} = (2H - 2f)g^{ik}g^{lj}h_{kl} \qquad \text{(A 5)}$$

$$\begin{aligned}
\frac{\partial h_{ij}}{\partial t} &= -\frac{\partial}{\partial t}\left\langle \frac{\partial^2\Gamma}{\partial p_i\partial p_j}, \sigma\right\rangle = \left\langle \frac{\partial^2}{\partial p_i\partial p_j}\left(-\frac{\partial\Gamma}{\partial t}\right), \sigma\right\rangle - \left\langle \frac{\partial^2\Gamma}{\partial p_i\partial p_j}, \frac{\partial\sigma}{\partial t}\right\rangle\\
&= \left\langle \frac{\partial^2}{\partial p_i\partial p_j}(-V_N\sigma), \sigma\right\rangle - \left\langle \frac{\partial^2\Gamma}{\partial p_i\partial p_j}, -\nabla_\Gamma V_N\right\rangle\\
&= \left\langle \frac{\partial^2}{\partial p_i\partial p_j}(H\sigma), \sigma\right\rangle - \left\langle \frac{\partial^2\Gamma}{\partial p_i\partial p_j}, \nabla_\Gamma H\right\rangle - \left\langle \frac{\partial^2}{\partial p_i\partial p_j}(f\sigma), \sigma\right\rangle + \left\langle \frac{\partial^2\Gamma}{\partial p_i\partial p_j}, \nabla_\Gamma f\right\rangle\\
&= \triangle_\Gamma h_{ij} - 2Hg^{lm}h_{il}h_{mj} + |A|^2 h_{ij} - \left\langle \frac{\partial^2}{\partial p_i\partial p_j}(f\sigma), \sigma\right\rangle + \left\langle \frac{\partial^2\Gamma}{\partial p_i\partial p_j}, \nabla_\Gamma f\right\rangle \qquad \text{(A 6)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial |A|^2}{\partial t} &= \frac{\partial}{\partial t}(g^{ik}g^{jl}h_{ij}h_{kl}) = g^{ik}_{,t}g^{jl}h_{ij}h_{kl} + g^{ik}g^{jl}_{,t}h_{ij}h_{kl} + g^{ik}g^{jl}(h_{ij})_{,t}h_{kl} + g^{ik}g^{jl}h_{ij}(h_{kl})_{,t}\\
&= \triangle_\Gamma |A|^2 - 2|\nabla_\Gamma A|^2 + 2|A|^4 + (I) + (II) \qquad \text{(A 7)}
\end{aligned}$$

where

$$\begin{aligned}
(I) &= -2fh^{ik}g^{jl}h_{ij}h_{kl} - 2fh^{jl}g^{ik}h_{ij}h_{kl} \quad \text{so that} \quad |(I)| \leqslant C(F)|A|^3\\
(II) &= g^{ik}g^{jl}h_{kl}\left(-\left\langle \frac{\partial^2}{\partial p_i\partial p_j}(f\sigma), \sigma\right\rangle + \left\langle \frac{\partial^2\Gamma}{\partial p_i\partial p_j}, \nabla_\Gamma f\right\rangle\right)\\
&\quad + g^{ik}g^{jl}h_{ij}\left(-\left\langle \frac{\partial^2}{\partial p_k\partial p_l}(f\sigma), \sigma\right\rangle + \left\langle \frac{\partial^2\Gamma}{\partial p_k\partial p_l}, \nabla_\Gamma f\right\rangle\right).
\end{aligned}$$

Note that

$$\begin{aligned}
\left\langle \frac{\partial^2}{\partial p_k\partial p_l}(f\sigma), \sigma\right\rangle &= \left\langle \frac{\partial^2 f}{\partial p_k\partial p_l}\sigma + \frac{\partial f}{\partial p_k}\frac{\partial\sigma}{\partial p_l} + \frac{\partial f}{\partial p_l}\frac{\partial\sigma}{\partial p_k} + f\frac{\partial^2\sigma}{\partial p_k\partial p_l}, \sigma\right\rangle\\
&= \frac{\partial^2 f}{\partial p_k\partial p_l} - f\left\langle \frac{\partial\sigma}{\partial p_k}, \frac{\partial\sigma}{\partial p_l}\right\rangle
\end{aligned}$$

so that

$$|(II)| \leqslant C(F)\,|A| + C(F).$$

Hence we have

$$\frac{\partial\,|A|^2}{\partial t} \leqslant \triangle_\Gamma\,|A|^2 - 2\,|\nabla_\Gamma A|^2 + 2\,|A|^4 + C(F)\,|A|^3 + C(F). \tag{A 8}$$

Finally, we need the evolution equation for z

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial t}\,\langle \sigma, \nu\rangle^{-1} = -z^2\,\langle \partial_t \sigma, \nu\rangle = z^2\,\langle -\nabla_\Gamma H, \nu\rangle + z^2\,\langle \nabla_\Gamma f, \nu\rangle \tag{A 9}$$

so that

$$\frac{\partial z}{\partial t} \leqslant \triangle_\Gamma z - |A|^2\,z - \frac{2}{z}\,|\nabla_\Gamma z|^2 + C(F)z^2. \tag{A 10}$$

Now we are ready to prove the stated Theorems.

A.1 Proof of Theorem 2.7 – Existence of classical solution

The main point here is that the initial data is only assumed to be Lipschitz. In order to prove the existence of classical solution, we need *a priori* estimates for the second derivatives or equivalently, the second fundamental form. This is provided by the following lemma on the *interior in time estimate* for the curvature.

Lemma A.1 *Let $\{\Gamma_t : t \geqslant 0\}$ be a classical solution of (1.8) such that $\|z\|_{\mathbb{L}^\infty(\mathbb{R}^n\times\mathbb{R}_+)} \leqslant N_* < \infty$. Then, for all $T > 0$, there exists a constant $C(N_*, F, T)$ such that for $0 \leqslant t \leqslant T$,*

$$\|\,|A|^2\,(\cdot, t)\|_{\mathbb{L}^\infty(\mathbb{R}^n)} \leqslant C(N_*, F, T)\frac{1}{t}. \tag{A 11}$$

Proof The proof follows very much the strategy of [13, Theorem 3.1]. Hence only the key steps will be outlined.

Let φ be a positive increasing function (to be determined). Then

$$\begin{aligned}
&(\partial_t - \triangle_\Gamma)\left[|A|^2\,\varphi(z^2)\right]\\
&\quad= |A|^2\,\varphi'(z^2)2zz_t + \varphi(z^2)(|A|^2)_t - \nabla_\Gamma\,\left[|A|^2\varphi'(z^2)2z\nabla_\Gamma z + \varphi(z^2)\nabla_\Gamma(|A|^2)\right]\\
&\quad= |A|^2\,\varphi'(z^2)2z\,[z_t - \triangle_\Gamma z] + \varphi(z^2)[(|A|^2)_t - \triangle_\Gamma|A|^2]\\
&\qquad - 4\varphi'(z^2)z\,\left\langle \nabla_\Gamma|A|^2, \nabla_\Gamma z\right\rangle - |A|^2\,\varphi''(z^2)4z^2\,|\nabla_\Gamma z|^2 - |A|^2\,\varphi'(z^2)2\,|\nabla_\Gamma z|^2\\
&\quad\leqslant 2\left[\varphi(z^2) - \varphi'(z^2)z^2\right]|A|^4 - 2\varphi(z^2)\,|\nabla_\Gamma A|^2 - (6\varphi'(z^2) + 4\varphi''(z^2)z^2)\,|A|^2\,|\nabla_\Gamma z|^2\\
&\qquad - 2\left\langle \nabla_\Gamma|A|^2, \nabla_\Gamma\varphi(z^2)\right\rangle + C(F)z^3\varphi'(z^2)\,|A|^2 + C(F)\varphi(z^2)[|A|^3 + 1]\\
&\quad\leqslant 2(\varphi - \varphi'z^2)\,|A|^4 - \varphi^{-1}\langle \nabla_\Gamma\varphi, \nabla_\Gamma(|A|^2\,\varphi)\rangle - (6\varphi'(1 - \varphi^{-1}\varphi'z^2) + 4\varphi''z^2)\,|A|^2\,|\nabla_\Gamma z|^2\\
&\qquad + C(F)z^3\varphi'\,|A|^2 + C(F)\varphi(z^2)[|A|^3 + 1].
\end{aligned}$$

Upon choosing $\varphi(s) = (s/(1-ks))$ where k is some small positive number (to be determined), we have

$$(\partial_t - \triangle_\Gamma)[|A|^2 \varphi] \leqslant -k[|A|^2 \varphi]^2 - \varphi^{-1}\langle \nabla_\Gamma \varphi, \nabla_\Gamma (|A|^2 \varphi)\rangle + C(F) z^3 \varphi' |A|^2 + C(F)\varphi[|A|^3 + 1].$$

As $\|z\|_\infty \leqslant N_* < \infty$, we can choose $k = k(N_*)$ small enough that $\varphi(z^2)$ and its derivatives are all uniformly bounded. Note that $\varphi(z^2)$ is also bounded from below as $z \geqslant 1$. The presence of $-k[|A|^2 \varphi]^2$ is crucial. It is the reflection of the fact that the equation is uniformly parabolic (as the gradient is assumed to be uniformly bounded). It can be used to absorb the $|A|^2$ and $|A|^3$ terms. By introducing $B = |A|^2 \varphi(z^2)$ and changing the constants, we thus arrive at

$$(\partial_t - \triangle_\Gamma) B \leqslant -kB^2 - \varphi^{-1} \langle \nabla_\Gamma \varphi, \nabla_\Gamma B\rangle + C(F, N_*).$$

Now consider the equation satisfied by the quantity tB

$$(\partial_t - \triangle_\Gamma)(tB) = t(\partial_t - \triangle_\Gamma)B + B \leqslant -ktB^2 - t\varphi^{-1} \langle \nabla_\Gamma \varphi, \nabla_\Gamma B\rangle + tC(N_*, F) + B.$$

Furthermore, the quantity $\varphi(z^2)$ satisfies

$$(\partial_t - \triangle_\Gamma)\, \varphi \leqslant -2z^2 \varphi' |A|^2 + C(N_*, F) z^3 \varphi'.$$

Thus we have

$$\begin{aligned}(\partial_t - \triangle_\Gamma)(tB + \varphi) &\leqslant -ktB^2 + B - \varphi^{-1} \langle \nabla_\Gamma \varphi, \nabla_\Gamma (tB + \varphi)\rangle + C(N_*, F)(|A|^2 + 1) + tC(N_*, F) \\ &\leqslant -ktB^2 + C(N_*, F)B - \varphi^{-1} \langle \nabla_\Gamma \varphi, \nabla_\Gamma (tB + \varphi)\rangle + tC(N_*, F) \\ &\leqslant -kB\,[tB - C(N_*, F)B] - \varphi^{-1} \langle \nabla_\Gamma \varphi, \nabla_\Gamma (tB + \varphi)\rangle + tC(N_*, F)\end{aligned}$$

(where in the above we have made used of the fact that $|\nabla_\Gamma \varphi|^2 \leqslant C(N_*)\,|A|^2$).

Now suppose $\sup_{t\in[0,T]}[tB + \varphi(z^2)]$ equals some constant $M > 0$. Assume that the sup is attained at p_* and t_*. Then we have

$$0 \leqslant -k\left(\frac{M - \varphi(z^2)}{t}\right)\left[M - \varphi(z^2) - C\right] + TC|_{(p_*, t_*)}$$

which leads to a contradiction upon choosing $M = M(N_*, T, F)$ large enough. The argument can be localized in space as done in [13] or we can also use the similar device as in p. 10 by choosing appropriate $(p_*^{(j)}, t_*^{(j)})$'s such that $tB + \varphi$ converges to the sup. The same proof then goes through.

The desired interior in time estimate (A 11) is thus established. □

With the above *a priori* bounds, Theorem 2.7 can be proved using approximation of the initial data. For smooth initial data, the result follows by Schauder Fixed Point Theorem. The estimates of Corollary 2.5 lead to uniform gradient bound which then gives a curvature bound which depends only on the gradient. The local-in-time existence and uniqueness of classical solutions then follow easily from standard arguments. The global-in-time existence follows from the combination of uniform oscillation and gradient bounds as explained in Remark 2.6-1.

A.2 Proof of Theorem 2.8 – Gradient decay estimate

The technique is initiated by [18] for the elliptic case. The computation here follows closely to that of [15, Theorem 5.2], making use of the graph equation (1.8) (or (A 2)).

We first recall the notations of (A 1). Furthermore, let $\|D^2u\|^2 = \sum_{ij} u^2_{x_ix_j}$. Then (A 9) takes the following analogous form:

$$z_t = g^{ij}z_{x_ix_j} - \frac{1}{z}g^{ij}g^{kl}u_{x_kx_i}u_{x_lx_j} - \frac{2}{z}g^{ij}\eta^k\eta^l u_{x_ix_k}u_{x_jx_l} + \eta^k\left(\delta z f\right)_{x_k} \tag{A 12}$$

or more compactly written as

$$\begin{aligned} z_t = {} & g^{ij}z_{x_ix_j} - \frac{\|D^2u\|^2}{z} + \frac{\langle \nabla u, \nabla z\rangle^2}{z^3} \\ & + \delta\left\{\frac{\langle \nabla u, \nabla z\rangle}{z}f(x,u) + \langle \nabla u, \nabla_x f(x,u)\rangle + |\nabla u|^2 f_u(x,u)\right\}. \end{aligned} \tag{A 13}$$

Furthermore, the symmetric matrix $\tilde{G} = (g^{ij})$ satisfies

$$\tilde{G} = I - \eta\otimes\eta = (1-|\eta|^2)I + |\eta|^2\left(I - \frac{\eta}{|\eta|}\otimes\frac{\eta}{|\eta|}\right) \geqslant (1-|\eta|^2)I = \frac{1}{z^2}I \tag{A 14}$$

so that $\tilde{G}$ is positive definite with its smallest eigenvalues equal to $\frac{1}{z^2}$.

Now we proceed to prove the theorem. Without loss of generality, we restrict our attention to $\Omega_{R,T} = \{(x,t) : 0 \leqslant |x| \leqslant R, 0 < t < T\}$. By adding a constant to u, we can assume $-3M \leqslant u \leqslant -M < 0$ so that $\|u\|_{\mathbb{L}^\infty(\mathbb{R}\times\mathbb{R}_+)} \leqslant 3M$. Let $u(0,T) = -m < 0$.

Define $h(x,t) = \rho(x,t)z(x,t)$ where

$$\rho(x,t) = e^{K\phi(x,t)} - 1 \quad \text{and} \quad \phi(x,t) = \left[\frac{u(x,t)}{2m} + \frac{t}{T}\left(1 - \frac{|x|^2}{R^2}\right)\right]^+$$

and the constant K is to be determined.

Consider the expression $Lh = g^{ij}h_{x_ix_j} - h_t$ which equals $\rho Lz + zL\rho + 2(\rho)_{x_i}z_{x_i} - \eta^i\eta^j(\rho)_{x_i}z_{x_j} - \eta^i\eta^j(\rho)_{x_j}z_{x_i}$. As $(\rho)_{x_i} = \frac{h_{x_i}-\rho z_{x_i}}{z}$, we have

$$Lh - 2\frac{g^{ij}}{z}z_jh_i = \rho\left(Lz - 2\frac{g^{ij}}{z}z_iz_j\right) + zL\rho.$$

Now estimate $Lz - 2\frac{g^{ij}}{z} z_{x_i} z_{x_j}$ and $L\rho$. Using (A 12) and (A 14), the former is estimated as

$$\begin{aligned} &Lz - 2\frac{g^{ij}}{z} z_{x_i} z_{x_j} \\ &= \frac{1}{z} g^{ij} g^{kl} u_{x_k x_i} u_{x_l x_j} + \frac{2}{z} g^{ij} \eta^k \eta^l u_{x_i x_k} u_{x_l x_j} - \eta^k (\delta z f)_k - 2\frac{g^{ij}}{z} \eta^k \eta^l u_{x_k x_i} u_{x_l x_j} \\ &\geqslant \frac{1}{z^5} \|D^2 u\|^2 - \delta[\langle [D^2 u]\eta, \eta \rangle f + \langle \nabla u, \nabla_x f\rangle + |\nabla u|^2 f_u]. \end{aligned}$$

Furthermore, as $|\langle [D^2 u]\eta, \eta\rangle| \leqslant \frac{\|D^2 u\| |\nabla u|^2}{z^2} = \frac{\|D^2 u\|}{z^{5/2}} |\nabla u|^2 z^{\frac{1}{2}}$, we have $\delta \langle D^2 u \eta, \eta\rangle f \leqslant \frac{1}{2} \frac{\|D^2 u\|^2}{z^5} + \frac{1}{2}\delta^2 z^5 C(F)$ which gives

$$Lz - 2\frac{g^{ij}}{z} z_i z_j \geqslant \frac{1}{2} \frac{\|D^2 u\|^2}{z^5} - C(F)\delta z^2 (1 + \delta z^3). \tag{A 15}$$

For $L\rho$, we have

$$\begin{aligned} L\rho &= K^2 e^{K\phi} g^{ij} (\phi)_{x_i} (\phi)_{x_j} + K e^{K\phi} \left(g^{ij} (\phi)_{x_i x_j} - (\phi)_t \right) \\ &\geqslant \frac{K^2 e^{K\phi}}{z^2} |\nabla(\phi)|^2 + K e^{K\phi} \left(g^{ij} (\phi)_{x_i x_j} - (\phi)_t \right) \qquad \text{(from (A 14))}. \end{aligned}$$

Note that

$$(\phi)_{x_i} = \frac{u_{x_i}}{2m} - \frac{2t x_i}{T R^2}, \qquad (\phi)_{x_i x_j} = \frac{u_{x_i x_j}}{2m}, \qquad (\phi)_t = \frac{u_t}{2m} + \frac{1}{T}\left(1 - \frac{|x|^2}{R^2}\right).$$

Hence

$$L\rho \geqslant K^2 \frac{e^{K\phi}}{z^2} \left| \frac{\nabla u}{2m} - \frac{2tx}{T R^2} \right|^2 + K e^{K\phi} \left[-\frac{\delta z f}{2m} - \frac{1}{T}\left(1 - \frac{|x|^2}{R^2}\right)\right]. \tag{A 16}$$

Combining (A 15) and (A 16), we sequentially estimate Lh

$$\begin{aligned} &Lh - 2\frac{g^{ij}}{z} z_j h_i \\ &\geqslant z L\rho - \rho C(F)\delta z^2 (1 + \delta z^3) \\ &\geqslant \rho \left\{ \frac{K^2}{4m^2 z^2} \left[|\nabla u|^2 - \frac{16m^2}{R^2} \right] - K \left[\frac{\delta z C(F)}{2m} + \frac{1}{T}\left(1 - \frac{|x|^2}{R^2}\right)\right] - \delta C(F) z (1 + \delta z^3) \right\}. \end{aligned}$$

Define the set

$$\mathscr{D} = \left\{ (x,t) \in \Omega_{R,T} : z^2(x,t) \geqslant 2\left(1 + \frac{16m^2}{R^2}\right)\right\}. \tag{A 17}$$

On $\mathscr{D}$, we have

$$\begin{aligned} Lh - 2\frac{g^{ij}}{z} z_j h_i &\geqslant z e^{K\phi} \left\{ \frac{K^2}{8m^2} - K\left(\frac{\delta z C(F)}{2m} + \frac{1}{T}\right) - \delta C(F) z (1 + \delta z^3) \right\} \\ &\geqslant z e^{K\phi} \left\{ \frac{K^2}{16m^2} - 4\delta^2 z^2 C(F) - \frac{16m^2}{T^2} - \delta C(F) z (1 + \delta z^3) \right\}. \end{aligned}$$

Let $z^* = \sup_{t\in[0,T]} \|z(t)\|_\infty$. If we choose

$$K \geqslant \frac{4m}{\sqrt{3}}\left(\frac{4m}{T} + C(F)\sqrt{\delta z^*} + C(F)\delta z^* + C(F)\delta z^{*2}\right)$$

then $Lh - 2\frac{g^{ij}}{z} z_j h_i \geqslant 0$.

Observe that $(0,T) \in \operatorname{spt}\phi$. Assume $(0,T) \in \mathscr{D}$, otherwise $z(0,T) \leqslant \sqrt{2}\left(1+\frac{4m}{R}\right)$. By maximum principle, we have $h(0,T) \leqslant \max_{(x,t)\in\partial\mathscr{D}} h(x,t)$, i.e.

$$\left(e^{K(-\frac{1}{2}+1)} - 1\right) z(0,T) \leqslant \max_{(x,t)\in\partial\mathscr{D}} \left(e^{K\left[\frac{u}{2m}+\frac{t}{T}\left(1-\frac{|x|^2}{R^2}\right)\right]^+} - 1\right) z(x,t)$$
$$\leqslant \left(e^K - 1\right)\sqrt{2}\left(1+\frac{4m}{R}\right)$$

leading to

$$z(0,T) \leqslant \sqrt{2}\left(1+\frac{4m}{R}\right)\left\{\exp\left(\frac{2m}{\sqrt{3}}\left[\frac{4m}{T} + C(F)\sqrt{\delta z^*} + C(F)\delta z^* + C(F)\delta z^{*2}\right]\right) + 1\right\}. \tag{A 18}$$

Now let $N_0 = \|z(0)\|_\infty$. By equation (2.7), we have $z^* \leqslant \bar{Z} := N_0 + C(F)\sqrt{\delta}N_0^2(1+M)$ for $\delta \leqslant \delta_1(N_0, M, F)$. Hence the result will follow if we choose N_0 such that

$$\sqrt{2}\left(1+\frac{4M}{R}\right)\left\{\exp\left[\frac{2M}{\sqrt{3}}\left(\frac{4M}{T} + C(F)(\sqrt{\delta\bar{Z}} + \delta\bar{Z} + \delta\bar{Z}^2)\right)\right] + 1\right\} \leqslant \frac{N_0}{2}. \tag{A 19}$$

If δ is small enough, such a choice for N_0 is always possible and it can be bounded from below and above by two constants $N_1(T,M,F)$ and $N_2(T,M,F)$. The theorem is thus proved.

References

[1] Allen, S. & Cahn, J. (1979) A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Metall.* **27**, 1084–1095.

[2] Bhattacharya, K. & Craciun, B. (2003) Homogenization of a Hamilton–Jacobi equation associated with the geometric motion of an interface. *Proc. Royal. Soc. Edin.* **133A**, 773–805.

[3] Bhattacharya, K. & Craciun, B. (2004) Effective motion of a curvature-sensitive interface through a heterogeneous medium. *Interfaces Free Bound.* **6**, 151–173.

[4] Caffarelli, L. & De la Llave, R. (2001) Plane-like minimizers in periodic media. *Comm. Pure Appl. Math.* **54**(12), 1403–1441.

[5] Cahn, J. W., Mallet-Paret, J. & Van Vleck, E. S. (1999) Traveling wave solutions for systems of ODEs on a two-dimensional spatial lattice. *SIAM J. Appl. Math.* **59**, 455–493.

[6] Cardaliaguet, P., Lions, P. L. & Souganidis, P. E. (2007) A discussion about the homogenization of moving interfaces, preprint.

[7] Chen, Y. G., Giga, Y. & Goto, S. (1991) Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Diff. Geom.* **33**, 749–786.

[8] Crandall, M. G., Ishii, H. & Lions, P.-L. (1992) User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)* **27**, 1–67.

[9] CARDALIAGUET, P., DA LIO, F., FORCADEL, N. & MONNEAU, R. (2007) Dislocation dynamics: A non-local moving boundary. In *Free Boundary Problems*, Internat. Ser. Numer. Math., Vol. 154, Birkhäuser, Basel, pp. 125–135.

[10] DIRR, N., LUCIA, M. & NOVAGA, M. (2006) Γ-convergence of the Allen–Cahn energy with an oscillating forcing term. *Interfaces Free Bound.* **8**(1), 47–78.

[11] DIRR, N. & YIP, N. K. (2006) Pinning and de-pinning phenomena in front propagation in heterogeneous media. *Interfaces Free Bound.* **8**, 79–109.

[12] ECKER, K. & HUISKEN, G. (1989) Mean curvature evolution of entire graphs. *Ann. Math.* **130**, 453–471.

[13] ECKER, K. & HUISKEN, G. (1991) Interior estimates for hypersurfaces moving by mean curvature. *Invent. Math.* **105**, 547–569.

[14] EVANS, L. C. & SPRUCK, J. (1991) Motion of level sets by mean curvature. I. *J. Differential Geom.* **33**, 635–681.

[15] EVANS, L. C. & SPRUCK, J. (1992) Motion of level sets by mean curvature. III. *J. Geom. Anal.* **2**, 121–150.

[16] FRIEDMAN, A. (1958) Remarks on the maximum principle for parabolic equations and its applications. *Pacific J. Math.* **8**, 201–211.

[17] HUISKEN, G. (1984) Flow by mean curvature of convex surfaces into spheres. *J. Diff. Geom.* **20**, 237–266.

[18] KOREVAAR, N. (1986) An easy proof of the interior gradient bound for solutions to the prescribed mean curvature equation, Nonlinear functional analysis and its applications, Part 2. *Proc. Sympos. Pure Math. Part 2, Amer. Math. Soc.* **45**, 81–89,

[19] LIONS, P.-L. & SOUGANIDIS, P. E. (2005) Homogenization of degenerate second-order PDE in periodic and almost periodic environments and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22**, 667–677.

[20] NAMAH, G. & ROQUEJOFFRE, J. M. (1997) Convergence to periodic fronts in a class of semi-linear parabolic equations. *Nonlinear Diff. Eq. Appl.* **4**, 521–536.

[21] PHILLIPS, R. (2001) *Crystals, Defects and Microstructures*, Cambridge, UK, Cambridge University Press.

[22] SANDIER, E. & SERFATY S. (2004) Gamma-convergence of gradient flows with applications to Ginzburg–Landau. *Comm. Pure Appl. Math.* **57**, 1627–1672.