FAST-TRACK COMMUNICATION

Fuzzy control of a three-tank system

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Abstract: Fuzzy inverse reasoning performed using fuzzy relational equations can be employed to deduce control actions appropriate for a desired process output. The three-tank plant to be controlled has two inputs and hence the fuzzy relational equation describing the dynamics of the plant is a two-decision-variable equation. Algorithms are only available for solving equations that have a single decision variable. A method is proposed in this paper to decompose the two-decision-variable fuzzy relational equation into single-decision-variable equations so that existing algorithms can be applied to produce control actions for the plant.

Keywords: fuzzy logic, inverse reasoning, process control

NOTATION

az_i	opening of the ith valve
A	cross-sectional area of a tank
$\boldsymbol{A}, \boldsymbol{A}', \boldsymbol{C}, \boldsymbol{C}'$	fuzzy sets
$\boldsymbol{A}^i, \boldsymbol{B}^j, \boldsymbol{C}^k, \boldsymbol{D}^l$	fuzzy sets
g	gravitational constant
h_i	liquid level in the ith tank
$\dot{h}_i$	derivative of h_i with respect to time
$h_i^j, \dot{h}_i^j, q_i^j$	degrees of membership
$\boldsymbol{H}_i, \dot{\boldsymbol{H}}_i, \boldsymbol{Q}_i$	fuzzy variables corresponding to h_i, $\dot{h}_i$ and q_i
M_p	maximum overshoot
q_i	inlet flowrate for the ith tank
r_{ij}	element of $\mathbf{R}'$
r_{ijkl}	element of $\mathbf{R}$
$\mathbf{R}, \mathbf{R}', \mathbf{R}_i, \mathbf{R}'_i$	fuzzy relations or fuzzy relational matrices
S_l	cross-sectional area of the outlet pipe
S_n	cross-sectional area of the connecting pipes
$u_i(t)$	control action of the ith control loop
x_{ijk}	element of $\mathbf{X}$
$\mathbf{X}$	input matrix of a multiple-decision-variable fuzzy relational equation
$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}, \boldsymbol{Y}$	fuzzy variables
$y_i(t)$	output response of the ith control loop
$\boldsymbol{Y}_d$	fuzzified desired process output
ε	steady state error
τ	rise time

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1 INTRODUCTION

Fuzzy reasoning can be classified as fuzzy forward reasoning and fuzzy inverse reasoning. Forward reasoning involves finding a logical consequence of a given condition. It starts by comparing the given condition and the antecedent part of a fuzzy rule. If they match, the conclusion part of the fuzzy rule will be taken as a consequence of the given condition. Fuzzy inverse reasoning (FIR) aims to deduce a sufficient condition for a specified conclusion called a goal. It searches for a sufficient condition, $\boldsymbol{A}'$, through fuzzy relations such as $\boldsymbol{A} \Rightarrow \boldsymbol{C}$. If $\boldsymbol{A}'$ is found, the given goal is considered true. FIR can be expressed as follows [**1**, **2**]:

$$\begin{array}{l} \textit{goal}: \ y \textit{ is } \boldsymbol{C}' \\ \textit{fuzzy relation}: \ \textit{If } x \textit{ is } \boldsymbol{A} \textit{ Then } y \textit{ is } \boldsymbol{C} \\ \hline \textit{sufficient condition}: \ x \textit{ is } \boldsymbol{A}' \end{array} \tag{1}$$

Consider that the given goal is a desired process output and the sufficient condition to be derived is a control action. FIR can thus be used to produce a control action for a specified process output. The technique of determining control actions by using FIR is called FIR control or fuzzy backward reasoning control (FBRC) [**3**]. In FIR control, the fuzzy relation represents the dynamics of a controlled process, the output of which can be expressed using a fuzzy relational equation such as

$$\boldsymbol{X} \circ \mathbf{R} = \boldsymbol{Y}_{\mathrm{d}} \tag{2}$$

where $\mathbf{R}$ is a fuzzy relation $\boldsymbol{X} \Rightarrow \boldsymbol{Y}$ representing the process dynamics, $\boldsymbol{X}$ is a control action and $\boldsymbol{Y}_{\mathrm{d}}$ is a given goal. FIR is performed by solving the fuzzy relational equation in order to obtain a control action $\boldsymbol{X}$ that is sufficient to yield $\boldsymbol{Y}_{\mathrm{d}}$.

FIR ensures that the produced control actions are sufficient to generate the desired process output. Algorithms exist for solving linear fuzzy relational equations that contain one decision variable to obtain the appropriate control actions [**4**–**6**].

A three-tank plant is a multiple-input–multiple-output (MIMO) plant. When used to describe the dynamics of this plant, a fuzzy relation will contain more than one input in its antecedent part. Consequently, there will be multiple decision variables included in the corresponding fuzzy relational equation and existing algorithms cannot be applied to determine control actions.

This paper introduces an approach to FIR control that can deal with MIMO systems such as a three-tank plant. The idea is to decompose the fuzzy relational equation that describes an MIMO plant into single-decision-variable equations and apply an available algorithm to deduce control actions. The remainder of this paper is organized as follows. Section 2 describes the three-tank plant employed to illustrate the proposed approach. Section 3 discusses the decomposition of MIMO processes where there are interactions between the different process variables. Section 4 presents the experimental results obtained.

2 THREE-TANK PLANT

The plant is illustrated in Fig. 1. The tanks all have the same cross-sectional area. They are linked by connecting pipes. Liquid levels in the tanks are regulated by manipulating the inlet flows of which there are two, one into tank I (the left-most tank) and the other into tank II (the right-most tank). Two pumps, P_1 and P_2, drawing liquid from a reservoir control the flowrates of the inlets. The flows in the pipes between tank I and tank II and the middle tank, tank III, are manipulated using ball valves V_1 and V_3. The plant output flow is from a pipe connected to tank II. Ball valve V_2 controls the outlet flow. Disturbances to the plant are in the form of leakages from the tanks which are controlled by ball valves V_4, V_5 and V_6 respectively. Pressure sensors are used to measure the levels of liquid in the tanks.

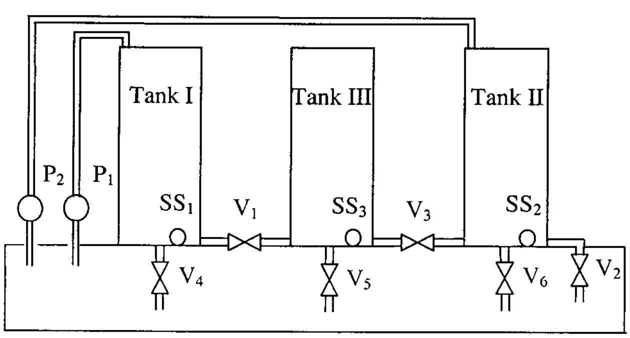

Fig. 1 A three-tank plant

When the tanks are coupled together through the connecting pipes, the effects of q_1 and q_2 will interact with one another. For example, assume that the plant is in a steady state initially. A new desired level for tank I is set that is higher than the current level. The level h_2 for tank II is to remain the same. The variable q_1 must increase to bring the level h_1 in tank I to its new set point. This is the *direct* effect of q_1 on h_1. However, because the tanks are coupled, q_1 will also disturb h_2. A reduction in q_2 is required to compensate for the increase in h_2 due to q_1. The decrease in q_2, in turn, reduces h_1. This is called the *indirect* effect of q_1 on h_1. These interactions can be seen from the following mathematical model of the plant:

$$A \frac{\mathrm{d}h_1}{\mathrm{d}t} = q_1 - az_1 S_{\mathrm{n}} \operatorname{sgn}(h_1 - h_3)\sqrt{2g|h_1 - h_3|} \tag{3a}$$

$$A \frac{\mathrm{d}h_3}{\mathrm{d}t} = az_1 S_{\mathrm{n}} \operatorname{sgn}(h_1 - h_3)\sqrt{2g|h_1 - h_3|} - az_3 S_{\mathrm{n}} \operatorname{sgn}(h_3 - h_2)\sqrt{2g|h_3 - h_2|} \tag{3b}$$

$$A \frac{\mathrm{d}h_2}{\mathrm{d}t} = q_2 - az_3 S_{\mathrm{n}} \operatorname{sgn}(h_2 - h_3)\sqrt{2g|h_2 - h_3|} - az_2 S_{\mathrm{l}}\sqrt{2gh_2} \tag{3c}$$

where az_i represents the opening of the ith valve, A is the cross-sectional area of a tank, S_{n} is the cross-sectional area of the connecting pipes and S_{l} is the cross-sectional area of the outlet pipe. The interactions between q_1 and q_2 make it difficult to implement FIR control for the three-tank plant. This will be discussed in the next section.

3 DIMENSION REDUCTION FOR FUZZY INVERSE REASONING CONTROL

The plant can be viewed as a two-input–two-output process. The two inputs are q_1 and q_2 and the two outputs are $\dot{h}_1$ and $\dot{h}_2$. The fuzzy relational equation that describes the dynamics of the process has two decision variables and is not a linear equation. It needs to be transformed into linear single-decision-variable equations before an existing fuzzy inverse reasoning algorithm can be applied.

A commonly used method when dealing with a complex problem is to assume that some unknown factors are known to reduce the complexity of the problem. If, in a two-decision-variable equation $[\boldsymbol{X}_1 \;\; \boldsymbol{X}_2] \circ \mathbf{R} = \boldsymbol{Y}$, $\boldsymbol{X}_2$ is assumed known, a fuzzy relation between $\boldsymbol{X}_1$ and $\boldsymbol{Y}$ can then be constructed, denoted as $\mathbf{R}'$. Consequently, the fuzzy relational equation becomes $\boldsymbol{X}_1 \circ \mathbf{R}' = \boldsymbol{Y}$, which is a linear single-decision-variable equation. However, when there is interaction between $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, this straightforward simplification cannot be made.

The concept of a *decoupler* [7] in conventional control can be employed to decompose interactions in an MIMO plant. A decoupler cancels the effect of the control actions of one control loop, $u_k(t)$, on the outputs of other loops, $y_i(t)\ (i \neq k)$, and hence decomposes interactions between $u_k(t)$ and $u_i(t)\ (i \neq k)$. The decoupler works by providing an additional action due to $u_k(t)$ on $y_i(t)\ (i \neq k)$ to compensate for the influence of $u_k(t)$ on $y_i(t)\ (i \neq k)$. In the time domain, it maintains $y_i(t)$ equal to $y_i(t-1)$ when $u_k(t)\ (i \neq k)$ changes.

Two FIR controllers have been designed for the three-tank plant based on the concept of the decoupler. One has q_1 as the input variable and $\dot{h}_1$ as the output variable. It also has two auxiliary variables, h_1 and h_3. The fuzzy relation employed by this controller is a relation from q_1, h_1 and h_3 to $\dot{h}_1$, called $\mathbf{R}_1$. The fuzzy relational equation for the controller is

$$[\boldsymbol{Q}_1 \;\; \boldsymbol{H}_1 \;\; \boldsymbol{H}_3] \circ \mathbf{R}_1 = \dot{\boldsymbol{H}}_1 \tag{4}$$

where $\boldsymbol{Q}_1$, $\boldsymbol{H}_1$, $\boldsymbol{H}_3$ and $\dot{\boldsymbol{H}}_1$ are fuzzy variables corresponding to q_1, h_1, h_3 and $\dot{h}_1$. When a new set point $h_{1\text{sp}}$ is chosen, $\dot{h}_1(t+1) \simeq h_{1\text{sp}} - h_1(t)$ is calculated and a goal $\dot{\boldsymbol{H}}_1$ is formed. By solving equation (4), control actions q_1 can be deduced. When $\dot{h}_1(t+1)$ is computed, the auxiliary variable h_3 is kept unchanged so that the controller works by providing an additional effect to counteract the influence of q_1 on h_3, thereby cancelling the indirect effects of q_1 on h_1.

The auxiliary fuzzy variables $\boldsymbol{H}_1$ and $\boldsymbol{H}_3$ are computed from the measured data, h_1 and h_3. Equation (4) is then rewritten as a linear equation, namely

$$\boldsymbol{Q}_1 \circ \mathbf{R}_1'|_{\boldsymbol{H}_1, \boldsymbol{H}_3} = \dot{\boldsymbol{H}}_1 \tag{5}$$

where $\mathbf{R}_1'$ is calculated from $\mathbf{R}_1$, $\boldsymbol{H}_1$ and $\boldsymbol{H}_3$.

A three-dimensional matrix is defined as $\mathbf{X} = [\boldsymbol{Q}_1 \;\; \boldsymbol{H}_1 \;\; \boldsymbol{H}_3]$. Entry $x_{ijk} = q_1^i \wedge h_1^j \wedge h_3^k$ stands for the antecedent part of some fuzzy rule, 'IF $\boldsymbol{Q}_1$ is $\boldsymbol{A}^i$ and $\boldsymbol{H}_1$ is $\boldsymbol{B}^j$ and $\boldsymbol{H}_3$ is $\boldsymbol{C}^k$, THEN $\dot{\boldsymbol{H}}_1$ is $\boldsymbol{D}^l$, where q_1^i, h_1^j and h_3^k are degrees of membership of fuzzy sets $\boldsymbol{A}^i$, $\boldsymbol{B}^j$ and $\boldsymbol{C}^k$ respectively.

The fuzzy relational matrix $\mathbf{R}_1$ is a four-dimensional matrix. Element r_{ijkl} of $\mathbf{R}_1$ is the truth degree of the given fuzzy rule.

With equation (4), $\dot{h}_1^l$, the degree of membership of fuzzy set $\boldsymbol{D}^l$, is computed:

$$\begin{aligned}
\dot{h}_1^l = {} & [(x_{111} \wedge r_{111l}) \vee (x_{112} \wedge r_{112l}) \vee \cdots \vee (x_{11K} \wedge r_{11Kl})] \\
& \vee [(x_{121} \wedge r_{121l}) \vee (x_{122} \wedge r_{122l}) \vee \cdots \vee (x_{12K} \wedge r_{12Kl})] \\
& \vee \cdots \\
& \vee [(x_{1J1} \wedge r_{1J1l}) \vee (x_{1J2} \wedge r_{1J2l}) \vee \cdots \vee (x_{1JK} \wedge r_{1JKl})] \\
& \vee \cdots \\
& \cdots \\
& \vee [(x_{I11} \wedge r_{I11l}) \vee (x_{I12} \wedge r_{I12l}) \vee \cdots \vee (x_{I1K} \wedge r_{I1Kl})] \\
& \vee [(x_{I21} \wedge r_{I21l}) \vee (x_{I22} \wedge r_{I22l}) \vee \cdots \vee (x_{I2K} \wedge r_{I2Kl})] \\
& \vee \cdots \\
& \vee [(x_{IJ1} \wedge r_{IJ1l}) \vee (x_{IJ2} \wedge r_{IJ2l}) \vee \cdots \vee (x_{IJK} \wedge r_{IJKl})]
\end{aligned} \tag{6}$$

Replacing x_{ijk} with $q_1^i \wedge h_1^j \wedge h_3^k$ and employing the associativity and commutativity of the t-norm and t-conorm ($\wedge$ and $\vee$) to rearrange equation (6) gives

$$\begin{aligned}
\dot{h}_1^l = {} & \{[q_1^1 \wedge (h_1^1 \wedge h_3^1 \wedge r_{111l})] \vee \cdots \vee [q_1^1 \wedge (h_1^1 \wedge h_3^K \wedge r_{11Kl})]\} \\
& \vee \cdots \\
& \vee \{[q_1^1 \wedge (h_1^J \wedge h_3^1 \wedge r_{1J1l})] \vee \cdots \vee [q_1^1 \wedge (h_1^J \wedge h_3^K \wedge r_{1JKl})]\} \\
& \vee \cdots \\
& \cdots \\
& \vee \{[q_1^I \wedge (h_1^1 \wedge h_3^1 \wedge r_{I11l})] \vee \cdots \vee [q_1^I \wedge (h_1^1 \wedge h_3^K \wedge r_{I1Kl})]\} \\
& \vee \cdots \\
& \vee \{[q_1^I \wedge (h_1^J \wedge h_3^1 \wedge r_{IJ1l})] \vee \cdots \vee [q_1^I \wedge (h_1^J \wedge h_3^K \wedge r_{IJKl})]\}
\end{aligned} \tag{7}$$

Equation (7) is rewritten as

$$(q_1^1 \quad q_1^2 \quad \cdots \quad q_1^I) \circ \begin{pmatrix} r'_{1l} \\ r'_{2l} \\ \vdots \\ r'_{Il} \end{pmatrix} = \dot{h}_1^l \tag{8}$$

Applying the above procedure to all elements of $\dot{\boldsymbol{H}}_1$, $\dot{h}_1^l$ $(l = 1, \ldots, L)$, yields the two-dimensional fuzzy relational matrix

$$\mathbf{R}'_1 = \begin{pmatrix} r'_{11} & r'_{12} & \cdots & r'_{1L} \\ r'_{21} & r'_{22} & & r'_{2L} \\ \vdots & & & \vdots \\ r'_{I1} & r'_{I2} & \cdots & r'_{IL} \end{pmatrix}$$

With $\mathbf{R}'_1$ completely defined, given a new set point $h_{1\mathrm{sp}}$, it is possible to apply an algorithm for solving linear fuzzy relational equations to equation (5) to produce control actions q_1.

The other FIR controller possesses a similar structure. It has q_2 as the input variable and $\dot{h}_2$ as the output variable. The auxiliary variables are h_2 and h_3. Inverse reasoning is again implemented with the decoupling technique to derive control actions q_2 from a specified set point $h_{2\mathrm{sp}}$.

4 EXPERIMENTAL RESULTS

The designed FIR controllers were applied to regulate the liquid levels under different operating conditions. First, the plant was set up as follows: the three tanks were empty and fully connected, the outlet valve opening was in the medium range, there was no leakage and the set points $h_{1\mathrm{sp}}$ and $h_{2\mathrm{sp}}$ were 200 mm. With the system starting from a steady state, the set points were suddenly changed to $h_{1\mathrm{sp}} = h_{2\mathrm{sp}} = 400$ mm. A disturbance was simulated by abruptly increasing the inlet flowrate for tank II by 20 per cent and holding it there for about 120 s when the plant was in the steady state.

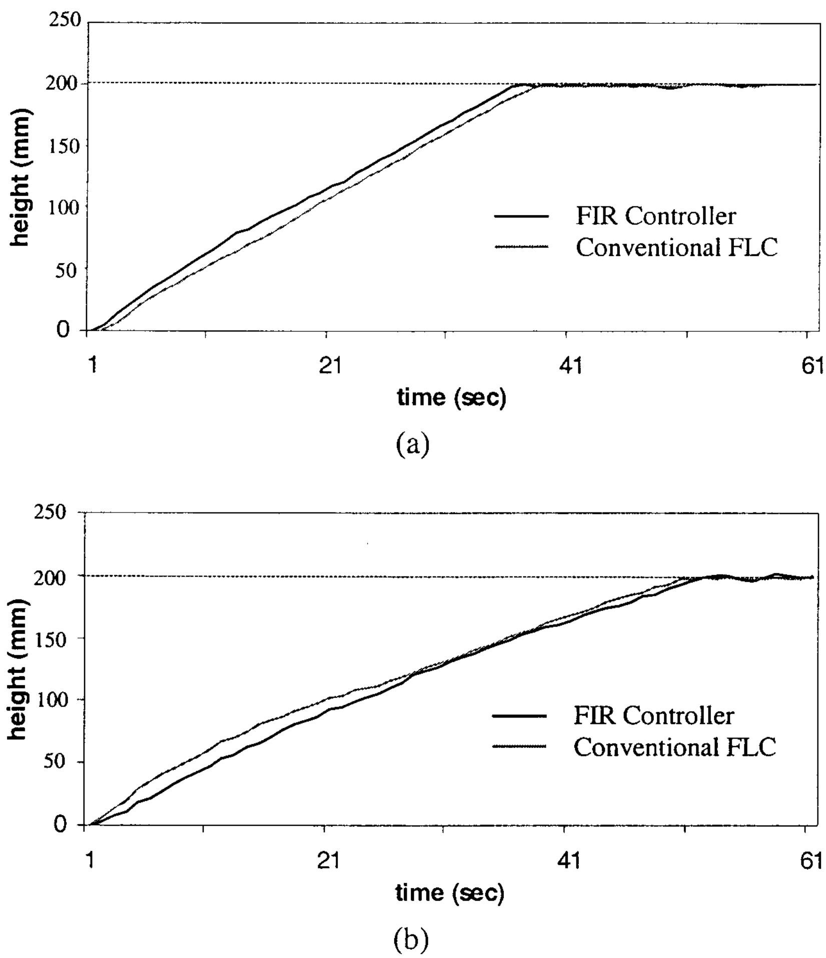

Fig. 2 Responses of the three-tank plant with the set points raised from 0 to 200 mm

The following design specifications were set for all the different operating conditions:

Steady state error (ε) $\leqslant 2.50\%$ (9a)

Rise time (τ) $\leqslant 90$ s (9b)

Maximum overshoot (M_p) $\leqslant 5\%$ (9c)

Figure 2 shows time domain plots of the liquid levels h_1 and h_2. It can be seen from these plots that the FIR controllers could successfully regulate the liquid levels and satisfy the design specifications.

Figure 3 presents the results when the set points were changed. Compared with the previous case, the plant needed a longer time to reach the higher set point. The rise time of the closed-loop system, denoted as τ, is not proportional to the height of the set point. This is because of the non-linear property of the ball valve. The plant was stable at the various set points. The steady state errors were less than 1.20 per cent when the set points were 200 mm and below 0.65 per cent when the set points were 400 mm in both tank I and tank II.

The results with a disturbance added to tank II are depicted in Fig. 4. Figure 4a illustrates that the controllers produced small control actions to reject the disturbance. Figure 4b shows the effects of the disturbance on the liquid level of tank II. It can be seen that the disturbance caused little change in the liquid level.

For comparison, a conventional fuzzy logic controller (FLC) was developed based on human expertise in manipulating the liquid levels in the same three-tank plant. With this controller, the responses of the plant when the set points were 200 and 400 mm were as shown in Figs 2 and 3 respectively. Although the FLC performed similarly to the FIR controller, it took 2 days for an engineer with a strong fuzzy logic control background to gain the required knowledge and convert it into fuzzy rules. On the other hand, the construction of the FIR controller required only less than 15 min. It is expected that, for more complex plants, the difference in development times will be even larger, which clearly demonstrates the advantage of the FIR approach.

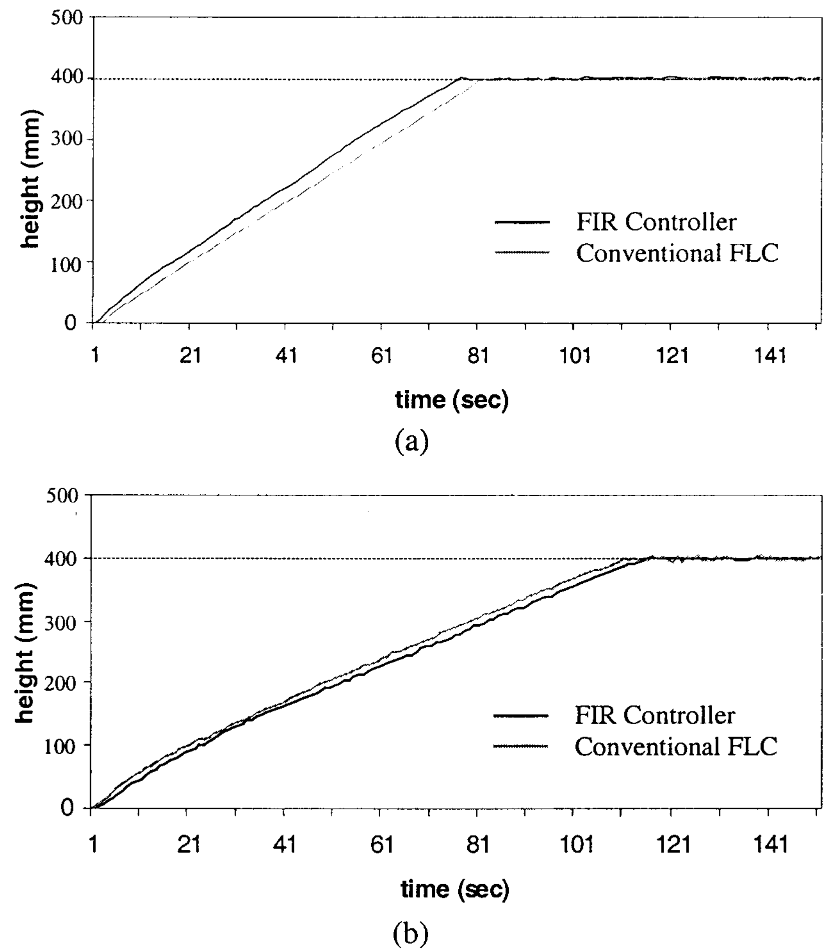

Fig. 3 Responses of the three-tank plant with the set points raised from 0 to 400 mm

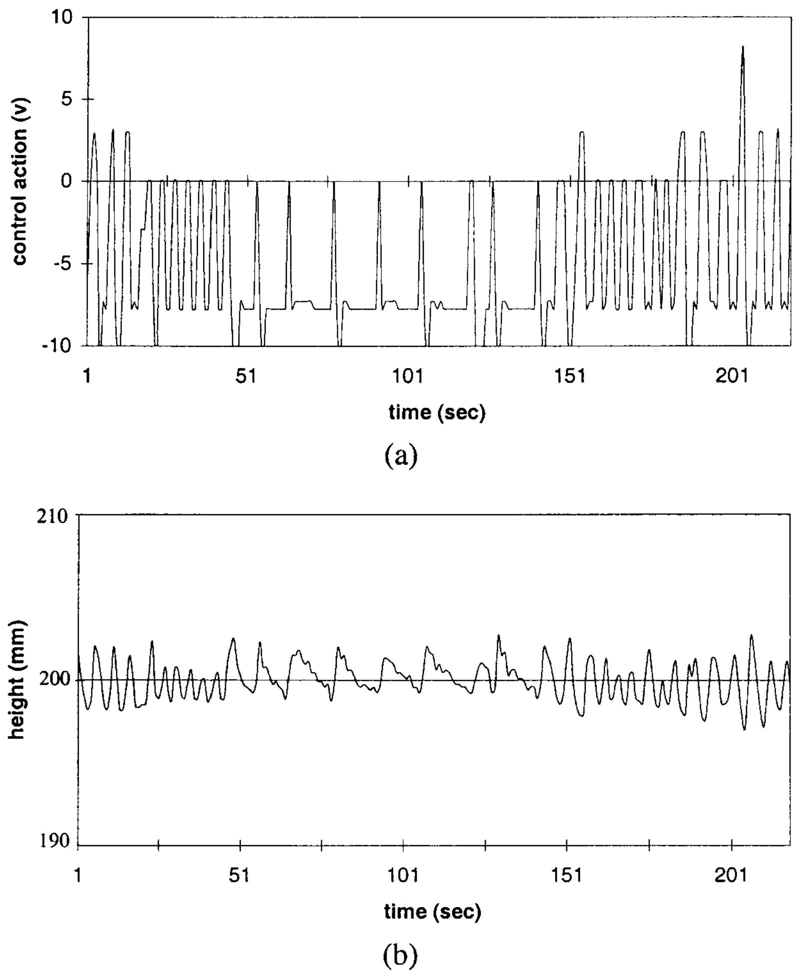

Fig. 4 Responses of the three-tank plant to a disturbance

5 CONCLUSION

The three-tank plant is an MIMO plant and therefore the fuzzy relational equation describing its dynamics is a multiple-decision-variable equation. The proposed FIR control approach first involves transforming this fuzzy relational equation into a number of linear single-decision-variable equations. These equations are then solved using one of the available algorithms to deduce appropriate control inputs to the plant. Experiments have shown that FIR controllers can provide satisfactory steady state and transient responses and successfully reject disturbances and handle set point changes.

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