European Congress on Computational Methods in Applied Sciences and Engineering
ECCOMAS 2004
P. Neittaanmäki, T. Rossi, K. Majava, and O. Pironneau (eds.)
O. Nevanlinna and R. Rannacher (assoc. eds.)
Jyväskylä, 24—28 July 2004

VARIATIONAL FORMULATION OF FRACTIONAL STEP METHODS IN SPH FLUID MECHANICS APPLICATIONS

Sivakumar Kulasegaram[*] and Javier Bonet[†]

[*] Cardiff School of Engineering
P.O. Box 925, Cardiff CF14 0YF Cardiff University, U.K.
e-mail: KulasegaramS@cardiff.ac.uk

[†]Civil and Computational Engineering Centre
Singleton Park, Swansea SA2 8PP University of Wales Swansea, U.K.
e-mails: J.Bonet@Swansea.ac.uk

Key words: Meshless Methods, SPH, Variational formulations, Fractional Step Method, Dyanmic problems

Abstract. *A fractional step projection method is introduced here to achieve incompressibility in the SPH method. Fractional step method is a popular approach used to solve incompressible fluid dynamics problems in traditional grid based methods and involves a two-step process to achieve incompressibility. Essentially, the velocity update over a time step is split into a component that does not take into account the divergence free condition, plus a pressure correction term which restores the incompressibility of the velocity. This paper mainly focuses on the variational formulation of fractional step method for SPH fluid applications by presenting a detailed description of the formulation of governing equations.*

1 INTRODUCTION

Smooth Particle Hydrodynamics (SPH) is a fully Lagrangian, particle based technique which does not require grid and can be used to simulate the motion of fluids and solids. SPH was originally invented to model astrophysical phenomena and more recently has been extended by many researchers to simulate various engineering applications [1-4]. In fluid applications, incompressibility in SPH was generally approximated by assuming large sound speed, and hence the real fluid was simulated as an artificial fluid with slightly more compressible characteristics [2,3]. The results using this approach have been impressive but requirement of a large sound speed leads to smaller time step constraint in order to satisfy stability.

In the present work, incompressibility in SPH is introduced by employing a fractional step projection method. Fractional step methods are frequently used in incompressible computational fluid dynamics [5]. This is a popular approach used to solve incompressible fluid dynamics problems in traditional grid based methods and involves a two-step process to achieve incompressibility [5,6]. First, the momentum equation is integrated in time to predict an intermediate velocity. This intermediate velocity field will, in general not satisfy continuity so the second step is to project this intermediate velocity onto the space of divergence free vector fields, thus transforming the intermediate velocity into an incompressible velocity field. Typically, the pressure correction term involves the implicit solution of a Poisson type of problem.

This paper mainly focuses on the variational formulation of fractional step method for SPH fluid applications. A detailed description of the formulation of governing equations is presented with a number of numerical simulations to demonstrate the capabilities of the proposed computational model.

2 SMOOTH PARTICLE HYDRODYNAMICS METHODOLOGY

In meshfree methods such as SPH, any problem variable and its gradient are generally interpolated from values at a discrete number of particles by using the following approximations:

$$(\boldsymbol{x}) = \sum_{b=1}^{N} V_b \;{}_b W_b(\boldsymbol{x}) \tag{1}$$

$$\nabla \;(\boldsymbol{x}) = \sum_{b=1}^{N} V_b \;{}_b \nabla W_b(\boldsymbol{x}) \tag{2}$$

where V_b denotes the volume of material associated to a given particle and W_b represents the ‘kernel’ or interpolation function, which usually has a bell shape with a compact support as shown in Figure 1. Most commonly used kernel function in SPH is a cubic spline kernel function given by,

$$W(x) = \frac{c}{h^d}\begin{cases} 1-\frac{3}{2}\;{}^{2}+\frac{3}{4}\;{}^{3} & if \quad \leq 1 \\ \frac{1}{4}(2- \quad)^3 & if \;\; 1< \quad \leq 2 \;\; ; \qquad = \frac{\|\boldsymbol{x}\|}{h} \\ 0 & if \quad > 2 \end{cases} \tag{3}$$

where d is the number of dimensions of the problem and c is a scaling factor to normalise the kernel function. Here, the length parameter h has a similar interpretation to the element size in finite element method.

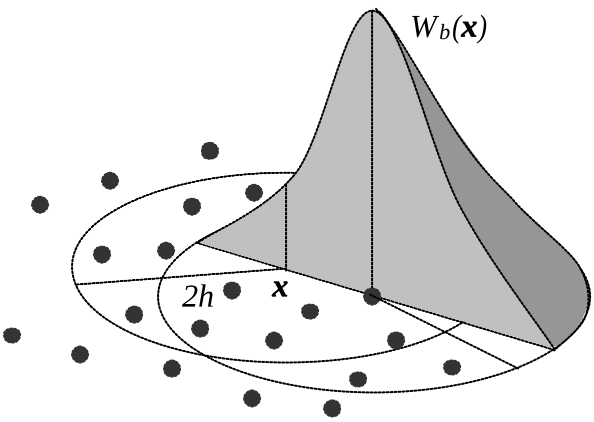

Figure 1: Particle interpolation and kernel function

For instance, applying equation (1) to density of a continuum leads to the classical SPH equation

$$(\boldsymbol{x}) = \sum_{b=1}^{N} m_b W_b(\boldsymbol{x}) \tag{4}$$

In this way, the SPH representation of the governing equations can be built from fundamental equations of motion.

3 VARIATIONAL FORMULATION OF CENTRAL DIFFERENCES

To introduce the variational formulation of the fractional step method, consider first the standard case of the commonly used central difference time integrator. For this purpose, recall first that the equations of motion of a Hamiltonian system represent the stationary conditions of the action integral:

$$S = \int_{t_0}^{t_N} L\,dt; \qquad L = K - \Pi \tag{5}$$

where L denotes the Lagrangian, K is the kinetic energy and $\Pi = \Pi_{\text{int}} + \Pi_{\text{ext}}$ is the total potential energy with, typically, an internal elastic component and an external component due to applied forces.

Consider now a sequence of timesteps $t_n = t_{n-1} + \Delta t,\ n = 0,1,...,N$. The position of the solid at each step is defined by a mapping $\boldsymbol{x}_n = \phi(\boldsymbol{X}, t_n)$. The discrete Lagrangian between two steps is now defined as:

$$L_{n,n+1}(\boldsymbol{x}_n, \boldsymbol{x}_{n+1}) = \tfrac{1}{2} M\left(\boldsymbol{v}_{n+\frac{1}{2}}, \boldsymbol{v}_{n+\frac{1}{2}}\right) - \Pi(\boldsymbol{x}_n) \tag{6}$$

where the intermediate velocity is $\boldsymbol{v}_{n+\frac{1}{2}} = (\boldsymbol{x}_{n+1} - \boldsymbol{x}_n)/\Delta t$ and the mass bilinear form is:

$$M(\boldsymbol{u}, \boldsymbol{v}) = \int_{V_0} (\boldsymbol{u} \cdot \boldsymbol{v})\ dV_0 \tag{7}$$

The action integral is now approximated as:

$$S(\boldsymbol{x}_0, ..., \boldsymbol{x}_N) = \sum_{n=0}^{N-1} \Delta t\, L_{n,n+1}(\boldsymbol{x}_n, \boldsymbol{x}_{n+1}) \tag{8}$$

The stationary conditions of S with respect to a variation $\delta\boldsymbol{v}$ the body position at n are:

$$D_n S[\delta\boldsymbol{v}] = \Delta t\, D_2 L_{n-1,n}[\delta\boldsymbol{v}] + \Delta t\, D_1 L_{n,n+1}[\delta\boldsymbol{v}] = 0 \tag{9}$$

where D_i denotes directional derivative with respect to i-th variable. Substituting for the Lagrangian expressions from equation (6) leads, after some simple algebra, to the standard explicit central difference time integration scheme:

$$M\left(\delta\boldsymbol{v}, \frac{\boldsymbol{v}_{n+1/2} - \boldsymbol{v}_{n-1/2}}{\Delta t}\right) = F(\delta\boldsymbol{v}; \boldsymbol{x}_n) - T(\delta\boldsymbol{v}; \boldsymbol{x}_n) \tag{10}$$

where the external and internal forces are respectively,

$$\begin{aligned} F(\delta\boldsymbol{v}; \boldsymbol{x}_n) &= -D\Pi_{\text{ext}}(\boldsymbol{x}_n)[\delta\boldsymbol{v}] \\ T(\delta\boldsymbol{v}; \boldsymbol{x}_n) &= D\Pi_{\text{int}}(\boldsymbol{x}_n)[\delta\boldsymbol{v}] \end{aligned} \tag{11}$$

Identical explicit equations are in fact obtained if the Lagrangian is approximated as:

$$L_{n,n+1}(\boldsymbol{x}_n, \boldsymbol{x}_{n+1}) = \tfrac{1}{2} M\left(\boldsymbol{v}_{n+\frac{1}{2}}, \boldsymbol{v}_{n+\frac{1}{2}}\right) - \Pi(\boldsymbol{x}_{n+1}) \tag{12}$$

or indeed,

$$L_{n,n+1}(\boldsymbol{x}_n, \boldsymbol{x}_{n+1}) = \tfrac{1}{2} M\left(\boldsymbol{v}_{n+\frac{1}{2}}, \boldsymbol{v}_{n+\frac{1}{2}}\right) - \tfrac{1}{2}\Pi(\boldsymbol{x}_n) - \tfrac{1}{2}\Pi(\boldsymbol{x}_{n+1}) \tag{13}$$

4 FRACTIONAL STEP VARIATIONA FORMULATION

In order to separate the volumetric from the isochoric components of the deformation during the time increment, consider the internal energy decomposition into volume preserving and volumetric components as: $\Pi_{\text{int}} = \Pi_{\text{iso}} + \Pi_{\text{vol}}$. In general the volumetric component will be a function of volume ratio $J = \dfrac{V}{V_0}$. In order to introduce the pressure as an extra variable, however, the volumetric strain energy can be expressed in terms of the complementary energy via the Legendre relationship:

$$\Pi_{\text{vol}}(\boldsymbol{x}) = \int_{V_0} p(J-1)\,dV_0 - \hat{\Pi}_{\text{vol}}(p) \tag{14}$$

where, typically, the complementary volumetric energy is:

$$\hat{\Pi}(p) = \tfrac{1}{2}\int_{V_0} \frac{1}{k} p^2\,dV_0 \tag{15}$$

The discrete Lagrangian between any two steps is now expressed as:

$$\begin{aligned} L_{n,n+1}(\boldsymbol{x}_n, \boldsymbol{x}_{n+1}, p_{n+\frac{1}{2}}) = {} & \tfrac{1}{2} M\left(\boldsymbol{v}_{n+\frac{1}{2}}, \boldsymbol{v}_{n+\frac{1}{2}}\right) - \Pi_{\text{iso}}(\boldsymbol{x}_n) - \Pi_{\text{ext}}(\boldsymbol{x}_n) \\ & - \int_{V_0} \tfrac{1}{2} p_{n+\frac{1}{2}}(J_{n+1} + J_n - 2)\,dV_0 + \hat{\Pi}_{\text{vol}}(p_{n+\frac{1}{2}}) \end{aligned} \tag{16}$$

Note that a central approximation for the volumetric components has been used. The stationary conditions of the action integral with respect to position at step n can now be obtained with the help of the expression $DJ[\boldsymbol{v}] = J\,\text{div}[\boldsymbol{v}]$, and lead to:

$$\begin{aligned} M\left(\boldsymbol{v}_n, \frac{\boldsymbol{v}_{n+\frac{1}{2}} - \boldsymbol{v}_{n-\frac{1}{2}}}{\Delta t}\right) = {} & F(\boldsymbol{v}_n; \boldsymbol{x}_n) - T'(\boldsymbol{v}_n; \boldsymbol{x}_n) \\ & - \int_{V_n} \tfrac{1}{2}(p_{n-\frac{1}{2}} + p_{n+\frac{1}{2}})\,\text{div}\,\boldsymbol{v}_n\,dV_n \end{aligned} \tag{17}$$

where T' represent the deviatoric component of the internal forces. Note also that the divergence of $\boldsymbol{v}_n$ is taken at the known configuration n. The above expression can now be re-arranged in a more traditional fractional step format as:

$$M\left(\boldsymbol{v}_n, \frac{\boldsymbol{v}^*_{n+\frac{1}{2}} - \boldsymbol{v}_{n-\frac{1}{2}}}{\Delta t}\right) = F(\boldsymbol{v}_n; \boldsymbol{x}_n) - T'(\boldsymbol{v}_n; \boldsymbol{x}_n) - \tfrac{1}{2}\int_{V_n} p_{n-\frac{1}{2}}\,\text{div}\,\boldsymbol{v}_n\,dV_n \tag{18}$$

$$M\left(\boldsymbol{v}_n, \frac{\boldsymbol{v}_{n+\frac{1}{2}} - \boldsymbol{v}^*_{n+\frac{1}{2}}}{\Delta t}\right) = -\tfrac{1}{2}\int_{V_n} p_{n+\frac{1}{2}}\,\text{div}\,\boldsymbol{v}_n\,dV_n \tag{19}$$

Note that the first of the above equations is explicit, whereas the second equation will require the solution of a set of equations for the pressure increment. These equations are derived from the stationary conditions of the action integral with respect to the pressure, which lead to an additional set of constitutive equations as:

$$M_k(\ p, p_{n+1/2}) = \int_{V_0} \tfrac{1}{2}(J_n + J_{n+1} - 2)\ p dV_0 \tag{20}$$

where the notation $M_k(pq) = \int_{V_0} \frac{1}{k} pq dV_0$ has been used. Note that for incompressible materials $k = \infty$ and the above expression enforces that the average volume ratio should be one.

5 SPH DISCRETIZATION

Consider now discretized version of equation (16) as,

$$\boldsymbol{L}_{n,n+1} = \sum_a \frac{1}{2} m_a \boldsymbol{v}_a \cdot \boldsymbol{v}_a - \Pi_{iso}\left(\boldsymbol{x}_a^n\right) - \Pi_{ext}\left(\boldsymbol{x}_a^n\right) - \sum_a \frac{1}{2} p_a^{n+\frac{1}{2}}\left(J_a^{n+1} + J_a^n - 2\right)\frac{m_a}{\boldsymbol{r}_a^0} + \hat{\Pi}_{vol}\left(p_a^{n+\frac{1}{2}}\right) \tag{21}$$

For fully incompressible case (as $k \to \infty$) equation (15) yields,

$$\hat{\Pi}_{vol}\left(p_a^{n+\frac{1}{2}}\right) = 0 \tag{22}$$

Therefore (21) can be simplified as,

$$\boldsymbol{L}_{n,n+1} = \sum_a \frac{1}{2} m_a \boldsymbol{v}_a^{n+\frac{1}{2}} \cdot \boldsymbol{v}_a^{n+\frac{1}{2}} - \Pi_{iso}\left(\boldsymbol{x}_a^n\right) - \Pi_{ext}\left(\boldsymbol{x}_a^n\right) - \sum_a \frac{1}{2} p_a^{n+\frac{1}{2}}\left(J_a^{n+1} + J_a^n - 2\right)\frac{m_a}{\boldsymbol{r}_a^0} \tag{23}$$

Similarly, equation (23) for time steps between n-1 and n can be written as,

$$\boldsymbol{L}_{n-1,n} = \sum_a \frac{1}{2} m_a \boldsymbol{v}_{\boldsymbol{a}}^{n-\frac{1}{2}} \cdot \boldsymbol{v}_{\boldsymbol{a}}^{n-\frac{1}{2}} - \Pi_{iso}\left(\boldsymbol{x}_a^{n-1}\right) - \Pi_{ext}\left(\boldsymbol{x}_a^{n-1}\right) - \sum_a \frac{1}{2} p_a^{n-\frac{1}{2}}\left(J_a^n + J_a^{n-1} - 2\right)\frac{m_a}{\boldsymbol{r}_a^0} \tag{24}$$

Consider now the stationary condition given by,

$$D\boldsymbol{L}_{n,n+1}\left[d\boldsymbol{v}_a^n\right] + D\boldsymbol{L}_{n-1,n}\left[d\boldsymbol{v}_a^n\right] = 0 \tag{25}$$

Substituting equations (23) and (24) in (25) gives,

$$\sum_a m_a \frac{\left(\boldsymbol{v}_a^{n+\frac{1}{2}} - \boldsymbol{v}_a^{n-\frac{1}{2}}\right)}{\Delta t} \cdot d\boldsymbol{v}_a^n = -D\Pi_{iso}\left[d\boldsymbol{v}_a^n\right] - D\Pi_{ext}\left[d\boldsymbol{v}_a^n\right] + \sum_a \frac{1}{2}\frac{m_a}{\left(r_a^n\right)^2}\left(p_a^{n+\frac{1}{2}} + p_a^{n-\frac{1}{2}}\right) D r_a^n\left[d\boldsymbol{v}_a^n\right] \tag{26}$$

Using the notations as in equation (17), the above equation can be re-written as,

$$\sum_a m_a \frac{\left(\boldsymbol{v}_a^{n+\frac{1}{2}} - \boldsymbol{v}_a^{n-\frac{1}{2}}\right)}{\Delta t} \cdot d\boldsymbol{v}_a^n = F\left(d\boldsymbol{v}_a^n;\boldsymbol{x}_a^n\right) - T'\left(d\boldsymbol{v}_a^n;\boldsymbol{x}_a^n\right) + \frac{1}{2}\sum_a \frac{m_a}{\left(\boldsymbol{r}_a^n\right)^2}\left(p_a^{n+\frac{1}{2}} + p_a^{n-\frac{1}{2}}\right) D\boldsymbol{r}_a^n\left[d\boldsymbol{v}_a^n\right] \tag{27}$$

Now to evaluate $D\boldsymbol{r}_a^n\left[d\boldsymbol{v}_a^n\right]$, consider the density equation (4) at time level n as,

$$\boldsymbol{r}_a^n = \sum_b m_b W_a\left(\boldsymbol{x}_b^n, h_a\right) \tag{28}$$

Linearising density in equation (28) with respect to velocity gives,

$$D\boldsymbol{r}_a^n\left[d\boldsymbol{v}_a^n\right] = \sum_b m_b \nabla W_a\left(\boldsymbol{x}_b^n, h_a\right)\cdot\left(d\boldsymbol{v}_b^n - d\boldsymbol{v}_a^n\right) \tag{29}$$

Substituting $D\boldsymbol{r}_a^n\left[d\boldsymbol{v}_a^n\right]$ from (29) into equation (27) yields,

$$\begin{aligned}\sum_a m_a \frac{\left(\boldsymbol{v}_a^{n+\frac{1}{2}} - \boldsymbol{v}_a^{n-\frac{1}{2}}\right)}{\Delta t} \cdot d\boldsymbol{v}_a^n = F\left(d\boldsymbol{v}_a^n; x_a^n\right) - T'\left(d\boldsymbol{v}_a^n; x_a^n\right) + \\ \frac{1}{2}\sum_a\sum_b m_a m_b \left(\frac{p_b^{n-\frac{1}{2}}}{\left(\boldsymbol{r}_b^n\right)^2} + \frac{p_a^{n-\frac{1}{2}}}{\left(\boldsymbol{r}_a^n\right)^2}\right)\nabla W_b\left(x_a^n, h_a\right)\cdot d\boldsymbol{v}_a^n + \\ \frac{1}{2}\sum_a\sum_b m_a m_b \left(\frac{p_b^{n+\frac{1}{2}}}{\left(\boldsymbol{r}_b^n\right)^2} + \frac{p_a^{n+\frac{1}{2}}}{\left(\boldsymbol{r}_a^n\right)^2}\right)\nabla W_b\left(x_a^n, h_a\right)\cdot d\boldsymbol{v}_a^n\end{aligned} \tag{30}$$

Hence, for a given node 'a' the above equilibrium equation can be written as,

$$\begin{aligned}m_a \frac{\left(\boldsymbol{v}_a^{n+\frac{1}{2}} - \boldsymbol{v}_a^{n-\frac{1}{2}}\right)}{\Delta t} = F_a^n - T_a'^n + \\ \frac{1}{2}\sum_b m_a m_b \left(\frac{p_b^{n-\frac{1}{2}}}{\left(\boldsymbol{r}_b^n\right)^2} + \frac{p_a^{n-\frac{1}{2}}}{\left(\boldsymbol{r}_a^n\right)^2}\right)\nabla W_b\left(x_a^n, h_a\right) + \\ \frac{1}{2}\sum_b m_a m_b \left(\frac{p_b^{n+\frac{1}{2}}}{\left(\boldsymbol{r}_b^n\right)^2} + \frac{p_a^{n+\frac{1}{2}}}{\left(\boldsymbol{r}_a^n\right)^2}\right)\nabla W_b\left(x_a^n, h_a\right)\end{aligned} \tag{31}$$

Rewriting equation (31) by taking $p^{n+\frac{1}{2}} = p^{n-\frac{1}{2}} + \Delta p^{n+\frac{1}{2}}$ gives,

$$m_a \frac{\left(\tilde{\boldsymbol{v}}_a^{n+\frac{1}{2}} - \boldsymbol{v}_a^{n-\frac{1}{2}}\right)}{\Delta t} = F_a^n - T_a'^n + \sum_b m_a m_b \left(\frac{p_b^{n-\frac{1}{2}}}{\left(\boldsymbol{r}_b^n\right)^2} + \frac{p_a^{n-\frac{1}{2}}}{\left(\boldsymbol{r}_a^n\right)^2}\right) \nabla W_b\left(x_a^n, h_a\right) \tag{32}$$

and,

$$m_a \frac{\left(\boldsymbol{v}_a^{n+\frac{1}{2}} - \tilde{\boldsymbol{v}}_a^{n+\frac{1}{2}}\right)}{\Delta t} = \frac{1}{2} \sum_b m_a m_b \left(\frac{\Delta p_b^{n+\frac{1}{2}}}{\left(\boldsymbol{r}_b^n\right)^2} + \frac{\Delta p_a^{n+\frac{1}{2}}}{\left(\boldsymbol{r}_a^n\right)^2}\right) \nabla W_b\left(x_a^n, h_a\right) \tag{33}$$

As noted in the previous section, equation (32) is explicit, whereas the equation (33) will require solution for the pressure increment $\Delta p^{n+\frac{1}{2}}$. To evaluate pressure increment , consider the density equation (4) as follows.

$$r_a^n = \sum_c m_c W_a\left(x_c^n, h_a\right) \tag{34}$$

Differentiating (34) with respect to time yields,

$$\frac{\partial r_a^n}{\partial t} = \sum_c m_c \nabla W_a\left(\boldsymbol{x}_c^n, h_a\right) \cdot \left(\boldsymbol{v}_c^n - \boldsymbol{v}_a^n\right) \tag{35}$$

For incompressible flow $\left(\dfrac{\partial r_a^n}{\partial t} = 0\right)$,

$$\sum_c m_c \nabla W_a\left(\boldsymbol{x}_c^n, h_a\right) \cdot \left(\boldsymbol{v}_c^n - \boldsymbol{v}_a^n\right) = 0 \tag{36}$$

The above equation can be re-written as,

$$\sum_c m_c \nabla W_a\left(\boldsymbol{x}_c^n, h_a\right) \cdot \left[\left(\boldsymbol{v}_c^{n+\frac{1}{2}} + \boldsymbol{v}_c^{n-\frac{1}{2}}\right) - \left(\boldsymbol{v}_a^{n+\frac{1}{2}} + \boldsymbol{v}_a^{n-\frac{1}{2}}\right)\right] = 0 \tag{37}$$

Simplifying equation (37) results in,

$$\sum_c m_c \nabla W_a\left(\boldsymbol{x}_c^n, h_a\right) \cdot \left(\boldsymbol{v}_c^{n+\frac{1}{2}} - \boldsymbol{v}_a^{n+\frac{1}{2}}\right) = -\sum_c m_c \nabla W_a\left(\boldsymbol{x}_c^n, h_a\right) \cdot \left(\boldsymbol{v}_c^{n-\frac{1}{2}} - \boldsymbol{v}_a^{n-\frac{1}{2}}\right) \tag{38}$$

From equation (33) $\boldsymbol{v}_a^{n+\frac{1}{2}}$ can be evaluated as,

$$\boldsymbol{v}_a^{n+\frac{1}{2}} = \tilde{\boldsymbol{v}}_a^{n+\frac{1}{2}} + \frac{\Delta t}{2} \sum_b m_b \left(\frac{\Delta p_b^{n+\frac{1}{2}}}{\left(\boldsymbol{r}_b^n\right)^2} + \frac{\Delta p_a^{n+\frac{1}{2}}}{\left(\boldsymbol{r}_a^n\right)^2} \right) \nabla W_b\left(x_a^n, h_a\right) \tag{39}$$

Using equation (39), $\boldsymbol{v}_c^{n+\frac{1}{2}} - \boldsymbol{v}_a^{n+\frac{1}{2}}$ can be evaluated as,

$$\begin{aligned} \boldsymbol{v}_c^{n+\frac{1}{2}} - \boldsymbol{v}_a^{n+\frac{1}{2}} = \tilde{\boldsymbol{v}}_c^{n+\frac{1}{2}} - \tilde{\boldsymbol{v}}_a^{n+\frac{1}{2}} &+ \frac{\Delta t}{2} \sum_d m_d \left(\frac{\Delta p_d^{n+\frac{1}{2}}}{\left(\boldsymbol{r}_d^n\right)^2} + \frac{\Delta p_c^{n+\frac{1}{2}}}{\left(\boldsymbol{r}_c^n\right)^2} \right) \nabla W_d\left(x_c^n, h_c\right) \\ &- \frac{\Delta t}{2} \sum_b m_b \left(\frac{\Delta p_b^{n+\frac{1}{2}}}{\left(\boldsymbol{r}_b^n\right)^2} + \frac{\Delta p_a^{n+\frac{1}{2}}}{\left(\boldsymbol{r}_a^n\right)^2} \right) \nabla W_b\left(x_a^n, h_a\right) \end{aligned} \tag{40}$$

From (40) , set of equations for solving pressure increment can be derived as,

$$\sum_b K_{ab} \Delta p_b^{n+\frac{1}{2}} = -m_a \sum_c m_c \nabla W_a\left(\boldsymbol{x}_c^n, h_a\right) \cdot \left(\boldsymbol{v}_c^{n-\frac{1}{2}} - \boldsymbol{v}_a^{n-\frac{1}{2}} + \tilde{\boldsymbol{v}}_c^{n+\frac{1}{2}} - \tilde{\boldsymbol{v}}_a^{n+\frac{1}{2}} \right) \tag{41}$$

and K_{ab} can be decomposed as, $K_{ab} = K_{ab}^1 + K_{ab}^2 + K_{ab}^3 + K_{ab}^4$ where,

$$\begin{aligned} K_{ab}^1 &= \frac{\Delta t}{2\left(\boldsymbol{r}_b^n\right)^2} \sum_c m_a m_c m_b \nabla W_a\left(\boldsymbol{x}_c^n\right) \cdot \nabla W_b\left(x_c^n\right) \\ K_{ab}^2 &= \frac{\Delta t}{2\left(\boldsymbol{r}_b^n\right)^2} \sum_c m_a m_c m_b \nabla W_a\left(\boldsymbol{x}_b^n\right) \cdot \nabla W_c\left(x_b^n\right) \\ K_{ab}^3 &= \frac{\Delta t}{2\left(\boldsymbol{r}_b^n\right)^2} \sum_c m_a m_b m_c \nabla W_a\left(\boldsymbol{x}_c^n\right) \cdot \nabla W_b\left(x_a^n\right) \\ K_{ab}^4 &= \frac{(\Delta t) m_a d_{ab}}{2\left(\boldsymbol{r}_b^n\right)^2} \left(\sum_c m_c \nabla W_c\left(\boldsymbol{x}_a^n\right) \right) \cdot \left(\sum_d m_d \nabla W_d\left(x_a^n\right) \right) \end{aligned} \tag{42}$$

Hence, the equations (32),(33) and (41) can be employed to enforce incompressibility and to solve fluid dynamics problems in SPH method.

6 CONCLUSIONS

The formulation discussed above should provide a novel approach for the simulation of incompressible fluid dynamics problems using SPH method. Further work is clearly needed in order to assess the range of problems for which the extra cost induced by the fractional step method is compensated by the larger step size permitted. A number of numerical example will be presented to illustrate the capability of the methodology described here.

REFERENCES

[1] J.J.Monaghan. Simulating Free Surface Flows with SPH. *J. Comput. Phys.*, **110**, 399-406, 1994.

[2] J.Bonet and T.S.Lok. Variational and Momentum Aspects of Smooth Particle Hydrodynamics Formulations. *Comput. Methods. Appl. Mech. Engrg.*, **180**, 97-115, 1999.

[3] P.W.Randles and L.D.Libersky. Smooth Particle Hydrodynamics: Some Recent Improvements an Applications. *Comput. Methods. Appl. Mech. Engrg.*, **139**, 375-408, 1996.

[4] J.Bonet and S.Kulasegaram. Correction and Stabilization of Smooth Particle Hydrodynamics Methods with Applications in Metal Forming Simulations. *Int. J. Numer. Methods Engng.*, **47**, 1189-1214, 2000.

[5] A.J.Chorin. Numerical solution of the Navier-Stokes equations. *J. Math. Comp.*, **22**, 745-762, 1968

[6] J.B.Bell, P.Colella and H.Glaz. A Second-Order Projection Method for the Incompressible Navier-Stokes Equations. *J. Comput. Phys.*, **85**, 257, 1989.