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Further insights into "The stability of supply chains"

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Abstract

Riddalls and Bennett (2002) presented a stability criterion for a supply chain model in continuous time. In this short technical note we show this solution is in error and provide the correct stability criterion. The results are substantiated by deriving the correct criterion from with several independent theoretical analyses and validating the solution via numerical simulation. The weight of evidence suggests that the new derivations lead to correct predictions of the stability boundary for this important and widely used supply chain model.

1. Introduction

In a recent paper in this journal Riddalls and Bennett (2002) conducted a continuous time analysis of a supply chain ordering decision. This ordering decision is known as the APIOBPCS (Automatic Pipeline, Inventory and Order Based Production Control System) model, John, Naim and Towill (1994). Naim and Towill (1995) and Riddalls and Bennett (2002) showed that the APIOPBCS model is identical to the anchoring and adjustment heuristic that Sterman (1989) showed matched players behaviour whilst participating in the Beer Game, a table-top simulation of a supply chain. The APIOPBCS model has also been widely used in industry. For example is actively used to control the flow of over 6000 products in a real company with a fictitious name (WMC) as reported by Lewis et al (1995). Furthermore variants of the APIOBPCS model are also an integral part of commercial production control software packages, Olsmats, Edghill and Towill (1988).

The important contribution of Riddalls and Bennett (2002) was to re-cast the APIOBPCS model into a Smith Predictor and use Bellman and Cooke's (1963) theorem to obtain the stability criteria of the APIOBPCS model when the production/ distribution delay was a pure time delay. Upon investigation we found Riddalls and Bennett's (hereafter referred to as R&B2002) contribution to be in error, and this paper details our reasoning and corrects the situation.

The structure of our note is as follows. Section 2 defines the APIOBPCS model, and provides an overview of the stability details. Section 3 details some simulation results

that uncovered the discrepancy with R&B2002. Section 4 corrects the R&B2002 derivation of the stability criteria from Bellman and Cooke's theorem. Sections 5, 6 and 7 confirm the result with other analytical approaches including an exact solution to the characteristic equation in terms of the Lambert W function, the Nyquist stability criterion and the Routh Hurwitz array. Section 8 presents a summary of numerical and graphical results, whilst Section 9 concludes this note.

2. The APIOBPCS ordering decision

The APIOBPCS ordering decision is typically represented as a block diagram in the Laplace domain as shown by Figure 1, John, Naim and Towill (1994). R&B2002 made some slight modifications to the APIOBPCS model. Whereas, Figure 1 uses exponential smoothing as a forecasting mechanism in the ordering policy, R&B2002 employed a moving average forecasting mechanism. However, it is well known that feed-forward loops in a block diagram do not affect the stability of a system, so this difference will not have any impact on our comparative study. This can also be verified by a differential equation approach where the demand forecasting mechanism contributes only inhomogeneous terms to the differential equations. Hence the divergence of the solution (i.e., the instability) is solely due to the homogeneous terms, and so the stability is independent of the form of the demand forecasting mechanism.

R&B2002 use a slightly different notation for the gains in the two feedback loops, but this is directly translated into our notation by the following relationships: $\alpha = \alpha^i = 1/Ti$ and $\alpha^{WIP} = 1/Tw$. Furthermore, they also use for the delay, $h=Tp$ and the term ORATE corresponds to $p(t)$ in the time domain. The important fact is that from a stability point of view this model is exactly the same as the R&B2002 model.
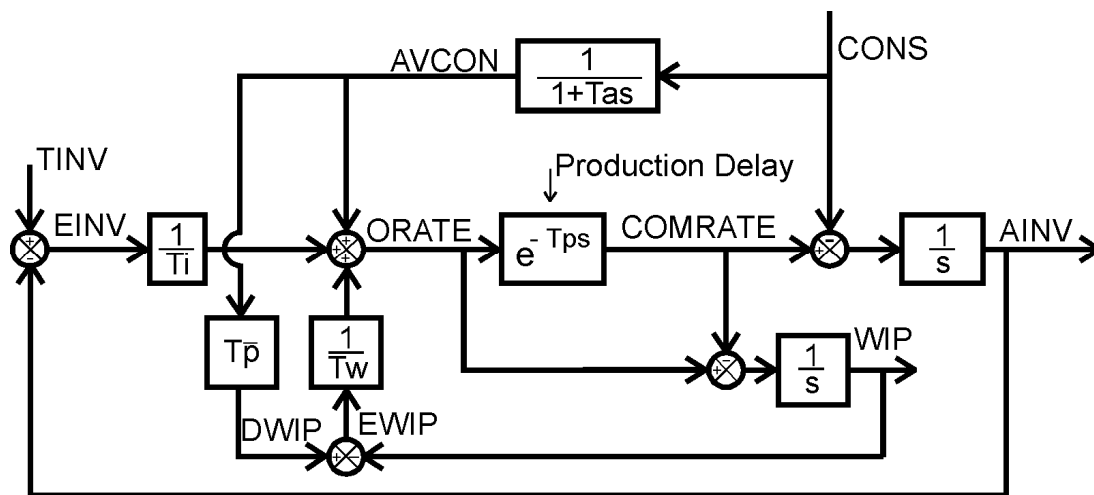

Figure 1. Block diagram of the APIOBPCS ordering decision

Note that when $Tw = \infty$ then the policy is referred to as the IOBPCS model (Towill, 1982). IOBPCS is shorthand for Inventory and Order Based Production Control System.

3. Simulation based verification of R&B2002's stability criteria

R&B2002 present the following relationship (Eq 1) that describes the maximum lead-time, h^*, for which the system is stable given values for α and β, where ($\beta = \alpha^{WIP} / \alpha^i = Ti / Tw$). (Equation 11 in R&B2002.)

$$h^* = \frac{arccos(\beta /(\beta - 1))}{\alpha\sqrt{1 - 2\beta^2}} \quad \Rightarrow \quad MaxTp = \frac{Ti.ArcCos\left[\dfrac{Ti}{Ti - Tw}\right]}{\sqrt{1 - \dfrac{2Ti^2}{Tw^2}}} \tag{1}$$

They also state that the APIOPBCS model is stable Independent of the Delay (IoD) if;

$$\beta = \frac{Ti}{Tw} > \frac{1}{2} \tag{2}$$

If equations (1) and (2) provide the stability boundary, then they should determine the maximum pure time delay (h^*, Tp) for which the system is stable given particular values of Ti and Tw. We had developed a Matlab simulation of the system and our investigation showed that the system was **not** critically stable on the boundary as defined above. It appears that the above criteria consistently under estimate the actual boundary. Table 1 summarizes some simulation results, and shows that on the average equations (1) and (2) underestimate h^*, (or Tp) by 25% for this particular sample. We do note however, that for the special case of $Tw = \infty$, (i.e. the IOPBCS design) the equations do provide the correct criteria.

4. The correct stability criterion via the Bellman and Cooke's theorem

Equation (10) of R&B2002 is:

$$\frac{dp(t)}{dt} + \alpha\beta p(t) + \alpha(1 - \beta)p(t - h) = 0 \tag{3}$$

From here the standard approach is to derive the characteristic equation by assuming a solution of the form $p = Ae^{st}$, which gives:

$$s + \alpha\beta + \alpha(1 - \beta)e^{-hs} = 0 \tag{4}$$

The solution is thus stable when $s < 0$. The stability of the roots of this equation are given by Theorem 13.8 of Bellman & Cooke (1963):

All the roots of $pe^z + q - ze^z = 0$ (p and q real) have negative real parts if and only if

(a) $p < 1$ *and*

(b) $p < -q < \sqrt{a^2 + p^2}$

where a is the root of $a = p\tan(a)$ such that $0 < a < \pi$, and if p = 0, we take $a = \pi / 2$

We now rearrange the characteristic equation so as to apply the above theorem. Multiply equation (4) by $-he^{sh}$, and set $z = sh$:

$$-h\alpha\beta e^z - h\alpha(1-\beta) - ze^z = 0 \qquad (5)$$

Equivalence with the Bellman & Cooke theorem is obtained by setting:

$$q = -h\alpha(1-\beta) \quad p = -h\alpha\beta \qquad (6)$$

We can now apply the Bellman & Cooke theorem to equation (5).
(a) $p < 1$
       (p is negative in equation (6), so (a) is satisfied identically)

(b) $p < -q < \sqrt{a^2 + p^2}$

The first part of the inequality is: $p < -q$, which gives:

$$-\beta\alpha h < \alpha h(1-\beta) \qquad -\beta < 1-\beta \qquad (7)$$
$$0 < 1$$

Therefore, the values of p and q in equation (6) also identically satisfy this part of the theorem. The second part of the Bellman & Cooke condition (b) is:

$$-q < \sqrt{a^2 + p^2} \qquad (8)$$

Dividing by p, squaring, re-arranging and taking the square root of both sides yields the following:

$$-\frac{q}{p} < \sqrt{\frac{a^2}{p^2} + 1} \qquad (9)$$

From the Bellman & Cooke theorem, $a = p\,tan(a)$, so

$$-\frac{q}{p} < \frac{1}{cos(a)} \qquad (10)$$

Substituting for p and q gives:

$$\frac{\alpha h(1-\beta)}{-\alpha\beta h} < \frac{1}{cos(a)} \qquad a = arc\,cos\left[\frac{\beta}{\beta-1}\right] \qquad (11)$$

Squaring the Bellman & Cooke condition and substituting for p and q gives:

$$\alpha^2 h^2 (1-\beta)^2 = a^2 + \alpha^2 h^2 \beta^2$$

$$\alpha^2 h^2 \left[(1-\beta)^2 - \beta^2 \right] = a^2 \tag{12}$$

$$\alpha^2 h^2 \left[1 - 2\beta \right] = a^2$$

$$h = \frac{a}{\alpha\sqrt{1-2\beta}} = \frac{arccos(\beta/(\beta-1))}{\alpha\sqrt{1-2\beta}} \tag{13}^1$$

Again this is not quite what is provided in R&B2002. They have:

$$h = \frac{arccos(\beta/(\beta-1))}{\alpha\sqrt{1-2\beta^2}} \tag{14}$$

For ease of comparison R&B2002's incorrect stability criteria (equation (14)) is plotted in Figure 2.   There also appears to be some confusion about the stable IoD criteria that has been incorporated into Figure 5 of R&B2002. The correct formula (our equation (13)) is also shown in *Figure 2* and it *agrees with all of the other results presented in this paper* (except the Padé approximation).

5. An exact solution to the characteristic equation

We now present the first of our methods of validating equation (13). It is not generally realized that one can solve equation (4) exactly. The key is to rearrange the equation as follows:

$$(s + \alpha\beta)e^{hs} = -h\alpha(1-\beta) \tag{15}$$

$$(sh + \alpha\beta h)e^{hs+\alpha\beta h} = -h\alpha(1-\beta)e^{\alpha\beta h} \tag{16}$$

The Lambert W function is defined as $W(z)e^{W(z)} = z$ and is known as the "golden ratio of the exponentials". Comparison with equation (16) yields:

$$W(z) = sh + \alpha\beta h \qquad z = -h\alpha(1-\beta)e^{\alpha\beta h} \tag{17}$$

Again, the system is stable when the real part of $s < 0$ (Corless et al, 1996) which from (17) is:

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[1] We note that in equation (13) that h becomes imaginary when $\beta > 1/2$. It is interesting, therefore, to examine the behaviour of the equation when $\beta = \dfrac{1}{2} - \varepsilon$.

$$h \to \frac{arccos\left(1 - \dfrac{2}{1+2\varepsilon}\right)}{\sqrt{2}\alpha\sqrt{\varepsilon}} \Rightarrow h \to \frac{arccos(-1)}{0} = \frac{\pi}{0} = \infty$$

We see that $h \to \infty$ as $\varepsilon \to 0$ and $\beta \to 1/2$. This is behaviour can be clearly seen in the corrected Figure 2.

$$s = \frac{W(z)}{h} - \alpha\beta < 0 \qquad \Rightarrow \qquad W(z) < h\alpha\beta \tag{18}$$

We emphasize that equation (18) is an ***exact*** solution to the stability problem defined by the characteristic equation. The predictions of Equation (18) agree exactly with the simulations in Table 1, and the other methods discussed below. Additionally Appendix B, uses the same procedure on the inventory response to further verify Equation 18.

As $Tw \to \infty$, $\beta \to 0$. An interesting property of $W(z)$ is that its real part is negative for $z > -\pi/2$ (Corless et al., 1996). Also, from (17), $z \to -\alpha$. The condition for stability when $Tw \to \infty$ is:

$$z = -\alpha > -\pi/2 \qquad \Rightarrow \qquad Ti > \frac{2Tp}{\pi} \tag{19}$$

6. Nyquist's stability criteria for IOBPCS

As a second cross-check, we employ the Nyquist Stability Technique (Towill 1971), as outlined in Appendix A. We have found the following criteria that ensures stability for a policy with only an inventory feedback loop (i.e. when $Tw = \infty$). Simulation shows this is completely accurate, see Table 1, but unfortunately it does not describe the complete solution space. We note that this result also agrees with equation (19).

$$Ti > \frac{2Tp}{\pi} \tag{20}$$

7. Stability of APIOPBCS via Padé approximation and the Routh Hurwitz array

In order to gain a better understanding of the general stability criteria in continuous time we have replaced the pure time delay for a second order Padé approximation and found the following stability boundary as detailed in Appendix C.

$$Ti > \frac{Tp^2Tw + 7TpTw^2}{3\left(Tp^2 + 5TpTw + 4Tw^2\right)} \tag{21}$$

Comparison of our second order Padé approximation stability criterion to the true time delay results obtained via simulation and is also shown in Table 2 (when $Tw \neq \infty$). Eq 21 over-estimates the stability boundary, but the match is close for low values of β.

8. Numerical results

Settings		Simulation	R&B2002		Bellman and Cooke		Lambert W Function		Padé Approximation	
Tp	Tw	Ti	Tp via Eq 1	Eq 1 % error	Tp via Eq 13	Eq 13 % error	Ti via Eq 18 and B4	Eq 18 and B4 % error	Ti via Eq 21	Eq 21 % error
1	1	**0.306**	0.689	-31.1	1	0	0.306	0	0.266	-12.9
3	1	**0.436**	1.359	-54. 7	3	0	0.436	0	0.357	-18.0
5	1	**0.469**	1.662	-66.7	5	0	0.469	0	0.370	-21.0
1	3	**0.470**	0.847	-15.2	1	0	0.470	0	0.423	-10.0
3	3	**0.919**	2.068	-31.0	3	0	0.919	0	0.8	-12.9
5	3	**1.126**	2.943	-41.1	5	0	1.126	0	0.956	-15.1
1	5	**0.525**	0.897	-10.2	1	0	0.525	0	0.476	-9.3
3	5	**1.165**	2.319	-22.7	3	0	1.165	0	1.033	-11.3
5	5	**1.532**	3.447	-31.0	5	0	1.532	0	1.333	-12.9
1	∞	**0.636**	1	0	1	0	0.636	0	0.636*	0
3	∞	**1.909**	3	0	3	0	1.909	0	1.909*	0
5	∞	**3.181**	5	0	5	0	3.181	0	3.183*	0
		Average Error	-25%		0%		0%		-10%	

Table 1. Accuracy of the various stability criteria considered
(* Found using the Eq 19/20)

For interest we have plotted R&B2002 stability criterion and compared it to the correct version in Figure 2 below.
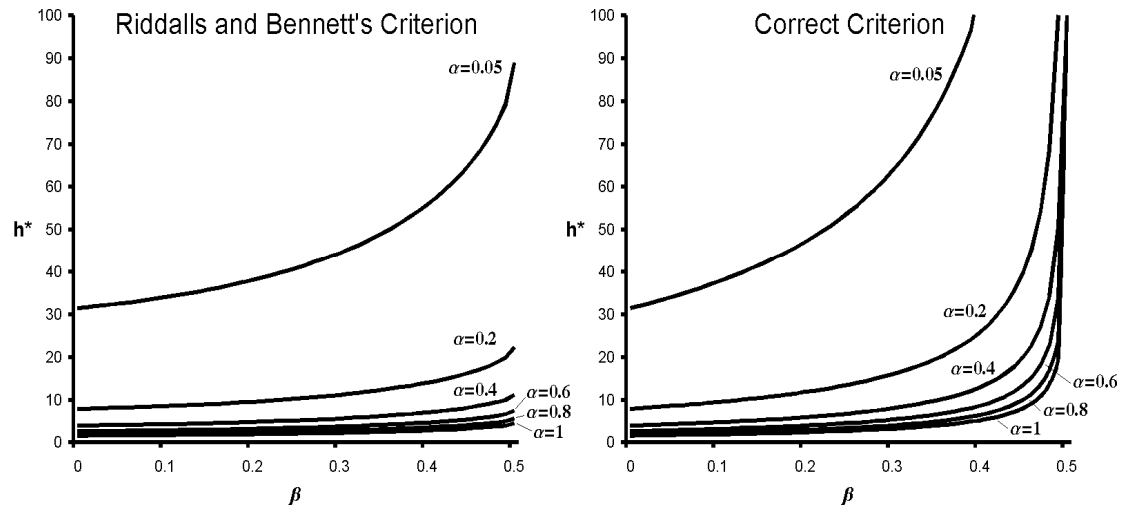

Figure 2. Graphical comparison the R&B2002 and the correct stability criterion

9. Conclusion

After an investigation of the continuous time APIOBPCS stability boundary with a simulation approach, we found discrepancies with the R&B2002 stability criterion. This led us to rectify their analysis, which now agrees exactly with our simulation results. Furthermore, we have independently verified the corrected criterion by

exactly solving the characteristic equation for stability in terms of the Lambert W Function. The $Tw \to \infty$ limit agrees with the Nyquist stability criterion. As should be expected, these results are also broadly in agreement with a second order Padé approximation.

Appendix A. Stability of IOBPCS via the Nyquist criteria

The characteristic equation of IOBPCS with a pure time delay is given by;

$$\frac{y}{x} = \frac{e^{-Tps}}{Tis} \tag{A1}$$

This is easily checked in the frequency domain via the Nyquist Stability Criterion. Hence;

$$\frac{e^{-jwTp}}{jwTi} = -1 \tag{A2}$$

The criterion is then expressed as:

$$\left| \frac{e^{-jwTp}}{jwTi} \right| = -1 \quad \angle \quad \frac{e^{-jwTp}}{jwTi} = -180° \tag{A3}$$

Now $jwTi$ gives a constant phase lag of $90°$. So ω_c, the critical frequency, occurs when the phase lag from the exponent is $90°$, i.e. $\pi/2$. The phase shift from the exponent must therefore be $\omega_c Tp = \pi/2$. Hence the critical $\omega_c = \pi/2Tp$. Substituting back into the equation for the amplitude ratio, bearing in mind that the amplitude of the exponent is unity at all frequencies, we have $|1/\omega_c Ti| = 1$. Substituting for ω_c we have:

$$\left[\frac{Ti}{Tp} \right]_{crit} = \frac{2}{\pi} \tag{A4}$$

Which is the fundamental requirement for stability in the continuous time IOBPCS model, that is the APIOBPCS when $Tw = \infty$.

Appendix B. Stability of APIOPBCS by the Lambert W Function

The characteristic equation determining the stability of the policy with a pure time delay in continuous time is obtained from the solution to the differential equations. The following outlines the approach. The inventory is depleted by the demand, *d(t)*, (here assumed to be a step function) and increased by the receipts *r(t)*, which are the orders, *o(t)*, (*p(t)* in Eq 13) just time delayed by the lead-time.

$$\frac{dns(t)}{dt} = r(t) - d(t) = o(t - Tp) - d \tag{B1}$$

The APIOPBCS ordering policy can be written as:

$$o(t) = \frac{tns - \left(ns(t) + \dfrac{Ti}{Tw} \int_{t-Tp}^{t} o(t)dt \right)}{Ti} \tag{B2}$$

Here, *tns* is the target net stock. The WIP term is represented by the sum of the orders during the fulfilment delay, *Tp*. The solution to these equations can be obtained by assuming the following form $o(t) = Ae^{qt}$. One can then integrate equation (B1) to find *ns(t)*. Substituting the expressions for *o(t)* and *ns(t)* into equation (B2) results in a characteristic equation for *q* that can be solved in terms of the Lambert W function:

$$q = -\frac{1}{Tw} + \frac{W[z]}{Tp} \qquad z = e^{Tp/Tw} \frac{Tp}{Tw}\left(1 - \frac{Tw}{Ti}\right) \tag{B3}$$

W(z) is complex if $z < -1/e$. This can occur if *Tw* > *Ti*. If *q* has a positive real part, then the inventory diverges for large *t*. We observe that the real part of *W(z)* changes sign at $z = \pi/2$, and is negative for $z > -\pi/2$. Therefore, *q = 0* gives the critical stability criterion, for APIOBPCS with a pure time delay. Collecting together the two equations in (B3) we have the following stability test.

$$q = -\frac{1}{Tw} + \frac{W\left[e^{Tp/Tw} Tp \left(1 - \dfrac{Tw}{Ti}\right) \Big/ Tw \right]}{Tp} \tag{B4}$$

Unfortunately, Eq (B4) is very difficult to manipulate algebraically, and we have to resort to numerical and/or graphical techniques to obtain the stability boundary, explicitly in terms of *Ti*, *Tw* or *Tp*. Never-the-less, the expression is analytically exact, and precisely verifies the results in Table 1.

Appendix C. Stability of APIOPBCS via a second order Padé approximation

The characteristic equation determining the stability of APIOBPCS in continuous time is obtained from the loop equation (Eq C1) for the OUT policy that yields the relationship shown in C2;

$$\frac{y}{x} = \frac{1}{Tis}\left[P \Big/ \left(1 + \frac{1-P}{Tws}\right) \right] \tag{C1}$$

$$\left[1 + P \Big/ Tis\left(1 - \frac{1-P}{Tws}\right) \right] = 0 \tag{C2}$$

If P, the production delay, is an exponential lag then equation B2 is difficult to solve via customary methods of investigating stability in the frequency domain. In this approach we shall exploit the representation of P by the second order Padé approximation, Truxal (1995, p550). Therefore,

$$P = e^{-Tps} \approx \left\{ \frac{1 - \dfrac{Tps}{3}}{1 + 2\dfrac{Tps}{3} + \dfrac{Tp^2 s^2}{6}} \right\} \tag{C3}$$

Substitution of this approximation for e^{-Tps} in Eq B2 will lead to a rational polynomial in "s" that can be tackled using traditional methods. After substitution we obtain the following characteristic equation for stability testing.

$$C(s) = 6Tw + s(6TiTp + 6TiTw - 2TpTw) + \\ s^2(TiTp^2 + 4TiTpTw) + s^3 TiTp^2 Tw \tag{C4}$$

This equation is of the form

$$C(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 \tag{C5}$$

and we know, from the Routh Hurwitz array, Disney and Towill (2002), that for stability $a_1 a_2 > a_3 a_0$. This establishes the following relationship;

$$\begin{pmatrix} TiTp(Tp + 4Tw) \\ (-2TpTw + 6Ti(Tp + Tw)) \end{pmatrix} > 6TiTp^2 Tw^2 \tag{C6}$$

Further re-arranging yields the following stability criteria for the value of *Ti* as a function of *Tp* and *Tw*.

$$Ti > \frac{Tp^2 Tw + 7TpTw^2}{3(Tp^2 + 5TpTw + 4Tw^2)} \tag{C7}$$

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