Renormalized and Entropy Solutions of Tumor Growth Model with Nonlinear Acid Production

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Abstract. This paper establishes the existence of renormalized and entropy solutions for a system of nonlinear reaction-diffusion equations which describes the tumor growth along with acidification and interaction. Under the assumptions of L^1 data and no growth conditions with zero Dirichlet boundary conditions, we prove the existence of renormalized and entropy solutions for the considered mathematical model.

Keywords: acid-mediated tumor growth, renormalized solution, entropy solution.

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1 Introduction

Acid-mediated tumor invasion model confines a mechanism linking altered glucose metabolism with the ability of tumor cells to form invasive cancers. Glucose metabolism and increased glucose uptake observed in the majority of clinical cancers which are critical for development of the invasive phenotype. Tumor cells are resistance to acid induced toxicity which survive and proliferate in low pH micro environments, invade the damaged adjacent normal tissues. First, acidification of the tumor micro environment is shown by Gatenby et al. [15] and Martin et al. [17] to increase invasiveness and metastasis of cancer cells using mathematical model. Acidification induced by the result of glycolysis both in the presence of oxygen through Warburg effect and intermittent hypoxia which produce toxicity in the surrounding normal tissue provides the empty space for tumor cell proliferation and invasion. Gatenby and Gawlinski [14] used the acid-mediated invasion hypothesis in a reaction-diffusion

framework which plays an important role in tumor progression. The mathematical representation of a reaction-diffusion system at the tissue scale in which many mathematical models have been developed to explore the relationships between tumor invasion, tissue acidity and cellular metabolism and energy requirements, see, for example, Bertuzzi et al. [6], Bianchini and Fasano [7], Ganesan and Lingeshwaran [13], McGillen et al. [19], Smallbone et al. [26], Tao et al. [27] and Venkatasubramanian et al. [28].

Acidification factor plays a major role in the development of tumor growth models. Therefore, in this work, we consider a PDE model for acid-mediated tumor growth extended from [14] with nonlinear acidification term to capture a wider range of tumor behaviors. Here, we also have included the interacting phenomena, that is, as tumor cell density increases with time, metabolism which produces H^+ ions leads to destruction of normal cells surrounding the tumor and thus a reduction in normal cell density. Thus, the nonlinear reaction-diffusion mathematical model is given by

$$
\begin{cases}
\dfrac{\partial u_1}{\partial t} - \mathrm{div}(\sigma_1(u_2)d_1(u_1, \nabla u_1)) = r_1(\rho_1(u_1) - u_2\mu_1(u_1)) \\
\qquad\qquad -\pi(u_1, u_3) + f(x, t) \ \text{ in } \ Q_T, \\
\dfrac{\partial u_2}{\partial t} - \mathrm{div}(\sigma_2(u_1)d_2(u_2, \nabla u_2)) = r_2(\rho_2(u_2) - u_1\mu_2(u_2)) + g(x, t) \, \text{in } Q_T, \\
\dfrac{\partial u_3}{\partial t} - \sigma_3 \mathrm{div}(d_3(u_3, \nabla u_3)) = r_3\rho_2(u_2) - \alpha_3 u_3 + h(x, t) \ \text{ in } \ Q_T
\end{cases}
\tag{1.1}
$$

with initial and boundary conditions

$$
\begin{aligned}
u_i(x, 0) &= u_{i,0}(x), \ \ i = 1, 2, 3 \ \ \text{ in } \ \Omega, \\
u_i(x, t) &= 0, \ \ i = 1, 2, 3 \qquad \text{ on } \ \Sigma_T,
\end{aligned}
$$

where $Q_T = \Omega \times (0, T)$, $\Sigma_T = \partial\Omega \times (0, T)$, Ω is an open bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ (no smoothness is assumed on the boundary $\partial\Omega$). The mathematical model consists of three unknown variables: normal cell density $u_1(x, t)$, tumor cell density $u_2(x, t)$ and acid concentration $u_3(x, t)$. The homogeneous Dirichlet boundary condition means that the model (1.1) is self-contained and has no population on the boundary $\partial\Omega$. The density dependent diffusion coefficients for the normal cells and tumor cells are given by $\sigma_1(u_2)$ and $\sigma_2(u_1)$ respectively. Further, the excess H^+ ions diffuse chemically with constant diffusion rate with σ_3. The normal cells and tumor cells obey the logistic growth $\rho_1(u_1) = u_1(1 - \frac{u_1}{K_1})$, $\rho_2(u_2) = u_2(1 - \frac{u_2}{K_2})$ with growth rates r_1 and r_2 and carrying capacities K_1 and K_2. The competition relationship between the normal cells and tumor cells with the rates a_1 and a_2 are given by $\mu_1(u_1) = \frac{a_1 u_1}{K_1}$ and $\mu_2(u_2) = \frac{a_2 u_2}{K_2}$. The interaction of healthy cells with the excess H^+ ions is given by $\pi(u_1, u_3) = \alpha_1 u_1 u_3$ which leads to a death rate proportional to the concentration of H^+ ions and denote the constant of proportionality by α_1. Acidification caused at a rate r_3 proportional to the tumor cell density and an uptake term with constant proportionality α_3 is included to measure the mechanisms for increasing extracellular pH. Moreover, $f(x, t), g(x, t)$ and $h(x, t)$ denote the source terms of the respective equations. Further, we have assumed that all the coefficients are positive and $\sigma_i(\cdot) \in \mathbb{C}^2(\mathbb{R})$, $i = 1, 2$.

The notion of renormalized solution is introduced by DiPerna and Lions for Boltzmann equations in [11]. Further, the same framework has extended for elliptic equations, parabolic equations and conservation laws, for example, see [5, 8, 9, 10, 18, 22, 23] and also see the references therein. Furthermore, a new class of study, equivalence between renormalized and entropy solutions studied for parabolic equations by many researchers, for example, see [1, 3, 12]. Moreover, in the literature, considerable amount of works are available for the existence and uniqueness of biological models using various mathematical techniques, for example, see, [2, 4, 24, 25] and the references therein. As far as, acid-mediated cancer invasion model is concerned, only few papers available in the literature. Local and global existence of solutions of the model governed by acid-mediated tumor invasion established in [20]. Acid-mediated invasion model for tumor-stromal interactions under no flux boundary condition is concerned in [16] and the global existence and uniqueness proved using contraction mapping principle. A mathematical model focusing on the effect of heat shock proteins on the tumor cell migration is proposed and the local existence of a unique positive solution is obtained in [21]. On the otherhand, existence of renormalized and entropy solutions for the system of parabolic equations concerned, only few papers available in the literature, see, [3, 4, 24, 25]. Therefore, in contrast to the above mentioned papers, in this work, the main novel point is to establish the existence of renormalized and entropy solutions of the model governed by the acid-mediated tumor growth under no growth conditions and integrable data.

The paper is organized as follows. In Section 2, we state the main theorem, that is, existence of renormalized solution of the parabolic system (1.1). Then, we introduce the regularized system of (1.1) and establish the existence of weak solutions of the regularized system. Further, we prove the existence of renormalized solution of (1.1) using the lemmas established in that section. Finally, in Section 3, we prove that renormalized solution of (1.1) is also an entropy solution.

2 Renormalized solutions for cancer invasion system

In this section, first we define the renormalized solution for the given parabolic system (1.1). After stating the main result of the work, we introduce regularized system for (1.1) and then we establish the existence of weak solutions of the regularized problem using the Faedo-Galerkin approximation method. Furthermore, we state and prove the certain lemmas which are useful to prove the existence of renormalized solutions of (1.1).

DEFINITION 1. A renormalized solution of (1.1) is a set of functions (u_1, u_2, u_3) satisfying the following conditions: $u_1(x, t)$, $u_2(x, t)$ and $u_3(x, t)$ are all non-negative for a.e. $(x, t) \in Q_T$. For $i = 1, 2, 3$

$$u_i \in L^\infty(0, T; L^1(\Omega)) \cap L^2(0, T; H_0^1(\Omega)), \quad T_k(u_i) \in L^2(0, T; H_0^1(\Omega))$$

$$\text{for any } k \geq 0, \quad \int_{(n \leq |u_i| \leq n+1)} d_i(u_i, \nabla u_i) \nabla u_i \mathrm{d}x \mathrm{d}t \to 0 \ \text{ as } \ n \to \infty.$$

For all $S(u_i) \in C^\infty(\mathbb{R})$ with supp S' is compact,

$$\frac{\partial S(u_1)}{\partial t} - \mathrm{div}(\sigma_1(u_2)S'(u_1)d_1(u_1, \nabla u_1)) + \sigma_1(u_1)d_1(u_1, \nabla u_1)S''(u_1)\nabla u_1$$
$$= r_1\rho_1(u_1)S'(u_1) - r_1u_2\mu_1(u_1)S'(u_1) - \pi(u_1, u_3)S'(u_1) + fS'(u_1) \text{ in } \mathcal{D}'(Q_T),$$

$$\frac{\partial S(u_2)}{\partial t} - \mathrm{div}(\sigma_2(u_1)S'(u_2)d_2(u_2, \nabla u_2)) + \sigma_2(u_1)d_2(u_2, \nabla u_2)S''(u_2)\nabla u_2$$
$$= r_2\rho_2(u_2)S'(u_2) - r_2u_1\mu_2(u_2)S'(u_2) + gS'(u_2) \quad \text{in } \mathcal{D}'(Q_T),$$

$$\frac{\partial S(u_3)}{\partial t} - \sigma_3\mathrm{div}(S'(u_3)d_3(u_3, \nabla u_3)) + \sigma_3 d_3(u_3, \nabla u_3)S''(u_3)\nabla u_3$$
$$= r_3\rho_2(u_2)S'(u_3) - \alpha_3 u_3 S'(u_3) + hS'(u_3) \quad \text{in } \mathcal{D}'(Q_T)$$

and the initial conditions

$$S(u_i(x,0)) = S(u_{i,0}(x)) \quad \text{in } \Omega.$$

Here, $\mathcal{D}(Q_T)$ denotes the set of all infinitely differentiable functions on Q_T with compact support and the distributions on Q_T are denoted by $\mathcal{D}'(Q_T)$. We assume the following hypotheses to establish the existence of weak-renormalized solutions of the system (1.1), as in [8, 9]:

(H1) $d_i(\eta, \zeta)\zeta \geq \gamma_i|\zeta|^2$, for every $\zeta \in \mathbb{R}^N$ and $\gamma_i > 0$, $i = 1, 2, 3$ is given real number.

(H2)   For any $k > 0$, there exists $C_k > 0$ such that $|d_1(\eta, \zeta)| \leq C_k(1 + |\zeta|)$.

(H3) $(d_i(s, \zeta) - d_i(s, \zeta'))(\zeta - \zeta') \geq 0$ for $\zeta, \zeta' \in \mathbb{R}^N$, $i = 1, 2, 3$.

(H4) $u_{i,0}(x), \in L^1(\Omega)$, $i = 1, 2, 3$.

(H5) $f, g, h \in L^1(Q_T)$.

Further, we introduce the truncation function at the height k,

$$T_k(z) = \begin{cases} k, & \text{if} \quad z \geq k, \\ z, & \text{if} \quad |z| \leq k, \\ -k, & \text{if} \quad z \leq -k, \end{cases}$$

$$\widetilde{T}_k(z) = \int_0^z T_k(s)\mathrm{d}s = \begin{cases} z^2/2, & \text{if} \quad |z| \leq k, \\ k|z| - k^2/2, & \text{if} \quad |z| \geq k. \end{cases}$$

Moreover, throughout this work, we use a generic constant C instead of different constants.

Theorem 1.   *Under the hypotheses* $(H1) - (H5)$*, there exists at least one renormalized solution of the system* (1.1) *in the sense of Definition 1.*

For $\varepsilon > 0$, let us introduce the following approximations of the data:

(H6) $u_{i,0}^\varepsilon \in L^2(\Omega)$, and $u_{i,0}^\varepsilon \to u_{i,0}$, $i = 1, 2, 3$ a.e. in Ω and strongly in $L^1(\Omega)$ as ε tends to zero.

(H7) $f^\varepsilon, g^\varepsilon, h^\varepsilon \in L^2(Q_T)$ and $f^\varepsilon \to f$, $g^\varepsilon \to g$, $h^\varepsilon \to h$ a.e. in Q_T and strongly in $L^1(Q_T)$ as ε tends to zero.

Consider the following approximation problem of the system (1.1) for $\varepsilon > 0$:

$$
\begin{cases}
\dfrac{\partial u_1^\varepsilon}{\partial t} - \operatorname{div}(\sigma_1(u_2^\varepsilon)d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)) = r_1\rho_1^\varepsilon(u_1^\varepsilon) - r_1 u_2^\varepsilon \mu_1^\varepsilon(u_1^\varepsilon) \\
\qquad\qquad - \pi^\varepsilon(u_1^\varepsilon, u_3^\varepsilon) \text{ in } Q_T, \\
\dfrac{\partial u_2^\varepsilon}{\partial t} - \operatorname{div}(\sigma_2(u_1^\varepsilon)d_2(u_2^\varepsilon, \nabla u_2^\varepsilon)) = r_2\rho_2^\varepsilon(u_2^\varepsilon) - r_2 u_1^\varepsilon \mu_2^\varepsilon(u_2^\varepsilon) + g^\varepsilon \text{ in } Q_T, \\
\dfrac{\partial u_3^\varepsilon}{\partial t} - \sigma_3 \operatorname{div}(d_3(u_3^\varepsilon, \nabla u_3^\varepsilon)) = r_3\rho_2^\varepsilon(u_2^\varepsilon) - \alpha_3 u_3^\varepsilon + h^\varepsilon \text{ in } Q_T
\end{cases}
\tag{2.1}
$$

with initial and boundary conditions

$$
\begin{aligned}
u_i^\varepsilon(x,0) &= u_{i,0}^\varepsilon(x), \quad i = 1,2,3 \quad \text{in } \Omega, \\
u_i^\varepsilon(x,t) &= 0, \quad i = 1,2,3 \qquad\quad \text{on } \Sigma_T,
\end{aligned}
$$

where $\rho_i^\varepsilon(u_i) = \rho_i(u_i^\varepsilon)/(1 + \varepsilon\rho_i(u_i^\varepsilon))$, $\mu_i^\varepsilon(u_i^\varepsilon) = \mu(u_i^\varepsilon)/(1 + \varepsilon\mu(u_i^\varepsilon))$, for $i = 1,2$ and $\pi^\varepsilon(u_1^\varepsilon, u_3^\varepsilon) = \pi(u_1^\varepsilon, u_3^\varepsilon)/(1 + \varepsilon\pi(u_1^\varepsilon, u_3^\varepsilon))$.

Lemma 1. *Under the hypotheses $(H1-H3)$ and $(H6-H7)$ the approximation system (2.1) admits unique weak solution*

$$
u_i^\varepsilon \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega)) \cap C([0,T], L^2(\Omega)),
$$

with $\frac{\partial u_i^\varepsilon}{\partial t} \in L^2(0,T; H^{-1}(\Omega))$ such that for any $\phi_i \in L^2(0,T; H_0^1(\Omega))$, $i = 1,2,3$,

$$
\int_0^T \langle \partial_t u_1^\varepsilon, \phi_1 \rangle \mathrm{d}t + \int_{Q_T} \sigma_1(u_2^\varepsilon)d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)\nabla\phi_1 \mathrm{d}x\mathrm{d}t = r_1\int_{Q_T} \rho_1^\varepsilon(u_1^\varepsilon)\phi_1 \mathrm{d}x\mathrm{d}t
$$

$$
- r_1\int_{Q_T} u_2^\varepsilon \mu_1^\varepsilon(u_1^\varepsilon)\phi_1 \mathrm{d}x\mathrm{d}t - \int_{Q_T} \pi^\varepsilon(u_1^\varepsilon, u_3^\varepsilon)\phi_1 \mathrm{d}x\mathrm{d}t + \int_{Q_T} f^\varepsilon\phi_1 \mathrm{d}x\mathrm{d}t,
$$

$$
\int_0^T \langle \partial_t u_2^\varepsilon, \phi_2 \rangle \mathrm{d}t + \int_{Q_T} \sigma_2(u_1^\varepsilon)d_2(u_2^\varepsilon, \nabla u_2^\varepsilon)\nabla\phi_2 \mathrm{d}x\mathrm{d}t
$$

$$
= r_2\int_{Q_T} \rho_2^\varepsilon(u_2^\varepsilon)\phi_2 \mathrm{d}x\mathrm{d}t - r_2\int_{Q_T} u_1^\varepsilon \mu_2^\varepsilon(u_2^\varepsilon)\phi_2 \mathrm{d}x\mathrm{d}t + \int_{Q_T} g^\varepsilon\phi_2 \mathrm{d}x\mathrm{d}t,
$$

$$
\int_0^T \langle \partial_t u_3^\varepsilon, \phi_3 \rangle \mathrm{d}t + \sigma_3\int_{Q_T} d_3(u_3^\varepsilon, \nabla u_3^\varepsilon)\nabla\phi_3 \mathrm{d}x\mathrm{d}t
$$

$$
= r_3\int_{Q_T} \rho_2^\varepsilon(u_2^\varepsilon)\phi_3 \mathrm{d}x\mathrm{d}t - \alpha_3\int_{Q_T} u_3^\varepsilon\phi_3 \mathrm{d}x\mathrm{d}t + \int_{Q_T} h^\varepsilon\phi_3 \mathrm{d}x\mathrm{d}t,
$$

hold.

Proof.   To use the Faedo-Galerkin approximation method, let us consider an appropriate spectral problem, see [4], in which the corresponding eigenfunctions $e_l(x)$ form an orthogonal basis in $H_0^1(\Omega)$ and orthonormal basis in $L^2(\Omega)$. Our aim is to identify the finite dimensional approximation solutions for the system (2.1) as sequences $\{u_{i,n}^\varepsilon\}$, $i = 1,2,3$ defined for $n \geq 1$, $t \geq 0$ and $x \in \overline{\Omega}$ by

$$
u_{i,n}^\varepsilon(x,t) = \sum_{l=1}^n c_{i,n,l}(t)e_l(x), \quad i = 1,2,3
$$

with the initial conditions

$$u_{i,n}^{\varepsilon}(x,0) = u_{i,0,n}(x) := \sum_{l=1}^{n} c_{i,n,l}(0)e_l(x).$$

Further, it should be remarked that the above form of solutions should satisfy the required boundary conditions. Next, we have to determine the set of coefficients $\{c_{i,n,l}\}_{l=1}^{n}, i = 1, 2, 3$ such that, for $m = 1, 2, \ldots, n$

$$
\begin{cases}
\displaystyle\int_{\Omega} \partial_t u_{1,n}^{\varepsilon} e_m \mathrm{d}x + \int_{\Omega} \sigma_1(u_{2,n}^{\varepsilon})d_1(u_{1,n}^{\varepsilon}, \nabla u_{1,n}^{\varepsilon})\nabla e_m \mathrm{d}x = r_1 \int_{\Omega} \rho_1^{\varepsilon}(u_{1,n}^{\varepsilon})e_m \mathrm{d}x \\
\quad - r_1 \displaystyle\int_{\Omega} u_{2,n}^{\varepsilon}\mu_1^{\varepsilon}(u_{1,n}^{\varepsilon})e_m \mathrm{d}x - \int_{\Omega} \pi^{\varepsilon}(u_{1,n}^{\varepsilon}, u_{3,n}^{\varepsilon})e_m \mathrm{d}x + \int_{\Omega} f^{\varepsilon} e_m \mathrm{d}x, \\
\displaystyle\int_{\Omega} \partial_t u_{2,n}^{\varepsilon} e_m \mathrm{d}x + \int_{\Omega} \sigma_2(u_{1,n}^{\varepsilon})d_2(u_{2,n}^{\varepsilon}, \nabla u_{2,n}^{\varepsilon})\nabla e_m \mathrm{d}x \\
\quad = r_2 \displaystyle\int_{\Omega} \rho_2^{\varepsilon}(u_{2,n}^{\varepsilon})e_m \mathrm{d}x - r_2 \int_{\Omega} u_{1,n}^{\varepsilon}\mu_2^{\varepsilon}(u_{2,n}^{\varepsilon})e_m \mathrm{d}x + \int_{\Omega} g^{\varepsilon} e_m \mathrm{d}x, \\
\displaystyle\int_{\Omega} \partial_t u_{3,n}^{\varepsilon} e_m \mathrm{d}x + \sigma_3 \int_{\Omega} d_3(u_{3,n}^{\varepsilon}, \nabla u_{3,n}^{\varepsilon})\nabla e_m \\
\quad = r_3 \displaystyle\int_{\Omega} \rho_2^{\varepsilon}(u_{2,n}^{\varepsilon})e_m \mathrm{d}x - \alpha_3 \int_{\Omega} u_{3,n}^{\varepsilon} e_m \mathrm{d}x + \int_{\Omega} h^{\varepsilon} e_m \mathrm{d}x.
\end{cases}
\tag{2.2}
$$

Now (2.2) can be rewritten in the following form:

$$
\begin{cases}
c_{1,n,m}'(t) = -\displaystyle\int_{\Omega} \sigma_1(u_{2,n}^{\varepsilon})d_1(u_{1,n}^{\varepsilon}, \nabla u_{1,n}^{\varepsilon})\nabla e_m \mathrm{d}x + r_1 \int_{\Omega} \rho_1^{\varepsilon}(u_{1,n}^{\varepsilon})e_m \mathrm{d}x \\
\quad - r_1 \displaystyle\int_{\Omega} u_{2,n}^{\varepsilon}\mu_1^{\varepsilon}(u_{1,n}^{\varepsilon})e_m \mathrm{d}x - \int_{\Omega} \pi^{\varepsilon}(u_{1,n}^{\varepsilon}, u_{3,n}^{\varepsilon})e_m \mathrm{d}x + \int_{\Omega} f^{\varepsilon} e_m \mathrm{d}x \\
\quad =: G_1^m\left(t, \{c_{1,n,l}\}_{l=1}^{n}, \{c_{2,n,l}\}_{l=1}^{n}, \{c_{3,n,l}\}_{l=1}^{n}\right), \\
c_{2,n,m}'(t) = -\displaystyle\int_{\Omega} \sigma_2(u_{1,n}^{\varepsilon})d_2(u_{2,n}^{\varepsilon}, \nabla u_{2,n}^{\varepsilon})\nabla e_m \mathrm{d}x + r_2 \int_{\Omega} \rho_2(u_{2,n}^{\varepsilon})e_m \mathrm{d}x \\
\quad - r_2 \displaystyle\int_{\Omega} u_{1,n}^{\varepsilon}\mu_2^{\varepsilon}(u_{2,n}^{\varepsilon})e_m \mathrm{d}x + \int_{\Omega} g^{\varepsilon} e_m \mathrm{d}x \\
\quad =: G_2^m\left(t, \{c_{1,n,l}\}_{l=1}^{n}, \{c_{2,n,l}\}_{l=1}^{n}\right), \\
c_{3,n,m}'(t) = -\sigma_3 \displaystyle\int_{\Omega} d_3(u_{3,n}^{\varepsilon}, \nabla u_{3,n}^{\varepsilon})\nabla e_m \mathrm{d}x + r_3 \int_{\Omega} \rho_2^{\varepsilon}(u_{2,n}^{\varepsilon})e_m \mathrm{d}x \\
\quad - \alpha_3 \displaystyle\int_{\Omega} u_{3,n}^{\varepsilon} e_m \mathrm{d}x + \int_{\Omega} h^{\varepsilon} e_m \mathrm{d}x =: G_3^m\left(t, \{c_{2,n,l}\}_{l=1}^{n}, \{c_{3,n,l}\}_{l=1}^{n}\right).
\end{cases}
\tag{2.3}
$$

Let $\kappa \in (0, T)$ and set $U = [0, \kappa]$. Choose $R > 0$ large enough so that the ball $B_R \subset \mathbb{R}^N$ contains $\{c_{i,n,l}(0)\}, i = 1, 2, 3$ and set $V = \overline{B_R}$. The components $G_i^m, i = 1, 2, 3$ can be bounded on $U \times V$ as follows:

$$\left| G_1^m\left(t, \{c_{1,n,l}\}_{l=1}^{n}, \{c_{2,n,l}\}_{l=1}^{n}, \{c_{3,n,l}\}_{l=1}^{n}\right) \right| \leq \left\| \sigma_1\left(\sum_{l=1}^{n} c_{2,n,l}(t)e_l(x)\right) \right\|_{L^{\infty}(\Omega)}$$

$$\times \left(\int_{\Omega} \left| d_1\left(\sum_{l=1}^{n} c_{1,n,l}(t)e_l(x), \sum_{l=1}^{n} c_{1,n,l}(t)\nabla e_l(x)\right) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}}$$

$$\left(\int_\Omega |\nabla e_m|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} + \frac{(r_1+1)}{\varepsilon}\mathrm{meas}(\Omega)\left(\int_\Omega |e_m|^2 \, \mathrm{d}x\right)^{\frac{1}{2}}$$

$$+ \frac{r_1}{\varepsilon}\left(\int_\Omega \left|\sum_{l=1}^n c_{2,n,l}(t)e_l(x)\right|^2 \, \mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega |e_m|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} + \int_\Omega f^\varepsilon e_m \, \mathrm{d}x$$

$$\leq C(R,n),$$

where the constant $C(R,n) > 0$ depends only on R and n. Similarly it is easy to obtain that

$$|G_2^m\left(t, \{c_{1,n,l}\}_{l=1}^n, \{c_{2,n,l}\}_{l=1}^n\right)| \leq C(R,n),$$
$$|G_3^m\left(t, \{c_{2,n,l}\}_{l=1}^n, \{c_{3,n,l}\}_{l=1}^n\right)| \leq C(R,n),$$

where the constant $C(R,n) > 0$ depends only on R, n.

According to the standard ODE theory, one can show that the absolutely continuous functions $\{c_{i,n,l}\}_{l=1}^n$, $i = 1,2,3$ satisfy (2.3) and the initial conditions for $t \in [0, \kappa']$ a.e. where $\kappa' > 0$. Moreover, we have

$$c_{1,n,l}(t) = c_{1,n,l}(0)$$
$$+ \int_0^t G_1^l\left(\tau, \{c_{1,n,m}(\tau)\}_{m=1}^n, \{c_{2,n,m}(\tau)\}_{m=1}^n, \{c_{3,n,m}(\tau)\}_{m=1}^n\right) \mathrm{d}\tau,$$

$$c_{2,n,l}(t) = c_{2,n,l}(0) + \int_0^t G_2^l\left(\tau, \{c_{1,n,m}(\tau)\}_{m=1}^n, \{c_{2,n,m}(\tau)\}_{m=1}^n\right) \mathrm{d}\tau,$$

$$c_{3,n,l}(t) = c_{3,n,l}(0) + \int_0^t G_3^l\left(\tau, \{c_{2,n,m}(\tau)\}_{m=1}^n, \{c_{3,n,m}(\tau)\}_{m=1}^n\right) \mathrm{d}\tau.$$

This proves that the sequences are well defined and approximate solutions to the system (2.3) on $[0, \kappa')$. Set

$$\phi_{i,n}(x,t) = \sum_{l=1}^n b_{i,n,l}(t)e_l(x), \quad i = 1,2,3,$$

where the coefficients $b_{i,n,l}, i = 1,2,3$ are absolutely continuous functions. Then from (2.2) the approximate solutions satisfy the weak formulation

$$\begin{cases} \int_\Omega \partial_t u_{1,n}^\varepsilon \phi_{1,n}\mathrm{d}x + \int_\Omega \sigma_1(u_{2,n}^\varepsilon)d_1(u_{1,n}^\varepsilon, \nabla u_{1,n}^\varepsilon)\nabla\phi_{1,n}\mathrm{d}x \\ \qquad = r_1\int_\Omega \rho_1^\varepsilon(u_{1,n}^\varepsilon)\phi_{1,n}\mathrm{d}x - r_1\int_\Omega u_{2,n}^\varepsilon\mu_1^\varepsilon(u_{1,n}^\varepsilon)\phi_{1,n}\mathrm{d}x \\ \qquad - \int_\Omega \pi^\varepsilon(u_{1,n}^\varepsilon, u_{3,n}^\varepsilon)\phi_{1,n}\mathrm{d}x + \int_\Omega f^\varepsilon\phi_{1,n}\mathrm{d}x, \\ \int_\Omega \partial_t u_{2,n}^\varepsilon \phi_{2,n}\mathrm{d}x + \int_\Omega \sigma_2(u_{1,n}^\varepsilon)d_2(u_{2,n}^\varepsilon, \nabla u_{2,n}^\varepsilon)\nabla\phi_{2,n}\mathrm{d}x \qquad\qquad (2.4) \\ \qquad = r_2\int_\Omega \rho_2^\varepsilon(u_{2,n}^\varepsilon)\phi_{2,n}\mathrm{d}x - r_2\int_\Omega u_{1,n}^\varepsilon\mu_2^\varepsilon(u_{2,n}^\varepsilon)\phi_{2,n}\mathrm{d}x + \int_\Omega g^\varepsilon\phi_{2,n}\mathrm{d}x, \\ \int_\Omega \partial_t u_{3,n}^\varepsilon \phi_{3,n}\mathrm{d}x + \sigma_3\int_\Omega d_3(u_{3,n}^\varepsilon, \nabla u_{3,n}^\varepsilon)\nabla\phi_{3,n}\mathrm{d}x \\ \qquad = r_3\int_\Omega \rho_2^\varepsilon(u_{2,n}^\varepsilon)\phi_{3,n}\mathrm{d}x - \alpha_3\int_\Omega u_{3,n}^\varepsilon\phi_{3,n}\mathrm{d}x + \int_\Omega h^\varepsilon\phi_{3,n}\mathrm{d}x. \end{cases}$$

Take $\phi_{i,n} = u_{i,n}^\varepsilon$, $i = 1, 2, 3$ respectively in (2.4) and use Gronwall's lemma with Young's inequality to get

$$\left\| u_{i,n}^\varepsilon \right\|_{L^\infty(0,\tilde{T};L^2(\Omega))} + \left\| u_{i,n}^\varepsilon \right\|_{L^2(0,\tilde{T};H_0^1(\Omega))} \leq C, \quad \left\| \partial_t u_{i,n}^\varepsilon \right\|_{L^2(0,\tilde{T};H^{-1}(\Omega))} \leq C,$$

where the constant $C > 0$ and $\tilde{T}$ is an arbitrary time in the existence interval $[0, \kappa')$ of Galerkin solutions. Similar approach developed in [2, 4] is used to prove the global existence of solutions. Therefore, as $n \to \infty$, for $i = 1, 2, 3$ we have,

$$\begin{aligned}
u_{i,n}^\varepsilon &\rightharpoonup u_i^\varepsilon \quad \text{weakly-* in } L^\infty(0, T; L^2(\Omega)), \\
u_{i,n}^\varepsilon &\rightharpoonup u_i^\varepsilon \quad \text{weakly in } L^2(0, T; H_0^1(\Omega)), \\
d_i(u_{i,n}^\varepsilon, \nabla u_{i,n}^\varepsilon) &\rightharpoonup \eta_i \quad \text{weakly in } L^2(Q_T), \\
\partial_t u_{i,n}^\varepsilon &\rightharpoonup \partial_t u_i^\varepsilon \quad \text{weakly in } L^2(0, T; H^{-1}(\Omega)).
\end{aligned}$$

Using similar type of arguments as in [4] we can show that $d_i(u_i^\varepsilon, \nabla u_i^\varepsilon) = \eta_i$, $i = 1, 2, 3$. Since the solutions $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ and from the above *a priori* estimates we conclude that $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon \in C([0, T]; L^2(\Omega))$. □

Lemma 2. *Under the hypotheses (H6), (H7) the functions $T_k(u_i^\varepsilon)$, $\frac{\partial S(u_i^\varepsilon)}{\partial t}$, $i = 1, 2, 3$, are bounded in $L^2(0, T; H_0^1(\Omega))$ and $L^1(Q_T) + L^2(0, T; H^{-1}(\Omega))$ respectively.*

Proof. Taking $T_k(u_1^\varepsilon)$ as a test function in the first equation of (2.1) and integrating over $Q_t = \Omega \times (0, t)$, we get

$$\int_{Q_t} u_{1s}^\varepsilon T_k(u_1^\varepsilon) \mathrm{d}x\mathrm{d}s + \int_{Q_t} \sigma_1(u_1^\varepsilon) d_1(u_1^\varepsilon, \nabla u_1^\varepsilon) \nabla T_k(u_1^\varepsilon) \mathrm{d}x\mathrm{d}s$$

$$= r_1 \int_{Q_t} \rho_1^\varepsilon(u_1^\varepsilon) T_k(u_1^\varepsilon) \mathrm{d}x\mathrm{d}s - r_1 \int_{Q_t} u_2^\varepsilon \mu_1^\varepsilon(u_1^\varepsilon) T_k(u_1^\varepsilon) \mathrm{d}x\mathrm{d}s$$

$$- \int_{Q_t} \pi^\varepsilon(u_1^\varepsilon, u_3^\varepsilon) T_k(u_1^\varepsilon) \mathrm{d}x\mathrm{d}s + \int_{Q_t} f^\varepsilon T_k(u_1^\varepsilon) \mathrm{d}x\mathrm{d}s, \qquad (2.5)$$

$$\int_\Omega \tilde{T}_k(u_1^\varepsilon)(t) \mathrm{d}x - \int_\Omega \widetilde{T}_k(u_{1,0}^\varepsilon) \mathrm{d}x + \gamma_1 \int_{Q_t} \sigma_1(u_2^\varepsilon) |\nabla T_k(u_1^\varepsilon)|^2 \mathrm{d}x\mathrm{d}s$$

$$\leq r_1 \int_{Q_t} \rho_1^\varepsilon(u_1^\varepsilon) T_k(u_1^\varepsilon) \mathrm{d}x\mathrm{d}s - r_1 \int_{Q_t} u_2^\varepsilon \mu_1^\varepsilon(u_1^\varepsilon) T_k(u_1^\varepsilon) \mathrm{d}x\mathrm{d}s$$

$$- \int_{Q_t} \pi^\varepsilon(u_1^\varepsilon, u_3^\varepsilon) T_k(u_1^\varepsilon) \mathrm{d}x\mathrm{d}s + \int_{Q_t} f^\varepsilon T_k(u_1^\varepsilon) \mathrm{d}x\mathrm{d}s. \qquad (2.6)$$

Similarly, from the second and third equations of (2.1), we get

$$\int_\Omega \tilde{T}_k(u_2^\varepsilon)(t) \mathrm{d}x - \int_\Omega \widetilde{T}_k(u_{2,0}^\varepsilon) \mathrm{d}x + \gamma_2 \int_{Q_t} \sigma_2(u_1^\varepsilon) |\nabla T_k(u_2^\varepsilon)|^2 \mathrm{d}x\mathrm{d}s \qquad (2.7)$$

$$\leq r_2 \int_{Q_t} \rho_2^\varepsilon(u_2^\varepsilon) T_k(u_2^\varepsilon) \mathrm{d}x\mathrm{d}s - r_2 \int_{Q_t} u_1^\varepsilon \mu_2^\varepsilon(u_2^\varepsilon) T_k(u_2^\varepsilon) \mathrm{d}x\mathrm{d}s + \int_{Q_t} g^\varepsilon T_k(u_2^\varepsilon) \mathrm{d}x\mathrm{d}s,$$

$$\int_\Omega \tilde{T}_k(u_3^\varepsilon)(t)\mathrm{d}x - \int_\Omega \tilde{T}_k(u_{3,0}^\varepsilon)\mathrm{d}x + \sigma_3\gamma_3 \int_{Q_t} |\nabla T_k(u_3^\varepsilon)|^2 \mathrm{d}x\mathrm{d}s$$

$$\leq r_3 \int_{Q_t} \rho_2^\varepsilon(u_2^\varepsilon)T_k(u_3^\varepsilon)\mathrm{d}x\mathrm{d}s - \alpha_3 \int_{Q_t} u_3^\varepsilon T_k(u_3^\varepsilon)\mathrm{d}x\mathrm{d}s + \int_{Q_t} h^\varepsilon T_k(u_3^\varepsilon)\mathrm{d}x\mathrm{d}s. \quad (2.8)$$

Add (2.5)–(2.8) and use Young's inequality, the properties of the functions $f^\varepsilon, g^\varepsilon, h^\varepsilon, u_{i,0}^\varepsilon(x), i = 1, 2, 3$, the continuity of $\sigma_1(u_i^\varepsilon), i = 1, 2$ and the boundedness of approximate solutions with the definition of the functions $\tilde{T}_k(u_i^\varepsilon), i = 1, 2, 3$ we have

$$\int_{Q_T} |\nabla T_k(u_i^\varepsilon)|^2 \mathrm{d}x\mathrm{d}s \leq C \quad (2.9)$$

for any $C > 0$ and $i = 1, 2, 3$. From (2.9), we conclude that $T_k(u_i^\varepsilon)$, $i = 1, 2, 3$ is bounded in $L^2(0, T; H_0^1(\Omega))$. Multiplying first equation of (2.1) by $S'(u_1^\varepsilon)$, we get

$$\frac{\partial S(u_1^\varepsilon)}{\partial t} = \mathrm{div}(S'(u_1^\varepsilon)\sigma_1(u_2^\varepsilon)d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)) - \sigma_1(u_2^\varepsilon)d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)S''(u_1^\varepsilon)\nabla u_1^\varepsilon$$

$$+ r_1\rho_1^\varepsilon(u_1^\varepsilon)S'(u_1^\varepsilon) - r_1u_2^\varepsilon\mu_1(u_1^\varepsilon)S'(u_1^\varepsilon) - \pi^\varepsilon(u_1^\varepsilon, u_3^\varepsilon)S'(u_1^\varepsilon) + f^\varepsilon S'(u_1^\varepsilon). \quad (2.10)$$

For any $S \in C^\infty(\mathbb{R})$ with supp S' compact and from the definition of $T_k(u_1^\varepsilon)$, we can rewrite (2.10) in the following way,

$$\frac{\partial S(u_1^\varepsilon)}{\partial t} = \mathrm{div}(S'(u_1^\varepsilon)\sigma_1(u_2^\varepsilon)d_1(T_k(u_1^\varepsilon), \nabla T_k(u_1^\varepsilon)))$$

$$- \sigma_1(u_2^\varepsilon)d_1(T_k(u_1^\varepsilon), \nabla T_k(u_1^\varepsilon))S''(u_1^\varepsilon)\nabla T_k(u_1^\varepsilon) + r_1\rho_1^\varepsilon(T_k(u_1^\varepsilon))S'(u_1^\varepsilon)$$

$$- r_1u_2^\varepsilon\mu_1^\varepsilon(T_k(u_1^\varepsilon))S'(u_1^\varepsilon) - \pi^\varepsilon(T_k(u_1^\varepsilon), T_k(u_3^\varepsilon))S'(u_1^\varepsilon) + f^\varepsilon S'(u_1^\varepsilon). \quad (2.11)$$

From (2.11), we conclude that $\frac{\partial S(u_1^\varepsilon)}{\partial t}$ is bounded in $L^1(Q_T) + L^2(0, T; H^{-1}(\Omega))$. Similar arguments for $\frac{\partial S(u_i^\varepsilon)}{\partial t}$, $i = 2, 3$, proves the desired result. This completes the proof of the lemma. $\quad\square$

Lemma 3. *The solution triple $(u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ of the approximation system (2.1) is non-negative.*

Proof. For $i = 1, 2, 3$, consider $u_i^{-\varepsilon} = \sup(-u_i^\varepsilon, 0)$ and multiply (2.1) by $-T_k(u_i^{-\varepsilon})$ respectively and integrating over Ω, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \tilde{T}_k(u_1^{-\varepsilon})\mathrm{d}x + \gamma_1 \int_\Omega \sigma_1(u_2^{-\varepsilon})|\nabla T_k(u_1^{-\varepsilon})|^2 \mathrm{d}x \leq r_1 \int_\Omega \rho_1(u_1^{-\varepsilon})T_k(u_1^{-\varepsilon})\mathrm{d}x$$

$$+ r_1 \int_\Omega u_2^{-\varepsilon}\mu_1^\varepsilon(u_1^{-\varepsilon})T_k(u_1^{-\varepsilon})\mathrm{d}x - \int_\Omega \pi^\varepsilon(u_1^{-\varepsilon}, u_3^{-\varepsilon})T_k(u_1^{-\varepsilon})\mathrm{d}x + \int_\Omega f^\varepsilon T_k(u_1^{-\varepsilon})\mathrm{d}x,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \tilde{T}_k(u_2^{-\varepsilon})\mathrm{d}x + \gamma_2 \int_\Omega \sigma_2(u_1^{-\varepsilon})|\nabla T_k(u_2^{-\varepsilon})|^2 \mathrm{d}x$$

$$\leq r_2 \int_\Omega \rho_2^\varepsilon(u_2^{-\varepsilon})T_k(u_2^{-\varepsilon})\mathrm{d}x + r_2 \int_\Omega u_1^{-\varepsilon}\mu_2^\varepsilon(u_2^{-\varepsilon})T_k(u_2^{-\varepsilon})\mathrm{d}x + \int_\Omega g^\varepsilon T_k(u_2^{-\varepsilon})\mathrm{d}x,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tilde{T}_k(u_3^{-\varepsilon})\mathrm{d}x + \gamma_3\sigma_3\int_\Omega |\nabla T_k(u_3^{-\varepsilon})|^2\mathrm{d}x$$

$$\leq r_3 \int_\Omega \rho_2^\varepsilon(u_2^{-\varepsilon})T_k(u_3^{-\varepsilon})\mathrm{d}x - \alpha_3 \int_\Omega u_3^{-\varepsilon}T_k(u_3^{-\varepsilon})\mathrm{d}x + \int_\Omega h^\varepsilon T_k(u_2^{-\varepsilon})\mathrm{d}x.$$

From the boundedness of solutions $(u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$, continuity of $\sigma_1(u_i^\varepsilon), i = 1, 2$ and the non-negativity of $f^\varepsilon, g^\varepsilon, h^\varepsilon, u_{i,0}^\varepsilon, i = 1, 2, 3$ together with Young's inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tilde{T}_k(u_i^{-\varepsilon})(t)\mathrm{d}x \leq 0, \ i = 1, 2, 3.$$

This proves the required result. $\quad\square$

DEFINITION 2. We define the Lipschitz continuous function in the following form

$$\Theta_n(z) = T_{n+1}(z) - T_n(z) = \begin{cases} 0 & \text{if } |z| \leq n, \\ (|z| - n)\mathrm{sgn}(z) & \text{if } n \leq |z| \leq n+1, \\ \mathrm{sgn}(z) & \text{if } |z| \geq n+1. \end{cases}$$

Obviously the function $\Theta_n(z)$ satisfies $\|\Theta_n(z)\|_{L^\infty(\mathbb{R})} \leq 1$, for any $n \geq 1$ and $\Theta_n(z) \to 0$, for any $n \geq 1$ and also $\Theta_n(z) \to 0$, for any z when $n \to \infty$.

Lemma 4.   *The Lipschitz continuous function $\Theta_n(u_i), i = 1, 2, 3$, for some $n > 0$ and $\varepsilon > 0$ satisfies*

$$\lim_{n\to\infty}\overline{\lim_{\varepsilon\to 0}}\int_0^t\int_{(n\leq|u_i^\varepsilon|\leq n+1)} d_i(u_i^\varepsilon, \nabla u_i^\varepsilon)\nabla u_i^\varepsilon\mathrm{d}x\mathrm{d}s = 0 \quad and$$

$$\Theta_n(u_i) \to 0 \quad strongly \ in \ \ L^2(0, T; H_0^1(\Omega)) \ \ as \ n \to \infty \ \ and \ for \ i = 1, 2, 3.$$

Proof.   Treating $\Theta_n(u_1^\varepsilon)$ as a test function and multiplying the first equation of (2.1) by $\Theta_n(u_1^\varepsilon)$ and integrating over $Q \times (0, t)$, we have

$$\int_\Omega \tilde{\Theta}_n(u_1^\varepsilon)\mathrm{d}x - \int_\Omega \tilde{\Theta}_n(u_{1,0}^\varepsilon)\mathrm{d}x + \int_{Q_t} \sigma_1^\varepsilon(u_2^\varepsilon)d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)\nabla\Theta_n(u_1^\varepsilon)\mathrm{d}x\mathrm{d}s$$

$$= r_1 \int_{Q_t} \rho_1^\varepsilon(u_1^\varepsilon)\Theta_n(u_1^\varepsilon)\mathrm{d}x\mathrm{d}s - r_1\int_{Q_t} u_2^\varepsilon\mu_2^\varepsilon(u_1^\varepsilon)\Theta_n(u_1^\varepsilon)\mathrm{d}x\mathrm{d}s$$

$$- \int_{Q_t} \pi^\varepsilon(u_1^\varepsilon, u_3^\varepsilon)\Theta_n(u_1^\varepsilon)\mathrm{d}x\mathrm{d}s + \int_{Q_t} f^\varepsilon\Theta_n(u_1^\varepsilon)\mathrm{d}x\mathrm{d}s$$

for almost t in $(0, T)$. Since $\Theta_n(u_1^\varepsilon) \geq 0$ for all $x \in \Omega$, we obtain

$$\int_{Q_t} \sigma_1(u_2^\varepsilon)d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)\nabla\Theta_n(u_1^\varepsilon)\mathrm{d}x\mathrm{d}s \leq r_1\int_{Q_t} \rho_1^\varepsilon(u_1^\varepsilon)\Theta_n(u_1^\varepsilon)\mathrm{d}x\mathrm{d}s$$

$$- r_1\int_{Q_t} u_2^\varepsilon\mu_2^\varepsilon(u_1^\varepsilon)\Theta_n(u_1^\varepsilon)\mathrm{d}x\mathrm{d}s - \int_{Q_t} \pi^\varepsilon(u_1^\varepsilon, u_3^\varepsilon)\Theta_n(u_1^\varepsilon)\mathrm{d}x\mathrm{d}s$$

$$+ \int_{Q_t} f^\varepsilon\Theta_n(u_1^\varepsilon)\mathrm{d}x\mathrm{d}s + \int_\Omega \tilde{\Theta}_n(u_{1,0}^\varepsilon(x))\mathrm{d}x \tag{2.12}$$

for all t in $(0, T)$ and $\varepsilon < \frac{1}{n+1}$. For any subsequences u_i^ε (still denoted by u_i^ε), Lemma 2 and (2.12) confirm that,

$$\left.\begin{array}{ll}
u_i^\varepsilon \to u_i & \text{a.e. in } Q_T, \ i = 1, 2, 3, \\
T_k(u_i^\varepsilon) \rightharpoonup T_k(u_i) & \text{weakly in } L^2(0, T; H_0^1(\Omega)), \ i = 1, 2, 3, \\
\Theta_n(u_i^\varepsilon) \rightharpoonup \Theta_n(u_i) & \text{weakly in } L^2(0, T; H_0^1(\Omega)), \ i = 1, 2, 3,
\end{array}\right\} \quad (2.13)$$

as $\varepsilon \to 0$ for any $k > 0$ and $n \geq 1$. Using $(H2)$, we can show that for $i = 1, 2, 3$ $d_i\big(T_k(u_i^\varepsilon), \nabla T_k(u_i^\varepsilon)\big)$ is bounded in $L^2(Q_T)$. Therefore, for $i = 1, 2, 3$,

$$d_i(T_k(u_i^\varepsilon), \nabla T_k(u_i^\varepsilon)) \rightharpoonup \eta_{i,k} \ \text{ weakly in } L^2(Q_T) \text{ as } \varepsilon \to 0 \quad (2.14)$$

for $\eta_{i,k} \in L^2(Q_T)$. From (2.5)–(2.8) and Lemma 3, we have

$$\sum_{i=1}^3 \int_\Omega \tilde{T}_k(u_i^\varepsilon)(t)\mathrm{d}x \leq \int_{Q_t} \left(r_1\rho_1^\varepsilon(u_1^\varepsilon) - r_1 u_2^\varepsilon \mu_1^\varepsilon(u_1^\varepsilon) - \pi^\varepsilon(u_1^\varepsilon, u_3^\varepsilon) + f^\varepsilon\right) T_k(u_1^\varepsilon)\mathrm{d}x\mathrm{d}s$$

$$+ \int_{Q_t} \left(r_2\rho_2^\varepsilon(u_2^\varepsilon) - u_1^\varepsilon \mu_2^\varepsilon(u_2^\varepsilon) + g^\varepsilon\right) T_k(u_2^\varepsilon)\mathrm{d}x\mathrm{d}s$$

$$+ \int_{Q_t} \left(r_3\rho_2^\varepsilon(u_2^\varepsilon) - \alpha_3 u_3^\varepsilon + h^\varepsilon\right) T_k(u_3^\varepsilon)\mathrm{d}x\mathrm{d}s + \sum_{i=1}^3 \int_\Omega \tilde{T}_k(u_{i,0}^\varepsilon)(x)\mathrm{d}x.$$

From the boundedness of the solution triple $(u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$, Lemma 2 and use Young's inequality, we get

$$\sum_{i=1}^3 \int_\Omega \tilde{T}_k(u_i^\varepsilon)\mathrm{d}x \leq C_k + k\big(\|f^\varepsilon\|_{L^2(Q_T)} + \|g^\varepsilon\|_{L^2(Q_T)}$$

$$+ \|h^\varepsilon\|_{L^2(Q_T)} + \sum_{i=1}^3 \|u_{i,0}^\varepsilon\|_{L^2(\Omega)}\big),$$

where C_k is a constant independent of ε. Taking lim inf as ε tends to 0 in the above estimate and using the results (2.13) and Lemma 2, we have

$$\sum_{i=1}^3 \int_\Omega \tilde{T}_k(u_i)\mathrm{d}x \leq C_k + k(\|f\|_{L^1(Q_T)} + \|g\|_{L^1(Q_T)} + \|h\|_{L^1(Q_T)} + \sum_{i=1}^3 \|u_{i,0}\|_{L^1(\Omega)}).$$

From the definition of $\tilde{T}_k(u_i), i = 1, 2, 3$, we deduce that

$$\sum_{i=1}^3 k \int_\Omega |u_i(x,t)|\mathrm{d}x \leq C_k + \frac{k^2}{2}\mathrm{meas}(\Omega)$$

$$+ k(\|f\|_{L^1(Q_T)} + \|g\|_{L^1(Q_T)} + \|h\|_{L^1(Q_T)} + \sum_{i=1}^3 \|u_{i,0}^\varepsilon\|_{L^1(\Omega)}) \quad (2.15)$$

for almost all $t \in (0, T)$ and (2.15) shows that $u \in L^\infty(0, T; L^1(\Omega))$.

Now (2.12) along with (2.13), proves that

$$\overline{\lim_{\varepsilon \to 0}} \int_{Q_t} \sigma_1(u_2^\varepsilon) d_1(u_1^\varepsilon, \nabla u_1^\varepsilon) \nabla \Theta_n(u_1^\varepsilon) \mathrm{d}x \mathrm{d}s \le r_1 \int_{Q_t} \rho_1(u_1) \Theta_n(u_1) \mathrm{d}x \mathrm{d}s$$

$$- r_1 \int_{Q_t} u_2 \mu_2(u_1) \Theta_n(u_1) \mathrm{d}x \mathrm{d}s - \int_{Q_t} \pi(u_1, u_3) \Theta_n(u_1) \mathrm{d}x \mathrm{d}s$$

$$+ \int_{Q_t} f \Theta_n(u_1) \mathrm{d}x \mathrm{d}s + \int_{Q_t} \tilde{\Theta}_n(u_{1,0}(x)) \mathrm{d}x. \tag{2.16}$$

Using $(H1)$, $\nabla \Theta_n(u^\varepsilon) = \chi_{(n \le |u^\varepsilon| \le n+1)} \nabla u^\varepsilon$, the continuity of $\sigma_1(u_2^\varepsilon)$ and the convergence results in (2.13), we have

$$\gamma_1 \int_{Q_t} |\nabla \Theta_n(u_1)|^2 \mathrm{d}x \mathrm{d}s \le r_1 \int_{Q_t} \rho_1(u_1) \Theta_n(u_1) \mathrm{d}x \mathrm{d}s - r_1 \int_{Q_t} u_2 \mu_2(u_1) \Theta_n(u_1) \mathrm{d}x \mathrm{d}s$$

$$- \int_{Q_t} \pi(u_1, u_3) \Theta_n(u_1) \mathrm{d}x \mathrm{d}s + \int_{Q_t} f \Theta_n(u_1) \mathrm{d}x \mathrm{d}s + \int_{Q_t} \tilde{\Theta}_n(u_{1,0}(x)) \mathrm{d}x. \tag{2.17}$$

Since $\Theta_n(u_1) \to 0$ as $n \to \infty$, we have $\Theta_n(u_1) \to 0$ weakly in $L^2(0, T; H_0^1(\Omega))$. This leads to

$$\int_{Q_t} r_1 \rho_1(u_1) \Theta_n(u_1) \mathrm{d}x \mathrm{d}s \to 0, \quad \int_{Q_t} r_1 u_2 \mu_2(u_1) \Theta_n(u_1) \mathrm{d}x \mathrm{d}s \to 0,$$

$$\int_{Q_t} \pi(u_1, u_3) \Theta_n(u_1) \mathrm{d}x \mathrm{d}s \to 0, \quad \int_{Q_t} f \Theta_n(u_1) \mathrm{d}x \mathrm{d}s \to 0,$$

as $n \to \infty$. Also, $\tilde{\Theta}_n(u_{1,0}) \to 0$ a.e. in Ω as $n \to \infty$ and $\|\Theta_n(u_{1,0})\| \le \|u_{1,0}\|_{L^1(\Omega)}$ implies $\int_\Omega \tilde{\Theta}_n(u_{1,0}) \mathrm{d}x$ tends to 0 as $n \to \infty$, which follows from the Lebesgue convergence theorem. Hence passing the lim-inf in (2.16) and (2.17), we get

$$\lim_{n \to \infty} \overline{\lim_{\varepsilon \to 0}} \int_0^t \int_{(n \le |u_1^\varepsilon| \le n+1)} d_1(u_1^\varepsilon, \nabla u_1^\varepsilon) \nabla u_1^\varepsilon \mathrm{d}x \mathrm{d}s = 0,$$

$$\Theta_n(u_1) \to 0 \quad \text{strongly in} \quad L^2(0, T; H_0^1(\Omega)) \text{ as } n \to \infty.$$

Similarly, we can show that, for $i = 1, 2, 3$

$$\lim_{n \to \infty} \overline{\lim_{\varepsilon \to 0}} \int_0^t \int_{(n \le |u_i^\varepsilon| \le n+1)} d_i(u_i^\varepsilon, \nabla u_i^\varepsilon) \nabla u_i^\varepsilon \mathrm{d}x \mathrm{d}s = 0,$$

$$\Theta_n(u_i) \to 0 \quad \text{strongly in} \quad L^2(0, T; H_0^1(\Omega)) \text{ as } n \to \infty.$$

$\square$

DEFINITION 3. For i=1,2,3, the time regularization for the function $T_k(u_i)$ is defined as

$$(T_k(u_i))_\delta = \delta \int_{-\infty}^t e^{\delta(s-t)} T_k(\overline{u_i(x, s)}) \mathrm{d}s, \quad \text{where } \overline{u_i(x, s)} = \begin{cases} u_i(x, s), & \text{if } s > 0, \\ u_{i,0}(x), & \text{if } s < 0. \end{cases}$$

Let us consider the unique solution $(T_k(u_i))_\delta \in L^\infty(Q_T) \cap L^2(0,T;H_0^1(\Omega))$ of the monotone problem

$$\begin{cases} \dfrac{\partial}{\partial t}(T_k(u_i))_\delta + \delta((T_k(u_i))_\delta - T_k(u_i)) = 0 \ \ \text{in} \ \ Q_T, \\ (T_k(u_i(x,0)))_\delta = T_k(u_{i,0}(x)) \ \ \text{in} \ \ \Omega, \end{cases} \tag{2.18}$$

for $\delta > 0$ and $k > 0$. From (2.18) and Lemma 2, we have $\frac{\partial}{\partial t}(T_k(u_i))_\delta \in L^2(0,T;H_0^1(\Omega))$.

Remark 1. For $i = 1,2,3$, we have $(T_k(u_i))_\delta \to T_k(u_i)$ a.e. in Q_T, weak-* in $L^\infty(Q_T)$ and strongly in $L^2(0,T;H_0^1(\Omega))$ as $\delta \to \infty$ and also

$$\|(T_k(u_i))_\delta\|_{L^\infty(Q_T)} \leq \max(\|T_k(u_i)\|_{L^\infty(Q_T)}, \ \|T_k(u_{i,0})\|_{L^\infty(\Omega)}) \leq k,$$

for any $\delta > 0$ and $k \geq 0$.

Lemma 5. *Let $k \geq 0$ be fixed and S be an increasing $C^\infty(\mathbb{R})$ function such that $S(z) = z$ for $|z| \leq k$ and supp S' be compact. Then*

$$\lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \int_0^T \int_{Q_t} \frac{\partial S(u_i^\varepsilon)}{\partial t}(T_k(u_i^\varepsilon) - (T_k(u_i))_\delta)\mathrm{d}x\mathrm{d}s\mathrm{d}t \geq 0.$$

Proof. The proof is as similar as of Lemma 1 in [9]. □

Lemma 6. *For $i = 1,2,3$, and $\eta_{i,k}$ which is defined in (2.14), the subsequence of u_i^ε (still denoted by u_i^ε) satisfies, for $i = 1,2,3$,*

$$\overline{\lim_{\varepsilon\to 0}} \int_0^T \int_{Q_t} d_i(T_k(u_i^\varepsilon), \nabla T_k(u_i^\varepsilon))\nabla T_k(u_i^\varepsilon)\mathrm{d}x\mathrm{d}s\mathrm{d}t \leq \int_0^T \int_{Q_t} \eta_{i,k}\nabla T_k(u_i)\mathrm{d}x\mathrm{d}s\mathrm{d}t.$$

Proof. Let us introduce S_n to be a sequence of increasing $C^\infty(\mathbb{R})$ functions such that

$$\begin{aligned} S_n(z) &= z, \ \ \text{for} \ |z| \leq n, \\ \text{supp } S_n' &\subset [-(n+1),(n+1)], \quad \|S_n''\|_{L^\infty(\mathbb{R})} \leq 1. \end{aligned} \tag{2.19}$$

Multiply the first equation of (2.1) by $S_n'(u_1^\varepsilon)$ to get

$$\frac{\partial S_n(u_1^\varepsilon)}{\partial t} = \mathrm{div}(S_n'(u_1^\varepsilon)\sigma_1(u_2^\varepsilon)d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)) - \sigma_1(u_2^\varepsilon)d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)S_n''(u_1^\varepsilon)\nabla u_1^\varepsilon$$
$$+ r_1\rho_1^\varepsilon(u_1^\varepsilon)S_n'(u_1^\varepsilon) - r_1 u_2^\varepsilon \mu_1(u_1^\varepsilon)S_n'(u_1^\varepsilon) - \pi^\varepsilon(u_1^\varepsilon, u_3^\varepsilon)S_n'(u_1^\varepsilon) + f^\varepsilon S_n'(u_1^\varepsilon). \tag{2.20}$$

By similar procedure, we obtain that, for $i = 1,2,3$

$$\frac{\partial}{\partial t}S_n(u_i^\varepsilon) \in L^1(Q_T) + L^2(0,T;H^{-1}(\Omega)).$$

For fixed $k > 0$, $\delta > 0$ and $\varepsilon > 0$, set

$$\Psi_\delta^\varepsilon = T_k(u_i^\varepsilon) - (T_k(u_i))_\delta, \ \ i = 1,2,3. \tag{2.21}$$

Multiplying (2.20) by Ψ_δ^ε and integrating over $Q_t \times (0, T)$, we get

$$\int_Q \frac{\partial S_n(u_1^\varepsilon)}{\partial t} \Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t = -\int_Q S_n'(u_1^\varepsilon)\sigma_1(u_2^\varepsilon)d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)\nabla\Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t$$

$$- \int_Q S_n''(u_1^\varepsilon)\sigma_1(u_2^\varepsilon)d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)\nabla u_1^\varepsilon \Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t$$

$$+ r_1 \int_Q S_n'(u_1^\varepsilon)\rho_1^\varepsilon(u_1^\varepsilon)\Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t - r_1 \int_Q S_n'(u_1^\varepsilon)u_2^\varepsilon\mu_1^\varepsilon(u_1^\varepsilon)\Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t$$

$$- \int_Q S_n'(u_1^\varepsilon)\pi^\varepsilon(u_1^\varepsilon, u_3^\varepsilon)\Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t + \int_Q f^\varepsilon S_n'(u_1^\varepsilon)\Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t, \qquad (2.22)$$

where $Q = Q_t \times (0, T)$. From (2.21) for fixed $\delta > 0$, $\Psi_\delta^\varepsilon \rightharpoonup T_k(u_i) - (T_k(u_i))_\delta$, $i = 1, 2, 3$ weakly in $L^2(0, T; H_0^1(\Omega))$ as $\varepsilon \to 0$. By Remark 1, we conclude that $\|\Psi_\delta^\varepsilon\|_{L^\infty(\Omega)} \leq 2k$, for any $\varepsilon > 0$ and $\delta > 0$. Therefore, the boundedness of Ψ_δ^ε shows that for fixed $\delta > 0$, $\Psi_\delta^\varepsilon \to T_k(u_i) - (T_k(u_i))_\delta$, $i = 1, 2, 3$ a.e. in Q_T and $L^\infty(Q_T)$ weak-* as $\varepsilon \to 0$.

By the definition of S_n, we have supp $S_n'' \subset [-(n+1), (n+1)] \cup [n, n+1]$ for any $n \geq 1$. As a consequence

$$\left| \int_Q S_n''(u_1^\varepsilon)\sigma_1(u_2^\varepsilon)d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)\nabla u_1^\varepsilon \Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t \right| \leq T\|S_n''(u_1^\varepsilon)\|_{L^\infty((R))}$$

$$\times \|\sigma_1(u_2^\varepsilon)\|_{L^\infty(Q_T)}\|\Psi_\delta^\varepsilon\|_{L^\infty(Q_T)} \int_{((x,t); n \leq |u_1^\varepsilon| \leq n+1)} d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)\nabla u_1^\varepsilon \mathrm{d}x\mathrm{d}s,$$

for any $n \geq 1$, $\varepsilon \leq \frac{1}{n+1}$ and $\delta > 0$. From $\|\Psi_\delta^\varepsilon\|_{L^\infty(Q_T)} \leq 2k$ and (2.19), it can be easily obtained that

$$\overline{\lim_{\delta \to \infty}} \overline{\lim_{\varepsilon \to 0}} \left| \int_Q S_n''(u_1^\varepsilon)\sigma_1(u_2^\varepsilon)d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)\nabla u_1^\varepsilon \Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t \right|$$

$$\leq C \overline{\lim_{\varepsilon \to 0}} \int_{((x,t); n \leq |u_1^\varepsilon| \leq n+1)} d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)\nabla u_1^\varepsilon \mathrm{d}x\mathrm{d}s,$$

for any $n \geq 1$, where the constant C depends only on T and k. Hence, by Lemma 4, we achieve that

$$\lim_{n \to \infty} \overline{\lim_{\delta \to \infty}} \overline{\lim_{\varepsilon \to 0}} \left| \int_Q S_n''(u_1^\varepsilon)\sigma_1(u_2^\varepsilon)d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)\nabla u_1^\varepsilon \Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t \right| = 0. \qquad (2.23)$$

For some $n \geq 1$, we have

$$r_1 S_n'(u_1^\varepsilon)\rho_1^\varepsilon(u_1^\varepsilon)\Psi_\delta^\varepsilon = r_1 S_n'(u_1^\varepsilon)\rho_1^\varepsilon(T_{n+1}(u_1^\varepsilon))\Psi_\delta^\varepsilon$$

a.e. in Q_T, since supp $S_n' \subset [-(n+1), (n+1)]$. Definition of $\rho_1^\varepsilon(u_1^\varepsilon)$ and the result (2.13) lead to $S_n'(u^\varepsilon)\rho_1^\varepsilon(T_{n+1}(u_1^\varepsilon)) \to S_n'(u)\rho_1(T_{n+1}(u_1))$ a.e. in Q_T and in $L^\infty(Q_T)$ weak-* as $\varepsilon \to 0$. This proves

$$\lim_{\varepsilon \to 0} \int_Q r_1 S_n'(u_1^\varepsilon)\rho_1^\varepsilon(u_1^\varepsilon)\Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t = \int_Q r_1 S_n'(u_1)\rho_1(u_1)(T_k(u_1) - (T_k(u_1))_\delta)\mathrm{d}x\mathrm{d}s\mathrm{d}t,$$

for any $\delta > 0$. Using Remark 1, we set

$$\lim_{\delta \to \infty} \lim_{\varepsilon \to 0} \int_Q r_1 S'_n(u_1^\varepsilon) \rho_1^\varepsilon(u_1^\varepsilon) \Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t = 0.$$

Similarly, we can show that

$$\lim_{\delta \to \infty} \lim_{\varepsilon \to 0} \int_Q r_1 S'_n(u_1^\varepsilon) u_2^\varepsilon \mu_1(u_1^\varepsilon) \Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t = 0,$$

$$\lim_{\delta \to \infty} \lim_{\varepsilon \to 0} \int_Q S'_n(u_1^\varepsilon) \pi(u_1^\varepsilon, u_3^\varepsilon) \Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t = 0.$$

Since $fS'_n(u_1) \in L^1(Q_T)$, (2.13) and Remark 1 lead to,

$$\lim_{\delta \to \infty} \lim_{\varepsilon \to 0} \int_Q f^\varepsilon S'_n(u_1^\varepsilon) \Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t = 0. \tag{2.24}$$

Consequently from Lemma 5 and the definition of Ψ_δ^ε, we have

$$\lim_{\delta \to \infty} \lim_{\varepsilon \to 0} \int_Q \frac{\partial S_n(u_1^\varepsilon)}{\partial t} \Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t \geq 0 \quad \text{for any} \quad n \geq k. \tag{2.25}$$

From (2.23)–(2.24) along with (2.22) and (2.25), we get

$$\lim_{n \to \infty} \overline{\lim_{\delta \to \infty}} \overline{\lim_{\varepsilon \to 0}} \int_Q S'_n(u_1^\varepsilon) d_1(u_1^\varepsilon, \nabla u_1^\varepsilon) \nabla \Psi_\delta^\varepsilon \mathrm{d}x\mathrm{d}s\mathrm{d}t \leq 0. \tag{2.26}$$

Using the definition of S_n and (2.26) to get

$$\overline{\lim_{\varepsilon \to 0}} \int_Q d_1(u_1^\varepsilon, \nabla u_1^\varepsilon) \nabla T_k(u_1^\varepsilon) \mathrm{d}x\mathrm{d}s\mathrm{d}t$$

$$\leq \lim_{n \to \infty} \overline{\lim_{\delta \to \infty}} \overline{\lim_{\varepsilon \to 0}} \int_Q S'_n(u_1^\varepsilon) d_1(u_1^\varepsilon, \nabla u_1^\varepsilon) \nabla(T_k(u_1^\varepsilon))_\delta \mathrm{d}x\mathrm{d}s\mathrm{d}t \quad \text{for} \quad k \leq n.$$

We know that

$$S'_n(u_1^\varepsilon) d_1(u_1^\varepsilon, \nabla u_1^\varepsilon) = S'_n(u_1^\varepsilon) d_1(T_{n+1}(u_1^\varepsilon), \nabla T_{n+1}(u_1^\varepsilon)), \text{ for } \varepsilon < \frac{1}{n+1}.$$

From (2.14), we have $S'_n(u_1^\varepsilon) d_1(T_{n+1}(u_1^\varepsilon), \nabla(T_{n+1}(u_1^\varepsilon)) \rightharpoonup S'_n(u_1^\varepsilon) \eta_{1,n+1}$ weakly in $L^2(Q_T)$ as $\varepsilon \to 0$. This helps us to prove that

$$\overline{\lim_{\varepsilon \to 0}} \int_Q d_1(T_k(u_1^\varepsilon), \nabla T_k(u_1^\varepsilon)) \nabla T_k(u_1^\varepsilon) \mathrm{d}x\mathrm{d}s\mathrm{d}t$$

$$\leq \lim_{n \to \infty} \int_Q S'_n(u_1) \eta_{1,n+1} \nabla T_k(u_1) \mathrm{d}x\mathrm{d}s\mathrm{d}t$$

$$= \int_Q \eta_{1,n+1} \nabla T_k(u_1) \mathrm{d}x\mathrm{d}s\mathrm{d}t \text{ for any } n \leq 1. \tag{2.27}$$

For any $k \leq n$, we have

$$d_1(T_{n+1}(u_1^\varepsilon), \nabla T_{n+1}(u_1^\varepsilon))_{\chi\{|u_1^\varepsilon| \leq k\}} = d_1(T_k(u_1^\varepsilon), \nabla T_k(u_1^\varepsilon))_{\chi\{|u^\varepsilon| \leq k\}} \text{ a.e. in } Q_T.$$

Above equation with (2.13) and (2.14), implies that

$$\eta_{1,n+1}\chi_{\{|u_1^\varepsilon|\le k\}} = \eta_{1,k}\chi_{\{|u_1^\varepsilon|\le k\}} \quad \text{a.e. in } Q_T - \{|u_1| = k\} \quad \text{for} \quad k \le n \quad \text{as} \quad \varepsilon \to 0.$$

Therefore (2.27) becomes

$$\overline{\lim_{\varepsilon\to 0}} \int_Q d_1(T_k(u_1^\varepsilon), \nabla T_k(u_1^\varepsilon))\nabla T_k(u_1^\varepsilon)\mathrm{d}x\mathrm{d}s\mathrm{d}t = \int_Q \eta_{1,k}\nabla T_k(u_1)\mathrm{d}x\mathrm{d}s\mathrm{d}t.$$

Similarly, we get

$$\overline{\lim_{\varepsilon\to 0}} \int_Q d_i(T_k(u_i^\varepsilon), \nabla T_k(u_i^\varepsilon))\nabla T_k(u_i^\varepsilon)\mathrm{d}x\mathrm{d}s\mathrm{d}t = \int_Q \eta_{i,k}\nabla T_k(u_i)\mathrm{d}x\mathrm{d}s\mathrm{d}t.$$

□

Lemma 7. *For any $k \ge 0$ and $i = 1, 2, 3$, we have*

$$\overline{\lim_{\varepsilon\to 0}} \int_Q [d_i(T_k(u_i^\varepsilon), \nabla T_k(u_i^\varepsilon)) - d_i(T_k(u_i), \nabla T_k(u_i))]$$
$$\times [\nabla T_k(u_i^\varepsilon) - \nabla T_k(u_i)]\mathrm{d}x\mathrm{d}s\mathrm{d}t = 0.$$

Proof.   From the monotone character $(H3)$, for any $k \ge 0$, we have

$$\overline{\lim_{\varepsilon\to 0}} \int_Q [d_i(T_k(u_i^\varepsilon), \nabla T_k(u_i^\varepsilon)) - d_i(T_k(u_i), \nabla T_k(u_i))]$$
$$\times [\nabla T_k(u_i^\varepsilon) - \nabla T_k(u_i)]\mathrm{d}x\mathrm{d}s\mathrm{d}t \ge 0.$$

By $(H2)$ one can remark that $|d_i(T_k(u_i), \nabla T_k(u_i))| \le C_k(1+|\nabla T_k(u_i)|)$ a.e. in Q_T. Then by Lemma 4, (2.13) and (2.14), we have

$$\overline{\lim_{\varepsilon\to 0}} \int_Q [d_i(T_k(u_i^\varepsilon), \nabla T_k(u_i^\varepsilon)) - d_i(T_k(u_i), \nabla T_k(u_i))]$$
$$\times [\nabla T_k(u_i^\varepsilon) - \nabla T_k(u_i)]\mathrm{d}x\mathrm{d}s\mathrm{d}t = 0 \text{ for any } k \ge 0.$$

□

Lemma 8.   *For fixed $k \ge 0$ and $i = 1, 2, 3$, we have $\eta_{i,k} = d_i(T_k(u_i), \nabla T_k(u_i))$ a.e. in Q_T and*

$$d_i(T_k(u_i^\varepsilon), \nabla T_k(u_i^\varepsilon))\nabla T_k(u_i^\varepsilon) \rightharpoonup d_i(T_k(u_i), \nabla T_k(u_i))\nabla T_k(u_i)$$

weakly in $L^1(Q_T)$.

Proof.   For any $k > 0$ and $0 < \varepsilon < \frac{1}{k}$, Lemma 6, (2.13) implies that, for $i = 1, 2, 3$,

$$\overline{\lim_{\varepsilon\to 0}} \int_Q d_i(T_k(u_i^\varepsilon), \nabla T_k(u_i^\varepsilon))\nabla T_k(u_i^\varepsilon)\mathrm{d}x\mathrm{d}s\mathrm{d}t = \int_Q \eta_{i,k}\nabla T_k(u_i)\mathrm{d}x\mathrm{d}s\mathrm{d}t.$$

Minty's type of arguments, in view of (2.13), shows that $d_i(T_k(u_i), \nabla T_k(u_i)) = \eta_{i,k}$ for any $k \geq 0$. This proves the first result of Lemma.

For any $k \geq 0$, $T' < T$, Lemma 7 shows that,

$$[d_i(T_k(u_i^\varepsilon), \nabla T_k(u_i^\varepsilon)) - d_i(T_k(u_i), \nabla T_k(u_i))][\nabla T_k(u_i^\varepsilon) - \nabla T_k(u_i)] \to 0$$

strongly in $L^1(\Omega \times (0, T'))$, as $\varepsilon \to 0$. By (2.13), the first result of the present Lemma implies that

$$d_i(T_k(u_i^\varepsilon), \nabla T_k(u_i)) \rightharpoonup d_i(T_k(u_i), \nabla T_k(u_i)) \quad \text{weakly in } L^1(Q_T),$$

$$d_i(T_k(u_i), \nabla T_k(u_i^\varepsilon)) \rightharpoonup d_i(T_k(u_i), \nabla T_k(u_i)) \quad \text{weakly in } L^1(Q_T).$$

Hence $d_i(T_k(u_i^\varepsilon), \nabla T_k(u_i^\varepsilon)) \rightharpoonup d_i(T_k(u_i), \nabla T_k(u_i))$ weakly in $L^1(\Omega \times (0, T'))$ for any $T' < T$ as $\varepsilon \to 0$. According to the definition of $d_i(\eta, \zeta)$, the assumptions hold true for all time T. Hence $d_i(T_k(u_i^\varepsilon), \nabla T_k(u_i^\varepsilon)) \rightharpoonup d_i(T_k(u_i), \nabla T_k(u_i))$ weakly in $L^1(Q_T)$ holds. $\square$

Lemma 9. *For any $n \geq 0$ and $i = 1, 2, 3$,*

$$\int_{\{(x,t) \in Q_T; n \leq |u_i| \leq n+1\}} d_i(u_i, \nabla u_i) \nabla u_i \mathrm{d}x \mathrm{d}t \to 0 \quad as \quad n \to \infty.$$

Proof.

$$\lim_{\varepsilon \to 0} \int_{\{(x,t) \in Q_T; n \leq |u_i^\varepsilon| \leq n+1\}} d_i(u_i^\varepsilon, \nabla u_i^\varepsilon) \nabla u_i^\varepsilon \mathrm{d}x \mathrm{d}t = \lim_{\varepsilon \to 0} \int_{Q_T} d_i(u_i^\varepsilon, \nabla u_i^\varepsilon) \nabla (T_{n+1}(u_i^\varepsilon) - T_n(u_i^\varepsilon)) \mathrm{d}x \mathrm{d}t$$

$$= \int_{Q_T} d_i(u_i, \nabla u_i) \nabla (T_{n+1}(u_i)) \mathrm{d}x \mathrm{d}t - \int_{Q_T} d_i(u_i, \nabla u_i) \nabla (T_n(u_i)) \mathrm{d}x \mathrm{d}t$$

$$= \int_{\{(x,t) \in Q_T; n \leq |u_i| \leq n+1\}} d_1(u_i, \nabla u_i) \nabla u_i \mathrm{d}x \mathrm{d}t \quad \text{for any } n \geq 0.$$

Using Lemma 4 together with the above inequality, we get

$$\int_{\{(x,t) \in Q_T; n \leq |u_i| \leq n+1\}} d_i(u_i, \nabla u_i) \nabla u_i \mathrm{d}x \mathrm{d}t \to 0 \quad \text{as } n \to \infty.$$

$\square$

Now, we prove the main theorem of the work. We have used the Lemma 2 and Aubin type of lemma to prove the Theorem 1.

Proof. From the system (2.1), we have

$$\frac{\partial S(u_1^\varepsilon)}{\partial t} - \operatorname{div}(S'(u_1^\varepsilon) \sigma_1(u_2^\varepsilon) d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)) + \sigma_1(u_2^\varepsilon) d_1(u_1^\varepsilon, \nabla u_1^\varepsilon) S''(u_1^\varepsilon) \nabla u_1^\varepsilon$$

$$= r_1 \rho_1^\varepsilon(u_1^\varepsilon) S'(u_1^\varepsilon) - r_1 u_2^\varepsilon \mu_1^\varepsilon(u_1^\varepsilon) S'(u_1^\varepsilon) - \pi^\varepsilon(u_1^\varepsilon, u_3^\varepsilon) S'(u_1^\varepsilon) + f^\varepsilon S'(u_1^\varepsilon),$$

$$\frac{\partial S(u_2^\varepsilon)}{\partial t} - \operatorname{div}(S'(u_2^\varepsilon) \sigma_2(u_1^\varepsilon) d_2(u_2^\varepsilon, \nabla u_2^\varepsilon)) + \sigma_2(u_1^\varepsilon) d_2(u_2^\varepsilon, \nabla u_2^\varepsilon) S''(u_2^\varepsilon) \nabla u_2^\varepsilon$$

$$= r_2 \rho_2^\varepsilon(u_2^\varepsilon) S'(u_2^\varepsilon) - r_2 u_1^\varepsilon \mu_2^\varepsilon(u_2^\varepsilon) S'(u_2^\varepsilon) + g^\varepsilon S'(u_2^\varepsilon),$$

$$\frac{\partial S(u_3^\varepsilon)}{\partial t} - \sigma_3 \operatorname{div}(S'(u_3^\varepsilon) d_3(u_3^\varepsilon, \nabla u_3^\varepsilon)) + \sigma_3 d_3(u_3^\varepsilon, \nabla u_3^\varepsilon) S''(u_3^\varepsilon) \nabla u_3^\varepsilon$$

$$= r_3 \rho_2^\varepsilon(u_2^\varepsilon) S'(u_3^\varepsilon) - \alpha_3 u_3^\varepsilon S'(u_3^\varepsilon) + h^\varepsilon S'(u_3^\varepsilon).$$

Use (2.13), (2.14), $(H5) - (H7)$, Lemma 8 and (2.12) with boundedness of S to get

$$\frac{\partial S(u_1)}{\partial t} - \text{div}(S'(u_1)\sigma_1(u_2)d_1(u_1, \nabla u_1)) + \sigma_1(u_2)d_1(u_1, \nabla u_1)S''(u_1)\nabla u_1$$
$$= r_1\rho_1(u_1)S'(u_1) - r_1 u_2 \mu_1(u_1)S'(u_1) - \pi(u_1, u_3)S'(u_1) + fS'(u_1),$$

$$\frac{\partial S(u_2)}{\partial t} - \text{div}(S'(u_2)\sigma_2(u_1)d_2(u_2, \nabla u_2)) + \sigma_2(u_1)d_2(u_2, \nabla u_2)S''(u_2)\nabla u_2$$
$$= r_2\rho_2(u_2)S'(u_2) - r_2 u_1 \mu_2(u_2)S'(u_2) + gS'(u_2),$$

$$\frac{\partial S(u_3)}{\partial t} - \sigma_3\text{div}(S'(u_3)d_3(u_3, \nabla u_3)) + \sigma_3 d_3(u_3, \nabla u_3)S''(u_3)\nabla u_3$$
$$= r_3\rho_2(u_2)S'(u_3) - \alpha_3 u_3 S'(u_3) + hS'(u_3).$$

as ε tends to 0. By Lemma 2 and Aubin type of lemma one can easily find that $S(u_i^\varepsilon(x,0)) = S(u_{i,0}^\varepsilon(x))$ converges to $S(u_{i,0})$ strongly in $H^{-1,s}(\Omega)$, where $s < \inf(2, \frac{N}{N-1})$. Then (H4),(H6) and smoothness of S prove the strong convergence in $L^2(\Omega)$. Hence, we conclude that $S(u_i(x,0)) = S(u_{i,0}(x))$. From the above results, we conclude the existence of renormalized solutions of system (1.1). $\square$

3   Entropy solutions for cancer invasion system

In this section, we have established the second main result of the paper, that is, the renormalized solution is also an entropy solution.

DEFINITION 4. An entropy solution (1.1) is a triple (u_1, u_2, u_3) satisfying the following conditions: for $i = 1, 2, 3$, $u_i \in L^\infty(0, T; L^1(\Omega)) \cap C([0, T], L^1(\Omega))$. For any $k > 0$ and for all $\phi_i \in C^1(Q_T)$ with $\phi_i = 0$ in Σ_T,

$$\int_\Omega \tilde{T}_k(u_1 - \phi_1)(T)\text{d}x - \int_\Omega \tilde{T}_k(u_1 - \phi_1)(0)\text{d}x + \int_0^T \langle \phi_{1t}, T_k(u_1 - \phi_1) \rangle \text{d}t$$
$$+ \int_{Q_T} \sigma_1(u_2)d_1(u_1, \nabla u_1)\nabla T_k(u_1 - \phi_1)\text{d}x\text{d}t$$
$$= \int_{Q_T} (r_1\rho_1(u_1) - r_1 u_2 \mu_1(u_1) - \pi(u_1, u_3) + f)T_k(u_1 - \phi_1)\text{d}x\text{d}t,$$

$$\int_\Omega \tilde{T}_k(u_2 - \phi_2)(T)\text{d}x - \int_\Omega \tilde{T}_k(u_2 - \phi_2)(0)\text{d}x + \int_0^T \langle \phi_{2t}, T_k(u_2 - \phi_2) \rangle \text{d}t$$
$$+ \int_{Q_T} \sigma_2(u_1)d_2(u_2, \nabla u_2)\nabla T_k(u_2 - \phi_2)\text{d}x\text{d}t$$
$$= \int_{Q_T} (r_2\rho_2(u_2) - r_2 u_1 \mu_2(u_2) + g)T_k(u_1 - \phi_1)\text{d}x\text{d}t,$$

$$\int_\Omega \tilde{T}_k(u_3 - \phi_3)(T)\text{d}x - \int_\Omega \tilde{T}_k(u_3 - \phi_3)(0)\text{d}x + \int_0^T \langle \phi_{3t}, T_k(u_3 - \phi_3) \rangle \text{d}t$$
$$+ \int_{Q_T} \sigma_3 d_3(u_3, \nabla u_3)\nabla T_k(u_3 - \phi_3)\text{d}x\text{d}t$$

$$= \int_{Q_T} (r_3\rho_2(u_2) - \alpha_3 u_3 + h)T_k(u_3 - \phi_3)\mathrm{d}x\mathrm{d}t$$

hold.

Theorem 2. *Under the hypotheses* $(H1) - (H5)$, *the renormalized solution of system* (1.1) *is also an entropy solution in the sense of Definition 4.*

Proof. For $i = 1, 2, 3$, we use $T_k(u_i^\varepsilon - \phi_i)$ as the test functions respectively in the equations (2.1) and for $k > 0$, $\phi_i \in C^1(\overline{Q}_T)$ with $\phi_i = 0$ in Σ_T. Multiply (2.1) by $T_k(u_i^\varepsilon - \phi_i)$ respectively and integrating over Q_T, to get

$$\int_0^T \langle u_{1t}^\varepsilon, T_k(u_1^\varepsilon - \phi_1)\rangle \mathrm{d}t + \int_{Q_T} \sigma_1(u_2^\varepsilon)d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)\nabla T_k(u_1^\varepsilon - \phi_1)\mathrm{d}x\mathrm{d}t$$

$$= \int_{Q_T} \left(r_1\rho_1^\varepsilon(u_1^\varepsilon) - r_1 u_2^\varepsilon \mu_1^\varepsilon(u_1^\varepsilon) - \pi^\varepsilon(u_1^\varepsilon, u_3^\varepsilon) + f^\varepsilon\right) T_k(u_1^\varepsilon - \phi_1)\mathrm{d}x\mathrm{d}t,$$

$$\int_0^T \langle u_{2t}^\varepsilon, T_k(u_2^\varepsilon - \phi_2)\rangle \mathrm{d}t + \int_{Q_T} \sigma_2(u_1^\varepsilon)d_2(u_2^\varepsilon, \nabla u_2^\varepsilon)\nabla T_k(u_2^\varepsilon - \phi_2)\mathrm{d}x\mathrm{d}t$$

$$= \int_{Q_T} \left(r_2\rho_2^\varepsilon(u_2^\varepsilon) - r_2 u_1^\varepsilon \mu_2^\varepsilon(u_2^\varepsilon) + g^\varepsilon\right) T_k(u_2^\varepsilon - \phi_2)\mathrm{d}x\mathrm{d}t,$$

$$\int_0^T \langle u_{3t}^\varepsilon, T_k(u_3^\varepsilon - \phi_3)\rangle \mathrm{d}t + \sigma_3 \int_{Q_T} d_3(u_3^\varepsilon, \nabla u_3^\varepsilon)\nabla T_k(u_3^\varepsilon - \phi_3)\mathrm{d}x\mathrm{d}t$$

$$= \int_{Q_T} \left(r_3\rho_2^\varepsilon(u_2^\varepsilon) - \alpha_3 u_3^\varepsilon + h^\varepsilon\right) T_k(u_3^\varepsilon - \phi_3)\mathrm{d}x\mathrm{d}t.$$

Choose $L = k + \|\phi_1\|_{L^\infty(Q_T)}$ and $u_{1t}^\varepsilon = (u_1^\varepsilon - \phi_1)_t - \phi_{1t}$, we get

$$\int_{Q_T} d_1(u_1^\varepsilon, \nabla u_1^\varepsilon)\nabla T_k(u_1^\varepsilon - \phi_1)\mathrm{d}x\mathrm{d}t$$

$$= \int_{Q_T} d_1(T_L(u_1^\epsilon), \nabla T_L(u_1^\epsilon))\nabla T_k(u_1^\epsilon)(T_L(u_1^\epsilon) - \phi_1)\mathrm{d}x\mathrm{d}t,$$

$$\int_0^T \langle u_{1t}^\varepsilon, T_k(u_1^\varepsilon - \phi_1)\rangle \mathrm{d}t = \int_\Omega \tilde{T}_k(u_1^\varepsilon - \phi_1)(T)\mathrm{d}x - \int_\Omega \tilde{T}_k(u_1^\varepsilon - \phi_1)(0)\mathrm{d}x$$

$$+ \int_0^T \langle \phi_{1t}, T_k(T_L(u_1^\varepsilon) - \phi_1)\rangle \mathrm{d}t.$$

From the above equations, we get

$$\int_\Omega \tilde{T}_k(u_1^\varepsilon - \phi_1)(T)\mathrm{d}x - \int_\Omega \tilde{T}_k(u_1^\varepsilon - \phi_1)(0)\mathrm{d}x + \int_0^T \langle \phi_{1t}, T_k(T_L(u_1^\varepsilon) - \phi_1)\rangle \mathrm{d}t$$

$$+ \int_{Q_T} \sigma_1(u_2^\varepsilon)d_1(T_L(u_1^\varepsilon), \nabla T_L(u_1^\varepsilon))\nabla T_k(T_L(u_1^\varepsilon) - \phi_1)\mathrm{d}x\mathrm{d}t$$

$$= \int_{Q_T} \left(r_1\rho_1^\varepsilon(u_1^\varepsilon) - r_1 u_2^\varepsilon \mu_1^\varepsilon(u_1^\varepsilon) - \pi^\varepsilon(u_1^\varepsilon, u_3^\varepsilon) + f^\varepsilon\right) T_k(u_1^\varepsilon - \phi_1)\mathrm{d}x\mathrm{d}t.$$

Since $\tilde{T}_k$ is Lipschitz continuous, using (2.13), we have

$$\int_\Omega \tilde{T}_k(u_1^\varepsilon - \phi_1)(T)\mathrm{d}x \to \int_\Omega \tilde{T}_k(u_1 - \phi_1)(T)\mathrm{d}x,$$

$$\int_\Omega \tilde{T}_k(u_1^\varepsilon - \phi_1)(0)\mathrm{d}x \to \int_\Omega \tilde{T}_k(u_1 - \phi_1)(0)\mathrm{d}x, \quad \text{as} \ \ \varepsilon \to 0.$$

Using the strong convergence of f^ε, (2.13), (2.14) and the Lemma 8 as $\varepsilon \to 0$, we get

$$\int_\Omega \tilde{T}_k(u_1 - \phi_1)(T)\mathrm{d}x - \int_\Omega \tilde{T}_k(u_1 - \phi_1)(0)\mathrm{d}x + \int_0^T \langle \phi_{1t}, T_k(T_L(u_1) - \phi_1)\rangle \mathrm{d}t$$

$$+ \int_{Q_T} \sigma_1(u_2)d_1(u_1, \nabla u_1)\nabla T_k(u_1 - \phi_1)\mathrm{d}x\mathrm{d}t$$

$$= \int_{Q_T} (r_1\rho_1(u_1) - r_1 u_2 \mu_1(u_1) - \pi(u_1, u_3) + f)\, T_k(u_1 - \phi_1)\mathrm{d}x\mathrm{d}t.$$

Similarly, we get

$$\int_\Omega \tilde{T}_k(u_2 - \phi_2)(T)\mathrm{d}x - \int_\Omega \tilde{T}_k(u_2 - \phi_2)(0)\mathrm{d}x + \int_0^T \langle \phi_{2t}, T_k(u_2 - \phi_2)\rangle \mathrm{d}t$$

$$+ \int_{Q_T} \sigma_2(u_1)d_2(u_2, \nabla u_2)\nabla T_k(u_2 - \phi_2)\mathrm{d}x\mathrm{d}t$$

$$= \int_{Q_T} (r_2\rho_2(u_2) - r_2 u_1 \mu_2(u_2) + g)\, T_k(u_2 - \phi_2)\mathrm{d}x\mathrm{d}t,$$

$$\int_\Omega \tilde{T}_k(u_3 - \phi_3)(T)\mathrm{d}x - \int_\Omega \tilde{T}_k(u_3 - \phi_3)(0)\mathrm{d}x + \int_0^T \langle \phi_{3t}, T_k(u_3 - \phi_3)\rangle \mathrm{d}t$$

$$+ \sigma_3 \int_{Q_T} d_3(u_3, \nabla u_3)\nabla T_k(u_3 - \phi_3)\mathrm{d}x\mathrm{d}t$$

$$= \int_{Q_T} (r_3\rho_2(u_2) - \alpha_3 u_3 + h)\, T_k(u_3 - \phi_3)\mathrm{d}x\mathrm{d}t,$$

for all $k > 0$ and for $i = 1, 2, 3$, $\phi_i \in C^1(\overline{Q}_T)$ with $\phi_i = 0$ in Σ_T. This completes the proof of existence of entropy solutions of the system (1.1). $\square$

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