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# A History-Dependent Frictional Contact Problem with Wear for Thermoviscoelastic Materials

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**Abstract.** In this manuscript we study a contact problem between a deformable viscoelastic body and a rigid foundation. Thermal effects, wear and friction between surfaces are taken into account. A variational formulation of the problem is supplied and an existence and uniqueness result is proved. The idea of the proof rested on a recent result on history-dependent quasivariational inequalities. Finally, a perturbation of the data is initiated and a convergence result is demonstrated when the perturbation parameter converges to zero.

**Keywords:** viscoelastic material, thermal effects, friction, history-dependent quasivariational inequality, convergence result.

**AMS Subject Classification:** 74M15; 74D10; 74M10; 74F05.

## 1 Introduction

Analysis of mathematical models in Contact Mechanics is rapidly growing. These models are suggested for different materials using different boundary conditions modelling friction, lubrication, adhesion, wear, damage, etc. The aim of this paper is to model and establish the variational analysis of a contact problem for viscoelastic materials within the infinitesimal strain theory. The process is supposed to be the subject of thermal effects, friction and wear of contacting surfaces. Mathematical models in Contact Mechanics can be found in [1, 2, 7, 8, 9, 10, 12, 13, 14, 17, 18, 19, 21, 22, 23, 24] .

Wear of surfaces is the degradation phenomenon of the superficial layer caused by many factors such as pressure, lubrication, friction and corrosion. Moreover, wear is a loss of use as a result of plastic deformations, material removal or fractures. Also, it is often connected to friction and lubrication. This phenomenon falls within a largest science named Tribology. The latter is encountered in industrial applications and everyday life (adhesion between wheels and rails and chalk squealing on a blackboard are simple examples). Analysis of contact problems with wear can be found in [4, 5, 16, 25].

Frictional contact represents a phenomenon which is frequently encountered almost everywhere in people's daily life. When two contacting bodies are subjected to relative motion, a friction force opposing motion appears leading to a modification of contact surfaces caused by particle detachment. Recent models of frictional contact problems can be found in [17, 19, 22, 23, 24].

Contact and friction processes are invariably accompanied by heat generation which may be considerable. Thermal effects in contact processes affect the composition and stiffness of the contacting surfaces, and cause thermal stresses in the contacting bodies. Moreover, the contacting surfaces exchange heat and energy is lost to the surroundings. Models taking into account thermal effects can be found in [3], [15] and [16] .

Thermoviscoelastic materials are materials which behave according to viscoelastic constitutive laws with added thermal effects. We model the material's behavior with a constitutive law with long memory of the form

$$\sigma(t) = \mathcal{A}\,\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{G}\,\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{B}(t-s)\,\varepsilon(\mathbf{u}(s))\,ds - \mathcal{M}\,\xi(t), \tag{1.1}$$

in which $\mathcal{A}$ is the viscosity operator, $\mathcal{G}$ is the elastic operator and $\mathcal{B}$ is the relaxation tensor. Also, $\mathbf{u}$ denotes the displacement field, $\sigma$ represents the stress tensor, $\varepsilon(\mathbf{u})$ is the linearized strain tensor of infinitesimal deformations and $\varepsilon(\dot{\mathbf{u}})$ is the velocity of infinitesimal deformations. Moreover, $\xi$ is the temperature field and $\mathcal{M} = (m_{ij})$ represents the thermal expansion tensor. In the particular case without thermal effects, the constitutive equation (1.1) reduces to the following viscoelastic constitutive law with long memory

$$\sigma(t) = \mathcal{A}\,\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{G}\,\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{B}(t-s)\,\varepsilon(\mathbf{u}(s))\,ds.$$

The evolution of the temperature field $\xi$ is governed by the heat equation obtained from the conservation of energy, and defined by the following differential equation for the temperature

$$\dot{\xi}(t) - Div\,(\mathcal{K}\,\nabla\xi(t)) = q(t) - \mathcal{M}\,\nabla\dot{\mathbf{u}}(t),$$

where $\mathcal{K} = (k_{ij})$ is the thermal conductivity tensor, $Div(\mathcal{K}\,\nabla\xi) = (k_{ij}\,\xi_{,i})_{,i}$ and $q(t)$ represents the density of volume heat sources. We recall that we use dots for derivatives with respect to the time variable $t$. The pointwise heat exchange condition on the contact surface is given by

$$-k_{ij}\,\frac{\partial\xi}{\partial x_i}(t)\;\nu_j = k_e\,(\xi(t) - \theta_R(t)),$$

where $k_{ij}$ are the components of the thermal conductivity tensor, $\nu_j$ are the normal components of the outward unit normal $\nu$ described in section 2, $k_e$ is the heat exchange coefficient, $\xi$ is the the pointwise surface temperature and $\theta_R$ is the known temperature of the foundation.

The model is obtained by combining the thermoviscoelastic constitutive law with long memory term, wear and friction. Moreover, the process is studied on an unbounded interval of time $\mathbb{R}^+ = [0, +\infty)$ , which implies the use of the framework of Fréchet spaces of continuous functions, rather than the classical Banach spaces of continuous functions defined on a bounded interval of time. This leads to a new mathematical model governed by a history-dependent quasivariational inequality for the velocity field and a nonlinear equation for the temperature field. In this paper, we prove that the proposed model has a unique weak solution using history-dependent operators. Also, we study the continuous dependence of the weak solution of the problem with respect to the relaxation tensor, friction coefficient, volume forces and surface tractions.

Finally, to study the suggested model, we proceed as follows. In Section 2, some notations and preliminary material are introduced. In Section 3, we provide a description of the model of the contact process, we list assumptions on the data and derive a variational formulation of the model. In Section 4, we state and prove our main existence and uniqueness result, Theorem 2. Finally, in Section 5, a continuous dependence of the solution with respect to the data is proved.

## 2 Notations and preliminaries

In this short section, we present the notations we shall use and some preliminary material. We use the notation $\mathbb{N}$ for the set of positive integers and $\mathbb{R}^+$ will represent the set of nonnegative real numbers, i.e. $\mathbb{R}^+ = [0, +\infty)$. For $d \in \mathbb{N}$, we denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 1, 2, 3)$. The inner products and norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\mathbf{u} \,.\mathbf{v} = u_i v_i \ , \ \ \|\mathbf{v}\| = (\mathbf{v} \,.\mathbf{v})^{\frac{1}{2}} \ \ \forall \mathbf{u},\, \mathbf{v} \in \mathbb{R}^d,$$
$$\sigma \,.\tau = \sigma_{ij}\tau_{ij} \ , \ \ \|\tau\| = (\tau \,.\tau)^{\frac{1}{2}} \ \ \forall \sigma,\, \tau \in \mathbb{S}^d.$$

Here and below, the indices $i$ and $j$ run between 1 to $d$ and, unless stated otherwise, the summation convention over repeated indices is used. Also, an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e. g. $u_{i,j} = \partial u_i / \partial x_j$.

Let $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be a bounded domain with a Lipschitz continuous boundary $\Gamma$ and let $\Gamma_1$ be a measurable part of $\Gamma$ such that $meas\,(\Gamma_1) > 0$. We use $\mathbf{x} = (x_i)$ for a generic point in $\Omega$ and we denote by $\nu = (\nu_i)$ the outward unit normal at $\Gamma$, $\mathbf{u} = (u_i)$, $\sigma = (\sigma_{ij})$ .

Now, let consider the spaces

$$V = \left\{ \mathbf{v} = (v_i) \in H^1\,(\Omega)^d \,/\, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1 \right\},$$
$$Q = \left\{ \tau = (\tau_{ij}) \in L^2(\Omega)^{d\times d} /\ \tau_{ij} = \tau_{ji} \right\}.$$

We note that $V$ and $Q$ are real Hilbert spaces endowed with the inner products

$$(\mathbf{u},\mathbf{v})_V = \int_\Omega \varepsilon(\mathbf{u}).\varepsilon(\mathbf{v})\, dx, \qquad (\sigma,\tau)_Q = \int_\Omega \sigma.\tau\, dx,$$

and the associated norms $\|.\|_V$ and $\|.\|_Q$ respectively. Here $\varepsilon$ represents the deformation operator given by

$$\varepsilon(\mathbf{v}) = (\varepsilon_{ij}(\mathbf{v})), \quad \varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}(v_{i,j} + v_{j,i})\ \forall \mathbf{v} \in H^1(\Omega)^d.$$

Completeness of the space $(V, \|.\|_V)$ follows from the assumption $meas\,(\Gamma_1) > 0$, which allows the use of Korn's inequality.

For an element $\mathbf{v} \in V$, we still write $\mathbf{v}$ for the trace of $\mathbf{v}$ on the boundary and we denote by $v_\nu$ and $\mathbf{v}_\tau$ the normal and the tangential components of $\mathbf{v}$ on the boundary $\Gamma$ given by $v_\nu = \mathbf{v}.\nu$, $\mathbf{v}_\tau = \mathbf{v} - v_\nu \nu$. Let $\Gamma_3$ be a measurable part of $\Gamma$. Then, by the Sobolev trace Theorem, there exists a positive constant $c_0$ which depends on $\Omega$, $\Gamma_1$ and $\Gamma_3$ such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \le c_0 \|\mathbf{v}\|_V, \qquad \forall \mathbf{v} \in V. \tag{2.1}$$

For a regular function $\sigma \in Q$ we denote by $\sigma_\nu$ and $\sigma_\tau$ the normal and the tangential components of the vector $\sigma\nu$ on $\Gamma$, respectively, and we recall that

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

We also recall that the divergence operator is defined by $Div\ \sigma = (\sigma_{ij,j})$, the following Green's formula holds:

$$\int_\Omega \sigma.\varepsilon(\mathbf{v})\, dx + \int_\Omega Div\ \sigma.\mathbf{v}\, dx = \int_\Gamma \sigma\nu.\mathbf{v}\, da, \quad \forall\, \mathbf{v} \in V. \tag{2.2}$$

For functional reasons, it is convenient to shift the ambient temperature to zero on $\Gamma_1 \cup \Gamma_2$. We consider for this purpose $\theta = \xi - \theta_a$, by assuming $\theta_a \in H^1\left(\mathbb{R}^+, H^1(\Omega)\right)$. Therefore, we have

$$\xi(t) = \theta_a(t) \Rightarrow \theta(t) = 0 \quad \text{on} \ \ \Gamma_1 \cup \Gamma_2,$$

and the following change of variable will be used

$$\xi(t) = \theta(t) + \theta_a(t), \quad \xi_0 = \theta_0 + \theta_a(0).$$

Then, the following spaces for the temperature field are introduced

$$E = \left\{\gamma \in H^1(\Omega) \,/\, \gamma = 0 \text{ on } \Gamma_1 \cup \Gamma_2\right\}, \quad F = L^2(\Omega).$$

The spaces $E$ and $F$ endowed with their respective canonical inner products are Hilbert spaces. Identifying $F$ with its own dual, we obtain the Gelfand evolution triple $E \subset F \equiv F' \subset E'$, where the inclusion is continuous and dense.

Finally, we denote by $Q_\infty$ the space of fourth order tensor fields given by

$$Q_\infty = \{\mathcal{E} = (\mathcal{E}_{ijkl}) \ / \ \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega), \ \ 1 \le i,j,k,l \le d\},$$

which is a real Banach space with the norm

$$\|\mathcal{E}\|_{Q_\infty} = \max_{1\le i,j,k,l\le d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}\,.$$

A simple calculation shows that

$$\|\mathcal{E}\,\tau\|_Q \le d\,\|\mathcal{E}\|_{Q_\infty}\,\|\tau\|_Q\,, \quad \forall \mathcal{E}\in Q_\infty,\;\; \tau\in Q. \tag{2.3}$$

For each Banach space $X$ we use the notation $C(\mathbb{R}^+, X)$ for the space of $X$-valued continuous functions defined on $\mathbb{R}^+$ with values on $X$, and $C^1(\mathbb{R}^+, X)$ for the space of continuous differentiable functions defined on $\mathbb{R}^+$. Details can be found in [6] and [11] for instance.

The following results will be used in Section 4. We start by recalling a convergence criterion in $C(\mathbb{R}^+, X)$ of a sequence $(x_k)_k$ to an element $x$, which is given by the following,

$$\begin{cases} x_k \to x \text{ in } C(\mathbb{R}^+, X) \text{ as } k\to\infty \text{ if and only if} \\ \max\limits_{r\in[0,n]} \|x_k(r) - x(r)\|_X \to 0 \text{ as } k\to\infty \text{ for all } n\in\mathbb{N}. \end{cases} \tag{2.4}$$

Moreover, the convergence of a sequence $(x_k)_k$ to an element $x\in C^1(\mathbb{R}^+, X)$ is given by

$$\begin{cases} x_k \to x \text{ in } C^1(\mathbb{R}^+, X) \text{ as } k\to\infty \text{ if and only if} \\ x_k \to x \text{ in } C(\mathbb{R}^+, X) \text{ and } \dot{x}_k \to \dot{x} \text{ in } C(\mathbb{R}^+, X) \text{ as } k\to\infty, \end{cases} \tag{2.5}$$

where $\dot{x}$ denotes the time derivative of $x$ for all $x\in C(\mathbb{R}^+, X)$.

Next, we state a second result proved in [20] on history-dependent quasi-variational inequalities. To this end, we introduce the following setting. Let $X$ be a real Hilbert space with the inner product $(.,.)_X$ and the associated norm $\|.\|_X$ and let $Y$ be a normed space with the norm $\|.\|_Y$. Let $K$ be a subset of $X$ and consider the operators $A: K\to X$, $\mathcal{S}: C(\mathbb{R}^+, X)\to C(\mathbb{R}^+, Y)$, the functionals $\varphi: Y\times K\to\mathbb{R}$ and $j: X\times K\to\mathbb{R}$, in addition to the function $f:\mathbb{R}^+\to X$ such that

$$K \text{ is a nonempty closed convex subset of } X. \tag{2.6}$$

$$\begin{cases} \text{(a) There exists } m_A>0 \text{ such that} \\ (A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1-\mathbf{u}_2)_X \ge m_A\;\|\mathbf{u}_1-\mathbf{u}_2\|_X^2\,,\quad \forall \mathbf{u}_1,\,\mathbf{u}_2\in K. \\ \text{(b) There exists } L_A>0 \text{ such that} \\ \|A\mathbf{u}_1 - A\mathbf{u}_2\|_X \le L_A\;\|\mathbf{u}_1-\mathbf{u}_2\|_X\,,\quad \forall \mathbf{u}_1,\,\mathbf{u}_2\in K. \end{cases} \tag{2.7}$$

$$\begin{cases} \text{For every } n\in\mathbb{N}^* \text{ there exists } s_n>0 \text{ such that} \\ \|\mathcal{S}\mathbf{u}_1(t) - \mathcal{S}\mathbf{u}_2(t)\|_Y \le s_n\;\int_0^t \|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_X\;ds, \\ \forall \mathbf{u}_1,\,\mathbf{u}_2\in C(\mathbb{R}^+, X),\; \forall t\in[0,n]. \end{cases} \tag{2.8}$$

$$\begin{cases} \text{(a) The function } \varphi(\mathbf{y},.) : K\to\mathbb{R} \text{ is convex} \\ \text{and lower semicontinuous, for all } \mathbf{y}\in Y. \\ \text{(b) There exists } \alpha\ge 0 \text{ such that} \\ \varphi(\mathbf{y}_1,\mathbf{v}_2) - \varphi(\mathbf{y}_1,\mathbf{v}_1) + \varphi(\mathbf{y}_2,\mathbf{v}_1) - \varphi(\mathbf{y}_2,\mathbf{v}_2) \\ \quad \le \alpha\,\|\mathbf{y}_1-\mathbf{y}_2\|_Y\,\|\mathbf{v}_1-\mathbf{v}_2\|_X\,, \\ \forall \mathbf{y}_1,\,\mathbf{y}_2\in Y,\;\; \forall \mathbf{v}_1,\;\mathbf{v}_2\in K. \end{cases} \tag{2.9}$$

$$\begin{cases} \text{(a) The function } j(\mathbf{u},.) \text{ is convex and lower} \\ \text{semicontinuous on } K, \text{ for all } \mathbf{u} \in X. \\ \text{(b) There exists } \beta \geq 0 \text{ such that} \\ j(\mathbf{u}_1,\mathbf{v}_2) - j(\mathbf{u}_1, \mathbf{v}_1) + j(\mathbf{u}_2,\mathbf{v}_1) - j(\mathbf{u}_2,\mathbf{v}_2) \\ \quad \leq \beta \|\mathbf{u}_1 - \mathbf{u}_2\|_X \|\mathbf{v}_1 - \mathbf{v}_2\|_X, \\ \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in X, \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in K. \end{cases} \tag{2.10}$$

$$f \in C\left(\mathbb{R}^+, X\right). \tag{2.11}$$

$$\beta < m_A. \tag{2.12}$$

We have the following result.

**Theorem 1.** *Assume that (2.6)–(2.12) hold. Then there exists a unique function* $\mathbf{u} \in C(\mathbb{R}^+, K)$ *such that, for all* $t \in \mathbb{R}^+$,

$$\begin{aligned} &(A\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t))_X + \varphi(\mathcal{S}\mathbf{u}(t), \mathbf{v}) - \varphi(\mathcal{S}\mathbf{u}(t), \mathbf{u}(t)) \\ &\quad + J(\mathbf{u}(t), \mathbf{v}) - J(\mathbf{u}(t), \mathbf{u}(t)) \geq (f(t), \mathbf{v} - \mathbf{u}(t))_X, \quad \forall \mathbf{v} \in K. \end{aligned} \tag{2.13}$$

We note that (2.13) is a time-dependent variational inequality in which the functional $\varphi$ depends on the solution; such inequality is referred to as a quasivariational inequality. Moreover, following the terminology introduced in [20] we refer to an operator which satisfies condition (2.8) as a history-dependent operator. Therefore, (2.13) represents a history-dependent quasivariational inequality.

Finally, we assume that $X$ and $Y$ are two real Hilbert spaces with the inner products $(.,.)_X$ and $(.,.)_Y$, and the associated norms $\|.\|_X$ and $\|.\|_Y$, respectively. We denote by $X \times Y$ the product space of $X$ and $Y$ and we recall that $X \times Y$ is a real Hilbert space with the canonical inner product $(.,.)_{X\times Y}$ given by

$$(z_1, z_2)_{X\times Y} = (x_1, x_2)_X + (y_1, y_2)_Y, \quad \forall\, z_1 = (x_1, y_1),\ z_2 = (x_2, y_2) \in X \times Y.$$

The associated norm of the space $X \times Y$, denoted $\|.\|_{X\times Y}$, satisfies

$$\|z\|_{X\times Y} \leq \|x\|_X + \|y\|_Y \leq \sqrt{2}\ \|z\|_{X\times Y}, \quad \forall\, z = (x, y) \in X \times Y. \tag{2.14}$$

## 3 Problem statement and variational formulation

We consider a thermoviscoelastic body which occupies a bounded domain $\Omega \subset \mathbb{R}^d$, $(d = 1, 2, 3)$ with a Lipschitz continuous boundary $\Gamma$, divided into three measurable parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, such that $meas(\Gamma_1) > 0$. The body is acted upon by body forces of density $\mathbf{f}_0$ and surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2$. We assume that the body is clamped on $\Gamma_1$, and therefore, the displacement field vanishes there. The body may come in contact over $\Gamma_3$ with a moving rigid foundation. The contact is supposed to be bilateral, which means that it is maintained at all times and there is only relative sliding. A common example of such a situation is a conveyer belt or a chain connecting two rotating wheels. Moreover, in our model, the contacting surfaces evolve via their wear which is modelled by the wear function $\omega : \Gamma_3 \times \mathbb{R}^+ \to \mathbb{R}^+$ measuring the depth in the normal direction of the removed material. Following [16], it is negative when the foundation is worn out, and positive when the surface of the body wears out. Here, we assume that the foundation is rigid and the body wears out, and therefore $\omega$ is nonnegative. Since the body is in bilateral contact with the foundation,

$$u_\nu = -\omega. \tag{3.1}$$

Since $\omega \geq 0$, it follows that $u_\nu \leq 0$. It means that the effect of the wear is the recession of $\Gamma_3$. So, we describe the evolution of the shape of the contact zone as a result of wear, see [16].

To describe the evolution of the wear, we use the following rate form of Archard's law of surface wear, (see [16] or [25])

$$\dot{\omega} = -k_\omega \left|\sigma_\nu\right| \left\|\dot{\mathbf{u}}_\tau - \vartheta^*\right\|,$$

where $k_\omega$ is the wear coefficient supposed to be a very small positive constant. Here, since the contact is bilateral, the foundation is itself moving with prescribed velocity $\vartheta^* = \vartheta^*(t)$. Moreover, for the sake of simplicity, we assume that $\vartheta^*$ is a positive constant; i.e. it does not vary in time. We can see that the rate of the wear of the surface is proportional to the contact pressure and to the relative slip speed $\dot{\mathbf{u}}_\tau - \vartheta^*$. Since the normal stress on the contact surface is negative ($\sigma_\nu \leq 0$ on $\Gamma_3$), then the previous version of Archard's law becomes

$$\dot{\omega} = -k_\omega \sigma_\nu \left\|\dot{\mathbf{u}}_\tau - \vartheta^*\right\|.$$

Also, we assume that $\alpha^* = \|\vartheta^*\| > 0$ is large so that we can neglect $\|\dot{\mathbf{u}}_\tau\|$ besides $\|\vartheta^*\|$ and thus, we obtain

$$\dot{\omega} = -k_\omega \alpha^* \sigma_\nu. \tag{3.2}$$

We note that (3.2) implies $\dot{\omega} \geq 0$; that is the wear increases in time. Also, from (3.1), $\dot{u}_\nu \leq 0$. Now, let $\beta^* = 1/\ k_\omega\ \alpha^*$, which is supposed to be a positive constant. Then, using (3.1) and (3.2), we can eliminate the unknown function $\omega$ from the problem. We obtain the following boundary condition

$$\sigma_\nu = \beta^* \dot{u}_\nu,$$

which has the same form of the normal damped response condition, see [16]. We note that, once the solution of the problem is obtained, $\omega$ is then calculated using (3.2). Recalling that there is only sliding contact, we model the friction by the following Coulomb's law of dry friction

$$\|\sigma_\tau\| = \mu \left|\sigma_\nu\right|, \quad \sigma_\tau = -\lambda \dot{\mathbf{u}}_\tau, \quad \lambda \geq 0,$$

where $\mu > 0$ is the friction coefficient. Since $\sigma_\nu \leq 0$, it follows that

$$\|\sigma_\tau\| = -\mu\ \sigma_\nu, \quad \sigma_\tau = -\lambda\ \dot{\mathbf{u}}_\tau, \quad \lambda \geq 0.$$

We are interested in the deformation of the body for the entire time interval $\mathbb{R}^+ = [0, +\infty)$. The process is assumed to be quasistatic, i.e. the inertial effect in the equation of the motion is neglected. To simplify the notation, sometimes we do not indicate the dependence of a function on the spatial variable $\mathbf{x}$ or the time variable $t$. The classical formulation of the contact problem is as follows:

*Problem P.* Find a displacement field $\mathbf{u} = (u_i) : \Omega \times \mathbb{R}^+ \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times \mathbb{R}^+ \to \mathbb{S}^d$ and a temperature field $\theta : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{G}\,\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{B}(t-s)\,\varepsilon(\mathbf{u}(s))\,ds - \mathcal{M}\,\theta(t) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{3.3}$$

$$\dot{\theta}(t) - Div(\mathcal{K}\nabla\theta(t)) = q(t) - \mathcal{M}\nabla\dot{\mathbf{u}}(t) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{3.4}$$

$$Div\,\sigma(t) + \mathbf{f}_0(t) = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \tag{3.5}$$

$$\mathbf{u}(t) = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \tag{3.6}$$

$$\sigma(t).\nu = \mathbf{f}_2(t) \quad \text{on } \Gamma_2 \times \mathbb{R}^+, \tag{3.7}$$

$$-k_{ij}\frac{\partial\theta}{\partial x_i}(t)\,\nu_i = k_e(\theta(t) - \theta_R(t)) \quad \text{on } \Gamma_3 \times \mathbb{R}^+, \tag{3.8}$$

$$\sigma_\nu(t) = \beta^*\dot{u}_\nu(t) \quad \text{on } \Gamma_3 \times \mathbb{R}^+, \tag{3.9}$$

$$\|\sigma_\tau(t)\| = -\mu\sigma_\nu(t), \quad \sigma_\tau = -\lambda\dot{\mathbf{u}}_\tau(t), \quad \lambda \geq 0. \quad \text{on } \Gamma_3 \times \mathbb{R}^+, \tag{3.10}$$

$$\theta(t) = 0 \quad \text{on } (\Gamma_1 \cup \Gamma_2) \times \mathbb{R}^+, \tag{3.11}$$

$$\mathbf{u}(0) = \mathbf{u}_0,\ \theta(0) = \theta_0 \quad \text{in } \Omega. \tag{3.12}$$

We now describe the problem (3.3)–(3.12). First, equations (3.3)–(3.4) represent the thermoviscoelastic constitutive law and the evolution equation of the heat field respectively introduced in Section 1. Equation (3.5) is the balance equation for the stress field and it is used since the process is assumed to be quasistatic. Conditions (3.6)–(3.7) are the displacement-traction boundary conditions respectively.

Now, we comment on the conditions (3.8)–(3.11) which are the boundary conditions. Conditions (3.8) and (3.11) represent the temperature boundary conditions, where (3.8) is described in Section 1. The wear evolution and the bilateral contact yield condition (3.9). Condition (3.10) is the friction law where $(-\mu\ \sigma_\nu)$ is the friction bound function, $\mu \geq 0$, $\lambda \geq 0$.

Finally, (3.12) represents the initial conditions in which $\mathbf{u}_0$ and $\theta_0$ are the initial displacement and the initial temperature respectively.

We turn now to the variational formulation of the Problem $P$. To this end, we assume that the viscosity operator $\mathcal{A}$, the elasticity operator $\mathcal{G}$ and the relaxation tensor $\mathcal{B}$ satisfy

$$\begin{cases} \text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \quad \|\mathcal{A}(\mathbf{x}, \xi_1) - \mathcal{A}(\mathbf{x}, \xi_2)\| \leq L_{\mathcal{A}}\ \|\xi_1 - \xi_2\|, \\ \quad \forall\, \xi_1,\ \xi_2 \in \mathbb{S}^d, \text{ a. e. } \mathbf{x} \in \Omega. \\ \text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(\mathbf{x}, \xi_1) - \mathcal{A}(\mathbf{x}, \xi_2)).(\xi_1 - \xi_2) \geq m_{\mathcal{A}}\ \|\xi_1 - \xi_2\|^2, \\ \quad \forall\, \xi_1,\ \xi_2 \in \mathbb{S}^d, \text{ a. e. } \mathbf{x} \in \Omega. \end{cases}$$

$$\begin{cases} \text{(d) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x},\xi) \text{ is measurable on } \Omega, \\ \qquad \text{for any } \xi \in \mathbb{S}^d. \\ \text{(e) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x},\mathbf{0}) \text{ belongs to } Q. \end{cases} \tag{3.13}$$

$$\begin{cases} \text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\ \quad \|\mathcal{G}(\mathbf{x},\xi_1) - \mathcal{G}(\mathbf{x},\xi_2)\| \le L_{\mathcal{G}} \ \|\xi_1 - \xi_2\|, \\ \quad \forall\, \xi_1,\ \xi_2 \in \mathbb{S}^d, \text{ a. e. } \mathbf{x} \in \Omega. \\ \text{(c) The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x},\xi) \text{ is Lebesgue measurable on } \Omega, \\ \quad \text{for any } \xi \in \mathbb{S}^d. \\ \text{(e) The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x},\mathbf{0}) \text{ belongs to } Q. \end{cases} \tag{3.14}$$

$$\mathcal{B} \in C\left(\mathbb{R}^+, Q_\infty\right). \tag{3.15}$$

The densities of body forces and surface tractions are such that

$$\mathbf{f}_0 \in C\left(\mathbb{R}^+, L^2(\Omega)^d\right), \qquad \mathbf{f}_2 \in C\left(\mathbb{R}^+, L^2(\Gamma_2)^d\right). \tag{3.16}$$

Coefficients $\mu$ and $\beta^*$ satisfy the following conditions

$$\begin{cases} \mu \in L^\infty(\Gamma_3), \qquad \mu(x) \ge 0 \text{ a.e. on } \Gamma_3. \\ \beta^* \in L^\infty(\Gamma_3), \qquad \beta^*(x) \ge \beta_0 > 0 \text{ a.e. on } \Gamma_3. \end{cases} \tag{3.17}$$

The thermal tensors and the heat source density satisfy

$$\begin{cases} \mathcal{M} = (m_{ij}), \qquad m_{ij} = m_{ji} \in L^\infty(\Omega). \\ \mathcal{K} = (k_{ij}), \qquad k_{ij} = k_{ji} \in L^\infty(\Omega),\ k_{ij}\ \xi_i\ \xi_i \ge c_k\ \xi_i\ \xi_i, \\ \qquad \text{for some } c_k > 0, \text{ for all } (\xi_i) \in \mathbb{R}^d. \\ q \in L^2\left(\mathbb{R}^+, L^2(\Omega)\right). \end{cases} \tag{3.18}$$

Finally, the boundary and initial data verify

$$\mathbf{u}_0 \in V, \quad \theta_0 \in E,\ \theta_R \in L^2\left(\mathbb{R}^+, L^2(\Gamma_3)\right),\ k_e \in L^\infty\left(\Omega, \mathbb{R}^+\right). \tag{3.19}$$

In order to get a variational formulation of the Problem $P$, we assume in what follows that $(\mathbf{u}, \sigma, \theta)$ are sufficiently regular functions which satisfy (3.3)–(3.12). Let $\mathbf{v} \in V$ and $t > 0$ be given.

Then, using the Riesz representation Theorem we define, for all $\mathbf{u}, \mathbf{v} \in V$ and $t \in \mathbb{R}^+$, the function $\mathbf{f} : \mathbb{R}^+ \to V$ by

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t)\ \mathbf{v}\ dx + \int_{\Gamma_2} \mathbf{f}_2(t)\ \mathbf{v}\ da. \tag{3.20}$$

Moreover, we define the mapping $J : V \times V \to \mathbb{R}$ as follows

$$J(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \beta^* |u_\nu|\ \ (\mu \|\mathbf{v}_\tau\| + v_\nu)\ da, \tag{3.21}$$

for all $\mathbf{u}, \mathbf{v} \in V$ and $t \in \mathbb{R}^+$. We note that, from assumptions (3.16) and (3.17), we can see that the integrals defined in (3.20)–(3.21) are well-defined and, in addition,

$$\mathbf{f} \in C\left(\mathbb{R}^+, V\right). \tag{3.22}$$

Also, we note that for all $\mathbf{u}_1, \mathbf{u_2}, \mathbf{v}_1, \mathbf{v}_2 \in V$ we have

$$\begin{aligned} &J(\mathbf{u}_1, \mathbf{v}_2) - J(\mathbf{u}_1, \mathbf{v}_1) + J(\mathbf{u}_2, \mathbf{v}_1) - J(\mathbf{u}_2, \mathbf{v}_2) \\ &\quad = \int_{\Gamma_3} \beta^* \ (|u_{1\nu}| - |u_{2\nu}|) \ [\mu \ (\|\mathbf{v}_{2\tau} - \mathbf{v}_{1\tau}\|) + (v_{2\nu} - v_{1\nu})] \ da. \end{aligned}$$

Taking into account assumptions (3.17) combined with (2.1), we obtain

$$\begin{aligned} &J(\mathbf{u}_1, \mathbf{v}_2) - J(\mathbf{u}_1, \mathbf{v_1}) + J(\mathbf{u}_2, \mathbf{v_1}) - J(\mathbf{u}_2, \mathbf{v}_2) \\ &\quad \le c_0^2 \ \|\beta^*\|_{L^\infty(\Gamma_3)} \ \left(\|\mu\|_{L^\infty(\Gamma_3)} + 1\right) \ \|\mathbf{u}_1 - \mathbf{u}_2\|_V \ \|\mathbf{v}_1 - \mathbf{v}_2\|_V . \end{aligned} \tag{3.23}$$

Finally, using (2.2), it is staightforward to see that if $\mathbf{u}$, $\sigma$ and $\theta$ are sufficiently regular functions which satisfy (3.5)–(3.11), then we obtain the following variational formulation of the Problem $P$.

*Problem $P_V$.* Find a displacement field $\mathbf{u} : \mathbb{R}^+ \to V$, a stress field $\sigma : \mathbb{R}^+ \to Q$ and a temperature field $\theta : \mathbb{R}^+ \to E$ such that for all $t \in \mathbb{R}^+$,

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{G}\ \varepsilon(\mathbf{u}) + \int_0^t \mathcal{B}(t-s)\varepsilon(\mathbf{u}(s))ds - \mathcal{M}\theta(t), \tag{3.24}$$

$$\begin{aligned} &(\mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}(t)))_Q + (\mathcal{G}\varepsilon(\mathbf{u}(t)), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}(t)))_Q \\ &\quad + \left(\int_0^t \mathcal{B}(t-s)\varepsilon(\mathbf{u}(s))ds - \mathcal{M}\theta(t), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}(t))\right)_Q \\ &\quad + J(\dot{\mathbf{u}}(t), \mathbf{v}) - J(\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t)) \ge (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V, \text{ for all } \mathbf{v} \in V, \end{aligned} \tag{3.25}$$

$$\dot{\theta}(t) + K\theta(t) = R\ \dot{\mathbf{u}}(t) + Z(t) \ \text{ in } E', \tag{3.26}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \theta(0) = \theta_0, \tag{3.27}$$

where $K : E \to E'$, $R : V \to E'$ and $Z : \mathbb{R}^+ \to E'$ are given by

$$\begin{aligned} &(K\tau, \eta)_{E' \times E} = \sum_{i,j=1}^d \int_\Omega k_{ij} \ \frac{\partial \tau}{\partial x_j} \frac{\partial \eta}{\partial x_i} \ dx + \int_{\Gamma_3} \ k_e \ \tau\eta \ da, \\ &(R\mathbf{v}, \eta)_{E' \times E} = -\int_\Omega \ m_{ij} \ \frac{\partial v_i}{\partial x_j} \ \eta \ dx, \\ &(Z(t), \eta)_{E' \times E} = \int_{\Gamma_3} \ k_e (\theta_R(t) - \theta_a(t)) \eta \ da + \int_\Omega \left(q(t) - \dot{\theta}_a(t)\right) \eta \ dx \\ &\qquad - \sum_{i,j=1}^d \int_\Omega \ k_{ij} \ \frac{\partial \theta_a(t)}{\partial x_j} \ \frac{\partial \eta}{\partial x_i} dx, \end{aligned}$$

for all $\mathbf{v} \in V$, $\tau, \eta \in E$.

## 4 Existence and uniqueness result

Our main existence and uniqueness result of this section is the following.

**Theorem 2.** *Assume (3.13)–(3.19). Then, there exists $L_0 > 0$ depending only on $\Omega$, $\Gamma_1$, $\Gamma_3$ and $\mathcal{A}$ such that if $\|\beta^*\|_{L^\infty(\Gamma_3)} \left(\|\mu\|_{L^\infty(\Gamma_3)} + 1\right) < L_0$, Problem $P_V$ has a unique solution which satisfies*

$$\begin{aligned} &\mathbf{u} \in C^1(\mathbb{R}^+, V), \quad \sigma \in C(\mathbb{R}^+, Q), \\ &\theta \in C(\mathbb{R}^+, F) \cap L^2(\mathbb{R}^+, E) \cap W^{1,2}(\mathbb{R}^+, E'). \end{aligned} \tag{4.1}$$

The proof of Theorem 2 will be carried out in several steps. We assume in what follows that (3.13)–(3.19) hold and, everywhere below, we denote by $c$ a generic positive constant which is independent of time and whose value may change from one occurrence to another. We start with the following result for the unique solvability of equation (3.26).

**Lemma 1.** *Given $\mathbf{u} \in C^1(\mathbb{R}^+, V)$, there exists a unique function $\theta = \theta(\mathbf{u}) \in C(\mathbb{R}^+, F) \cap L^2(\mathbb{R}^+, E) \cap W^{1,2}(\mathbb{R}^+, E')$ such that for all $t \in \mathbb{R}^+$,*

$$\dot{\theta}(t) + K\theta(t) = R\,\dot{\mathbf{u}}(t) + Z(t) \;\; in\ E', \tag{4.2}$$

$$\theta(0) = \theta_0. \tag{4.3}$$

*Also, if $\theta_i = \theta(\mathbf{u}_i) \in C(\mathbb{R}^+, F)$ are solutions of (4.2)–(4.3) which correspond to $\mathbf{u}_i \in C^1(\mathbb{R}^+, V)$, for $i = 1, 2$, then we have for all $t \in \mathbb{R}^+$*

$$\|\theta_1(t) - \theta_2(t)\|_F^2 \le c \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 \; ds, \quad c > 0. \tag{4.4}$$

*Proof.* For $t \in \mathbb{R}^+$ fixed, we note that the operator $K : E \to E'$ is linear continuous and strongly monotone. Moreover, Friedrich's-Poincaré inequality yields

$$(K\tau, \tau)_{E' \times E} \ge c \;|\tau|_E^2, \quad c > 0.$$

Since we have the Gelfand triple $E \subset F \equiv F' \subset E'$, we use a classical result on first order evolution equations given in [18] to prove the unique solvability of (4.2)–(4.3) at any $t$.

Now, for $\mathbf{u}_i \in C^1(\mathbb{R}^+, V)$, $i = 1, 2$, we still use the notation $\theta_i = \theta(\mathbf{u}_i)$, $i = 1, 2$. Let $t \in \mathbb{R}^+$ be fixed. Then, we have

$$\begin{aligned} &\left(\dot{\theta}_1(t) - \dot{\theta}_2(t), \theta_1(t) - \theta_2(t)\right)_{E' \times E} + (K\theta_1(t) - K\theta_2(t), \theta_1(t) - \theta_2(t))_{E' \times E} \\ &\quad = (R\,\dot{\mathbf{u}}_1(t) - R\,\dot{\mathbf{u}}_2(t), \theta_1(t) - \theta_2(t))_{E' \times E}. \end{aligned} \tag{4.5}$$

We integrate (4.5) over $(0, t)$ and we use the strong monotonicity of $K$ and the Lipschitz continuity of $R : V \to E'$ to deduce that (4.4) holds at any $t \in \mathbb{R}^+$. □

In the next step in the proof of Theorem 2, we define the operators $\mathcal{S}_1 : C(\mathbb{R}^+, V) \to C(\mathbb{R}^+, V)$ by

$$\mathcal{S}_1 \mathbf{w}(t) = \int_0^t \mathbf{w}(s) \; ds + \mathbf{u}_0. \tag{4.6}$$

It is clear that $\mathcal{S}_1$ is a history-dependent operator since it verifies

$$\|\mathcal{S}_1\mathbf{w}_1(t) - \mathcal{S}_1\mathbf{w}_2(t)\|_V \leq \int_0^t \|\mathbf{w}_1(s) - \mathbf{w}_2(s)\|_V \; ds. \tag{4.7}$$

Also, let introduce the operator $\mathcal{S} : C(\mathbb{R}^+, V) \to C(\mathbb{R}^+, Q \times F)$ given by $\mathcal{S}\mathbf{w}(t) = (\mathcal{S}_2\mathbf{w}(t), \mathcal{S}_3\mathbf{w}(t))$ such that

$$\mathcal{S}_2\mathbf{w}(t) = \mathcal{G}\; \varepsilon(\mathcal{S}_1\mathbf{w}(t)) \; + \int_0^t \mathcal{B}(t-s)\; \varepsilon(\mathcal{S}_1\mathbf{w}(s))\; ds, \tag{4.8}$$

$$\mathcal{S}_3\mathbf{w}(t) = \; \theta(\mathcal{S}_1\mathbf{w}(t)). \tag{4.9}$$

Let $Y = Q \times F$ and let define the functional $\varphi : Y \times V \to \mathbb{R}$ by

$$\varphi(\mathbf{y}, \mathbf{v}) = (\tau - \mathcal{M}\; \eta, \varepsilon(\mathbf{v}))_Q, \tag{4.10}$$

for all $\mathbf{y} = (\tau, \eta) \in Y$, $\mathbf{v} \in V$.

This allows us to introduce the following variational problem.

*Problem $Q_V$*. Find a velocity field $\mathbf{w} : \mathbb{R}^+ \to V$ such that for all $t \in \mathbb{R}^+$,

$$\begin{aligned}(\mathcal{A}\varepsilon(\mathbf{w}(t)), \varepsilon(\mathbf{v}) - \varepsilon(\mathbf{w}(t)))_Q + \varphi(\mathcal{S}\mathbf{w}(t), \mathbf{v}) - \varphi(\mathcal{S}\mathbf{w}(t), \mathbf{w}(t)) \\ + J(\mathbf{w}(t), \mathbf{v}) - J(\mathbf{w}(t), \mathbf{w}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \mathbf{w}(t))_V, \quad \forall \mathbf{v} \in V.\end{aligned} \tag{4.11}$$

The link between the Problems $P_V$ and $Q_V$ is provided by the following equivalence result.

**Lemma 2.** *Let the triplet $(\mathbf{u}, \sigma, \theta)$ be a solution of Problem $P_V$ satisfying (4.1). Then, the velocity field $\mathbf{w} = \dot{\mathbf{u}}$ is a solution of Problem $Q_V$ and $\mathbf{w} \in C(\mathbb{R}^+, V)$. Conversely, if $\mathbf{w} \in C(\mathbb{R}^+, V)$ is a solution of Problem $Q_V$, then the triplet $(\mathbf{u}, \sigma, \theta)$ defined by*

$$\mathbf{u} = \mathcal{S}_1\mathbf{w},\; \theta = \mathcal{S}_3\mathbf{w}, \quad \sigma = \mathcal{A}\; \varepsilon(\mathbf{w}) + \mathcal{S}_2\mathbf{w} - \mathcal{M}\; \mathcal{S}_3\mathbf{w}, \tag{4.12}$$

*is a solution of Problem $P_V$ and it satisfies (4.1).*

*Proof.* Let $(\mathbf{u}, \sigma, \theta)$ be a solution of Problem $P_V$ which satisfies regularity (4.1) and let use the notation $\mathbf{w} = \dot{\mathbf{u}}$. From (3.27) we have

$$\mathbf{u}(t) = \int_0^t \mathbf{w}(s)\, ds + \mathbf{u}_0,$$

and then (4.6) implies

$$\mathbf{u}(t) = \mathcal{S}_1\mathbf{w}(t). \tag{4.13}$$

Next, by Lemma 1, $\theta(t) = \theta(\mathbf{u}(t))$. Using (4.13) and (4.9), we get

$$\theta(t) = \mathcal{S}_3\mathbf{w}(t). \tag{4.14}$$

We use now (4.13)–(4.14) and (4.8) as well as (4.10) and the notation $\mathbf{w} = \dot{\mathbf{u}}$ in (3.25) to see that $\mathbf{w}$ is a solution of Problem $Q_V$. Since $\mathbf{u} \in C^1(\mathbb{R}^+, V)$, then $\mathbf{w} \in C(\mathbb{R}^+, V)$.

Conversely, let $\mathbf{w} \in C(\mathbb{R}^+, V)$ be a solution of Problem $Q_V$ and let define the triplet $(\mathbf{u}, \sigma, \theta)$ by (4.12). Let $t \in \mathbb{R}^+$. By using definition of $\mathbf{u}$ in (4.12) with (4.6), it follows that $\mathbf{w} = \dot{\mathbf{u}}$ and that $\mathbf{u}(0) = \mathbf{u}_0$. Also, regularity of $\mathbf{w}$ yields $\mathbf{u} \in C^1(\mathbb{R}^+, V)$. Next, from the definition of $\theta$ in (4.12) and from (4.6) and (4.9), we deduce that $\theta(t) = \theta(\mathbf{u}(t))$. Therefore, Lemma 1 shows that (3.26) holds with regularity $\theta \in C(\mathbb{R}^+, F) \cap L^2(\mathbb{R}^+, E) \cap W^{1,2}(\mathbb{R}^+, E')$. Moreover, $\sigma$ is calculated from (4.12) and it is clear, using (3.13), that $\sigma \in C(\mathbb{R}^+, Q)$. Finally, we recall that $\mathbf{w} = \dot{\mathbf{u}}$ and we use (4.11), definition of $\sigma$ in (4.12) and (4.10) to find (3.25). It is now clear that the triplet $(\mathbf{u}, \sigma, \theta)$ is a solution of Problem $P_V$ with regularity (4.1). □

The last step in the proof of Theorem 2 consists in the proof of the unique solvability of Problem $Q_V$ which is provided in the following result.

**Lemma 3.** *Problem $Q_V$ has a unique solution $\mathbf{w} \in C(\mathbb{R}^+, V)$.*

*Proof.* We apply Theorem 2 with $K = X = V$, $Y = Q \times F$. Let define the operator $A : C(\mathbb{R}^+, V) \to C(\mathbb{R}^+, V)$ by

$$(A\mathbf{w}, \mathbf{v}) = (\mathcal{A}\varepsilon(\mathbf{w}), \varepsilon(\mathbf{v}))_Q \qquad \forall\ \mathbf{w}, \mathbf{v} \in V.$$

We use (3.13) (b) and (3.13) (c) to find that

$$\|A\mathbf{w}_1 - A\mathbf{w}_2\|_V \le L_{\mathcal{A}}\ \|\mathbf{w}_1 - \mathbf{w}_2\|_V,$$
$$(A\mathbf{w}_1 - A\mathbf{w}_2, \mathbf{w}_1 - \mathbf{w}_2)_V \ge m_{\mathcal{A}} \|\mathbf{w}_1 - \mathbf{w}_2\|_V^2,$$

which shows that condition (2.7) of Theorem 2 is satisfied.

Moreover, by using assumptions on $\mathcal{M}$ given in (3.18), it is easy to show that the functional $\varphi$ given in (4.10) is convex and lower semi continuous on $V$, which means that it satisfies condition (2.9) (a). In addition, for all $\mathbf{y}_1 = (\tau_1, \eta_1)$, $\mathbf{y}_2 = (\tau_2, \eta_2) \in Y$, $\mathbf{v_1}$, $\mathbf{v_2} \in V$ we have

$$\begin{aligned}&\varphi(\mathbf{y}_1, \mathbf{v}_2) - \varphi(\mathbf{y}_1, \mathbf{v_1}) + \varphi(\mathbf{y}_2, \mathbf{v_1}) - \varphi(\mathbf{y}_2, \mathbf{v}_2)\\ &\qquad = (\tau_2 - \tau_1, \varepsilon(\mathbf{v}_1) - \varepsilon(\mathbf{v}_2))_Q - (\mathcal{M}\eta_2 - \mathcal{M}\eta_1, \varepsilon(\mathbf{v}_1) - \varepsilon(\mathbf{v}_2))_Q.\end{aligned}$$

By assumptions on $\mathcal{M}$ given in (3.18) and (2.14) we can see that

$$\begin{aligned}&\varphi(\mathbf{y}_1, \mathbf{v}_2) - \varphi(\mathbf{y}_1, \mathbf{v_1}) + \varphi(\mathbf{y}_2, \mathbf{v_1}) - \varphi(\mathbf{y}_2, \mathbf{v}_2)\\ &\quad \le \|\tau_2 - \tau_1\|_Q\ \|\mathbf{v}_2 - \mathbf{v}_1\|_V + \|\mathcal{M}\|\ \|\eta_2 - \eta_1\|_{L^2(\Omega)}\ \|\mathbf{v}_2 - \mathbf{v}_1\|_V\\ &\quad \le \sqrt{2}\ \max\{1, \|\mathcal{M}\|\ \}\ \ \|\mathbf{y}_2 - \mathbf{y}_1\|_Y \|\mathbf{v}_2 - \mathbf{v}_1\|_V\ ,\end{aligned} \tag{4.15}$$

which means that condition (2.9) (b) is satisfied with $\alpha = \sqrt{2}\max\{1, \|\mathcal{M}\|\}$.

For the functional $J$, we use (3.17) and (2.1) to deduce that $J$ satisfies condition (2.10) (a). Also, we can see from (3.23) that condition (2.10) (b) is satisfied with $\beta = c_0^2\ \|\beta^*\|_{L^\infty(\Gamma_3)} \left(\|\mu\|_{L^\infty(\Gamma_3)} + 1\right)$.

Next, let $n \in \mathbb{N}^*$, $t \in [0, n]$ and let $\mathbf{w}_1, \mathbf{w}_2 \in C(\mathbb{R}^+, V)$. Then, the definition of the operator $\mathcal{S}$ yields

$$\begin{aligned}&\|\mathcal{S}\mathbf{w}_1(t) - \mathcal{S}\mathbf{w}_2(t)\|_{Q\times L^2(\Omega)}\\ &\qquad \le \|\mathcal{S}_2\mathbf{w}_1(t) - \mathcal{S}_2\mathbf{w}_2(t)\|_Q + \|\mathcal{S}_3\mathbf{w}_1(t) - \mathcal{S}_3\mathbf{w}_2(t)\|_F.\end{aligned} \tag{4.16}$$

However, from (4.8) and by using assumptions (3.14)–(3.15) and inequality (2.3), we obtain for all $\mathbf{v} \in V$,

$$(\mathcal{S}_2\mathbf{w}_1(t) - \mathcal{S}_2\mathbf{w}_2(t), \varepsilon(\mathbf{v}))_Q \le L_{\mathcal{G}} \|\varepsilon(\mathcal{S}_1\mathbf{w}_1(t)) - \varepsilon(\mathcal{S}_1\mathbf{w}_2(t))\|_Q$$
$$\times \|\varepsilon(\mathbf{v})\|_Q + \max_{r\in[0,n]} \|\mathcal{B}(r)\|_{Q_\infty} \int_0^t \|\varepsilon(\mathcal{S}_1\mathbf{w}_1(s)) - \varepsilon(\mathcal{S}_1\mathbf{w}_2(s))\|_Q \, ds \, \|\varepsilon(\mathbf{v})\|_Q,$$

which implies

$$\|\mathcal{S}_2\mathbf{w}_1(t) - \mathcal{S}_2\mathbf{w}_2(t)\|_Q \le L_{\mathcal{G}} \|\mathcal{S}_1\mathbf{w}_1(t) - \mathcal{S}_1\mathbf{w}_2(t)\|_Q$$
$$+ \max_{r\in[0,n]} \|\mathcal{B}(r)\|_{Q_\infty} \int_0^t \|\mathcal{S}_1\mathbf{w}_1(s) - \mathcal{S}_1\mathbf{w}_2(s)\|_Q \, ds.$$

From the previous inequality and (4.7), we can say that, for any $n \in \mathbb{N}^*$, there exists an $n$-dependent constant $r_n > 0$ such that

$$\|\mathcal{S}_2\mathbf{w}_1(t) - \mathcal{S}_2\mathbf{w}_2(t)\|_Q \le r_n \int_0^t \|\mathbf{w}_1(s) - \mathbf{w}_2(s)\|_V \, ds \qquad \forall t \in [0,n]. \quad (4.17)$$

Also, using (4.9) we can see that

$$\|\mathcal{S}_3\mathbf{w}_1(t) - \mathcal{S}_3\mathbf{w}_2(t)\|_F = \|\theta(\mathcal{S}_1\mathbf{w}_1(t)) - \theta(\mathcal{S}_1\mathbf{w}_2(t))\|_F.$$

From (4.13), it follows that $\mathcal{S}_1\mathbf{w}_1(t) = \mathbf{u}_1(t)$, $\mathcal{S}_1\mathbf{w}_2(t) = \mathbf{u}_2(t)$. Then, we use (4.6) to see that $\dot{\mathbf{u}}_1(t) = \mathbf{w}_1(t)$, $\dot{\mathbf{u}}_2(t) = \mathbf{w}_2(t)$. We now apply (4.4) to obtain

$$\|\mathcal{S}_3\mathbf{w}_1(t) - \mathcal{S}_3\mathbf{w}_2(t)\|_F^2 \le c \int_0^t \|\mathbf{w}_1(s) - \mathbf{w}_2(s)\|_V^2 \, ds.$$

Since $\mathbf{w}_1$, $\mathbf{w}_2 \in C(\mathbb{R}^+, V)$ we find that for all $t \in [0,n]$

$$\|\mathcal{S}_3\mathbf{w}_1(t) - \mathcal{S}_3\mathbf{w}_2(t)\|_F^2 \le c \int_0^t \|\mathbf{w}_1(s) - \mathbf{w}_2(s)\|_V \, ds.$$

Moreover, we can see that there exists $n^* \in \mathbb{N}$ such that

$$\|\mathcal{S}_3\mathbf{w}_1(t) - \mathcal{S}_3\mathbf{w}_2(t)\|_F \le n^* \|\mathcal{S}_3\mathbf{w}_1(t) - \mathcal{S}_3\mathbf{w}_2(t)\|_F^2.$$

Then,

$$\|\mathcal{S}_3\mathbf{w}_1(t) - \mathcal{S}_3\mathbf{w}_2(t)\|_F \le cn^* \int_0^t \|\mathbf{w}_1(s) - \mathbf{w}_2(s)\|_V \, ds, \qquad (4.18)$$

From (4.16), (4.17) and (4.18), we conclude that

$$\|\mathcal{S}\mathbf{w}_1(t) - \mathcal{S}\mathbf{w}_2(t)\|_{Q\times F} \le s_n \int_0^t \|\mathbf{w}_1(s) - \mathbf{w}_2(s)\|_V \, ds, \qquad (4.19)$$

where $s_n = c\, n^* + r_n > 0$. Therefore condition (2.8) is satisfied. Finally, (3.22) implies (2.11). Also, we suppose that

$$c_0^2 \|\beta^*\|_{L^\infty(\Gamma_3)} \left(\|\mu\|_{L^\infty(\Gamma_3)} + 1\right) < m_{\mathcal{A}},$$

and then condition (2.12) is satisfied. Taking $L_0 = \frac{m_{\mathcal{A}}}{c_0^2}$, Theorem 2 is then a direct consequence of Lemmas 1 and 3. We recall that $c_0$ depends on $\Omega$, $\Gamma_1$ and $\Gamma_3$ and, therefore, $L_0$ depends on $\Omega$, $\Gamma_1$, $\Gamma_3$ *and* $\mathcal{A}$. □

## 5 A convergence result

In this section, we study the dependence of the solution of Problem $P_V$ with respect to perturbations of the data. For a parameter $\rho > 0$, we prove that the solution $(\mathbf{u}_\rho, \sigma_\rho, \theta_\rho)$ of the perturbated Problem $P_V^\rho$ converges to the solution $(\mathbf{u}, \sigma, \theta)$ of the Problem $P_V$ obtained in Theorem 2. To this end, we assume in what follows that (3.13)–(3.23) still hold as well as the condition mentioned in Theorem 2 which is given by

$$\|\beta^*\|_{L^\infty(\Gamma_3)} \left(\|\mu\|_{L^\infty(\Gamma_3)} + 1\right) < L_0,$$

where $L_0 > 0$.

For the sake of simplicity, we restrict the study of the continuous dependence of the solution for the relaxation tensor $\mathcal{B}$, the coefficient of friction $\mu$ and the densities of the body forces and surface tractions $\mathbf{f}_0$ and $\mathbf{f}_2$, respectively. Then, for $\rho > 0$, let $\mathcal{B}_\rho$, $\mu_\rho$, $\mathbf{f}_{0\rho}$ and $\mathbf{f}_{2\rho}$ be the perturbations of $\mathcal{B}$, $\mu$, $\mathbf{f}_0$ and $\mathbf{f}_2$, respectively, satisfying assumptions (3.15), (3.16) and (3.17). Also, for all $\mathbf{u}, \mathbf{v} \in V$ and $t \in \mathbb{R}^+$, let define the mapping $J_\rho : V \times V \to \mathbb{R}$ and the function $\mathbf{f}_\rho : \mathbb{R}^+ \to V$ by

$$J_\rho(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \beta^* |u_\nu| \left(\mu_\rho \|\mathbf{v}_\tau\| + v_\nu\right) da, \tag{5.1}$$

$$(\mathbf{f}_\rho(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_{0\rho}(t)\, \mathbf{v}\, dx + \int_{\Gamma_2} \mathbf{f}_{2\rho}(t)\ \mathbf{v} da.$$

Moreover, we assume that there exists $m_0 \geq 0$ such that

$$c_0^2\ \|\beta^*\|_{L^\infty(\Gamma_3)} \left(\|\mu_\rho\|_{L^\infty(\Gamma_3)} + 1\right) \leq m_0 < m_{\mathcal{A}} \quad \text{for each } \rho > 0. \tag{5.2}$$

The perturbation of the Problem $P_V$ is given by the following.

*Problem* $P_V^\rho$. Find a displacement field $\mathbf{u}_\rho : \mathbb{R}^+ \to V$, a stress field $\sigma_\rho : \mathbb{R}^+ \to Q$ and a temperature field $\theta_\rho : \mathbb{R}^+ \to E$ such that for all $t \in \mathbb{R}^+$,

$$\sigma_\rho(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}_\rho) + \mathcal{G}\varepsilon(\mathbf{u}_\rho) + \int_0^t \mathcal{B}_\rho(t-s)\ \varepsilon(\mathbf{u}_\rho(s))\, ds - \mathcal{M}\ \theta_\rho(t),$$

$$\begin{aligned}
&(\mathcal{A}\varepsilon(\dot{\mathbf{u}}_\rho(t)), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_\rho(t)))_Q + (\mathcal{G}\ \varepsilon(\mathbf{u}_\rho(t)), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_\rho(t)))_Q \\
&\quad + \left(\int_0^t \mathcal{B}_\rho(t-s)\varepsilon(\mathbf{u}_\rho(s))\ ds - \mathcal{M}\ \theta_\rho(t), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_\rho(t))\right)_Q \\
&\quad + J(\dot{\mathbf{u}}_\rho(t), \mathbf{v}) - J\left(\dot{\mathbf{u}}_\rho(t), \dot{\mathbf{u}}_\rho(t)\right) \geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}_\rho(t))_V, \forall \mathbf{v} \in V, \\
&\dot{\theta}_\rho(t) + K\theta_\rho(t) = R\ \dot{\mathbf{u}}_\rho(t) + Z(t)\ \text{ in } E', \\
&\mathbf{u}_\rho(0) = \mathbf{u}_0, \quad \theta_\rho(0) = \theta_0.
\end{aligned}$$

Then, from Theorem 2, Problem $P_V^\rho$ has a unique solution $(\mathbf{u}_\rho, \sigma_\rho, \theta_\rho)$ satisfying the regularity

$$\begin{aligned}
&\mathbf{u}_\rho \in C^1(\mathbb{R}^+, V), \quad \sigma_\rho \in C(\mathbb{R}^+, Q), \\
&\theta_\rho \in C(\mathbb{R}^+, F) \cap L^2(\mathbb{R}^+, E) \cap W^{1,2}(\mathbb{R}^+, E').
\end{aligned}$$

Now, let consider the following assumptions

$$\mathcal{B}_\rho \to \mathcal{B} \text{ in } C\left(\mathbb{R}^+, Q_\infty\right) \text{ as } \rho \to 0. \tag{5.3}$$

$$\mu_\rho \to \mu \text{ in } L^\infty\left(\Gamma_3\right) \text{ as } \rho \to 0. \tag{5.4}$$

$$\mathbf{f}_{0\rho} \to \mathbf{f}_0 \text{ in } C\left(\mathbb{R}^+, L^2\left(\Omega\right)^d\right) \text{ as } \rho \to 0. \tag{5.5}$$

$$\mathbf{f}_{2\rho} \to \mathbf{f}_2 \text{ in } C\left(\mathbb{R}^+, L^2\left(\Gamma_2\right)^d\right) \text{ as } \rho \to 0. \tag{5.6}$$

These assumptions allow us to give the following convergence result.

**Theorem 3.** *Assume that (5.3)–(5.6) hold. Then, the solution* $\left(\mathbf{u}_\rho, \sigma_\rho, \theta_\rho\right)$ *of the Problem* $P_V^\rho$ *converges to the solution* $(\mathbf{u}, \sigma, \theta)$ *of the Problem* $P_V$*; i.e.*

$$\mathbf{u}_\rho \to \mathbf{u} \text{ in } C^1\left(\mathbb{R}^+, V\right), \quad \sigma_\rho \to \sigma \text{ in } C\left(\mathbb{R}^+, Q\right) \quad \text{as } \rho \to 0.$$
$$\theta_\rho \to \theta \text{ in } C\left(\mathbb{R}^+, F\right) \quad \text{as } \rho \to 0.$$

*Proof.* Let $\rho > 0$ and let use the notations $\mathbf{w}(t) = \dot{\mathbf{u}}(t)$ and $\mathbf{w}_\rho(t) = \dot{\mathbf{u}}_\rho(t)$ for the velocity fields. Since there is a perturbation of the relaxation tensor $\mathcal{B}$, we use definitions (4.8)–(4.9) to define the operator $\mathcal{S}_\rho : C(\mathbb{R}^+, V) \to C(\mathbb{R}^+, Q \times F)$ by

$$\mathcal{S}_\rho \mathbf{w}(t) = \left(\mathcal{S}_{2\rho}\mathbf{w}(t), \mathcal{S}_3\mathbf{w}(t)\right), \tag{5.7}$$

$$\mathcal{S}_{2\rho}\mathbf{w}(t) = \mathcal{G}\varepsilon(\mathcal{S}_1\mathbf{w}(t)) + \int_0^t \mathcal{B}_\rho(t-s)\varepsilon(\mathcal{S}_1\mathbf{w}(s))ds, \tag{5.8}$$

$$\mathcal{S}_3\mathbf{w}(t) = \theta\left(\mathcal{S}_1\mathbf{w}(t)\right).$$

We can see from Lemma 2 that the solution $\left(\mathbf{u}_\rho, \sigma_\rho, \theta_\rho\right)$ of the Problem $P_V^\rho$ satisfies

$$\mathbf{u}_\rho = \mathcal{S}_1\mathbf{w}_\rho, \quad \sigma_\rho = \mathcal{A}\varepsilon(\mathbf{w}_\rho) + \mathcal{S}_{2\rho}\mathbf{w}_\rho - \mathcal{M}\mathcal{S}_3\mathbf{w}_\rho, \tag{5.9}$$

$$\theta_\rho = \mathcal{S}_3\mathbf{w}_\rho, \tag{5.10}$$

$$\begin{aligned}&\left(\mathcal{A}\varepsilon(\mathbf{w}_\rho(t)), \varepsilon(\mathbf{v}) - \varepsilon\left(\mathbf{w}_\rho(t)\right)\right)_Q + \varphi\left(\mathcal{S}_\rho\mathbf{w}_\rho(t), \mathbf{v}\right) - \varphi\left(\mathcal{S}_\rho\mathbf{w}_\rho(t), \mathbf{w}_\rho(t)\right)\\ &\quad + J\left(\mathbf{w}_\rho(t), \mathbf{v}\right) - J\left(\mathbf{w}_\rho(t), \mathbf{w}_\rho(t)\right) \geq \left(\mathbf{f}_\rho(t), \mathbf{v} - \mathbf{w}_\rho(t)\right)_V, \forall \mathbf{v} \in V.\end{aligned} \tag{5.11}$$

Now, let $\rho > 0$ and $t \in [0, n]$. Taking $\mathbf{v} = \mathbf{w}(t)$ in (5.11) and $\mathbf{v} = \mathbf{w}_\rho(t)$ in (4.11), we add the resulting inequalities to obtain

$$\begin{aligned}&\left(\mathcal{A}\varepsilon(\mathbf{w}_\rho(t)) - \mathcal{A}\varepsilon(\mathbf{w}(t)), \varepsilon\left(\mathbf{w}_\rho(t)\right) - \varepsilon\left(\mathbf{w}(t)\right)\right)_Q \leq \varphi\left(\mathcal{S}\mathbf{w}(t), \mathbf{w}_\rho(t)\right)\\ &\quad - \varphi\left(\mathcal{S}\mathbf{w}(t), \mathbf{w}(t)\right) + \varphi\left(\mathcal{S}_\rho\mathbf{w}_\rho(t), \mathbf{w}(t)\right) - \varphi\left(\mathcal{S}_\rho\mathbf{w}_\rho(t), \mathbf{w}_\rho(t)\right)\\ &\quad + J\left(\mathbf{w}(t), \mathbf{w}_\rho(t)\right) - J\left(\mathbf{w}(t), \mathbf{w}(t)\right) + J\left(\mathbf{w}_\rho(t), \mathbf{w}(t)\right)\\ &\quad - J\left(\mathbf{w}_\rho(t), \mathbf{w}_\rho(t)\right) + \left(\mathbf{f}_\rho(t) - \mathbf{f}(t), \mathbf{w}_\rho(t) - \mathbf{w}(t)\right)_V.\end{aligned} \tag{5.12}$$

Next, we proceed to estimate each term of the previous inequality. First, from assumption (3.13) (c), we find that

$$m_{\mathcal{A}}\left\|\mathbf{w}_\rho(t) - \mathbf{w}(t)\right\|_V^2 \leq \left(\mathcal{A}\varepsilon(\mathbf{w}_\rho(t)) - \mathcal{A}\varepsilon(\mathbf{w}(t)), \varepsilon\left(\mathbf{w}_\rho(t)\right) - \varepsilon\left(\mathbf{w}(t)\right)\right)_Q. \tag{5.13}$$

Also, we use (4.15) to see that

$$\varphi\left(\mathcal{S}\mathbf{w}\left(t\right),\mathbf{w}_{\rho}\left(t\right)\right)-\varphi\left(\mathcal{S}\mathbf{w}\left(t\right),\mathbf{w}\left(t\right)\right)+\varphi\left(\mathcal{S}_{\rho}\mathbf{w}_{\rho}\left(t\right),\mathbf{w}\left(t\right)\right)$$
$$-\varphi\left(\mathcal{S}_{\rho}\mathbf{w}_{\rho}\left(t\right),\mathbf{w}_{\rho}\left(t\right)\right)\leq\sqrt{2}\max\left\{1,\|\mathcal{M}\|\right\}$$
$$\times\left\|\mathcal{S}_{\rho}\mathbf{w}_{\rho}\left(t\right)-\mathcal{S}\mathbf{w}\left(t\right)\right\|_{Y}\left\|\mathbf{w}_{\rho}\left(t\right)-\mathbf{w}\left(t\right)\right\|_{V}. \tag{5.14}$$

However, from definition of $\mathcal{S}\mathbf{w}\left(t\right)$ and (5.7), it follows that

$$\left\|\mathcal{S}_{\rho}\mathbf{w}_{\rho}\left(t\right)-\mathcal{S}\mathbf{w}\left(t\right)\right\|_{Y}\leq\left\|\mathcal{S}_{2\rho}\mathbf{w}_{\rho}\left(t\right)-\mathcal{S}_{2}\mathbf{w}\left(t\right)\right\|_{Q}+\left\|\mathcal{S}_{3}\mathbf{w}_{\rho}\left(t\right)-\mathcal{S}_{3}\mathbf{w}\left(t\right)\right\|_{L^{2}(\Omega)}.$$

By (4.8), (5.8) and by using (3.14)–(3.15), we deduce that

$$\left\|\mathcal{S}_{2\rho}\mathbf{w}_{\rho}\left(t\right)-\mathcal{S}_{2}\mathbf{w}\left(t\right)\right\|_{Q}\leq L_{\mathcal{G}}\left\|\mathcal{S}_{1}\mathbf{w}_{\rho}\left(t\right)-\mathcal{S}_{1}\mathbf{w}\left(t\right)\right\|_{V}$$
$$+\left\|\int_{0}^{t}\left(\mathcal{B}_{\rho}\left(t-s\right)\ \varepsilon\left(\mathcal{S}_{1}\mathbf{w}_{\rho}\left(s\right)\right)-\mathcal{B}\left(t-s\right)\ \varepsilon\left(\mathcal{S}_{1}\mathbf{w}\left(s\right)\right)\right)\,ds\right\|_{Q}.$$

We write,

$$\mathcal{B}_{\rho}\left(t-s\right)\ \varepsilon\left(\mathcal{S}_{1}\mathbf{w}_{\rho}\left(s\right)\right)-\mathcal{B}\left(t-s\right)\ \varepsilon\left(\mathcal{S}_{1}\mathbf{w}\left(s\right)\right)=\mathcal{B}_{\rho}\left(t-s\right)$$
$$\times\left(\varepsilon\left(\mathcal{S}_{1}\mathbf{w}_{\rho}\left(s\right)\right)-\varepsilon\left(\mathcal{S}_{1}\mathbf{w}\left(s\right)\right)\right)+\left(\mathcal{B}_{\rho}\left(t-s\right)-\mathcal{B}\left(t-s\right)\right)\ \varepsilon\left(\mathcal{S}_{1}\mathbf{w}\left(s\right)\right).$$

Then, we obtain

$$\left\|\mathcal{S}_{2\rho}\mathbf{w}_{\rho}\left(t\right)-\mathcal{S}_{2}\mathbf{w}\left(t\right)\right\|_{Q}\leq L_{\mathcal{G}}\left\|\mathcal{S}_{1}\mathbf{w}_{\rho}\left(t\right)-\mathcal{S}_{1}\mathbf{w}\left(t\right)\right\|_{V}+\max_{r\in[0,n]}\left\|\mathcal{B}_{\rho}\left(r\right)\right\|_{Q_{\infty}}$$
$$\times\int_{0}^{t}\left\|\mathcal{S}_{1}\mathbf{w}_{\rho}\left(s\right)-\mathcal{S}_{1}\mathbf{w}\left(s\right)\right\|_{V}ds+\max_{r\in[0,n]}\left\|\mathcal{B}_{\rho}\left(r\right)-\mathcal{B}\left(r\right)\right\|_{Q_{\infty}}\int_{0}^{t}\left\|\mathcal{S}_{1}\mathbf{w}\left(s\right)\right\|_{V}ds.$$

We recall that $\mathcal{S}_{1}\mathbf{w}\left(t\right)=\mathbf{u}\left(t\right)$. From the last inequality and (4.6), we can see that for any $t\in[0,n]$, there exists an $n$-dependent constant $d_{n}>0$ such that

$$\left\|\mathcal{S}_{2\rho}\mathbf{w}_{\rho}\left(t\right)-\mathcal{S}_{2}\mathbf{w}\left(t\right)\right\|_{Q}\leq d_{n}\ \left(\int_{0}^{t}\left\|\mathbf{w}_{\rho}\left(s\right)-\mathbf{w}\left(s\right)\right\|_{V}\ ds\right)$$
$$+\max_{r\in[0,n]}\left\|\mathcal{B}_{\rho}\left(r\right)-\mathcal{B}\left(r\right)\right\|_{Q_{\infty}}\int_{0}^{t}\left\|\mathbf{u}\left(s\right)\right\|_{V}\ ds. \tag{5.15}$$

Moreover, applying (4.4) and proceeding in the same way followed to find (4.18), we obtain

$$\left\|\mathcal{S}_{3}\mathbf{w}_{\rho}\left(t\right)-\mathcal{S}_{3}\mathbf{w}\left(t\right)\right\|_{F}\leq c\int_{0}^{t}\left\|\mathbf{w}_{\rho}\left(s\right)-\mathbf{w}\left(s\right)\right\|_{V}\ ds. \tag{5.16}$$

Combining (5.15) and (5.16) yields

$$\left\|\mathcal{S}_{\rho}\mathbf{w}_{\rho}\left(t\right)-\mathcal{S}\mathbf{w}\left(t\right)\right\|_{Y}\leq h_{n}\int_{0}^{t}\left\|\mathbf{w}_{\rho}\left(s\right)-\mathbf{w}\left(s\right)\right\|_{V}ds$$
$$+\max_{r\in[0,n]}\left\|\mathcal{B}_{\rho}\left(r\right)-\mathcal{B}\left(r\right)\right\|_{Q_{\infty}}\int_{0}^{t}\left\|\mathbf{u}\left(s\right)\right\|_{V}ds, \tag{5.17}$$

where $h_n = c + d_n > 0$. As a result of (5.14) and (5.17), we deduce that

$$\varphi\left(\mathcal{S}\mathbf{w}(t), \mathbf{w}_\rho(t)\right) - \varphi\left(\mathcal{S}\mathbf{w}(t), \mathbf{w}(t)\right) + \varphi\left(\mathcal{S}_\rho \mathbf{w}_\rho(t), \mathbf{w}(t)\right)$$
$$- \varphi\left(\mathcal{S}_\rho \mathbf{w}_\rho(t), \mathbf{w}_\rho(t)\right) \leq \sqrt{2}\ \max\{1, \|\mathcal{M}\|\} \max_{r\in[0,n]} \|\mathcal{B}_\rho(r) - \mathcal{B}(r)\|_{Q_\infty}$$
$$\times \|\mathbf{w}_\rho(t) - \mathbf{w}(t)\|_V \int_0^t \|\mathbf{u}(s)\|_V\, ds + h_n \sqrt{2}\max\{1, \|\mathcal{M}\|\}$$
$$\times \|\mathbf{w}_\rho(t) - \mathbf{w}(t)\|_V \int_0^t \|\mathbf{w}_\rho(s) - \mathbf{w}(s)\|_V\, ds. \tag{5.18}$$

Moreover, we use (3.21), (3.23) and (5.1); after some standard computations it follows that

$$J\left(\mathbf{w}(t), \mathbf{w}_\rho(t)\right) - J\left(\mathbf{w}(t), \mathbf{w}(t)\right) + J\left(\mathbf{w}_\rho(t), \mathbf{w}(t)\right) - J\left(\mathbf{w}_\rho(t), \mathbf{w}_\rho(t)\right)$$
$$= \int_{\Gamma_3} \beta^*\left(\|\mathbf{w}_{\rho,\tau}(t)\| - \|\mathbf{w}_\tau(t)\|\right)\ \left(\mu\,|w_\nu(t)| - \mu_\rho\,|w_{\rho,\nu}(t)|\right)$$
$$+ \int_{\Gamma_3} \beta^*\left(|w_\nu(t)| - |w_{\rho,\nu}(t)|\right)\ .\left(w_{\rho,\nu}(t) - w_\nu(t)\right).$$

Then, by writing

$$\mu\,|w_\nu(t)| - \mu_\rho\,|w_{\rho,\nu}(t)| = (\mu - \mu_\rho)\,|w_\nu(t)| + \mu_\rho\left(|w_\nu(t)| - |w_{\rho,\nu}(t)|\right),$$

and by using (2.1), we obtain

$$J\left(\mathbf{w}(t), \mathbf{w}_\rho(t)\right) - J\left(\mathbf{w}(t), \mathbf{w}(t)\right) + J\left(\mathbf{w}_\rho(t), \mathbf{w}(t)\right) - J\left(\mathbf{w}_\rho(t), \mathbf{w}_\rho(t)\right)$$
$$\leq\ c_0^2\,\|\beta^*\|_{L^\infty(\Gamma_3)}\left(\|\mu_\rho\|_{L^\infty(\Gamma_3)} + 1\right)\|\mathbf{w}_\rho(t) - \mathbf{w}(t)\|_V^2$$
$$+ c_0^2\,\|\beta^*\|_{L^\infty(\Gamma_3)}\ \|\mu_\rho - \mu\|_{L^\infty(\Gamma_3)}\,\|\mathbf{w}(t)\|_V\,\|\mathbf{w}_\rho(t) - \mathbf{w}(t)\|_V\,. \tag{5.19}$$

Finally, Cauchy-Schwartz inequality yields

$$\left(\mathbf{f}_\rho(t) - \mathbf{f}(t), \mathbf{w}_\rho(t) - \mathbf{w}(t)\right)_V \leq \|\mathbf{f}_\rho(t) - \mathbf{f}(t)\|_V\,\|\mathbf{w}_\rho(t) - \mathbf{w}(t)\|_V\,. \tag{5.20}$$

Let use the notations

$$\xi_{\rho n} = \sqrt{2}\max\{1, \|\mathcal{M}\|\} \max_{r\in[0,n]} \|\mathcal{B}_\rho(r) - \mathcal{B}(r)\|_{Q_\infty} \int_0^n \|\mathbf{u}(s)\|_V\, ds, \tag{5.21}$$

$$\phi_\rho = c_0^2\,\|\beta^*\|_{L^\infty(\Gamma_3)}\,\|\mu_\rho - \mu\|_{L^\infty(\Gamma_3)} \max_{s\in[0,n]} \|\mathbf{w}(s)\|_V\,, \tag{5.22}$$

$$\delta_\rho = \max_{s\in[0,n]} \|\mathbf{f}_\rho(s) - \mathbf{f}(s)\|_V\,. \tag{5.23}$$

Then, we combine (5.21)–(5.23) with inequalities (5.12)–(5.13) and (5.18)–(5.20) to deduce that

$$\left[m_A - c_0^2\,\|\beta^*\|_{L^\infty(\Gamma_3)}\left(\|\mu_\rho\|_{L^\infty(\Gamma_3)} + 1\right)\right]\|\mathbf{w}_\rho(t) - \mathbf{w}(t)\|_V$$
$$\leq (\xi_{\rho n} + \phi_\rho + \delta_\rho) + h_n\sqrt{2}\max\{1, \|\mathcal{M}\|\} \textstyle\int_0^t \|\mathbf{w}_\rho(s) - \mathbf{w}(s)\|_V\, ds.$$

Moreover, condition (5.2) yields

$$(m_A - m_0) \|\mathbf{w}_\rho(t) - \mathbf{w}(t)\|_V \le (\xi_{\rho n} + \phi_\rho + \delta_\rho) + h_n \sqrt{2} \max\{1, \|\mathcal{M}\|\} \int_0^t \|\mathbf{w}_\rho(s) - \mathbf{w}(s)\|_V \, ds.$$

We apply a Gronwall argument to the previous inequality to find

$$\|\mathbf{w}_\rho(t) - \mathbf{w}(t)\|_V \le \frac{1}{m_A - m_0} (\xi_{\rho n} + \phi_\rho + \delta_\rho) \, e^{\frac{h_n \sqrt{2} \max\{1, \|\mathcal{M}\|\}}{m_A - m_0} t},$$

and therefore,

$$\max_{t\in[0,n]} \|\mathbf{w}_\rho(t) - \mathbf{w}(t)\|_V \le \frac{1}{m_A - m_0} (\xi_{\rho n} + \phi_\rho + \delta_\rho) \; e^{\frac{h_n \sqrt{2} \max\{1, \|\mathcal{M}\|\}}{m_A - m_0} n}. \tag{5.24}$$

Then, we use the convergence definition (2.4) and assumptions (5.3)–(5.6) to see that

$$\xi_{\rho n} \to 0, \quad \phi_\rho \to 0, \quad \delta_\rho \to 0 \qquad \text{as } \rho \to 0. \tag{5.25}$$

From (5.24)–(5.25), we conclude that

$$\max_{t\in[0,n]} \|\mathbf{w}_\rho(t) - \mathbf{w}(t)\|_V \to 0 \quad \text{as } \rho \to 0. \tag{5.26}$$

Let denote by $\mathbf{u}_\rho(t) = \mathcal{S}_1 \; \mathbf{w}_\rho(t)$, $\mathbf{u}(t) = \mathcal{S}_1 \; \mathbf{w}(t)$ the displacement fields corresponding to the velocity fields $\mathbf{w}_\rho(t)$ and $\mathbf{w}(t)$, where $\mathcal{S}_1$ is the operator given in (4.6). Therefore, (4.7) and (5.26) yield

$$\max_{t\in[0,n]} \|\mathbf{u}_\rho(t) - \mathbf{u}(t)\|_V \to 0 \quad \text{as } \rho \to 0.$$

Also, we recall that $\dot{\mathbf{u}}_\rho(t) = \mathbf{w}_\rho(t)$, $\dot{\mathbf{u}} = \mathbf{w}(t)$ and we use (5.26), the previous inequality as well as (2.5) to get

$$\mathbf{u}_\rho \to \mathbf{u} \;\text{ in } C^1\left(\mathbb{R}^+, V\right) \;\text{ as } \rho \to 0.$$

Finally, from (5.9)–(5.10) and definitions of $\sigma$ and $\theta$ in (4.12), we deduce that

$$\|\sigma_\rho(t) - \sigma(t)\|_Q + \|\theta_\rho(t) - \theta(t)\|_F \le L_\mathcal{A} \; \|\mathbf{w}_\rho(t) - \mathbf{w}(t)\|_V + \|\mathcal{S}_{2\rho}\mathbf{w}_\rho(t) - \mathcal{S}_2\mathbf{w}(t)\|_Q + \|\mathcal{M} + 1\| \, \|\mathcal{S}_3\mathbf{w}_\rho(t) - \mathcal{S}_3\mathbf{w}(t)\|_F .$$

Then, (2.14) implies

$$\|\sigma_\rho(t) - \sigma(t)\|_Q + \|\theta_\rho(t) - \theta(t)\|_F \le L_\mathcal{A} \; \|\mathbf{w}_\rho(t) - \mathbf{w}(t)\|_V + \sqrt{2} \; \max\{1, \|\mathcal{M}\| + 1\} \, \|\mathcal{S}_\rho\mathbf{w}_\rho(t) - \mathcal{S}\mathbf{w}(t)\|_Y .$$

We use now (5.17) and (5.21) to obtain

$$\|\sigma_\rho(t) - \sigma(t)\|_Q + \|\theta_\rho(t) - \theta(t)\|_F \le \xi_{\rho n} + L_\mathcal{A} \; \|\mathbf{w}_\rho(t) - \mathbf{w}(t)\|_V + h_n\sqrt{2} \; \max\{1, \|\mathcal{M}\| + 1\} \int_0^t \|\mathbf{w}_\rho(s) - \mathbf{w}(s)\|_V \, ds. \tag{5.27}$$

We combine (5.26), (5.27) and the convergence $\xi_{\rho n} \to 0$, as $\rho \to 0$ given in (5.25) to conclude that

$$\max_{t\in[0,n]} \left\|\sigma_\rho(t) - \sigma(t)\right\|_Q \to 0 \qquad \text{as } \rho \to 0, \tag{5.28}$$

$$\max_{t\in[0,n]} \left\|\theta_\rho(t) - \theta(t)\right\|_F \to 0 \qquad \text{as } \rho \to 0. \tag{5.29}$$

We use (5.28)–(5.29) and the convergence definition (2.4) to get the following convergences

$$\begin{aligned} &\sigma_\rho \to \sigma \quad \text{in } C(\mathbb{R}^+, Q) \qquad \text{as } \rho \to 0,\\ &\theta_\rho \to \theta \quad \text{in } C(\mathbb{R}^+, F) \qquad \text{as } \rho \to 0, \end{aligned}$$

which concludes the proof of Theorem 3. □

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