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An Improved Finite Element Approximation and Superconvergence for Temperature Control Problems*

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Abstract. In this paper, we consider an improved finite element approximation for temperature control problems, where the state and the adjoint state are discretized by piecewise linear functions while the control is not discretized directly. The numerical solution of the control is obtained by a projection of the adjoint state to the set of admissible controls. We derive a priori error estimates and superconvergence of second-order. Moreover, we present some numerical examples to illustrate our theoretical results.

Keywords: optimal control, finite element method, approximation, convergence analysis.

AMS Subject Classification: 49J20; 65M60; 80M10.

1 Introduction

Temperature control problem plays a very important role in our everyday life. For example, a refrigerator works through a cooling system very similar to the one in an air conditioning unit. It has refrigerant that it cycles through a compressor and evaporator to take heat out of the refrigerator. Normally, the system only cycles on periodically to maintain a temperature range. The system is governed by the following equation:

$$\begin{cases} -\operatorname{div}(A\nabla y) + \phi(y) = f, & x \in \Omega, \\ y|_{\partial\Omega} = 0, \end{cases}$$

where Ω is a convex domain with a Lipschitz boundary $\partial\Omega$, y is the temperature distribution which is maintained equal to zero along the boundary, $\phi(y)$ denotes the thermal radiation or positive temperature feedback due to chemical reactions (see e.g., [16]) and f is a source term.

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Then we may control the temperature distribution y to come close to a given target y_d by acting with an additional distributed source term u . The corresponding temperature control problem is formulated as follows:

$$\begin{cases} \min_{u \in K} \left\{ \frac{1}{2} \|y - y_d\|^2 + \frac{1}{2} \|u\|^2 \right\}, \\ -\operatorname{div}(A \nabla y) + \phi(y) = f + Bu, \quad x \in \Omega, \\ y|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where K is a closed convex subset in $L^2(\Omega)$, the coefficient matrix $A = (a_{ij})_{n \times n} \in (W^{1,\infty}(\Omega))^{n \times n}$, such that $(A\xi) \cdot \xi \geq c|\xi|^2, \forall \xi \in \mathbb{R}^n, f \in L^2(\Omega)$, and B is a linear continuous operator. Moreover, K is defined by

$$K = \{v(x) \in L^2(\Omega) : a \leq v(x) \leq b, \text{ a.e. } x \in \Omega\},$$

where a and b are constants.

The literature on finite element solving optimal control problems is huge. For a linear elliptic optimal control problem, a priori error estimates were investigated in [10], a posteriori error estimates based on recovery techniques have been obtained in [9, 17], a posteriori error estimates of residual type have been derived in [12], a superconvergence result can be seen in [14], a variational discretization method was introduced in [13], and Chen et al. considered its Legendre Galerkin spectral approximation in [4].

For semilinear elliptic optimal control problems, a priori error estimates were investigated in [1], a posteriori error estimates have been obtained in [11], a superconvergence can be seen in [3], Borzì considered a second-order discretization and multigrid solution in [2]. Recently, some adaptive algorithm and superconvergence results can be found in [6, 7, 15]. Notice that all the control variables are first-order convergence and the superconvergence of the state and the adjoint state variables are 1.5 order in the above works.

To the best of our knowledge, our work is the first to consider a numerical approximation for the semilinear elliptic temperature control problem in which both the a priori error estimates and the superconvergence are of the second-order.

The paper is organized as follows: In Section 2, we introduce an improved finite element approximation for the model problem. In Section 3, we derive a priori error estimates. In Section 4, we obtain the superconvergence property. We present some numerical examples to confirm our theoretical results in the last section.

2 An Improved Finite Element for the Model Problem

For ease of exposition, we set $W = H_0^1(\Omega), U = L^2(\Omega), \|\cdot\| = \|\cdot\|_{L^2(\Omega)}, \|\cdot\|_m = \|\cdot\|_{H^m(\Omega)}$ and

$$\begin{aligned} a(y, w) &= \int_{\Omega} (A \nabla y) \cdot \nabla w, \quad \forall y, w \in W, \\ (u, w) &= \int_{\Omega} u \cdot w, \quad \forall u, w \in U. \end{aligned}$$

It follows from the assumptions on the coefficient matrix A that

$$a(y, y) \geq c\|y\|_1^2, \quad |a(y, w)| \leq C\|y\|_1\|w\|_1, \quad \forall y, w \in W. \tag{2.1}$$

Then the standard weak formula for the state equation reads as follows:

$$a(y, w) + (\phi(y), w) = (f + Bu, w), \quad \forall w \in W, \tag{2.2}$$

where we assume that the function $\phi(\cdot) \in W^{2,\infty}(-R, R)$ for any $R > 0$ and $\phi'(\cdot) \geq 0$.

Then the model problem (1.1) can be restated as: Find $(y, u) \in W \times K$, such that

$$\begin{cases} \min_{u \in K} \left\{ \frac{1}{2}\|y - y_d\|^2 + \frac{1}{2}\|u\|^2 \right\}, \\ a(y, w) + (\phi(y), w) = (f + Bu, w), \quad \forall w \in W. \end{cases} \tag{2.3}$$

It is well known (see e.g., [10]) that the control problem (2.3) has a solution $(y, u) \in W \times K$, and that if the pair $(y, u) \in W \times K$ is the solution of (2.3), then there is an adjoint state $p \in W$ such that the triplet $(y, p, u) \in W \times W \times K$ satisfies the following optimal conditions:

$$a(y, w) + (\phi(y), w) = (f + Bu, w), \quad \forall w \in W, \tag{2.4}$$

$$a(q, p) + (\phi'(y)p, q) = (y_d - y, q), \quad \forall q \in W, \tag{2.5}$$

$$(u - B^*p, v - u) \geq 0, \quad \forall v \in K, \tag{2.6}$$

where B^* is the adjoint operator of B .

We introduce the following pointwise projection operator:

$$\Pi_{[a,b]}(g(x)) = \max(a, \min(b, g(x))). \tag{2.7}$$

As in [13, 14], it is easy to prove the following lemma:

Lemma 1. *Let (y, p, u) be the solution of (2.4)–(2.6). Then we have*

$$u = \Pi_{[a,b]}(B^*p). \tag{2.8}$$

Let \mathcal{T}^h be a regular triangulation of Ω , such that $\bar{\Omega} = \cup_{\tau \in \mathcal{T}^h} \bar{\tau}$. Let $h = \max_{\tau \in \mathcal{T}^h} \{h_\tau\}$, where h_τ denotes the diameter of the element τ . Associated with \mathcal{T}^h is a finite dimensional subspace S^h of $C(\bar{\Omega})$, such that $\chi|_\tau$ are polynomials of m -order ($m \leq 1$) for all $\chi \in S^h$ and $\tau \in \mathcal{T}^h$. Let $W^h = \{v_h \in S^h : v_h|_{\partial\Omega} = 0\}$. It is easy to see that $W^h \subset W$.

Then an improved finite element approximation scheme of (2.3) is as follows: Find $(y_h, u_h) \in W^h \times K$, such that

$$\begin{cases} \min_{u_h \in K} \left\{ \frac{1}{2}\|y_h - y_d\|^2 + \frac{1}{2}\|u_h\|^2 \right\}, \\ a(y_h, w_h) + (\phi(y_h), w_h) = (f + Bu_h, w_h), \quad \forall w_h \in W^h. \end{cases} \tag{2.9}$$

It follows (see e.g., [11]) that the control problem (2.9) has a solution $(y_h, u_h) \in W^h \times K$, and that if the pair $(y_h, u_h) \in W^h \times K$ is the solution of (2.9), then there is an adjoint state $p_h \in W^h$ such that the triplet $(y_h, p_h, u_h) \in W^h \times W^h \times K$ satisfies the following optimal conditions:

$$a(y_h, w_h) + (\phi(y_h), w_h) = (f + Bu_h, w_h), \quad \forall w_h \in W^h, \tag{2.10}$$

$$a(q_h, p_h) + (\phi'(y_h)p_h, q_h) = (y_d - y_h, q_h), \quad \forall q_h \in W^h, \tag{2.11}$$

$$(u_h - B^*p_h, v - u_h) \geq 0, \quad \forall v \in K. \tag{2.12}$$

Similar to Lemma 1, as in [13, 14], it is easy to show the following lemma:

Lemma 2. *Let (y_h, p_h, u_h) be the solution of (2.10)–(2.12). Then we have*

$$u_h = \Pi_{[a,b]}(B^*p_h). \tag{2.13}$$

3 A Priori Error Estimates

We introduce some intermediate variables. Let $(y(u_h), p(u_h))$ satisfy the following system:

$$a(y(u_h), w) + (\phi(y(u_h)), w) = (f + Bu_h, w), \quad \forall w \in W, \tag{3.1}$$

$$a(q, p(u_h)) + (\phi'(y(u_h))p(u_h), q) = (y_d - y(u_h), q), \quad \forall q \in W. \tag{3.2}$$

Just for ease of exposition, let

$$J(u) = \frac{1}{2}\|y - y_d\|^2 + \frac{1}{2}\|u\|^2,$$

and $J'(u)$ is the Fréchet derivative of $J(u)$ at u . It is easy to prove that

$$(J'(u), v) = (u - B^*p, v), \quad \forall v \in K,$$

$$(J'(u_h), v) = (u_h - B^*p(u_h), v), \quad \forall v \in K.$$

Lemma 3. [5] *Let π_h be the standard Lagrange interpolation operator. For $m = 0$ or 1 , $q > \frac{n}{2}$ and $\forall v \in W^{2,q}(\Omega)$, we have*

$$|v - \pi_h v|_{W^{m,q}(\Omega)} \leq Ch^{2-m}|v|_{W^{2,q}(\Omega)}.$$

Lemma 4. *Let (y_h, p_h, u_h) and $(y(u_h), p(u_h))$ be the solutions of (2.10)–(2.12) and (3.1)–(3.2), respectively. Assume that $p(u_h), y(u_h) \in H^2(\Omega)$. Then there exists a constant C independent of h such that*

$$\|y(u_h) - y_h\|_1 + \|p(u_h) - p_h\|_1 \leq Ch. \tag{3.3}$$

Proof. According to (2.1), (2.10), (3.1), $\phi'(\cdot) \geq 0$ and Cauchy’s inequality with ε , we have

$$\begin{aligned} & c\|y(u_h) - y_h\|_1^2 \\ & \leq a(y(u_h) - y_h, y(u_h) - y_h) + (\phi(y(u_h)) - \phi(y_h), y(u_h) - y_h) \\ & = a(y(u_h) - \pi_h y(u_h), y(u_h) - y_h) + (\phi(y(u_h)) - \phi(y_h), y(u_h) - \pi_h y(u_h)) \\ & \leq C\|y(u_h) - y_h\|_1 \|y(u_h) - \pi_h y(u_h)\|_1 + C\|y(u_h) - y_h\| \|\pi_h y(u_h) - y_h\| \\ & \leq 2\varepsilon\|y(u_h) - y_h\|_1^2 + \frac{C}{2\varepsilon}\|y(u_h) - \pi_h y(u_h)\|_1^2. \end{aligned} \tag{3.4}$$

Note that $y(u_h) \in H^2(\Omega)$, by using Lemma 3, we obtain

$$\|y(u_h) - \pi_h y(u_h)\|_1 \leq Ch \|y(u_h)\|_2. \tag{3.5}$$

Let $2\varepsilon < c$, it follows from (3.4)–(3.5) that

$$\|y(u_h) - y_h\|_1 \leq Ch. \tag{3.6}$$

Similarly, we can prove that

$$\|p(u_h) - p_h\|_1 \leq Ch. \tag{3.7}$$

Then (3.3) follows from (3.6)–(3.7). \square

We introduce the following auxiliary problems:

$$\begin{aligned} -\operatorname{div}(A^* \nabla \xi) + \Phi \xi &= F_1, & \text{in } \Omega, & \quad \xi|_{\partial\Omega} = 0, & \tag{3.8} \\ -\operatorname{div}(A \nabla \zeta) + \phi'(y(u_h)) \zeta &= F_2, & \text{in } \Omega, & \quad \zeta|_{\partial\Omega} = 0, & \tag{3.9} \end{aligned}$$

where

$$\Phi = \begin{cases} \frac{\phi(y(u_h)) - \phi(y_h)}{y(u_h) - y_h}, & y(u_h) \neq y_h, \\ \phi'(y_h), & y(u_h) = y_h. \end{cases}$$

From the regularity estimates (see e.g., [16]), we obtain

$$\|\xi\|_2 \leq C \|F_1\|, \quad \|\zeta\|_2 \leq C \|F_2\|.$$

By using the Aubin–Nitsche technique, it is easy to prove the following lemma:

Lemma 5. *Let (y_h, p_h, u_h) and $(y(u_h), p(u_h))$ be the solutions of (2.10)–(2.12) and (3.1)–(3.2), respectively. Assume that $p(u_h), y(u_h) \in H^2(\Omega)$. Then there exists a constant C independent of h such that*

$$\|y(u_h) - y_h\| + \|p(u_h) - p_h\| \leq Ch^2. \tag{3.10}$$

Lemma 6. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (2.4)–(2.6) and (2.10)–(2.12), respectively. Assume that all the conditions in Lemma 5 are valid. Then there exists a constant C independent of h such that*

$$\|u - u_h\| \leq Ch^2. \tag{3.11}$$

Proof. It is clear that

$$(J'(v) - J'(u), v - u) \geq c \|v - u\|^2, \quad \forall v, u \in K. \tag{3.12}$$

By using (2.6) and (2.12), we have

$$\begin{aligned} c \|u - u_h\|^2 &\leq (J'(u) - J'(u_h), u - u_h) \\ &= (u - B^* p, u - u_h) - (u_h - B^* p(u_h), u - u_h) \\ &\leq (B^* p(u_h) - B^* p_h, u - u_h) \\ &\leq C \|p(u_h) - p_h\| \|u - u_h\|. \end{aligned} \tag{3.13}$$

From (3.10) and (3.13), we derive (3.11). \square

By combining Lemmas 3–6, we can prove the following main result.

Theorem 1. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (2.4)–(2.6) and (2.10)–(2.12), respectively. Assume that all the conditions in Lemmas 4–6 are valid. Then, we have*

$$\|u - u_h\| + \|y - y_h\| + \|p - p_h\| \leq Ch^2. \quad (3.14)$$

4 Superconvergence Property

We introduce an elliptic projection operator $P_h : W \rightarrow W^h$, which satisfies: for any $\phi \in W$

$$a(\phi - P_h\phi, w_h) = 0, \quad \forall w_h \in W^h. \quad (4.1)$$

It has the following approximation property (see e.g., [18]):

$$\|\phi - P_h\phi\| \leq Ch^2\|\phi\|_2, \quad \forall \phi \in H^2(\Omega). \quad (4.2)$$

Then we consider the superconvergence property between elliptic projections and numerical solutions of the state and the adjoint state.

Theorem 2. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (2.4)–(2.6) and (2.10)–(2.12), respectively. Then there exists a constant C independent of h such that*

$$\|P_h y - y_h\|_1 + \|P_h p - p_h\|_1 \leq Ch^2. \quad (4.3)$$

Proof. From (2.4) and (2.10), we get

$$a(y - y_h, w_h) + (\phi(y) - \phi(y_h), w_h) = (B(u - u_h), w_h), \quad \forall w_h \in W^h.$$

According to the definition of P_h , we have

$$a(P_h y - y_h, w_h) + (\phi(y) - \phi(y_h), w_h) = (B(u - u_h), w_h), \quad \forall w_h \in W^h.$$

By selecting $w_h = P_h y - y_h$ and using (2.1) and the assumed conditions on $\phi(\cdot)$, we obtain

$$\begin{aligned} c\|P_h y - y_h\|_1^2 &\leq a(P_h y - y_h, P_h y - y_h) + (\phi(P_h y) - \phi(y_h), P_h y - y_h) \\ &= (\phi(P_h y) - \phi(y), P_h y - y_h) + (B(u - u_h), P_h y - y_h) \\ &\leq C(\|P_h y - y\| + \|u - u_h\|)\|P_h y - y_h\| \\ &\leq Ch^2\|P_h y - y_h\|. \end{aligned}$$

Hence,

$$\|P_h y - y_h\|_1 \leq Ch^2. \quad (4.4)$$

Similarly, we can prove that

$$\|P_h p - p_h\|_1 \leq Ch^2. \quad (4.5)$$

Then (4.3) follows from (4.4)–(4.5). \square

Table 1. Numerical results, Example 1.

Mesh	$\ u - u_h\ $	$\ y - y_h\ $	$\ p - p_h\ $	$\ P_h y - y_h\ _1$	$\ P_h p - p_h\ _1$
10×10	3.10200e-02	5.27023e-02	5.03458e-02	5.43093e-02	6.16469e-02
20×20	7.78194e-03	1.38042e-02	1.31083e-02	1.41742e-02	1.62494e-02
40×40	2.00103e-03	3.49306e-03	3.31053e-03	3.58245e-03	4.12064e-03
80×80	5.01440e-04	8.75888e-04	8.29739e-04	8.98110e-04	1.03390e-03
160×160	1.25846e-04	2.19143e-04	2.07566e-04	2.24679e-04	2.58711e-04

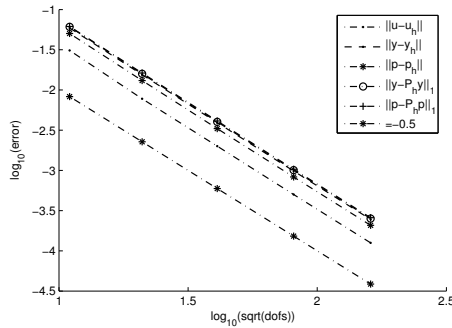


Figure 1. The order of convergence, Example 1.

5 Numerical Experiment

In this section, optimal control problems were solved numerically with codes developed by using AFEPack. This package is freely available and the details can be found in [8]. The discretization has been described in Section 2.

Let $\Omega = [0, 1] \times [0, 1]$ and $B = I$, we solve the following semilinear optimal control problem:

$$\begin{cases} \min_{u \in K} \left\{ \frac{1}{2} \|y - y_d\|^2 + \frac{1}{2} \|u\|^2 \right\}, \\ -\operatorname{div}(A \nabla y) + \phi(y) = f + Bu, \quad x \in \Omega, \\ y|_{\partial\Omega} = 0. \end{cases}$$

Example 1. The data are as follows:

$$\begin{aligned} A &= I, \quad \phi(y) = y^3, \quad a = -0.5, \quad b = 0.5, \\ p(x) &= \sin(2\pi x_1) \sin(2\pi x_2), \quad y(x) = p(x), \\ u(x) &= \min(0.5, \max(-0.5, p(x))), \\ f(x) &= -\operatorname{div}(A \nabla y(x)) + \phi(y(x)) - u(x), \\ y_d(x) &= y(x) - \operatorname{div}(A^* \nabla p(x)) + \phi'(y(x)) p(x). \end{aligned}$$

Some numerical results based on a sequence of uniformly refined meshes are shown in Table 1.

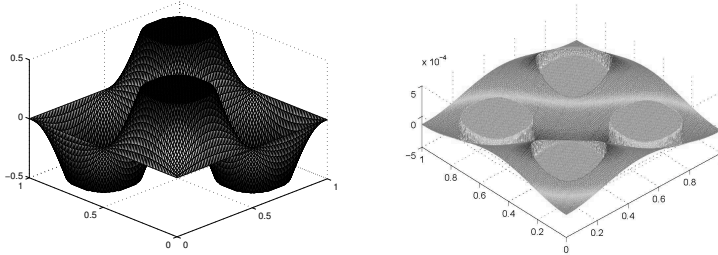


Figure 2. The exact solution u (left) and the error $u - u_h$ (right), Example 1.

Table 2. Numerical results, Example 2.

Mesh	$\ u - u_h\ $	$\ y - y_h\ $	$\ p - p_h\ $	$\ P_h y - y_h\ _1$	$\ P_h p - p_h\ _1$
10×10	2.65120e-02	5.30353e-02	5.38607e-02	5.59887e-02	5.59896e-02
20×20	6.23877e-03	1.39308e-02	1.41608e-02	1.40492e-02	1.41264e-02
40×40	1.56441e-03	3.52693e-03	3.52612e-03	3.56435e-03	3.58241e-03
80×80	3.92635e-04	8.64530e-04	8.69441e-04	8.76090e-04	8.80613e-04
160×160	9.82587e-05	2.11316e-04	2.15043e-04	2.19025e-04	2.20152e-04

We show the relationship between $\log_{10}(\text{error})$ and $\log_{10}(\text{sqrt}(\text{dofs}))$ in Figure 1. It is clear that $\|u - u_h\| = O(h^2)$, $\|y - y_h\| = O(h^2)$, $\|p - p_h\| = O(h^2)$, $\|y - P_h y\|_1 = O(h^2)$, $\|p - P_h p\|_1 = O(h^2)$. In Figure 2, we plot the exact solution u and the error $u - u_h$.

Example 2. The data are as follows:

$$\begin{aligned}
 A &= I, \quad \phi(y) = e^y, \quad a = -1, \quad b = 0.5, \\
 p(x) &= (1 - 4x_1x_2)\sin(2\pi x_1)\sin(2\pi x_2), \quad y(x) = p(x), \\
 u(x) &= \min(0.5, \max(-1, p(x))), \\
 f(x) &= -\text{div}(A\nabla y(x)) + \phi(y(x)) - u(x), \\
 y_d(x) &= y(x) - \text{div}(A^*\nabla p(x)) + \phi'(y(x))p(x).
 \end{aligned}$$

In Table 2, we have shown some numerical results on a sequential meshes. The second - order convergence of errors $\|u - u_h\|$, $\|y - y_h\|$, $\|p - p_h\|$, $\|y - P_h y\|_1$, and $\|p - P_h p\|_1$ can be seen in Figure 3. The exact solution u and the error $u - u_h$ are shown in Figure 4.

The above numerical examples confirm $\|u - u_h\| = O(h^2)$, $\|y - P_h y\|_1 = O(h^2)$, $\|p - P_h p\|_1 = O(h^2)$. But in [3], the authors have just derived the estimates $\|u_h - G_h u_h\| = O(h^{\frac{3}{2}})$, $\|y - P_h y\|_1 = O(h^{\frac{3}{2}})$, $\|p - P_h p\|_1 = O(h^{\frac{3}{2}})$. Hence, our method is much more efficient.

6 Conclusions

We have investigated an improved finite element approximation for semilinear temperature control problem and have obtained a priori error estimates and

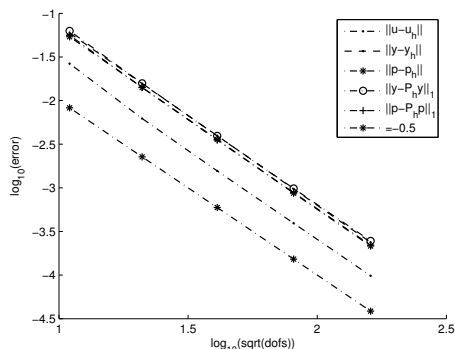


Figure 3. The order of convergence, Example 2.

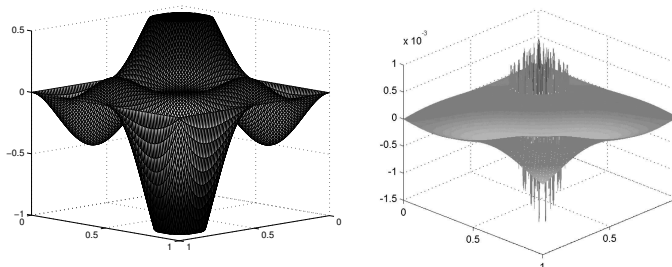


Figure 4. The exact solution u (left) and the error $u - u_h$ (right), Example 2.

superconvergence of the second-order. Our analysis for the semilinear temperature control problem seems to be new, and these results can be extended to general convex problems.

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