CORE
Metadata, citation and similar papers at core.ac.uk
Provided by DSpace@MIT

MIT-CTP-4278
June 2011

arXiv:1107.0008v2 [hep-th] 8 Sep 2011

Double Field Theory of Type II Strings

Olaf Hohm, Seung Ki Kwak, and Barton Zwiebach

Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge, MA 02139, USA
ohohm@mit.edu, sk_kwak@mit.edu, zwiebach@mit.edu

Abstract

We use double field theory to give a unified description of the low energy limits of type IIA and type IIB superstrings. The Ramond-Ramond potentials fit into spinor representations of the duality group $O(D, D)$ and field-strengths are obtained by acting with the Dirac operator on the potentials. The action, supplemented by a $\text{Spin}^+(D, D)$-covariant self-duality condition on field strengths, reduces to the IIA and IIB theories in different frames. As usual, the NS-NS gravitational variables are described through the generalized metric. Our work suggests that the fundamental gravitational variable is a hermitian element of the group $\text{Spin}(D, D)$ whose natural projection to $O(D, D)$ gives the generalized metric.

Contents

1 Introduction and summary

T-duality transformations along circles of compactified type II superstrings show that type IIA and type IIB superstrings are, in fact, the same theory for toroidal backgrounds of odd dimension (see [1] and references therein). This naturally leads to the question of whether there exists a formulation of type II theories that makes this feature manifest. In this paper we will address this question, reporting on results that have recently been announced in [2].

The understanding of the T-duality transformation rules for the Ramond-Ramond (RR) fields has been the subject of many works in a number of formalisms [3–8]. Experience with bosonic strings, or with the NS-NS sector of type II superstrings, has shown that the duality group is $O(d,d)$, where d is the number of toroidal dimensions [9,10]. In double field theory — an approach to make T-duality manifest for the massless sector of string theory by doubling the coordinates [11–14] — it has been useful to work with the group $O(D,D)$, where D is the total number of *spacetime* dimensions. (See [15] for earlier work by Siegel and [16–27] for related papers.) Conservatively, one can focus on the elements of $O(D,D)$ that act only on d compact space dimensions. In bosonic double field theory, however, the full $O(D,D)$ is a symmetry when all spacetime coordinates are non-compact and doubled. The symmetry is manifest, acting both on the fields and on the coordinates.

In an important work, Fukuma, Oota and Tanaka [28] discussed the IIA and IIB supergravity limits of superstrings compactified on a torus T^d. The authors verified that the dimensionally reduced theory arising from the RR sector contains scalars, one-forms, and higher forms, each of which fit into the *spinor* representation of $O(d,d)$. The kinetic operator was shown to use the spin representative of the familiar $O(d,d)$ matrix of scalar fields that arise from the metric and b-field components along the compact directions. The required spin representatives of $O(d,d)$ elements were discussed in the earlier work of Brace, Morariu, and Zumino [29] in their study of RR backgrounds in the matrix model. The relevance of $O(d,d)$ spinors for dimensionally reduced RR fields was first noted by Hull and Townsend [4].

In this paper we construct the double field theory of the RR massless sector of superstring theory. The NS-NS massless sector is described by the same theory that describes the massless sector of the bosonic string [11–14]. The fields are a duality invariant dilaton d and the generalized metric $\mathcal{H}_{MN}$, that encodes the metric and b-fields in a matrix called $\mathcal{H}$:

$$\mathcal{H}_{MN} \;=\; \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij}-b_{ik}g^{kl}b_{lj} \end{pmatrix} \equiv \mathcal{H}\;. \tag{1.1}$$

Here $M,N,\ldots = 1,\ldots,2D$ denote fundamental $O(D,D)$ indices. The double field theory action then takes the Einstein-Hilbert-like form

$$S = \int dx\, d\tilde{x}\, e^{-2d}\, \mathcal{R}(\mathcal{H}, d)\ , \tag{1.2}$$

where $\mathcal{R}(\mathcal{H}, d)$ is an $O(D,D)$ invariant scalar. In here all fields depend on the doubled coordinates $X^M = (\tilde{x}_i, x^i)$, and after setting $\tilde{\partial}^i = 0$ the action (1.2) reduces to the conventional low-energy action for the massless NS-NS fields. The action also features a gauge symmetry with an $O(D,D)$ vector parameter $\xi^M = (\tilde{\xi}_i, \xi^i)$ that combines the diffeomorphism parameter ξ^i and the Kalb-Ramond gauge parameter $\tilde{\xi}_i$:

$$\begin{aligned} \delta_\xi \mathcal{H}_{MN} &= \widehat{\mathcal{L}}_\xi \mathcal{H}_{MN} \equiv \xi^P \partial_P \mathcal{H}_{MN} + \left(\partial_M \xi^P - \partial^P \xi_M\right) \mathcal{H}_{PN} + \left(\partial_N \xi^P - \partial^P \xi_N\right) \mathcal{H}_{MP}\ , \\ \delta d &= \xi^M \partial_M d - \frac{1}{2} \partial_M \xi^M\ . \end{aligned} \tag{1.3}$$

Here $\widehat{\mathcal{L}}_\xi$ defines a generalized Lie derivative and $\partial_M = (\tilde{\partial}^i, \partial_i)$. The gauge invariance of the action requires the $O(D,D)$ covariant constraints

$$\partial^M \partial_M A = \eta^{MN} \partial_M \partial_N A = 0\ , \qquad \partial^M A\, \partial_M B = 0\ , \qquad \eta^{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \tag{1.4}$$

for all fields and parameters A and B, where η is the $O(D,D)$ invariant metric. This constraint implies that locally one can always find an $O(D,D)$ transformation that rotates into a frame in which the fields depend only on half of the coordinates, e.g., only on the x^i or the $\tilde{x}_i$.

Let us now turn to the RR sector, which requires some new ingredients. The first one is that the RR gauge fields fit naturally into the spinor representation of $O(D,D)$. In the case of interest, the physical dimension is $D = 10$ and we have a spinor of $O(10,10)$. The spinor representation of $O(D,D)$ of dimension 2^D is real (or Majorana) and reducible. This dimension equals the sum of the number of components of all the forms in a D-dimensional spacetime. An additional Weyl condition yields two spinor representations of opposite chirality, each of dimension 2^{D-1}, that can be matched with even and odd forms and therefore with the RR fields in the type II theories. The RR potentials of the IIA and IIB theories do not include all odd and all even forms, but duality relations can be naturally imposed on the field strengths to reduce the spectrum to the desired one. This 'democratic' formulation of the type II supergravities uses field strengths of degrees $2, 4, 6$, and 8 for type IIA and field strengths of degrees $1, 3, 5, 7$, and 9 for type IIB [28].

The type II theories are formulated in a ten-dimensional spacetime with Lorentzian signature. In fact, the requisite self-duality condition of type IIB is consistent only with this signature. A number of features arise from this choice of signature that require a careful discussion of the relevant duality groups, in particular of the 'spin' groups that provide the double covers of the orthogonal duality groups. The RR fields, as mentioned above, fit into a spinor of $O(D,D)$, but the so-called 'spinor' representation of $O(D,D)$ is only defined up to signs. A true representation exists for the group $\mathrm{Pin}(D,D)$, which provides a double cover of $O(D,D)$, or for the group $\mathrm{Spin}(D,D)$, which provides a double cover of $SO(D,D)$. Just like $SO(D,D)$ is a subgroup of $O(D,D)$, $\mathrm{Spin}(D,D)$ is a subgroup of $\mathrm{Pin}(D,D)$. Because of the double covering, each element in $O(D,D)$ has two lifts to $\mathrm{Pin}(D,D)$ and similarly each element in $SO(D,D)$ has two lifts to $\mathrm{Spin}(D,D)$. Moreover, there is a group homomorphism $\rho :\ \mathrm{Pin}(D,D) \to O(D,D)$

that also takes $\text{Spin}(D,D)$ to $SO(D,D)$. If S is an element in $\text{Pin}(D,D)$, then $(-S)$ is also an element and both S and $(-S)$ map to the same $O(D,D)$ element under ρ.

T-dualities about single circles are elements of $O(D,D)$ that are not in $SO(D,D)$: they are represented by matrices of determinant minus one. Their lifts are transformations in $\text{Pin}(D,D)$ that are not in $\text{Spin}(D,D)$ and have the effect of changing the Weyl condition of a spinor. Since the chirality of the spinor that encodes the RR forms must be fixed in order to write down the theory, the duality group is $\text{Spin}(D,D)$. Calling χ the spinor that encodes the RR forms we have the duality transformations

$$\text{Duality transformations:} \quad \chi \;\to\; S\,\chi\,, \qquad S \,\in\, \text{Spin}(D,D)\,. \tag{1.5}$$

In the doubled space it is natural to define a Dirac operator

$$\not{\partial} \;\equiv\; \frac{1}{\sqrt{2}}\Gamma^M\partial_M \;=\; \frac{1}{\sqrt{2}}\,(\Gamma^i\partial_i + \Gamma_i\tilde{\partial}^i)\,, \tag{1.6}$$

where Γ^M are gamma matrices of $O(D,D)$. Using the Clifford algebra and the constraint (1.4), we readily verify that $\not{\partial}\not{\partial} = 0$. We show that $\not{\partial}$ is duality invariant and as a result (1.5) implies

$$\not{\partial}\chi \;\to\; S\,\not{\partial}\chi\,, \qquad S \,\in\, \text{Spin}(D,D)\,. \tag{1.7}$$

Since $\not{\partial}$ is first order in derivatives, $\not{\partial}\chi$ is naturally interpreted as the field strength associated to the RR potentials, to which it indeed reduces for $\tilde{\partial}^i = 0$.

Following the insights of [28] it is natural to consider the spin group representative of $\mathcal{H}$ to discuss the coupling of the RR fields to the NS-NS fields. The generalized metric $\mathcal{H}$ is a symmetric matrix that is also an $O(D,D)$ element. Since the determinant of $\mathcal{H}$ is plus one, we actually have $\mathcal{H} \in SO(D,D)$. The group $SO(D,D)$ has two disconnected components: the subgroup $SO^+(D,D)$ that contains the identity and a coset denoted by $SO^-(D,D)$. One can check that in Lorentzian signature $\mathcal{H}$ is actually in $SO^-(D,D)$. The associated spin representatives are in $\text{Spin}^-(D,D)$; they are elements S and $-S$, such that $\rho(\pm S) = \mathcal{H}$. It turns out to be impossible to choose a spin representative in a single-valued and continuous way over the space of possible $\mathcal{H}$. We illustrate this with an explicit example of a closed path in the space of $\mathcal{H}$ configurations (*i.e.* a closed path in $SO^-(D,D)$) for which forcing a continuous choice of representative results in an open path in $\text{Spin}^-(D,D)$, a path in which the initial and final elements differ by a sign. We note that this phenomenon occurs whenever a *timelike* T-duality is employed, and therefore does not arise in Euclidean signature where $\mathcal{H} \in SO^+(D,D)$ and a lift to $\text{Spin}^+(D,D)$ can be chosen continuously.

In light of the above topological subtlety we suggest that instead of viewing $\mathcal{H}$ as the fundamental gravitational field, from which a spin representative needs to be constructed, we view the spin element itself as the dynamical field, denoted by $\mathbb{S} \in \text{Spin}^-(D,D)$. The generalized metric can then be defined uniquely by the homomorphism: $\mathcal{H} = \rho(\mathbb{S})$. The condition that $\mathcal{H}$ is symmetric requires that $\mathbb{S}$ be hermitian, $\mathbb{S} = \mathbb{S}^\dagger$. Under the duality transformation (1.5) we declare that

$$\text{Duality transformations:} \qquad \mathbb{S} \;\to\; \mathbb{S}' \;=\; (S^{-1})^\dagger\,\mathbb{S}\,S^{-1} \qquad S \in \text{Spin}(D,D)\,. \tag{1.8}$$

This transformation is consistent with that of the generalized metric, namely, $\rho(S)$ is an $SO(D,D)$ transformation that takes $\mathcal{H} = \rho(\mathbb{S})$ to $\mathcal{H}' = \rho(\mathbb{S}')$.

We can now discuss the double field theory action for type II theories, whose independent fields are $\mathbb{S}$, χ and d. It is the sum of the action (1.2) for the NS-NS sector and a new action for the RR sector:

$$\begin{aligned} S &= \int dx d\tilde{x} \left(e^{-2d}\, \mathcal{R}(\mathcal{H}, d) + \frac{1}{4} (\partial\!\!\!/ \chi)^\dagger\, \mathbb{S}\, \partial\!\!\!/ \chi \right), \\ \mathcal{H} &= \rho(\mathbb{S}), \quad \mathbb{S} \in \mathrm{Spin}^-(D,D)\,, \quad \mathbb{S}^\dagger = \mathbb{S}\,. \end{aligned} \tag{1.9}$$

The RR action is quadratic in the field strengths $\partial\!\!\!/ \chi$, and $\mathbb{S}$ is actually needed to produce the Hodge dual that then leads to conventional kinetic terms. The duality invariance of the RR action is manifest on account of (1.7) and (1.8). The definition of the theory also requires the field strength $\partial\!\!\!/ \chi$ to satisfy a self-duality constraint that can be written in a manifestly duality covariant way,

$$\partial\!\!\!/ \chi = -C^{-1} \mathbb{S}\, \partial\!\!\!/ \chi\,. \tag{1.10}$$

Here the charge conjugation matrix C satisfies $C^{-1}\Gamma^M C = (\Gamma^M)^\dagger$. While the action is invariant under $\mathrm{Spin}(D,D)$, the self-duality constraint breaks the duality symmetry down to $\mathrm{Spin}^+(D,D)$. This is not unexpected since the epsilon tensor in the duality relations is only left invariant by the orientation-preserving transformations $GL^+(D) \subset SO^+(D,D)$. It should be emphasized that the action is originally $\mathrm{Pin}(D,D)$ invariant. The Weyl condition on the spinor reduces the duality symmetry of the action to $\mathrm{Spin}(D,D)$. Finally, the self-duality constraint reduces the symmetry of the theory to $\mathrm{Spin}^+(D,D)$.

The RR potentials have the usual abelian gauge symmetries in which the form fields are shifted by exact forms. This symmetry also takes a manifestly duality covariant form,

$$\delta_\lambda \chi = \partial\!\!\!/ \lambda\,, \tag{1.11}$$

and leaves (1.9) invariant because $\partial\!\!\!/^2 = 0$. More nontrivially, the invariance of the theory under the gauge symmetries parameterized by ξ^M requires that χ transform as

$$\delta_\xi \chi = \widehat{\mathcal{L}}_\xi \chi \equiv \xi^M \partial_M \chi + \frac{1}{2} \partial_M \xi_N \Gamma^M \Gamma^N \chi\,. \tag{1.12}$$

In here we defined the generalized Lie derivative $\widehat{\mathcal{L}}_\xi$ acting on a spinor. To complete the analysis we require a gauge transformation of the gravitational field $\mathbb{S}$ that satisfies two consistency conditions: (i) together with (1.12) it must leave the action invariant, and (ii) it must imply the gauge transformation (1.3) for $\mathcal{H}$ that is required for gauge invariance of the NS-NS part of the action. We find that these two conditions are satisfied by

$$\delta_\xi \mathbb{S} = \xi^M \partial_M \mathbb{S} + \frac{1}{2} C \big[\Gamma^{PQ}, C^{-1}\mathbb{S}\big] \partial_P \xi_Q\,. \tag{1.13}$$

In order to evaluate the action in different T-duality frames, i.e., for different solutions of the constraint (1.4), and to compare with the conventional formulation in terms of fields like g and b, we need to choose a particular parametrization of the field $\mathbb{S}$. We start from the parametrization (1.1) of the generalized metric $\mathcal{H} = \rho(\mathbb{S})$ implied by $\mathbb{S}$. A spin representative

$S_{\mathcal{H}}$ can then be defined locally, and we parametrize the field $\mathbb{S}$ by setting $\mathbb{S} = S_{\mathcal{H}}$. It turns out, however, that once a parametrization has been chosen in terms of g and b, the original $\mathrm{Spin}(D, D)$ symmetry of the action cannot be fully realized as transformations of g and b since they change the sign of the RR double field theory action for timelike T-dualities. If the full $\mathrm{Spin}(D, D)$ is to be a symmetry we must view $\mathbb{S}$ as the fundamental field. A manifestation of the sign phenomenon is that evaluating the action in T-duality frames related via timelike T-dualities results in RR actions that differ by an overall sign, a result that turns out to be consistent with proposals in the literature. In order to explain this, let us discuss the evaluation of the action in different T-duality frames.

Suppose we have chosen a chirality of χ and a parametrization of $\mathbb{S}$ such that the theory reduces for $\tilde{\partial}^i = 0$ to type IIA. All other solutions of (1.4) can be obtained from this one by an $O(D, D)$ transformation. For the bosonic double field theory, or for the NS-NS part of the type II theory, it has been shown in [13] that the action reduces in all frames to the same theory: the conventional low-energy action of bosonic string theory, but written in terms of different field variables, which are related by the corresponding T-duality transformations. In type II theories, however, this changes, because generally T-duality relates different type II theories to each other. If, for instance, the theory reduces in one frame to type IIA, we will see that it reduces in any other frame obtained by an *odd* number of spacelike T-duality inversions to type IIB, and vice versa. If, on the other hand, the frames are related by an even number of spacelike T-duality inversions, the theory reduces in both frames to the same theory, either IIA or IIB. We next consider the case of a frame that is obtained by a timelike T-duality transformation. First, let us review the status of timelike T-duality as discussed in the literature.

If one considers the reduction of the ten-dimensional low-energy type IIA or IIB theory on a timelike circle, one finds that each RR p-form gives rise to a form of the same degree in the nine-dimensional Euclidean theory, together with a $(p-1)$-form, which originates from the timelike component. The latter form enters with the wrong sign kinetic term. Consequently, the timelike circle reductions of type IIA and type IIB do not give rise to the same theory in nine dimensions; they give two theories that differ by an overall sign in the RR kinetic terms. Therefore, the IIA and IIB theories cannot be T-dual on a timelike circle. It has been proposed by Hull that on a timelike circle the proper T-dual pairs are type IIA and type IIB*, or type IIB and type IIA* [30]. In the low-energy description the type IIA* and type IIB* differ from the IIA and IIB theories just by the overall sign of the RR kinetic terms, such that the timelike circle reductions of IIA and IIB*, and of IIB and IIA*, give rise to the same theory.

If we start from a T-duality frame in which the double field theory reduces to type IIA (IIB), we indeed find that the same theory reduces to IIB* (IIA*) in any frame obtained by a timelike T-duality transformation. In summary, the manifestly T-duality invariant double field theory defined by (1.9) and (1.10) unifies these four different type II theories in that each of them arises in particular T-duality frames.

This paper is organized as follows. In sec. 2 we review the properties of the spinor representation of $O(D, D)$ and of its double covering group. Due to the aforementioned topological subtleties, we find it necessary to delve in some detail into the mathematical issues. In sec. 3 we discuss the field that is interpreted as the spinor representative of the generalized metric.

The duality covariant form of the action and duality relations is introduced in sec. 4, while their evaluation in particular T-duality frames is done in sec. 5 and 6. We conclude with a brief discussion in sec. 7. A number of technical proofs as well as an example illustrating the topological obstructions in the construction of the spin representative of the generalized metric are given in an appendix.

2 O(D,D) spinor representation

In this section we review properties of the T-duality group $O(D,D)$ and its spinor representation or, more precisely, the properties of its two-fold covering group $\mathrm{Pin}(D,D)$ and its representations. Convenient references for this section are [28], [31], and [32].

2.1 O(D,D), Clifford algebras, and Pin(D,D)

In order to fix our conventions, we start by recalling some basic properties of $O(D,D)$. This group is defined to be the group leaving the metric of signature $(\mathbf{1}_D,-\mathbf{1}_D)$ invariant. We choose a basis where the metric takes the form

$$\eta \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,, \tag{2.1}$$

and we denote it by η^{MN} or η_{MN} which, viewed as matrices, are equal. The indices M,N run over the $2D$ values $1,2,\ldots,2D$. The preservation of η implies that group elements $h\in O(D,D)$, viewed as matrices, satisfy

$$\eta^{MN} \;=\; h^M{}_P\, h^N{}_Q\, \eta^{PQ} \quad\Leftrightarrow\quad \eta \;=\; h\,\eta\, h^T \,. \tag{2.2}$$

This implies that $\det(h)=\pm1$. The subgroup of $O(D,D)$ whose elements have determinant plus one is denoted by $SO(D,D)$. While the group $O(D,D)$ has four connected components, $SO(D,D)$ has two connected components. In $SO(D,D)$ the component connected to the identity is the subgroup denoted as $SO^+(D,D)$. It can be shown that in the basis where the metric takes the diagonal form $\mathrm{diag}(\mathbf{1}_D,-\mathbf{1}_D)$, the two $D\times D$ block-diagonal matrices of any $SO^+(D,D)$ element have positive determinant. The other component of $SO(D,D)$ is denoted by $SO^-(D,D)$. It is not a subgroup of $SO(D,D)$ but rather a coset of $SO^+(D,D)$.

The Lie algebra of $O(D,D)$ is spanned by generators $T^{MN}=-T^{NM}$ satisfying

$$\left[T^{MN},T^{KL}\right] \;=\; \eta^{MK}\,T^{LN}-\eta^{NK}\,T^{LM}-\eta^{ML}\,T^{KN}+\eta^{NL}\,T^{KM}\,. \tag{2.3}$$

Any group element connected to the identity can be written as an exponential of Lie algebra generators,

$$h^M{}_N \;=\; \left[\exp\left(\tfrac{1}{2}\Lambda_{PQ}T^{PQ}\right)\right]^M{}_N \,, \tag{2.4}$$

where

$$(T^{MN})^K{}_L \;=\; 2\eta^{K[M}\delta^{N]}{}_L \,, \tag{2.5}$$

is the fundamental representation of the Lie algebra (2.3). We use the anti-symmetrization convention $X_{[MN]}\equiv\frac{1}{2}(X_{MN}-X_{NM})$.

We turn now to the spinor representation of $O(D,D)$ and to the groups $\mathrm{Spin}(D,D)$ and $\mathrm{Pin}(D,D)$, whose properties will be instrumental below. The (reducible) spinor representation of $O(D,D)$ has dimension 2^D and can be chosen to be real or Majorana. Imposing an additional Weyl condition will yield two spinor representations of opposite chirality, both of dimension 2^{D-1}. These can be identified with even and odd forms and thus with the RR fields in type II.

To begin with, we introduce the Clifford algebra $C(D,D)$ associated to the quadratic form $\eta\,(\,\cdot\,,\,\cdot\,)$ on $\mathbb{R}^{2D}$. With basis vectors Γ_M, $M = 1,\ldots,2D$, we have

$$\eta_{MN} = \eta\,(\Gamma_M, \Gamma_N) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{2.6}$$

The main relation of the Clifford algebra states that for any $V \in \mathbb{R}^{2D}$

$$V \cdot V = \eta(V,V)\, \mathbf{1} , \tag{2.7}$$

where $\mathbf{1}$ is the unit element and the dot indicates the product in the algebra. This algebra is generated by the unit and basis vectors Γ_M. Writing $V = V^M \Gamma_M$, substitution in (2.7) gives

$$\big\{\Gamma_M, \Gamma_N\big\} \;\equiv\; \Gamma_M \cdot \Gamma_N + \Gamma_N \cdot \Gamma_M = \; 2\,\eta_{MN} . \tag{2.8}$$

Using the quadratic form η_{MN} and its inverse η^{MN} to raise and lower indices, we can write arbitrary vectors as $V = V^M\Gamma_M = V_M\Gamma^M$, which then allows to write (2.8) with all indices raised.

An explicit representation of the Clifford algebra (and below of the Pin group) can be conveniently constructed using fermionic oscillators ψ^i and ψ_i, $i = 1,\ldots,D$, satisfying

$$\{\psi_i, \psi^j\} \;=\; \delta_i{}^j \;, \qquad \{\psi_i, \psi_j\} \;=\; 0 \;, \qquad \{\psi^i, \psi^j\} \;=\; 0 \;, \tag{2.9}$$

where

$$(\psi_i)^\dagger \;=\; \psi^i \;. \tag{2.10}$$

Defining

$$\Gamma_i \;=\; \sqrt{2}\psi_i \;, \qquad \Gamma^i \;=\; \sqrt{2}\psi^i \;, \tag{2.11}$$

the oscillators realize the algebra (2.8). Spinor states can be defined introducing a Clifford vacuum $|0\rangle$ annihilated by the ψ_i for all i:

$$\psi_i|0\rangle = 0\,, \quad \forall\, i\,. \tag{2.12}$$

From this, we derive a convenient identity that will be useful below,

$$\psi_j\psi^{i_1}\cdots\psi^{i_p}|0\rangle \;=\; p\,\delta_j{}^{[i_1}\psi^{i_2}\cdots\psi^{i_p]}|0\rangle \;. \tag{2.13}$$

A spinor χ in the 2^D-dimensional space can then be identified with a general state

$$|\chi\rangle \;=\; \sum_{p=0}^{D}\frac{1}{p!}\,C_{i_1\ldots i_p}\,\psi^{i_1}\ldots\psi^{i_p}|0\rangle \;, \tag{2.14}$$

where the coefficients are completely antisymmetric tensors. Thus, there is a natural identification of the spinor representation with the p-forms on $\mathbb{R}^D$. We define $\langle 0|$ to be the the 'dagger' of the state $|0\rangle$ and declare:

$$\langle 0|0\rangle = 1\,. \tag{2.15}$$

For more general states,

$$(\psi^{i_1}\ldots\psi^{i_p}|0\rangle)^\dagger = \langle 0|\psi_{i_p}\ldots\psi_{i_i}\,. \tag{2.16}$$

We work on a real vector space, so the $\dagger$ operation does not affect the numbers multiplying the vectors. In the notation where dagger takes $|a\rangle$ to $\langle a|$ and vice versa, we can quickly show that $\langle a|b\rangle = \langle b|a\rangle$. We see from these definitions that in the spinor representation $(\Gamma^i)^\dagger$ is indeed equal to Γ_i. Since all matrix elements are real, the dagger operation is just transposition.

Let us now turn to the definition of the groups $\mathrm{Spin}(D,D)$ and $\mathrm{Pin}(D,D)$, which act on the spinor states. These groups are, respectively, double covers of the groups $SO(D,D)$ and $O(D,D)$. To describe these groups we need to introduce an anti-involution $\star$ of the Clifford algebra $C(D,D)$, which is defined by

$$(V_1\cdot V_2\ldots\cdot V_k)^\star \equiv (-1)^k\, V_k\cdot\ldots V_2\cdot V_1\,. \tag{2.17}$$

Note that for any vector V in $\mathbb{R}^{2D}$, $V^\star = -V$. For arbitrary elements S,T of the Clifford algebra one has $(S+T)^\star = S^\star + T^\star$ and $(S\cdot T)^\star = T^\star\cdot S^\star$. The group $\mathrm{Pin}(D,D)$ is now defined as follows:

$$\mathrm{Pin}(D,D) \;:=\; \big\{S\in C(D,D)\,|\,S\cdot S^\star = \pm\mathbf{1}\,,\; V\in\mathbb{R}^{2D}\;\Rightarrow\; S\cdot V\cdot S^{-1}\in\mathbb{R}^{2D}\big\}\,. \tag{2.18}$$

The first condition implies for all group elements that $S^\star$ is, up to a sign, the inverse of S. The second condition indicates that acting by conjugation with S on any vector $V\in\mathbb{R}^{2D}$ results in a vector in $\mathbb{R}^{2D}$. One readily checks that $S\in\mathrm{Pin}(D,D)$ implies $S^\star\in\mathrm{Pin}(D,D)$. In what follows we will omit the dot indicating Clifford multiplication whenever no confusion can arise. We finally note that the Lie algebras of $O(D,D)$ and $\mathrm{Pin}(D,D)$ are isomorphic, and in spinor representation the generators are given by

$$T^{MN} \;=\; \frac{1}{2}\Gamma^{MN} \;\equiv\; \frac{1}{4}\big[\Gamma^M,\Gamma^N\big]\,, \tag{2.19}$$

which satisfy (2.3).

Next, we define a group homomorphism

$$\rho:\;\mathrm{Pin}(D,D)\to O(D,D)\,, \tag{2.20}$$

with kernel $\{\mathbf{1},-\mathbf{1}\}$, that encodes the two-fold covering of $O(D,D)$. It is defined via its action on a vector $V\in\mathbb{R}^{2D}$ according to

$$\rho(S)V \;=\; SVS^{-1}\,. \tag{2.21}$$

It is easily seen that this is a homomorphism, i.e., for arbitrary $S_1,S_2\in\mathrm{Pin}(D,D)$

$$\rho(S_1S_2) \;=\; \rho(S_1)\rho(S_2)\,. \tag{2.22}$$

Moreover, ρ indeed maps into $O(D,D)$, for it preserves the quadratic form,

$$\begin{aligned}\eta(\rho(S)V,\rho(S)V)\,\mathbf{1} \;&=\; \eta(SVS^{-1},SVS^{-1})\,\mathbf{1} \;=\; SVS^{-1}\cdot SVS^{-1}\\ &=\; S\cdot(V\cdot V)\cdot S^{-1} = S\cdot\mathbf{1}\cdot S^{-1}\eta(V,V) = \eta\,(V,V)\,\mathbf{1}\,,\end{aligned} \tag{2.23}$$

where the Clifford algebra relation (2.7) has been used. Finally, ρ is surjective, i.e., for any $h\in O(D,D)$ there is an $S_h\in \text{Pin}(D,D)$ such that $\rho(S_h)=h$. More precisely, by the two-fold covering, both S_h and $-S_h$ are mapped to h under ρ.

The map ρ can be written in a basis using $V=V^M\Gamma_M$ for the original vector and $V'=V'^M\Gamma_M$, with $V'^M = h^M{}_N V^N$, for the rotated vector, where $h^M{}_N$ is an $O(D,D)$ element. With this, the map in (2.21) becomes

$$\rho(S)V \;=\; V' \;=\; SVS^{-1} \quad\to\quad h^M{}_N V^N\Gamma_M \;=\; S\,V^M\Gamma_M\,S^{-1}\,. \tag{2.24}$$

Relabeling and canceling out the vector components we find

$$S\,\Gamma_M\,S^{-1} \;=\; \Gamma_N\,h^N{}_M\,. \tag{2.25}$$

Here $\rho(S)=h$, and h — with matrix representative $h^M{}_N$ — is the $O(D,D)$ element associated with S. We rewrite the above equation by raising the indices. Using the invariance property $\eta_{MN}(h^{-1})^N{}_K = \eta_{KN}h^N{}_M$, we find

$$S\,\Gamma^M\,S^{-1} \;=\; (h^{-1})^M{}_N\Gamma^N\,. \tag{2.26}$$

Rewritten as $h^M{}_N\,S\,\Gamma^N\,S^{-1} \;=\; \Gamma^M$, this is the familiar statement that gamma matrices are invariant under the combined action of $\text{Pin}(D,D)$ on the spinor and vector indices.

Let us now turn to the definition of the subgroup $\text{Spin}(D,D)$ of $\text{Pin}(D,D)$. It is obtained if in (2.18) we have $S\in C(D,D)^{\text{even}}$, which is the Clifford subalgebra spanned by elements with an even number of products of basis vectors. In this case the homomorphism ρ above restricts to a homomorphism

$$\rho:\ \text{Spin}(D,D)\to SO(D,D)\,, \tag{2.27}$$

with kernel $\{\mathbf{1},-\mathbf{1}\}$. If, in addition to restricting to $C(D,D)^{\text{even}}$, the normalization condition is changed to $SS^\star=\mathbf{1}$, the resulting group is $\text{Spin}^+(D,D)$ and ρ would map to $SO^+(D,D)$.

Let us consider a set of useful elements S of $\text{Pin}(D,D)$. We write the elements using the oscillators ψ_i and ψ^i,[1]

$$\begin{aligned} S_b \;&\equiv\; e^{-\frac{1}{2}b_{ij}\psi^i\psi^j}\,,\\ S_r \;&\equiv\; \frac{1}{\sqrt{\det r}}\,e^{\psi^i R_i{}^j\psi_j}\,,\quad (r=(r_i{}^j)=e^{R_i{}^j}\in GL^+(D))\,,\\ S_i \;&\equiv\; \psi^i+\psi_i\,,\quad (i=1,\ldots,D)\,,\end{aligned} \tag{2.28}$$

where $GL^+(D)$ is the group of $D\times D$ matrices with strictly positive determinant. It is instructive and straightforward to verify that the first condition in (2.18) holds. Noting that $(e^x)^\star = e^{x^\star}$ we have

$$(S_b)^\star = e^{-\frac{1}{2}b_{ij}\psi^j\psi^i} = e^{\frac{1}{2}b_{ij}\psi^i\psi^j} = (S_b)^{-1}\,. \tag{2.29}$$

[1] Here we are closely following [28] with a slightly different notation.

We note that $S_b \in \mathrm{Spin}^+(D,D)$. For S_r we have

$$\begin{aligned} (S_r)^\star &= \frac{1}{\sqrt{\det r}}\, e^{\psi_j R_i{}^j \psi^i} &=& \frac{1}{\sqrt{\det r}}\; e^{-\psi^i R_i{}^j \psi_j + R_k{}^k} \\ &= \frac{1}{\sqrt{\det r}}\; e^{-\psi^i R_i{}^j \psi_j} e^{\mathrm{tr} R} &=& \frac{\det r}{\sqrt{\det r}}\; e^{-\psi^i R_i{}^j \psi_j} \\ &= \sqrt{\det r} e^{-\psi^i R_i{}^j \psi_j} &=& (S_r)^{-1} \,, \end{aligned} \tag{2.30}$$

which implies that S_r is in $\mathrm{Spin}^+(D,D)$. Since S_i is linear in gamma matrices, $S_i^\star = -S_i$. We thus have

$$S_i S_i^\star = -S_i S_i = -(\psi^i + \psi_i)(\psi^i + \psi_i) = -\psi^i \psi_i - \psi_i \psi^i = -\mathbf{1}\,. \tag{2.31}$$

It follows that $S_i \in \mathrm{Pin}(D,D)$, while *even* powers of the S_i are in $\mathrm{Spin}(D,D)$.

Using the definition (2.21) we can calculate the $O(D,D)$ elements associated with these $\mathrm{Spin}(D,D)$ elements. For this we expand (2.25) to find

$$\begin{aligned} S\,\Gamma_i\, S^{-1} &= \Gamma_k\, h^k{}_i + \Gamma^k\, h_{ki}\,, \\ S\,\Gamma^i\, S^{-1} &= \Gamma_k\, h^{ki} + \Gamma^k\, h_k{}^i\,, \end{aligned} \tag{2.32}$$

and we build the h matrix as follows

$$h^M{}_N = \begin{pmatrix} h_i{}^k & h_{ik} \\ h^{ik} & h^i{}_k \end{pmatrix}\,. \tag{2.33}$$

Applying the above to (2.28) one finds the $O(D,D)$ matrices associated to the Pin elements:

$$h_b \;\equiv\; \rho(S_b) \;=\; \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}\,, \quad b^T = -b\,, \tag{2.34}$$

$$h_r \;\equiv\; \rho(S_r) \;=\; \begin{pmatrix} r & 0 \\ 0 & (r^{-1})^T \end{pmatrix}\,, \quad r \in GL^+(D)\,, \tag{2.35}$$

$$h_i \;\equiv\; \rho(S_i) \;=\; -\begin{pmatrix} 1-e_i & -e_i \\ -e_i & 1-e_i \end{pmatrix}\,, \quad (e_i)_{jk} \equiv \delta_{ij}\delta_{ik}\,, \quad (i = 1,\ldots,D)\,. \tag{2.36}$$

The group elements h_b, h_r and even powers of the h_i generate the component $SO^+(D,D)$ connected to the identity. It is convenient to also record that

$$\rho\big(e^{\frac{1}{2}b_{ij}\psi_i\psi_j}\big) \;=\; \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\,, \qquad \rho\big(\psi^i - \psi_i\big) \;=\; -\begin{pmatrix} 1-e_i & e_i \\ e_i & 1-e_i \end{pmatrix}\,. \tag{2.37}$$

We note that (2.35) provides an embedding $r \to h_r$ of $GL^+(D)$ into $SO^+(D,D)$, preserving the group structure,

$$h_r\, h_s \;=\; h_{rs}\,, \tag{2.38}$$

and thereby, via (2.28), an embedding into $\mathrm{Spin}^+(D,D)$. In order to represent $GL^-(D,D)$ in $\mathrm{Spin}(D,D)$, we note that this group can be identified with the coset $GL^+(D)\, h_-$, with an arbitrary $h_- \in GL^-(D)$. An example for such an element h_- is given by the transformation that changes the orientation in one direction, and for this we consider:

$$\begin{aligned} \rho(\psi^i\psi_i - \psi_i\psi^i) &= \rho\Big((\psi^i - \psi_i)(\psi^i + \psi_i)\Big) = \rho(\psi^i - \psi_i)\rho(\psi^i + \psi_i) \\ &= \begin{pmatrix} 1-e_i & e_i \\ e_i & 1-e_i \end{pmatrix} \begin{pmatrix} 1-e_i & -e_i \\ -e_i & 1-e_i \end{pmatrix} = \begin{pmatrix} 1-2e_i & 0 \\ 0 & 1-2e_i \end{pmatrix}, \end{aligned} \tag{2.39}$$

where we used (2.36) and (2.37) . This shows that

$$\rho(\psi^i\psi_i - \psi_i\psi^i) \;=\; h_- \;=\; \mathrm{diag}(k_i, k_i)\,, \qquad i \text{ not summed}\,, \tag{2.40}$$

with the diagonal $D \times D$ matrix $k_i \equiv \mathrm{diag}(1, \ldots, -1, \ldots, 1)$ that has a -1 in the i-th diagonal entry. We will use this result below to define a spinor representative of a metric g with Lorentzian signature.

2.2 Conjugation in Pin(D, D)

We turn next to the definition of the charge conjugation matrix. The charge conjugation matrix C can be viewed as an element of Pin(D, D) in general and as an element of Spin(D, D) for even D. It is defined in terms of the oscillators by

$$C \;\equiv\; \begin{cases} C_+ \;\equiv\; (\psi^1 + \psi_1)(\psi^2 + \psi_2)\cdots(\psi^D + \psi_D)\,, & \text{if } D \text{ odd}\,, \\ C_- \;\equiv\; (\psi^1 - \psi_1)(\psi^2 - \psi_2)\cdots(\psi^D - \psi_D)\,, & \text{if } D \text{ even}\,. \end{cases} \tag{2.41}$$

Noticing that with i not summed $(\psi^i \pm \psi_i)(\psi^i \pm \psi_i) \;=\; \pm\{\psi^i\,, \psi_i\} \;=\; \pm 1$, simple calculations show that

$$C_+(C_+)^\star = (-1)^D\,, \qquad C_-(C_-)^\star = 1\,. \tag{2.42}$$

It is useful to note that the charge conjugation matrix is proportional to its inverse,

$$C^{-1} \;=\; (-1)^{D(D-1)/2}\,C\,. \tag{2.43}$$

Since C and C^{-1} just differ by a sign, all expressions of the form $C \ldots C^{-1}$ can be rewritten as $C^{-1} \ldots C$. It is straightforward to show that

$$\begin{aligned} C_+\psi_i(C_+)^{-1} &= -(-1)^D\psi^i\,, \qquad & C_+\psi^i(C_+)^{-1} &= -(-1)^D\psi_i\,, \\ C_-\psi_i(C_-)^{-1} &= (-1)^D\psi^i\,, \qquad & C_-\psi^i(C_-)^{-1} &= (-1)^D\psi_i\,. \end{aligned} \tag{2.44}$$

It then follows from (6.41) that in *all* dimensions

$$C\,\psi_i\,C^{-1} \;=\; \psi^i\,, \qquad C\,\psi^i\,C^{-1} \;=\; \psi_i\,. \tag{2.45}$$

As $\psi^i = (\psi_i)^\dagger$, these relations can be written in terms of gamma matrices as follows

$$C\,\Gamma^M\,C^{-1} = (\Gamma^M)^\dagger\,, \quad \text{or} \quad C\,\Gamma_M\,C^{-1} = (\Gamma_M)^\dagger\,. \tag{2.46}$$

Introducing the $O(D, D)$ element

$$J^\bullet{}_\bullet \;=\; J \;\equiv\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.47}$$

we can use (2.25) to write the second equation in (2.46) as

$$C\,\Gamma_M\,C^{-1} = \Gamma_N\,(\rho(C))^N{}_M \;=\; (\Gamma_M)^\dagger = \Gamma_N J^N{}_M\ . \tag{2.48}$$

We thus learn that

$$\rho(C) \;=\; J\,. \tag{2.49}$$

Since C and C^{-1} just differ by a sign, $\rho(C^{-1}) = J$ and equation (2.46) also implies that

$$C^{-1}\,\Gamma^M\,C \;=\; (\Gamma^M)^\dagger\,. \tag{2.50}$$

More generally we define the action of dagger by stating that $\mathbf{1}^\dagger = \mathbf{1}$, and that on vectors V dagger is realized by C conjugation:

$$V^\dagger \equiv CVC^{-1} = J\,V\,. \tag{2.51}$$

On general elements of the Clifford algebra we define dagger using

$$(V_1\cdot V_2\cdot \ldots \cdot V_n)^\dagger \equiv V_n^\dagger \cdot \ldots \cdot V_2^\dagger \cdot V_1^\dagger\,, \tag{2.52}$$

so that for general elements $(S_1\cdot S_2)^\dagger = S_2^\dagger \cdot S_1^\dagger$. A short calculation gives

$$C^\dagger \;=\; C^{-1}\,. \tag{2.53}$$

It is straightforward to verify that $S \in \mathrm{Pin}(D,D)$ implies $S^\dagger \in \mathrm{Pin}(D,D)$. It is then natural to ask how the homomorphism ρ behaves under the dagger conjugation.

To answer this and related questions it is convenient to describe the dagger operation in $C(D,D)$ in terms of C conjugation and the anti-involution τ defined by

$$\tau(V_1\cdot V_2\cdot \ldots \cdot V_n) = V_n\cdot \ldots \cdot V_2\cdot V_1\,, \tag{2.54}$$

which satisfies $\tau(S_1S_2) = \tau(S_2)\tau(S_1)$. Indeed, it is clear that

$$S^\dagger = C\,\tau(S)\,C^{-1}\,. \tag{2.55}$$

We now prove that the action of τ in $\mathrm{Pin}(D,D)$ maps under ρ to the inverse operation in $O(D,D)$:

$$\rho(\tau(S)) = \rho(S)^{-1}\,. \tag{2.56}$$

We begin with the defining relation (2.21) applied to S^{-1}:

$$S^{-1}VS \;=\; \rho(S^{-1})V\ . \tag{2.57}$$

Now take the τ action on both sides. Noticing that the right-hand side is left unchanged we get, because for any vector $\tau(V) = V$,

$$\begin{aligned}\tau(S)V\tau(S^{-1}) \;&=\; \rho(S^{-1})V \to \quad \tau(S)V\tau(S)^{-1} \;=\; \rho(S^{-1})V\\ \to \quad \rho(\tau(S))V \;&=\; \rho(S)^{-1}V\,,\end{aligned} \tag{2.58}$$

thus establishing (2.56). It is now easy to calculate $\rho(S^\dagger)$ using (2.55). Indeed, taking ρ of this equation gives

$$\rho(S^\dagger) = J\rho(\tau(S))J = J\rho(S)^{-1}J\,, \tag{2.59}$$

where we recognized that $\rho(C^{-1}) = J$ and used (2.56). Recalling that $O(D,D)$ elements h satisfy $hJh^T = J$, we have $h^T = Jh^{-1}J$. Thus the right-hand side above is simply $\rho(S)^T$, showing that

$$\rho(S^\dagger) = \rho(S)^T\,. \tag{2.60}$$

For elements S of $\mathrm{Spin}(D,D)$, $\tau(S) = S^\star$, thus (2.55) becomes

$$S^\dagger = C\,S^\star\,C^{-1}\,, \quad S \in \mathrm{Spin}(D,D)\,. \tag{2.61}$$

Using that $S^\star = \pm S^{-1}$ for $S \in \mathrm{Spin}^\pm(D,D)$, this implies

$$\begin{aligned} S^\dagger \;&=\; C\,S^{-1}\,C^{-1} \qquad \text{for} \qquad S \;\in\; \mathrm{Spin}^+(D,D)\,,\\ S^\dagger \;&=\; -C\,S^{-1}\,C^{-1} \quad \text{for} \qquad S \;\in\; \mathrm{Spin}^-(D,D)\,. \end{aligned} \tag{2.62}$$

In particular, for the spin generators S_b and S_r we get

$$\begin{aligned} S_b^\dagger \;&=\; CS_b^{-1}C^{-1}\,,\\ S_r^\dagger \;&=\; CS_r^{-1}C^{-1}\,. \end{aligned} \tag{2.63}$$

Since $\tau(S_i) = S_i$, for the final generator we have

$$S_i^\dagger \;=\; CS_iC^{-1}\,. \tag{2.64}$$

2.3 Chiral spinors

We close this section with a brief discussion of the chirality conditions to be imposed on the spinors. To this end it is convenient to introduce a 'fermion number operator' N_F, defined by

$$N_F \;=\; \sum_k \psi^k\psi_k\,. \tag{2.65}$$

It acts on a spinor state that is of degree p in the oscillators as follows

$$\begin{aligned} N_F|\chi\rangle_p \;&\equiv\; N_F\Big(\frac{1}{p!}C_{i_1\ldots i_p}\psi^{i_1}\cdots\psi^{i_p}|0\rangle\Big)\\ &=\; \sum_k p\frac{1}{p!}C_{i_1\ldots i_p}\psi^k\delta_k{}^{[i_1}\psi^{i_2}\cdots\psi^{i_p]}|0\rangle \;=\; p|\chi\rangle_p\,, \end{aligned} \tag{2.66}$$

where (2.13) has been used. Thus, acting with $(-1)^{N_F}$ on a general spinor state (2.14), one obtains

$$(-1)^{N_F}\chi \;=\; \sum_{p=0}^{D}(-1)^p\,\frac{1}{p!}\,C_{i_1\ldots i_p}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle\,. \tag{2.67}$$

We conclude that the eigenstates of $(-1)^{N_F}$ consist of a χ that is a linear combination of only even forms, with eigenvalue $+1$, or of a χ that is a linear combination of only odd forms, with eigenvalue -1. Given an arbitrary spinor χ, one can project onto the two respective chiralities,

$$\chi_\pm \;\equiv\; \frac{1}{2}\big(1\pm(-1)^{N_F}\big)\chi \qquad\Rightarrow\qquad (-1)^{N_F}\chi_\pm \;=\; \pm\,\chi_\pm\,. \tag{2.68}$$

Then χ_+ has positive chirality, consisting only of even forms, and χ_- has negative chirality, consisting only of odd forms. The operator $(-1)^{N_F}$ is the analogue of the γ^5 matrix in four dimensions.

Finally, we note that the chirality is preserved under an arbitrary Spin(D, D) transformation. In fact, since the group elements of Spin(D, D) contain only an even number of fermionic oscillators, they map even forms into even forms and odd forms into odd forms. In contrast, a general Pin(D, D) transformation can act with an odd number of oscillators and thereby map spinors of positive chirality to spinors of negative chirality and vice versa. Thus, when fixing the chirality, as for the action to be introduced below, we break the symmetry from Pin(D, D) to Spin(D, D).

3 Spin representative of the generalized metric

In this section we discuss the spin representative $S_{\mathcal{H}}$ of the generalized metric $\mathcal{H}_{MN}$. We determine its transformation behavior under gauge symmetries and T-duality. More fundamentally, we will adopt the point of view that $S_{\mathcal{H}}$ is just a particular parametrization of the fundamental field $\mathbb{S}$.

3.1 The generalized metric in Spin(D, D)

We take the fundamental field to be $\mathbb{S}$, satisfying

$$\mathbb{S} \;=\; \mathbb{S}^\dagger\,, \qquad \mathbb{S}\,\in\,\text{Spin}^-(D,D)\,. \tag{3.1}$$

The generalized metric $\mathcal{H}_{MN}$ will then be *defined* as

$$\mathcal{H}\;\equiv\;\rho(\mathbb{S}) \qquad\Rightarrow\qquad \mathcal{H}^T \;=\; \rho(\mathbb{S}^\dagger)\;=\;\mathcal{H}\,,\quad \mathcal{H}\,\in\,SO^-(D,D)\,. \tag{3.2}$$

Moreover, we constrain $\mathcal{H}$ and thereby $\mathbb{S}$ by requiring that the upper-left $D\times D$ block matrix encoding g^{-1} has Lorentzian signature. An immediate consequence of (3.1) follows with (2.62)

$$\mathbb{S}\;=\;\mathbb{S}^\dagger\;=\;-C\,\mathbb{S}^{-1}\,C^{-1}\,. \tag{3.3}$$

Equivalently, recalling that $C=\pm C^{-1}$,

$$\mathbb{S}\,C\,\mathbb{S}\;=\;-C\,. \tag{3.4}$$

It is also possible to adopt the opposite point of view, i.e., to take the group element $\mathcal{H}$ as given and then determine a corresponding spin group representative $S_{\mathcal{H}}$ as a derived object.

However, as we will discuss in more detail below and in the appendix, this cannot be done in a consistent way globally over the space of $\mathcal{H}$. In the following we first determine a spin representative $S_{\mathcal{H}}$ locally from $\mathcal{H}$, but we stress that this should be viewed as just a particular parameterization of $\mathbb{S}$ — in the same sense that the explicit form of $\mathcal{H}_{MN}$ in terms of g and b is just a particular parametrization of $\mathcal{H}$.

We start by writing the $O(D,D)$ matrix $\mathcal{H}_{MN}$ as a product of simple group elements,[2]

$$\mathcal{H} \;=\; \begin{pmatrix} g^{-1} & -g^{-1}b \\ bg^{-1} & g-bg^{-1}b \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \;\equiv\; h_b^T\, h_{g^{-1}}\, h_b\,. \tag{3.5}$$

The matrices defined in the last equation are analogous to the matrices defined in (2.34) and (2.35). More precisely, this is true for h_b while for h_g (or $h_{g^{-1}} = h_g^{-1}$) eq. (2.35) is only valid if g has euclidean signature, because then $g \in GL^+(D)$. Here, however, we assume that g has Lorentzian signature $(-+\cdots+)$. Accordingly, $\mathcal{H}$ is indeed an element of $SO^-(D,D)$.

In order to find the corresponding spinor representative for h_g and thereby for $\mathcal{H}$, it is convenient to introduce vielbeins in the usual way,

$$g_{ij} \;=\; e_i{}^\alpha\, e_j{}^\beta\, k_{\alpha\beta}\;, \qquad k_{\alpha\beta} \;=\; \mathrm{diag}(-1,1,\ldots,1)\;, \tag{3.6}$$

where $\alpha,\beta,\ldots = 1,\ldots,D$ are flat Lorentz indices with invariant metric $k_{\alpha\beta}$. In matrix notation, we also write

$$g \;=\; e\,k\,e^T\;. \tag{3.7}$$

We can choose e to have positive determinant, and thus its spin representative can be chosen to be S_e as defined in (2.28). Using (2.40), the spin representative of $\mathrm{diag}(k,k)$ can be taken to be

$$S_k \;=\; \psi^1\psi_1 - \psi_1\psi^1\;, \tag{3.8}$$

where the label one denotes the timelike direction. We note that

$$S_k = S_k^\dagger = S_k^{-1} = -S_k^\star\,. \tag{3.9}$$

Since $S_k S_k^\star = -1$, we confirm that $S_k \in \mathrm{Spin}^-(D,D)$.

Thus, we can choose the spinor representative of g to be

$$S_g \;\equiv\; S_e\, S_k\, S_e^\dagger \;=\; \frac{1}{\det{(e)}}\, e^{\psi^i E_i{}^j \psi_j}(\psi^1\psi_1 - \psi_1\psi^1)e^{\psi^i (E^T)_i{}^j\psi_j}\;, \tag{3.10}$$

where $e_i{}^\alpha = \exp(E)_i{}^\alpha$, and we used $(E^T)_i{}^j = E_j{}^i$. From its definition it follows that

$$S_g^\dagger \;=\; S_g\,. \tag{3.11}$$

Similarly,

$$S_g^{-1} \;\equiv\; (S_e^{-1})^\dagger\, S_k\, S_e^{-1} \;=\; \det e\, e^{-\psi^i (E^T)_i{}^j\psi_j}(\psi^1\psi_1 - \psi_1\psi^1)e^{-\psi^i E_i{}^j\psi_j}\;. \tag{3.12}$$

[2]We note that our conventions differ slightly from those in [14] in that what we denote by $\mathcal{H}$ has been denoted $\mathcal{H}^{-1}$ there. All other conventions, however, are the same.

We note that S_g is an element of $\text{Spin}^-(D,D)$ because it is the product of $S_k \in \text{Spin}^-(D,D)$ times elements of $\text{Spin}^+(D,D)$. From this and (2.61) we also infer that

$$S_g^\dagger \;=\; S_g \;=\; C\,S_g^\star\,C^{-1} \;=\; -C\,S_g^{-1}\,C^{-1}\ . \tag{3.13}$$

We can finally define the element $S_{\mathcal{H}}$ of $\text{Spin}(D,D)$ as follows

$$S_{\mathcal{H}} \;\equiv\; S_b^\dagger\,S_g^{-1}\,S_b \;=\; e^{\frac{1}{2}b_{ij}\psi_i\psi_j}\,S_g^{-1}\,e^{-\frac{1}{2}b_{ij}\psi^i\psi^j}\ . \tag{3.14}$$

Using (3.11) we infer that

$$S_{\mathcal{H}}^\dagger \;=\; S_{\mathcal{H}}\,. \tag{3.15}$$

By construction, the image of $S_{\mathcal{H}}$ under the group homomorphism ρ is precisely $\mathcal{H}$:

$$\rho(S_{\mathcal{H}}) = \rho(S_b)^T\rho(S_g^{-1})\rho(S_b) \;=\; h_b^T h_g^{-1} h_b = \mathcal{H}\,. \tag{3.16}$$

Since $S_b, S_b^\dagger \in \text{Spin}^+(D,D)$ and $S_g^{-1} \in \text{Spin}^-(D,D)$, we have $S_{\mathcal{H}} \in \text{Spin}^-(D,D)$. As a result, $S_{\mathcal{H}}$ satisfies the identities (3.3) and (3.4) and therefore gives a consistent parametrization of $\mathbb{S}$.

The flat Minkowski background $g = k$ with zero b-field gives a generalized metric that we denote as $\mathcal{H}_0 \equiv \text{diag}(k,k)$. Since $S_g = S_k$ and $S_b = \mathbf{1}$, we have

$$S_{\mathcal{H}_0} = S_k^{-1} = S_k = \psi^1\psi_1 - \psi_1\psi^1\,. \tag{3.17}$$

3.2 Duality transformations

We discuss now the transformation behavior of $\mathbb{S}$ under some arbitrary element $S \in \text{Pin}(D,D)$. Since we view $\mathbb{S}$ as an elementary field we can postulate such a transformation. The transformation of $\mathbb{S}$, however, must be consistent with the transformation of the associated $\mathcal{H} = \rho(\mathbb{S})$. Writing also $\mathcal{H}' = \rho(\mathbb{S}')$, we want to postulate a transformation for which

$$\mathbb{S} \;\xrightarrow{\;S\;}\; \mathbb{S}' \;\;\text{implies}\;\; \mathcal{H} \;\xrightarrow{\;\rho(S)\;}\; \mathcal{H}'\,. \tag{3.18}$$

In words, the $O(D,D)$ transformation $\rho(S)$ associated with $S \in \text{Pin}(D,D)$ relates the corresponding generalized metrics. The generalized metric appears explicitly in the NS-NS action.

Recall that under an $O(D,D)$ transformation h the generalized metric transforms as

$$\mathcal{H}'_{MN} \;=\; \mathcal{H}_{PQ}(h^{-1})^P{}_M(h^{-1})^Q{}_N\,. \tag{3.19}$$

In matrix notation, we will write $\mathcal{H}$ transformations as follows:

$$\mathcal{H}' = h\circ\mathcal{H} \;\equiv\; (h^{-1})^T\,\mathcal{H}\,h^{-1}\ . \tag{3.20}$$

For an element $S \in \text{Pin}(D,D)$ we postulate the following $\mathbb{S}$ transformation:

$$\mathbb{S}'(X') \;=\; (S^{-1})^\dagger\,\mathbb{S}(X)\,S^{-1}\ . \tag{3.21}$$

Here $X' = hX$, where $h = \rho(S)$. The compatibility with (3.20) is verified by taking ρ on both sides. Suppressing the coordinate arguments, we indeed find

$$\begin{aligned} \mathcal{H}' &= \rho(\mathbb{S}') = \rho\big((S^{-1})^\dagger\, \mathbb{S}\, S^{-1}\big) = \rho\big((S^{-1})^\dagger\big)\, \rho(\mathbb{S})\, \rho(S^{-1}) \\ &= (\rho(S)^{-1})^T\, \mathcal{H}\, \rho(S)^{-1} = (h^{-1})^T\, \mathcal{H}\, h^{-1} = h \circ \mathcal{H}\,. \end{aligned} \tag{3.22}$$

We infer that $\mathcal{H}'$ satisfies (3.20).

Independently of the postulated transformation rule (3.21), we can ask how $S_{\mathcal{H}}$, defined in (3.14) in terms of $\mathcal{H}$, transforms under a duality transformation generated by an element $S \in \text{Pin}(D, D)$. This transformation is simply given by

$$S: \quad S_{\mathcal{H}} \ \to \ S_{\mathcal{H}'}\,, \quad \text{where} \ \ \mathcal{H}' = \rho(S) \circ \mathcal{H}\,. \tag{3.23}$$

It is of interest to compare

$$(S^{-1})^\dagger\, S_{\mathcal{H}}\, S^{-1} \quad \longleftrightarrow \quad S_{\mathcal{H}'}\,. \tag{3.24}$$

Under ρ they both map to $\mathcal{H}'$, thus the two can be equal or can differ by a sign. Perhaps surprisingly, there is a sign factor that depends nontrivially on $\rho(S)$ and on $\mathcal{H}$. We will write

$$(S^{-1})^\dagger\, S_{\mathcal{H}}\, S^{-1} = \sigma_{\rho(S)}(\mathcal{H})\ S_{\rho(S)\circ\mathcal{H}}\,. \tag{3.25}$$

In the remainder of this section we determine this sign factor.

The case of zero b field and flat Minkowski background, $\mathcal{H}_0 = \text{diag}(k, k)$, is readily analyzed. We consider a factorized T-duality h_i with spin representative $S_i = \psi^i + \psi_i = S_i^{-1} = S_i^\dagger$. Under this transformation $\mathcal{H}_0$ remains invariant, since it corresponds to a diagonal metric with entries of absolute value one. We then have, using (3.17),

$$(S_i^{-1})^\dagger\, S_{\mathcal{H}_0}\, S_i^{-1} = (\psi^i + \psi_i)\, (\psi^1\psi_1 - \psi_1\psi^1)\, (\psi^i + \psi_i)\,. \tag{3.26}$$

It is manifest that the right-hand side is equal to $S_{\mathcal{H}_0}$ when $i \neq 1$, and a small calculation shows that is equal to $-S_{\mathcal{H}_0}$ when $i = 1$:

$$(S_i^{-1})^\dagger\, S_{\mathcal{H}_0}\, S_i^{-1} = (-1)^{\delta_{i,1}}\, S_{\mathcal{H}_0}\,. \tag{3.27}$$

We see that the sign is negative for a timelike T-duality, while the sign is positive for spacelike T-dualities.

There is a large set of $O(D, D)$ transformations h for which the sign in (3.25) is plus. As we show in the Appendix

$$(S_h^{-1})^\dagger S_{\mathcal{H}}\, S_h^{-1} = +S_{h\circ\mathcal{H}}\,, \qquad \text{when} \quad h \in GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)}\,. \tag{3.28}$$

The group $GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)}$ is that generated by successive applications of $GL(D)$ transformations and b-shifts, transformations h_b of the form indicated in (2.34), which define the abelian subgroup $\mathbb{R}^{\frac{1}{2}D(D-1)}$.

It is the T-dualities that produce sign changes. We therefore consider the sign factor in

$$(S_i^{-1})^\dagger\, S_{\mathcal{H}}\, S_i^{-1} = \sigma_i(\mathcal{H})\, S_{h_i\circ\mathcal{H}}\,. \tag{3.29}$$

As we can see, the sign factor depends on the particular $\mathcal{H}$ appearing on the left-hand side above. Our strategy in Appendix A.2 is to determine $O(D,D)$ transformations h that acting on $\mathcal{H}$ do not change the sign factor. We will show that if $h \in GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)}$ and $h_i\, h\, h_i \in GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)}$, then

$$\sigma_i(h \circ \mathcal{H}) = \sigma_i(\mathcal{H})\,. \tag{3.30}$$

It turns out that b-shifts satisfy the above conditions. Since at any point X the b-field of an $\mathcal{H}$ can be removed completely by a b-shift, we learn that the sign factor depends only on the metric g in $\mathcal{H}$:

$$\sigma_i(\mathcal{H}) = \sigma_i(g)\,. \tag{3.31}$$

We then find a restricted class of $GL(D)$ transformations that also satisfy the conditions for invariance of the sign factor. With these we are able to show that the metric g can be put in diagonal form, with entries ± 1. The sign factor then becomes calculable, just like we had for the case of $\mathcal{H}_0$. Our final result is:

$$\sigma_i(\mathcal{H}) \;=\; \mathrm{sgn}(g_{ii})\,. \tag{3.32}$$

It follows from this equation that for the flat Minskowski metric the duality transformation J about all of the spacetime coordinates gives the sign factor: $\sigma_J(\mathcal{H}_0) = -1$. At the end of Appendix A.2 we prove that this result holds for a general background $\mathcal{H}$ whose metric has Lorentzian signature:

$$\sigma_J(\mathcal{H}) = -1\,. \tag{3.33}$$

It is possible to give some intuition for the appearance of the minus signs under duality transformations. For more details see Appendix A.3, where an example is worked out as well. Since a sign cannot change continuously, $\sigma_i(\mathcal{H}+\delta\mathcal{H}) = \sigma_i(\mathcal{H})$ as long as the variation $\delta\mathcal{H}$ does not generate singularities in the fields (g,b) or their T-dual versions $h_i \circ (g,b)$ in equation (3.29). Consider now a continuous family $\mathcal{H}(\alpha)$ parameterized by α in which the metric component $g_{ii}(\alpha)$ changes sign at some point α^*. Consider also the related family $h_i \circ \mathcal{H}(\alpha)$ obtained by T-duality about the i-th direction. Under this duality the new metric, indicated by primes, is

$$g'_{ii}(\alpha) \;=\; \frac{1}{g_{ii}(\alpha)}\,. \tag{3.34}$$

It follows that g'_{ii} diverges and is discontinuous at $\alpha = \alpha^*$. Note, however, that the generalized metric $h_i \circ \mathcal{H}(\alpha)$ is regular throughout, since it is obtained from the regular $\mathcal{H}(\alpha)$ by the action of the regular matrix h_i. The discontinuity of g'_{ii} implies a discontinuity in the vielbein e' and a discontinuity in $S_{g'} = S_{e'} S_k S^{\dagger}_{e'}$. This results in a discontinuity of $S_{h_i \circ \mathcal{H}(\alpha)}$. Since $h_i \circ \mathcal{H}(\alpha)$ is continuous, the only discontinuity in $S_{h_i \circ \mathcal{H}(\alpha)}$ consistent with the homomorphism ρ is a change of sign. The right-hand side of (3.29) changes sign at the point where the original metric component changes sign. This is consistent with our result (3.32).

The issues of signs are not an artifact of our definition of $S_{\mathcal{H}}$. In Appendix A.3 we construct a continuous family of regular generalized metrics $\mathcal{H}(\alpha)$ for which $\mathcal{H}(0) = \mathcal{H}(\pi/2)$, so that $\mathcal{H}(\alpha)$ with $\alpha \in [0, \pi/2]$ is a closed path in the space of generalized metrics. If we define the representative $S_{\mathcal{H}(0)}$ and then continuously deform this representative along the path we find that at the end of the path the representative is $-S_{\mathcal{H}(0)}$. The lift to the spin group cannot be done continuously over the space of generalized metrics. If we do the lift using our definition of $S_{\mathcal{H}}$ from $\mathcal{H}$ we find that for some intermediate α the metric $g(\alpha)$ and the b-field $b(\alpha)$ become singular, while $\mathcal{H}(\alpha)$ remains regular. At this point the definition of $S_{\mathcal{H}}$ gives a discontinuity.

There seems to be some tension between the defined duality transformation of $\mathbb{S}$ in (3.21), which has no signs, and the duality transformation (3.25) of its particular parametrization $S_{\mathcal{H}}$, which shows some signs. The sign-free transformation of $\mathbb{S}$ implies that the double field theory action is fully invariant under all duality transformations, including those, like timelike T-dualities, that give a sign in (3.25). Once we choose a parametrization by setting $\mathbb{S} = S_{\mathcal{H}}$, the sign factors in (3.25) have two consequences. First, it follows that the $\mathrm{Spin}(D, D)$ invariance of the action cannot be fully realized through transformations of the conventional fields g and b. More precisely, it can only be realized for $SO(D, D)$ transformations that do not involve a timelike T-duality. This means that if we take timelike T-dualities seriously, we inevitably have to view $\mathbb{S}$ as the fundamental field. Second, when comparing the double field theory evaluated in one T-duality frame (as $\tilde{\partial}^i = 0$) to the same theory evaluated in another T-duality frame obtained by a timelike T-duality transformation (as $\partial_i = 0$), the conventional effective RR action changes sign. This sign change corresponds precisely to the transition from type II to type II* theories expected for timelike T-dualities. Correspondingly, the freedom in the choice of parametrization for $\mathbb{S}$, namely $\pm S_{\mathcal{H}}$, has no physical significance in that it merely fixes for which coordinates (x or $\tilde{x}$) we obtain the type II and for which we obtain the type II* theory. Similarly, the actual sign of the RR term in the double field theory action (1.9) has no physical significance. Therefore, we find a consistent picture, though certain invariances of the action cannot be fully realized on the conventional gravitational fields.

3.3 Gauge transformations

In this section we determine the gauge transformation of the spinor representative $\mathbb{S}$ in such a way that it us consistent with the known gauge variation of the generalized metric $\mathcal{H}_{MN}$. This variation, given in (1.3), can be rewritten as:

$$\delta_\xi \mathcal{H}^M{}_P \;=\; \xi^L \partial_L \mathcal{H}^M{}_P + \left(\partial^M \xi_K - \partial_K \xi^M\right) \mathcal{H}^K{}_P + \left(\partial_P \xi^K - \partial^K \xi_P\right) \mathcal{H}^M{}_K \;, \tag{3.35}$$

where we used that the metric η_{MN} that lowers indices is gauge invariant. We have positioned the indices of the generalized metric as in $\mathcal{H}^\bullet{}_\bullet$ to emphasize its role as an $O(D, D)$ group element. We also recall that $\mathcal{H}^M{}_K \mathcal{H}^K{}_N = \delta^M{}_N$. The matrix $\mathcal{H}$ used so far represents $\mathcal{H}_{\bullet\bullet}$.

It turns out to be convenient to write the gauge variation in terms of the spin variable $\mathcal{K}$ defined by

$$\mathcal{K} \;\equiv\; C^{-1} \mathbb{S}\,. \tag{3.36}$$

This combination will be used to prove the gauge invariance of the action in section 4.2.2. While $\mathbb{S}$ is a spin representative of $\mathcal{H}_{\bullet\bullet}$, we now check that $\mathcal{K}$ is the spin representative of $\mathcal{H}^\bullet{}_\bullet$. Indeed recalling that $\rho(C^{-1}) = J$ with J defined in (2.47), we have

$$\rho(\mathcal{K}) = \rho(C^{-1})\rho(\mathbb{S}) = J\mathcal{H}_{\bullet\bullet} = \mathcal{H}^\bullet{}_\bullet\,, \tag{3.37}$$

since J is identical to the matrix η^{-1} that raises indices. We write this conclusion as

$$S_{\mathcal{H}^\bullet{}_\bullet} = \pm\,\mathcal{K}\,. \tag{3.38}$$

We will show that the gauge transformation of $\mathcal{K}$ compatible with that of $\mathcal{H}^{\bullet}{}_{\bullet}$ takes the form

$$\delta_\xi \mathcal{K} \;=\; \xi^M \partial_M \mathcal{K} + \frac{1}{2}\big[\Gamma^{PQ},\, \mathcal{K}\,\big]\partial_P \xi_Q \;, \tag{3.39}$$

where $\Gamma^{PQ} \equiv \frac{1}{2}[\Gamma^P, \Gamma^Q]$. We will prove the above in a different but equivalent form, which reads

$$\delta_\xi \mathcal{K} \;=\; \xi^M \partial_M \mathcal{K} + \frac{1}{2}\left(\Gamma^{PQ} - \Gamma^{RS}\,\mathcal{H}^P{}_R\,\mathcal{H}^Q{}_S\right)\mathcal{K}\;\partial_P \xi_Q \;. \tag{3.40}$$

This, in turn, can be written more suggestively as

$$(\delta_\xi \mathcal{K})\,\mathcal{K}^{-1} \;=\; \xi^M (\partial_M \mathcal{K})\mathcal{K}^{-1} + \frac{1}{2}\left(\Gamma^{PQ} - \Gamma^{RS}\,\mathcal{H}^P{}_R\,\mathcal{H}^Q{}_S\right)\partial_P \xi_Q \;. \tag{3.41}$$

To see that (3.40) is equivalent to (3.39) we use that (2.25) implies for any $h \in O(D,D)$

$$S\,\Gamma_{PQ}\,S^{-1} \;=\; \Gamma_{RS}\,h^R{}_P\,h^S{}_Q \;, \quad \rho(S) \;=\; h\,. \tag{3.42}$$

Specialized to the $O(D,D)$ element $\mathcal{H}^M{}_N$ this yields

$$S_{\mathcal{H}^{\bullet}{}_{\bullet}}\,\Gamma_{PQ}\,(S_{\mathcal{H}^{\bullet}{}_{\bullet}})^{-1} \;=\; \Gamma_{RS}\,\mathcal{H}^R{}_P\,\mathcal{H}^S{}_Q \;, \tag{3.43}$$

and with use of (3.38) we find

$$\mathcal{K}\,\Gamma_{PQ}\,\mathcal{K}^{-1} \;=\; \Gamma_{RS}\,\mathcal{H}^R{}_P\,\mathcal{H}^S{}_Q \quad\;\; \to \quad \mathcal{K}\,\Gamma^{PQ} \;=\; \Gamma^{RS}\,\mathcal{H}^P{}_R \mathcal{H}^Q{}_S\,\mathcal{K}\,. \tag{3.44}$$

This final identity demonstrates the equivalence of (3.40) and (3.39).

The strategy in our construction will be to express the gauge transformations as Lie algebra identities that can be realized both in the fundamental and spin representations of $O(D,D)$. To begin, we consider the transport term δ^t_ξ in the transformation (3.35) of the generalized metric, written as follows

$$(\delta^t_\xi \mathcal{H}^M{}_P)(\mathcal{H}^{-1})^P{}_N \;=\; \xi^L \partial_L \mathcal{H}^M{}_P (\mathcal{H}^{-1})^P{}_N\,. \tag{3.45}$$

This equality of Lie-algebra elements is here realized in the fundamental representation. In the spin representation, where the group element $\mathcal{H}^{\bullet}{}_{\bullet}$ is represented by $\mathcal{K}$ we would have

$$(\delta^t_\xi \mathcal{K})\mathcal{K}^{-1} \;=\; \xi^L (\partial_L \mathcal{K})\,\mathcal{K}^{-1}\,. \tag{3.46}$$

This proves that the transport term in (3.40) is required by consistency. Calling $\Delta_\xi \mathcal{H}^M{}_P$ the non-transport terms in the transformation, we now have

$$\begin{aligned}
\Delta_\xi \mathcal{H}^M{}_P\,(\mathcal{H}^{-1})^P{}_N \;&=\; \Delta_\xi \mathcal{H}^M{}_P\,\mathcal{H}^P{}_N \\
&=\; (\partial^M \xi_N - \partial_N \xi^M) + (\partial_P \xi^K - \partial^K \xi_P)\mathcal{H}^M{}_K \mathcal{H}^P{}_N \\
&=\; \left(\partial_P \xi_Q - \partial_R \xi_S\,\mathcal{H}^R{}_P\,\mathcal{H}^S{}_Q\right)\left(\eta^{MP}\delta^Q{}_N - \eta^{MQ}\delta^P{}_N\right)\,,
\end{aligned} \tag{3.47}$$

where the last equality is readily checked by expansion of the product. We now recognize the last factor in the last line of the above equation as $(T^{PQ})^M{}_N$, the Lie algebra generator in the fundamental representation, as introduced in (2.5). We thus have

$$\Delta_\xi \mathcal{H}^M{}_P\,(\mathcal{H}^{-1})^P{}_N \;=\; \left(\partial_P \xi_Q - \partial_R \xi_S\,\mathcal{H}^R{}_P\,\mathcal{H}^S{}_Q\right)(T^{PQ})^M{}_N\,. \tag{3.48}$$

Passing to the spin representation with matrices (2.19) we find

$$\begin{aligned}(\Delta_\xi\mathcal{K})\,\mathcal{K}^{-1} &= \frac{1}{2}\left(\partial_P\xi_Q - \partial_R\xi_S\,\mathcal{H}^R{}_P\,\mathcal{H}^S{}_Q\right)\Gamma^{PQ} \\ &= \frac{1}{2}\left(\Gamma^{PQ} - \Gamma^{RS}\,\mathcal{H}^P{}_R\,\mathcal{H}^Q{}_S\right)\partial_P\xi_Q\,.\end{aligned} \tag{3.49}$$

This coincides exactly with the non-transport term in (3.41) and concludes our proof that the postulated gauge transformation (3.39) of $\mathcal{K}$ is consistent with that of the generalized metric.

4 Action, duality relations, and gauge symmetries

In this section we introduce the $O(D,D)$ covariant double field theory formulation of the RR action and the duality relations. We prove T-duality invariance and gauge invariance, and we determine the $O(D,D)$ covariant form of the field equations.

4.1 Action, duality relations, and $O(D,D)$ invariance

The dynamical field we will use to write an action is a spinor of $\mathrm{Pin}(D,D)$ written as in (2.14):

$$\chi \;\equiv\; |\chi\rangle \;=\; \sum_{p=0}^{D}\frac{1}{p!}\,C_{i_1\ldots i_p}\,\psi^{i_1}\ldots\psi^{i_p}|0\rangle\;. \tag{4.1}$$

Here the component forms $C_{i_1\ldots i_p}(x,\tilde{x})$ are the dynamical fields and, as is usual in double field theory, they are real functions of the full collection of $2D$ coordinates x and $\tilde{x}$. We will assume χ to have a definite chirality. Thus, as discussed in sec. 2.3, it consists either of only odd forms or even forms. The bra associated with this ket is called $\chi^\dagger$ and is defined by

$$\chi^\dagger \;\equiv\; \langle\chi| \;=\; \sum_{p=0}^{D}\frac{1}{p!}\,C_{i_1\ldots i_p}\,\langle 0|\psi_{i_p}\ldots\psi_{i_1}\;. \tag{4.2}$$

We conventionally define the conjugate spinor using the C matrix defined in section 2.2:

$$\bar{\chi} \;\equiv\; \chi^\dagger C\;. \tag{4.3}$$

We will make use of a Dirac operator on spinors that behaves just as an exterior derivative on the associated forms:

$$\partial\!\!\!/ \equiv \frac{1}{\sqrt{2}}\,\Gamma^M\partial_M = \;\psi^i\partial_i + \psi_i\tilde{\partial}^i\,, \tag{4.4}$$

where we used (2.11). The $\partial\!\!\!/$ operator behaves like the exterior derivative d in that its repeated action gives zero:

$$\partial\!\!\!/^2 \;=\; \frac{1}{2}\Gamma^M\Gamma^N\partial_M\partial_N \;=\; \frac{1}{4}\{\Gamma^M,\Gamma^N\}\partial_M\partial_N \;=\; \frac{1}{2}\eta^{MN}\partial_M\partial_N \;=\; 0\,, \tag{4.5}$$

by the constraint (1.4). The $\partial\!\!\!/$ operator will be used to define field strengths in a $\mathrm{Pin}(D,D)$ covariant way. It is clear that acting on forms that do not depend on $\tilde{x}$, the only term that survives, $\psi^i\partial_i$, both differentiates with respect to x and increases the degree of the form by one. More details will be given in section 5.

We turn now to a discussion of the double field theory action. We claim that the RR action is $S = \int dx d\tilde{x} \mathcal{L}$, where the Lagrangian density $\mathcal{L}$ is simply given by

$$\mathcal{L} \;=\; \frac{1}{4} (\partial\!\!\!/ \chi)^\dagger \, \mathbb{S} \, \partial\!\!\!/ \chi \, . \tag{4.6}$$

The above Lagrangian is manifestly real: $\mathcal{L}^\dagger = \mathcal{L}$ because the spinor χ is Grassmann even and $\mathbb{S}$ is Hermitian. The Lagrangian can be written using conjugate spinors and the kinetic operator $\mathcal{K} = C^{-1}\mathbb{S}$. We claim that the above Lagrangian is equal to

$$\mathcal{L} \;=\; \frac{1}{8} \, \partial_M \bar{\chi} \, \Gamma^M \, \mathcal{K} \, \Gamma^N \partial_N \chi \, . \tag{4.7}$$

Indeed, using the conjugate spinor (4.3) and (2.46) this second version is written as

$$\begin{aligned} \mathcal{L} \;&=\; \frac{1}{8} \, \partial_M \chi^\dagger C \, \Gamma^M \, C^{-1} \, \mathbb{S} \, \sqrt{2} \partial\!\!\!/ \chi \;=\; \frac{1}{8} \, \partial_M \chi^\dagger \, (\Gamma^M)^\dagger \, \mathbb{S} \, \sqrt{2} \partial\!\!\!/ \chi \\ &=\; \frac{1}{8} \, \sqrt{2} (\partial\!\!\!/ \chi)^\dagger \, \mathbb{S} \, \sqrt{2} \partial\!\!\!/ \chi \;=\; \frac{1}{4} (\partial\!\!\!/ \chi)^\dagger \, \mathbb{S} \, \partial\!\!\!/ \chi \, . \end{aligned} \tag{4.8}$$

The properties of bar conjugation allow us to recognize that

$$\overline{\partial\!\!\!/ \chi} = \frac{1}{\sqrt{2}} (\Gamma^M \partial_M \chi)^\dagger C = \frac{1}{\sqrt{2}} \, \partial_M \chi^\dagger (\Gamma^M)^\dagger C = \frac{1}{\sqrt{2}} \, \partial_M \bar{\chi} C^{-1} (\Gamma^M)^\dagger C = \frac{1}{\sqrt{2}} \, \partial_M \bar{\chi} \Gamma^M \, , \tag{4.9}$$

and therefore we can write the action more compactly as

$$\mathcal{L} \;=\; \frac{1}{4} \, \overline{\partial\!\!\!/ \chi} \, \mathcal{K} \, \partial\!\!\!/ \chi \, . \tag{4.10}$$

Our first task now is to establish the global Spin(D, D) invariance of this Lagrangian (the $dx d\tilde{x}$ measure is $O(D, D)$ invariant). This is the maximal invariance group that is consistent with the fixed chirality of χ. Under the action of a Spin(D, D) element S, whose associated $O(D, D)$ element is $h = \rho(S)$, the spinor field χ transforms as follows:

$$\chi \to \chi' \;=\; S \, \chi \, . \tag{4.11}$$

Implicit in here is that the coordinates the fields depend on are also transformed: primed fields depend on primed coordinates $X'^M = h^M{}_N X^N$. Note also that the daggered state transforms as

$$\chi^\dagger \;\to\; \chi^\dagger \, S^\dagger \, . \tag{4.12}$$

We also have that

$$\partial\!\!\!/ \chi = \frac{1}{\sqrt{2}} \Gamma^M \partial_M \chi \to \frac{1}{\sqrt{2}} \, \Gamma^M (h^{-1})^N{}_M \partial_N S \chi \;=\; \frac{1}{\sqrt{2}} \, S \, [S^{-1} \Gamma^M S] (h^{-1})^N{}_M \partial_N \chi \, . \tag{4.13}$$

We now use (2.26) to find

$$\partial\!\!\!/ \chi \;\to\; \frac{1}{\sqrt{2}} \, S \, h^M{}_P \Gamma^P (h^{-1})^N{}_M \partial_N \chi = \frac{1}{\sqrt{2}} \, S \, \Gamma^N \partial_N \chi \, , \tag{4.14}$$

and therefore we have

$$\partial\!\!\!/ \chi \;\to\; S \, \partial\!\!\!/ \chi \, . \tag{4.15}$$

We have thus leaned that $\partial\!\!\!/\chi$ transforms just like χ. In other words, the Dirac operator $\partial\!\!\!/$ is $\mathrm{Spin}(D,D)$ invariant. Recalling the transformation of $\mathbb{S}$ in (3.21) : $\mathbb{S} \rightarrow (S^{-1})^{\dagger}\,\mathbb{S}\,S^{-1}$, the invariance of the Lagrangian (4.6) is essentially manifest:

$$\mathcal{L} \;=\; \frac{1}{4}(\partial\!\!\!/\chi)^{\dagger}\,\mathbb{S}\,\partial\!\!\!/\chi \;\;\rightarrow\;\; \frac{1}{4}(\partial\!\!\!/\chi)^{\dagger}S^{\dagger}\,(S^{-1})^{\dagger}\,\mathbb{S}\,S^{-1}\,S\partial\!\!\!/\chi \;=\; \mathcal{L}\,. \tag{4.16}$$

The action must be supplemented by duality constraints among the field strengths. We can write $\mathrm{Spin}^{+}(D,D)$ covariant versions of the duality relations that relate all RR field strengths:[3]

$$\partial\!\!\!/\chi \;=\; -\mathcal{K}\,\partial\!\!\!/\chi\,. \tag{4.17}$$

According to (4.15), the left-hand side transforms covariantly with $S\in\mathrm{Spin}(D,D)$. The right-hand side transforms in the same way, since

$$-\,\mathcal{K}\,\partial\!\!\!/\chi \;\;\rightarrow\;\; -C^{-1}(S^{-1})^{\dagger}\,\mathbb{S}\,S^{-1}S\partial\!\!\!/\chi \;=\; -S\,C^{-1}\,\mathbb{S}\,\partial\!\!\!/\chi \;=\; -S\,\mathcal{K}\,\partial\!\!\!/\chi\,, \tag{4.18}$$

where we used that (2.62) implies $C^{-1}(S^{-1})^{\dagger} = SC^{-1}$ for $S\in\mathrm{Spin}^{+}(D,D)$. Thus, the duality relations are actually only invariant under $\mathrm{Spin}^{+}(D,D)$. This is to be expected since already for conventional duality relations the presence of an epsilon tensor breaks the symmetry to the group $GL^{+}(D)$ of parity-preserving transformations.

The relations (4.17) require a consistency condition. Acting on both sides of (4.17) with $\mathcal{K}$, we see that consistency requires $\mathcal{K}^2 = 1$, which in turn implies

$$\mathcal{K}^2 \;=\; C^{-1}\,\mathbb{S}\,C^{-1}\,\mathbb{S} \;=\; C(\mathbb{S}\,C\,\mathbb{S}) \;=\; C(-C) \;=\; -(-1)^{\frac{1}{2}D(D-1)} \;=\; 1\,, \tag{4.19}$$

where we used (3.4) and (2.43). Thus, the duality relations are self-consistent in dimensions for which $\frac{1}{2}D(D-1)$ is odd. For $D\leq 10$, these are

$$D \;=\; \big\{\,10\,,7\,,6\,,3\,,2\,\big\}\,. \tag{4.20}$$

We note that the even dimensions above are precisely those for which conventional self-duality relations can be imposed consistently. Indeed, the middle degree forms corresponding to the self-dual field strengths are then odd, and for them $\star^2 = 1$ in Lorentzian signature. As we will show in sec. 5.1.3 the component form of (4.17) contains one self-duality relation in even dimensions, so this result is to be expected. In the following we will focus on $D=10$, but we note that $D=2,6$ can be seen as type II toy models. The possible significance of theories with odd D will not be discussed here.

We close by giving the equations of motion of χ, which are readily derived from (4.7),

$$\partial\!\!\!/\big(\mathcal{K}\,\partial\!\!\!/\chi\big) \;=\; 0\,. \tag{4.21}$$

As it should be, the equation of motion is the integrability condition for the duality relations: acting with a $\partial\!\!\!/$ on both sides of (4.17), and using $\partial\!\!\!/^{2}=0$, we recover the field equation.

[3] For the special case of type IIA, a similar $O(D,D)$-covariant form of the duality relations has also been proposed in the second reference of [23].

4.2 Gauge invariance

In this subsection we give the gauge transformation of the RR fields. The p-form gauge transformations are manifestly invariances of the Lagrangian and of the duality constraints. For the gauge transformations parameterized by ξ^M the transformation of χ is nontrivial and so are the checks of gauge invariance of the Lagrangian and the duality constraints.

4.2.1 Gauge transformations

We start by introducing the double field theory version of the abelian gauge symmetries of the p-form gauge fields. These are parameterized by a spacetime dependent spinor λ:

$$\delta_\lambda \chi = \partial\!\!\!/ \lambda \,. \tag{4.22}$$

Since λ encodes a set of forms and $\partial\!\!\!/$ acts as an exterior derivative, the above transformations are the familiar ones. It follows that

$$\delta_\lambda \partial\!\!\!/ \chi = \partial\!\!\!/ \partial\!\!\!/ \lambda = 0 \,, \tag{4.23}$$

and this implies the gauge invariance of the Lagrangian density (4.6) and of the duality constraint (4.17).

For the gauge parameter ξ^M that encodes the diffeomorphism and Kalb-Ramond gauge symmetries, we postulate the gauge transformation

$$\begin{aligned} \delta_\xi \chi \;&=\; \widehat{\mathcal{L}}_\xi \chi \;\equiv\; \xi^M \partial_M \chi \;+\; \frac{1}{\sqrt{2}} \partial\!\!\!/ \xi^M \Gamma_M \, \chi \\ &=\; \xi^M \partial_M \chi \;+\; \frac{1}{2}\, \partial_N \xi_M \Gamma^N \Gamma^M \chi \,. \end{aligned} \tag{4.24}$$

In the second form it is simple to verify that a gauge parameter of the form $\xi_M = \partial_M \Theta$ is trivial in that it generates no gauge transformations:

$$\delta_{\partial\Theta} \chi \;=\; \partial^M \Theta \, \partial_M \chi \;+\; \frac{1}{2} \partial_N \partial_M \Theta \, \Gamma^N \Gamma^M \chi \;=\; \frac{1}{2} \partial_N \partial_M \Theta \, \eta^{MN} \chi \;=\; 0 \,. \tag{4.25}$$

A short calculation gives the gauge transformation of the conjugate spinor $\bar{\chi}$:

$$\delta_\xi \bar{\chi} \;=\; \xi^M \partial_M \bar{\chi} \;+\; \frac{1}{2}\, \partial_N \xi_M \, \bar{\chi} \, \Gamma^M \Gamma^N \,. \tag{4.26}$$

Let us now turn to the gauge algebra. We claim that the gauge transformations parametrized by λ and ξ^M close as follows

$$\big[\delta_\lambda \,, \delta_\xi \big] = \delta_{\widehat{\mathcal{L}}_\xi \lambda} \,. \tag{4.27}$$

To check this we consider the left-hand side acting on χ:

$$\begin{aligned} \big[\delta_\lambda \,, \delta_\xi \big] \chi = \delta_\lambda \delta_\xi \chi \;&=\; \delta_\lambda \big(\xi^M \partial_M \chi \;+\; \frac{1}{2} \partial_N \xi_M \Gamma^N \Gamma^M \chi \big) \\ &=\; \xi^M \partial_M \partial\!\!\!/ \lambda \;+\; \frac{1}{2} \partial_N \xi_M \Gamma^N \Gamma^M \partial\!\!\!/ \lambda \,. \end{aligned} \tag{4.28}$$

The right-hand side of the expected algebra is:

$$\begin{aligned}
\delta_{\widehat{\mathcal{L}}_\xi \lambda}\chi \;&= \partial\!\!\!/ \widehat{\mathcal{L}}_\xi \lambda = \partial\!\!\!/ \big(\xi^M \partial_M \lambda \;+\; \frac{1}{2}\partial_N \xi_M \Gamma^N \Gamma^M \lambda\big) \\
&= \frac{1}{\sqrt{2}}\partial_P \big(\xi^M \partial_M \Gamma^P \lambda \;+\; \frac{1}{2}\partial_N \xi_M \Gamma^P \Gamma^N \Gamma^M \lambda\big) \\
&= \xi^M \partial_M \partial\!\!\!/ \lambda \;+\; \frac{1}{\sqrt{2}}\partial_P \xi^M \Gamma^P \partial_M \lambda + \;\frac{1}{2\sqrt{2}}\partial_N \xi_M \Gamma^P \Gamma^N \Gamma^M \partial_P \lambda\,,
\end{aligned} \tag{4.29}$$

where the term with two derivatives on ξ vanishes by the use of $\partial\!\!\!/^2 = 0$. Using the commutator

$$[\,\Gamma^P\,, \Gamma^N \Gamma^M\,] = 2\eta^{PN}\Gamma^M - 2\eta^{PM}\Gamma^N\,, \tag{4.30}$$

one can readily show that

$$\frac{1}{2\sqrt{2}}\,\partial_N \xi_M [\,\Gamma^P\,, \Gamma^N \Gamma^M\,] \partial_P \lambda \;=\, -\frac{1}{\sqrt{2}}\,\partial_P \xi^M\, \Gamma^P \partial_M \lambda\,, \tag{4.31}$$

where we used the constraint and relabeled the indices. Then, returning to (4.29),

$$\delta_{\widehat{\mathcal{L}}_\xi \lambda}\chi \;=\; \xi^M \partial_M \partial\!\!\!/ \lambda \;+\; \frac{1}{2}\,\partial_N \xi_M \Gamma^N \Gamma^M \partial\!\!\!/ \lambda\,. \tag{4.32}$$

This agrees with (4.28) confirming the closure of the gauge algebra. We have also verified that, as expected, $[\delta_{\xi_1}, \delta_{\xi_2}] = -\delta_{[\xi_1,\xi_2]_{\rm C}}$, where $[\,\cdot\,,\,\cdot]_{\rm C}$ is the C-bracket discussed in [14].

4.2.2 Gauge invariance of the action and the duality constraints

The action is manifestly invariant under p-form gauge transformations. Here we check the invariance under δ_ξ. We use the Lagrangian in (4.7):

$$\mathcal{L} \;=\; \frac{1}{4}\,\overline{\partial\!\!\!/ \chi}\;\mathcal{K}\;\partial\!\!\!/ \chi\,. \tag{4.33}$$

As usual, when we vary the Lagrangian, which has the index structure of a scalar, we obtain a transport term and a 'non-covariant' term

$$\delta_\xi \mathcal{L} \;=\; \xi^M \partial_M \mathcal{L} + \Delta_\xi \mathcal{L}\,. \tag{4.34}$$

Since Δ_ξ acts as a derivation and commutes with bar-conjugation,

$$\Delta_\xi \mathcal{L} = \frac{1}{4}\Big((\overline{\Delta_\xi \partial\!\!\!/ \chi})\;\mathcal{K}\;\partial\!\!\!/ \chi \;+\; \overline{\partial\!\!\!/ \chi}\;(\Delta_\xi \mathcal{K})\;\partial\!\!\!/ \chi \;+\; \overline{\partial\!\!\!/ \chi}\;\mathcal{K}\;\Delta_\xi \partial\!\!\!/ \chi\Big)\,. \tag{4.35}$$

For the action to be gauge invariant, $\Delta_\xi \mathcal{L}$ must be such that $\delta_\xi \mathcal{L}$ in (4.34) is a total derivative. Since $\Delta_\xi \mathcal{K}$ can be read from (3.39), we only have to calculate $\Delta_\xi \partial\!\!\!/ \chi$. We begin my noting that

$$\delta_\xi\big(\partial\!\!\!/ \chi\big) \;=\; \frac{1}{\sqrt{2}}\Gamma^M \partial_M\Big(\xi^P \partial_P \chi \;+\; \frac{1}{2}\partial_P \xi_Q \Gamma^P \Gamma^Q \chi\Big)\,. \tag{4.36}$$

The noncovariant piece in this transformation includes all terms in the right-hand side except for $\xi^P \partial_P \partial\!\!\!/ \chi$. Therefore we have

$$\Delta_\xi\big(\partial\!\!\!/ \chi\big) \;=\; \frac{1}{\sqrt{2}}\Big(\partial_M \xi^P \Gamma^M \partial_P \chi + \;\frac{1}{2}\,\partial_P \xi_Q \Gamma^M \Gamma^P \Gamma^Q \partial_M \chi\Big)\,, \tag{4.37}$$

since the term with two derivatives on ξ vanishes. A short computation using (4.30) to bring Γ^M next to the spinor gives the final answer

$$\Delta_\xi(\partial\!\!\!/\chi) \;=\; \frac{1}{2}\,\partial_P\xi_Q\,\Gamma^P\Gamma^Q\partial\!\!\!/\chi\,. \tag{4.38}$$

Bar conjugation immediately yields,

$$\overline{\Delta_\xi(\partial\!\!\!/\chi)} \;=\; \frac{1}{2}\,\partial_P\xi_Q\;\overline{\partial\!\!\!/\chi}\,\Gamma^Q\Gamma^P\,. \tag{4.39}$$

Using the above variations and (3.39) we find that (4.35) gives

$$\Delta_\xi\mathcal{L} = \frac{1}{8}\,\partial_P\xi_Q\;\overline{\partial\!\!\!/\chi}\,\Big(\Gamma^Q\Gamma^P\,\mathcal{K} + \big[\Gamma^{PQ},\,\mathcal{K}\big] +\;\mathcal{K}\,\Gamma^P\Gamma^Q\Big)\,\partial\!\!\!/\chi\,. \tag{4.40}$$

A short calculation shows that the factor in parenthesis equals $2\eta^{PQ}\mathcal{K}$. As a result we find

$$\Delta_\xi\mathcal{L} = \frac{1}{4}\,\partial_M\xi^M\;\overline{\partial\!\!\!/\chi}\,\mathcal{K}\,\partial\!\!\!/\chi \;=\; \partial_M\xi^M\,\mathcal{L}\,. \tag{4.41}$$

Back in (4.34) we get $\delta_\xi\mathcal{L} \;=\; \xi^M\partial_M\mathcal{L} + (\partial_M\xi^M)\mathcal{L} \;=\; \partial_M(\xi^M\mathcal{L})$, which confirms the gauge invariance of the action.

Finally, we have to prove gauge covariance of the duality constraints $\partial\!\!\!/\chi = -\mathcal{K}\partial\!\!\!/\chi$. We now take the gauge variation δ_ξ of both sides of the duality constraint. The transport terms on both sides are identical, using the duality constraint. So only the non-covariant terms matter, and we can evaluate Δ_ξ on both sides of the constraint, finding

$$\Delta_\xi\partial\!\!\!/\chi \;=\; -(\Delta_\xi\mathcal{K})\partial\!\!\!/\chi \;-\; \mathcal{K}\Delta_\xi\partial\!\!\!/\chi\,. \tag{4.42}$$

Our task is to verify that this holds, using the duality constraint. Bringing all terms to one side we must check that

$$\Delta_\xi\partial\!\!\!/\chi \;+\; (\Delta_\xi\mathcal{K})\partial\!\!\!/\chi \;+\; \mathcal{K}\Delta_\xi\partial\!\!\!/\chi \;=\; 0\,. \tag{4.43}$$

Using our earlier results we find that the left-hand side is equal to

$$\frac{1}{2}\,\partial_P\xi_Q\,\Big(\Gamma^P\Gamma^Q + [\Gamma^{PQ},\mathcal{K}\,] +\;\mathcal{K}\,\Gamma^P\Gamma^Q\Big)\partial\!\!\!/\chi\,. \tag{4.44}$$

Expanding the commutator and using the duality constraint we find that the above becomes

$$\frac{1}{2}\,\partial_P\xi_Q\,\Big((\Gamma^P\Gamma^Q - \Gamma^{PQ}) +\;\mathcal{K}\,(\Gamma^P\Gamma^Q - \Gamma^{PQ})\Big)\partial\!\!\!/\chi \;=\; \frac{1}{2}\,\partial_P\xi_Q\,\eta^{PQ}(1+\;\mathcal{K})\partial\!\!\!/\chi \;=\; 0\,. \tag{4.45}$$

This concludes our proof.

4.3 General variation of $\mathbb{S}$ and gravitational equations of motion

In this section we determine the general variation of the action under a variation of $\mathbb{S}$ in order to determine the contribution of the new action to the field equations. This is non-trivial since $\mathbb{S}$ is a *constrained* field in that it takes values in $\mathrm{Spin}(D,D)$. The corresponding problem for the constrained variable given by the generalized metric $\mathcal{H}$ has been discussed in [14], and the method employed there can be elevated to $\mathbb{S}$, as we discuss next.

In [14], sec. 4, it was shown that a general variation of the constrained variable $\mathcal{H}$ can be parametrized in terms of a symmetric but otherwise unconstrained matrix $\mathcal{M}^{MN}$ as follows

$$\begin{aligned}\delta\mathcal{H}^{MN} &= \frac{1}{4}\left[\left(\delta^M{}_P+\mathcal{H}^M{}_P\right)\left(\delta^N{}_Q-\mathcal{H}^N{}_Q\right)+\left(\delta^M{}_P-\mathcal{H}^M{}_P\right)\left(\delta^N{}_Q+\mathcal{H}^N{}_Q\right)\right]\mathcal{M}^{PQ} \\ &= \frac{1}{2}\Big[\,\mathcal{M}^{MN}-\mathcal{H}^M{}_P\,\mathcal{M}^{PQ}\,\mathcal{H}^N{}_Q\,\Big]\,.\end{aligned} \tag{4.46}$$

Lowering the N index,

$$\delta\mathcal{H}^M{}_N \;=\; \frac{1}{2}\Big[\,\mathcal{M}^M{}_N-\mathcal{H}^M{}_P\,\mathcal{M}^{PQ}\,\mathcal{H}_{NQ}\,\Big]\,. \tag{4.47}$$

As in section 3.3 we now form the Lie-algebra element

$$\begin{aligned}(\delta\mathcal{H}^M{}_P)\mathcal{H}^P{}_N &= \frac{1}{2}\Big(\mathcal{M}^M{}_R\mathcal{H}^R{}_N-\mathcal{H}^M{}_R\mathcal{M}^R{}_N\,\Big) \\ &= \frac{1}{2}\,\mathcal{M}_{PR}\mathcal{H}^R{}_Q\Big(\,\eta^{MP}\delta^Q{}_N-\eta^{MQ}\delta^P{}_N\Big) \\ &= \frac{1}{2}\,\mathcal{M}_{PR}\mathcal{H}^R{}_Q\big(T^{PQ}\big)^M{}_N\,,\end{aligned} \tag{4.48}$$

where we made repeated use of the symmetry properties of $\mathcal{H}$ and $\mathcal{M}$ and used (2.5). In the spin representation this equation yields

$$(\delta\mathcal{K})\,\mathcal{K}^{-1} \;=\; \frac{1}{4}\mathcal{M}_{PR}\mathcal{H}^R{}_Q\,\Gamma^{PQ} \;=\; \frac{1}{4}\,\mathcal{M}_{MN}\mathcal{H}^M{}_P\,\Gamma^{NP}\,, \tag{4.49}$$

after some index relabeling. Our final result for the variation is therefore

$$\delta\mathcal{K} \;=\; \frac{1}{4}\,\mathcal{M}_{MN}\mathcal{H}^M{}_P\,\Gamma^{NP}\,\mathcal{K}\,. \tag{4.50}$$

This, with $\mathcal{H}^{\bullet}{}_{\bullet}=\rho(\mathcal{K})$, is the general variation of $\mathcal{K}$ consistent with its group property $\mathcal{K}\in\mathrm{Spin}(D,D)$. It is consistent with the variation (4.47), and thus the variation of the NS-NS action is unmodified as compared to the discussion in [14].

Next, we apply (4.50) in order to compute the variation of the RR action

$$\delta\mathcal{L} \;=\; \frac{1}{4}\,\overline{\partial\!\!\!/\chi}\,\delta\,\mathcal{K}\,\partial\!\!\!/\chi \;=\; \frac{1}{16}\mathcal{M}_{MN}\,\mathcal{H}^M{}_P\,\overline{\partial\!\!\!/\chi}\,\Gamma^{NP}\mathcal{K}\,\partial\!\!\!/\chi\,. \tag{4.51}$$

Since $\mathcal{M}$ is an arbitrary symmetric matrix, we read off that the contribution to the field equations is given by the symmetric 'stress-tensor'

$$\mathcal{E}^{MN} \;=\; \frac{1}{16}\mathcal{H}^{(M}{}_P\,\overline{\partial\!\!\!/\chi}\,\Gamma^{N)P}\,\mathcal{K}\,\partial\!\!\!/\chi\,. \tag{4.52}$$

It is possible to verify that, as required, the above symmetric tensor is real $(\mathcal{E}^{MN})^\dagger=\mathcal{E}^{MN}$. This calculation makes use of $C^\dagger=C^{-1}$ and (3.44). It is also important to note that $\mathcal{E}^{MN}$ transforms covariantly under duality:

$$\mathcal{E}'^{MN}(X') \;=\; h^M{}_P\,h^N{}_Q\,\mathcal{E}^{PQ}(X)\,. \tag{4.53}$$

The explicit check makes use of (3.42) and the duality properties of $\mathcal{H}$.

Taking the variation of the NS-NS action into account, which leads to the tensor $\mathcal{R}_{MN}$ defined in eq. (4.58) of [14], this leads to the $O(D,D)$ covariant form of the type II field equations,

$$\mathcal{R}_{MN} + \mathcal{E}_{MN} \;=\; 0 \;, \tag{4.54}$$

supplemented by the duality constraint (4.17). In fact, the duality constraint allows us to simplify $\mathcal{E}^{MN}$ considerably:

$$\mathcal{E}^{MN} \;=\; -\frac{1}{16}\,\mathcal{H}^{(M}{}_{P}\,\overline{\partial\!\!\!/\chi}\,\Gamma^{N)P}\,\partial\!\!\!/\chi\;. \tag{4.55}$$

One may try to verify again the reality of this stress-tensor. A short calculation shows that it is only real whenever $CC = -1$. This is precisely the constraint for consistent duality constraints, as discussed at the end of section 4.1. Since we work with real numbers throughout, a non-real stress-tensor can only be equal to zero.

5 Action and duality relations in the standard frame

In this section we examine the form of the action and duality relations when choosing the 'standard' duality frame $\tilde{\partial}^i = 0$, and we show that they reduce to the conventional democratic formulation of type II theories. For this we have to assume that we are in a region with a well-defined metric, so that we can choose the parametrization $\mathbb{S} = S_{\mathcal{H}}$. The physical significance of this particular parametrization will be discussed in the next section.

5.1 Action and duality relations in $\tilde{\partial} = 0$ frame

In this section we evaluate the action and duality relations in the standard frame $\tilde{\partial} = 0$. We begin by deriving some relations which will turn out to be useful for this analysis.

5.1.1 Preliminaries

Let us derive some useful identities for the action of S_g on general spinor states. To this end we need to determine the action of an exponential of fermionic oscillators. We find

$$\begin{aligned} e^{\psi^i R_i{}^j \psi_j}\,\psi^k|0\rangle \;&=\; \Big(1 + R_i{}^j\psi^i\psi_j + \frac{1}{2}R_i{}^j R_p{}^q \psi^i\psi_j\psi^p\psi_q + \cdots\Big)\psi^k|0\rangle \\ &=\; \Big(\psi^k + \psi^i R_i{}^j\{\psi_j,\psi^k\} + \frac{1}{2}R_i{}^j R_p{}^q\psi^i\{\psi_j,\psi^p\}\{\psi_q,\psi^k\} + \cdots\Big)|0\rangle \\ &=\; \Big(\delta_l{}^k + R_l{}^k + \frac{1}{2}R_l{}^j R_j{}^k + \cdots\Big)\psi^l|0\rangle \;=\; \big(\exp R\big)_l{}^k\,\psi^l|0\rangle\;. \end{aligned} \tag{5.1}$$

In order to determine now the action of $S_g = S_e S_k S_e^\dagger$ on general states, we compute the action of the respective factors. For S_e, we introduce $e = \exp(E)$ and we have

$$S_e\,\psi^i\,|0\rangle \;=\; \frac{1}{\sqrt{\det e}}\,e^{\psi^j E_j{}^k \psi_k}\,\psi^i\,|0\rangle \;=\; \frac{1}{\sqrt{\det e}}\big(\exp E\big)_j{}^i\,\psi^j\,|0\rangle \;=\; \frac{1}{\sqrt{\det e}}\,e_j{}^i\psi^j\,|0\rangle\;. \tag{5.2}$$

For $S_e^\dagger$ we find an expression with unusual index position

$$S_e^\dagger\,\psi^i\,|0\rangle \;=\; \frac{1}{\sqrt{\det e}}\,e_i{}^j\,\psi^j\,|0\rangle\;. \tag{5.3}$$

The action of S_k can be easily computed,

$$S_k\,\psi^p\,|0\rangle \;=\; \left(\psi^1\psi_1-\psi_1\psi^1\right)\psi^p\,|0\rangle \;=\; -k_{pq}\,\psi^q|0\rangle\;, \tag{5.4}$$

using the flat Lorentz metric $k=\mathrm{diag}(-1,1,\ldots,1)$ defined in (3.6). Using (5.2), (5.3) and (5.4), the action of S_g is then given by

$$\begin{aligned}
S_g\,\psi^i\,|0\rangle \;&=\; S_eS_kS_e^\dagger\,\psi^i\,|0\rangle \;=\; \frac{1}{\sqrt{\det e}}\,S_eS_k\,e_i{}^j\,\psi^j\,|0\rangle \;=\; -\frac{1}{\sqrt{\det e}}\,S_e\,e_i{}^j\,k_{jp}\,\psi^p\,|0\rangle\\
&=\; -\frac{1}{\det e}\,(e_i{}^j\,k_{jp}\,e_q{}^p)\psi^q\,|0\rangle \;=\; -\frac{1}{\det e}\,(e_i{}^j\,e_q{}^p\,k_{jp}\,)\psi^q\,|0\rangle\\
&=\; -\frac{1}{\det e}\,g_{iq}\,\psi^q\,|0\rangle \;=\; -\frac{1}{\sqrt{|\det g|}}\,g_{iq}\,\psi^q\,|0\rangle\;,
\end{aligned} \tag{5.5}$$

where we used the definition of the metric in (3.6) and wrote $\det e=\sqrt{|\det g|}$. Similarly, for S_g^{-1} one finds

$$S_g^{-1}\,\psi^i\,|0\rangle \;=\; -\sqrt{|\det g|}\;g^{ij}\,\psi^j\,|0\rangle\;\;, \tag{5.6}$$

where g^{ij} is, as usual, the inverse of the metric g_{ij}.

All of the above relations straightforwardly extend to the case where S_g acts on multiple fermionic oscillators, for which eqs. (5.5) and (5.6) are generalized to

$$\begin{aligned}
S_g^{-1}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle \;&=\; -\sqrt{|\det g|}\,g^{i_1j_1}\cdots g^{i_pj_p}\;\psi^{j_1}\cdots\psi^{j_p}\,|0\rangle\;,\\
S_g\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle \;&=\; -\frac{1}{\sqrt{|\det g|}}\,g_{i_1j_1}\cdots g_{i_pj_p}\,\psi^{j_1}\cdots\psi^{j_p}\,|0\rangle\;.
\end{aligned} \tag{5.7}$$

With these ingredients we are now ready to evaluate the action.

5.1.2 The action

We start by writing the action in the duality frame $\tilde{\partial}=0$. For this choice, the field strength

$$|F\rangle \equiv \not{\partial}|\chi\rangle\,, \tag{5.8}$$

reduces to

$$\begin{aligned}
|F\rangle\Big|_{\tilde{\partial}=0} \;&=\; \sum_{p=0}^{D}\frac{1}{p!}\partial_jC_{i_1\ldots i_p}\,\psi^j\psi^{i_1}\cdots\psi^{i_p}|0\rangle \;=\; \sum_{p=1}^{D}\frac{1}{(p-1)!}\partial_{[i_1}C_{i_2\ldots i_p]}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle\\
&=\; \sum_{p=1}^{D}\frac{1}{p!}F_{i_1\ldots i_p}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle\;,
\end{aligned} \tag{5.9}$$

where we performed an index shift and relabeled the indices. Thus, the components are given by the conventional field strengths

$$F_{i_1\ldots i_p} \;=\; p\,\partial_{[i_1}C_{i_2\ldots i_p]}\;. \tag{5.10}$$

It is sometimes useful to avoid explicit indices and combinatorial factors by using the language of differential forms. In general, we identify a spinor state $|G_p\rangle$ with a p-form $G^{(p)}$ as follows

$$|G_p\rangle = \frac{1}{p!}\, G_{i_1\cdots i_p}\, \psi^{i_1}\cdots\psi^{i_p}|0\rangle \;\longleftrightarrow\; G^{(p)} = \frac{1}{p!}\, G_{i_1\cdots i_p}\, dx^{i_1}\wedge\cdots\wedge dx^{i_p}\,. \tag{5.11}$$

Whenever we speak of a p-form $G^{(p)}$ and its components $G_{i_1\ldots i_p}$, we will assume a normalization that includes the $p!$ coefficient shown above. It is now straightforward to translate (5.10) to form language:

$$F^{(p)} = dC^{(p-1)}\,. \tag{5.12}$$

We now collect all field strengths of different degrees into a single form $F = \sum_p F^{(p)}$ and do the same for the potentials $C = \sum_p C^{(p)}$. We then have that (5.12), or for that matter (5.10), for all relevant p is summarized by

$$F = dC\,. \tag{5.13}$$

In order to evaluate the action we need to choose a parameterization for $\mathbb{S}$, which we take to be $S_{\mathcal{H}}$,

$$\mathbb{S} \;=\; S_{\mathcal{H}} \;=\; e^{\frac{1}{2}b_{ij}\psi_i\psi_j}\, S_g^{-1}\, e^{-\frac{1}{2}b_{ij}\psi^i\psi^j}\,. \tag{5.14}$$

The b-dependent terms in $S_{\mathcal{H}}$ suggest the definition of modified field strengths, related to the original field strengths $|F\rangle = \partial\!\!\!/|\chi\rangle$ by the addition of Chern-Simons like terms:

$$|\widehat{F}\rangle \;\equiv\; e^{-\frac{1}{2}b_{ij}\psi^i\psi^j}|F\rangle \;=\; \sum_{p=1}^{D}\frac{1}{p!}\widehat{F}_{i_1\ldots i_p}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle\,. \tag{5.15}$$

This relation is summarized in form language by

$$\widehat{F} \;=\; e^{-b^{(2)}}\wedge F \;=\; e^{-b^{(2)}}\wedge dC\,, \quad \text{with} \quad b^{(2)} \equiv \frac{1}{2}\, b_{ij}\, dx^i\wedge dx^j\,. \tag{5.16}$$

Explicitly, for example,

$$\begin{aligned}\widehat{F}^{(3)} \;&=\; F^{(3)} - b^{(2)}\wedge F^{(1)}\\ \widehat{F}^{(5)} \;&=\; F^{(5)} - b^{(2)}\wedge F^{(3)} \;+\; \frac{1}{2}b^{(2)}\wedge b^{(2)}\wedge F^{(1)}\,, \qquad \text{etc.}\end{aligned} \tag{5.17}$$

The bra corresponding to $|\widehat{F}\rangle$ is given by

$$\langle\widehat{F}| \;=\; \sum_{p=1}^{D}\frac{1}{p!}\langle 0|\psi_{i_p}\cdots\psi_{i_1}\,\widehat{F}_{i_1\ldots i_p}\,. \tag{5.18}$$

Next, we can evaluate the Lagrangian (4.6) using (5.14), (5.15) and (5.18), which yields

$$\mathcal{L} \;=\; \frac{1}{4}\langle\widehat{F}|\, S_g^{-1}\,|\widehat{F}\rangle \;=\; \frac{1}{4}\sum_{p,q=1}^{D}\frac{1}{q!p!}\widehat{F}_{i_1\ldots i_p}\widehat{F}_{j_1\ldots j_q}\langle 0|\psi_{i_p}\cdots\psi_{i_1}\, S_g^{-1}\,\psi^{j_1}\cdots\psi^{j_q}|0\rangle\,. \tag{5.19}$$

Using now (5.7) for the action of S_g^{-1} and the normalization

$$\langle 0|\psi_{i_p}\cdots\psi_{i_1}\psi^{m_1}\ldots\psi^{m_q}|0\rangle \;=\; \delta_{pq}\, p!\,\delta_{i_1}{}^{[m_1}\cdots\delta_{i_p}{}^{m_p]}\,, \tag{5.20}$$

following from $\langle 0|0\rangle = 1$, the action reduces to

$$\mathcal{L} \;=\; -\frac{1}{4}\sqrt{g}\sum_{p=1}^{D}\frac{1}{p!}\,g^{i_1 j_1}\cdots g^{i_p j_p}\,\widehat{F}_{i_1\ldots i_p}\widehat{F}_{j_1\ldots j_p}\;, \tag{5.21}$$

where we used the short-hand notation $\sqrt{g} = \sqrt{|\det g|}$. This can also be written as

$$\mathcal{L} \;=\; -\frac{1}{4}\sqrt{g}\sum_{p=1}^{D}|\widehat{F}^{(p)}|^2 \quad , \tag{5.22}$$

where we define for any p-form $\omega^{(p)}$:

$$|\omega^{(p)}|^2 \equiv \frac{1}{p!}\,g^{i_1 j_1}\cdots g^{i_p j_p}\,\omega_{i_1\ldots i_p}\omega_{j_1\ldots j_p}\;. \tag{5.23}$$

The result in (5.22) is the required sum of kinetic terms for all p-form gauge fields (of odd or even degree, depending on the chirality of χ), which appear in the democratic formulation. This action needs to be supplemented by the duality relations, ensuring that we propagate only the physical degrees of freedom of type II. We consider these next.

5.1.3 Self-duality relations in terms of field strengths

Here we show that for $\tilde{\partial} = 0$ the self-duality conditions $\not{\partial}\chi = -\mathcal{K}\not{\partial}\chi$, c.f. eq. (4.17), reduce to

$$\widehat{F}^{(p)} \;=\; (-1)^{\frac{(D-p)(D-p-1)}{2}}\,*\,\widehat{F}^{(D-p)}\;. \tag{5.24}$$

These are conventional duality relations for p-form field strengths. In here we use the following definition of the Hodge-dual form:

$$(*A)_{i_1\cdots i_p} \equiv \frac{1}{(D-p)!}\,g_{i_1 j_1}\cdots g_{i_p j_p}\varepsilon^{k_{p+1}\cdots k_D\, j_1\cdots j_p}A_{k_{p+1}\cdots k_D}\;. \tag{5.25}$$

Our conventions for the epsilon symbols are as follows:

$$\begin{aligned} \epsilon^{1\,2\ldots D} \;&=\; +1\;, \qquad \varepsilon^{i_1\ldots i_D} \;=\; \frac{1}{\sqrt{g}}\,\epsilon^{i_1\ldots i_D}\;,\\ \epsilon_{1\,2\ldots D} \;&=\; -1\;, \qquad \varepsilon_{i_1\ldots i_D} \;=\; \sqrt{g}\,\epsilon_{i_1\ldots i_D}\;, \end{aligned} \tag{5.26}$$

i.e., ϵ is a tensor density, while ε is a (pseudo-)tensor. As usual, lowering the indices on $\varepsilon^{i_1\cdots i_D}$ with g_{ij} yields $\varepsilon_{i_1\ldots i_D}$, and ε and ϵ coincide on flat space. We note the familiar relation for the square of the Hodge star on forms of degree p in a D-dimensional spacetime with signature s:

$$**\omega^{(p)} = (-1)^{p(D-p)}s\,\omega^{(p)}\,. \tag{5.27}$$

We can ask when is (5.24) consistent with repeated application of the Hodge star operation. A calculation gives the condition

$$s\,(-1)^{\frac{1}{2}D(D-1)} \;=\; 1\,. \tag{5.28}$$

Not surprisingly, in Lorentzian signature this agrees with the result in (4.19). Finally, for $D = 10$, the duality constraints (5.24) take the form

$$\widehat{F}^{(p)} \;=\; -(-1)^{\frac{1}{2}p(p+1)}\,*\,\widehat{F}^{(D-p)}\;. \tag{5.29}$$

We can now begin our calculation. Let us first introduce the short-hand notation

$$\mathbf{B} \;=\; \tfrac{1}{2} b_{ij} \psi^i \psi^j \,, \qquad \mathbf{B}^\dagger \;=\; -\tfrac{1}{2} b_{ij}\, \psi_i \psi_j \,, \tag{5.30}$$

which allows us to write $S_{\mathcal{H}}$ in (3.14) as follows

$$S_{\mathcal{H}} \;=\; e^{-\mathbf{B}^\dagger} S_g^{-1}\, e^{-\mathbf{B}} \,. \tag{5.31}$$

The self-duality conditions $\partial\!\!\!/\chi = -\mathcal{K}\partial\!\!\!/\chi$ can now be written as

$$e^{-\mathbf{B}} |\partial\!\!\!/\chi\rangle \;=\; -e^{-\mathbf{B}}\, C^{-1}\, e^{-\mathbf{B}^\dagger} S_g^{-1}\, e^{-\mathbf{B}} |\partial\!\!\!/\chi\rangle \,, \tag{5.32}$$

where we multiplied the factor $e^{-\mathbf{B}}$ from the left to form the modified field strengths $|\widehat{F}\rangle$ defined in (5.15):

$$|\widehat{F}\rangle \;=\; -e^{-\mathbf{B}}\, C^{-1}\, e^{-\mathbf{B}^\dagger}\, S_g^{-1}\, |\widehat{F}\rangle \,. \tag{5.33}$$

Using (2.45) we readily verify that

$$C e^{-\mathbf{B}} C^{-1} = e^{-C\mathbf{B}C^{-1}} = e^{-\frac{1}{2} b_{ij} \psi_i \psi_j} = e^{\mathbf{B}^\dagger} \,, \tag{5.34}$$

and, as a result,

$$|\widehat{F}\rangle \;=\; -C^{-1}\, S_g^{-1} |\widehat{F}\rangle \,. \tag{5.35}$$

A further simplification is possible using (3.13) in the form $S_g = -C^{-1} S_g^{-1} C$, giving

$$|\widehat{F}\rangle \;=\; S_g\, C^{-1} |\widehat{F}\rangle \,. \tag{5.36}$$

Finally, we recall that in the dimensions with self-consistent duality constraints (4.19) we have $C^{-1} = -C$ and therefore

$$|\widehat{F}\rangle \;=\; -\, S_g\, C |\widehat{F}\rangle \,. \tag{5.37}$$

This is the simplest possible form of the duality constraints.

We can now examine (5.37) in terms of component fields, as defined in (5.15). We find

$$\sum_{p=1}^{D} \frac{1}{p!} \widehat{F}_{i_1 \ldots i_p}\, \psi^{i_1} \cdots \psi^{i_p} |0\rangle \;=\; -\sum_{p=1}^{D} \frac{1}{p!}\, \widehat{F}_{i_1 \ldots i_p}\; S_g\; \psi_{i_1} \cdots \psi_{i_p}\, C\, |0\rangle \,, \tag{5.38}$$

where we used (2.45). Next, we show that the charge conjugation matrix in (5.38) effectively acts like an epsilon symbol. In fact, by multiple application of the oscillator algebra one can verify that

$$\begin{aligned} \psi_{i_1} \cdots \psi_{i_p}\, C\, |0\rangle \;&=\; \psi_{i_1} \cdots \psi_{i_p} \psi^1 \psi^2 \cdots \psi^D |0\rangle \\ &=\; \frac{1}{(D-p)!} \epsilon^{i_p i_{p-1} \cdots i_1\, j_{p+1} \cdots j_D}\, \psi^{j_{p+1}} \cdots \psi^{j_D} |0\rangle \\ &=\; \frac{1}{(D-p)!} (-1)^{\frac{p(p-1)}{2}} \epsilon^{i_1 i_2 \cdots i_p\, j_{p+1} \cdots j_D}\, \psi^{j_{p+1}} \cdots \psi^{j_D} |0\rangle \,. \end{aligned} \tag{5.39}$$

Back in (5.38) and defining $\tilde{p} = D - p$ we have

$$
\begin{aligned}
&\sum_{p=1}^{D} \frac{1}{p!} \widehat{F}_{i_1 \ldots i_p}\, \psi^{i_1} \cdots \psi^{i_p} |0\rangle \\
&= -\sum_{p=1}^{D} (-1)^{\frac{p(p-1)}{2}} \frac{1}{p!\,\tilde{p}!}\, \widehat{F}_{i_1 \ldots i_p}\, \epsilon^{i_1 i_2 \cdots i_p\, j_1 \cdots j_{\tilde{p}}}\, S_g\, \psi^{j_1} \cdots \psi^{j_{\tilde{p}}} |0\rangle \\
&= \sum_{p=1}^{D} (-1)^{\frac{p(p-1)}{2}} \frac{1}{p!\,\tilde{p}!} \frac{1}{\sqrt{g}}\, \epsilon^{i_1 i_2 \cdots i_p\, j_1 \cdots j_{\tilde{p}}} \widehat{F}_{i_1 \ldots i_p}\, g_{j_1 k_1} \cdots g_{j_{\tilde{p}} k_{\tilde{p}}}\, \psi^{k_1} \cdots \psi^{k_{\tilde{p}}} |0\rangle \\
&= \sum_{p=1}^{D} (-1)^{\frac{p(p-1)}{2}} \frac{1}{\tilde{p}!} \frac{1}{p!}\, g_{k_1 j_1} \cdots g_{k_{\tilde{p}} j_{\tilde{p}}}\, \varepsilon^{i_1 i_2 \cdots i_p\, j_p \cdots j_{\tilde{p}}}\, \widehat{F}_{i_1 \ldots i_p}\, \psi^{k_1} \cdots \psi^{k_{\tilde{p}}} |0\rangle \\
&= \sum_{p=1}^{D} (-1)^{\frac{p(p-1)}{2}} \frac{1}{\tilde{p}!}\, (*\widehat{F})_{k_1 \cdots k_{\tilde{p}}} \psi^{k_1} \cdots \psi^{k_{\tilde{p}}} |0\rangle \\
&= \sum_{p=1}^{D} (-1)^{\frac{(D-p)(D-p-1)}{2}} \frac{1}{p!}\, (*\widehat{F})_{i_1 \cdots i_p}\, \psi^{i_1} \cdots \psi^{i_p} |0\rangle\ .
\end{aligned}
\tag{5.40}
$$

In obtaining this result we made use of (5.7), the definition (5.25) and some simple manipulations. Thus, we have shown that the duality constraint implies the claimed duality relations (5.24).

5.2 Conventional gauge symmetries

Let us now verify that the gauge transformations parameterized by ξ^M and λ reduce to the conventional gauge symmetries of type II theories in the frame $\tilde{\partial}^i = 0$. We start with the p-form gauge symmetries (4.22) whose parameter we write in components as

$$
|\lambda\rangle \;=\; \sum_{p=0}^{D} \frac{1}{p!} \lambda_{i_1 \ldots i_p}\, \psi^{i_1} \cdots \psi^{i_p} |0\rangle\ . \tag{5.41}
$$

For $\tilde{\partial} = 0$ this implies

$$
\delta_\lambda |\chi\rangle \;=\; \partial\!\!\!/ |\lambda\rangle \;=\; \psi^j \partial_j |\lambda\rangle \;=\; \sum_{p=1}^{D} \frac{1}{(p-1)!} \partial_{[i_1} \lambda_{i_2 \ldots i_p]}\, \psi^{i_1} \cdots \psi^{i_p} |0\rangle\ , \tag{5.42}
$$

from which we read off

$$
\delta_\lambda C_{i_1 \ldots i_p} \;=\; p\, \partial_{[i_1} \lambda_{i_2 \ldots i_p]}\ . \tag{5.43}
$$

These are the conventional p-form gauge transformations. In form language they read

$$
\delta_\lambda\, C \;=\; d\lambda\, . \tag{5.44}
$$

Let us now discuss the gauge transformations parameterized by $\xi^M = (\tilde{\xi}_i, \xi^i)$. We first claim that the C forms transform as p-forms under diffeomorphisms parameterized by ξ^i. To see this, we compute

$$\delta_\xi|\chi\rangle \;=\; \Big(\xi^j\partial_j + \partial_j\xi^k\,\psi^j\psi_k\Big)\sum_{p=0}^{D}\frac{1}{p!}\,C_{i_1\ldots i_p}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle\;. \tag{5.45}$$

The transport term just gives rise to the transport term of the component fields. The second term can be evaluated using (2.13), which then implies for the components

$$\delta_\xi C_{i_1\ldots i_p} \;=\; \xi^j\partial_j C_{i_1\ldots i_p} + p\,\partial_{[i_1}\xi^j\,C_{|j|i_2\ldots i_p]} \;\equiv\; \mathcal{L}_\xi C_{i_1\ldots i_p}\;. \tag{5.46}$$

This is the usual diffeomorphism symmetry which infinitesimally acts via the Lie derivative.

We now consider the $\tilde{\xi}_i$ parameters, which are parameters for the b-field gauge transformations. It turns out that the C forms transform non-trivially under this symmetry. In order to see this we compute for $\tilde{\partial}=0$

$$\begin{aligned}\delta_{\tilde{\xi}}|\chi\rangle \;&=\; \partial_k\tilde{\xi}_l\,\psi^k\psi^l|\chi\rangle \;=\; \sum_{p=0}^{D}\frac{1}{p!}\partial_{[i_1}\tilde{\xi}_{i_2}\,C_{i_3\ldots i_{p+2}]}\psi^{i_1}\cdots\psi^{i_{p+2}}|0\rangle\\ &=\; \sum_{p=2}^{D}\frac{1}{(p-2)!}\partial_{[i_1}\tilde{\xi}_{i_2}\,C_{i_3\ldots i_p]}\psi^{i_1}\cdots\psi^{i_p}|0\rangle\;,\end{aligned} \tag{5.47}$$

where we performed an index shift $p\to p+2$ in the last equation. We thus read off

$$\delta_{\tilde{\xi}}C_{i_1\ldots i_p} \;=\; p(p-1)\partial_{[i_1}\tilde{\xi}_{i_2}\,C_{i_3\ldots i_p]}\;. \tag{5.48}$$

In the language of forms the above equation reads

$$\delta_{\tilde{\xi}}C \;=\; d\tilde{\xi}\wedge C\,. \tag{5.49}$$

Note that this implies that

$$\delta_{\tilde{\xi}}C^{(0)} = \delta_{\tilde{\xi}}C^{(1)} = 0\,,\quad \delta_{\tilde{\xi}}C^{(2)} = d\tilde{\xi}\cdot C^{(0)}\,,\quad \ldots\;,\quad \delta_{\tilde{\xi}}C^{(p)} = d\tilde{\xi}\wedge C^{(p-2)}\,. \tag{5.50}$$

Recalling that

$$\delta_{\tilde{\xi}}\,b^{(2)} = d\tilde{\xi}\,, \tag{5.51}$$

it is straightforward to define new potentials $\hat{A}$ that do not transform under $\tilde{\xi}$:

$$\hat{A} \;\equiv\; e^{-b^{(2)}}\wedge C\,. \tag{5.52}$$

Indeed it is simple to verify that

$$\delta_{\tilde{\xi}}\,\hat{A} = 0\,. \tag{5.53}$$

One can also write the C fields in terms of the $\hat{A}$ fields easily:

$$C \;=\; e^{b^{(2)}}\wedge\hat{A}\,. \tag{5.54}$$

The $\hat{A}$ potentials are hatted to distinguish them from conventional type II potentials to be discussed below.

5.3 Democratic formulation

The democratic formulation of type II theories introduces an action for all even and odd forms, which is then supplemented by duality relations between the corresponding field strengths. The resulting equations of motion are equivalent to the standard equations of motion by virtue of the Bianchi identities of the field strengths [28, 33]. Here we briefly introduce this formulation and show the equivalence with the conventional formulation.

5.3.1 Review and comments on the standard formulation

The standard 10-dimensional low energy action for type II theories is given by

$$S \;=\; S_{\mathrm{NS-NS}} + S_{\mathrm{RR}}\;, \tag{5.55}$$

where $S_{\mathrm{NS-NS}}$ is the same for both type IIA and type IIB and written as

$$S_{\mathrm{NS-NS}} = \int d^{10}x\sqrt{-g}e^{-2\phi}\left[R + 4(\partial\phi)^2 - \frac{1}{2}|H^{(3)}|^2\right]\;. \tag{5.56}$$

The RR actions S_{RR} for type IIA and type IIB are given by, respectively,

$$S_{\mathrm{RR}}^{(\mathrm{IIA})} \;=\; -\frac{1}{2}\int d^{10}x\sqrt{-g}\left(|\widehat{F}^{(2)}|^2 + |\widehat{F}^{(4)}|^2\right) + \frac{1}{2}\int b^{(2)}\wedge dA^{(3)}\wedge dA^{(3)}\;, \tag{5.57}$$

$$S_{\mathrm{RR}}^{(\mathrm{IIB})} \;=\; -\frac{1}{2}\int d^{10}x\sqrt{-g}\left(|\widehat{F}^{(1)}|^2 + |\widehat{F}^{(3)}|^2 + \frac{1}{2}|\widehat{F}^{(5)}|^2\right) + \;\frac{1}{2}\int b^{(2)}\wedge dA^{(4)}\wedge dA^{(2)}\,, \tag{5.58}$$

with the additional self-duality condition $\widehat{F}^{(5)} = *\widehat{F}^{(5)}$ for type IIB, which has to be imposed on the field equations after varying the action. We also note that the type II* theories take the same form, with the overall sign of the kinetic terms for the RR fields (but not of the Chern-Simons terms) reversed. The field strengths $\widehat{F}^{(n)}$ are defined in terms of the original RR potentials $A^{(n)}$ as

$$\begin{array}{ll}
\widehat{F}^{(1)} \equiv dA^{(0)} & \widehat{F}^{(2)} \equiv dA^{(1)} \\
\widehat{F}^{(3)} \equiv dA^{(2)} + H^{(3)}\wedge A^{(0)} & \widehat{F}^{(4)} \equiv dA^{(3)} + H^{(3)}\wedge A^{(1)} \\
\widehat{F}^{(5)} \equiv dA^{(4)} + \tfrac{1}{2}H^{(3)}\wedge A^{(2)} - \tfrac{1}{2}b^{(2)}\wedge dA^{(2)}\,. &
\end{array} \tag{5.59}$$

The field strengths above must be invariant under p-form gauge transformations of the potentials. But the presence of A-forms without an exterior derivative acting on them implies that the p-form gauge transformations of A's are a bit nontrivial:

$$\begin{array}{ll}
\delta_\lambda A^{(0)} = \;0 & \delta_\lambda A^{(1)} = \;d\lambda^{(0)} \\
\delta_\lambda A^{(2)} = \;d\lambda^{(1)} & \delta_\lambda A^{(3)} = \;d\lambda^{(2)} - b^{(2)}\wedge d\lambda^{(0)} \\
\delta_\lambda A^{(4)} = \;d\lambda^{(3)} - \tfrac{1}{2}\,b^{(2)}\wedge d\lambda^{(1)}\,. &
\end{array} \tag{5.60}$$

One can readily verify that $\delta_\lambda\widehat{F}^{(p)} = 0$ and that the Chern-Simons terms are invariant because the integrands change by a d-exact form. Since the $\widehat{F}$'s involve the field $b^{(2)}$, the A potentials are not invariant under the $b^{(2)}$ gauge transformations, $\delta_{\tilde{\xi}}b^{(2)} = d\xi$. The invariance of the $\widehat{F}$'s requires

$$\begin{array}{ll}
\delta_{\tilde{\xi}}A^{(0)} = \;0 & \delta_{\tilde{\xi}}A^{(1)} = \;0 \\
\delta_{\tilde{\xi}}A^{(2)} = \;0 & \delta_{\tilde{\xi}}A^{(3)} = \;0 \\
\delta_{\tilde{\xi}}A^{(4)} = \;\tfrac{1}{2}\,d\tilde{\xi}\wedge A^{(2)}\,. &
\end{array} \tag{5.61}$$

One can readily verify that $\delta_{\tilde{\xi}}\widehat{F}^{(p)} = 0$ and that the Chern-Simons terms are invariant because the integrands change by a d-exact form (use $dA^{(2)} \wedge dA^{(2)} = 0$, for the IIB case).

A set of modified RR potentials $C^{(n)}$ are constructed by combining the NS-NS 2-form $b^{(2)}$ and the original RR potentials $A^{(n)}$:

$$\begin{aligned}
&C^{(0)} \equiv A^{(0)} && C^{(1)} \equiv A^{(1)} \\
&C^{(2)} \equiv A^{(2)} + b^{(2)} \wedge A^{(0)} && C^{(3)} \equiv A^{(3)} + b^{(2)} \wedge A^{(1)} \\
&C^{(4)} \equiv A^{(4)} + \tfrac{1}{2} b^{(2)} \wedge A^{(2)} + \tfrac{1}{2} b^{(2)} \wedge b^{(2)} \wedge A^{(0)} \,. &&
\end{aligned} \tag{5.62}$$

These transformations have one peculiar feature. The field C fails to be equal to $e^{b^{(2)}} \wedge A$ because of the terms in $C^{(4)}$. As we will argue below, this is because matters can be simplified by using a different $A^{(4)}$ field. The inverse relations are

$$\begin{aligned}
&A^{(0)} = C^{(0)} && A^{(1)} = C^{(1)} \\
&A^{(2)} = C^{(2)} - b^{(2)} \wedge C^{(0)} && A^{(3)} \equiv C^{(3)} - b^{(2)} \wedge C^{(1)} \\
&A^{(4)} \equiv C^{(4)} - \tfrac{1}{2} b^{(2)} \wedge C^{(2)} \,. &&
\end{aligned} \tag{5.63}$$

We now claim that the C fields defined above coincide with the C fields we have been using in this paper; the fields that transform naturally under T-duality and have conventional p-form gauge transformations. Indeed, a short calculation shows that the p-form gauge transformations in (5.60) imply

$$\delta_\lambda C = d\lambda \,, \tag{5.64}$$

in agreement with (5.44). Moreover, the $\tilde{\xi}$ gauge transformations in (5.61) imply that the $\tilde{\xi}$ gauge transformations of the C fields are summarized by

$$\delta_{\tilde{\xi}} C = d\tilde{\xi} \wedge C \,, \tag{5.65}$$

in agreement with (5.49). Finally, the field strengths $\widehat{F}$ take a simple form in terms of the C forms

$$\widehat{F} = e^{-b^{(2)}} \wedge dC \;, \tag{5.66}$$

in agreement with (5.16). The desired properties $\delta_\lambda \widehat{F} = 0$ and $\delta_{\tilde{\xi}}\widehat{F} = 0$ are now manifest.

We noted earlier in (5.52) that the potentials

$$\hat{A} = e^{-b^{(2)}} \wedge C \,, \tag{5.67}$$

are invariant under $\delta_{\tilde{\xi}}$. Comparing with (5.63) we see that

$$\hat{A}^{(p)} = A^{(p)} \,, \quad p \neq 4 \,, \tag{5.68}$$

and for the case $p = 4$ a short calculation shows that

$$A^{(4)} = \hat{A}^{(4)} + \frac{1}{2} b^{(2)} \wedge \hat{A}^{(2)} \,. \tag{5.69}$$

This equation is consistent with (5.61) and $\delta_{\tilde{\xi}}\hat{A}^{(4)} = 0$. Moreover, using (5.54) and (5.66) we find

$$\widehat{F} = e^{-b^{(2)}} \wedge d\Big(e^{-b^{(2)}} \wedge A\Big) \,, \tag{5.70}$$

which quickly yields

$$\widehat{F} = d\hat{A} + H^{(3)} \wedge \hat{A} \,. \tag{5.71}$$

The above means that a formulation with potentials $\hat{A}$ is somewhat more efficient than the conventional formulation. Indeed, a small calculation shows that the type IIB Chern-Simons term, expressed in terms of $\hat{A}^{(4)}$ and $\hat{A}^{(2)}$ takes exactly the same form as before, thus

$$S^{(\mathrm{IIA})}_{\mathrm{RR}} = -\frac{1}{2}\int d^{10}x\sqrt{-g}\left(|\widehat{F}^{(2)}|^2 + |\widehat{F}^{(4)}|^2\right) + \frac{1}{2}\int b^{(2)} \wedge d\hat{A}^{(3)} \wedge d\hat{A}^{(3)} \,, \tag{5.72}$$

$$S^{(\mathrm{IIB})}_{\mathrm{RR}} = -\frac{1}{2}\int d^{10}x\sqrt{-g}\left(|\widehat{F}^{(1)}|^2 + |\widehat{F}^{(3)}|^2 + \frac{1}{2}|\widehat{F}^{(5)}|^2\right) + \frac{1}{2}\int b^{(2)} \wedge d\hat{A}^{(4)} \wedge d\hat{A}^{(2)} \,. \tag{5.73}$$

Here, collecting information,

$$\widehat{F} = d\hat{A} + H^{(3)} \wedge \hat{A} \,, \quad \delta_\lambda \hat{A} = e^{-b^{(2)}} \wedge d\lambda \,, \quad \delta_{\tilde{\xi}} \hat{A} = 0 \,. \tag{5.74}$$

The advantage of this formulation is that the haphazard δ_λ transformations of the A's now takes a closed form expression and the $\delta_{\tilde{\xi}}$ symmetry is manifest.

The formulation in terms of C potentials is also elegant as it brings out the duality properties more clearly. This time the Chern-Simons terms are a bit more complex, however. One finds

$$\begin{aligned}
S^{(\mathrm{IIA})}_{\mathrm{RR}} &= -\frac{1}{2}\int d^{10}x\sqrt{-g}\left(|\widehat{F}^{(2)}|^2 + |\widehat{F}^{(4)}|^2\right) \\
&\quad + \frac{1}{2}\int b^{(2)} \wedge d\Big(C^{(3)} - b^{(2)} \wedge C^{(1)}\Big) \wedge d\Big(C^{(3)} - b^{(2)} \wedge C^{(1)}\Big) \,, \\
S^{(\mathrm{IIB})}_{\mathrm{RR}} &= -\frac{1}{2}\int d^{10}x\sqrt{-g}\left(|\widehat{F}^{(1)}|^2 + |\widehat{F}^{(3)}|^2 + \frac{1}{2}|\widehat{F}^{(5)}|^2\right) \\
&\quad + \frac{1}{2}\int b^{(2)} \wedge d\Big(C^{(4)} - \frac{1}{2}b^{(2)} \wedge C^{(2)}\Big) \wedge d\Big(C^{(2)} - b^{(2)} \wedge C^{(0)}\Big) \,, \\
\widehat{F} &= e^{-b^{(2)}} \wedge dC \,, \quad \delta_\lambda C = d\lambda \,, \quad \delta_{\tilde{\xi}} C = d\tilde{\xi} \wedge C \,.
\end{aligned} \tag{5.75}$$

5.3.2 Democratic formulation and equivalence

Let us now turn to the democratic formulation. The democratic action features kinetic terms for all forms, but no explicit Chern-Simons terms,

$$S^{(\mathrm{IIA}),\mathrm{dem}}_{\mathrm{RR}} = -\frac{1}{4}\int d^{10}x\sqrt{-g}\sum_{n=2,4,6,8}|\widehat{F}^{(n)}|^2 = \frac{1}{4}\int \sum_{n=2,4,6,8}\widehat{F}^{(n)} \wedge *\widehat{F}^{(n)} \,, \tag{5.76}$$

$$S^{(\mathrm{IIB}),\mathrm{dem}}_{\mathrm{RR}} = -\frac{1}{4}\int d^{10}x\sqrt{-g}\sum_{n=1,3,5,7,9}|\widehat{F}^{(n)}|^2 = \frac{1}{4}\int \sum_{n=1,3,5,7,9}\widehat{F}^{(n)} \wedge *\widehat{F}^{(n)} \,. \tag{5.77}$$

Note that the normalization of the action has a factor of 1/2 relative to similar (non Chern-Simons) terms in the standard formulation (5.57). The above actions are supplemented by the duality relations

$$\begin{aligned}
*\widehat{F}^{(1)} &= \widehat{F}^{(9)}\,, & *\widehat{F}^{(2)} &= -\widehat{F}^{(8)}\,,\\
*\widehat{F}^{(3)} &= -\widehat{F}^{(7)}\,, & *\widehat{F}^{(4)} &= \widehat{F}^{(6)}\,,\\
*\widehat{F}^{(5)} &= \widehat{F}^{(5)}\,, & *\widehat{F}^{(6)} &= -\widehat{F}^{(4)}\,,\\
*\widehat{F}^{(7)} &= -\widehat{F}^{(3)}\,, & *\widehat{F}^{(8)} &= \widehat{F}^{(2)}\,,\\
*\widehat{F}^{(9)} &= \widehat{F}^{(1)}\,.
\end{aligned} \tag{5.78}$$

The above are indeed the duality constraints we obtained before at the component level, as can be readily checked using (5.29). Moreover, our action, evaluated for $\tilde{\partial} = 0$, is also identical to the above democratic action. This can be seen in (5.22), where we also recall in (5.23) the definition of $|\ldots|^2$ on forms. By showing that the democratic formulation agrees with the standard formulation we will have shown that our double field theory type II action, for $\tilde{\partial} = 0$, agrees with the type II theories. We note that the democratic formulation of the type II* theories is completely analogous; it introduces a RR action with a kinetic term for each potential together with duality constraints, both with a reversed overall sign.

The claim is that the field equations of the standard action are the same as those of the democratic action after imposing these duality relations. We present this equivalence for type IIA. The story for type IIB is completely analogous.

The field equation for g_{ij} is relatively straightforward. Since the NS-NS action is the same for the standard formulation and the democratic formulation, it is sufficient to examine the energy-momentum tensor for both formulations. The energy-momentum tensor T_{ij} in the standard formulation does not receive contributions from the Chern-Simons terms. We simply have

$$T_{ij} = \mathcal{E}_{ij}(\widehat{F}^{(2)}) + \mathcal{E}_{ij}(\widehat{F}^{(4)})\,, \tag{5.79}$$

where

$$\mathcal{E}_{ij}(\widehat{F}^{(n)}) \;\equiv\; \frac{1}{(n-1)!}\widehat{F}_{ik_1k_2\cdots k_{n-1}}\widehat{F}_j{}^{k_1k_2\cdots k_{n-1}} - \frac{1}{2}g_{ij}|\widehat{F}^{(n)}|^2\,. \tag{5.80}$$

The energy-momentum tensor resulting from the democratic action is given by

$$T_{ij} = \frac{1}{2}\sum_{n=2,4,6,8}\mathcal{E}_{ij}(\widehat{F}^{(n)})\,, \tag{5.81}$$

where the 1/2 factor is due to the normalization, as mentioned above (5.78). From the identity $\mathcal{E}_{ij}(\widehat{F}^{(n)}) = \mathcal{E}_{ij}(*\widehat{F}^{(n)})$ and the duality relations, we infer that the energy-momentum tensor in the democratic action is equal to that in the standard action. Both formulations give the same Einstein equations.

In the standard formulation (c.f. (5.56) and (5.57)) the field equation for $b^{(2)}$ is

$$d(e^{-2\phi} * H^{(3)}) + \widehat{F}^{(2)} \wedge *\widehat{F}^{(4)} - \frac{1}{2}\widehat{F}^{(4)} \wedge \widehat{F}^{(4)} \;=\; 0\,, \tag{5.82}$$

and in the democratic formulation (5.76) with (5.78) the field equation for $b^{(2)}$ reads

$$d(e^{-2\phi} * H^{(3)}) + \frac{1}{2}\widehat{F}^{(2)} \wedge *\widehat{F}^{(4)} + \frac{1}{2}\widehat{F}^{(4)} \wedge *\widehat{F}^{(6)} + \frac{1}{2}\widehat{F}^{(6)} \wedge *\widehat{F}^{(8)} \;=\; 0\,, \tag{5.83}$$

which is equivalent to (5.82) after imposing the duality relations (5.78).

The most nontrivial checks in the equivalence of the two formulations are the field equations for $C^{(n)}$. In the standard formulation we have nontrivial Bianchi identities from (5.66):

$$d\widehat{F}^{(n)} = -H^{(3)} \wedge \widehat{F}^{(n-2)} \,. \tag{5.84}$$

The field equations for $C^{(1)}$ and $C^{(3)}$ in the standard formulation read, respectively,

$$\begin{aligned} 0 &= d\left(- * \widehat{F}^{(2)} + b^{(2)} \wedge *\widehat{F}^{(4)} + \frac{1}{2} b^{(2)} \wedge b^{(2)} \wedge \widehat{F}^{(4)} + \frac{1}{6} b^{(2)} \wedge b^{(2)} \wedge b^{(2)} \wedge \widehat{F}^{(2)}\right) \,, \\ 0 &= d\left(*\widehat{F}^{(4)} + b^{(2)} \wedge \widehat{F}^{(4)} + \frac{1}{2} b^{(2)} \wedge b^{(2)} \wedge \widehat{F}^{(2)}\right) \,. \end{aligned} \tag{5.85}$$

In the democratic formulation the field equations for all odd forms $C^{(1)}, C^{(3)}, C^{(5)}, C^{(7)}$ are respectively given by

$$\begin{aligned} 0 &= d\left(- * \widehat{F}^{(2)} + b^{(2)} \wedge *\widehat{F}^{(4)} - \frac{1}{2} b^{(2)} \wedge b^{(2)} \wedge *\widehat{F}^{(6)} + \frac{1}{6} b^{(2)} \wedge b^{(2)} \wedge b^{(2)} \wedge *\widehat{F}^{(8)}\right) \,, \\ 0 &= d\left(*\widehat{F}^{(4)} - b^{(2)} \wedge *\widehat{F}^{(6)} + \frac{1}{2} b^{(2)} \wedge b^{(2)} \wedge *\widehat{F}^{(8)}\right) \,, \\ 0 &= d\left(- * \widehat{F}^{(6)} + b^{(2)} \wedge *\widehat{F}^{(8)}\right) \,, \\ 0 &= d\left(*\widehat{F}^{(8)}\right) \,. \end{aligned} \tag{5.86}$$

By imposing the duality relations (5.78) the last two equations become the Bianchi identities for $\widehat{F}^{(4)}$ and $\widehat{F}^{(2)}$ in (5.84) and the first two equations are equivalent to the field equations (5.85) for $C^{(1)}$ and $C^{(3)}$. In summary, for the common potentials the equations of motion agree after use of duality relations. For the potentials in the democratic formulation that are absent in the standard formulation, the democratic equations of motion arise from the Bianchi identities of the potentials in the standard formulation. The analysis given above explicitly shows that in the democratic formulation the field equations are equivalent to those of the standard formulation.

6 IIA versus IIB

Here we consider double field theory evaluated in frames with $\tilde{\partial}^i \neq 0$. In the first part, we review the results of [13] for the NS-NS sector and give an intuitive picture of how this generalizes to the RR sector. In the second part, we give a more explicit treatment of the RR action when evaluated in different T-duality frames.

6.1 Review of NS-NS sector and motivation for RR fields

In the previous section we have seen that for fields with no $\tilde{x}$ dependence or, equivalently, setting $\tilde{\partial}^i = 0$, the proposed double field theory reduces to the type IIA or type IIB theory in the democratic formulation, depending on the chosen chirality of χ. It is equally consistent with the strong constraint, however, to keep the $\tilde{x}$ dependence of fields while dropping the x dependence by setting $\partial_i = 0$. We will see that if the theory reduces to type IIA when setting $\tilde{\partial}^i = 0$, the same theory reduces to type IIA* when setting $\partial_i = 0$, and vice versa. Similarly, for the opposite chirality of χ, in one frame the theory reduces to type IIB and in the other frame to type IIB*.

More generally, we can consider intermediate frames that originate from the $\tilde{x}_i = 0$ frame by an arbitrary $O(D,D)$ transformation. Specifically, with the subgroup $O(n-1,1)\times O(d,d) \subset O(D,D)$ acting on coordinates $(x^\mu, x^a, \tilde{x}_a)$, with $\mu = 0,\ldots,n-1$ and $a = 1,\ldots,d$, we can consider the $O(d,d)$ transformation that maps the $\tilde{x}_a = 0$ frame to the $x^a = 0$ frame. Here we find that the resulting theory is equivalent to the original one if d is even or to the theory with opposite chirality if d is odd. In other words, for d odd, if we start with a chirality such that the theory reduces to IIA for $\tilde{x}_a = 0$, the same theory reduces to type IIB for $x^a = 0$, and vice versa.

In order to set the stage to discuss the above claims, let us first review the transition from the $\tilde{x} = 0$ frame to the $x = 0$ frame for the pure NS-NS sector. This matter was analyzed in sec. 3.2 of [13]. The two T-duality frames $\tilde{\partial}^i = 0$ and $\partial_i = 0$ are mapped into each other by the $O(D,D)$ transformation J that exchanges x and $\tilde{x}$,

$$J^M{}_N \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{6.1}$$

The action evaluated in one duality frame is equivalent to the action evaluated in the other duality frame, but written in terms of field variables that are redefined according to the $O(D,D)$ transformation (6.1). To make this more explicit, we introduce

$$\tilde{\mathcal{H}} \;\equiv\; J\,\mathcal{H}\,J \;=\; \mathcal{H}^{-1} . \tag{6.2}$$

In components, we obtain

$$\tilde{\mathcal{H}} \;=\; \begin{pmatrix} g_{ij} - b_{ik}g^{kl}b_{lj} & b_{ik}g^{kj} \\ -g^{ik}b_{kj} & g^{ij} \end{pmatrix} . \tag{6.3}$$

If we view $\tilde{\mathcal{H}}$ as the generalized metric associated with a new metric g' and a new antisymmetric field b', following (1.1) we would write

$$\tilde{\mathcal{H}} \;=\; \begin{pmatrix} g'^{ij} & -g'^{ik}b'_{kj} \\ b'_{ik}g'^{kj} & g'_{ij} - b'_{ik}g'^{kl}b'_{lj} \end{pmatrix} = \begin{pmatrix} \tilde{g}_{ij} & -\tilde{g}_{ik}\tilde{b}^{kj} \\ \tilde{b}^{ik}\tilde{g}_{kj} & \tilde{g}^{ij} - \tilde{b}^{ik}\tilde{g}_{kl}\tilde{b}^{lj} \end{pmatrix} , \tag{6.4}$$

where in the second step we defined the tilde fields by

$$\tilde{g}^{ij} \equiv g'_{ij} \quad\to\quad \tilde{g}_{ij} = g'^{\,ij} , \qquad \text{and} \quad \tilde{b}^{ij} \equiv b'_{ij} . \tag{6.5}$$

Note that the change of index position in passing from primed to tilde variables makes the right-hand sides of (6.3) and (6.4) have consistent index positions:

$$\tilde{\mathcal{H}} \;=\; \begin{pmatrix} g_{ij} - b_{ik}g^{kl}b_{lj} & b_{ik}g^{kj} \\ -g^{ik}b_{kj} & g^{ij} \end{pmatrix} \;\equiv\; \begin{pmatrix} \tilde{g}_{ij} & -\tilde{g}_{ik}\tilde{b}^{kj} \\ \tilde{b}^{ik}\tilde{g}_{kj} & \tilde{g}^{ij} - \tilde{b}^{ik}\tilde{g}_{kl}\tilde{b}^{lj} \end{pmatrix} . \tag{6.6}$$

The dilaton is invariant under this inversion duality: $\tilde{d} = d$.

Let us verify directly that the field redefinition in (6.6) is equivalent to the change of variable induced by T-duality, following eqs. (3.20)–(3.22) of [13]. In there, we considered the fundamental field $\mathcal{E}_{ij} = g_{ij} + b_{ij}$ represented by the matrix $\mathcal{E}$ and the T-dual field $\tilde{\mathcal{E}} = \mathcal{E}^{-1}$, writing

$$\tilde{\mathcal{E}}^{ij} \;\equiv\; \left(\mathcal{E}^{-1}\right)^{ij} \;\equiv\; \tilde{g}^{ij} + \tilde{b}^{ij} \qquad \Rightarrow \qquad \mathcal{E}_{ik}\tilde{\mathcal{E}}^{kj} \;=\; \delta_i{}^j \;, \tag{6.7}$$

where $\tilde{g}^{ij}$ and $\tilde{b}^{ij}$ are the symmetric and antisymmetric parts of $\tilde{\mathcal{E}}^{ij}$, respectively. Consequently, $\tilde{g}^{ij}$ is interpreted as the metric and $\tilde{g}_{ij}$ denotes the inverse metric. The duality transformations of the metric imply that they satisfy [13]:

$$\tilde{g}_{ij} \;=\; \mathcal{E}_{ki}\, g^{kl}\, \mathcal{E}_{lj} \;, \qquad g^{ij} \;=\; \tilde{\mathcal{E}}^{ik}\, \tilde{g}_{kl}\, \tilde{\mathcal{E}}^{jl} \;. \tag{6.8}$$

Writing these equations in terms of g and b (or their dual variables $\tilde{g}$ and $\tilde{b}$), we recover (6.6) for the diagonal matrix entries. For the off-diagonal entries we compute, for instance,

$$\begin{aligned} -\tilde{g}_{ik}\tilde{b}^{kj} \;&=\; -\tilde{g}_{ik}\big(\tilde{\mathcal{E}}^{kj} - \tilde{g}^{kj}\big) \;=\; -\tilde{g}_{ik}\tilde{\mathcal{E}}^{kj} + \delta_i{}^j \;=\; -\mathcal{E}_{pi}g^{pq}\mathcal{E}_{qk}\tilde{\mathcal{E}}^{kj} + \delta_i{}^j \\ &=\; -\mathcal{E}_{pi}g^{pj} + \delta_i{}^j \;=\; -(g_{pi} + b_{pi})g^{pj} + \delta_i{}^j \;=\; b_{ip}g^{pj} \;, \end{aligned} \tag{6.9}$$

confirming the equality of the off-diagonal entries in (6.6).

We note that the field redefinitions (6.5) interchange upper with lower indices in order to work consistently with the *lower* indices of the dual coordinates $\tilde{x}_i$. In particular, the diffeomorphisms in the dual coordinates are generated by $\tilde{\xi}_i$ in that the gauge transformations (see (2.37) and (2.38) of [13]) reduce for $\partial_i = 0$ to

$$\delta_{\tilde{\xi}}\tilde{\mathcal{E}}^{ij} \;=\; \tilde{\xi}_k\tilde{\partial}^k\tilde{\mathcal{E}}^{ij} + \tilde{\partial}^i\tilde{\xi}_k\,\tilde{\mathcal{E}}^{kj} + \tilde{\partial}^j\tilde{\xi}_k\,\tilde{\mathcal{E}}^{ik} \;. \tag{6.10}$$

Viewing $\tilde{\mathcal{E}}^{ij}$ with upper indices as a *covariant* rather than a contravariant tensor, this is the conventional transformation of such a tensor under infinitesimal diffeomorphisms.

The double field theory action $S_{\rm NS-NS}$ for the NS-NS fields is, of course, the same as the double field theory action $S_{\rm DFT}$ for the low energy bosonic string. We thus write

$$S_{\rm NS-NS}\Big|_{\tilde{\partial}=0} \;=\; S_{\rm DFT}\Big|_{\tilde{\partial}=0} \;=\; S\Big[g, b, d, \partial\Big] \;, \tag{6.11}$$

with S a function of the four arguments written above. In the dual frame $\partial = 0$, our discussion above implies that we have

$$S_{\rm NS-NS}\Big|_{\partial=0} \;=\; S_{\rm DFT}\Big|_{\partial=0} \;=\; S\Big[\tilde{g}, \tilde{b}, \tilde{d}, \tilde{\partial}\Big] \;. \tag{6.12}$$

The replacements in the arguments of S are, explicitly,

$$g_{ij} \to \tilde{g}^{ij} \;, \quad g^{ij} \to \tilde{g}_{ij} \;, \quad b_{ij} \to \tilde{b}^{ij} \;, \quad \partial_i \to \tilde{\partial}^i \;. \tag{6.13}$$

Let us now see how this generalizes in presence of the RR fields. Before we give a general discussion in the next section, it will be instructive to first examine more explicitly, along the lines reviewed above, what happens in the frame $\partial_i = 0$ with $\tilde{\partial}^i \neq 0$. Let us first evaluate the field strength $|F\rangle$ in this frame,

$$
\begin{aligned}
|F\rangle\Big|_{\partial_i=0} &= \tilde{\partial\!\!\!/}|\chi\rangle \;=\; \psi_j\tilde{\partial}^j\sum_{p=0}^{D}\frac{1}{p!}C_{i_1\cdots i_p}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle \\
&= \sum_{p=1}^{D}\frac{1}{p!}\,\tilde{\partial}^j C_{ji_2\ldots i_p}\,p\,\psi^{i_2}\cdots\psi^{i_p}|0\rangle \;=\; \sum_{p=1}^{D}\frac{1}{(p-1)!}\,\tilde{\partial}^j C_{ji_1\ldots i_{p-1}}\,\psi^{i_1}\cdots\psi^{i_{p-1}}|0\rangle\;. \\
&= \sum_{p=0}^{D-1}\frac{1}{p!}\tilde{\partial}^j C_{ji_1\ldots i_p}\psi^{i_1}\cdots\psi^{i_p}|0\rangle\;.
\end{aligned}
\tag{6.14}
$$

At first sight this looks rather different from the conventional field strength of a p-form, but it can actually be brought to the form of a 'dual field strength' if we introduce a dual potential $\tilde{C}$ according to

$$
C_{i_1\ldots i_p} \;=\; \alpha_p\,\epsilon_{i_1\ldots i_p j_1\ldots j_{D-p}}\,\tilde{C}^{j_1\ldots j_{D-p}}\;, \tag{6.15}
$$

where the numerical coefficients α_p will be fixed below. We recall that the epsilon symbol is constant and equal to ± 1, i.e., it is not a tensor but rather a density. In terms of this new variable, (6.14) reads

$$
\begin{aligned}
|F\rangle\Big|_{\partial_i=0} &= \sum_{p=0}^{D-1}\frac{\alpha_{p+1}}{p!}\epsilon_{ji_1\ldots i_p j_1\ldots j_{D-p-1}}\tilde{\partial}^j\tilde{C}^{j_1\ldots j_{D-p-1}}\psi^{i_1}\cdots\psi^{i_p}|0\rangle \\
&\equiv \sum_{p=0}^{D-1}\frac{\alpha_{p+1}(-1)^p}{p!(D-p)}\,\epsilon_{i_1\ldots i_p j_1\ldots j_{D-p}}\tilde{F}^{j_1\ldots j_{D-p}}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle\;,
\end{aligned}
\tag{6.16}
$$

where we introduced in analogy to (5.10)

$$
\tilde{F}^{j_1\ldots j_p} \;=\; p\,\tilde{\partial}^{[j_1}\tilde{C}^{j_2\ldots j_p]}\;. \tag{6.17}
$$

We should stress that (6.15) does not involve any metric and so this is *not* the Hodge dual. Consequently, $\tilde{C}$ is not a covariant tensor in the usual sense. However, what we actually have to verify is that, as in (6.10), this is a tensor in the T-dual sense that it transforms under $\tilde{\xi}_i$ rather than ξ^i with a Lie derivative. To see this, we examine the gauge transformation (4.24)

$$
\delta_{\tilde{\xi}}|\chi\rangle \;=\; \tilde{\xi}_j\tilde{\partial}^j|\chi\rangle + \tilde{\partial}^j\tilde{\xi}_k\,\psi_j\psi^k|\chi\rangle\;. \tag{6.18}
$$

The transport term gives manifestly rise to the correct structure, so we focus on the second term, denoted by $\bar{\delta}_{\tilde{\xi}}$, which yields

$$
\begin{aligned}
\bar{\delta}_{\tilde{\xi}}|\chi\rangle &= \sum_{p=0}^{D}\frac{p+1}{p!}\tilde{\partial}^j\tilde{\xi}_{[j}\,C_{i_1\ldots i_p]}\psi^{i_1}\cdots\psi^{i_p}|0\rangle \\
&= \sum_{p=0}^{D}\frac{\alpha_p(p+1)}{p!}\tilde{\partial}^j\tilde{\xi}_{[j}\,\epsilon_{i_1\ldots i_p]k_1\ldots k_{D-p}}\tilde{C}^{k_1\ldots k_{D-p}}\psi^{i_1}\cdots\psi^{i_p}|0\rangle\;.
\end{aligned}
\tag{6.19}
$$

To simplify this, we use that a fully antisymmetric tensor with $D+1$ indices in D dimensions vanishes identically,

$$\begin{aligned} 0 &= (D+1)\tilde{\partial}^j \tilde{\xi}_{[j}\, \epsilon_{i_1\ldots i_p k_1\ldots k_{D-p}]} \\ &= (p+1)\tilde{\partial}^j \tilde{\xi}_{[j}\, \epsilon_{i_1\ldots i_p]k_1\ldots k_{D-p}} - (D-p)\tilde{\partial}^j \tilde{\xi}_{[k_1}\, \epsilon_{|i_1\ldots i_p j|k_2\ldots k_{D-p}]}\ . \end{aligned} \tag{6.20}$$

Using this in (6.19), one obtains

$$\bar{\delta}_{\tilde{\xi}}|\chi\rangle \;=\; \sum_{p=0}^{D} \frac{\alpha_p(D-p)}{p!} \epsilon_{i_1\ldots i_p k_1\ldots k_{D-p}} \tilde{\partial}^{k_1} \tilde{\xi}_j\, \tilde{C}^{jk_2\ldots k_{D-p}} \psi^{i_1}\cdots\psi^{i_p}|0\rangle\ , \tag{6.21}$$

where we relabeled $k_1 \leftrightarrow j$. In total, we read off

$$\delta_{\tilde{\xi}} \tilde{C}^{i_1\ldots i_{D-p}} \;=\; \tilde{\xi}_j \tilde{\partial}^j \tilde{C}^{i_1\ldots i_{D-p}} + (D-p)\tilde{\partial}^{[i_1} \tilde{\xi}_k\, \tilde{C}^{|k|i_2\ldots i_{D-p}]} \;\equiv\; \mathcal{L}_{\tilde{\xi}} \tilde{C}^{i_1\ldots i_{D-p}}\ . \tag{6.22}$$

This is the dual Lie derivative with respect to $\tilde{\xi}_i$ of a dual p-form, where we note that upper indices are now covariant indices and so the signs in (6.22) are the conventional ones, c.f. (5.46) and (6.10).

6.2 RR action in different T-duality frames

So far we have seen explicitly that the field strengths in the dual frame $\partial_i = 0$, $\tilde{\partial}^i \neq 0$, take the conventional form when written in terms of the right 'T-dual' variables $\tilde{C}^{i_1\cdots i_p}$. We will now prove more generally that the action and duality relations in the frame $\partial_i = 0$ yield the T-dual type II theory written in terms of the T-dual variables (6.6) for the NS-NS fields and $\tilde{C}$ for the RR fields. Since the $O(D,D)$ transformation inverts all space-time dimensions, it contains a timelike T-duality and thus maps, say, IIA and IIA* into each other.

To proceed, we describe the field redefinition (6.15) by introducing the following tilde variable of the $O(D,D)$ spinor,

$$\tilde{\chi} \;=\; S_J\, \chi\,, \qquad S_J \;=\; C\ . \tag{6.23}$$

This corresponds to the action of the spinor representative of the $O(D,D)$ transformation $J = J^{-1}$ that exchanges x^i and $\tilde{x}_i$, which for convenience we have chosen to be C, but we stress that this field redefinition does not affect the coordinate arguments.

We can then verify that the field redefinition $\chi \to \tilde{\chi}$ indeed amounts to the duality transformation (6.15). In fact, with (2.45) and (5.39) we obtain

$$\begin{aligned} \tilde{\chi} \;&\equiv\; \sum_p \frac{1}{p!}\, \tilde{C}^{i_1\ldots i_p}\, \psi^{i_1}\cdots\psi^{i_p}|0\rangle \;=\; C\chi \;=\; \sum_p \frac{1}{p!}\, C_{i_1\ldots i_p}\, \psi_{i_1}\cdots\psi_{i_p} C|0\rangle \\ &=\; \sum_p \frac{1}{p!(D-p)!}(-1)^{\frac{1}{2}p(p-1)}\, C_{i_1\ldots i_p}\, \epsilon^{i_1\ldots i_p j_1\ldots j_{D-p}}\, \psi^{j_1}\cdots\psi^{j_{D-p}}|0\rangle\ . \end{aligned} \tag{6.24}$$

This equation determines the tilde variables in terms of the original ones:

$$\tilde{C}^{i_1\ldots i_p} \;=\; (-1)^{\frac{1}{2}(D-p)(D+p-1)} \frac{1}{(D-p)!} \epsilon^{i_1\ldots i_p j_1\ldots j_{D-p}} C_{j_1\ldots j_{D-p}}\ , \tag{6.25}$$

where we performed an index shift. It can be checked with the standard identity

$$\epsilon^{i_1\dots i_p j_1\dots j_{D-p}}\,\epsilon_{i_1\dots i_p k_1\dots k_{D-p}} \;=\; -p!(D-p)!\,\delta^{[j_1}{}_{k_1}\cdots\delta^{j_{D-p}]}{}_{k_{D-p}}\,, \tag{6.26}$$

following from (5.26), that this coincides with (6.15) for $\alpha_p = (-1)^{\frac{1}{2}p(p-1)+1}/(D-p)!$.

In terms of the tilde variables (6.23) we have, using (2.26),

$$\begin{aligned}\not{\partial}\chi \;&=\; \frac{1}{\sqrt{2}}\Gamma^M\partial_M(S_J^{-1}\tilde{\chi}) \;=\; \frac{1}{\sqrt{2}}\Gamma^M S_J^{-1}\partial_M\tilde{\chi}\\ &=\; \frac{1}{\sqrt{2}}J^M{}_N S_J^{-1}\Gamma^N\partial_M\tilde{\chi} \;=\; S_J^{-1}\frac{1}{\sqrt{2}}\Gamma^N(J^M{}_N\partial_M)\tilde{\chi} \;=\; S_J^{-1}\hat{\not{\partial}}\tilde{\chi}\,,\end{aligned} \tag{6.27}$$

where we introduced a redefined derivative and Dirac operator,

$$\hat{\not{\partial}} \;\equiv\; \frac{1}{\sqrt{2}}\Gamma^N\hat{\partial}_N\,, \qquad \hat{\partial}_N \;\equiv\; J^M{}_N\partial_M\,. \tag{6.28}$$

Recalling that the matrix $J^M{}_N$ has only the non-vanishing matrix elements J^{ij} and J_{ij} that are equal to Kronecker deltas we find that

$$\hat{\not{\partial}} \;=\; \psi^i\tilde{\partial}^i + \psi_i\partial_i\,. \tag{6.29}$$

As expected, the ∂_i and $\tilde{\partial}^i$ derivatives have been exchanged. For the Lagrangian we now find

$$\mathcal{L} \;=\; \frac{1}{4}(\not{\partial}\chi)^\dagger\,S_{\mathcal{H}}\,\not{\partial}\chi \;=\; \frac{1}{4}(\hat{\not{\partial}}\tilde{\chi})^\dagger(S_J^{-1})^\dagger\,S_{\mathcal{H}}\,S_J^{-1}\hat{\not{\partial}}\tilde{\chi} \;=\; -\frac{1}{4}(\hat{\not{\partial}}\tilde{\chi})^\dagger\,S_{\tilde{\mathcal{H}}}\,\hat{\not{\partial}}\tilde{\chi}\,, \tag{6.30}$$

where we used the sign factor in (3.33). We see that in tilde-variables the RR action takes the same form as in the original variables, up to a sign. It can also be checked that the duality constraints in the dual frame take the form

$$\hat{\not{\partial}}\tilde{\chi} \;=\; C^{-1}S_{\tilde{\mathcal{H}}}\,\hat{\not{\partial}}\tilde{\chi}\,, \tag{6.31}$$

which differs from the constraints in the original frame by a sign factor.

It follows now that setting $\partial_i = 0$ in the evaluation of the Lagrangian as written in the first form in (6.30) is equivalent to setting $\hat{\not{\partial}} \;=\; \psi^i\tilde{\partial}^i$ in the evaluation of the Lagrangian as written in the last form in (6.30). But this latter evaluation is identical to our original computation in sec. 5, with ∂_i derivatives replaced by $\tilde{\partial}^i$ derivatives and $C_{i_1\dots i_p}$ replaced by $\tilde{C}^{i_1\dots i_p}$. Of course, this time we get an extra minus sign.

Due to this sign change in the RR action we conclude that if the theory reduces for $\tilde{\partial}^i = 0$ to IIA, the same theory reduces for $\partial_i = 0$ to IIA*, but written in terms of the T-dual variables. We thus have, for instance,

$$S_{\mathrm{DFT_{II}}}\Big|_{\tilde{\partial}=0} \;=\; S_{\mathrm{IIA}}\Big[g,b,d,C,\partial\Big]\,, \qquad S_{\mathrm{DFT_{II}}}\Big|_{\partial=0} \;=\; S_{\mathrm{IIA}^\star}\Big[\tilde{g},\tilde{b},\tilde{d},\tilde{C},\tilde{\partial}\Big]\,, \tag{6.32}$$

where we indicated by $S_{\mathrm{DFT_{II}}}$ the full double field theory action of type II, while S_{IIA} and $S_{\mathrm{IIA}^\star}$ are the low-energy actions of IIA and IIA*, respectively. Moreover, the corresponding duality constraints differ by a sign. This is the expected sign given that the stress-tensor from the RR sector in the dual frame must have a sign opposite to the one in the original frame.

Similarly, if the chosen chirality is such that the theory reduces in the $\tilde{\partial}^i = 0$ frame to type IIB, the same theory reduces in the $\partial_i = 0$ frame to type IIB*. We finally note that had we chosen the equally valid parametrization $\mathbb{S} = -S_{\mathcal{H}}$, we would have obtained either IIA* or IIB* in the frame $\tilde{\partial}^i = 0$ and the conventional IIA or IIB theories in the opposite frame.

It is instructive to reconsider the above analysis in somewhat more explicit terms by performing an expansion of the RR action in tilde derivatives $\tilde{\partial}$,

$$S_{\rm RR} \;=\; S_{\rm RR}^{(0)} + S_{\rm RR}^{(1)} + S_{\rm RR}^{(2)}\;, \tag{6.33}$$

where the superscript denotes the number of $\tilde{\partial}$. For simplicity, let us assume that the b-field vanishes. The first term $S_{\rm RR}^{(0)}$ is a conventional type II action as discussed in sec. 5. The remaining terms can be straightforwardly computed using that, by the linearity of the Dirac operator $\not{\partial}$, the full field strength (5.8) is simply the sum of (5.9) and (6.16),

$$|F\rangle \;=\; \sum_{p=0}^{D}\frac{1}{p!}\,\mathcal{F}_{i_1\ldots i_p}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle\;, \tag{6.34}$$

where

$$\mathcal{F}_{i_1\ldots i_p} \;\equiv\; F_{i_1\ldots i_p} + \beta_p\,\epsilon_{i_1\ldots i_p j_1\ldots j_{D-p}}\tilde{F}^{j_1\ldots j_{D-p}}\;, \tag{6.35}$$

and $\beta_p = (-1)^p\alpha_{p+1}/(D-p)$. In here F is the conventional field strength, depending on derivatives ∂, and $\tilde{F}$ is the field strength in terms of the dual variables, depending on the dual derivatives $\tilde{\partial}$. Precisely as in sec. 5, one then finds for the full RR-action

$$S_{\rm RR} \;=\; -\frac{1}{4}\sum_p\frac{1}{p!}\sqrt{g}\,g^{i_1j_1}\cdots g^{i_pj_p}\,\mathcal{F}_{i_1\ldots i_p}\mathcal{F}_{j_1\ldots j_p}\;. \tag{6.36}$$

Insertion of (6.35) then gives

$$\begin{aligned} S_{\rm RR}^{(0)} \;&=\; -\frac{1}{4}\sum_p\frac{1}{p!}\sqrt{g}\,g^{i_1j_1}\cdots g^{i_pj_p}\,F_{i_1\ldots i_p}F_{j_1\ldots j_p}\;,\\ S_{\rm RR}^{(2)} \;&=\; +\frac{1}{4}\sum_p\frac{1}{p!}\frac{1}{\sqrt{g}}\,g_{i_1j_1}\cdots g_{i_pj_p}\,\tilde{F}^{i_1\ldots i_p}\tilde{F}^{j_1\ldots j_p}\;. \end{aligned} \tag{6.37}$$

For the second equation we shifted the summation index p and used the identity

$$\sqrt{g}\,g^{i_1j_1}\cdots g^{i_pj_p}\,\epsilon_{i_1\ldots i_pk_1\ldots k_{D-p}}\,\epsilon_{j_1\ldots j_pl_1\ldots l_{D-p}} \;=\; -\frac{1}{\sqrt{g}}\,p!(D-p)!\,g_{[k_1|l_1|}\cdots g_{k_{D-p}]l_{D-p}}\;, \tag{6.38}$$

which follows from (6.26) and (5.26). We stress that the minus sign on the right-hand side of this identity is due to the Lorentzian signature. It is this sign that is responsible for the relative sign between $S^{(0)}$ and $S^{(2)}$ in (6.37). We have thus re-derived the sign change of (6.30) for the special case of vanishing b-field. Let us note that, as discussed in sec. 3.2, not all invariances of the original action are still present once we parametrize $\mathbb{S}$ in terms of the conventional fields. For instance, the transformation J maps, using (6.24),

$$\partial_i \to \tilde{\partial}^i\;,\quad C_{i_1\ldots i_p} \to \tilde{C}^{i_1\ldots i_p} \qquad\Rightarrow\qquad F_{i_1\ldots i_p} \to \tilde{F}^{i_1\ldots i_p}\;. \tag{6.39}$$

Therefore, $S^{(0)}$ in (6.37) is transformed without a sign change, i.e., $S^{(0)}$ is mapped to $-S^{(2)}$ and so (6.33) is not invariant.

We close this section with a brief discussion of intermediate frames, which we illustrate with the simplest case of one T-duality inversion. Thus, we split the indices as $x^i = (x^1, x^a)$ and assume that the non-trivial derivatives are $(\tilde{\partial}^1, \partial_a)$, where '1' denotes the special direction. As above, we consider a field redefinition that takes the form of the T-duality inversion,

$$\begin{aligned}\chi' \;=\; S_1\chi \;&=\; (\psi^1+\psi_1)\sum_p \frac{1}{p!}\left(C_{a_1\ldots a_p}\psi^{a_1}\cdots\psi^{a_p} + pC_{1a_1\ldots a_{p-1}}\psi^1\psi^{a_1}\cdots\psi^{a_{p-1}}\right)|0\rangle \\ &=\; \sum_p \frac{1}{p!}\left(C_{a_1\ldots a_p}\psi^1\psi^{a_1}\ldots\psi^{a_p} + pC_{1\,a_1\ldots a_{p-1}}\psi^{a_1}\cdots\psi^{a_{p-1}}\right)|0\rangle \\ &\equiv\; \sum_p \frac{1}{p!}C'_{i_1\ldots i_p}\psi^{i_1}\cdots\psi^{i_p}|0\rangle\;.\end{aligned} \tag{6.40}$$

This implies that the redefined $C^{(p)}$ are given in terms of the original ones by

$$C'_{i_1\ldots i_p} \;=\; \begin{cases} C_{a_2\ldots a_p} & \text{if }\; i_1=1,\;\; i_2=a_2\,,\;\ldots\;,i_p=a_p \\ C_{1a_1\ldots a_p} & \text{if }\; i_1=a_1\,,\;\ldots\;,i_p=a_p\,.\end{cases} \tag{6.41}$$

Put differently, the new p-forms are obtained from the original ones by adding or deleting the special index. It follows that this redefinition interchanges even and odd forms and thus changes the chirality of χ. The field strength then reads

$$\partial\!\!\!/\chi \;=\; \big(\psi^a\partial_a+\psi_1\tilde{\partial}^1\big)\big(\psi^1+\psi_1\big)\chi' \;=\; \big(\psi^1+\psi_1\big)\big(\psi^a(-\partial_a)+\psi^1\tilde{\partial}^1\big)\chi' \;=\; S_1\,\psi^i\partial'_i\chi'\;, \tag{6.42}$$

where we recognized the transformed (primed) derivatives $\partial'_i = (\tilde{\partial}^1, -\partial_a)$, recalling that the transformation h_i in (2.36) changes the overall sign of the coordinates x^a.

In precise analogy to (6.30), we can now conclude that the action in the frame with $\tilde{\partial}^1,\,\partial_a\neq 0$ takes the same form as in the frame $\tilde{\partial}^i=0$, just with all field variables replaced by primed variables. Since the primed variables have the opposite chirality, it follows that if the theory reduces for $\tilde{\partial}^i=0$ to, say, type IIA, in the new frame it reduces to type IIB if g_{11} is positive and to type IIB* if g_{11} is negative. More generally, if we evaluate the theory in any frame that results from the $\tilde{\partial}^i=0$ frame by an $O(d,d)$ transformation, we obtain the corresponding T-dual theory.

7 Discussion and conclusions

In this paper we introduced a double field theory formulation for the low-energy limit of type II strings. T-duality relates different type II theories, a feature that does not occur in bosonic string theory. In the double field theory built here each of the type II theories can be obtained by choosing different 'slicings' within the doubled coordinates. Consistent slicings are those allowed by the $O(D,D)$ covariant strong constraint $\partial^M\partial_M=0$ that originates from the $L_0-\bar{L}_0=0$ constraint of closed string theory. If we consider two slicings related by an odd number of spacelike T-duality inversions and one yields type IIA, the other must yield type IIB. The double field theory necessarily features the so-called type IIA* and type IIB* theories, which are related to the conventional type II theories via T-dualities along *timelike* directions.

Despite this unification, the actual invariance group of the theory is only $\mathrm{Spin}^+(D,D)$ and therefore does not contain any of the T-duality transformations that relate different type II theories. This means that the $\mathrm{Pin}(D,D)$ transformations that are not in $\mathrm{Spin}^+(D,D)$ must be viewed as dualities rather than invariances. More precisely, while we fix the chirality of the spinor χ from the outset, the opposite chirality is obtained by the field redefinition induced by the appropriate T-duality transformation. The situation is similar to theories that depend on a background but which are nevertheless background-independent in the sense that any shift of the background can be absorbed into a field redefinition. Just as one may then ask for a manifestly background independent formulation, we may now wonder if there is a formulation with full $\mathrm{Pin}(D,D)$ invariance. This would presumably require the introduction of a spinor without a chirality condition, together with an additional gauge symmetry to remove the new unphysical degrees of freedom.

Further generalizations of this work are possible. It would be interesting to see if this type II double field theory allows for an enhancement of the global symmetry to a U-duality group, such that the NS-NS and RR fields transform in an irreducible representation. Results on reformulations of 11-dimensional supergravity may be relevant, see [34] and [22, 27]. Moreover, exceptional groups are of particular interest since they naturally combine fundamental and spinor representation, and in this context the Kac-Moody algebras E_{11} [23, 35] and E_{10} [36, 37] have been proposed. Being infinite-dimensional, they easily accommodate the massless fields of various string theories, but they also give rise to an infinite set of further representations for which a physical interpretation has yet to be found.

The work here may also contain pointers for a yet to be constructed string field theory of type II strings. This is an outstanding problem since these remain the only string theories for which no string field action is known. Finally, there might be applications to generalized Kaluza-Klein type reductions or to the construction of T-duality invariant world-volume theories of branes. We leave these and other questions for future research.

Acknowledgments

For useful discussions we would like to thank Marco Gualtieri, Juan Maldacena, Ashoke Sen and Paul Townsend. We thank Chris Hull for comments, and understand from him that he has worked in closely related directions.

This work is supported by the U.S. Department of Energy (DoE) under the cooperative research agreement DE-FG02-05ER41360. The work of OH is supported by the DFG – The German Science Foundation, and the work of SK is supported in part by a Samsung Scholarship.

A Duality transformations of $S_{\mathcal{H}}$

In this appendix we discuss the transformation behavior of $S_{\mathcal{H}}$ in some detail. We first give the general proofs of the transformation rules stated in the main text, and then give an example to illustrate these rules.

A.1 $GL(D)$ and b-shifts

Our goal is to determine the sign factor σ appearing in the transformation of $S_{\mathcal{H}}$ under $O(D,D)$,

$$(S^{-1})^{\dagger}\, S_{\mathcal{H}}\, S^{-1} \;=\; \sigma_{\rho(S)}(\mathcal{H})\; S_{\rho(S)\circ\mathcal{H}}\,. \tag{A.1}$$

We start by considering the 'geometric subgroup'. It consists of $GL(D)$ transformations and the abelian subgroup $\mathbb{R}^{\frac{1}{2}D(D-1)}$ of b-shifts, which together form the semi-direct product $GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)}$. We show that for this subgroup no sign factor arises:
Theorem: Given an arbitrary $\mathcal{H}$, for any $h_r \in GL(D)$ and $h_b \in \mathbb{R}^{\frac{1}{2}D(D-1)}$

$$\sigma_{h_r}(\mathcal{H}) \;=\; \sigma_{h_b}(\mathcal{H}) \;=\; 1\,. \tag{A.2}$$

We can then immediately conclude that $\sigma_h(\mathcal{H}) = 1$ for any $h \in GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)}$.

In the remainder of this subsection we will prove this theorem. We first present the proof for b-shifts, and then discuss $GL^{+}(D)$ and $GL^{-}(D)$, respectively.

<u>b-shifts:</u> The $O(D,D)$ element which shifts $b \to b - \Delta b$ and its corresponding Spin(D,D) element are given by, respectively,

$$h_{\Delta b} \;=\; \begin{pmatrix} 1 & -\Delta b \\ 0 & 1 \end{pmatrix}\,, \qquad S_{\Delta b} = e^{-\frac{1}{2}\Delta b_{ij}\psi^i\psi^j}\,. \tag{A.3}$$

Then the duality transformation of $S_{\mathcal{H}}$ under b-shifts can be written as

$$\begin{aligned}(S_{\Delta b}^{-1})^{\dagger}\, S_{\mathcal{H}}\, S_{\Delta b}^{-1} \;&=\; e^{-\frac{1}{2}\Delta b_{ij}\psi_i\psi_j} S_{\mathcal{H}}\, e^{\frac{1}{2}\Delta b_{ij}\psi^i\psi^j} \;=\; e^{-\frac{1}{2}\Delta b_{ij}\psi_i\psi_j}\, e^{\frac{1}{2}b_{ij}\psi_i\psi_j}\; S_g^{-1}\, e^{-\frac{1}{2}b_{ij}\psi^i\psi^j}\; e^{\frac{1}{2}\Delta b_{ij}\psi^i\psi^j} \\ &=\; e^{\frac{1}{2}(b_{ij}-\Delta b_{ij})\psi_i\psi_j}\; S_g^{-1}\, e^{-\frac{1}{2}(b_{ij}-\Delta b_{ij})\psi^i\psi^j} \;=\; S_{\mathcal{H}'}\,.\end{aligned} \tag{A.4}$$

We conclude $\sigma_{h_b}(\mathcal{H}) = 1$.

<u>$GL^{+}(D)$:</u> An arbitrary $O(D,D)$ element in $GL^{+}(D)$ and its corresponding Spin(D,D) element can be written as

$$h_r \;=\; \begin{pmatrix} r & 0 \\ 0 & (r^{-1})^t \end{pmatrix}\,, \qquad S_r \;=\; \frac{1}{\sqrt{\det r}} e^{\psi^i R_i{}^j \psi_j}\,, \tag{A.5}$$

with $\det r > 0$. Under this $O(D,D)$ transformation, g and b transform covariantly,

$$g \;\to\; r\, g\, r^t\,, \qquad b \;\to\; r\, b\, r^t\,. \tag{A.6}$$

This transformation of the metric g is induced by the covariant transformation $e \to re$ of the vielbein. The duality transformation of $S_{\mathcal{H}}$ under $GL^{+}(D)$ is then

$$(S_r^{-1})^{\dagger}\, S_{\mathcal{H}}\, S_r^{-1} \;=\; (S_r^{-1})^{\dagger} S_b^{\dagger}\, S_g^{-1}\, S_b S_r^{-1} \;=\; \Big[(S_r^{-1})^{\dagger} S_b^{\dagger}\, S_r^{\dagger}\Big]\, (S_r^{-1})^{\dagger} S_g^{-1} S_r^{-1}\, \big[\, S_r\, S_b\, S_r^{-1}\big]\,. \tag{A.7}$$

We first evaluate the terms in the square parentheses. We only need to evaluate the second parenthesis since the term in the first parenthesis is just its hermitian conjugate,

$$S_r\, S_b\, S_r^{-1} \;=\; S_r\, e^{-\frac{1}{2}b_{ij}\psi^i\psi^j}\, S_r^{-1} \;=\; e^{-\frac{1}{2}b_{ij}(\psi^k r_k{}^i)(\psi^l r_l{}^j)} \;=\; e^{-\frac{1}{2}(rbr^t)_{kl}\psi^k\psi^l} \;=\; S_{b'}\,, \tag{A.8}$$

where we used

$$S_r \psi^i S_r^{-1} \;=\; \psi^k r_k{}^i\,, \qquad S_r \psi_i S_r^{-1} \;=\; \psi_k (r^{-1})_i{}^k\,. \tag{A.9}$$

Thus we see that the b-field transforms exactly as required by (A.6). It remains to inspect the following term in (A.7)

$$(S_r^{-1})^\dagger S_g^{-1} S_r^{-1} \;=\; (S_r^{-1})^\dagger (S_e^{-1})^\dagger\, S_k\, S_e^{-1} S_r^{-1}\;. \tag{A.10}$$

We write now S_e in terms of the oscillators as

$$S_e \;=\; \frac{1}{\sqrt{\det e}} e^{\psi^i E_i{}^j \psi_j}\;, \tag{A.11}$$

where $\exp(E) = e$. To simplify the computation of (A.10), it is convenient to note that with $\overline{A} \equiv \psi^i A_i{}^j \psi^j$ we have

$$[\overline{A}, \overline{B}] = \overline{[A, B]} \quad\Rightarrow\quad e^{\overline{R}} e^{\overline{E}} \;=\; \overline{(e^R e^E)} \;=\; \overline{re} \;=\; \overline{e^{\log(re)}}\;. \tag{A.12}$$

Thus,

$$S_r S_e \;=\; \frac{1}{\sqrt{\det r}}\, e^{\psi^i R_i{}^j \psi_j} \frac{1}{\sqrt{\det e}}\, e^{\psi^i E_i{}^j \psi_j} \;=\; \frac{1}{\sqrt{\det{(re)}}} e^{\psi^i (\log(re))_i{}^j \psi_j} \;=\; S_{re} \;=\; S_{e'}\;, \tag{A.13}$$

where $\log(re)$ is defined by $e^{\log(re)} = re$. Using this in (A.10) gives

$$(S_r^{-1})^\dagger S_g^{-1} S_r^{-1} \;=\; S_{g'}^{-1}\;. \tag{A.14}$$

In total, combining this and (A.8) we obtain

$$(S_r^{-1})^\dagger\, S_{\mathcal{H}}\, S_r^{-1} \;=\; S_{b'}^\dagger\, S_{g'}^{-1}\, S_{b'} \;=\; S_{\mathcal{H}'}\;, \tag{A.15}$$

which proves $\sigma_{h_r}(\mathcal{H}) = 1$ for $h_r \in GL^+(D)$.

<u>$GL^-(D)$:</u> An arbitrary $GL^-(D)$ matrix and its corresponding Spin(D,D) element are given by, respectively,

$$h_r \;=\; \begin{pmatrix} r & 0 \\ 0 & (r^{-1})^t \end{pmatrix}\;, \qquad S_r \;=\; (\psi^i \psi_i - \psi_i \psi^i) \frac{1}{\sqrt{|\det r|}} e^{\psi^k R_k{}^l \psi_l}\;, \tag{A.16}$$

with $\det r < 0$. The index i is fixed but arbitrary; in particular, there is no sum over i. $R_i{}^j$ is defined by

$$e^R \;=\; r_+\;, \qquad \text{s.t.}\quad r \;=\; k_i\, r_+\;, \tag{A.17}$$

where $k_i = \text{diag}(1, \cdots, -1, \cdots, 1)$ is the diagonal matrix that has a -1 in the diagonal entry i and $r_+ \in GL^+(D)$. Under this $O(D,D)$ transformation, g and b transform covariantly as in (A.6). We have to keep in mind, however, that in writing the metric as $g = eke^t$, we require e be positive definite, and thus we cannot write $e' = re$. One way to resolve this is to define a positive definite e' as

$$e' \;=\; r\, e\, k_i\;. \tag{A.18}$$

Since $k = k_i k k_i$, this definition of e' correctly gives $g' = rek(re)^t = rgr^t$. The duality transformation of $S_{\mathcal{H}}$ under $GL^-(D)$ is then

$$\begin{aligned}(S_r^{-1})^\dagger S_{\mathcal{H}} S_r^{-1} \;&=\; (S_r^{-1})^\dagger S_b^\dagger\, S_g^{-1}\, S_b S_r^{-1} \\ &=\; \Big[(S_r^{-1})^\dagger S_b^\dagger\, S_r^\dagger\Big]\, (S_r^{-1})^\dagger S_g^{-1} S_r^{-1}\, \big[\,S_r\, S_b\, S_r^{-1}\big]\;.\end{aligned} \tag{A.19}$$

It is straightforward to see that, as in the case of $GL^+(D)$, $S_r\, S_b\, S_r^{-1} = S_{b'}$. The remaining part is more subtle. We first compute

$$
\begin{aligned}
(S_r^{-1})^\dagger S_g^{-1} S_r^{-1} &= (S_r^{-1})^\dagger (S_e^{-1})^\dagger\, S_k\, S_e^{-1} S_r^{-1} \\
&= (S_r^{-1})^\dagger (S_e^{-1})^\dagger (\psi^i\psi_i - \psi_i\psi^i)\, S_k\, (\psi^i\psi_i - \psi_i\psi^i) S_e^{-1} S_r^{-1}\,.
\end{aligned}
\tag{A.20}
$$

Here we used $S_k = (\psi^i\psi_i - \psi_i\psi^i)\, S_k\, (\psi^i\psi_i - \psi_i\psi^i)$, which can be straightforwardly verified both for the case that i is equal to the timelike direction and for the case that it is different from the timelike direction. This guarantees that the proof is independent of the particular factorization in (A.17). Since

$$
(\psi^i\psi_i - \psi_i\psi^i)\, \psi^j\, (\psi^i\psi_i - \psi_i\psi^i) \;=\; \psi^l (k_i)_l{}^j\,, \quad (\psi^i\psi_i - \psi_i\psi^i)\, \psi_j\, (\psi^i\psi_i - \psi_i\psi^i) \;=\; \psi_l (k_i)_j{}^l\,, \tag{A.21}
$$

we obtain as in (A.13)

$$
\begin{aligned}
S_r S_e (\psi^i\psi_i - \psi_i\psi^i) &= (\psi^i\psi_i - \psi_i\psi^i) \frac{1}{\sqrt{|\det r|}} e^{\psi^i R_i{}^j \psi_j} \frac{1}{\sqrt{\det e}}\, e^{\psi^i E_i{}^j \psi_j} (\psi^i\psi_i - \psi_i\psi^i) \\
&= (\psi^i\psi_i - \psi_i\psi^i) \frac{1}{\sqrt{|\det (re)|}} e^{\psi^i (\log(r_+ e))_i{}^j \psi_j} (\psi^i\psi_i - \psi_i\psi^i) \\
&= \frac{1}{\sqrt{|\det (re)|}} e^{\psi^l (k_i)_l{}^p (\log(r_+ e))_p{}^q \psi_m (k_i)_q{}^m} \;=\; S_{e'}\,.
\end{aligned}
\tag{A.22}
$$

Summarizing, we have shown

$$
(S_r^{-1})^\dagger S_{\mathcal{H}} S_r^{-1} \;=\; S_{b'}^\dagger\, S_{g'}^{-1}\, S_{b'} \;=\; S_{\mathcal{H}'}\,, \tag{A.23}
$$

which proves $\sigma_{h_r}(\mathcal{H}) = 1$ for $h_r \in GL^-(D)$.

A.2 T-dualities

We turn now to the sign factors in (A.1) for factorized T-dualities. Using the shorthand notation $\sigma_i(\mathcal{H}) = \sigma_{h_i}(\mathcal{H})$, we will prove that $\sigma_i(\mathcal{H}) = 1$ if the ith direction is spacelike and $\sigma_i(\mathcal{H}) = -1$ if this direction is timelike.

We start by establishing a simple lemma that allows to distinguish elements of the geometric subgroup just discussed from genuine T-duality transformations.

Lemma 1: An $O(D,D)$ matrix of the form

$$
h \;=\; \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix}\,, \tag{A.24}
$$

where $\star$ stands for nonzero blocks, is an element of $GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)}$ and can be written as the product of a $GL(D)$ element and an element of $\mathbb{R}^{\frac{1}{2}D(D-1)}$.

Proof: The group properties of $O(D,D)$ imply for a general matrix of the form (A.24)

$$
h \;=\; \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \qquad \Rightarrow \qquad d \;=\; (a^{-1})^T\,, \qquad a^{-1}b \;\; \text{antisymmetric}\,. \tag{A.25}
$$

Then the matrix takes the form

$$\begin{pmatrix} a & b \\ 0 & (a^{-1})^t \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & (a^{-1})^t \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} , \tag{A.26}$$

proving the claim.

Let us now assume that

$$(S_i^{-1})^\dagger \, S_{\mathcal{H}} \, S_i^{-1} \;=\; \sigma_i(\mathcal{H}) \, S_{h_i \circ \mathcal{H}} \, . \tag{A.27}$$

Using $S_i = S_i^{-1} = S_i^\dagger$, this can also be written as

$$S_i \, S_{\mathcal{H}} \, S_i \;=\; \sigma_i(\mathcal{H}) \, S_{h_i \circ \mathcal{H}} \, . \tag{A.28}$$

Letting h denote an $O(D,D)$ transformation, the above equation implies that

$$S_i \, S_{h \circ \mathcal{H}} \, S_i \;=\; \sigma_i(h \circ \mathcal{H}) \, S_{h_i h \circ \mathcal{H}} \, . \tag{A.29}$$

We want to determine the equivalence class of $h \in O(D,D)$ satisfying $\sigma_i(h \circ \mathcal{H}) = \sigma_i(\mathcal{H})$. A sufficient condition is given by the following lemma:

Lemma 2: If $h \in GL(D) \times \mathbb{R}^{\frac{1}{2}D(D-1)}$ and $h_i \, h \, h_i \in GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)}$, then

$$\sigma_i(h \circ \mathcal{H}) = \sigma_i(\mathcal{H}) \, . \tag{A.30}$$

Proof: We write

$$h_\star \equiv h_i \, h \, h_i \; \in GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)} \, , \tag{A.31}$$

and note that

$$S_{h_\star} \;=\; \pm S_i \, S_h \, S_i \, . \tag{A.32}$$

Since $h \in GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)}$,

$$(S_h^{-1})^\dagger S_{\mathcal{H}} \, S_h^{-1} \;=\; + S_{h \circ \mathcal{H}} \, . \tag{A.33}$$

We calculate the left-hand side of (A.29), using (A.33) in the first step,

$$\begin{aligned} S_i \, S_{h \circ \mathcal{H}} \, S_i \;&=\; S_i \, (S_h^{-1})^\dagger \; S_{\mathcal{H}} \; S_h^{-1} \, S_i \\ &=\; S_i \, (S_h^{-1})^\dagger S_i \, \big(S_i \, S_{\mathcal{H}} \, S_i \big) \; S_i \, S_h^{-1} \, S_i \\ &=\; \big((S_i \, S_h \, S_i)^{-1} \big)^\dagger \, \big(\sigma_i(\mathcal{H}) \, S_{h_i \circ \mathcal{H}} \big) \; \big(S_i \, S_h \, S_i \big)^{-1} \, , \end{aligned} \tag{A.34}$$

where we made use of (A.80). Making use of (A.32),

$$S_i \, S_{h \circ \mathcal{H}} \, S_i \;=\; \sigma_i(\mathcal{H}) \, \big(S_{h_\star}^{-1} \big)^\dagger \, S_{h_i \circ \mathcal{H}} \; S_{h_\star}^{-1} \;=\; \sigma_i(\mathcal{H}) \; S_{h_\star h_i \circ \mathcal{H}} \, , \tag{A.35}$$

since $h_\star \in GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)}$. We now note that using (A.31)

$$h_\star \, h_i \circ \mathcal{H} = h_i \, h \, h_i \, h_i \circ \mathcal{H} = h_i \, h \circ \mathcal{H} \, , \tag{A.36}$$

and therefore we have obtained,

$$S_i \, S_{h\circ\mathcal{H}} \, S_i \;=\; \sigma_i(\mathcal{H}) \, S_{h_i h\circ\mathcal{H}} \,. \tag{A.37}$$

Comparing with (A.29) we conclude that (A.30) is true, as we wanted to prove.

As a first application we show that b-shifts satisfy the conditions of Lemma 2. Indeed, taking $h = h_b$ for some arbitrary $D \times D$ matrix b, a small computation confirms that

$$h_i \, h_b \, h_i \;=\; \begin{pmatrix} 1 + be_i & -b + e_i b + be_i \\ 0 & 1 + e_i b \end{pmatrix} \in GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)} \,. \tag{A.38}$$

It then follows that

$$\sigma_i\big(h_b \circ \mathcal{H}\big) = \sigma_i(\mathcal{H}) \,. \tag{A.39}$$

Since at any point X the b field in $\mathcal{H}$ can be removed completely by a b-shift, the sign $\sigma_i(\mathcal{H})$ is in fact a function $\sigma_i(g)$ of the metric only:

$$\sigma_i(\mathcal{H}) = \sigma_i(g) \,. \tag{A.40}$$

In order to determine now $\sigma_i(g)$, we use $GL(D)$ transformations that bring the metric into a simpler form. There is an important complication, however: for arbitrary $r \in GL(D)$ it is not generally true that $h_i h_r h_i$ is in $GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)}$, and thus Lemma 2 cannot be generally applied. For the Lemma to be applicable, the lower left block of the matrix $h_i h_r h_i$ must vanish. A small calculation shows that this requires

$$-\, e_i r (1 - e_i) - (1 - e_i)(r^{-1})^T \, e_i = 0 \quad (i \text{ not summed}) \,. \tag{A.41}$$

Using $e_i A e_i = A_{ii} e_i$ for any matrix A, and $e_i e_i = e_i$, we can rewrite the above condition as

$$-\, e_i (r - r_{ii} e_i) - \Big((r^{-1})^T - ((r^{-1})^T)_{ii} e_i \Big) e_i = 0 \,. \tag{A.42}$$

Consider the condition that the first term vanishes:

$$e_i (r - r_{ii} e_i) = 0 \,. \tag{A.43}$$

This requires the i-th row of r to vanish, except for the diagonal element r_{ii} that can be arbitrary. Without loss of generality, and to display more easily the matrices, let us take $i = 1$. The condition then gives

$$e_1 (r - r_{11} e_1) = 0 \quad \to \quad r = r_\star \equiv \begin{pmatrix} r_{11} & \vec{0}^T \\ \vec{V} & \hat{r} \end{pmatrix} \,, \tag{A.44}$$

letting $r_\star$ denote the solution of this constraint. We decomposed the matrix $r_\star$ into a 1×1 corner block with element r_{11}, a $(D-1)$ column vector $\vec{V}$, the vanishing $(D-1)$ row vector, and the $(D-1) \times (D-1)$ invertible matrix $\hat{r}$. A small calculation shows that

$$r_\star^{-1} = \begin{pmatrix} 1/r_{11} & \vec{0}^T \\ -\hat{r}^{-1}\vec{V}/r_{11} & \hat{r}^{-1} \end{pmatrix} \,. \tag{A.45}$$

This shows that $(r_\star^{-1})^T$ has a vanishing first column, except for its diagonal element, which implies that

$$\Big((r_\star^{-1})^T - ((r_\star^{-1})^T)_{11} e_1\Big) e_1 = 0\,. \tag{A.46}$$

Thus for $r_\star$ the second term in (A.42) vanishes as well. This shows that for $r_\star$ as in (A.44), we have $h_1 h_{r_\star} h_1 \in GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)}$, and the conditions of Lemma 2 apply. We thus have

$$\sigma_1(g) = \sigma_1\left(\, h_{r_\star} \circ g\right) = \sigma_1\left(g_\star\right)\,, \quad \text{with} \;\; g_\star = r_\star\, g\, r_\star^T\,. \tag{A.47}$$

Let us compute the rotated metric assuming a block decomposition for g:

$$g = \begin{pmatrix} g_{11} & \vec{A}^T \\ \vec{A} & \hat{g} \end{pmatrix}\,. \tag{A.48}$$

A small calculation gives

$$g_\star = r_\star\, g\, r_\star^T = \begin{pmatrix} r_{11} g_{11} r_{11} & r_{11}(g_{11}\vec{V} + \hat{r}\vec{A})^T \\ r_{11}(g_{11}\vec{V} + \hat{r}\vec{A}) & (g_{11}\vec{V} + \hat{r}\vec{A})\vec{V}^T + \vec{V}\vec{A}^T\hat{r}^T + \hat{r}\,\hat{g}\,\hat{r}^T \end{pmatrix}\,. \tag{A.49}$$

Choosing

$$r_{11} = \frac{1}{\sqrt{|g_{11}|}}\,, \quad \vec{V} = -\frac{1}{g_{11}}\,\hat{r}\vec{A}\,, \tag{A.50}$$

we find

$$g_\star = \begin{pmatrix} \text{sgn}(g_{11}) & \vec{0}^T \\ \vec{0} & \hat{r}\left(\hat{g} - \frac{1}{g_{11}}\vec{A}\,\vec{A}^T\right)\hat{r}^T \end{pmatrix}\,. \tag{A.51}$$

By the general result on diagonalization of quadratic forms, we can choose $\hat{r}$ in such a way that the lower-right block becomes a diagonal matrix with entries equal to plus or minus ones,

$$g_\star = \begin{pmatrix} \text{sgn}(g_{11}) & \vec{0}^T \\ \vec{0} & \hat{k} \end{pmatrix}\,, \quad \hat{k} \text{ diagonal with } \pm 1 \text{ entries.} \tag{A.52}$$

By Sylvester's theorem of inertia, the matrix $g_\star$ has a single -1 entry. Thus either g_{11} is negative and $g_\star = k$, with k the Minkowski metric, or $g_{11} > 0$ and $g_\star = k_i$, for some $i \neq 1$. In either case we know how to determine the sign factor:

$$\sigma_1(g) = \sigma_1(g_\star) = \text{sgn}(g_{11})\,. \tag{A.53}$$

Since our choice of the first coordinate was just irrelevant, this holds for a factorized T-duality about any coordinate. Our final result is therefore

$$\sigma_i(\mathcal{H}) = \text{sgn}(g_{ii})\,. \tag{A.54}$$

Equivalently, $\sigma_i = 1$ for a T-duality along a coordinate direction x^i that is space-like, and $\sigma_i = -1$ for a T-duality along a coordinate direction that is timelike.

In order to use (A.54) for the successive application of several T-dualities, we have to keep in mind that each action of h_i transforms the full metric g_{ij} non-trivially and therefore the full sign factor cannot be inferred from the signs of the diagonal entries of the initial metric g_{ij}.

For the special case of the transformation J, i.e., T-dualities along all coordinates, however, we can show that $\sigma_J(\mathcal{H}) = -1$ as follows.

Under the $O(D,D)$ transformation J, $\mathcal{H}$ transforms as

$$\mathcal{H}' \;=\; J\mathcal{H}J\,. \tag{A.55}$$

We define $h \equiv h_e^{-1}h_b$ and rewrite $\mathcal{H}$ and $\mathcal{H}'$ as

$$\mathcal{H} \;=\; h^T\mathcal{H}_0 h\,, \qquad \mathcal{H}' \;=\; h^{-1}\mathcal{H}_0(h^{-1})^T\,. \tag{A.56}$$

With the corresponding spin representative $S_h = S_e^{-1}S_b$ of h we then have, by definition,

$$S_{\mathcal{H}} \;=\; S_h^\dagger S_{\mathcal{H}_0}S_h\,, \qquad S_{\mathcal{H}'} \;=\; S_{h^{-1}}S_{\mathcal{H}_0}S_{h^{-1}}^\dagger\,. \tag{A.57}$$

Using that $S_J^{-1} = S_J^\dagger$ we have

$$(S_J^{-1})^\dagger S_{\mathcal{H}}\,S_J^{-1} \;=\; S_J S_h^\dagger S_{\mathcal{H}_0}S_h\,S_J^\dagger \;=\; \Big[(S_h)^{-1}S_hS_JS_h^\dagger\Big]\,S_{\mathcal{H}_0}\,\Big[S_hS_J^\dagger S_h^\dagger(S_h^\dagger)^{-1}\Big]\,. \tag{A.58}$$

J is an invariant matrix, $hJh^T = J$, and thus in $\mathrm{Pin}(D,D)$ we have

$$S_hS_JS_h^\dagger \;=\; \pm S_J\,. \tag{A.59}$$

We can thus simplify (A.58)

$$(S_J^{-1})^\dagger S_{\mathcal{H}}\,S_J^{-1} \;=\; S_h^{-1}S_JS_{\mathcal{H}_0}S_J^\dagger(S_h^\dagger)^{-1} \;=\; -S_h^{-1}S_{\mathcal{H}_0}(S_h^{-1})^\dagger \;=\; -S_{\mathcal{H}'}\,, \tag{A.60}$$

where we used $S_JS_{\mathcal{H}_0}S_J^\dagger \;=\; -S_{\mathcal{H}_0}$ in the second equality and (A.57) for the last equality. We have thus shown that $\sigma_J(\mathcal{H}) = -1$.

A.3 Example

Next, we present an instructive example concerning the above rules of sign factors. We construct a closed loop in the space of $\mathcal{H}$ in $SO^-(D,D)$ that cannot be lifted to a closed loop in $\mathrm{Spin}^-(D,D)$.

Consider for $D = 2$ the one-parameter family of $SO^+(D,D)$ transformations parameterized by α:

$$h(\alpha) \;=\; \exp\big[\alpha\,T\big]\,, \qquad \alpha\in\big[\,0,\frac{\pi}{2}\big]\,, \tag{A.61}$$

where T is the Lie algebra generator

$$T \;\equiv\; T^{14}+T^{12}+T^{32}+T^{34} \;=\; \begin{pmatrix} 0 & 1 & 0 & 1\\ -1 & 0 & -1 & 0\\ 0 & 1 & 0 & 1\\ -1 & 0 & -1 & 0\end{pmatrix}\,, \tag{A.62}$$

and T^{MN} are the standard fundamental generators (2.5). A computation gives:

$$h(\alpha) = \begin{pmatrix} \cos^2\alpha & \frac{1}{2}\sin 2\alpha & -\sin^2\alpha & \frac{1}{2}\sin 2\alpha\\ -\frac{1}{2}\sin 2\alpha & \cos^2\alpha & -\frac{1}{2}\sin 2\alpha & -\sin^2\alpha\\ -\sin^2\alpha & \frac{1}{2}\sin 2\alpha & \cos^2\alpha & \frac{1}{2}\sin 2\alpha\\ -\frac{1}{2}\sin 2\alpha & \sin^2\alpha & -\frac{1}{2}\sin 2\alpha & \cos^2\alpha\end{pmatrix}\,. \tag{A.63}$$

Since $T^t = -T$, we have $h(\alpha)^t = h(\alpha)^{-1}$. For later use we also note that (A.61) can be defined for arbitrary α, which then has the periodicity $h(\alpha) = h(\alpha+\pi)$. This family of transformations was designed so that for $\alpha = \pi/2$ we get the product of the two T-dualities h_1 and h_2:

$$h(\tfrac{\pi}{2}) \;=\; \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \;=\; h_1\, h_2\ . \tag{A.64}$$

Consider now the 'flat' generalized metric $\mathcal{H}_0 = \mathrm{diag}(k,k) \in SO^-(D,D)$. The $SO^+(D,D)$ transformations $h(\alpha)$ acting on this generalized metric give us a family of rotated metrics,

$$\mathcal{H}(\alpha) \;=\; (h(\alpha)^{-1})^t\, \mathcal{H}_0\, h(\alpha)^{-1} \;=\; h(\alpha)\, \mathcal{H}_0\, h(\alpha)^{-1}\,. \tag{A.65}$$

A computation of the matrix product gives

$$\mathcal{H}(\alpha) = \begin{pmatrix} -\cos^2 2\alpha & \frac{1}{2}\sin 4\alpha & \sin^2 2\alpha & \frac{1}{2}\sin 4\alpha \\ \frac{1}{2}\sin 4\alpha & \cos^2 2\alpha & \frac{1}{2}\sin 4\alpha & -\sin^2 2\alpha \\ \sin^2 2\alpha & \frac{1}{2}\sin 4\alpha & -\cos^2 2\alpha & \frac{1}{2}\sin 4\alpha \\ \frac{1}{2}\sin 4\alpha & -\sin^2 2\alpha & \frac{1}{2}\sin 4\alpha & \cos^2 2\alpha \end{pmatrix}\,. \tag{A.66}$$

As it turns out, the transformation h_1h_2 leaves $\mathcal{H}_0$ invariant, thus $\mathcal{H}(\alpha)$ traces a closed curve as $\alpha \in [0,\pi/2]$:

$$\mathcal{H}(0) = \mathcal{H}(\tfrac{\pi}{2}) = \mathcal{H}_0\,. \tag{A.67}$$

For general α, the metric and b field read off from $\mathcal{H} = \mathcal{H}_{\bullet\bullet}$ are

$$g_{ij}(\alpha) \;=\; \begin{pmatrix} -1 & \tan 2\alpha \\ \tan 2\alpha & 1 \end{pmatrix}\,, \qquad b_{ij}(\alpha) \;=\; \begin{pmatrix} 0 & \tan 2\alpha \\ -\tan 2\alpha & 0 \end{pmatrix}\,. \tag{A.68}$$

For $\alpha = \frac{\pi}{4}$ both the metric and the b field components become infinite, even though the generalized metric is still perfectly regular:

$$\mathcal{H}(\tfrac{\pi}{4}) \;=\; \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\,. \tag{A.69}$$

At this singular point we expect that our explicit formula for $S_{\mathcal{H}}$ is affected by some kind of 'phase transition'.

Next we turn to the study of the corresponding elements in Spin(D,D). Since $h(\alpha)$ is in the component of the group connected to the identity, its spinor representative follows directly from (A.62),

$$S(\alpha) \;\equiv\; S_{h(\alpha)} \;=\; \exp\big[\,\alpha\,\hat{\Gamma}\,\big]\,, \qquad \hat{\Gamma} \;\equiv\; \frac{1}{2}\Big(\Gamma^{14} + \Gamma^{12} + \Gamma^{32} + \Gamma^{34}\Big)\,. \tag{A.70}$$

Recalling that $\Gamma^{MN} = \frac{1}{2}(\Gamma^M\Gamma^N - \Gamma^N\Gamma^M)$ and that $\Gamma^M = \sqrt{2}\,(\psi_1,\psi_2,\psi^1,\psi^2)$ we infer

$$\hat{\Gamma} \;=\; \psi_1\psi^2 + \psi_1\psi_2 + \psi^1\psi_2 + \psi^1\psi^2 \;=\; (\psi^1+\psi_1)(\psi^2+\psi_2)\,. \tag{A.71}$$

As $\hat{\Gamma}^2 = -\mathbf{1}$, we get in closed form

$$S(\alpha) \;=\; \cos\alpha \cdot \mathbf{1} + \sin\alpha \cdot \hat{\Gamma}\,. \tag{A.72}$$

We can now investigate its action on the spinor representative for $\mathcal{H}_0$, which we choose to be

$$S_{\mathcal{H}_0} \;=\; \psi^1\psi_1 - \psi_1\psi^1\,, \tag{A.73}$$

where we denoted the timelike direction by 1. We then define

$$S_{\mathcal{H}}(\alpha) \;\equiv\; (S(\alpha)^{-1})^\dagger\, S_{\mathcal{H}_0}\, S(\alpha)^{-1}\,, \tag{A.74}$$

such that $S_{\mathcal{H}}(0) = S_{\mathcal{H}_0}$. Taking the ρ homomorphism of (A.74) we conclude that $S_{\mathcal{H}}(\alpha)$, so defined, is

$$S_{\mathcal{H}}(\alpha) \;=\; \pm\, S_{\mathcal{H}(\alpha)}\,. \tag{A.75}$$

Since the plus sign holds for $\alpha = 0$ and both sides appear to be defined by continuous deformations, it is puzzling that the sign becomes minus at some point. This is what we want to understand.

The explicit calculation of (A.74) gives with (A.72)

$$S_{\mathcal{H}}(\alpha) \;=\; \cos(2\alpha)\Big[\psi^1\psi_1 - \psi_1\psi^1 \;-\; \tan(2\alpha)(\psi^1 - \psi_1)(\psi^2 + \psi_2)\Big]\,. \tag{A.76}$$

Recalling that $\mathcal{H}(\frac{\pi}{2}) = \mathcal{H}_0$, we observe that

$$S_{\mathcal{H}}(\tfrac{\pi}{2}) \;=\; -(\psi^1\psi_1 - \psi_1\psi^1) \;=\; -S_{\mathcal{H}_0} = -S_{\mathcal{H}(\frac{\pi}{2})}\,. \tag{A.77}$$

We have gotten now a minus sign in (A.75). Alternatively, while $\alpha \in [0, \pi/2]$ gives a closed loop for $\mathcal{H}(\alpha)$ it gives an open loop for $S_{\mathcal{H}}(\alpha)$. The aim of the following discussion is to see how this minus sign arises.

We should compare $S_{\mathcal{H}}(\alpha)$ with the family $S_{\mathcal{H}(\alpha)}$, which can be defined independently as:

$$S_{\mathcal{H}(\alpha)} \;\equiv\; S^\dagger_{b(\alpha)}\, S^{-1}_{g(\alpha)}\, S_{b(\alpha)}\,, \quad \text{with} \quad S^{-1}_{g(\alpha)} \;=\; (S^{-1}_{e(\alpha)})^\dagger\, S_k\, S^{-1}_{e(\alpha)}\,. \tag{A.78}$$

In this definition one must extract $g(\alpha)$ and $b(\alpha)$ from $\mathcal{H}(\alpha)$ and use $g(\alpha)$ to define a vielbein $e(\alpha)$ from $g(\alpha) = e(\alpha)ke(\alpha)^t$. The potential difficulty here is the possibility that divergent g's can lead to discontinuous e's and thus a discontinuous definition of $S_{\mathcal{H}}$.

We begin the calculation by computing the vielbein using $g = e\,k\,e^T$ and the metric from (A.68). The vielbein is not unique, but one representative is

$$e(\alpha) \;=\; \begin{pmatrix} |\sec 2\alpha| & \tan 2\alpha \\ 0 & 1 \end{pmatrix}\,. \tag{A.79}$$

Next, we need a matrix E such that $e = \exp(E)$:

$$E \;=\; \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} \quad\to\quad \exp(E) = \begin{pmatrix} e^u & \frac{v}{u}(e^u - 1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} |\sec 2\alpha| & \tan 2\alpha \\ 0 & 1 \end{pmatrix}\,. \tag{A.80}$$

We then compute

$$S_e^{-1} \;=\; \sqrt{\det e}\, e^{-\psi^1 E_1{}^1\psi_1 - \psi^1 E_1{}^2\psi_2} \;=\; \sqrt{\det e}\, e^{-u\psi^1\psi_1 - v\psi^1\psi_2}\ . \tag{A.81}$$

The exponential can be worked out explicitly, giving

$$S_e^{-1} \;=\; \sqrt{\det e}\,\big(1 - e^{-u}(e^u - 1))(\psi^1\psi_1 + vu^{-1}\psi^1\psi_2)\big)\ . \tag{A.82}$$

Therefore, using (A.80) we find

$$S_e^{-1} = |\sec 2\alpha|^{1/2}\Big(1 - |\cos 2\alpha|\Big[(|\sec 2\alpha| - 1)\psi^1\psi_1 + \tan 2\alpha\,\psi^1\psi_2\Big]\Big)\ . \tag{A.83}$$

We then obtain for the metric

$$S_g^{-1} \;=\; |\sec 2\alpha|\Big(\cos^2 2\alpha\,\psi^1\psi_1 - \psi_1\psi^1 - \sin 2\alpha\cos 2\alpha(\psi^1\psi_2 + \psi^2\psi_1) + \sin^2 2\alpha\,\psi^2\psi_1\psi^1\psi_2\Big)\ . \tag{A.84}$$

The b-field contributions are given by

$$S_b \;=\; e^{-\frac{1}{2}b_{ij}\psi^i\psi^j} \;=\; 1 - b_{12}\psi^1\psi^2\ , \tag{A.85}$$

while all higher terms vanish in $D = 2$. Using (A.68),

$$S_b \;=\; 1 - \tan(2\alpha)\,\psi^1\,\psi^2\ , \qquad S_b^\dagger \;=\; 1 - \tan(2\alpha)\,\psi_2\,\psi_1\ . \tag{A.86}$$

After some further calculation we get

$$\begin{aligned} S_{\mathcal{H}(\alpha)} \;&=\; |\cos(2\alpha)|(\psi^1\psi_1 - \psi_1\psi^1) - \frac{|\cos(2\alpha)|}{\cos(2\alpha)}\sin(2\alpha)(\psi^1 - \psi_1)(\psi^2 + \psi_2) \\ &=\; \mathrm{sgn}(\cos(2\alpha))\, S_{\mathcal{H}}(\alpha)\ , \end{aligned} \tag{A.87}$$

where sgn denotes the sign of its argument, and we compared with (A.76). This result is perfectly consistent with the sign change found in (A.77). For small values of $\alpha > 0$, the sign is positive and so this agrees with (A.75) using the $+$ sign. At $\alpha = \frac{\pi}{4}$, $S_{\mathcal{H}(\alpha)}$ is discontinuous. For $\alpha > \frac{\pi}{4}$, (A.75) holds for the minus sign, as it should be in order to be consistent with the final relative sign at $\alpha = \frac{\pi}{2}$.

Let us finally reconsider the above analysis in a different approach. Specifically, since we saw above that the sign change occurs at a singular point for which g degenerates and $S_{\mathcal{H}}$ becomes ill-defined, it is natural to inquire what happens if one employs a definition that only requires $\mathcal{H}$ to be regular, but not necessarily decomposable into h_g and h_b. Such a definition can indeed be given by separating $\mathcal{H}(\alpha)$ into two pieces:

$$\mathcal{H}(\alpha) = \mathcal{H}_0(\mathcal{H}_0\mathcal{H}(\alpha)) \equiv \mathcal{H}_0\hat{\mathcal{H}}(\alpha)\ , \tag{A.88}$$

where $\mathcal{H}_0$ is an $O(D,D)$ element disconnected from the identity while $\hat{\mathcal{H}}$ is an $O(D,D)$ element connected to the identity. Thus, $\hat{\mathcal{H}}$ can be written in exponential form. Indeed, with the explicit forms (A.63) and (A.66) one can verify

$$\hat{\mathcal{H}}(\alpha) \;=\; \exp\big[-2\alpha\,T\big]\ . \tag{A.89}$$

Therefore, its spinor representative can also be defined as an exponential which, after choosing $S_{\mathcal{H}_0}$, gives a spinor representative for $\mathcal{H}(\alpha)$ according to (A.88).

At first sight, this leads to a well-defined and smooth spinor representative for all α. There is a subtlety, however, which is due to the following periodicity of $\hat{\mathcal{H}}$,

$$\hat{\mathcal{H}}(\alpha) \;=\; \hat{\mathcal{H}}(-\tfrac{\pi}{2}+\alpha)\;, \tag{A.90}$$

following analogously to the periodicity of $h(\alpha)$ noted after (A.63). Consequently, given an $\mathcal{H}(\alpha)$, there is no unique parameter value α that reproduces this generalized metric according to (A.89), and therefore there is no unique spinor representative of $\hat{\mathcal{H}}$. More precisely, if we attempt to define the exponential form of $S_{\hat{\mathcal{H}}}$ by replacing T by $\hat{\Gamma}$ in (A.89), there are actually two choices,

$$S_{\hat{\mathcal{H}}(\alpha)} \;=\; \begin{cases} \exp\big[-2\alpha\,\hat{\Gamma}\big] \\ \exp\big[(-2\alpha+\pi)\,\hat{\Gamma}\big] \end{cases} . \tag{A.91}$$

Since, using (A.72), $\exp[\pi\hat{\Gamma}] = -1$, these two choices differ precisely by a sign. This has the consequence that there is no continuos and single-valued way to choose the spin representative over the complete path of $\mathcal{H}(\alpha)$. In fact, since the path is closed, single-valuedness requires $S_{\hat{\mathcal{H}}(0)} = S_{\hat{\mathcal{H}}(\frac{\pi}{2})} = \mathbf{1}$. This, in turn, can only be achieved if we choose in (A.91) the first parametrization of $S_{\hat{\mathcal{H}}}$ for $\alpha = 0$ and the second parametrization for $\alpha = \frac{\pi}{2}$. Thus, at some point in the interval $(0, \frac{\pi}{2})$ we need to change the parametrization, leading to a non-continuous $S_{\hat{\mathcal{H}}}$ and $S_{\mathcal{H}}$. (In the previous approach, this point was at $\alpha = \frac{\pi}{4}$.) Thus, we conclude that while in this approach the 'point of discontinuity' can be chosen arbitrarily in the interval $(0, \frac{\pi}{2})$, the associated sign change as in (A.77) is unavoidable.

References

[1] J. Polchinski, "String Theory," vol. 2. Cambridge University Press (1998).

[2] O. Hohm, S. K. Kwak and B. Zwiebach, "Unification of Type II Strings and T-duality," arXiv:1106.5452 [hep-th].

[3] M. Dine, P. Y. Huet and N. Seiberg, "Large and Small Radius in String Theory," Nucl. Phys. B **322**, 301 (1989).

[4] C. M. Hull, P. K. Townsend, "Unity of superstring dualities," Nucl. Phys. **B438**, 109-137 (1995). [hep-th/9410167].

[5] E. Bergshoeff, C. M. Hull and T. Ortin, "Duality in the type II superstring effective action," Nucl. Phys. B **451**, 547 (1995) [arXiv:hep-th/9504081].

[6] S. F. Hassan, "SO(d,d) transformations of Ramond-Ramond fields and space-time spinors," Nucl. Phys. B **583**, 431 (2000) [arXiv:hep-th/9912236].
"T duality, space-time spinors and RR fields in curved backgrounds," Nucl. Phys. B **568**, 145 (2000) [arXiv:hep-th/9907152].

[7] M. Cvetic, H. Lu, C. N. Pope and K. S. Stelle, "T duality in the Green-Schwarz formalism, and the massless / massive IIA duality map, Nucl. Phys. B **573**, 149 (2000) [arXiv:hep-th/9907202].
B. Kulik and R. Roiban, "T duality of the Green-Schwarz superstring," JHEP **0209**, 007 (2002) [arXiv:hep-th/0012010].
I. A. Bandos and B. Julia, "Superfield T duality rules," JHEP **0308**, 032 (2003) [arXiv:hep-th/0303075].

[8] R. Benichou, G. Policastro and J. Troost, "T-duality in Ramond-Ramond backgrounds," Phys. Lett. B **661**, 192 (2008) [arXiv:0801.1785 [hep-th]].

[9] K. A. Meissner and G. Veneziano, "Symmetries of cosmological superstring vacua," Phys. Lett. B **267** (1991) 33; "Manifestly O(d,d) invariant approach to space-time dependent string vacua," Mod. Phys. Lett. A **6** (1991) 3397 [arXiv:hep-th/9110004].

[10] J. Maharana and J. H. Schwarz, "Noncompact symmetries in string theory," Nucl. Phys. B **390** (1993) 3 [arXiv:hep-th/9207016].

[11] C. Hull, B. Zwiebach, "Double Field Theory," JHEP **0909**, 099 (2009). [arXiv:0904.4664 [hep-th]].

[12] C. Hull, B. Zwiebach, "The Gauge algebra of double field theory and Courant brackets," JHEP **0909**, 090 (2009). [arXiv:0908.1792 [hep-th]].

[13] O. Hohm, C. Hull and B. Zwiebach, "Background independent action for double field theory," JHEP **1007** (2010) 016 [arXiv:1003.5027 [hep-th]].

[14] O. Hohm, C. Hull and B. Zwiebach, "Generalized metric formulation of double field theory," JHEP **1008** (2010) 008 [arXiv:1006.4823 [hep-th]].

[15] W. Siegel, "Superspace duality in low-energy superstrings," Phys. Rev. D **48**, 2826 (1993) [arXiv:hep-th/9305073],
"Two vierbein formalism for string inspired axionic gravity," Phys. Rev. D **47**, 5453 (1993) [arXiv:hep-th/9302036].

[16] A. A. Tseytlin, "Duality Symmetric Formulation Of String World Sheet Dynamics," Phys. Lett. B **242**, 163 (1990); "Duality Symmetric Closed String Theory And Interacting Chiral Scalars," Nucl. Phys. B **350**, 395 (1991).

[17] S. K. Kwak, "Invariances and Equations of Motion in Double Field Theory," JHEP **1010** (2010) 047 [arXiv:1008.2746 [hep-th]].

[18] O. Hohm, S. K. Kwak, "Frame-like Geometry of Double Field Theory," J. Phys. A **A44**, 085404 (2011). [arXiv:1011.4101 [hep-th]].

[19] O. Hohm, "T-duality versus Gauge Symmetry," arXiv:1101.3484 [hep-th].

[20] O. Hohm, "On factorizations in perturbative quantum gravity," JHEP **1104**, 103 (2011). [arXiv:1103.0032 [hep-th]].

[21] O. Hohm and S. K. Kwak, "Double Field Theory Formulation of Heterotic Strings," arXiv:1103.2136 [hep-th].

[22] D. S. Berman and M. J. Perry, "Generalized Geometry and M theory," arXiv:1008.1763 [hep-th].
D. S. Berman, H. Godazgar and M. J. Perry, "SO(5,5) duality in M-theory and generalized geometry," Phys. Lett. B **700**, 65 (2011) [arXiv:1103.5733 [hep-th]].

[23] P. West, "E_{11}, generalised space-time and IIA string theory," Phys. Lett. B **696**, 403 (2011) [arXiv:1009.2624 [hep-th]].
A. Rocen and P. West, "E11, generalised space-time and IIA string theory: the R-R sector," arXiv:1012.2744 [hep-th].

[24] I. Jeon, K. Lee and J. H. Park, "Differential geometry with a projection: Application to double field theory," JHEP **1104** (2011) 014, arXiv:1011.1324 [hep-th],
"Double field formulation of Yang-Mills theory," arXiv:1102.0419 [hep-th],
"Stringy differential geometry, beyond Riemann," arXiv:1105.6294 [hep-th].

[25] N. B. Copland, "Connecting T-duality invariant theories," [arXiv:1106.1888 [hep-th]].

[26] D. Andriot, M. Larfors, D. Lust, P. Patalong, "A ten-dimensional action for non-geometric fluxes," [arXiv:1106.4015 [hep-th]].

[27] D. C. Thompson, "Duality Invariance: From M-theory to Double Field Theory," [arXiv:1106.4036 [hep-th]].

[28] M. Fukuma, T. Oota and H. Tanaka, "Comments on T-dualities of Ramond-Ramond potentials on tori," Prog. Theor. Phys. **103** (2000) 425 [arXiv:hep-th/9907132].

[29] D. Brace, B. Morariu and B. Zumino, "T duality and Ramond-Ramond backgrounds in the matrix model," Nucl. Phys. B **549**, 181 (1999) [arXiv:hep-th/9811213].
"Dualities of the matrix model from T duality of the Type II string," Nucl. Phys. B **545**, 192 (1999) [arXiv:hep-th/9810099].

[30] C. M. Hull, "Timelike T duality, de Sitter space, large N gauge theories and topological field theory," JHEP **9807** (1998) 021. [hep-th/9806146].

[31] M. Gualtieri, "Generalized complex geometry," PhD Thesis (2004). arXiv:math/0401221v1 [math.DG]

[32] W. Fulton and J. Harris, "Representation theory: a first course," Springer Verlag, New York Inc. (1991).

[33] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, A. Van Proeyen, "New formulations of D = 10 supersymmetry and D8 - O8 domain walls," Class. Quant. Grav. **18**, 3359-3382 (2001). [hep-th/0103233].

[34] C. Hillmann, "Generalized E(7(7)) coset dynamics and D=11 supergravity," JHEP **0903**, 135 (2009). [arXiv:0901.1581 [hep-th]].

[35] P. C. West, "E(11) and M theory," Class. Quant. Grav. **18**, 4443 (2001) [arXiv:hep-th/0104081].
"E(11), SL(32) and central charges," Phys. Lett. B **575** (2003) 333 [arXiv:hep-th/0307098].

[36] T. Damour, M. Henneaux and H. Nicolai, "E(10) and a 'small tension expansion' of M theory," Phys. Rev. Lett. **89**, 221601 (2002) [arXiv:hep-th/0207267].

[37] A. Kleinschmidt and H. Nicolai, "E(10) and SO(9,9) invariant supergravity," JHEP **0407** (2004) 041 [arXiv:hep-th/0407101].