# CURVES OVER EVERY GLOBAL FIELD VIOLATING THE LOCAL-GLOBAL PRINCIPLE

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Abstract. There is an algorithm that takes as input a global field $k$ and produces a curve over $k$ violating the local-global principle. Also, given a global field $k$ and a nonnegative integer $n$, one can effectively construct a curve $X$ over $k$ such that $\#X(k) = n$.

## 1. Introduction

Let $k$ be a global field, by which we mean a finite extension of either $\mathbb{Q}$ or $\mathbb{F}_p(t)$ for some prime $p$. Let $\Omega_k$ be the set of nontrivial places of $k$. For each $v \in \Omega_k$, let $k_v$ be the completion of $k$ at $v$. By variety, we mean a separated scheme of finite type over a field. A curve is a variety of dimension 1. Call a variety nice if it is smooth, projective, and geometrically integral. Say that a $k$-variety $X$ satisfies the local-global principle if the implication

$$X(k_v) \neq \emptyset \text{ for all } v \in \Omega_k \implies X(k) \neq \emptyset$$

holds.

Nice genus-0 curves (and more generally, quadrics in $\mathbb{P}^n$) satisfy the local-global principle: this follows from the Hasse-Minkowski theorem for quadratic forms. The first examples of varieties violating the local-global principle were genus-1 curves, such as the smooth projective model of $2y^2 = 1 - 17x^4$, over $\mathbb{Q}$, discovered by Lind [Lin40] and Reichardt [Rei42].

Our goal is to prove that there exist curves over every global field violating the local-global principle. We can also produce curves having a prescribed positive number of $k$-rational points. In fact, such examples can be constructed effectively:

**Theorem 1.1.** *There is an algorithm that takes as input a global field $k$ and a nonnegative integer $n$, and outputs a nice curve $X$ over $k$ such that $\#X(k) = n$ and $X(k_v) \neq \emptyset$ for all $v \in \Omega_k$.*

*Remark* 1.2. For the sake of definiteness, let us assume that $k$ is presented by giving the minimal polynomial for a generator of $k$ as an extension of $\mathbb{Q}$ or $\mathbb{F}_p(t)$. The output can be described by giving a finite list of homogeneous polynomials that cut out $X$ in some $\mathbb{P}^n$. For more details on representation of number-theoretic and algebraic-geometric objects, see [Len92, §2] and [BGJGP05, §5].

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## 2. Proof

**Lemma 2.1.** *Given a global field $k$, one can effectively construct a nice curve $Z$ over $k$ such that $Z(k)$ is finite, nonempty, and computable.*

*Proof.* First suppose that $\operatorname{char} k = 0$. Let $E$ be the elliptic curve $X_1(11)$ over $k$. By computing a Selmer group, compute an integer $r$ strictly greater than the rank of the finitely generated abelian group $E(k)$. Let $Z = X_1(11^r)$ over $k$. By [DS05, Theorem 6.6.6], the Jacobian $J_Z$ of $Z$ is isogenous to a product of $E^r$ with another abelian variety over $k$ (geometrically, these $r$ copies of $E$ in $J_Z$ arise from the degeneracy maps $Z \to E$ indexed by $s \in \{1, \dots, r\}$ that in moduli terms send $(A, P)$ to $(A/\langle 11^s P\rangle, 11^{s-1}P)$ where $A$ is an elliptic curve and $P$ is a point on $A$ of exact order $11^r$). So the Dem'janenko-Manin method [Dem66, Man69] yields an upper bound on the height of points in $Z(k)$. In particular, $Z(k)$ is finite and computable. It is also nonempty, since the cusp $\infty$ on $X_1(11^r)$ is a rational point.

If $\operatorname{char} k > 0$, let $Z$ be any nonisotrivial curve of genus greater than 1 such that $Z(k)$ is nonempty: for instance, let $a$ be a transcendental element of $k$, and use the curve $C_a$ in the first paragraph of the proof of Theorem 1.4 in [PP08]. Then $Z(k)$ is finite by [Sam66, Théorème 4], and computable because of the height bound proved in [Szp81, §8, Corollaire 2]. □

**Lemma 2.2.** *Given a global field $k$ and a nonnegative integer $n$, one can effectively construct a nice curve $Y$ over $k$ such that $Y(k)$ is finite, computable, and of size at least $n$.*

*Proof.* Construct $Z$ as in Lemma 2.1. Let $\kappa(Z)$ denote the function field of $Z$. Find a closed point $P \in Z - Z(k)$ whose residue field is separable over $k$.

If $\operatorname{char} k = 0$, the Riemann-Roch theorem, which can be made constructive, together with a little linear algebra, lets us find $f \in \kappa(Z)$ taking the value 1 at each point of $Z(k)$, and having a simple pole at $P$. If $\operatorname{char} k = p > 2$, instead find $t \in \kappa(Z)$ such that $t$ has a pole at $P$ and nowhere else, and such that $t$ takes the value 1 at each point of $Z(k)$; then let $f = t + g^p$ for some $g \in \kappa(Z)$ such that $g$ has a pole at $P$ of odd order greater than the order of the pole of $t$ at $P$ and no other poles, such that $g$ is zero at each point of $Z(k)$, and such that $t + g^p$ is nonzero at each zero of $dt$; this ensures that $f$ has an odd order pole at $P$ and no other poles, and is 1 at each point of $Z(k)$, and has only simple zeros (since $f$ and $df = dt$ do not simultaneously vanish). In either case, $f$ has an odd order pole at $P$, so $\kappa(Z)(\sqrt{f})$ is ramified over $\kappa(Z)$ at $P$, so the regular projective curve $Y$ with $\kappa(Y) = \kappa(Z)(\sqrt{f})$ is geometrically integral. A local calculation shows that $Y$ is also smooth, so $Y$ is nice. Equations for $Y$ can be computed by resolving singularities of an initial birational model. The points in $Z(k)$ split in $Y$, so $\#Y(k) = 2\#Z(k)$, and $Y(k)$ is computable. Iterating this paragraph eventually produces a curve $Y$ with enough points.

If $\operatorname{char} k = 2$, use the same argument, but instead adjoin to $\kappa(Z)$ a solution $\alpha$ to $\alpha^2 - \alpha = f$, where $f \in \kappa(Z)$ has a pole of high odd order at $P$, no other poles, and a zero at each point of $Z(k)$. □

*Proof of Theorem 1.1.* Given $k$ and $n$, apply Lemma 2.2 to find $Y$ over $k$ with $Y(k)$ finite, computable, and of size at least $n + 4$. Write $Y(k) = \{y_1, \dots, y_m\}$. Find a closed point $P \in Y - Y(k)$ with residue field separable over $k$.

Suppose that $\operatorname{char} k \neq 2$. Compute $a, b \in k^\times$ whose images in $k^\times/k^{\times 2}$ are $\mathbb{F}_2$-independent. Let $S$ be the set of places $v \in k$ such that $a$, $b$, and $ab$ are all nonsquares in $k_v$. By Hensel's lemma, if $v \nmid 2, \infty$ and $v(a) = v(b) = 0$, then $v \notin S$. So $S$ is finite and computable. Let $w \in \Omega_k - S$. Weak approximation [AW45, Theorem 1], whose proof is constructive, lets us find $c \in k^\times$ such that $c$ is a square in $k_v$ for all $v \in S$ and $w(c)$ is odd. The purpose of $w$ is to ensure that $c$ is not a square in $k$. Find $f \in \kappa(Y)^\times$ such that $f$ has an odd order pole at $P$ and a simple zero at each of $y_1, \ldots, y_n$, and such that $f(y_{n+1}) = a$, $f(y_{n+2}) = b$, $f(y_{n+3}) = ab$, and $f(y_{n+4}) = \cdots = f(y_m) = c$. If $\operatorname{char} k = p > 2$, the same argument as in the proof of Lemma 2.2 lets us arrange in addition that $f$ has no poles other than $P$, and that all zeros of $f$ are simple. Construct the nice curve $X$ whose function field is $\kappa(Y)(\sqrt{f})$. Then $X \to Y$ maps $X(k)$ bijectively to $\{y_1, \ldots, y_n\}$, so $X(k)$ is computable and of size $n$. Also, for each $v \in \Omega_k$, at least one of $a, b, ab, c$ is a square in $k_v$, so $X(k_v) \neq \emptyset$.

If $\operatorname{char} k = 2$, use the same argument, with the following modifications. For any extension $L$ of $k$, define the additive homomorphism $\wp \colon L \to L$ by $\wp(t) = t^2 - t$. Construct $a, b \in k$ such that the images of $a$ and $b$ in $k/\wp(k)$ are $\mathbb{F}_2$-independent. Let $S$ be the set of places $v \in k$ such that $a$, $b$, and $a + b$ are all outside $\wp(k_v)$. As before, $S$ is finite and computable. Choose $w \in \Omega_k - S$. Use weak approximation to find $c \in k$ such that $c \in \wp(k_v)$ for all $v \in S$ but $c \notin \wp(k_w)$. Find $f \in \kappa(Y)$ such that $f$ has a pole of high odd order at $P$, a simple pole at $y_1, \ldots, y_n$, and no other poles, and such that $f(y_{n+1}) = a$, $f(y_{n+2}) = b$, $f(y_{n+3}) = a + b$, and $f(y_{n+4}) = \cdots = f(y_m) = c$. Construct the nice curve $X$ whose function field is obtained by adjoining to $\kappa(Y)$ a solution $\alpha$ to $\alpha^2 - \alpha = f$. □

## 3. Other constructions of curves violating the local-global principle

3.1. **Lefschetz pencils in a Châtelet surface.** J.-L. Colliot-Thélène has suggested another approach to constructing curves violating the local-global principle, which we now sketch. For any global field $k$, there exists a Châtelet surface over $k$ violating the local-global principle: see [Poo09, Proposition 5.1] and [Vir09, Theorem 1.1]. Let $V$ be such a surface. Choose a projective embedding of $V$. By [Kat73, Théorème 2.5], after replacing $V$ by a $d$-uple embedding for some $d \geq 1$, there is a Lefschetz pencil of hyperplane sections of $V$, fitting together into a family $\tilde{V} \to \mathbb{P}^1$, where $\tilde{V}$ is the blowup of $V$ along the intersection of $V$ with the axis of the pencil. Since $\tilde{V} \to V$ is a birational morphism, the Lang-Nishimura theorem (see [Nis55], [Lan54, Theorem 3], and also [CTCS80, Lemme 3.1.1]) shows that $\tilde{V}$ has a $k$-point if and only if $V$ does, and the same holds with $k$ replaced by any completion $k_v$. By definition of Lefschetz pencil, each geometric fiber of the pencil is either an integral curve or a union of two nice curves intersecting transversely in a single point. By requiring $d \geq 3$ above, we can ensure that each geometric fiber is also 2-connected, which means that whenever it decomposed as a sum $D_1 + D_2$ of two nonzero effective divisors, the intersection number $D_1.D_2$ is at least 2 (the 2-connectedness follows from [VdV79, Theorem I]; that paper is over $\mathbb{C}$, but the argument works in arbitrary characteristic). This rules out the possibility of a geometric fiber with two components, so every geometric fiber is integral. The "fibration method" (see, e.g., [CTSSD87], [CT98, 2.1], [CTP00, Lemma 3.1]) shows that there is a finite set of places $S$ such that for every place $v \notin S$ and every point $t \in \mathbb{P}^1(k)$, the fiber of $\tilde{V} \to \mathbb{P}^1$ above $t$ has a $k_v$-point. For $v \in S$, the set $\tilde{V}(k_v)$ is nonempty, and its image in $\mathbb{P}^1$ contains a nonempty open subset $U_v$ of $\mathbb{P}^1(k_v)$. By weak approximation, we can find $t \in \mathbb{P}^1(k)$ such that $t \in U_v$ for all $v \in S$, and such that the fiber of $\tilde{V} \to \mathbb{P}^1$ above $t$ is smooth. That fiber violates the local-global principle.

With a little work, this construction can be made effective. On the other hand, this approach does not seem to let one construct curves with a prescribed positive number of points.

3.2. **Atkin-Lehner twists of modular curves.** Theorem 1 of [Cla08] constructs a natural family of curves over $\mathbb{Q}$ violating the local-global principle: namely, for any squarefree integer $N$ with $N > 131$ and $N \neq 163$, there is a positive-density set of primes $p$ such that the twist of $X_0(N)$ by the main Atkin-Lehner involution $w_N$ and the quadratic extension $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$ violates the local-global principle over $\mathbb{Q}$. See [Cla08] for details, and for a connection to the inverse Galois problem. The proof involves Faltings' theorem [Fal83], so it does not yield an effective construction of a suitable pair $(N, p)$.

On the other hand, as P. Clark explained to me, a variant of this construction is effective, and works over an arbitrary global field $k$. His idea is to replace $X_0(N)$ above with a modular curve $X$ having both $\Gamma_0(N)$ and $\Gamma_1(M)$ level structures, for suitable $M$ and $N$ depending on $k$, and to apply Merel's theorem (or a characteristic $p$ analogue) to $X_1(M)$ to control $X(k)$. See [Cla09] for details.

*Remark* 3.1. One can also find counterexamples to the local-global principle over $\mathbb{Q}$ among Atkin-Lehner *quotients* of Shimura curves: see [RSY05] and [PY07].

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