Metric uniformization and spectral bounds for graphs

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Abstract

We present a method for proving upper bounds on the eigenvalues of the graph Laplacian. A main step involves choosing an appropriate "Riemannian" metric to uniformize the geometry of the graph. In many interesting cases, the existence of such a metric is shown by examining the combinatorics of special types of flows. This involves proving new inequalities on the crossing number of graphs.

In particular, we use our method to show that for any positive integer k, the k^{th} smallest eigenvalue of the Laplacian on an n-vertex, bounded-degree planar graph is $O(k/n)$. This bound is asymptotically tight for every k, as it is easily seen to be achieved for square planar grids. We also extend this spectral result to graphs with bounded genus, and graphs which forbid fixed minors. Previously, such spectral upper bounds were only known for the case $k = 2$.

1 Introduction

Eigenvalues of the Laplacian on graphs and manifolds have been studied for over forty years in combinatorial optimization and geometric analysis. In combinatorial optimization, spectral methods are a class of techniques that use the eigenvectors of matrices associated with the underlying graphs. These matrices include the adjacency matrix, the Laplacian, and the random-walk matrix of a graph. One of the earliest applications of spectral methods is to graph partitioning, pioneered by Hall [27] and Donath and Hoffman [18, 19] in the early 1970s. The use of the graph Laplacian for partitioning was introduced by Fiedler [22, 23, 24], who showed a connection between the second-smallest eigenvalue of the Laplacian of a graph and its connectivity. Since their inception, spectral methods have been used for solving a wide range of optimization problems, from graph coloring [7, 3] to image segmentation [44, 48] to web search [32, 10].

Analysis of the Fiedler value. In parallel with the practical development of spectral methods, progress on the mathematical front has been extremely fruitful, involving a variety of connections between various graph properties and corresponding graph spectra.

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In 1970, independent from the work of Hall and of Donath and Hoffman, Cheeger [14] proved that the isoperimetric number of a continuous manifold can be bounded from above by the square root of the smallest non-trivial eigenvalue of its Laplacian. Cheeger's inequality was then extended to graphs by Alon [2], Alon and Milman [4], and Sinclair and Jerrum [45]. They showed that if the Fiedler value of a graph — the second smallest eigenvalue of the Laplacian of the graph — is small, then partitioning the graph according to the values of the vertices in the associated eigenvector will produce a cut where the ratio of cut edges to the number of vertices in the cut is similarly small.

Spielman and Teng [46] proved a spectral theorem for planar graphs, which asserts that the Fiedler value of every bounded-degree planar graph with n vertices is $O(1/n)$. They also showed that the Fiedler value of a finite-element mesh in d dimensions with n vertices is $O(n^{-2/d})$. Kelner [30] then proved that the Fiedler value of a bounded-degree graph with n vertices and genus g is $O((g+1)/n)$. The proofs in [46, 30] critically use the inherent geometric structure of the planar graphs, meshes, and graphs with bounded genus. Recently, Biswal, Lee, and Rao [8] developed a new approach for studying the Fiedler value; they resolved most of the open problems in [46]. In particular, they proved that the Fiedler value of a bounded-degree graph on n vertices without a K_h minor is $O((h^6 \log h)/n)$. These spectral theorems together with Cheeger's inequality on the Fiedler value immediately imply that one can use the spectral method to produce a partition as good as the best known partitioning methods for planar graphs [39], geometric graphs [40], graphs with bounded genus [25], and graphs free of small complete minors [5].

Higher eigenvalues and our contribution. Although previous work in the graph setting focuses mostly on $k = 2$ (the Fiedler value of a graph), higher eigenvalues and eigenvectors are used in many heuristic algorithms [6, 12, 13, 48].

In this paper, we prove the following theorem on higher graph spectra, which concludes a long line of work on upper bounds for the eigenvalues of planar graphs.

Theorem 1.1 (Planar and bounded-genus graphs)**.** *Let G be a bounded-degree n-vertex planar graph. Then the k^{th} smallest eigenvalue of the Laplacian on G is $O(k/n)$.*

More generally, if G can be embedded on an orientable surface of genus g, then the k^{th} smallest eigenvalue of the Laplacian is at most

$$O\left((g+1)(\log(g+1))^2 \frac{k}{n}\right). \tag{1}$$

The asymptotic dependence on k and n is seen to be tight even for the special case of square planar grids; see Remark 5.1. Our spectral theorem provides a mathematical justification of the experimental observation that when k is small, the k^{th} eigenvalues of the graphs arising in many application domains are small as well. We hope our result will lead to new progress in the analysis of spectral methods.

We remark that the $(\log(g+1))^2$ factor of (1) comes from a certain geometric decomposability property of genus-g graphs (see Theorem 2.2) and is most likely non-essential. Without this factor, the bound is tight up to a universal constant, as shown by the construction of [25].

A well-known generalization of graphs which can be drawn on a manifold of fixed genus involves the notion of a graph minor. Given finite graphs H and G, one says that H is a *minor of* G if H can be obtained from G by a sequence of edge contractions and vertex deletions. A family $\mathcal{F}$ of graphs is said to be *minor-closed* if whenever $G \in \mathcal{F}$ and H is a minor of G, then $H \in \mathcal{F}$ as well. By the famous graph minor theorem of Robertson and Seymour [43], every such family $\mathcal{F}$

is characterized by a finite list of forbidden minors. For instance, by Wagner's theorem [50], the family of planar graphs is precisely the family of graphs which do not have $K_{3,3}$ or K_5 as a minor. We prove the following (see the end of Section 5.1).

Theorem 1.2 (Minor-closed families)**.** *If $\mathcal{F}$ is any minor-closed family of graphs which does not contain all graphs, then there is a constant $c_{\mathcal{F}} > 0$ such that for all $G \in \mathcal{F}$ with n vertices and maximum degree $d_{\max}$, and all $1 \le k \le n$,*

$$\lambda_k(G) \le c_{\mathcal{F}} \cdot d_{\max} \frac{k}{n} .$$

The Riemannian setting and conformal uniformization. The spectra of the Laplacian on compact Riemannian surfaces of fixed genus is also well-studied. Let M be a compact Riemannian manifold of genus g, and let $\lambda_k(M)$ be the k^{th} smallest eigenvalue of the Laplace operator on M.[1]

Hersch [29] showed that $\lambda_2(M) \le O(1/\mathrm{vol}(M))$ for Riemannian metrics on the 2-sphere, i.e. for the $g = 0$ case. This was extended by Yang and Yau [52] to a bound of the form $\lambda_2(M) \le O((g+1)/n)$ for all $g \ge 0$. Yau asked whether, for every $g \ge 0$, there was a constant c_g such that

$$\lambda_k(M) \le c_g \frac{k}{\mathrm{vol}(M)}, \qquad (2)$$

for all $k \ge 1$. The question was resolved by Korevaar [33] who proved that one can take $c_g = O(g+1)$. As mentioned at the end of the section, we prove that bounds in the graph setting yield bounds in the setting of surfaces, and thus our result also gives a new proof of (2) with the slightly worse constant $c_g = O((g+1)(\log(g+1))^2)$.

An important point is that the bounds for planar and genus-g graphs—in addition to the work discussed above for Riemannian surfaces—are proved using some manifestation of conformal uniformization. In the graph case, this is via the Koebe-Andreev-Thurston circle packing theorem, and in the manifold case, by the uniformization theorem. The methods of Hersch, Yang-Yau, and Spielman-Teng start with a representation of the manifold or graph on the 2-sphere, and then apply an appropriate Möbius transformation to obtain a test vector that bounds λ_2. There is no similar method known for bounding λ_3, and indeed Korevaar's approach to (2) is significantly more delicate and uses very strongly the geometry of the standard 2-sphere.

However, the spectra of graphs may be more subtle than the spectra of surfaces. We know of a reduction in only one direction: Bounds on graph eigenvalues can be used to prove bounds for surfaces; see Section 5.2. For graphs with large diameter, the analysis of graph spectra resembles the analysis for surfaces. For example, Chung [15] gave an upper bound of $O(1/D^2)$ on the the Fiedler value, where D is the diameter of the graph. Grigor'yan and Yau [26] extended Korevaar's analysis to bounded genus graphs that have a strong volume measure — in particular, these graphs have diameter $\Omega(\sqrt{n})$.

Bounded-degree planar graphs (and bounded genus graphs), however, may have diameter as small as $O(\log n)$, making it impossible to directly apply these diameter-based spectral analyses. Our work builds on the method of Biswal, Lee, and Rao [8], which uses multi-commodity flows to define a deformation of the graph geometry. Essentially, we try to construct a metric on the graph

[1] In Riemannian geometry, the convention is to number the eigenvalues starting from λ_0, but we use the graph theory convention to make direct comparison easier.

which is "uniform" in a metrically defined sense. We then show that sufficiently uniform metrics allow us to recover eigenvalue bounds.

To construct metrics with stronger uniformity properties, which can be used to capture higher eigenvalues, we study a new flow problem, which we define in Section 1.2 and call *subset flows*; this notion may be independently interesting. As we discuss in the next section, these flows arise as dual objects of certain kinds of optimal spreading metrics on the graph. We use techniques from the theory of discrete metric spaces to build test vectors from spreading metrics, and we develop new combinatorial methods to understand the structure of optimal subset flows.

Our spectral theorem not only provides a discrete analog for Korevaar's theorem on higher eigenvalues, but also extends the higher-eigenvalue bounds to graphs with a bounded forbidden minor, a family that is more combinatorially defined. Because the Laplacian of a manifold can be approximated by that of a sufficiently fine mesh graph (see Section 5.2), our result also provides a new proof of Korevaar's theorem, with a slightly worse constant.

1.1 Outline of our approach

For the sake of clarity, we restrict ourselves for now to a discussion of the case where $G = (V, E)$ is a bounded-degree planar graph. Let $n = |V|$, and for $1 \leq k \leq n$, let λ_k be the k^{th} smallest eigenvalue of the Laplacian on G (see Section 1.2.1 for a discussion of graph Laplacians). We first review the known methods for bounding $\lambda_2 = \lambda_2(G)$.

Bounding λ_2. By the variational characterization of eigenvalues, giving an upper bound on λ_2 requires finding a certain kind of mapping of G into the real line (see Section 1.2). Spielman and Teng [46] obtain an initial geometric representation using the Koebe-Andreev-Thurston circle packing theorem for planar graphs. Because of the need for finding a test vector which is orthogonal to the first eigenvector (i.e., the constant function), one has to post-process this representation before it will yield a bound on λ_2. They use a topological argument to show the existence of an appropriate Möbius transformation which achieves this. (As we discussed, a similar step was used by Hersch [29] in the manifold setting.) Even in the arguably simpler setting of manifolds, no similar method is known for bounding λ_3, due to the lack of a rich enough family of circle-preserving transformations.

Our approach begins with the arguments of Biswal, Lee, and Rao [8]. Instead of finding an external geometric representation, those authors begin by finding an appropriate intrinsic deformation of the graph, expressed via a non-negative vertex-weighting $\omega : V \to [0, \infty)$, which induces a corresponding shortest-path metric[2] on G,

$$\mathsf{dist}_\omega(u, v) = \text{length of shortest } u\text{-}v \text{ path},$$

where the length of a path P is given by $\sum_{v \in P} \omega(v)$. The proper deformation ω is found via variational methods, by minimizing the ratio,

$$\frac{\sqrt{\sum_{v \in V} \omega(v)^2}}{\sum_{u,v \in V} \mathsf{dist}_\omega(u, v)}. \tag{3}$$

[2]Strictly speaking, this is only a pseudometric since $\mathsf{dist}_\omega(u, v) = 0$ is possible for $u \neq v$, but we ignore this distinction for the sake of the present discussion.

The heart of the analysis involves studying the geometry of the minimal solutions, via their dual formulation in terms of certain kinds of multi-commodity flows. Finally, techniques from the theory of metric embeddings are used to embed the resulting metric space $(V, \mathsf{dist}_\omega)$ into the real line, thus recovering an appropriate test vector to bound λ_2.

Controlling λ_k for $k \geq 3$. In order to bound higher eigenvalues, we need to produce a system of many linearly independent test vectors. The first problem one encounters is that the optimizer of (3) might not contain enough information to produce more than a single vector if the geometry of the ω-deformed graph is degenerate, e.g. if $V = C \cup C'$ for two large, well-connected pieces C, C' where C and C' are far apart, but each has small diameter. (Intuitively, there are only two degrees of freedom, the value of the eigenfunction on C and the value on C'.)

Spreading metrics and padded partitions. To combat this, we would like to impose the constraint that no large set collapses in the metric dist_ω, i.e. that for some $k \geq 1$ and any subset $C \subseteq V$ with $|C| \geq n/k$, the diameter of C is large. In order to produce such an ω by variational techniques, we have to specify this constraint (or one like it) in a convex way. We do this using the *spreading metric* constraints which are well-known in mathematical optimization (see, e.g. [20]). The spreading constraint on a subset $S \subseteq V$ takes the form,

$$\frac{1}{|S|^2} \sum_{u,v \in S} \mathsf{dist}_\omega(u,v) \geq \varepsilon \sqrt{\sum_{u \in V} \omega(u)^2}, \tag{4}$$

for some $\varepsilon > 0$.

Given such a spreading weight ω for sets of size $\approx n/k$, we show in Section 2 how to obtain a bound on λ_k by producing k smooth, disjointly suppported bump functions on $(V, \mathsf{dist}_\omega)$, which then act as our k linearly independent test vectors. The bound depends on the value ε from (4), as well as a certain geometric decomposability property of the space $(V, \mathsf{dist}_\omega)$. The bump functions are produced using padded metric partitions (see, e.g. [35] and [36]), which are known to exist for all planar graphs from the seminal work of Klein, Plotkin, and Rao [31].

The spreading deformation, duality, and subset flows. At this point, to upper bound λ_k, it suffices to find a spreading weight ω for subsets of size $\approx n/k$, with ε (from (4)) as large as possible. To the end, in Section 2.3, we write a convex program whose optimal solution yields a weight ω with the largest possible value of ε. The dual program involves a new kind of multi-commodity flow problem, which we now describe.

Consider a probability distribution μ on subsets $S \subseteq V$. For a flow F in G (see Section 1.2 for a review of multi-commodity flows), we write $F[u,v]$ for the total amount of flow sent from u to v, for any $u, v \in V$. In this case a *feasible μ-flow* is one which satisfies, for every $u, v \in V$,

$$F[u,v] = \mathbb{P}_{S \sim \mu}[u, v \in S],$$

where we use the notation $S \sim \mu$ to denote that S is chosen according to the distribution μ. In the language of demands, every set S places a demand of $\mu(S)$ between every pair $u, v \in S$. For instance, the classical all-pairs multi-commodity flow problem would be specified by choosing μ which concentrates all its weight on the entire vertex set V.

Given such a μ, the corresponding "subset flow" problem is to find a feasible μ-flow F so that the total ℓ_2-norm of the congestion of F at vertices is minimized (see Section 2.3 for a formal definition of the ℓ_2-congestion). Finally, by duality, bounding λ_k requires us to prove *lower bounds* on the congestion of *every possible* μ-flow with μ concentrated on sets of size $\approx n/k$.

An analysis of optimal subset flows: New crossing number inequalities. In the case of planar graphs G, we use a randomized rounding argument to relate the existence of a feasible μ-flow in G with small ℓ_2-congestion to the ability to draw certain kinds of graphs in the plane without too many edge crossings. This was done in [8], where the relevant combinatorial problem involved the number of edge crossings necessary to draw dense graphs in the plane, a question which was settled by Leighton [38], and Ajtai, Chvátal, Newborn, and Szemerédi [1].

In the present work, we have to develop new crossing *weight* inequalities for a "subset drawing" problem. Let $H = (U, F)$ be a graph with non-negative edge weights $W : F \to [0, \infty)$. Given a drawing of H in the plane, we define the crossing weight of the drawing as the total weight of all edge crossings, where two edges $e, e' \in F$ incur weight $W(e) \cdot W(e')$ when they cross. Write $\mathsf{cr}(H; W)$ for the minimal crossing weight needed to draw H in the plane. In Section 4, we prove a generalization of the following theorem (it is stated there in the language of flows), which forms the technical core of our eigenvalue bound.

Theorem 1.3 (Subset crossing theorem)**.** *There exists a constant $C \geq 1$ such that if μ is any probability distribution on subsets of $[n]$ with $\mathbb{E}_{S\sim\mu}|S|^2 \geq C$, then the following holds. For $u, v \in [n]$, let*

$$W(u, v) = \mathbb{P}_{S\sim\mu}[u, v \in S] \,.$$

Then we have,

$$\mathsf{cr}(K_n; W) \gtrsim \frac{1}{n} \left(\mathbb{E}_{S\sim\mu}|S|^2\right)^{5/2} ,$$

where K_n is the complete graph on $\{1, 2, \ldots, n\}$.

Observe that the theorem is asymptotically tight for all values of $\mathbb{E}|S|^2$. It is straightforward that one can draw an r-clique in the plane using only $O(r^4)$ edge crossings. Thus if we take μ to be uniform on k disjoint subsets of size n/k, then the crossing weight is on the order of $k \cdot (1/k)^2 \cdot (n/k)^4 = n^4/k^5$, which matches the lower bound $\frac{1}{n}(\mathbb{E}|S|^2)^{5/2} = \frac{1}{n}(n/k)^5$. The proof involves some delicate combinatorial and analytic arguments, and is discussed at the beginning of Section 4.

More general families: Bounded genus and excluded minors. Clearly the preceding discussion was specialized to planar graphs. A similar approach can be taken for graphs of bounded genus (orientable or non-orientable) using the appropriate generalization of Euler's formula.

To handle general minor-closed families, we can no longer deal with the notion of drawings, and we have to work directly with multi-commodity flows in graphs. To do this, we use the corresponding "flow crossing" theory developed in [8], with some new twists to handle the regime where the total amount of flow being sent is very small (this happens when bounding λ_k for large values of k, e.g. $k \geq \sqrt{n}$).

1.2 Preliminaries

We often use the asymptotic notation $A \lesssim B$ to denote $A = O(B)$. We use $A \asymp B$ to denote the conjunction of $A \lesssim B$ and $A \gtrsim B$. For a graph G, we use $V(G)$ and $E(G)$ to denote the edge and vertex sets of G, respectively. We write $\mathbb{R}_+ = [0, \infty)$.

1.2.1 Laplacian spectrum

Let $G = (V, E)$ be a finite, undirected graph. We use $u \sim v$ to denote $\{u, v\} \in E$. We consider the linear space $\mathbb{R}^V = \{f : V \to \mathbb{R}\}$ and define the Laplacian $\mathcal{L} : \mathbb{R}^V \to \mathbb{R}^V$ as the symmetric, positive-definite linear operator given by

$$(\mathcal{L}f)(v) = \sum_{u:u\sim v} (f(v) - f(u)),$$

which in matrix form could be written as $\mathcal{L} = D - A$ where A is the adjacency matrix of G and D the diagonal matrix whose entries are the vertex degrees. We wish to give upper bounds on the k^{th} eigenvalue of $\mathcal{L}$ for each k. To do this we consider the seminorm given by

$$\|f\|_{\mathcal{L}}^2 = \langle f, \mathcal{L}f\rangle = \sum_{u\sim v}(f(u) - f(v))^2,$$

and restrict it to k-dimensional subspaces $U \subset \mathbb{R}^V$. By the spectral theorem, the maximum ratio $\|f\|_{\mathcal{L}}^2/\|f\|^2$ over U is minimized when U is spanned by the k eigenvectors of least eigenvalue, in which case its value is λ_k. Therefore if we exhibit a k-dimensional subspace U in which $\|f\|_{\mathcal{L}}^2 \le c$ for all unit vectors f, it follows that $\lambda_k \le c$. In particular, this yields the following simple lemma.

Lemma 1.4. *For any $k \ge 1$, suppose that $f_1, f_2, \dots, f_k \in \mathbb{R}^V$ is a collection of non-zero vectors such that for all $1 \le i < j \le k$, $\operatorname{supp}(f_i) \cap \operatorname{supp}(f_j) = \emptyset$. Then,*

$$\lambda_k \le \max\left\{\frac{\|f_i\|_{\mathcal{L}}^2}{\|f_i\|^2} : i \in \{1, 2, \dots, k\}\right\}.$$

1.2.2 Flows

Let $G = (V, E)$ be a finite, undirected graph, and for every pair $u, v \in V$, let $\mathcal{P}_{uv}$ be the set of all paths between u and v in G. Let $\mathcal{P} = \bigcup_{u,v\in V} \mathcal{P}_{uv}$. Then a *flow in* G is a mapping $F : \mathcal{P} \to [0, \infty)$. For any $u, v \in V$, let $F[u, v] = \sum_{p\in\mathcal{P}_{uv}} F(p)$ be the amount of flow sent between u and v.

Our main technical theorem concerns a class of flows we call *subset flows*. Let μ be a probability distribution on subsets of V. Then F is a μ*-flow* if it satisfies $F[u, v] = \mathbb{P}_{S\sim\mu}[u, v \in S]$ for all $u, v \in V$. For $r \le |V|$, we write $\mathcal{F}_r(G)$ for the set of all μ-flows in G with $\operatorname{supp}(\mu) \subseteq \binom{V}{r}$.

We say a flow F is an *integral flow* if it is supported on only one path p in each $\mathcal{P}_{uv}$, and a *unit flow* if $F[u, v] \in \{0, 1\}$ for every $u, v \in V$. An edge-weighted graph H is one which comes equipped with a non-negative weight function $w : E(H) \to [0, \infty)$ on edges. We say that a flow F in G is an H*-flow* if there exists an injective mapping $\phi : V(H) \to V$ such that for all $\{u, v\} \in E(H)$, we have $F[\phi(u), \phi(v)] \ge w(u, v)$. In this case, H is referred to as the *demand graph* and G as the *host graph.*

We define the *squared ℓ_2-congestion*, or simply *congestion*, of a flow F by $\mathsf{con}(F) = \sum_{v\in V} C_F(v)^2$, where $C_F(v) = \sum_{p\in\mathcal{P}:v\in p} F(p)$. This congestion can also be written as

$$\mathsf{con}(F) = \sum_{p,p'\in\mathcal{P}} \sum_{v\in p\cap p'} F(p)F(p')$$

and is therefore bounded below by a more restricted sum, the *intersection number*:

$$\mathsf{inter}(F) = \sum_{\substack{u,v,u',v' \\ |\{u,v,u',v'\}|=4}} \sum_{\substack{p\in\mathcal{P}_{uv} \\ p'\in\mathcal{P}_{u'v'}}} \sum_{x\in p\cap p'} F(p)F(p').$$

2 Eigenvalues and spreading weights

We will now reduce the problem of proving upper bounds on the eigenvalues of a graph G, to the problem of proving lower lower bounds on the congestion of subset flows in G. In the present section, if (X, d) is a metric space, and $x \in X, R \geq 0$, we will use the notation

$$B(x, R) = \{y \in X : d(x, y) \leq R\}.$$

2.1 Padded partitions

Let (X, d) be a finite metric space. We will view a partition P of X as a collection of subsets, and also as a function $P : X \to 2^X$ mapping a point to the subset that contains it. We write $\beta(P, \Delta)$ for the infimal value of $\beta \geq 1$ such that

$$\left|\left\{x \in X : B(x, \Delta/\beta) \subseteq P(x)\right\}\right| \geq \frac{|X|}{2}.$$

Let $\mathcal{P}_\Delta$ be the set of all partitions P such that for every $S \in P$, $\mathrm{diam}(S) \leq \Delta$. Finally, we define

$$\beta_\Delta(X, d) = \inf\left\{\beta(P, \Delta) : P \in \mathcal{P}_\Delta\right\}.$$

The following theorem is a consequence [41] of the main theorem of Klein, Plotkin, and Rao [31], with the dependence of r^2 due to [21].

Theorem 2.1. *Let $G = (V, E)$ be a graph without a $K_{r,r}$ minor and (V, d) be any shortest-path semimetric on G, and let $\Delta > 0$. Then $\beta_\Delta(V, d) = O(r^2)$.*

In particular, if G is planar then $\beta_\Delta(V, d)$ is upper bounded by an absolute constant, and if G is of genus $g > 0$ then $\beta_\Delta(V, d) = O(g)$, since the genus of K_h is $\Omega(h^2)$ [28, p. 118]. The paper [37] proves the following strengthening (which is tight, up to a universal constant).

Theorem 2.2. *Let $G = (V, E)$ be a graph of orientable genus g, and (V, d) be any shortest-path semimetric on G, and let $\Delta > 0$. Then $\beta_\Delta(V, d) = O(\log g)$.*

2.2 Spreading vertex weights

Consider a non-negative weight function $\omega : V \to \mathbb{R}_+$ on vertices, and extend ω to subsets $S \subseteq V$ via $\omega(S) = \sum_{v \in V} \omega(v)$. We associate a vertex-weighted shortest-path metric by defining

$$\mathsf{dist}_\omega(u, v) = \min_{p \in \mathcal{P}_{uv}} \omega(p).$$

Say that ω is (r, ε)-spreading if, for every $S \subseteq V$ with $|S| = r$, we have

$$\frac{1}{|S|^2} \sum_{u,v \in S} \mathsf{dist}_\omega(u, v) \geq \varepsilon \sqrt{\sum_{v \in V} \omega(v)^2}.$$

Write $\varepsilon_r(G, \omega)$ for the maximal value of ε for which ω is (r, ε)-spreading.

Theorem 2.3 (Higher eigenvalues)**.** *Let $G = (V, E)$ be any n-vertex graph with maximum degree $d_{\max}$, and let λ_k be the kth Laplacian eigenvalue of G. For any $k \geq 1$, the following holds. For any weight function $\omega : V \to \mathbb{R}_+$ with*

$$\sum_{v \in V} \omega(v)^2 = 1, \tag{5}$$

we have

$$\lambda_k \leq \frac{64\, d_{\max}}{\varepsilon^2 n} \left(\beta_{\varepsilon/2}(V, \mathsf{dist}_\omega)\right)^2,$$

where $\varepsilon = \varepsilon_{\lfloor n/4k \rfloor}(G, \omega)$.

Proof. Let ω be an $(\lfloor n/4k \rfloor, \varepsilon)$-spreading weight function. Let $V = C_1 \cup C_2 \cup \cdots \cup C_m$ be a partition of V into sets of diameter at most $\varepsilon/2$, and define for every $i \in [m]$,

$$\hat{C}_i = \left\{ x \in C_i : B(x, \varepsilon/(2\beta)) \subseteq C_i \right\},$$

where $\beta = \beta_{\varepsilon/2}(V, \mathsf{dist}_\omega)$. By the definition of β, there exists a choice of $\{C_i\}$ with

$$|\hat{C}_1 \cup \hat{C}_2 \cup \cdots \cup \hat{C}_m| \geq n/2.$$

Now, for any set $A \subseteq V$ with $\mathrm{diam}(A) \leq \varepsilon/2$, we see that

$$\frac{1}{|A|^2} \sum_{u,v \in A} \mathsf{dist}_\omega(u, v) \leq \frac{\varepsilon}{2} = \frac{\varepsilon}{2} \sqrt{\sum_{v \in V} \omega(v)^2}. \tag{6}$$

Since $\mathrm{diam}(C_i) \leq \varepsilon/2$, if $|C_i| > n/4k$, then we could pass to a subset of C_i of size exactly $\lfloor n/4k \rfloor$ which satisfies (6), but this would violate the $(\lfloor n/4k \rfloor, \varepsilon)$-spreading property of ω. Hence we know that $|C_i| \leq n/4k$ for each $i = 1, 2, \ldots, m$.

Thus by taking disjoint unions of the sets $\{\hat{C}_i\}$ which are each of size at most $n/4k$, we can find sets $S_1, S_2, \ldots, S_{2k}$ with

$$\frac{n}{2k} \geq |S_i| \geq \frac{n}{4k}. \tag{7}$$

For each $i \in [2k]$, let $\tilde{S}_i$ be the $\varepsilon/(2\beta)$-neighborhood of S_i. Observe that the sets $\{\tilde{S}_i\}$ are pairwise disjoint, since by construction each is contained in a union of C_i's, which are themselves pairwise disjoint.

Now define, for every $i \in [2k]$, define

$$W(\tilde{S}_i) = \sum_{u \in \tilde{S}_i} \sum_{v : uv \in E} [\omega(u) + \omega(v)]^2$$

Clearly, we have

$$\sum_{i=1}^{2k} W(\tilde{S}_i) \leq 2 \sum_{uv \in E} [\omega(u) + \omega(v)]^2 \leq 4 d_{\max} \sum_{v \in V} \omega(v)^2 = 4 d_{\max},$$

where the latter equality is (5). Hence if we renumber the sets so that $\left\{ \tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_k \right\}$ have the smallest $W(\tilde{S}_i)$ values, then for each $i = 1, 2, \ldots, k$, we have $W(\tilde{S}_i) \leq \frac{4 d_{\max}}{k}$.

Finally, we define functions $f_1, f_2, \dots, f_k : V \to \mathbb{R}$ by

$$f_i(x) = \max\left\{0, \frac{\varepsilon}{2\beta} - \mathsf{dist}_\omega(x, S_i)\right\}$$

so that f_i is supported on $\tilde{S}_i$, and $f_i(x) = \varepsilon/(2\beta)$ for $x \in S_i$.

Since each f_i is 1-Lipschitz and has $\mathrm{supp}(f_i) \subseteq \tilde{S}_i$, we have

$$\begin{aligned}
\|f_i\|_{\mathcal{L}}^2 = \sum_{uv\in E} |f_i(u) - f_i(v)|^2 &= \sum_{u\in\tilde{S}_i}\sum_{v:uv\in E} |f_i(u) - f_i(v)|^2 \\
&\le \sum_{u\in\tilde{S}_i}\sum_{v:uv\in E} \mathsf{dist}_\omega(u,v)^2 \\
&= \sum_{u\in\tilde{S}_i}\sum_{v:uv\in E} [\omega(u)+\omega(v)]^2 \\
&= W(\tilde{S}_i) \le \frac{4d_{\max}}{k}.
\end{aligned}$$

Furthermore the functions have disjoint support and satisfy,

$$\|f_i\|^2 \ge \left(\frac{\varepsilon}{2\beta}\right)^2 |S_i| \ge \frac{\varepsilon^2}{16\beta^2}\frac{n}{k},$$

where in the final inequality we have used (7).

Combining the preceding two estimates shows that for each f_i,

$$\frac{\|f_i\|_{\mathcal{L}}^2}{\|f_i\|^2} \le \frac{d_{\max}}{64n}\left(\frac{\beta}{\varepsilon}\right)^2,$$

and the proof is complete by Lemma 1.4. $\square$

2.3 Spreading weights and subset flows

We now show a duality between the optimization problem of finding a spreading weight ω and the problem of minimizing congestion in subset flows. The following theorem is proved by a standard Lagrange multipliers argument.

Theorem 2.4 (Duality)**.** *Let $G = (V, E)$ be a graph and let $r \le |V|$. Then*

$$\max\left\{\varepsilon_r(G,\omega)\,\middle|\,\omega : V \to \mathbb{R}_+\right\} = \frac{1}{r^2}\min\left\{\sqrt{\mathsf{con}(F)}\,\middle|\,F \in \mathcal{F}_r(G)\right\}.$$

Proof. We shall write out the optimizations $\max_\omega \varepsilon_r(G,\omega)$ and $\frac{1}{r^2}\min_F \sqrt{\mathsf{con}(F)}$ as convex programs, and show that they are dual to each other. The equality then follows from Slater's condition [9, Ch. 5]:

Fact 2.5 (Slater's condition for strong duality)**.** *When the feasible region for a convex program* (**P**) *has non-empty interior, the values of* (**P**) *and its dual* (**P***) *are equal.*

We begin by expanding $\max_\omega \varepsilon_r(G,\omega)$ as a convex program $(\mathbf{P})$. Let $P \in \{0,1\}^{\mathcal{P}\times V}$ be the path incidence matrix; $Q \in \{0,1\}^{\mathcal{P}\times\binom{V}{2}}$ the path connection matrix; and $R \in \{0,1\}^{\binom{V}{r}\times\binom{V}{2}}$ a normalized set containment matrix, respectively defined as

$$P_{p,v} = \begin{cases} 1 & v \in p \\ 0 & \text{else} \end{cases} \qquad Q_{p,uv} = \begin{cases} 1 & p \in \mathcal{P}_{uv} \\ 0 & \text{else} \end{cases} \qquad R_{S,uv} = \begin{cases} 1/r^2 & \{u,v\} \subset S \\ 0 & \text{else.} \end{cases}$$

Then the convex program $(\mathbf{P}) = \max_\omega \varepsilon_r(G,\omega)$ is

$$\begin{array}{rlll} \text{minimize} & -\varepsilon & & \\ \text{subject to} & \varepsilon\mathbf{1} \preceq Rd & Qd \preceq Ps & s^\top s \le 1 \\ & d \succeq 0 & s \succeq 0 & \end{array}.$$

Introducing the non-negative Lagrange multipliers μ, λ, ν, the Lagrangian function is

$$L(d,s,\mu,\lambda,\nu) = -\varepsilon + \mu^\top(\varepsilon\mathbf{1} - Rd) + \lambda^\top(Qd - Ps) + \nu(s^\top s - 1)$$

so that $(\mathbf{P})$ and its dual $(\mathbf{P}^*)$ may be written as

$$\begin{aligned} (\mathbf{P}) &= \inf_{\varepsilon,d,s} \sup_{\mu,\lambda,\nu} L(d,s,\mu,\lambda,\nu) \\ (\mathbf{P}^*) &= \sup_{\mu,\lambda,\nu} \inf_{\varepsilon,d,s} L(d,s,\mu,\lambda,\nu). \end{aligned}$$

Now we simplify $(\mathbf{P}^*)$. Rearranging terms in L, we have

$$\begin{aligned} (\mathbf{P}^*) &= \sup_{\mu,\lambda,\nu} \inf_{\varepsilon,d,s} (\mu^\top\mathbf{1} - 1)\varepsilon + (\lambda^\top Q - \mu^\top R)d + (\nu s^\top s - \lambda^\top P s) - \nu \\ &= \sup_{\mu,\lambda,\nu} \inf_{\varepsilon} (\mu^\top\mathbf{1} - 1)\varepsilon + \inf_d (\lambda^\top Q - \mu^\top R)d + \inf_s (\nu s^\top s - \lambda^\top P s) - \nu. \end{aligned}$$

Now the infima $\inf_\varepsilon(\mu^\top\mathbf{1} - 1)\varepsilon$ and $\inf_d(\lambda^\top Q - \mu^\top R)d$ are either 0 or $-\infty$, so at the optimum they must be zero and $\mu^\top\mathbf{1} - 1 \ge 0$, $\lambda^\top Q - \mu^\top R \succeq 0$. With these two constraints, the optimization reduces to $\sup_{\lambda,\nu}\inf_s(\nu s^\top s - \lambda^\top P s) - \nu$. At the optimum, the gradient of the infimand is zero, so $s = \frac{P^\top\lambda}{2\nu}$ and the infimum is $-\frac{\|P^\top\lambda\|_2^2}{4\nu}$. Then at the maximum, $\nu = \frac{1}{2}\|P^\top\lambda\|_2$, so that the supremand is $-\|P^\top\lambda\|_2$. We have shown that $(\mathbf{P}^*)$ is the convex program

$$\begin{array}{rlll} \text{maximize} & -\left\|P^\top\lambda\right\|_2 & & \\ \text{subject to} & \lambda^\top Q \succeq \mu^\top R & \mu^\top\mathbf{1} \ge 1 \\ & \lambda \succeq 0 & \mu \succeq 0 \end{array}.$$

This program is precisely (the negative of) the program to minimize vertex 2-congestion of a subset flow in $\mathcal{F}_r(G)$, where the subset weights are normalized to have unit sum. The proof is complete. $\square$

3 Congestion measures

In this section, we develop concepts that will enable us to give lower bounds on the congestion $\mathsf{con}(F)$ of all subset flows F in a given graph G. The reader may wish to consult with Section 1.2.2 to recall the relevant definitions.

Definition 3.1. *Let G be an arbitrary host graph, and H an edge-weighted demand graph. Define the G-*congestion of H *by*

$$\mathsf{con}_G(H) = \min_{F \text{ an } H\text{-flow in } G} \mathsf{con}(F)$$

*and the G-*intersection number of H *by*

$$\mathsf{inter}_G(H) = \min_{F \text{ an } H\text{-flow in } G} \mathsf{inter}(F),$$

and the integral G-intersection number of H *by*

$$\mathsf{inter}^*_G(H) = \min_{F \text{ an integral } H\text{-flow in } G} \mathsf{inter}(F).$$

Note that even if H is a unit-weighted graph and $\mathsf{inter}^*_G(H) = 0$, this does not imply that G contains an H-minor. This is because the intersection number involves quadruples of four distinct vertices. For example, if H is a triangle, then $\mathsf{inter}^*_G(H) = 0$ for any G, even when G is a tree (and thus does not have a triangle as a minor). However, we recall the following (which appears as Lemma 3.2 in [8]).

Lemma 3.2 ([8])**.** *If H is a unit-weighted, bipartite demand graph in which every node has degree at least two, then for any graph G, $\mathsf{inter}^*_G(H) = 0$ implies that G contains an H-minor.*

The next lemma is proved via randomized rounding.

Lemma 3.3 (Rounding)**.** *For any graph G and unit flow F, there is an integral unit flow F^* with $F^*[u, v] = F[u, v]$ for all $u, v \in V(G)$, and such that*

$$\mathsf{inter}(F^*) \le \mathsf{inter}(F).$$

Consequently for every G and unit-weighted H,

$$\mathsf{inter}^*_G(H) = \mathsf{inter}_G(H) \le \mathsf{con}_G(H). \tag{8}$$

Proof. We produce an integral flow F^* randomly by rounding F. For each pair of endpoints u, v, choose independently a path p_{uv} in $\mathcal{P}_{uv}$ with $\mathbb{P}[p_{uv} = p] = F(p)$ for each p. Then

$$\mathbb{E}[\mathsf{inter}(F^*)] \;=\; \sum_{\substack{u,v,u',v' \\ |\{u,v,u',v'\}|=4}} \mathbb{E}\big[\, |p_{uv} \cap p_{u'v'}| \,\big] \;=\; \sum_{\substack{u,v,u',v' \\ |\{u,v,u',v'\}|=4}} \sum_{\substack{p \in \mathcal{P}_{uv} \\ p' \in \mathcal{P}_{u'v'}}} \sum_{x \in p \cap p'} F(p)F(p') \;=\; \mathsf{inter}(F)$$

so that with positive probability we must have $\mathsf{inter}(F^*) \le \mathsf{inter}(F)$. Equation (8) follows because $\mathsf{inter}(F) \le \mathsf{con}(F)$ always. $\square$

Definition 3.4. *Given a host graph G, we say that inter_G is a (c, a)-*congestion measure *if for all unit-weighted graphs $H = (V, E)$, we have the inequality*

$$\mathsf{inter}_G(H) \ge \frac{|E|^3}{c|V^2|} - a|V|\,. \tag{9}$$

In particular, $\mathsf{inter}_G(K_n) \ge \frac{n^4}{8c} - an$.

Lemma 3.5. *Suppose that for some G and $k = k(G)$, every unit-weighted H obeys*

$$\mathsf{inter}^*_G(H) = \mathsf{inter}_G(H) \geq |E(H)| - k|V(H)| - k^2. \tag{10}$$

Then it follows that for every unit-weighted H,

$$\mathsf{inter}_G(H) \geq \frac{1}{27}\frac{|E(H)|^3}{k^2|V(H)|^2} - k|V(H)| \tag{11}$$

so that inter_G is an $(27k^2, k)$-congestion measure.

Proof. It suffices to consider $|E(H)| \geq 3k|V(H)|$ since otherwise the right-hand side of inequality (11) is negative.

Fix any H-flow F in G. Sample the nodes of H independently with probability p each to produce a new demand graph H' and flow $F' = F|_{H'}$. Then $\mathsf{inter}(F') \geq \mathsf{inter}_G(H') \geq |E(H')| - k|V(H')| - k^2$, and by taking expectations we have

$$p^4\mathsf{inter}(F) \geq p^2|E(H)| - pk|V(H)| - k^2.$$

Choosing $p = 3k|V(H)|/|E(H)|$ and using the fact that $|E(H)|/|V(H)|^2 < 1$ we obtain (11). □

The next proof follows employs the techniques of [8].

Corollary 3.6. *If G is planar, then inter_G is an $(O(1), 3)$-congestion measure. If G has genus $g > 0$, then inter_G is an $(O(g), O(\sqrt{g}))$-congestion measure. If G is K_h-minor-free, then inter_G is an $(O(h^2 \log h), O(h\sqrt{\log h}))$-congestion measure.*

Proof. Suppose that H is a unit-weighted demand graph. If F is an integral H-flow with $\mathsf{inter}(F) > 0$, then some path in F and corresponding edge of H can be removed to yield an integral H'-flow F' with $\mathsf{inter}(F') \leq \mathsf{inter}(F) - 1$. Therefore to prove (10) it suffices to consider H with $\mathsf{inter}_G(H) = 0$ and show that $|E(H)| \leq k|V(H)| + k^2$. Then Lemma 3.5 will imply inter_G is an $(O(k^2), k)$-congestion measure.

When G is planar, an H-flow F in G with $\mathsf{inter}(F) = 0$ gives a drawing of H in the plane without crossings, so that H itself is planar. Then an elementary application of the Euler characteristic gives

$$|E(H)| \leq 3|V(H)| - 6 < 3|V(H)|.$$

When G is of genus at most $g > 0$, the same argument gives

$$|E(H)| \leq 3|V(H)| + 6(g - 1),$$

which suffices for $k = O(\sqrt{g})$.

For K_h-minor-free G and H with $\mathsf{inter}_G(H) = 0$, if H is bipartite with minimum degree 2, then Lemma 3.2 implies that H is K_h-minor-free, so that $|E(H)| \leq c_{KT}|V(H)|h\sqrt{\log h}$ by the theorem of Kostochka [34] and Thomason [47].

For general H, we can first take a partition to obtain a bipartite subgraph H' with $|E(H')| \geq |E(H)|/2$. We then remove isolated vertices from H', and iteratively remove vertices of degree one and the associated edges to obtain a bipartite subgraph H'' with minimum degree two, and

$$|E(H'')| \geq |E(H')| - |V(H')| \geq |E(H)|/2 - |V(H)|\,. \tag{12}$$

Now $\mathsf{inter}_G(H) = 0$ together with Lemma 3.2 implies that

$$|E(H'')| \le 2c_{KT} h\sqrt{\log h}|V(H'')|$$

which together with (12), implies that $|E(H)| \le O(h\sqrt{\log h})|V(H)|$. □

In the next section, we will also require the following lemma.

Lemma 3.7. *Let μ be any probability distribution over subsets of V. Writing H_μ for the graph on V with edge weights $H_\mu(u,v) = \mathbb{P}_{S\sim\mu}[u,v \in S]$, we have*

$$\mathsf{inter}_G(H_\mu) \ge \mathbb{E}_{S\sim\mu, S'\sim\mu}\left[\mathsf{inter}_G(K_{|S\cap S'|})\right],$$

where by K_n we intend the unit-weighted complete graph on n vertices.

Proof. Let F be any H_μ-flow, and let the vertices of H_μ be identified with the corresponding vertices of G (recall that every H-flow in G comes with an injection from $V(H)$ into $V(G)$).

Now, define, for every $u, v \in V$ with $F[u,v] \neq 0$, $S \subseteq V$, and $p \in \mathcal{P}_{uv}$, the flow F^S by,

$$F^S(p) = \frac{F(p)}{F[u,v]}\mu(\{S\}),$$

and observe that since $F[u,v] = \mathbb{P}_{S\sim\mu}[u,v\in S]$, we have $F = \sum_{S\subseteq V} F^S$.

In this case, we can write

$$\begin{aligned}
\mathsf{inter}(F) &= \sum_{\substack{u,v,u',v' \\ |\{u,v,u',v'\}|=4}} \sum_{\substack{p\in\mathcal{P}_{uv} \\ p'\in\mathcal{P}_{u'v'}}} |p\cap p'|\, F(p)F(p') \\
&= \sum_{\substack{u,v,u',v' \\ |\{u,v,u',v'\}|=4}} \sum_{\substack{p\in\mathcal{P}_{uv} \\ p'\in\mathcal{P}_{u'v'}}} |p\cap p'| \left(\sum_S F^S(p)\right)\left(\sum_S F^S(p')\right) \\
&= \sum_{\substack{u,v,u',v' \\ |\{u,v,u',v'\}|=4}} \sum_{\substack{p\in\mathcal{P}_{uv} \\ p'\in\mathcal{P}_{u'v'}}} |p\cap p'| \sum_{S,S'} F^S(p)F^{S'}(p') \\
&= \sum_{S,S'} \sum_{\substack{u,v\in S \\ u',v'\in S' \\ |\{u,v,u',v'\}|=4}} \sum_{\substack{p\in\mathcal{P}_{uv} \\ p'\in\mathcal{P}_{u'v'}}} |p\cap p'| F^S(p)F^{S'}(p') \\
&= \sum_{S,S'} \sum_{\substack{u,v\in S \\ u',v'\in S' \\ |\{u,v,u',v'\}|=4}} \sum_{\substack{p\in\mathcal{P}_{uv} \\ p'\in\mathcal{P}_{u'v'}}} |p\cap p'| \mu(\{S\})\frac{F(p)}{F[u,v]}\mu(\{S'\})\frac{F(p')}{F[u',v']} \\
&\ge \sum_{S,S'} \mu(\{S\})\mu(\{S'\}) \sum_{\substack{u,v,u',v'\in S\cap S' \\ |\{u,v,u',v'\}|=4}} \sum_{\substack{p\in\mathcal{P}_{uv} \\ p'\in\mathcal{P}_{u'v'}}} |p\cap p'| \frac{F(p)}{F[u,v]}\frac{F(p')}{F[u',v']} \\
&\ge \mathbb{E}_{S\sim\mu, S'\sim\mu}\left[\mathsf{inter}_G(K_{|S\cap S'|})\right],
\end{aligned}$$

where we have used the fact that the double sum in the penultimate line contains precisely the intersection number of a unit-weighted complete-graph flow on $S \cap S'$. □

4 Congestion for subset flows

We now prove our main estimate on the congestion incurred by subset flows in terms of a graph's congestion measure.

The proof of Theorem 4.1 below involves some delicate combinatorial and analytic arguments. The difficulty lies in controlling the extent to which μ is a mixture of three different types of "extremal" distributions:

1. μ is uniformly distributed on all sets of size r,
2. μ is concentrated on a single set of size r,
3. μ is uniform over n/r disjoint sets, each of size r.

In the actual proof, we deal with the corresponding cases: (1') μ is "uniformly spread" over edges, i.e. $\mathbb{P}_{S\sim\mu}[u, v \in S]$ is somewhat uniform over choices of $u, v \in V$. In this case, we have to take a global approach, showing that not only are there many intra-set crossings, but also a lot of crossing weight is induced by crossing edges coming from different sets. (2') $\mathbb{P}_{S\sim\mu}[u \in S]$ is unusually large for all $u \in V'$ with $|V'| \ll |V|$. In this case, there is a "density increment" on the induced subgraph $G[V']$, and we can apply induction. Finally, if we are in neither of the cases (1') or (2'), we are left to show that, in some sense, the distribution μ must be similar to case (3) above, in which case we can appeal to the classical dense crossing bounds applied to the complete graph on $S \cap S'$ where $S, S' \sim \mu$ are chosen i.i.d.

Theorem 4.1. *There is a universal constant $c_0 > 0$ such that the following holds. Let μ be any probability distribution on subsets of $[n]$. For $u, v \in [n]$, define*

$$F(u, v) = \mathbb{P}_{S\sim\mu}[u, v \in S]$$

and let H_μ be the graph on $[n]$ weighted by F. For any graph G such that inter_G is a (c, a)-congestion measure, we have

$$\mathsf{inter}_G(H_\mu) \gtrsim \frac{1}{cn}\left(\mathbb{E}|S|^2\right)^{5/2} - c_0\frac{a}{n}\mathbb{E}|S|^2$$

Corollary 4.2. *If μ is supported on $\binom{[n]}{r}$ for some r, then $\mathsf{inter}_G(H_\mu) \gtrsim \frac{r^5}{cn} - c_0\frac{ar^2}{n}$. In particular, if $r \gtrsim (a \cdot c)^{1/3}$, then*

$$\mathsf{inter}_G(H_\mu) \gtrsim \frac{r^5}{cn}.$$

Proof of Theorem 4.1. We will freely use the fact that

$$\mathbb{E}|S|^2 = \sum_{u,v} F(u, v).$$

Also, put $F(u) = \mathbb{P}_{S\sim\mu}[u \in S]$ for $u \in [n]$.

The proof will proceed by induction on n, and will be broken into three cases. Let

$$\beta = \sqrt{\frac{1}{n^2}\sum_{u,v} F(u, v)},$$

and put $E(\alpha', \alpha) = \{(u,v) : \alpha' \le F(u,v) \le \alpha\}$. Define the set of "heavy vertices" as

$$H_K = \{u : F(u) \ge K\beta\},$$

for some constant $K \ge 1$ to be chosen later. Let $E_H = \{(u,v) : u, v \in H_K\}$ and $E_{HL} = E(0,\beta) \cup E_H$.

Case I (Light edges): $\displaystyle\sum_{(u,v)\in E(0,\beta)} F(u,v) \ge \frac{1}{4}\sum_{u,v} F(u,v).$

The desired conclusion comes from applying the following claim.

Claim 4.3. *For every $\beta \in [0,1]$, we have*

$$\mathsf{inter}_G(H_\mu) \gtrsim \frac{\left(\sum_{(u,v)\in E(0,\beta)} F(u,v)\right)^3}{\beta c n^2} - 2\beta^2 a n. \tag{13}$$

Proof. First, observe that by (9), the subgraph consisting of the edges in $E(\alpha,\beta)$ contributes at least

$$\alpha^2 \frac{|E(\alpha,\beta)|^3}{cn^2} - \beta^2 a n$$

to $\mathsf{inter}_G(H_\mu)$ for every $\alpha, \beta \in [0,1]$. Therefore letting $E_i = E\left(2^{-i-1}\beta, 2^{-i}\beta\right)$, we have

$$\mathsf{inter}_G(H_\mu) \gtrsim \frac{1}{cn^2}\sum_{i=0}^{\infty} 2^{-2i}\beta^2 |E_i|^3 - an\sum_{i=0}^{\infty} 2^{-2i}\beta^2.$$

Let $F_i = \sum_{(u,v)\in E_i} F(u,v)$ so that $|E_i| \ge (2^i/\beta)F_i$, and then

$$\mathsf{inter}_G(H_\mu) \gtrsim \frac{1}{\beta c n^2}\sum_{i=0}^{\infty} 2^i F_i^3 - 2\beta^2 a n,$$

but also $\sum_{i=0}^{\infty} F_i = \sum_{u,v\in E(0,\beta)} F(u,v)$. Thus (13) is proved by noting that

$$\sum_{i=0}^{\infty} F_i = \sum_{i=0}^{\infty}\left(2^{-i/3}\cdot 2^{i/3}F_i\right) \le \left(\sum_{i=0}^{\infty} 2^{-i/2}\right)^{2/3}\left(\sum_{i=0}^{\infty} 2^i F_i^3\right)^{1/3} < 2.27\left(\sum_{i=0}^{\infty} 2^i F_i^3\right)^{1/3},$$

using Hölder's inequality. ∎

Case II (Heavy endpoints): $\displaystyle\sum_{(u,v)\in E_H} F(u,v) \ge \frac{1}{4}\sum_{u,v} F(u,v)$**.**

Observe that

$$\sum_{u\in[n]} F(u) = \mathbb{E}_{S\sim\mu}|S| \le \sqrt{\mathbb{E}_{S\sim\mu}|S|^2} = \sqrt{\sum_{u,v} F(u,v)} = \beta n,$$

hence $|H_K| \le n/K$ by Markov's inequality.

Apply the statement of the Theorem inductively to the distribution over subsets of $V(H_K)$ corresponding to the random set $S \cap V(H_K)$, to conclude that

$$\mathsf{inter}_G(H_\mu) \gtrsim \frac{K}{cn}\left(\sum_{(u,v)\in E_H} F(u,v)\right)^{5/2} - c_0\frac{a}{n}\sum_{(u,v)\in E_H} F(u,v). \tag{14}$$

Consequently, by choosing $K = 32$, under the assumption of this case,

$$K\left(\frac{\sum_{(u,v)\in E_H} F(u,v)}{\sum_{u,v} F(u,v)}\right)^{5/2} \geq 1$$

and the conclusion again follows.

Case III (Heavy edges, light endpoints): $\displaystyle\sum_{(u,v)\in E_{HL}} F(u,v) \geq \frac{1}{2}\sum_{u,v} F(u,v).$

By definition, $E_{HL} = \{(u,v) : F(u,v) > \beta, \{u,v\} \nsubseteq H_K\}$. Let $\kappa = (16ac)^{1/3}$, so that $\frac{\kappa^4}{8c} \geq 2a\kappa$. By Lemma 3.7 and then since inter_G is a (c,a)-congestion measure, we have

$$\begin{aligned}
\mathsf{inter}_G(H_\mu) &\geq \mathbb{E}_{S\sim\mu,S'\sim\mu}\left[\mathsf{inter}_G(K_{|S\cap S'|})\right] \\
&\geq \mathbb{E}_{S\sim\mu,S'\sim\mu}\left[\mathsf{inter}_G(K_{|S\cap S'|})\mathbf{1}_{|S\cap S'|\geq\kappa}\right] \\
&\geq \frac{1}{8c}\mathbb{E}_{S\sim\mu,S'\sim\mu}\left[|S\cap S'|^4\,\mathbf{1}_{|S\cap S'|\geq\kappa}\right] - a\mathbb{E}_{S\sim\mu,S'\sim\mu}\left[|S\cap S'|\,\mathbf{1}_{|S\cap S'|\geq\kappa}\right] \\
&\geq \frac{1}{16c}\mathbb{E}_{S\sim\mu,S'\sim\mu}\left[|S\cap S'|^4\,\mathbf{1}_{|S\cap S'|\geq\kappa}\right] \\
&= \frac{1}{16c}\sum_{u\in[n]}\mathbb{P}[u\in S\cap S']\,\mathbb{E}_{S\sim\mu,S'\sim\mu}\left[|S\cap S'|^3\,\mathbf{1}_{|S\cap S'|\geq\kappa}\,\middle|\, u\in S\cap S'\right] \\
&= \frac{1}{16c}\sum_{u\in[n]}(\mathbb{P}[u\in S])^2\,\mathbb{E}_{S\sim\mu,S'\sim\mu}\left[|S\cap S'|^3\,\mathbf{1}_{|S\cap S'|\geq\kappa}\,\middle|\, u\in S\cap S'\right] \\
&\geq \frac{1}{16c}\sum_{u:\beta\leq F(u)\leq K\beta} F(u)^2\,\mathbb{E}_{S\sim\mu,S'\sim\mu}\left[|S\cap S'|^3\,\mathbf{1}_{|S\cap S'|\geq\kappa}\,\middle|\, u\in S\cap S'\right] \\
&\geq \frac{\beta^2}{16c}\sum_{u:\beta\leq F(u)\leq K\beta}\mathbb{E}_{S\sim\mu,S'\sim\mu}\left[|S\cap S'|^3\,\middle|\, u\in S\cap S'\right] - \frac{K^2\beta^2}{16c}n\kappa^3.
\end{aligned}$$

Since $\frac{K^2\beta^2}{16c}n\kappa^3 = \frac{K^2a}{n}\mathbb{E}|S|^2$, to finish the proof we need only show that

$$\sum_{u:\beta\leq F(u)\leq K\beta}\mathbb{E}_{S\sim\mu,S'\sim\mu}\left[|S\cap S'|^3\,\middle|\, u\in S\cap S'\right] \gtrsim n\left(\mathbb{E}|S|^2\right)^{3/2}. \tag{15}$$

Now for each $u\in[n]$ with $F(u) = \mathbb{P}[u\in S] > 0$, let μ_u denote the distribution μ conditioned on $u\in S$. Let I_{vS} denote the indicator of the event $\{v\in S\}$, so that $\mathbb{P}[v\in S \mid u\in S] = \mathbb{E}_{S\sim\mu_u}[I_{vS}]$.

In this case,

$$\begin{aligned}
\mathbb{E}_{S\sim\mu,S'\sim\mu}\left[|S\cap S'|^3 \,\middle|\, u\in S\cap S'\right] &= \sum_{v,v',v''\in[n]} \mathbb{E}_{S\sim\mu_u,S'\sim\mu_u}\left[I_{vS}I_{v'S}I_{v''S}I_{vS'}I_{v'S'}I_{v''S'}\right] \\
&= \mathbb{E}_{S\sim\mu_u,S'\sim\mu_u}\left[\Big(\sum_v I_{vS}I_{vS'}\Big)^3\right] \\
&\geq \Big(\mathbb{E}_{S\sim\mu_u,S'\sim\mu_u}\Big[\sum_v I_{vS}I_{vS'}\Big]\Big)^3 \\
&= \Big(\sum_v \left(\mathbb{E}_{S\sim\mu_u}[I_{vS}]\right)^2\Big)^3 .
\end{aligned}$$

Therefore the left hand side of (15) is at least

$$\begin{aligned}
\sum_{u:\beta\leq F(u)\leq K\beta}\left(\sum_v \mathbb{P}[v\in S\mid u\in S]^2\right)^3 &\geq \frac{1}{K^6}\sum_{u:\beta\leq F(u)\leq K\beta}|\{v: F(u,v)/F(u)\geq 1/K\}|^3 \\
&\geq \frac{1}{K^6}\sum_{u:\beta\leq F(u)\leq K\beta}|\{v: F(u,v)\geq \beta\}|^3 \qquad (16)\\
&\geq \frac{1}{K^6}\sum_{u:\beta\leq F(u)\leq K\beta}|\{v: (u,v)\in E_{HL}\}|^3 ,
\end{aligned}$$

since each of the edges in E_{HL} appears at least once in the sum (16), because every edge $(u,v)\in E_{HL}$ has either $F(u)\leq K\beta$ or $F(v)\leq K\beta$.

In particular, for such edges, $F(u,v)\leq K\beta$, which means that

$$|E_{HL}| \geq \frac{\sum_{(u,v)\in E_{HL}} F(u,v)}{K\beta}. \qquad (17)$$

Thus by the power-mean inequality, the left hand side of (15) is at least

$$\begin{aligned}
\frac{1}{K^6}\sum_{u:\beta\leq F(u)\leq K\beta}|\{v:(u,v)\in E_{HL}\}|^3 &\geq \frac{1}{K^6n^2}\left(\sum_{u:\beta\leq F(u)\leq K\beta}|\{v:(u,v)\in E_{HL}\}|\right)^3 \\
&\geq \frac{1}{K^6n^2}|E_{HL}|^3
\end{aligned}$$

and when $\sum_{(u,v)\in E_{HL}} F(u,v) \geq \frac{1}{2}\sum_{u,v}F(u,v)$ it follows from (17) that this is at least

$$\frac{1}{8K^9n^2\beta^3}\left(\sum_{u,v}F(u,v)\right)^3 \gtrsim n\left(\mathbb{E}|S|^2\right)^{3/2},$$

completing the proof. □

5 Eigenvalues of graphs and surfaces

5.1 Graphs

We can now prove our main theorem.

Theorem 5.1. *If G is an n-node graph, then for every $1 \le k \le n$, we have the following bounds. If G is planar, then*

$$\lambda_k \le O\left(d_{\max}\frac{k}{n}\right). \tag{18}$$

If G is of genus $g > 0$, then

$$\lambda_k \le O\left(d_{\max}\frac{k}{n}g(\log g)^2\right).$$

If G is K_h-minor-free, then

$$\lambda_k \le O\left(d_{\max}\frac{k}{n}h^6 \log h\right).$$

Proof. We prove the planar case; the other cases follow similarly. Let $G = (V, E)$ be planar with maximum degree $d_{\max}$ and $n = |V|$. First, by Theorem 2.3, we see that for any weight function $\omega : V \to \mathbb{R}_+$ and every $k \ge 1$,

$$\lambda_k \lesssim \frac{d_{\max}}{\varepsilon^2 n}\left(\beta_{\varepsilon/2}(V, \mathsf{dist}_\omega)\right)^2,$$

where $\varepsilon = \varepsilon_{\lfloor n/4k\rfloor}(G, \omega)$. Since G is planar, by Theorem 2.1, we have $\beta_{\varepsilon/2}(V, \mathsf{dist}_\omega) = O(1)$ for any ω, hence

$$\lambda_k \lesssim \frac{d_{\max}}{\left(\varepsilon_{\lfloor n/4k\rfloor}(G, \omega)\right)^2 n}. \tag{19}$$

Using Corollaries 4.2 and 3.6, we see that for some constant $c_0 \ge 1$ and any $c_0 \le r \le |V|$, if $F \in \mathcal{F}_r(G)$, i.e. F if a μ-flow with $\mathrm{supp}(\mu) \subseteq \binom{V}{r}$, then

$$\mathsf{con}(F) \gtrsim \frac{r^5}{n}.$$

Now, by Theorem 2.4, this implies that for $r \ge c_0$, there exists a weight $\omega_r : V \to \mathbb{R}_+$ with $\varepsilon_r(G, \omega_r) \gtrsim \frac{1}{r^2}\sqrt{r^5/n} = \sqrt{r/n}$.

If $\lfloor n/4k\rfloor < c_0$, then (18) holds trivially using the bound $\lambda_k \le 2\, d_{\max}$ for all $1 \le k \le n$. Finally, using (19), for $r = \lfloor n/4k\rfloor \ge c_0$, we have

$$\lambda_k \lesssim \frac{d_{\max}}{\left(\varepsilon_r(G, \omega_r)\right)^2 n} \lesssim \frac{d_{\max}}{r} \lesssim d_{\max}\frac{k}{n},$$

completing the proof.

Remark 5.1 (Asymptotic dependence on k)**.** *We remark that the asymptotic dependence on k in Theorem 5.1 is tight. First, consider the eigenvalues $\lambda'_1 \le \cdots \le \lambda'_n$ for the n-node path graph P_n. It is a straightforward calculation to verify that the eigenvalues are precisely the set*

$$\{2 - 2\cos(2\pi k/n) : 1 \le k \le n/2\},$$

and each such eigenvalue has multiplicity at most 2. In particular, $\lambda'_k \asymp \frac{k^2}{n^2}$ for all $k \ge 2$.

Now, since the $n \times n$ grid graph G_n is the Cartesian product graph $P_n \times P_n$, it is easy to verify that the eigenvalues are precisely

$$\{\lambda_{i,j} = \lambda'_i + \lambda'_j : 1 \leq i, j \leq n\}.$$

In particular, since $\lambda_{i,j} \asymp \max(i^2, j^2)/n^2$, we have $\lambda_k(G_n) \asymp \frac{k}{n^2} \asymp \frac{k}{|G_n|}$. □

Finally, we use the Robertson-Seymour structure theorem to prove Theorem 1.2.

Proof of Theorem 1.2. If $\mathcal{F}$ is any minor-closed family of graphs that does not contain all graphs, then by the deep Robertson-Seymour structure theory [42], there exists some number $h \in \mathbb{N}$ such that no graph in $\mathcal{F}$ has K_h as a minor. An application of Theorem 5.1 finishes the proof. □

5.2 Surfaces

In this section, we shall show how our result implies a bound on the eigenvalues of the Laplacian of a compact Riemannian surface.

Theorem 5.2. *Let (M, g) be a compact, orientable Riemannian surface of genus g and area A, and let Δ_M be its Laplacian. The k^{th} smallest Neumann eigenvalue of Δ_M is at most*

$$O\left(k(g+1)\log^2(g+1)/A\right).$$

Intuitively, this theorem follows by applying the eigenvalue bound for genus g graphs from Theorem 5.1 to a sequence of successively finer meshes that approximate M.

Our proof will begin with the combinatorial Hodge theory of Dodziuk [16], which produces a sequence of finite-dimensional operators $\Delta_M^{(1)}, \Delta_M^{(2)}, \ldots$ whose eigenvalues converge to those of Δ_M. Unfortunately, the objects that this produces will not be the Laplacians of unweighted graphs of bounded degree. However, we will show that, when applied to a sufficiently nice triangulation, the operators produced by Dodziuk's theory can be approximated well enough by such graph Laplacians to establish our desired result.

5.2.1 The Whitney Map and Combinatorial Hodge Theory

We begin by recalling the basic setup of Dodziuk's combinatorial Hodge theory [16]. Let $\chi : K \to M$ be a finite triangulation of M with vertices $p_1, \ldots, p_n \in K$. For all $q \in \mathbb{N}$, let $L^2\Lambda^q = L^2\Lambda(M)$ be the space of square integrable q-forms on M, and let $C^q = C^q(K)$ be the space of real simplicial cochains on K. We will identify each simplex σ of K with the corresponding cochain, which allows us to write elements of $C^q(K)$ as formal sums of the q-simplices in K. For any triangle $\sigma \in K$, we will use $\mathrm{area}(\sigma)$ and $\mathrm{diam}(\sigma)$ to denote that area and diameter of $\chi\sigma$ with respect to the Riemannian metric on M.

For each p_i, let $\beta_i : K \to \mathbb{R}$ equal the p_i^{th} barycentric coordinate on simplices in $\mathrm{St}(p_i)$, the open star of p_i, and 0 on $K \setminus \mathrm{St}(p_i)$. This lets us define barycentric coordinate functions $\mu_i = \chi^*\beta_i$ on M.

Let $\sigma = [p_{i_0}, \ldots, p_{i_q}]$ be a q-simplex in K with $i_0 \leq \cdots \leq i_q$. We define the *Whitney map* $W : C^q(K) \to L^2\Lambda$ to be the linear map that takes each such simplex to

$$W\sigma = q! \sum_{k=0}^{q} (-1)^k \mu_{i_k} d\mu_{i_0} \wedge \cdots \wedge \widehat{d\mu_{i_k}} \wedge \cdots \wedge d\mu_{i_q}.$$

Whitney [51] showed that the above definition gives a well-defined element of $L^2\Lambda^q$, even though the μ_i are not differentiable on the boundaries of top-dimensional simplices.

The Riemannian metric endows $L^2\Lambda^q$ with the inner product

$$(f, g) = \int_M f \wedge *g,$$

where $*$ is the Hodge star operator. Using the Whitney map, this lets us define an inner product on C^q by setting

$$(a, a') = (Wa, Wa')$$

for $a, a' \in C^q$. Let d^c be the simplicial coboundary operator. Dodziuk defined the combinatorial codifferential δ^c to be the adjoint of d^c with respect to this inner product, and he defined the combinatorial Laplacian $\Delta_q^c : C^q \to C^q$ by

$$\Delta_q^c = d^c\delta^c + \delta^c d^c.$$

In the remainder of this paper, we will only use the Laplacian on functions, which we will denote by $\Delta^c := \Delta_0^c$.

To obtain a sequence of successively finer triangulations, we will use Whitney's *standard subdivision* procedure [51]. For a complex K, this produces a new complex SK in which each q-dimensional simplex of K is divided into 2^q smaller simplices. In contrast to barycentric subdivision, it is constructed in a way that prevents the simplices from becoming arbitrarily poorly conditioned under repeated subdivision.

Let $S_0K = K$, and inductively define $S_{n+1}K = S(S_nK)$. Dodziuk showed the following convergence result about the discrete Laplacians on functions:[3]

Theorem 5.3 (Dodziuk)**.** *Let $\lambda_i^{(n)}$ be the i^{th} smallest eigenvalue of $\Delta^c(S_nK)$, and let λ_i be the i^{th} smallest eigenvalue of Δ_M. Then $\lambda_i^{(n)} \to \lambda_i$ as $n \to \infty$.*

5.2.2 Relating the Combinatorial and Graph Laplacians

To relate the combinatorial Laplacian to a graph Laplacian, we will construct a triangulation in which all of the triangles have approximately the same volume, are fairly flat, and have vertex angles bounded away from 0. We will then show that the eigenvalues of combinatorial Laplacians arising from such a triangulation and its standard subdivisions can be bounded in terms of those of the Laplacian of an unweighted graph of bounded degree.

Lemma 5.4. *There exist strictly positive universal constants C_1, C_2, C_3, and θ such that, for any $\epsilon > 0$, every compact Riemannian surface M has a triangulation K with the following properties:*

1. *For every triangle $\sigma \in K$, $\mathrm{diam}(\sigma) < \epsilon$, the interior angles of σ all lie in $[\theta, \pi - \theta]$, and*

$$\frac{1}{C_2} \leq \frac{\mathrm{area}(\sigma)}{\mathrm{diam}(\sigma)^2} \leq C_2.$$

[3] Dodziuk and Patodi [17] later proved an analogous result for the Laplacians on q-forms, for arbitrary q.

2. For any two triangles $\sigma_1, \sigma_2 \in K$,

$$1/C_1 \leq \frac{\text{area}(\sigma_1)}{\text{area}(\sigma_2)} \leq C_1,$$

and

$$1/C_1 \leq \frac{\text{diam}(\sigma_1)}{\text{diam}(\sigma_2)} \leq C_1.$$

3. The edges of K are embedded as geodesics, and every vertex of K has degree at most C_3.

Furthermore, these properties are satisfied by $S_n K$ for all $n \geq 0$.

Proof. The existence of such a triangulation is established by Buser, Seppälä, and Silhol [11], following an argument originally due to Fejes Tóth [49]. They do not explicitly state the degree bound, but it follows immediately from the fact that the angles are bounded away from zero. The fact that these properties remain true under subdivision follows from the basic properties of standard subdivision given by Whitney [51]. □

Proof of Theorem 5.2. For a given ϵ, let K_ϵ be a triangulation with the properties guaranteed by Lemma 5.4, and let $G = (V, E)$ be the 1-skeleton of K_ϵ. Let $f : V \to \mathbb{R}$, and let $f_i = f(p_i)$. We will show that, for sufficiently small ϵ,

$$\frac{(f, \Delta^c f)}{(f, f)} \lesssim \frac{|V|}{A} \frac{\|f\|^2_{\mathcal{L}_G}}{\|f\|_2^2} \tag{20}$$

for all f, and that this remains true when K_ϵ is replaced by $S_n K_\epsilon$ for any n. By the variational characterization of eigenvalues, this implies that $\lambda_k(\Delta^c) \lesssim \frac{|V|}{A} \lambda_k(\mathcal{L}_G)$. By applying Theorem 5.1 to $\mathcal{L}_G$, we obtain

$$\lambda_k(\Delta^c) \lesssim \frac{|V|}{A} \lambda_k(\mathcal{L}_G) \lesssim \frac{|V|}{A} \frac{k(g+1)\log^2 g}{|V|} = \frac{k(g+1)\log^2 g}{A}.$$

This bound remains true as we subdivide K_ϵ, so Theorem 5.2 now follows from Theorem 5.3. It thus suffices to prove equation (20).

Let $\sigma = [p_{i_0}, p_{i_1}, p_{i_2}]$ be a triangle in K_ϵ. We can write the restriction of Wf to σ in barycentric coordinates as

$$Wf|_\sigma = f_1 \mu_{i_1} + f_2 \mu_{i_2} + f_3 \mu_{i_3}.$$

When ϵ is sufficiently small compared to the minimum radius of curvature of M, we have

$$\int_\sigma \mu_i \mu_j \, dV = \begin{cases} (1 \pm o(1))\text{area}(\sigma)/6 & \text{if } i = j \\ (1 \pm o(1))\text{area}(\sigma)/12 & \text{if } i \neq j \end{cases},$$

where dV is the volume element on M, and the $o(1)$ indicates a function that goes to zero with ϵ.

This gives

$$\begin{aligned}
\int_\sigma (Wf)\wedge *(Wf) &= \int_\sigma (Wf)^2 dV \\
&= \int_\sigma (f_1\mu_{i_1} + f_2\mu_{i_2} + f_3\mu_{i_3})^2\, dV \\
&\asymp \frac{\text{area}(\sigma)}{6}(f_1^2 + f_2^2 + f_3^2 + f_1 f_2 + f_1 f_3 + f_2 f_3) \\
&= \frac{\text{area}(\sigma)}{12}\left(f_1^2 + f_2^2 + f_3^2 + (f_1+f_2+f_3)^2\right) \\
&\geq \frac{\text{area}(\sigma)}{12}(f_1^2 + f_2^2 + f_3^2).
\end{aligned}$$

Let A_ϵ be the maximum area of a triangle in K_ϵ. Since all triangles have the same area up to a multiplicative constant, and each vertex appears in only a constant number of triangles, summing this over all of the triangles in K_ϵ gives

$$(f,f) = \int_M (Wf)\wedge *(Wf) \gtrsim A_\epsilon \sum_{i=1}^n f_i^2 = A_\epsilon \|f\|_2^2. \tag{21}$$

When restricted to σ, we have

$$d^c f|_\sigma = (f_1 - f_0)[p_{i_0}, p_{i_1}] + (f_2 - f_1)[p_{i_1}, p_{i_2}] + (f_2 - f_0)[p_{i_0}, p_{i_2}],$$

so

$$Wd^c f|_\sigma = (f_1 - f_0)\left(\mu_{i_0} d\mu_{i_1} - \mu_{i_1} d\mu_{i_0}\right) + (f_2 - f_1)\left(\mu_{i_1} d\mu_{i_2} - \mu_{i_2} d\mu_{i_1}\right) + (f_2 - f_0)\left(\mu_{i_0} d\mu_{i_2} - \mu_{i_2} d\mu_{i_0}\right). \tag{22}$$

By again assuming that ϵ is sufficiently small and using the fact that the triangles in K_ϵ are all well-conditioned, we obtain by a simple calculation the estimate

$$\int_\sigma d\mu_{i_k} * d\mu_{i_k} \lesssim \left(\frac{1}{\text{diam}(\sigma)}\right)^2 \cdot \text{area}(\sigma) \asymp 1$$

for each $k \in \{0,1,2\}$, where the asymptotic equality of the last two quantities follows from property 1 of Lemma 5.4. Applying this and Cauchy-Schwartz to equation (22), and using the fact that the μ_{i_j} are bounded above by 1, gives

$$\int_\sigma (Wd^c f)\wedge *(Wd^c f) \lesssim (f_1 - f_0)^2 + (f_2 - f_1)^2 + (f_2 - f_0)^2.$$

Summing this over all of the triangles and using Lemma 5.4 then yields

$$(df, df) = \int_M (Wd^c f)\wedge *(Wd^c f) \lesssim \sum_{(i,j)\in E} (f_i - f_j)^2 = \|f\|_{\mathcal{L}}^2. \tag{23}$$

The total area of M equals A, and the area of each triangle is within a constant factor of A_ϵ, so $|V| \asymp A/A_\epsilon$. If we combine this with the inequalities in (21) and (23), we obtain

$$\frac{(f, \Delta^c f)}{(f,f)} = \frac{(df, df)}{(f,f)} \lesssim \frac{\|f\|_{\mathcal{L}}^2}{A_\epsilon \|f\|_2^2} \asymp \frac{|V|}{A}\frac{\|f\|_{\mathcal{L}}^2}{\|f\|_2^2}.$$

This proves equation (20), which completes the proof of Theorem 5.2. □

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