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SUSY Ward identities, Superamplitudes, and Counterterms

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Abstract

Ward identities of SUSY and R-symmetry relate n-point amplitudes in supersymmetric theories. We review recent work in which these Ward identities are solved in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity. The solution, valid at both tree and loop level, expresses any N^KMHV superamplitude in terms of a basis of ordinary amplitudes. Basis amplitudes are classified by semi-standard tableaux of rectangular $\mathcal{N}$-by-K Young diagrams. The SUSY Ward identities also impose constraints on the matrix elements of candidate ultraviolet counterterms in $\mathcal{N} = 8$ supergravity, and they can be studied using superamplitude basis expansions. This leads to a novel and quite comprehensive matrix element approach to counterterms, which we also review.

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Contents

1 Introduction

Supersymmetry and R-symmetry Ward identities impose linear relations among individual amplitudes in supersymmetric theories. The first question addressed in this review is how to solve the Ward identities in $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity and specify a basis of amplitudes that determines all others in the same N^KMHV class. For MHV amplitudes, the answer is simple: any one amplitude determines the entire class. However, for $K \geq 1$ very little information was known until recently. We review the results of [1], where the SUSY and R-symmetry Ward identities were solved to give an expansion of the general N^KMHV superamplitude in terms of a minimal basis of component amplitudes that are independent under these Ward identities. In the second part of this review, we apply this expansion to the analysis of potential counterterms in $\mathcal{N}=8$ supergravity [2]. Imposing the additional requirement of locality on the manifestly SUSY and R-invariant expansion of superamplitudes is at the heart of this *matrix-element approach to counterterms*. Just as recursion relations focus on on-shell scattering amplitudes instead of Lagrangians, the center of attention is shifted from counterterm operators to their matrix elements.

The first approach to SUSY Ward identities for on-shell amplitudes was the 1977 work of Grisaru and Pendleton [3] (see also [4, 5]). They discussed the structure of these identities and solved them for 6-point NMHV amplitudes in $\mathcal{N}=1$ SUSY. Six basis amplitudes were needed to determine all 60 NMHV amplitudes.[1]

A general solution to the $\mathcal{N}=4$ and $\mathcal{N}=8$ Ward identities was recently presented in [1] and will be reviewed in Sec. 4-5 below. The solution exploits the properties of superamplitudes which compactly encode all individual n-point amplitudes at each N^KMHV level. The Ward identities can be elegantly imposed as constraints on the superamplitudes which are then expressed as sums of simple manifestly SUSY- and R-invariant Grassmann polynomials, each multiplied by an ordinary amplitude. This set of ordinary amplitudes comprise a basis for the superamplitude. Only the Ward identities of non-anomalous Poincaré SUSY and $SU(\mathcal{N})_R$ symmetry are used, so the results apply both to tree and loop amplitudes. The dual conformal and Yangian symmetries of the $\mathcal{N}=4$ theory are important and have led to much new information about planar amplitudes of the theory.[2] Those symmetries were not included in the analysis of [1], so that the results are valid for both planar and nonplanar amplitudes of $\mathcal{N}=4$ SYM and also for $\mathcal{N}=8$ supergravity.

Let us provide a preview of the structure of superamplitudes and their basis expansion with details discussed in Sec. 3-5. Superamplitudes [8–10] $\mathcal{A}_n$ are generating functions for ordinary amplitudes whose bookkeeping Grassmann variables η_{ia} are labeled by particle number $i=1,\dots,n$ and by the $SU(\mathcal{N})_R$-symmetry index $a=1,\dots,\mathcal{N}$. At level N^KMHV, the superamplitudes are Grassmann polymomials of order $\mathcal{N}(K+2)$. Their coefficients are the actual scattering amplitudes.

Supercharges Q^a and $\widetilde{Q}_a$ defined by the simple expressions

$$\widetilde{Q}_a = \sum_{j=1}^{n} |j\rangle\, \eta_{ja}\,, \qquad Q^a = \sum_{j=1}^{n} |j]\, \frac{\partial}{\partial \eta_{ja}} \tag{1.1}$$

act directly on the superamplitudes, giving the Ward identities

$$\widetilde{Q}_a\, \mathcal{A}_n = 0, \qquad Q^a \mathcal{A}_n = 0\,. \tag{1.2}$$

It is these SUSY Ward identities combined with important constraints due to R-symmetry which are solved in [1]. The solutions derived for superamplitudes take the schematic form

$$\mathcal{A}_n^{\mathrm{N}^K\mathrm{MHV}} \;=\; \sum_I A_I\, Z_I\,. \tag{1.3}$$

The index I enumerates the set of independent SUSY and R-symmetry invariant Grassmann polynomials Z_I

[1]The solution of [3] was rederived using modern spinor-helicity methods in [6].
[2]See [7] for a review in this issue.

of degree $\mathcal{N}(K+2)$. They are constructed from two simple and familiar ingredients, which are explained in more detail below. First, each Z_I contains a factor of the well known Grassmann delta-function, $\delta^{(2\mathcal{N})}(\widetilde{Q})$, which expresses the conservation of $\widetilde{Q}_a$. It is a degree $2\mathcal{N}$ polynomial which is annihilated by both Q^a and $\widetilde{Q}_a$. The other ingredient is that each Z_I contains $\mathcal{N}K$ factors of the first-order polynomial

$$m_{ijk,a} \equiv [ij]\eta_{ka} + [jk]\eta_{ia} + [ki]\eta_{ja} \,, \tag{1.4}$$

in which i, j, k label three external lines of the n-point amplitude. Every polynomial $m_{ijk,a}$ is annihilated by Q^a. The polynomial (1.4) is the essential element of the well-known 3-point anti-MHV superamplitude.

The basis amplitudes A_I in (4.10) are matrix elements for specific particle processes within each N^{K}MHV sector. Finding the basis can be formulated as a group theoretic problem, and it has a neat solution. The number of amplitudes in the basis is the dimension of the irreducible representation of $SU(n-4)$ corresponding to a rectangular Young diagram with K rows and $\mathcal{N}$ columns! The independent amplitudes are precisely labeled by the semi-standard tableaux of this Young diagram.

As an example, consider the 6-point NMHV amplitude $\mathcal{A}_6^{\text{NMHV}}$ in $\mathcal{N} = 4$ SYM. There are 5 basis amplitudes which can be chosen to be the 6-point matrix elements:

$$\langle -+++--\rangle \,, \quad \langle \lambda^- \lambda^+ ++--\rangle \,, \quad \langle s\,\bar{s}++--\rangle \,, \quad \langle \lambda^+ \lambda^- ++--\rangle \,, \quad \langle +-++--\rangle \,. \tag{1.5}$$

The last 4 particles in each amplitude are 'standardized' by SUSY to be gluons of positive and negative helicity. In the first two positions we must allow any combination that leads to an NMHV amplitude, *i.e.* pairs of gluons, gluinos, and scalars. For $\mathcal{N} = 8$ supergravity, the analogous basis contains 9 basis amplitudes which we can again specify to contain 'standardized' gravitons as the last 4 particles and pairs of gravitions, gravitinos, etc. on the first two lines.

Basis amplitudes containing four gluons '$++--$' on four fixed lines are particularly convenient to write down the superamplitude in closed form. Using a computer-based implementation of this superamplitude, however, one can choose any other set with the same number of linearly independent amplitudes. Linear independence, in this case, is best verified numerically. At the 6-point NMHV level, for example, a suitable basis of 5 linearly independent gauge theory amplitudes is the split-helicity gluon amplitude $\langle +++---\rangle$ together with 4 of its cyclic permutations, specifically

$$\langle +++---\rangle, \quad \langle -+++--\rangle, \quad \langle --+++-\rangle, \quad \langle ---+++\rangle, \quad \langle +---++\rangle \,. \tag{1.6}$$

In $\mathcal{N} = 8$ supergravity, the pure graviton amplitude $M_6(+++---)$ together with 8 permutations of its external lines represents a suitable basis. It is striking that the basis of planar $\mathcal{N} = 4$ SYM ($\mathcal{N} = 8$ supergravity) at the 6-point NMHV level reduces to momentum permutations of a single all-gluon (all-graviton) amplitude.

The *second major topic* of this review is the application of the basis expansions of superamplitudes to candidate counterterms of the form $\sqrt{-g}D^{2k}R^n$ in the loop expansion of perturbative $\mathcal{N} = 8$ supergravity. The matrix element method complements and extends other approaches to counterterms which work with on-shell superspace [11–13], information from string theory [14, 15, 17], and light-cone superspace [18]. The leading matrix elements of a potential counterterm must be local and gauge invariant, and this means that they are polynomials in the spinor brackets $\langle i\,j\rangle$, $[k\,l]$ associated with the external momenta. Matrix elements of candidate counterterms at loop order L are strongly constrained by the overall scale dimension and the helicities of their external particles. In many cases one can show quite simply that there are no local SUSY and R-invariant superamplitudes that satisfy these constraints. Then the corresponding operator is not supersymmetrizable and cannot appear as an independent counterterm. On the other hand, when the constraints are satisfied, the method explicitly constructs the matrix elements of a linearized supersymmetric completion of the operator.

In addition to SUSY and R-symmetry, the spontaneously broken $E_{7(7)}$ symmetry [19] of $\mathcal{N} = 8$ supergravity gives additional constraints on counterterm matrix elements with external scalar particles. In particular, counterterm matrix elements must vanish in the single-soft scalar limit. These constraints were analyzed to exclude the potential 3-, 5-, and 6-loop counterterms R^4, D^4R^4, and D^6R^4 in the recent papers [20,21] (see also [17,22,23]), which are reviewed in Sec. 6.5 below.

The net result of the matrix element approach to counterterms, combined with the results of [13], is that there are no admissible counterterms in $\mathcal{N}=8$ supergravity at loop order $L<7$. The method does not exclude counterterms at loop order $L\geq 7$, but it shows that the only possible independent $L=7$ loop counterterm is $D^8R^4+\dots$, whose leading matrix elements involve 4 external particles [21]; higher-point operators such as D^4R^6 (for which we present simple explicit superamplitude expressions in Sec. 6.3) and R^8 are compatible with SUSY and R-symmetry, but have non-vanishing single-soft scalar limits and thus violate continuous $E_{7(7)}$ symmetry [21] (see also [24]). This implies that a computation of the 4-point amplitude is sufficient to determine whether or not $\mathcal{N}=8$ supergravity is finite at 7-loop order.

In Sec. 7, we discuss the structure of superamplitudes with reduced R symmetry. We focus on amplitudes that are invariant under an $SU(4)\times SU(4)$ subgroup of $SU(8)$; these are relevant both for the study of single-soft scalar limits in $\mathcal{N}=8$ supergravity and for closed string tree amplitudes with massless external states in 4 dimensions.

2 SUSY Ward identities

Particle states of the $\mathcal{N}=4$ and $\mathcal{N}=8$ theories transform in anti-symmetric products of the fundamental representation of the R-symmetry groups $SU(4)$ and $SU(8)$. Thus the gluons, the 4 gluinos, and the 6 scalars of the $\mathcal{N}=4$ theory can be described by annihilation operators which carry anti-symmetrized upper indices:

$$B,\quad B^a,\quad B^{ab},\quad B^{abc},\quad B^{abcd}\,, \tag{2.1}$$

with $1\leq a,b,\ldots\leq 4$. The tensor rank r is related to the particle helicity h by $2h=2-r$. The 256 particle states of $\mathcal{N}=8$ supergravity are described analogously by annihilation operation operators of tensor rank $0\leq r\leq 8$. Helicity and rank are related by $2h=4-r$.

We now discuss the S-matrix elements and Ward identities for the simpler $\mathcal{N}=4$ theory. The extension to $\mathcal{N}=8$ is straightforward. One can suppress indices and simply use $B^{\cdots}$ for any annihilation operator from the set in (2.1). A generic n-point amplitude may then be denoted by

$$A_n(1,2,\ldots,n)\;=\;\big\langle B_1^{\cdots}B_2^{\cdots}\,\cdots B_n^{\cdots}\big\rangle\,. \tag{2.2}$$

$SU(4)$ invariance requires that the total number of (suppressed) indices is a multiple of 4, i.e. $\sum_{i=1}^n r_i\;=\;4m$. Furthermore each index value $a=1,2,3,4$ must appear m times among the operators $B_i^{\cdots}$. The integer m determines the N^{K}MHV class by $K=m-2$.

In general one considers complex null momenta p^μ, described by a bi-spinor $p^{\alpha\dot\beta}=|p]^\alpha\langle p|^{\dot\beta}$. For real momenta, when angle and square spinors are related by complex conjugation, each $A_n(1,2,\ldots,n)$ describes a physical amplitude in which particles in the final state have positive null p^μ and particles in the initial state have negative null p^μ. In scattering theory the S-matrix describes particle states in the limit of infinite past and future in which wave packets separate and interactions can be neglected. Therefore the SUSY charges that act on asymptotic states are determined by the free field limit of the transformation rules of the field theory.

In this section it is convenient to define chiral supercharges $Q^a\equiv-\epsilon^\alpha\,Q^a_\alpha$ and $\widetilde{Q}_a\equiv\epsilon_{\dot\alpha}\widetilde{Q}^{\dot\alpha}_a$, which include contraction with the anti-commuting parameters $\epsilon^\alpha,\epsilon_{\dot\alpha}$ of SUSY transformations. The commutators of the operators Q^a and $\widetilde{Q}_a$ with the various annihilators are given by:

$$\begin{aligned}
\big[\widetilde{Q}_a, B\big] &= 0\,, & \big[Q^a, B\big] &= [p\,\epsilon]\, B^a\,,\\
\big[\widetilde{Q}_a, B^b\big] &= \langle \epsilon\, p\rangle\, \delta^b_a\, B\,, & \big[Q^a, B^b\big] &= [p\,\epsilon]\, B^{ab}\,,\\
\big[\widetilde{Q}_a, B^{bc}\big] &= \langle \epsilon\, p\rangle\, 2!\, \delta^{[b}_a\, B^{c]}\,, & \big[Q^a, B^{bc}\big] &= [p\,\epsilon]\, B^{abc}\,,\\
\big[\widetilde{Q}_a, B^{bcd}\big] &= \langle \epsilon\, p\rangle\, 3!\, \delta^{[b}_a B^{cd]}\,, & \big[Q^a, B^{bcd}\big] &= [p\,\epsilon]\, B^{abcd}\,,\\
\big[\widetilde{Q}_a, B^{bcde}\big] &= \langle \epsilon\, p\rangle\, 4!\, \delta^{[b}_a B^{cde]}\,, & \big[Q^a, B^{bcde}\big] &= 0\,,
\end{aligned} \tag{2.3}$$

Note that $\widetilde{Q}_a$ raises the helicity of all operators and involves the spinor angle bracket $\langle \epsilon\, p\rangle$. Similarly, Q^a lowers the helicity and spinor square brackets $[p\,\epsilon]$ appear.

The Ward identities that relate S-matrix elements are obtained from

$$0 = \Big\langle\, [\, \widetilde{Q}_a\,,\, B_1^{\cdots} B_2^{\cdots} \cdots B_n^{\cdots}\,]\, \Big\rangle\,, \tag{2.4}$$

$$0 = \Big\langle\, [\, Q^a\,,\, B_1^{\cdots} B_2^{\cdots} \cdots B_n^{\cdots}\,]\, \Big\rangle\,. \tag{2.5}$$

The overall expressions vanish because supercharges annihilate the vacuum state. One then obtains concrete relations among amplitudes by moving the supercharges to the right, using the appropriate entry from (2.3) to evaluate $[\widetilde{Q}_a, B_i^{\cdots}]$ or $[Q^a, B_i^{\cdots}]$. To obtain a non-trivial relation, the product of operators $B_1^{\cdots} \cdots B_n^{\cdots}$ must contain an odd number of fermions. There are further constraints from $SU(4)$ invariance. In (2.4). the distinguished index value a must appear $m+1$ times among the $B_i^{\cdots}$, while other index values appear m times each. Similarly, in (2.5), the index value a must appear $m-1$ times and other index values m times each. One thus sees that the Ward identities relate amplitudes within an N^{K}MHV class.

At the MHV level, the SUSY Ward identities give very simple and transparent relations. For example, consider

$$\begin{aligned}
0 &= \Big\langle\, [\, \widetilde{Q}_1\,,\, -\,-\,B^1 + \cdots +\,]\, \Big\rangle \\
&= \langle \epsilon 1\rangle \big\langle B^{234} - B^1 + \cdots + \big\rangle \;+\; \langle \epsilon 2\rangle \big\langle - B^{234} B^1 + \cdots + \big\rangle \;+\; \langle \epsilon 3\rangle \big\langle - - + + \cdots + \big\rangle\,,
\end{aligned} \tag{2.6}$$

where we used that negative helicity gluons '$-$' transform as $[\widetilde{Q}_1, B_i^{1234}] = \langle \epsilon i\rangle B_i^{234}$, while positive helicity gluons '+' are annihilated by the supercharge, $[\widetilde{Q}_1, B_i] = 0$. There are three contributions on the right hand side of (2.6): the first two are gluon pair amplitudes and the last one is the n-gluon MHV amplitude. However, there are two linearly independent choices of the SUSY spinor $\langle \epsilon|$. If we choose $\langle \epsilon| \sim \langle 2|$, then (2.6) yields the relation

$$\big\langle B^{234} - B^1 + \cdots + \big\rangle \;=\; \frac{\langle 2\,3\rangle}{\langle 1\,2\rangle}\, \big\langle - - + + \cdots + \big\rangle\,. \tag{2.7}$$

between a gluino pair amplitude and the n-gluon amplitude. If we choose $\langle \epsilon| \sim \langle 1|$, then we find a similar relation for the other gluon pair amplitude. For every set of operators in $\langle [\widetilde{Q}_1, B_1^{\cdots} \cdots B_n^{\cdots}]\rangle$ in which the index $a = 1$ appears three times and the indices $2, 3, 4$ twice each, the Ward identity contains three terms. By choice of $\langle \epsilon|$ one obtains two independent relations similar to (2.7). By combining the various relations, one can show that any MHV n-point amplitude can be expressed as a rational function of angle brackets times the n-gluon amplitude $\langle - - + + \cdots + \rangle$. Another fact about MHV amplitudes is that the Q^a Ward identities are automatically satisfied when the relations from the $\widetilde{Q}_a$ WI's are incorporated. These key properties of the MHV sector are best seen from the MHV generating function discussed in the next section.

The situation in the NMHV sector is very different, as we can see by examining the Ward identity

$$\begin{aligned}
0 &= \Big\langle\, [\, \widetilde{Q}_1\,,\, - - - B^1 + +\,]\, \Big\rangle \\
&= \langle \epsilon 1\rangle \big\langle B^{234} - - B^1 + + \big\rangle \;+\; \langle \epsilon 2\rangle \big\langle - B^{234} - B^1 + + \big\rangle \;+\; \langle \epsilon 3\rangle \big\langle - - B^{234} B^1 + + \big\rangle \;+\; \langle \epsilon 4\rangle \big\langle - - - + + + \big\rangle\,.
\end{aligned} \tag{2.8}$$

Now there are four terms, while $\langle\epsilon|$ can take two independent values. Thus, one obtains two independent equations for four amplitudes, which is a linear system of "defect two". Every NMHV compatible choice of the operators in $\langle[\widetilde{Q}_1, B_1^{\cdots} \cdots B_n^{\cdots}]\rangle$ produces a similar pair of linear equations. Thus one obtains a large coupled set of such relations, and the overall rank of the system is difficult to ascertain. This problem is indeed best addressed in the language of superamplitudes, which we introduce in the next section. Please read on.

3 Superamplitudes and their Symmetries

3.1 Superamplitudes and supersymmetry constraints

The annihilation operators of the 16 massless states — gluons, gluinos, scalars — of the $\mathcal{N}=4$ supermultiplet can be encoded in the 'on-shell superfield'

$$\Phi = B + \eta_a\, B^a - \frac{1}{2!}\eta_a\,\eta_b\, B^{ab} - \frac{1}{3!}\eta_a\,\eta_b\,\eta_c\, B^{abc} + \eta_1\,\eta_2\,\eta_3\,\eta_4\, B^{1234}\,, \tag{3.1}$$

in which the bookkeeping Grassmann variables η_a are labeled by $SU(4)$ indices $a, b, \ldots = 1,2,3,4$. The supercharges $\tilde{q}_a = \langle\epsilon p\rangle\eta_a$ and $q^a = [\epsilon p]\frac{\partial}{\partial\eta_a}$ act on Φ by multiplication or differentiation. They 'move' operators to the right or left in (3.1) to reproduce the commutation relations (2.3). The anticommutator of the two supercharges is $[\tilde{q}_a, q^b] = \delta_a^b\langle\epsilon|p_i|\epsilon]$ and thus realizes Poincaré SUSY.

The amplitudes for all n-point processes within a given N^KMHV class are collected into *superamplitudes* $\mathcal{A}_n(\Phi_1,\ldots,\Phi_n)$, which are polynomials in the η_{ia}. The superamplitudes we discuss here must be $SU(4)$ invariant. In particular, an N^KMHV superamplitude is a degree $4(K{+}2)$ polynomial in the η_{ia} in which each index value $a = 1,2,3,4$ appears $(K{+}2)$ times in every monomial term. Any desired amplitude can be projected out from $\mathcal{A}_n$ by acting with the differential operators [6] that select the desired external state $B_i^{\cdots}$ from each Φ_i. The total derivative order is $4(K{+}2)$.

The construction for $\mathcal{N}=8$ supergravity is completely analogous: the 256 massless states are encoded into superfields using Grassmann variables η_a labelled by the global R-symmetry group $SU(8)$. The N^KMHV superamplitudes are degree $8(K+2)$ polynomials in the η_{ia}'s. In the rest of this section we study the maximally supersymmetric gauge and gravity theories ($\mathcal{N}=4$ and $\mathcal{N}=8$) jointly.

In [1] it is shown that the SUSY Ward identities (1.2) can be satisfied if superamplitudes are constructed from two basic ingredients. The first ingredient is the well-known Grassmann δ-function

$$\delta^{(2\mathcal{N})}(\widetilde{Q}) \;\equiv\; \delta^{(2\mathcal{N})}\Big(\sum_{i=1}^{n}|i\rangle\eta_{ia}\Big) \;=\; \frac{1}{2^{\mathcal{N}}}\prod_{a=1}^{\mathcal{N}}\sum_{i,j}^{n}\langle i\,j\rangle\,\eta_{ia}\eta_{ja}\,. \tag{3.2}$$

$\delta^{(2\mathcal{N})}(\widetilde{Q})$ is fully supersymmetric. Indeed, it is clear that $\widetilde{Q}_a\,\delta^{(2\mathcal{N})}\big(\widetilde{Q}\big) = 0$, while momentum conservation ensures that $Q_a\,\delta^{(2\mathcal{N})}\big(\widetilde{Q}\big) = 0$. We will show below that $\delta^{(2\mathcal{N})}(\widetilde{Q})$ is also $SU(\mathcal{N})$ invariant.

The $\delta^{(2\mathcal{N})}$-function is the only element needed to construct MHV superamplitudes. Note that it has the correct polynomial order, namely $2\mathcal{N}$. The n-point MHV superamplitude is simply given by

$$\mathcal{A}_n^{\mathrm{MHV}} \;=\; \delta^{(2\mathcal{N})}(\widetilde{Q})\;\frac{\langle + + \cdots + - - \rangle}{\langle n-1, n\rangle^{\mathcal{N}}}\,. \tag{3.3}$$

It has one 'basis amplitude,' namely the pure gluon/graviton MHV amplitude $A_n(+ + \cdots + - -)$. When the order-$2\mathcal{N}$ differential operator, which selects a given process, is applied, $\mathcal{N}$ angle brackets are produced from the $\delta^{(2\mathcal{N})}$-function, and the chosen amplitude is then $\langle..\rangle\cdots\langle..\rangle/\langle n{-}1, n\rangle^{\mathcal{N}}$ times the basis amplitude.

The second basic ingredient that is needed to construct N^KMHV superamplitudes is the simple polynomial $m_{ijk,a}$ of (1.4). The Schouten identity ensures $Q^a\, m_{ijk,b} = 0$, and this holds for *any* choice of three lines i, j, k, adjacent or non-adjacent, independent of momentum conservation.

We write the N^{K}MHV superamplitude

$$\mathcal{A}_n^{\mathrm{N}^K\mathrm{MHV}} \;=\; \delta^{(2\mathcal{N})}(\widetilde{Q})\; P_{\mathcal{N}\times K}\,, \tag{3.4}$$

where $P_{\mathcal{N}\times K}$ is a polynomial of degree $\mathcal{N}\times K$ in the η_{ia} variables. The delta-function (3.2) ensures that $\widetilde{Q}_a \mathcal{A}_n^{\mathrm{N}^K\mathrm{MHV}} = 0$. Since Q^a commutes with the delta-function, the only remaining SUSY constraint is $Q^a P_{\mathcal{N}\times K} = 0$. This is a non-trivial condition, but we show that its general solution can be expressed in terms of products of the polynomials $m_{ijk,a}$. The solution depends on the R-symmetry Ward identities, which we discuss next.

3.2 R-symmetry

To establish $SU(\mathcal{N})_R$ invariance of a function of the η_{ia}-variables it is sufficient to impose invariance under $SU(2)_R$ transformations acting on all choices of a pair of the $SU(\mathcal{N})_R$ indices $1,\ldots,\mathcal{N}$. To be specific, consider infinitesimal $SU(2)_R$ transformations in the ab-plane:

$$\sigma_1 : \begin{cases} \delta_R \eta_{ia} &= \theta \eta_{ib} \\ \delta_R \eta_{ib} &= \theta \eta_{ia} \end{cases}, \qquad \sigma_2 : \begin{cases} \delta_R \eta_{ia} &= -i\theta \eta_{ib} \\ \delta_R \eta_{ib} &= i\theta \eta_{ia} \end{cases}, \qquad \sigma_3 : \begin{cases} \delta_R \eta_{ia} &= \theta \eta_{ia} \\ \delta_R \eta_{ib} &= -\theta \eta_{ib} \end{cases}. \tag{3.5}$$

Here θ is the infinitesimal transformation parameter.

As a warm-up to further applications, we show that the $\delta^{(2\mathcal{N})}$-function (3.2) is $SU(\mathcal{N})_R$ invariant; this implies that MHV superamplitudes necessarily preserve the full R-symmetry. Since any monomial of the form $\eta_{i1}\,\eta_{j2}\cdots\eta_{l\mathcal{N}}$ is invariant under a σ_3-transformation, so is the $\delta^{(2\mathcal{N})}$-function. A σ_1-transformation in the 12-plane gives

$$\delta_R\big(\delta^{(2\mathcal{N})}(\widetilde{Q})\big) \;=\; \frac{\theta}{2^{\mathcal{N}-1}}\Big(\sum_{i,j=1}^{n}\langle ij\rangle\,\eta_{i1}\,\boldsymbol{\eta_{j2}}\;\sum_{k,l=1}^{n}\langle kl\rangle\,\boldsymbol{\eta_{k2}}\,\boldsymbol{\eta_{l2}}\Big)\Big(\prod_{a=3}^{\mathcal{N}}\sum_{k',l'=1}^{n}\langle k'l'\rangle\eta_{k'a}\,\eta_{l'a}\Big) \;+\;\ldots\;=\;0\,. \tag{3.6}$$

Anticommutation of the (highlighted) Grassmann variables antisymmetrizes the sum over j,k,l and $\langle ij\rangle\langle kl\rangle$ then vanishes by Schouten identity. The "$+\ldots$" stands for independent terms from δ_R acting on η_{k2} and η_{l2}. These terms can be treated the same way. Invariance under σ_2-transformations follows directly from $\sigma_{1,3}$ invariance and needs no further proof.

The R-symmetry constraints play an important role in the analysis of the SUSY Ward identities beyond the MHV level.

The analysis of the R-symmetry Ward identities also leads to a set of new *cyclic identities* for amplitudes. The identities encode relationships among amplitudes with the same types of external states, but with their R-symmetry indices distributed in different ways. An example is the following 4-term relation among $\mathcal{N}=4$ SYM NMHV amplitudes with gluinos λ and scalars s:

$$\begin{aligned} 0 \;=\; & A_6(\lambda^{123}\lambda^{\mathbf{3}}\lambda^{123}\lambda^{\mathbf{4}}s^{\mathbf{1}4}s^{\mathbf{2}4}) \;+\; A_6(\lambda^{123}\lambda^{\mathbf{4}}\lambda^{123}\lambda^{\mathbf{3}}s^{\mathbf{1}4}s^{\mathbf{2}4}) \\ & +\; A_6(\lambda^{123}\lambda^{\mathbf{4}}\lambda^{123}\lambda^{\mathbf{4}}s^{\mathbf{13}}s^{\mathbf{2}4}) \;+\; A_6(\lambda^{123}\lambda^{\mathbf{4}}\lambda^{123}\lambda^{\mathbf{4}}s^{\mathbf{1}4}s^{\mathbf{23}})\,. \end{aligned} \tag{3.7}$$

We call this a cyclic identity because the four boldfaced $SU(4)$ indices are cyclically permuted.

4 Basis expansion of superamplitudes in $\mathcal{N}=4$ SYM

We outline the strategy used to solve the SUSY- and R-symmetry Ward identities and construct a particular basis for the amplitudes at NMHV level for $\mathcal{N}=4$ SYM. We then present the representations of superamplitudes using this basis. We emphasize results below and leave full details to App. A.

4.1 Strategy for solving the SUSY Ward identities

The initial form of the $\mathcal{N}=4$ NMHV superamplitude is

$$\mathcal{A}_n^{\text{NMHV}} \;=\; \delta^{(8)}(\widetilde{Q})\; P_4\,. \tag{4.1}$$

Our task is to construct a minimal basis for all 4th order Grassmann polynomials P_4 that are $SU(4)$ invariant and satisfy $Q^a P_4 = 0$. Let's get to work.

1. First consider the constraints of $SU(4)$ R-symmetry invariance discussed in Sec. 3.2. The σ_3-transformations require P_4 to be a linear combination of $\eta_{i1}\,\eta_{j2}\,\eta_{k3}\,\eta_{l4}$ monomials, so we write

$$P_4 \;=\; \sum_{i,j,k,l=1}^{n} q_{ijkl}\,\eta_{i1}\,\eta_{j2}\,\eta_{k3}\,\eta_{l4}\,. \tag{4.2}$$

   The action of the σ_1-rotation in the 12-plane gives

$$\delta_R\left(q_{ijkl}\,\eta_{i1}\,\eta_{j2}\,\eta_{k3}\,\eta_{l4}\right) = \theta\, q_{\mathbf{ijkl}}(\eta_{\mathbf{i2}}\eta_{\mathbf{j2}} + \eta_{\mathbf{i1}}\eta_{\mathbf{j1}})\eta_{k3}\,\eta_{l4}\,. \tag{4.3}$$

   This quantity must vanish; hence the coefficients q must be symmetric in indices i and j, $q_{ijkl} = q_{jikl}$. A similar argument for any generator of $SU(4)_R$ implies that q_{ijkl} is a totally symmetric tensor.

2. The superamplitude (4.1) includes the $\delta^{(8)}$-function as a factor, so the 8 conditions it imposes can be used to eliminate a total of 8 distinct η_{ia}, namely any choice of two η_{ia} for each a. A convenient choice (which we make) is to eliminate the 4+4 Grassmann variables associated with lines $n{-}1$ and n. Then P_4 will then not depend on $\eta_{n-1,a}$ and η_{na}, and we write

$$P_4 = \frac{1}{\langle n{-}1,n\rangle^4}\sum_{i,j,k,l=1}^{n-2} c_{ijkl}\,\eta_{i1}\,\eta_{j2}\,\eta_{k3}\,\eta_{l4}\,. \tag{4.4}$$

   The c_{ijkl}'s are linear combinations of the q_{ijkl}'s; we will not need their detailed relationship. The coefficient $1/\langle n{-}1,n\rangle^4$ in (A.3) could be absorbed by a redefinition of the c_{ijkl}, but we keep it for later convenience. As in step 1, R-symmetry requires the c_{ijkl}'s to be fully symmetric, so the number of needed inputs at this stage is $(n-2)(n-1)n(n+1)/4!$.

   It is a consequence of our choice to eliminate $\eta_{n-1,a}$ and η_{na} that *all basis amplitudes have negative helicity gluons on lines $n{-}1$ and n.*

3. The polynomial P_4 in (A.3) satisfies the Ward identity $Q^a P_4 = 0$ if and only if the linear relations $\sum_{i=1}^{n-2}[\epsilon i]\,c_{ijkl} = 0$ hold for any triple jkl. Since ϵ^α is a 2-component spinor, there are two independent constraints which allow us to eliminate a choice of two lines s and t completely from the indices of the c_{ijkl}'s in P_4. This is analogous to the use of the $\widetilde{Q}_a$ Ward identities to eliminate two sets of η_{ia}-variables in step 2, and a consequence is that *lines s and t are positive helicity gluons in all basis amplitudes.* In the following we choose $s = n{-}3$ and $t = n{-}2$.

   We rewrite P_4 in terms of c_{ijkl}'s with $i,j,k,l \neq n{-}3, n{-}2$ and find that this naturally leads to the appearance of the polynomials $m_{ist,a}$, defined in (1.4). The result (see appendix for details) is the following form of the NMHV superamplitude:

$$\mathcal{A}_n^{\rm NMHV} \;=\; \sum_{1\le i\le j\le k\le l\le n-4} c_{ijkl}\, X_{(ijkl)} \qquad \text{with} \qquad X_{(ijkl)} \equiv \sum_{\mathcal{P}(i,j,k,l)} X_{ijkl}\,, \tag{4.5}$$

where the X_{ijkl} are η-polynomials of degree 12 that are annihilated by both Q^a and $\widetilde{Q}_a$:

$$X_{ijkl} \;\equiv\; \delta^{(8)}\big(\widetilde{Q}_a\big)\; \frac{m_{i,n\text{-}3,n\text{-}2;1}\, m_{j,n\text{-}3,n\text{-}2;2}\, m_{k,n\text{-}3,n\text{-}2;3}\, m_{l,n\text{-}3,n\text{-}2;4}}{[n-3,n-2]^4\langle n-1,n\rangle^4}\,. \tag{4.6}$$

The sum over permutations $\mathcal{P}(i,j,k,l)$ in the definition of $X_{(ijkl)}$ is over all *distinct* arrangements of fixed indices i,j,k,l. For instance, we have $X_{(1112)} = X_{1112}+X_{1121}+X_{1211}+X_{2111}$. Likewise, $X_{(1122)}$ contains the 6 distinct permutation of its indices, and $X_{(1123)}$ has 12 terms.[3]

4. The coefficients c_{ijkl} with $1 \le i,j,k,l \le n-4$ parameterize the most general SUSY and R-symmetry invariant NMHV superamplitude. The last step is to relate c_{ijkl} to actual amplitudes which then become the basis amplitudes. By direct application of the appropriate Grassmann derivatives, we find that each c_{ijkl} is identified as a single amplitude

$$c_{ijkl} \;=\; A_n\big(\{i,j,k,l\}++--\big) \;\equiv\; \big\langle \cdots B^1_i \cdots B^2_j \cdots B^3_k \cdots B^4_l \cdots +_{n\text{-}3} +_{n\text{-}2} -_{n\text{-}1} -_n \big\rangle\,, \tag{4.7}$$

with $1 \le i \le j \le k \le l \le n-4$. Let us clarify the notation: $A_n\big(\{i,j,k,l\}++--\big)$ means that line i carries $SU(4)_R$ index 1, line j carries index 2 etc. If $i=j$, this means that the line carries both indices 1 and 2, and the notation $B^1_i B^2_i$ should then be understood as B^{12}_i. Furthermore the dots indicate positive-helicity gluons in the unspecified positions, specifically any state $\neq i,j,k,l,n{-}1,n$ is a positive helicity gluon. For example, $A_{10}\big(\{1,1,2,4\}++--\big) = A_{10}(B^{12}B^3+B^4++|++--)$. For clarity, we have used a '|' to separate the first $n{-}4$ states from the last four gluon states, which are the same for all basis amplitudes.

Our final result for the manifestly SUSY and R-symmetric $\mathcal{N}=4$ SYM NMHV superamplitude is

$$\mathcal{A}_n^{\rm NMHV} \;=\; \sum_{1\le i\le j\le k\le l\le n-4} A_n\big(\{i,j,k,l\}++--\big)\; X_{(ijkl)}\,. \tag{4.8}$$

One might say that we have used the SUSY generators Q^a and $\widetilde{Q}_a$ to 'rotate' two states, n–3 and n–2, to be positive helicity gluons and two other states n–1 and n, to be negative helicity gluons. Any NMHV amplitude can be obtained from (4.8) by applying the 12th-order Grassmann derivative that corresponds to its external states. The amplitude will then be expressed as a linear combination of the $(n{-}4)(n{-}3)(n{-}2)(n{-}1)/4!$ independent basis amplitudes $A_n\big(\{i,j,k,l\}++--\big)$. The collection of these amplitudes is what we define as the *algebraic basis*.

Let us consider examples of superamplitudes in the basis (4.8). For $n=5$ we have to distribute the four $SU(4)$-indices on $n-4=1$ lines: there is only one choice, namely to put them all on line 1, which then must be a negative helicity gluon. Thus the 5-point NMHV superamplitude is described in terms of a single basis element $A_5\big(\{1,1,1,1\}{+}{+}{-}{-}\big) = \big\langle{-}{+}{+}{-}{-}\big\rangle$; this is of course not surprising, since the 5-point NMHV sector is equivalently described as anti-MHV. The superamplitude takes the form $\mathcal{A}_5^{\rm NMHV} = \big\langle -++--\big\rangle\, X_{1111}$.

Next, let us write the 6-point superamplitude in the basis (4.8). The four $SU(4)$ indices should now be distributed in all inequivalent ways on lines 1 and 2. There are five ways to do this — 1111, 1112, 1122, 1222 and 2222 — giving five basis amplitudes. The 6-point NMHV superamplitude can thus be written

$$\begin{aligned}\mathcal{A}_6^{\rm NMHV} \;=\; & \big\langle -+++--\big\rangle\, X_{1111} \;+\; \big\langle \lambda^{123}\lambda^4 ++--\big\rangle\, X_{(1112)} \;+\; \big\langle s^{12}\, s^{34}\; ++--\big\rangle\, X_{(1122)} \\ & +\; \big\langle \lambda^1\lambda^{234} ++--\big\rangle\, X_{(1222)} \;+\; \big\langle +-++--\big\rangle\, X_{2222}\,. \end{aligned} \tag{4.9}$$

[3]The number of distinct permutations of a set with repeated entries is a multinomial coefficient [25].

Here, we use a notation where λ denotes a gluino (B^a or B^{abc}) with the indicated $SU(4)_R$ indices, and s^{ab} denotes the scalar B^{ab}.

The amplitudes of the algebraic basis used in (4.8) are of the schematic form $\langle B_1^{\cdot\cdot\cdot} B_2^{\cdot\cdot\cdot} \cdots B_{n-4}^{\cdot\cdot\cdot} + + - - \rangle$. The states $B_i^{\cdot\cdot\cdot}$ can be any particles of the theory, subject to the NMHV level constraint that each $SU(4)$ index $a = 1, 2, 3, 4$ appears once among the $B_i^{\cdot\cdot\cdot}$. As in any vector space, there many other ways to specify a basis. One can choose any other set with the same number of amplitudes, provided that they are linearly independent under the SUSY and R-symmetry Ward identities. To verify linear independence of a putative set of basis amplitudes one can project them from the superamplitude (4.8) using the appropriate differential operators and then check that the matrix which relates the new set to the original basis has maximal rank. Due to algebraic complexity, this check is best done numerically using a computer-based implementation of the superamplitude.

At the 6-point NMHV level, for example, any choice of 5 linearly independent $\mathcal{N} = 4$ SYM amplitudes form a valid basis that completely determines the superamplitude. We have verified that the split-helicity amplitude $A_6(+ + + - - -)$ together with 4 of its cyclic permutations is a suitable basis of 6-point NMHV amplitudes. Similarly, there are pure-gluonic algebraic basis for NMHV amplitudes with $n = 7$ and $n = 8$ external legs. At $n = 9$, however, the 84 distinct gluonic amplitudes span a 69-dimensional subspace of the 70-dimensional algebraic basis. For $n > 9$ the dimension of the algebraic basis even exceeds the number of pure-gluon amplitudes, which immediately rules out the possibility of a purely gluonic basis.

4.2 Functional bases and single-trace amplitudes

The representation (4.8) contains a sum over basis amplitudes which are *algebraically* independent under the symmetries we have imposed. However, we have not yet included possible *functional* relations among amplitudes, that is relations which involve reordering of particle momenta. The cyclic and reflection symmetries of single trace color ordered amplitudes are examples of such relations.

For amplitudes in the single-trace sector, the cyclic permutations are functionally dependent; they can be computed from cyclic momentum relabelings. Thus the all-gluon *algebraic basis* of single-trace 6-point NMHV amplitudes discussed above reduces to a *functional basis containing the single amplitude* $\langle + + + - - - \rangle$. (Note that functional relations among amplitudes do not invalidate their use in an algebraic basis.)

For $n > 6$, the functional basis in the single-trace sector cannot consist of a single amplitude. Indeed, dihedral symmetry relates $2n$ amplitudes, which, for $n > 6$, is smaller than the number of algebraic basis amplitudes. For example, for $n = 7$ dihedral symmetry generates a set of at most 14 amplitudes from any one given amplitude, but $\binom{n-1}{4} = 15$ amplitudes are needed needed to form an algebraic basis. It is an open problem to find a simple expression of the superamplitude in terms of the minimal functional basis. However, it is possible to write down superamplitudes whose algebraic basis amplitudes are pairwise functionally related by dihedral symmetry. We refer the reader to [1] for details of this construction.

4.3 Beyond NMHV: superamplitudes and Young tableaux

The NMHV basis amplitudes $A_n(\{i, j, k, l\} + + - -)$ of (4.8) are labeled by four integers in the range $1 \le i \le j \le k \le l \le n-4$. These numbers are conveniently arranged in the semi-standard tableaux $\boxed{i \,|\, j \,|\, k \,|\, l}$ of the Young diagram with one row and four columns. It was shown in [1] that semi-standard Young tableaux provide the general organizing principle for N^KMHV superamplitudes. These superamplitudes can be written in the schematic form

$$\mathcal{A}_n^{\mathrm{N}^K\mathrm{MHV}} = \sum_I A_I \, Z_I \,, \tag{4.10}$$

in which index I enumerates the $SU(n-4)$ semi-standard tableaux of the rectangular Young diagram Y with K rows and $\mathcal{N}$ columns. The number of such semi-standard tableaux is the dimension d_Y of the $SU(n-4)$ irrep corresponding to the Young diagram Y. For each tableau there is a basis amplitude A_I

and a manifestly SUSY and $SU(4)_R$-invariant η-polynomial Z_I. To illustrate this structure, we discuss the N^{2}MHV superamplitudes of $\mathcal{N}=4$ SYM.

The basis amplitudes of the n-point N^{2}MHV superamplitude are labeled by $SU(n-4)$ semi-standard Young tableaux with two rows and four columns,

$$\begin{array}{|c|c|c|c|}\hline i_1 & j_1 & k_1 & l_1 \\ \hline i_2 & j_2 & k_2 & l_2 \\ \hline \end{array}\;. \tag{4.11}$$

Each row is non-decreasing ($i_A \le j_A \le k_A \le l_A$) and each column is strictly increasing ($i_1 < i_2$, etc.). Each tableau corresponds to a basis amplitude $A_n\big(\{\begin{smallmatrix} i_1 j_1 k_1 l_1 \\ i_2 j_2 k_2 l_2\end{smallmatrix}\}\,++--\big)$ with the specified gluons on the last four lines and with $SU(4)_R$ index 1 on lines i_1 and i_2, $SU(4)_R$ index 2 on lines j_1 and j_2, etc. For example,

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 2 & 5 \\ \hline \end{array}\;. \qquad \longleftrightarrow \qquad A_9\big(\{\begin{smallmatrix} 1113 \\ 2225\end{smallmatrix}\}\,++--\big) \;=\; A_9(\lambda^{123}\lambda^{123}\lambda^4+\lambda^4++--)\,. \tag{4.12}$$

From the hook rule [26] it follows that the

$$\#(\text{N}^2\text{MHV } n\text{-pt basis amplitudes}) \;=\; \dim_{SU(n-4)} \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline \end{array} \;=\; \tfrac{(n-5)(n-4)^2(n-3)^2(n-2)^2(n-1)}{4!\,5!}\,. \tag{4.13}$$

The N^{2}MHV superamplitude can be written in terms of basis amplitudes as

$$\mathcal{A}_n^{\text{N}^2\text{MHV}} \;=\; \frac{1}{16} \sum_{\substack{\text{semi-standard}\\ \text{tableaux Y}}} (-)^Y A_n\big(\{\begin{smallmatrix} i_1 j_1 k_1 l_1 \\ i_2 j_2 k_2 l_2\end{smallmatrix}\}\,++--\big)\; Z_{i_2 j_2 k_2 l_2}^{i_1 j_1 k_1 l_1}\,, \tag{4.14}$$

where the Z's are manifestly SUSY- and R-symmetry invariant η-polynomials similar to the X's in (4.6), but contain eight instead of four powers of $m_{ijk,a}$. The Z-polynomials and the sign factor $(-)^Y$ are defined in [1].

Let us comment on the detailed information contained in the semi-standard tableaux labels of the basis amplitudes. In the example (4.12), line labels $1,2,3,4,5$ appeared $3,3,1,0,1$ times, respectively. This is a particular (ordered) partition of the $8=3+3+1+0+1$ boxes of the Young diagram; each semi-standard tableau corresponding to a 3+3+1+0+1 partition of 8 corresponds to a process with the same particles types for the external states: states 1 and 2 are negative helicity gluinos, states 3 and 5 are positive helicity gluinos, and state 4 is a positive helicity gluon. How many independent basis amplitudes are there corresponding to this partition? — In other words, how many $SU(4)$-inequivalent ways are there to arrange the two sets of $SU(4)$-indices on the two negative helicity gluinos and the two positive helicity gluinos? The answer to this question is the combinatorial quantity called the *Kostka number*.[4] For the partition 3+3+1+0+1 of the 2-by-4 rectangular Young diagram, the Kostka number is 2: in addition to (4.12) there is a second basis amplitude with the same particle types on each external line, namely

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & 5 \\ \hline \end{array} \qquad \longleftrightarrow \qquad A_9\big(\{\begin{smallmatrix} 1112 \\ 2235\end{smallmatrix}\}\,++--\big) \;=\; A_9(\lambda^{123}\lambda^{124}\lambda^3+\lambda^4++--)\,. \tag{4.15}$$

Note that a different ordering of the partition is also possible, namely $8=3+3+1+1+0$. The Kostka number is independent of the ordering, so there are also two semi-standard tableaux associated with this second ordering; they are just obtained from those in (4.12) and (4.15) by exchanging 5 and 4. The corresponding basis amplitudes have a positive helicity gluino on line 4 and a positive helicity gluon on line 5. The structure outlined in this example generalizes to characterize all basis amplitudes of N^{K}MHV superamplitudes. Further details can be found in [1].

[4]The Kostka number $C_{\lambda;Y}$ depends on a Young tableaux Y with M boxes and a partition λ of M. The partition λ is a weight that dictates the number of times each number is used in the construction of the semi-standard tableaux of Y. The Kostka number $C_{\lambda;Y}$ counts the number of semi-standard tableaux of Y with weight λ.

5 Basis expansion of superamplitudes in $\mathcal{N}=8$ supergravity

The generalization of the above results to $\mathcal{N}=8$ supergravity is straightforward. The MHV sector is particularly simple because the superamplitude contains only one basis amplitude which we take to be the n-graviton amplitude $M_n(--+\cdots+)$. The superamplitude is the 16th order Grassmann polynomial

$$\mathcal{C}_n^{\rm MHV}=\delta^{(16)}\Big(\sum_i |i\rangle\eta_{ai}\Big)\frac{M_n(--+\cdots+)}{\langle 12\rangle^8}\,. \tag{5.1}$$

The amplitude $M_n(--+\cdots+)$ must be bose symmetric under exchange of helicity spinors for the two negative helicity particles and for any pair of positive helicity particles. However the superamplitude must have full S_n permutation symmetry, and so must the ratio $M_n(\ldots)/\langle 12\rangle^8$.

At the N^{K}MHV level, the amplitudes of the algebraic basis are now characterized by the $SU(n-4)$ semi-standard tableaux of a rectangular 8-by-K Young diagram. The SUSY- and R-invariant Grassmann polynomials Z_I multiplying each basis amplitude are order $8K$; they are constructed as in $\mathcal{N}=4$, but with twice as many η_{ia}'s. N^{K}MHV n-point superamplitudes must also have S_n permutation symmetry We now discuss the NMHV sector in more detail.

NMHV amplitudes in $\mathcal{N}=8$ supergravity

The identification of an algebraic basis in supergravity proceeds as in gauge theory and leads to a representation of NMHV superamplitudes analogous to (4.8) namely

$$\mathcal{M}_n^{\rm NMHV}=\sum_{1\le i\le j\le\cdots\le v\le n\text{-}4} c_{ijklpquv}\;X_{(ijklpquv)}\,, \tag{5.2}$$

with symmetrized versions of the Q^a- and $\widetilde{Q}_a$-invariant polynomial

$$X_{ijklpquv}\;=\;\delta^{(16)}(\widetilde{Q}_a)\;\frac{m_{i,n\text{-}3,n\text{-}2;1}\;m_{j,n\text{-}3,n\text{-}2;2}\;\cdots\;m_{v,n\text{-}3,n\text{-}2;8}}{[n-3,n-2]^8\langle n-1,n\rangle^8}\;. \tag{5.3}$$

As in $\mathcal{N}=4$ SYM, we can identify each coefficient $c_{ijklpquv}$ with an amplitude:

$$c_{ijklpquv}\;=\;M_n\big(\{i,j,k,l,p,q,u,v\}++--\big)\;\equiv\;\big\langle\cdots B_i^1\cdots B_j^2\cdots\cdots B_v^8\cdots++--\big\rangle\,. \tag{5.4}$$

The notation $\{i,j,k,l,p,q,u,v\}$ indicates that line i carries $SU(8)_R$ index 1, while line j carries $SU(8)_R$ index 2, etc. If indices are identical, say $i=j$, the line in question carries both $SU(8)_R$ indices 1 and 2.

In gravity, as opposed to gauge theory, there is no ordering of the external states. Therefore amplitudes with the same external particles and the same $SU(8)_R$ charges are all related by momentum relabeling. For example,

$$c_{22222222}\;=\;\langle +_1\,-_2\,+_3\,+_4\,-_5-_6\rangle=\langle -_2\,+_1\,+_3\,+_4\,-_5-_6\rangle=\big(c_{11111111}\;\text{ with }\;p_1\leftrightarrow p_2\big)\,. \tag{5.5}$$

Since there are a total of eight $SU(8)_R$ indices $1,2,\ldots,8$ distributed on these $n-4$ states, the number of functionally independent amplitudes cannot exceed the number of partitions of 8 into $n-4$ non-negative integers.

For example, for $n=6$ we have the partitions $[8,0]$, $[7,1]$, $[6,2]$, $[5,3]$ and $[4,4]$ corresponding to a reduced set of 5 functional basis amplitudes in the functional basis. The 6-point superamplitude is then

$$\begin{aligned}\mathcal{M}_6^{\rm NMHV}\;=\;&\Big\{\;\big\langle -+++--\big\rangle\,X_{11111111}\;+\;\big\langle\psi^-\psi^+ ++--\big\rangle\,X_{(11111112)}\\ &+\big\langle v^-v^+ ++--\big\rangle\,X_{(11111122)}\;+\;\big\langle\chi^-\chi^+ ++--\big\rangle\,X_{(11111222)}\\ &+\frac{1}{2}\big\langle\,\phi^{1234}\phi^{5678}++--\big\rangle\,X_{(11112222)}\Big\}\;+\;(1\leftrightarrow 2)\,.\end{aligned} \tag{5.6}$$

Particle types are indicated by $\psi^+ = B^1$, $\psi^- = B^{2345678}$ etc, in hopefully self-explanatory notation. The "$+ (1 \leftrightarrow 2)$" exchanges momentum labels 1 and 2 in the X-polynomials as well as in the basis amplitudes. The exchange does not introduce new basis functions, it only relabels momenta in the basis amplitudes written explicitly in (5.6).

For the n-point NMHV superamplitudes, the number of amplitudes in the functional basis is the number of partitions of the number 8 into $n-4$ bins:

$$\begin{array}{rcccccccc} n & = & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \geq 12 \\ \text{basis count} & = & 1 & 5 & 10 & 15 & 18 & 20 & 21 & 22. \end{array}$$

The entry in the second line is the number of n-point amplitudes one needs to compute in order to fully determine the n-point NMHV superamplitude. The saturation at $n = 12$ occurs because the longest partition of $n - 4 = 8$ is reached, namely the partition $[1, 1, 1, 1, 1, 1, 1, 1]$. This partition corresponds to a basis amplitude with 8 gravitinos, two positive-helicity gravitons and two negative-helicity gravitons. For $n > 12$, one only adds further positive-helicity gluons to each partition. This does not change the count of functional basis amplitudes.

Minimal functional basis

In the functional basis discussed above, we have considered the functional dependence between algebraic basis amplitudes of the form $\langle B_1^{\cdots} \cdots B_{n-4}^{\cdots} + + - - \rangle$. As in $\mathcal{N} = 4$ SYM, one can use a computer-based implementation of the superamplitude to check linear independence of a different set of algebraic basis amplitudes that are not all of this form. Although the superamplitude then generically takes a very complicated form in terms of these basis amplitudes, this approach is still convenient as the amplitudes in the new algebraic basis can exhibit a larger functional dependence. At the 6-point NMHV level, for example, we verified that the 9 algebraic basis amplitudes $\langle B^{\cdots} B^{\cdots} + + - - \rangle$ can be replaced by an algebraic basis that consists of the pure-graviton amplitude $\langle - - - + + + \rangle$, together with 8 inequivalent permutations of its external gravitons. Therefore, *at the 6-point NMHV level, the minimal functional basis consists of a single amplitude,* $\langle - - - + + + \rangle$. For $n > 6$, however, the dimension of the algebraic basis exceeds the number of pure-graviton amplitudes, which rules out the possibility of a pure-graviton functional basis.

Examples

Let us illustrate the solution to the Ward identities in a few explicit examples. Consider first the amplitude with 2 sets of 3 identical gravitinos, $\langle \psi^{1234567}\, \psi^8 \psi^8 \psi^8 \psi^{1234567} \psi^{1234567} \rangle$. Applying the corresponding Grassmann derivatives [6] to the superamplitude (5.6), we find

$$\begin{aligned} &\langle \psi^{1234567}\, \psi^8\, \psi^8\, \psi^8\, \psi^{1234567} \psi^{1234567} \rangle \\ &\qquad = \frac{1}{[34]\langle 56 \rangle} \Big\{ \langle 2|3+4|1] \, \big\langle - + + + - - \big\rangle + s_{234} \, \big\langle \psi^- \psi^+ + + - - \big\rangle \Big\} , \end{aligned} \tag{5.7}$$

where $s_{234} = -(p_2 + p_3 + p_4)^2$. This particular $\mathcal{N} = 8$ amplitude agrees with the 6-gravitino amplitude $\big\langle \psi^- \psi^+ \psi^+ \psi^+ \psi^- \psi^- \big\rangle$ in the truncation of the $\mathcal{N} = 8$ theory to $\mathcal{N} = 1$ supergravity. In fact the relation (5.7) is a special case of the "old" solution to the $\mathcal{N} = 1$ SUSY Ward identities [3, 6].

An example which does not reduce to $\mathcal{N} = 1$ supergravity is obtained by interchanging the $SU(8)_R$ indices 7 and 8 on states 1 and 2 in the 6-gravitino amplitude. The result is another 6-gravitino amplitude whose expression in terms of basis amplitudes is found to be

$$\begin{aligned} &\langle \psi^{1234568}\, \psi^7\, \psi^8\, \psi^8\, \psi^{1234567} \psi^{1234567} \rangle \\ &\qquad = -\frac{1}{[34]\langle 56 \rangle} \Big\{ s_{134} \, \big\langle \psi^- \psi^+ + + - - \big\rangle + \langle 1|3+4|2] \, \big\langle v^- \, v^+ \, + + - - \big\rangle \Big\} . \end{aligned} \tag{5.8}$$

This example could be interpreted as the solution to the SUSY Ward identities in $\mathcal{N} = 2$ supergravity.

Our final example contains two distinct scalars and four gravitinos:

$$\begin{aligned}&\langle\phi^{1238}\,\phi^{4568}\,\psi^{7}\,\psi^{8}\,\psi^{1234567}\psi^{1234567}\rangle \\ &\quad = \frac{\langle 2|1+4|3]}{[34]^2\langle 56\rangle}\Big\{\,[14]\,\big\langle\chi^-\chi^+ ++--\big\rangle + [24]\,\big\langle\phi^{1234}\phi^{5678}++--\big\rangle\Big\}\;-\;(1\leftrightarrow 2)\,.\end{aligned} \tag{5.9}$$

We have checked the solutions (5.7), (5.8), and (5.9) numerically at tree level using the MHV vertex expansion, which is valid [6] for the specific $\mathcal{N}=8$ amplitudes considered here. Of course the relations (5.7), (5.8), and (5.9) hold in general, at arbitrary loop order.

6 Application: superamplitude approach to counterterms

A theoretical development without application is like a bicycle without wheels. For this reason we now review the application [2] of the basis expansions of superamplitudes to study candidate counterterms for $\mathcal{N}=8$ supergravity. The $\mathcal{N}=8$ theory [19, 27] is the maximal supergravity theory in $D=4$ spacetime dimensions, and the idea was expressed quite early that it might have favorable ultraviolet properties. Recent support for this idea has come from the remarkable calculations of [28] based on the generalized unitarity method [29–31], which showed that 4-point amplitudes are UV finite in 3-loop and 4-loop order. It is interesting to ask whether this situation continues to higher number of external legs and higher loop order. If not, then we ask in which amplitudes and at which loop order might the first divergence occur?

In four dimensions, the coupling constant κ of perturbative quantum gravity theories has dimensions of length. Dimensional analysis then shows that the degree of divergence increases with the loop order L, but is independent of the number of external legs n. A logarithmic divergence at loop order L would require a local counterterm of dimension $\Delta = 2(L+1)$. Here, we define the dimension of an operator in a slightly non-standard manner as its "power-counting dimension", *i.e.* as the number of derivatives plus $\frac{1}{2}$ the number fermionic fields that it contains.[5] The counterterms of n-point graviton amplitudes must respect general coordinate invariance and thus take the form $\int d^4x\sqrt{-g}D^{2k}R^n$. This is a schematic form in which the index contractions and distribution of covariant derivatives on the curvature tensor are not specified. A counterterm of this form has dimension $2(k+n)$, and could describe a UV divergence at loop level $L=n+k-1$.

In $\mathcal{N}=8$ supergravity, the lower spin fields are unified with gravity, so counterterms must contain a supersymmetric completion involving those fields, which we denote very schematically by $\int d^4x\sqrt{-g}(D^{2k}R^n+\dots)$. Terms in the linearized SUSY completion contribute to n-graviton and other n-particle processes, while non-linear terms contribute to various processes with more than n external particles. Little is known about the component form of the supersymmetrization of these operators, nor is it needed in the approach [2] we now review.

The approach of [2] focuses on matrix elements of candidate counterterm operators. If an operator $D^{2k}R^n$ has at least a linearized supersymmetric completion then the n-particle matrix elements it generates must obey the SUSY Ward identities discussed in Secs. 2. Furthermore, and crucially, the leading n-point matrix elements of any counterterm must be local; this means that they must not have any poles in their dependence on momenta p_i; gauge invariance then implies that they are polynomials in the spinor brackets $\langle i\,j\rangle$, $[k\,l]$. The matrix elements must also be $SU(8)$ invariant.

In many cases, the requirements of locality, $SU(8)$ symmetry and SUSY are incompatible. This proves that no supersymmetrization exists, and the operator cannot occur in the perturbation series of $\mathcal{N}=8$ supergravity.

In other cases, the Ward identities and locality are compatible. The operator is then linearly $\mathcal{N}=8$ supersymmetrizable and $SU(8)$ symmetric. It is accepted as a potential candidate counterterm pending a

[5] External line factors carry dimension 1/2 for fermions, but are dimensionless for bosons.

study of the questions of nonlinear SUSY and the low energy theorems of the $E_{7(7)}$ symmetry. Low energy theorems were considered in [20,21] (see also [22]), which we review in Sec. 6.5. For operators compatible with locality, linearized SUSY and R-symmetry, our method constructs their general matrix elements explicitly. In particular, this allows us to determine the multiplicity[6] of the operator.

Two simple features facilitate the matrix element approach.

- There are constraints on matrix elements from dimensional analysis and particle helicities, and these become particularly powerful for local matrix elements. Spinor brackets $\langle i\, j\rangle$, $[k\, l]$ have mass dimension 1, and all terms in a possible supersymmetrization $D^{2k}R^n + \ldots$ have dimension $2(k+n)$. Thus any matrix element of this operator must be a sum of monomials which each contain $2(k+n)$ angle and square brackets. The particle helicity constraint arises from the little group scaling property [9]

$$m_n(\ldots, t_i|i\rangle, t_i^{-1}|i], \ldots) = t_i^{-2h_i}\, m_n(\ldots, |i\rangle, |i], \ldots)\,, \tag{6.1}$$

  which holds for each particle i. This determines the difference $a_i - s_i = -2h_i$ between the number of angle spinor $|i\rangle$ and square spinor $|i]$ factors in each term of m_n.

- The index contractions of an operator $D^{2k}R^n + \ldots$ can be organized according to the N^{K}MHV classification of its n-point matrix elements. This is possible because on-shell the Riemann tensor $R_{\mu\nu\rho\sigma}$ splits into a totally symmetric 4th rank spinor $R_{\alpha\beta\gamma\delta}$ and its conjugate $\bar{R}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}$, which communicate to gravitons of opposite helicity. Terms in $D^{2k}R^n$ with 2 factors of R and $(n-2)$ factors of $\bar{R}$ contribute to the MHV graviton matrix element while $R^3\,\bar{R}^{n-3}$ is the NMHV part and so on. This separation persists in the SUSY completion, because the SUSY Ward identities relate amplitudes within a given N^{K}MHV sector.

The import of this is that we can use the basis expansions of N^{K}MHV superamplitudes discussed in Sec. 4. The first step is to determine the basis amplitudes: each basis amplitude is constructed as the most general polynomial in angle and square brackets consistent with helicity-scaling, Bose-symmetry, and dimensional requirements. Two polynomials are identical if they are related by momentum conservation and Schouten identities. A systematic way to construct a complete set of polynomials subject to these requirements is to consider the polynomials as elements of a quotient ring $\mathbb{P}\big[\langle ij\rangle, [ij]\big]/\mathbb{I}$, where $\mathbb{P}\big[\langle ij\rangle, [ij]\big]$ is the ring of all polynomials in angle- and square brackets and $\mathbb{I}$ is the ideal generated by the polynomial conditions for momentum conservation and Schouten identities. Gröbner basis techniques can then be used to find a basis for the vector space of all polynomials, of fixed degree Δ and given little-group scaling weights (6.1), in the quotient ring. Linear combination of these basis elements then constitute the most general expression for a given basis amplitude.

The above construction ensures that the basis amplitudes are local. The next step is to demand that all amplitudes produced by the corresponding superamplitude are also local; this requires that the factors in the denominator of (5.1) and (5.2)-(5.3) cancel in all amplitudes. As we shall see, this is a nontrivial constraint.

If the poles in non-basis amplitudes do not cancel for any admissible choice of basis polynomials, then the operator under study does not have an acceptable supersymmetrization and is ruled out. If, on the other hand, the method determines one or more sets of basis amplitudes that do lead to a local and permutation symmetric superamplitude, then each set yields an independent linear supersymmetrization of the operator. In practice, it is difficult to explicitly verify locality of all non-basis amplitudes. However, it was shown in [2] that *any superamplitude with local basis matrix elements and full permutation symmetry produces local matrix elements for any process.* This reduces the difficult process of checking locality of non-basis amplitudes to the much more practical check of permutation symmetry of the full superamplitude.

The matrix element method cannot predict whether an accepted candidate counterterm corresponds to an actual divergence in the perturbative S-matrix of $\mathcal{N}=8$ supergravity. At loop levels $L = 3, 4$, the results

[6]This multiplicity is the number of independent linearized supersymmetrizations of an operator, including distinct index contractions. Operators are considered dependent if they are related by the linearized equations of motion; in that case their leading matrix elements are identical.

of [28] show evidence for cancelations beyond those associated with $\mathcal{N} = 8$ SUSY and this situation may persist. As we discuss in Sec. 6.5, the additional constraints from $E_{7(7)}$ explain and predict the absence of any UV divergences below 7-loop order.

6.1 Candidate MHV counterterms

Ruling out R^n for $n \geq 5$: To see how the method of [2] works, let us ask whether the operator R^n has a linearized supersymmetrization at the MHV level. Its n-point matrix element $m_n(--+\cdots+)$ must be a polynomial with the spinor powers $|1\rangle^4$, $|2\rangle^4$ and $|i]^4$, $i = 3, \ldots, n$, which are the minimal powers consistent with the helicity weights $-2h_i = 4, 4, -4, -4, \ldots$. With these minimal powers the total dimension $2n$ is saturated, so the basis matrix element in (5.1) must take the form

$$m_n(--+\cdots+) \;=\; \langle 12\rangle^4 f_n(|3], |4], \ldots |n])\,. \tag{6.2}$$

The function f_n is an order $2n-4$ polynomial in square brackets, and depends only on square spinors $|i]$ for positive helicity gravitons, *i.e.* $i \geq 3$. It must also be bose symmetric, but we will not need this information. The MHV superamplitude is obtained by inserting this matrix element in (5.1), which then reads

$$\mathcal{C}_n^{\rm MHV} = \delta^{(16)}\Big(\sum_i |i\rangle \eta_{ai}\Big)\frac{f_n(|3], |4], \ldots |n])}{\langle 12\rangle^4}\,. \tag{6.3}$$

The basis matrix element $m_n(--+\cdots+)$ is local, but we must test whether all other matrix elements obtained by differentiation of (5.1) are also local. We will examine the n-graviton matrix element with the negative helicity gravitons on lines 3 and 4. To 'project out' a negative helicity graviton on line i, one applies the 8th order Grassmann derivative

$$\Pi_i \equiv \prod_{a=1}^{8} \frac{\partial}{\partial \eta_{ia}}\,. \tag{6.4}$$

We thus find the permuted matrix element

$$m_n(++--+\cdots+) = \Pi_3\Pi_4\mathcal{C}_n^{\rm MHV} = \frac{\langle 34\rangle^8}{\langle 12\rangle^4} \times f_n(|3], |4], \ldots |n])\,. \tag{6.5}$$

We now show that the non-locality in $\langle 12\rangle$ does not cancel for $n \geq 5$. To do this we introduce a complex variable z and evaluate (6.5) for the shifted spinors

$$|i\rangle \to |\hat{i}\rangle \;=\; |i\rangle + z c_i |\xi\rangle\,, \qquad\qquad i = 1, 2, 5\,, \qquad \sum_i c_i |i] = 0\,, \tag{6.6}$$

with all other angle spinors and all square spinors unshifted.[7] The quantity $|\xi\rangle$ is an arbitrary reference spinor. The shift affects only the denominator in (6.5), so the right-hand side has an uncanceled 4th order pole in z. Therefore the amplitude $m_n(++--+\cdots+)$ is non-local, even with the input of the most general basis polynomial in (6.2) which satisfies the scaling constraints. R^n MHV counterterms for $n \geq 5$ are therefore ruled out!

The condition $\sum_i c_i|i] = 0$ in (6.6) is needed so that the shifted spinors satisfy momentum conservation. There are non-vanishing choices of the constants c_i only if at least 3 lines are shifted. Therefore the shift does not work when there are only 4 external particles, and cannot be used to rule out R^4. We will discuss the R^4 counterterm shortly.

[7] An "anti-holomorphic" shift similar to (6.6) has been used in [32–35] to prove the CSW recursion relations [36] (see [37] for a review in this issue).

Ruling out D^2R^n, D^4R^n, and D^6R^n for $n \geq 5$: Next consider potential MHV counterterms $D^{2k}R^n$. Their overall dimension is $2(k+n)$. To satisfy the scaling constraints we must construct basis polynomials with the minimal powers used in (6.2) plus $2k$ additional matched pairs $|i\rangle$, $|i]$ for any choice of up to $2k$ lines. The basis amplitude thus contains $4+k$ angle brackets. When shifted, it becomes a polynomial in z of order no greater than $4+k$. For $k<4$, this is not enough to cancel the 8th order pole from the factor $1/\langle 12\rangle^8$ in the superamplitude. The shift argument thus rules out the MHV counterterms D^2R^n, D^4R^n, D^6R^n for $n \geq 5$.

Ruling in D^8R^n The analysis has ruled out MHV operators $D^{2k}R^n$ for $k<4$. It may not be immediately clear whether the bound $k<4$ is a limitation of method or fact. We now settle that question in favor of fact by exhibiting that the bound is saturated: we do this by explicit construction of an MHV superamplitude for the counterterm D^8R^n. The 8 angle brackets required by scaling weights allow the factor $\langle 12\rangle^8$ which directly cancels the singular factors in (3.3), leaving the manifestly local superamplitude [2]

$$\mathcal{C}_n^{\rm MHV} = \delta^{(16)}\Big(\sum_i |i\rangle\eta_{ai}\Big)\Big[c_1\big([12]^2[23]^2\cdots[n1]^2 + \text{perms}\big) + c_2\big(([12][34]\cdots[n-1,n])^4 + \text{perms}\big)\Big]. \qquad (6.7)$$

The second term only exists if n is even, but the first is valid for all n. For $n=4$ these two terms are linearly dependent through the Schouten identity. For $n=6$ the two terms are independent, and there are no other independent contributions.[8]

$D^{2k}R^4$ are allowed for all $k \neq 1$: Consider a possible R^4 counterterm. There is only one local expression for its basis amplitude with the correct dimension and weights, and it is

$$m_4(1^-,2^-,3^+,4^+) = \langle 12\rangle^4[34]^4\,. \qquad (6.8)$$

This form also appears, for example, in [38]. The better known [39] form $m_4(1^-,2^-,3^+,4^+) = s\,t\,u\,M_4^{\rm tree}$ is equivalent to (6.8) using momentum conservation

$$\langle yx\rangle[xz] \;=\; -\sum_{i\neq x}\langle yi\rangle[iz]\,. \qquad (6.9)$$

The resulting superamplitude is

$$\mathcal{C}_4^{\rm MHV} = \delta^{(16)}\Big(\sum_i |i\rangle\eta_{ai}\Big)\frac{[3\,4]^4}{\langle 1\,2\rangle^4}\,. \qquad (6.10)$$

Using (6.9) one can show that *all* matrix elements obtained from it are local. Indeed, this means that $\mathcal{C}_4^{\rm MHV}$ is local.

For all $k \geq 0$, the allowed polynomial form of the basis matrix element of $D^{2k}R^4$ is $m_4(--++) = g_{D^{2k}R^4}\langle 1\,2\rangle^4[3\,4]^4$, so the superamplitude is

$$\mathcal{C}_4^{\rm MHV} = \delta^{(16)}\Big(\sum_i |i\rangle\eta_{ai}\Big) g_{D^{2k}R^4}(s,t,u)\frac{[3\,4]^4}{\langle 1\,2\rangle^4}\,. \qquad (6.11)$$

g is an order k symmetric polynomial in s,t,u, for example $g_{R^4}=1$, $g_{D^2R^4} = s+t+u = 0$, $g_{D^4R^4} = s^2+t^2+u^2$, $g_{D^6R^4} = s^3+t^3+u^3$ [15,2]. Since $g_{D^2R^4}=0$, the 4-loop counterterm D^2R^4 is ruled out. For all other k, $D^{2k}R^4$ is allowed by $\mathcal{N}=8$ SUSY and $SU(8)$-symmetry.

[8]Other structures may become available when n is sufficiently large.

6.2 Candidate NMHV counterterms

The extension of the matrix element method to the NMHV level is based on the superamplitudes (5.2), which are Grassmann polynomials of order 24. For each basis amplitude, we input the most general polynomial in spinor brackets consistent with helicity-scaling, Bose-symmetry, and dimensional requirements. The superamplitudes guarantee that individual matrix elements, obtained by Grassmann differentiation, are related by the appropriate SUSY Ward identities. Since the Ward identities are under control, we can proceed to study to test if the non-basis matrix elements produced from the superamplitudes are also local.

In [2], NHMV level R^n and D^2R^n counterterms were ruled out by a shift argument similar to that used at the MHV level in Sec. 6.1. Supersymmetric NMHV counterterms D^4R^n, on the other hand, can be constructed. The NMHV bound is weaker than in the MHV sector where independent D^4R^n and D^6R^n counterterms were also ruled out. In this review we discuss only the case $n = 6$.

6.2.1 No R^6 and D^2R^6 NMHV counterterms

The 6-point superamplitude was discussed in Sec. 5. It is convenient here to work with a more schematic form of (5.6). There are 9 terms in the symmetrized sum, so we write

$$\mathcal{C}_6^{\text{NMHV}} = \sum_{j=0}^{8} m^{(j)} X_{(j)} \,. \tag{6.12}$$

The $m^{(j)}$ indicate the basis amplitudes of (5.6) in sequential order, e.g. $m^{(0)} = m_6(-+++--)$, $m^{(1)} = m_6(\psi^-\psi^+ ++--), \ldots,\ m^{(8)} = m_6(+-++--)$. Note that $m^{(5)}, \ldots, m^{(8)}$ are in the $1 \leftrightarrow 2$ exchanged part of (5.6). The $X_{(j)}$ are the corresponding symmetrized 24th order Grassmann polynomials. They are the symmetrizations of the polynomials in (5.3), but with $n = 6$. They contain the singular factor $1/([34]\langle 56\rangle)^8$ which will be important shortly.

The basis matrix elements of the superamplitude describing a possible supersymmetrization of the operator $D^{2k}R^6$ must be local expressions of mass dimension $2(k+6)$, so the total number of angle and square spinors is $\sum_i (a_i + s_i) = 4(k+6)$. Helicity weights determine the difference $\sum_i (a_i - s_i) = -2\sum_i h_i = 0$ for any basis element of (5.6). Thus each basis matrix element is a product of $\sum_i a_i = 6 + k$ angle and $\sum_i s_i = 6 + k$ square brackets.

Using a suitable complex shift, we now show that when $k = 0, 1$ the potential pole factor $1/\langle 56\rangle^8$ *cannot cancel* in the permuted 6-graviton matrix element $m_6(--+++-)$ obtained from the superamplitude (5.6). We project out $m_6(--+++-)$ from the superamplitude by applying the Grassmann derivatives defined in (6.4) for negative helicity graviton lines, obtaining

$$m_6(--+++-) \;=\; \Pi_1\Pi_2\Pi_6\,\mathcal{C}_6^{\text{NMHV}} \;=\; \frac{1}{\langle 56\rangle^8}\sum_{j=0}^{8}\binom{8}{j}\langle 26\rangle^{8-j}\langle 16\rangle^{j}\, m^{(j)} \,. \tag{6.13}$$

The eight angle brackets in the numerator come from derivatives of the Grassmann $\delta^{(16)}$ in the X-polynomials (5.3). The factor $1/[34]^8$ in (5.6) cancels in (6.13) because differentiation of the $m_{ijk,a}$ polynomials produces compensating factors in all terms. The binomial coefficients appear because of the symmetrization of labels in the X-polynomials.

Consider now the effect of the holomorphic 3-line shift of angle spinors as in (6.6), but acting on the spinors $|3\rangle$, $|4\rangle$, and $|5\rangle$. Spinor brackets $\langle q\, q'\rangle$ are invariant under this shift unless they involve at least one spinor from the set $|3\rangle$, $|4\rangle$, $|5\rangle$. Shifted brackets are linear in z. The denominator of (6.13) has an 8th order pole in z, but the brackets $\langle 26\rangle$ and $\langle 16\rangle$ in the numerator do not shift. The only potential z dependence in the numerator comes from the $6+k$ spinor brackets in the basis matrix elements $m^{(j)}$. The poles cannot cancel in any linear combination of basis elements if they contain fewer than 8 shifted angle brackets. Thus the counterterm is ruled out if $6+k<8$; hence for $k=0,\,1$.

6.3 7-loops: Explicit NMHV superamplitudes for D^4R^6

In Sec. 6.2.1, we used a shift argument to rule out NMHV counterterms $D^{2k}R^n$ with $k\leq 1$. The result $k\leq 1$ is an actual bound for NMHV operators, not just a limitation of the method. Indeed, two independent supersymmetric 7-loop NMHV operators D^4R^6 were constructed in [2, 21], and it was also shown that precisely two such operators exist.

In fact it is quite simple to write down a new type of representation of the two superamplitudes. Since their matrix elements contain products of 8 angle and 8 square brackets, one can conjecture that they can be written in the form

$$\mathcal{C}_6 = \delta^{(16)}(\widetilde{Q})\delta^{(16)}(Q)P_{24}(\eta_{ia})\,, \tag{6.14}$$

where $P_{24}(\eta_{ia})$ is a 24th order $SU(8)$ invariant polynomial in the η's and $\delta^{(16)}(Q)$ is the Grassmann differential operator

$$\delta^{(16)}(Q) = \prod_{a=1}^{8}\sum_{i<j}[ij]\,\frac{\partial^2}{\partial\eta_{ia}\partial\eta_{ja}}\,. \tag{6.15}$$

It is not hard to write down two candidate polynomials for the explicit superamplitudes. Their independence was verified numerically. For example, one can choose

$$\begin{aligned}\mathcal{C}_{D^4R^6} &= \delta^{(16)}(\widetilde{Q})\delta^{(16)}(Q)\big[(\varphi_1,\varphi_2)(\varphi_3,\varphi_4)\,(\varphi_5,\varphi_6)+\text{perms}\big]\,,\\ \mathcal{C}_{D^4R^{6\prime}} &= \delta^{(16)}(\widetilde{Q})\delta^{(16)}(Q)\big[(\varphi_1,\,\varphi_2,\,\varphi_3,\,\varphi_4,\,\varphi_5,\,\varphi_6)+\text{perms}\big]\,.\end{aligned} \tag{6.16}$$

The sums in (6.16) run over all inequivalent permutations of the external state labels i of the φ_i, and the φ-products are defined as

$$\begin{aligned}(\varphi_i,\varphi_j) &\equiv \epsilon^{a_1a_2a_3a_4b_1b_2b_3b_4}\prod_{t=1}^{4}\eta_{ia_t}\eta_{jb_t}\,,\\ (\varphi_i,\varphi_j,\varphi_k,\varphi_l,\varphi_m,\varphi_n) &\equiv \epsilon^{a_1a_2b_1b_2b_3b_4c_1c_2}\epsilon^{c_3c_4d_1d_2d_3d_4e_1e_2}\epsilon^{e_3e_4f_1f_2f_3f_4a_3a_4}\prod_{t=1}^{4}\eta_{ia_t}\eta_{jb_t}\eta_{kc_t}\eta_{ld_t}\eta_{me_t}\eta_{nf_t}\,.\end{aligned} \tag{6.17}$$

Of course, the choice of contractions is not unique, and it is only through the independently established multiplicity count in [21] that we know the two contractions given in (6.16) to be sufficient.

6.4 Summary: Potential counterterms

We have excluded MHV and NMHV operators $D^{2k}R^n$, $n>4$, with $k<4$ and $k<2$, respectively. Since divergences in L-loop amplitudes correspond to counterterms of dimension $2L+2$, this translates to the bounds

$$\text{no MHV:}\quad L<n+3\,,\qquad\quad \text{no NMHV:}\quad L<n+1\,,\qquad\quad (n>4) \tag{6.18}$$

for the *exclusion* of $\mathcal{N}=8$ SUSY and $SU(8)$-invariant operators $D^{2k}R^n$ in each class. The explicit construction of the set of MHV superamplitudes (6.7) for D^8R^n and of the 7-loop NMHV superamplitudes (6.16) for D^4R^6 show that the bounds are optimal.

The bounds (6.18) also apply [2] to the existence of *non-gravitational* counterterms such as $D^{2k}\phi^m+\ldots$ whose supersymmetrizations do not include any purely gravitational terms. Furthermore, it was conjectured

L											
3	~~R^4~~ ($E_{7(7)}$)										
4	~~D^2R^4~~	~~R^5~~									
5	~~D^4R^4~~ ($E_{7(7)}$)	~~D^2R^5~~	~~R^6~~								
6	~~D^6R^4~~ ($E_{7(7)}$)	~~D^4R^5~~	~~D^2R^6~~	~~R^7~~							
7	D^8R^4	~~D^6R^5~~	~~D^4R^6~~ ($E_{7(7)}$)	~~D^2R^7~~	~~R^8~~ ($E_{7(7)}$)	~~$\varphi^2D^2R^7$~~	~~φ^2R^8~~ ($E_{7(7)}$)	~~$\varphi^4D^2R^7$~~	~~φ^4R^8~~ ($E_{7(7)}$)	~~$\varphi^6D^2R^7$~~	
8	$D^{10}R^4$	D^8R^5	D^6R^6	D^4R^7	D^2R^8	R^9	$\varphi^2D^2R^8$	φ^2R^9	$\varphi^4D^2R^8$	φ^4R^9	$\longrightarrow$ No N^4MHV
9	$D^{12}R^4$	$D^{10}R^5$	D^8R^6	D^6R^7	D^4R^8	D^2R^9	R^{10}	$\varphi^2D^2R^9$	φ^2R^{10}	$\varphi^4D^2R^9$	
10	$D^{14}R^4$	$D^{12}R^5$	$D^{10}R^6$	D^8R^7	D^6R^8	D^4R^9	D^2R^{10}	R^{11}	$\varphi^2D^2R^{10}$	φ^2R^{11}	
					$\longrightarrow$ No MHV		$\longrightarrow$ No NMHV		$\longrightarrow$ No N^2MHV		$\longrightarrow$ No N^3MHV

Table 1: Potential counterterms in $\mathcal{N} = 8$ supergravity. The crossed-out operators are excluded. At loop order $L < 7$, only three operators are allowed by SUSY and R-symmetry, namely R^4, D^4R^4 and D^6R^4. However, these are not compatible with nonlinear $E_{7(7)}$ symmetry and this means that there are no available counterterm operators below 7-loop order. For $L \geq 7$, both the matrix element method of this review and an analysis of the representations of the superalgebra $SU(2,2|4)$ were used. The second method is described in [21], and a more detailed version of the chart appears there.

in [2] and proven in [13], that the bound

$$\text{no N}^K\text{MHV:} \quad L < n + 3 - 2K\,, \qquad (n > 4) \tag{6.19}$$

holds for *all* N^KMHV operators of dimension $2L + 2$. Operators below this bound are not compatible with $\mathcal{N} = 8$ SUSY and $SU(8)$ R-symmetry. Charts of available operators and their multiplicities were given in [2, 21], and a concise chart that summarizes the available counterterms is given in table 1.

The 4-point operators R^4, D^4R^4 and D^6R^4, with the rather simple 4-point superamplitudes discussed in Sec. 6.1, are the only operators below the 7-loop level that are consistent with SUSY and $SU(8)_R$. However, we show in the next section that these operators are ruled out as possible counterterms by the nonlinear $E_{7(7)}$ symmetry. Thus the combined SUSY, R- and $E_{7(7)}$-symmetries leave no candidate counterterms for $L < 7$.

6.5 $E_{7(7)}$ constraints on counterterms

Since the operators R^4, D^4R^4, and D^6R^4 are compatible with SUSY and $SU(8)_R$ symmetry, more information is needed to rule out these operators as potential counterterms of $\mathcal{N} = 8$ supersymmetry. We show now, following the analysis of [2] and its extension in [21], that R^4, D^4R^4, and D^6R^4 are incompatible with continuous $E_{7(7)}$ symmetry. Recall that $E_{7(7)}$ symmetry is spontaneously broken to its maximally compact subgroup $SU(8)_R$; the 70 scalars in the spectrum of $\mathcal{N} = 8$ supergravity are the Goldstone bosons associated with this symmetry breaking. It has been argued [40] that the $E_{7(7)}$ symmetry is also a symmetry at loop level, and from this it follows from the "soft-pion theorem" that the matrix elements of an admissible counterterm must vanish when the momentum of any external scalar is taken to zero.[9] Our matrix-element approach to counterterms is thus ideally suited to address the question of $E_{7(7)}$-compatibility.

[9] Non-vanishing SSL's from external line insertions occur in pion physics, but not with the cubic vertices of $\mathcal{N} = 8$ [6, 42, 43].

The leading 4-point matrix elements of R^4, D^4R^4, and D^6R^4 take the form

$$\big\langle B_1^{\cdots} B_2^{\cdots} B_3^{\cdots} B_4^{\cdots} \big\rangle_{D^{2k}R^4} \;=\; g(s,t,u)\, \big\langle B_1^{\cdots} B_2^{\cdots} B_3^{\cdots} B_4^{\cdots} \big\rangle_{\rm SG}\,, \tag{6.20}$$

where $\langle \cdots \rangle_{\rm SG}$ is the tree-level $\mathcal{N}=8$ supergravity amplitude with the same choice of external states, and the function $g(s,t,u)$ is given by

$$g_{R^4} = 1\,, \qquad\qquad g_{D^4R^4} = \; s^2+t^2+u^2\,, \qquad\qquad g_{D^6R^4} = s^3+t^3+u^3\,. \tag{6.21}$$

Tree-level supergravity amplitudes have vanishing single-soft scalar limits (SSL's), so the SSL's of the 4-point matrix elements also vanish for all 4-point operators:

$$\lim_{p_\varphi\to 0} \big\langle \varphi \cdots \big\rangle_{D^{2k}R^4} \;=\; \lim_{p_\varphi \to 0} g(s,t,u)\, \big\langle \varphi \cdots \big\rangle_{\rm SG} \;=\; 0\,. \tag{6.22}$$

Thus we need to consider higher-point matrix elements of $D^{2k}R^4$ to rule out these operators.

Specifically, we study the soft scalar limit of the 6-point NMHV matrix elements $\langle ++--\varphi\bar\varphi\rangle_{D^{2k}R^4}$. The external states are two pairs of opposite helicity gravitons and two conjugate scalars. These matrix elements contain local terms from nth order field monomials in the nonlinear SUSY completion of $D^{2k}R^4$ as well as non-local pole diagrams in which one or more lines of the operator are off-shell and communicate to tree vertices from the classical Lagrangian. It is practically impossible to calculate these matrix elements with either Feynman rules (because the non-linear supersymmetrizations of $D^{2k}R^4$ are unknown) or recursion relations (because no valid ones are known). Instead we use the α'-expansion of the closed string tree amplitude to obtain the desired matrix elements.

At tree level, the closed string effective action takes the form

$$S_{\rm eff} \;=\; S_{\rm SG} - 2\alpha'^3\zeta(3)e^{-6\phi}R^4 - \zeta(5)\alpha'^5 e^{-10\phi}D^4R^4 + \tfrac{2}{3}\alpha'^6\zeta(3)^2 e^{-12\phi}D^6R^4 + \ldots\,. \tag{6.23}$$

Couplings of the dilaton ϕ break the $SU(8)$ symmetry of the supergravity theory to $SU(4)\times SU(4)$ when $\alpha'>0$, so matrix elements constructed from $S_{\rm eff}$ do not directly correspond to the desired $SU(8)$-invariant operators. As explained in [20], an $SU(8)$-*averaging procedure* can be used to extract the $SU(8)$ singlet contribution from the string matrix elements. Specifically, the $SU(8)$ average of the $\langle ++--\varphi\,\bar\varphi\rangle_{e^{-(2k+6)\phi}D^{2k}R^4}$ matrix elements from string theory is

$$\langle ++--\varphi\bar\varphi\rangle_{\rm avg} = \tfrac{1}{35}\langle ++--\varphi^{1234}\varphi^{5678}\rangle - \tfrac{16}{35}\langle ++--\varphi^{123|5}\varphi^{4|678}\rangle + \tfrac{18}{35}\langle ++--\varphi^{12|56}\varphi^{34|78}\rangle\,. \tag{6.24}$$

The 3 terms on the right side correspond to the 3 inequivalent ways to construct scalars from particles of the $\mathcal{N}=4$ gauge theory, namely from gluons, gluinos, and $\mathcal{N}=4$ scalars. There are 35 distinct embeddings of $SU(4)\times SU(4)$ in $SU(8)$. Averaging is sufficient to give the matrix elements of the $\mathcal{N}=8$ field theory operator R^4 and D^4R^4. For D^6R^4, a further correction is necessary and is discussed below in Sec. 6.5.3.

6.5.1 From open strings to closed strings

All closed string tree amplitudes in the work [20, 21] are obtained via KLT [41] from the open string tree amplitudes of [44]. The KLT relations express closed string amplitudes M_n as products of 'left' and 'right' sector open string amplitudes $A_n^{(L)}$ and $A_n^{(R)}$; schematically

$$M_n = \sum_{\mathcal{P}} f(s_{ij})\, A_n^{(L)}\, A_n^{(R)}\,. \tag{6.25}$$

The sum is over permutations of different orderings of the external states of the open string amplitudes. The functions $f(s_{ij})$ involve a product of $n-3$ factors $\sin(\alpha'\pi s_{ij})$, where s_{ij} are Mandelstam variables. The decomposition of $\mathcal{N}=8$ states into products of two $\mathcal{N}=4$ states (L- and R-movers) is described in detail in [6]. To obtain the 6-point closed string amplitude $\langle ++--\varphi\bar{\varphi}\rangle$ for the three independent pairs of conjugate scalars φ, $\bar{\varphi}$, the open-string amplitudes presented in [44] are not sufficient. Instead, SUSY Ward identities are needed to express the desired open string amplitudes in terms of the ones given[10] in [44]; but this is precisely the problem that we solved in Sec. 4! We can use the superamplitude (4.9) to express the basis amplitudes in terms of the open-string tree amplitudes amplitudes of [44]. Then we project out any other desired process from the superamplitude (4.9). Via KLT, we can then obtain any 6-point NMHV closed-string tree amplitude. The resulting closed string amplitudes confirm the structure and coefficients of (6.23).

6.5.2 3- and 5-loop counterterms R^4 and D^4R^4

At order α'^3 and α'^5, the $SU(8)$-average (6.24) of the string theory amplitudes directly gives the matrix elements of the unique $SU(8)$-invariant supersymmetrization of R^4 and D^4R^4, respectively. The result is a complicated non-local expression, but its single-soft scalar limit is very simple and local, viz.

$$\lim_{p_6\to 0}\langle ++--\varphi\bar{\varphi}\rangle_{R^4} = -\tfrac{6}{5}[12]^4\langle 34\rangle^4\,, \qquad \lim_{p_6\to 0}\langle ++--\varphi\bar{\varphi}\rangle_{D^4R^4} = -\tfrac{6}{7}[12]^4\langle 34\rangle^4\sum_{i<j}s_{ij}^2\,. \tag{6.26}$$

Since these SSL's are non-vanishing, the operators R^4 and D^4R^4 are incompatible with continuous $E_{7(7)}$ symmetry.

6.5.3 6-loop counterterm D^6R^4

The single-soft scalar limit of the $SU(8)$-singlet part of the closed string matrix element at order α'^6, obtained by the $SU(8)$-averaging (6.24), is

$$\lim_{p_6\to 0}\langle ++--\varphi\bar{\varphi}\rangle_{(e^{-12\phi}D^6R^4)_{\rm avg}} = -\tfrac{33}{35}[12]^4\langle 34\rangle^4\sum_{i<j}s_{ij}^3\,. \tag{6.27}$$

At order α'^6, it is important to realize that the 6-point NMHV closed string amplitudes receive contributions not only from $e^{-12\phi}D^6R^4$, but also from pole diagrams with two 4-point vertices of $e^{-6\phi}R^4$. Since no dimension 8 operator (R^4 nor $e^{-6\phi}R^4$) is present in $\mathcal{N}=8$ supergravity, its contributions must be removed from the string tree matrix element in order to extract the matrix elements of the supergravity operator D^6R^4. The removal process must be supersymmetric.

We first compute the R^4-R^4 pole contributions to the 6-graviton NMHV matrix element $\langle ---+++\rangle$ as follows. This amplitude has dimension 14. Factorization at the pole determines the simple form

$$\langle 12\rangle^4[45]^4\langle 3|P_{126}|6]^4/P_{126}^2 + 8 \text{ permutations}\,, \tag{6.28}$$

up to a local polynomial. The 9 terms correspond to the 9 distinct non-vanishing 3-particle pole diagrams. The result (6.28) is then checked by computation of the Feynman diagrams from the R^4 vertex [45]. As the non-linear supersymmetrization of R^4 may contribute additional local terms, we also consider adding the most general gauge-invariant and bose-symmetric polynomial of dimension 14 that can contribute to $\langle ---+++\rangle$, namely

$$\big(\langle 12\rangle\langle 23\rangle\langle 31\rangle[45][56][64]\big)^2 P_{123}^2\,. \tag{6.29}$$

To incorporate SUSY, we recall from the discussion of the "minimal functional basis" in Sec. 5, that there is an algebraic basis consisting of $\langle ---+++\rangle$ and 8 distinct permutations of the states. In this basis we write a superamplitude ansatz as the sum of the pole amplitude (6.28) plus a multiple of (6.29). We then impose full S_6 permutation symmetry on the ansatz. This fixes the coefficient of the polynomial (6.29) to vanish and determines the SUSY completion of the desired pole diagram uniquely.

[10] In [22], Ward identities were also used for this purpose.

Finally we project out the scalar-graviton matrix element from this superamplitude and subtract its (properly normalized) soft scalar limit from (6.27) to obtain

$$\lim_{p_6\to 0}\langle ++--\varphi\bar{\varphi}\rangle_{D^6R^4} = -\tfrac{60}{35}[12]^4\langle 34\rangle^4 \sum_{i<j} s_{ij}^3\,. \tag{6.30}$$

This is the single-soft scalar limit of the unique independent D^6R^4 operator in $\mathcal{N}=8$ supergravity. Since the limit does not vanish, the operator D^6R^4 is incompatible with continuous $E_{7(7)}$ symmetry.

As explained in Sec. 6.4, R^4, D^4R^4 and D^6R^4 are the only local supersymmetric and $SU(8)$-symmetric operators for loop levels $L\leq 6$ [2, 12, 13]. Hence $\mathcal{N}=8$ supergravity has no potential counterterms that satisfy the continuous $E_{7(7)}$ symmetry for $L\leq 6$. We stress that string theory is used as a tool to extract $SU(8)$-invariant matrix elements that must agree with the matrix elements of the $\mathcal{N}=8$ supergravity operators R^4, D^4R^4 and D^6R^4 because each of these operators is unique. No remnant of string-specific dynamics remains in the final results.

It is instructive to check whether the scalar matrix elements of the NMHV operators D^4R^6 have non-vanishing single-soft scalar limits. Scalar matrix elements can be projected from the two superamplitudes (6.16), and direct computation shows that SSL's do not vanish for any linear combination of the two. Thus these potential 7-loop, $n=6$ NMHV counterterms are also incompatible with the continuous $E_{7(7)}$ symmetry. The same conclusion holds for all other independent higher-point 7-loop operators R^8, φ^2R^8, φ^4R^8, ..., as was shown in [21] using a different, complementary method (see also [24]).

6.5.4 Matching of soft scalar limits to automorphism analysis

The non-vanishing of the single-soft scalar limits of the matrix elements for R^4, D^4R^4 and D^6R^4 show that these operators are not compatible with $E_{7(7)}$ symmetry. This conclusion was also suggested in a very different approach by Green, Miller, Russo and Vanhove [17], who studied the moduli-dependence in D-dimensions of the above three supersymmetric operators. They found that the moduli-dependent function $f(\varphi)$ that appears in the non-linear completion $f(\varphi)D^{2k}R^4$ of these operators has to obey a certain Laplacian eigenvalue equation. The results for the single-soft scalar limits were used in [20, 21] to compute the relative coefficient of the constant and the quadratic terms of the functions $f(\varphi)$, and it matched exactly the prediction of [17] for all three cases R^4, D^4R^4 and D^6R^4.

7 Superamplitudes without maximal R-symmetry

In this section we consider $\mathcal{N}=8$ SUSY superamplitudes that transform non-trivially under the $SU(8)_R$ symmetry. Our techniques apply to superamplitudes that preserve any subgroup of $SU(8)$, or even break $SU(8)$ completely, but we focus on superamplitudes that are invariant under an $SU(4)\times SU(4)$ subgroup. These superamplitudes are relevant for two reasons:

- As discussed in Sec. 6.5, $\mathcal{N}=8$ supergravity amplitudes with external scalars must obey the low energy theorems of the spontaneously broken $E_{7(7)}$ symmetry of the theory. These theorems require that the single soft scalar limits of any amplitude vanish, and this becomes a test for the matrix elements of candidate counterterms for the theory. If a supersymmetric counterterm is *not* $E_{7(7)}$-compatible, its matrix elements have single soft scalar limits that can be collected into $SU(4)\times SU(4)$ invariant superamplitudes, as we now explain.

  A scalar such as ϕ^{1234} transforms in the **70** representation of $SU(8)$, but it is a singlet of the $SU(4)$ subgroup which acts on the chosen set of indices 1234 or on the complementary set 5678. Hence every scalar is invariant under a particular $SU(4)\times SU(4)$ subgroup of $SU(8)$. Let us consider an amplitude $M_n = \langle B_1^{\cdots}\cdots B_{n\text{-}1}^{\cdots}\phi_n^{1234}\rangle$ containing a scalar and an unspecified set of $n-1$ particles (which may include other scalars). Overall $SU(8)$ invariance requires that the multiparticle state $B_1^{\cdots}\cdots B_{n\text{-}1}^{\cdots}$ is also invariant under the $SU(4)\times SU(4)$ subgroup that preserves 1234. The single soft scalar limit of M_n gives

$$\lim_{p_n\to 0} M_n = C_{n-1}(B_1^{\cdots}\cdots B_{n\text{-}1}^{\cdots})\,. \tag{7.1}$$

If non-vanishing, the amplitude C_{n-1} is $SU(4)\times SU(4)$-invariant *and* still subject to the SUSY Ward identities. Thus it makes sense to package the C_{n-1} in $SU(4)\times SU(4)$-invariant superamplitudes. These transform in the **70** of $SU(8)$.

- Consider a toroidal compactification of string theory to four dimensions where the massless spectrum of the closed string is that of $\mathcal{N}=8$ supergravity and the open strings states are those of $\mathcal{N}=4$ SYM. The symmetry group of *tree level* closed string amplitudes with massless external states is the $SU(4)\times SU(4)$ inherited from the T-duality group $SO(6,6)$. The $SU(4)\times SU(4)$ symmetry manifests itself directly in the KLT relations given above in (6.25). The amplitudes on the RHS of (6.25) are each $SU(4)$-invariant, so M_n is manifestly invariant under $SU(4)\times SU(4)$. In the strict supergravity limit, $\alpha'\to 0$, M_n must preserve the full $SU(8)$. The α'-corrections, however, explicitly break $SU(8)$ to $SU(4)\times SU(4)$ at tree level.[11] A prime example of an $SU(8)$-violating amplitude is $\big\langle --++\phi^{1234}\big\rangle$ which has two pairs of opposite helicity gravitons and a single scalar ϕ^{1234}. This amplitude has a leading non-vanishing contribution at order α'^3. A detailed discussion of this amplitude and the symmetries of string tree amplitudes can be found in [20].

Consider a general $SU(4)\times SU(4)$-invariant amplitude $C_n(B_1^{\cdots}B_2^{\cdots}\cdots B_n^{\cdots})$. Each particle carries up to 8 indices of the full set 12345678. Suppose that the indices 1234 and 5678 transform under the left and right $SU(4)$ factors of the product group, respectively. Then $SU(4)\times SU(4)$ invariance requires that each index in the set 1234 appears $k+2$ times among the particle labels and that each index in the set 5678 appears $\tilde{k}+2$ times.[12] This gives a natural $\mathrm{N}^{(k,\tilde{k})}$MHV classification of such amplitudes, characterized by a pair of integers $(k,\tilde{k})$. If the amplitude arises as the single soft scalar limit (7.1) of an $SU(8)$-invariant amplitude, then it has $\tilde{k}=k-1$. We consider amplitudes which are characterized by two *independent* integers k and $\tilde{k}$; these are relevant for the analysis of closed string tree amplitudes.

The $\mathrm{N}^{(k,\tilde{k})}$MHV amplitudes satisfy SUSY Ward identities of the *same* form as $SU(8)$-invariant supergravity amplitudes. Therefore we package all $SU(4)\times SU(4)$-invariant amplitudes in each class into superamplitudes $\mathcal{C}_n^{\mathrm{N}^{(k,\tilde{k})}\mathrm{MHV}}$. They are polynomials of degree $4(k+2)$ in the Grassmann variables $\eta_{ia},\ a=1,2,3,4$ and of degree $4(\tilde{k}+2)$ in the variables $\eta_{ia},\ a=5,6,7,8$. Next we discuss the construction of the simplest of these superamplitudes.

7.1 MHV, $\sqrt{\mathrm{N}}$MHV, and N'MHV superamplitudes

For amplitudes in the MHV (*i.e.* $\mathrm{N}^{(0,0)}$MHV) sector, the SUSY Ward identities, independent of R-symmetry, determine the unique superamplitude,

$$\mathcal{M}_n^{\mathrm{MHV}} \;=\; \frac{\delta^{(16)}\big(\widetilde{Q}\big)}{\langle n-1,n\rangle^8}\;\times\;\big\langle +\cdots+--\big\rangle\,, \tag{7.2}$$

with a single basis element. The $\delta^{(16)}$-function is automatically $SU(8)$-invariant, as shown in Sec. 3.2, so there are no MHV superamplitudes that violate $SU(8)$. From the point of view of KLT (6.25), each of the MHV superamplitudes on the open string side contain a $\delta^{(8)}$-function, giving $\delta^{(8)}\times\tilde{\delta}^{(8)}=\delta^{(16)}$.

The first true $SU(4)\times SU(4)$ superamplitude sits at the $\mathrm{N}^{(1,0)}\mathrm{MHV}\oplus\mathrm{N}^{(0,1)}$MHV level, which we call the "$\sqrt{\mathbf{N}}$**MHV sector**" for simplicity. We impose a $\mathbb{Z}_2$-exchange symmetry between the two $SU(4)$ factors.[13]

[11] The $SU(4)\times SU(4)$ subgroup is the one that leaves the string dilaton and axion invariant.

[12] $SU(8)$-invariant N^kMHV amplitudes vanish unless the external states, labeled with upper indices, saturate an integer number of 8-index Levi-Civita tensors $(\epsilon^{\cdots\cdots\cdots})^{k+2}$. Similarly, for $SU(4)\times SU(4)$ invariance, the tensor structure $(\epsilon^{\cdots\cdot})^{k+2}(\tilde{\epsilon}^{\cdots\cdot})^{\tilde{k}+2}$ characterizes the $\mathrm{N}^{(k,\tilde{k})}$MHV sector.

[13] This is motivated by the symmetry of the closed string amplitudes we consider here under exchange of left- and right-movers.

Clearly, $\sqrt{\text{N}}$MHV amplitudes break $SU(8)$. To construct the $\sqrt{\text{N}}$MHV superamplitude we define

$$Y_{ijkl} \;=\; [n{-}3,n{-}2]^{-4} \times\, m_{i,n\text{-}3,n\text{-}2;1}\; m_{j,n\text{-}3,n\text{-}2;2}\; m_{k,n\text{-}3,n\text{-}2;3}\, m_{l,n\text{-}3,n\text{-}2;4}\;\;. \tag{7.3}$$

Full symmetrization of its indices (see (4.5)) makes $Y_{(ijkl)}$ invariant under $SU(4)\times SU(4)$. We also need the analogous polynomial $\widetilde{Y}_{(ijkl)}$ that depends on the η_{ia} with $a = 5,6,7,8$. The n-point $\sqrt{\text{N}}$MHV superamplitude then takes the form

$$\mathcal{M}_n^{\sqrt{\text{N}}\text{MHV}} \;=\; \frac{\delta^{(16)}\big(\widetilde{Q}\big)}{\langle n-1,n\rangle^8} \times \sum_{1\le i\le j\le k\le l\le n-4} M_n(\{i,j,k,l\}++--)\,\Big[Y_{(ijkl)}+\widetilde{Y}_{(ijkl)}\Big]\,. \tag{7.4}$$

Note that Y and $\widetilde{Y}$ are multiplied by the same basis amplitudes due to the $\mathbb{Z}_2$-exchange symmetry. The basis amplitudes $M_n(\{i,j,k,l\}++--)$ have the indicated gravitons on the last four lines. Their particle content on the remaining lines is determined by the set $\{i,j,k,l\}$, which indicates that state i carries $SU(4)$ index 1, state j carries $SU(4)$ index 2, and so on. For example:

$$M_6(\{1,1,1,1\}++--) \,\equiv\, \big\langle \phi^{1234}+++--\big\rangle\,, \quad M_6(\{1,1,2,2\}++--) \,\equiv\, \big\langle v^{12}v^{34}++--\big\rangle\,. \tag{7.5}$$

In general, there are $\binom{n-1}{4}$ basis amplitudes at the $\sqrt{\text{N}}$MHV level. At the 5-point level, there is precisely one basis amplitude, namely $M_5(\{1,1,1,1\}++--)$, and the superamplitude is given by

$$\mathcal{M}_5^{\sqrt{\text{N}}\text{MHV}} = \frac{\delta^{(16)}\big(\widetilde{Q}\big)}{\langle 45\rangle^8}\,\big[Y_{1111}+\widetilde{Y}_{1111}\big] \times \big\langle \phi^{1234}++--\big\rangle\,. \tag{7.6}$$

The next $SU(8)$-violating sector is $\text{N}^{(2,0)}\text{MHV}\oplus\text{N}^{(0,2)}\text{MHV}$ sector, which we denote **N′MHV** for brevity. For $n = 6$, there is only one basis amplitude,[14] $M_6(\{1,1,1,1\},\{2,2,2,2\}++--)$ and the superamplitude is given by

$$\mathcal{M}_6^{\text{N}'\text{MHV}} \;=\; \frac{\delta^{(16)}\big(\widetilde{Q}\big)}{\langle n-1,n\rangle^8}\Big[Y_{1111}Y_{2222}+\widetilde{Y}_{1111}\widetilde{Y}_{2222}\Big] \times \big\langle \phi^{1234}\phi^{1234}++--\big\rangle\,. \tag{7.7}$$

The superamplitudes (7.6) and (7.7) manifestly violate $SU(8)$ and thus vanish in $\mathcal{N}=8$ supergravity.

7.2 The NMHV sector

The external particles of amplitudes in the $\text{N}^{(1,1)}$MHV (=NMHV) sector are exactly as in the NMHV amplitudes we studied in Sec. 4. The NMHV sector therefore includes both $SU(8)$- and $SU(4)\times SU(4)$-invariant superamplitudes. Amplitudes with particles of $SU(8)$-equivalent labels, such as

$$\big\langle \phi^{1234}\,\phi^{5678}\;++--\big\rangle\,, \quad \big\langle \phi^{12|56}\phi^{34|78}++--\big\rangle\,, \quad \big\langle \phi^{123|8}\phi^{4|567}++--\big\rangle\,, \tag{7.8}$$

must be identical if $SU(8)$-invariance is imposed, but they can be distinct in the case of $SU(4)\times SU(4)$ symmetry. The $SU(8)$-singlet NMHV superamplitudes were given in (5.2). It should be contrasted with the more general $SU(4)\times SU(4)$ NMHV superamplitude which takes the form

$$\mathcal{M}_n^{\text{NMHV}} \;=\; \frac{\delta^{(16)}\big(\widetilde{Q}\big)}{\langle n-1,n\rangle^8} \sum_{\substack{1\le i\le j\le k\le l\le n-4\\ 1\le p\le q\le u\le v\le n-4}} M_n(\{i,j,k,l|p,q,u,v\};++--)\,Y_{(ijkl)}\widetilde{Y}_{(pquv)}\,. \tag{7.9}$$

[14]For $n = 6$, this sector occurs in KLT from anti-MHV×MHV open string factors.

The set $\{i,j,k,l|p,q,u,v\}$ denotes the lines on which the $SU(4)\times SU(4)$ indices $1,2,3,4$ and $5,6,7,8$ are distributed. For example,

$$M_6(\{1,1,2,2|1,2,2,2\};++--) \;=\; \big\langle \chi^{12|5}\chi^{34|678}++--\big\rangle\,. \tag{7.10}$$

If, in addition, the $\mathbb{Z}_2$ symmetry is imposed, we have

$$M_n(\{i,j,k,l|p,q,u,v\};++--) \;=\; M_n(\{p,q,u,v|i,j,k,l\};++--)\,. \tag{7.11}$$

Due to the reduced constraints from R-symmetry, more basis amplitudes are required for $SU(4)\times SU(4)$ NMHV superamplitudes (7.9) than for $SU(8)$-invariant ones (5.2). For example, the algebraic basis for the 6-point $SU(4)\times SU(4)$ NMHV superamplitude, with $\mathbb{Z}_2$ symmetry, contains 15 basis amplitudes, whereas only 9 were needed with full $SU(8)$ R-symmetry. There are *functional* dependencies among the 15 basis amplitudes, because lines 1 and 2 can be exchanged due to permutation symmetry. Taking functional relations between basis amplitudes into account, we obtain a functional basis with $SU(4)\times SU(4)\times\mathbb{Z}_2$ symmetry. It consists of the following 9 amplitudes:

$$\begin{array}{lll}\big\langle +-++--\big\rangle\,, & \big\langle \psi^1\psi^{234|5678}++--\big\rangle\,, & \big\langle v^{12}v^{34|5678}++--\big\rangle\,,\\ \big\langle v^{1|8}v^{234|567}++--\big\rangle\,, & \big\langle \chi^{123}\chi^{4|5678}++--\big\rangle\,, & \big\langle \chi^{12|5}\chi^{34|678}++--\big\rangle\,,\\ \big\langle \phi^{1234}\phi^{5678}++--\big\rangle\,, & \big\langle \phi^{123|8}\phi^{4|567}++--\big\rangle\,, & \big\langle \phi^{12|56}\phi^{34|78}++--\big\rangle\,.\end{array} \tag{7.12}$$

Imposing full $SU(8)$ is equivalent to demanding

$$\begin{aligned}\big\langle v^{12}v^{34|5678}++--\big\rangle &= \big\langle v^{1|8}v^{234|567}++--\big\rangle\,,\\ \big\langle \chi^{123}\chi^{4|5678}++--\big\rangle &= \big\langle \chi^{12|5}\chi^{34|678}++--\big\rangle\,,\\ \big\langle \phi^{1234}\,\phi^{5678}\;++--\big\rangle &= \big\langle \phi^{12|56}\phi^{34|78}++--\big\rangle \;=\; \big\langle \phi^{123|8}\phi^{4|567}++--\big\rangle\,.\end{aligned} \tag{7.13}$$

These $SU(8)$ conditions reduce the number of functional basis elements to the 5 ones of (5.6).

7.3 Application to closed string tree amplitudes

We now use the $SU(4)\times SU(4)$ superamplitudes to describe tree-level closed string amplitudes in toroidal compactification to four dimensions. The symmetry group of tree level closed string amplitudes with massless external states is $SU(4)\times SU(4)\times\mathbb{Z}_2$, where the $\mathbb{Z}_2$-symmetry exchanges the L and R-movers.

The open-string amplitudes on the right-hand side of the KLT relations (6.25) are each $SU(4)$-invariant, so M_n is manifestly invariant under $SU(4)\times SU(4)$. In the strict supergravity limit, $\alpha'\to 0$, M_n must preserve the full $SU(8)$, but the α'-corrections explicitly break $SU(8)$ to $SU(4)\times SU(4)$. As discussed above, MHV amplitudes preserve the full $SU(8)$ symmetry, so the simplest possible $SU(8)$-violating amplitude belongs to the 5-point $\sqrt{\text{N}}$MHV sector. Expanding the closed string amplitude in small α', we find

$$\big\langle \phi^{1234}++--\big\rangle_{\text{closed}} \;=\; 6\,\zeta(3)\,\alpha'^3\,[23]^4\langle 45\rangle^4 + O(\alpha'^5)\,. \tag{7.14}$$

This basis amplitude alone determines the full 5-point $\sqrt{\text{N}}$MHV superamplitude (7.6).

The 6-point N$'$MHV superamplitude (7.7) is also determined by a single basis amplitude: expanding the closed string amplitude we find

$$\big\langle \phi^{1234}\phi^{1234}++--\big\rangle_{\text{closed}} = -24\,\zeta(3)\,\alpha'^3\,[34]^4\langle 56\rangle^4 + O(\alpha'^5)\,. \tag{7.15}$$

From the point of view of the string effective action (6.23), the origin of the leading results for the two

$SU(8)$-violating amplitudes (7.14) and (7.15) is the operator

$$-2\zeta(3)\,\alpha'^3 e^{-6\phi} R^4 = -2\zeta(3)\,\alpha'^3(1-6\phi+36\phi^2+\dots)R^4\,, \tag{7.16}$$

where the dilaton is $\phi = \frac{1}{2}(\phi^{1234}+\phi^{5678})$. Indeed one can match [20] the numerical coefficients of relevant local 4-, 5- and 6-point amplitudes to the numerical coefficients in (7.16).

Note that no pole terms contribute to the $SU(8)$-violating basis amplitudes (7.14) and (7.15). At the α'^3 order, Feynman pole diagrams involve a single insertion of an interaction vertex from (7.16) together with vertices from the supergravity theory. One can show that for the $n \le 6$ amplitudes we discuss, any $SU(8)$-violation comes from local interaction terms [20].

At the NMHV level, the most general $SU(4)\times SU(4)$-invariant NMHV superamplitude is characterized by the 9 basis amplitudes given in (7.12). The 6-point NMHV closed-string amplitude is not $SU(8)$ invariant, and therefore its basis amplitudes do not satisfy the constraints (7.13). However, it is possible to decompose the closed-string amplitude $\mathcal{M}_6^{\text{closed}}$ into an $SU(8)$-singlet piece, $\mathcal{M}_6^{\text{singlet}}$, which satisfies the $SU(8)$ conditions (7.13), and a remainder piece, $\mathcal{M}_6^{SU(4)\times SU(4)}$, which transforms non-trivially under $SU(8)$:

$$\mathcal{M}_6^{\text{closed}} \;=\; \mathcal{M}_6^{\text{singlet}} + \mathcal{M}_6^{SU(4)\times SU(4)}\,. \tag{7.17}$$

Interestingly, the $SU(8)$-violating piece has to be local because no $SU(8)$-violating pole term can contribute to the 6-point NMHV sector. One can thus use the method described in Sec. 7.2 to determine the matrix elements of the general $SU(4)\times SU(4)\times\mathbb{Z}_2$-preserving operator at order α'^3. First recall from Sec. 7.2 that the 6-point NMHV superamplitude $\mathcal{M}_6^{SU(4)\times SU(4)}$ is determined by the nine basis amplitudes (7.12). At order α'^3 of the closed string amplitude, these basis amplitudes have to be local and of dimension 8 (because R^4 contains 8 derivatives). Little group scaling implies that only the 3 scalar basis amplitudes of (7.12) can be non-vanishing; indeed, they must each be equal to $[34]^4\langle 56\rangle^4$ times a numerical coefficient. The ratio of the coefficients for the three scalar amplitudes is fixed by requiring that the resulting superamplitude $\mathcal{M}_6^{SU(4)\times SU(4)}$ is permutation invariant. All included, this uniquely fixes the superamplitude $\mathcal{M}_6^{SU(4)\times SU(4)}$ to be

$$\begin{aligned}\mathcal{M}_6^{SU(4)\times SU(4)} = -\tfrac{6}{5}\zeta(3)\,\alpha'^3 \times \delta^{(16)}\big(\widetilde{Q}\big)\,[34]^4\langle 56\rangle^{-4}\Big[&12\big(Y_{1111}\widetilde{Y}_{2222}+Y_{2222}\widetilde{Y}_{1111}\big)+2\,Y_{(1122)}\widetilde{Y}_{(1122)}\\ &-3\big(Y_{(1112)}\widetilde{Y}_{(1222)}+Y_{(1222)}\widetilde{Y}_{(1112)}\big)\Big]\;+\;O(\alpha'^5)\end{aligned} \tag{7.18}$$

The overall normalization is fixed by the explicit closed-string amplitude computation [20]. This superamplitude (7.18) encodes the $SU(8)$-*violating* component of ϕ^2R^4 in the NMHV sector. One can also check directly that no $SU(8)$-singlet contribution 'hides' in (7.18): if we average the superamplitude (7.18) over $SU(8)$, using the formula (6.24), we find that it is $\propto 12+2\times 18-3\times 16 = 0$.

The α'^3 term in $\mathcal{M}_6^{\text{singlet}}$ is precisely the superamplitude that encodes the 6-point NMHV matrix elements of the $SU(8)$-invariant operator R^4. In Sec. 6.5, we only computed its matrix element $\langle\varphi\bar{\varphi}{+}{+}{-}{-}\rangle$ to rule out R^4 as a potential counterterm in $\mathcal{N}=8$ supergravity. However, the entire 6-point NMHV superamplitude of R^4 can be determined from (7.17) in terms of closed-string tree amplitudes, using (7.18) to subtract off the $SU(8)$-violating contribution. The results for the single soft scalar limits agree with those obtained from the $SU(8)$-averaging procedure.

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A Derivation of solution to NMHV SUSY Ward identities

We provide in this appendix the missing steps in the derivation-outline presented in Sec. 4.1. In **step 2** of Sec. 4.1, all $\eta_{n-1,a}$ and $\eta_{n,a}$ are eliminated from P_4. This is done by first using the Schouten identity to rewrite the Grassmann $\delta^{(8)}$-function of (4.2) as

$$\delta^{(8)}\Big(\sum_{i=1}^{n}|i\rangle\,\eta_{ia}\Big) \;=\; \frac{1}{\langle n-1,n\rangle^4}\;\delta^{(4)}\Big(\sum_{i=1}^{n}\langle n-1,i\rangle\,\eta_{ia}\Big)\;\delta^{(4)}\Big(\sum_{j=1}^{n}\langle nj\rangle\,\eta_{ja}\Big)\,, \tag{A.1}$$

The two $\delta^{(4)}$-functions can be used to express $\eta_{n-1,a}$ and η_{na} in terms of the other η_{ia}'s, specifically

$$\eta_{n-1,a} = -\sum_{i=1}^{n-2}\frac{\langle ni\rangle}{\langle n,n-1\rangle}\,\eta_{ia}\,,\qquad \eta_{na} = -\sum_{i=1}^{n-2}\frac{\langle n-1,i\rangle}{\langle n-1,n\rangle}\,\eta_{ia}\,. \tag{A.2}$$

Inserting this into the P_4 of (4.2), we find

$$P_4 = \frac{1}{\langle n-1,n\rangle^4}\sum_{i,j,k,l=1}^{n-2} c_{ijkl}\,\eta_{i1}\,\eta_{j2}\,\eta_{k3}\,\eta_{l4}\,. \tag{A.3}$$

The c_{ijkl}'s are linear combinations of the q_{ijkl}'s, but we will not need their detailed relationship.

In **step 3** of Sec. 4.1, we presented the solution to the Q^a-Ward identities $Q^a\,P_4 = 0$. The action of Q^1 on P_4 gives

$$0 \;=\; Q^1 P_4 \;\propto\; \sum_{i,j,k,l=1}^{n-2}[\epsilon i]\,c_{ijkl}\,\eta_{j2}\,\eta_{k3}\,\eta_{l4} \;=\; \sum_{j,k,l=1}^{n-2}\Big[\sum_{i=1}^{n-2}[\epsilon i]\,c_{ijkl}\Big]\eta_{j2}\,\eta_{k3}\,\eta_{l4}\,. \tag{A.4}$$

The quantity in square brackets must vanish for any triple jkl, so the c_{ijkl} must satisfy

$$\sum_{i=1}^{n-2}[\epsilon i]\,c_{ijkl} \;=\; 0\,. \tag{A.5}$$

This is the relation quoted in **step 3** of Sec. 4.1.

Now select two arbitrary (but fixed) lines s and t among the remaining lines $1,\dots,n-2$. We choose the SUSY spinor $|\epsilon] \sim |t]$ and then $|\epsilon] \sim |s]$ and use (A.5) to express the coefficients c_{sjkl} and c_{tjkl} in terms of c_{ijkl} with $i \neq s,t$:

$$c_{sjkl} = -\sum_{i\neq s,t}^{n-2}\frac{[ti]}{[ts]}\,c_{ijkl}\,,\qquad c_{tjkl} = -\sum_{i\neq s,t}^{n-2}\frac{[si]}{[st]}\,c_{ijkl}\,. \tag{A.6}$$

The sums extend from $i = 1$ to $i = n-2$, excluding lines s and t. We can write similar relations for c_{iskl}, c_{itkl}, etc. using supercharges Q^a, $a = 2,3,4$, in the same way. We use the relations (A.6) to write P_4

in (A.3) as

$$\begin{aligned}\langle n-1,n\rangle^4 P_4 &= \sum_{j,k,l=1}^{n-2}\sum_{i\neq s,t}^{n-2} c_{ijkl}\;\eta_{i1}\eta_{j2}\,\eta_{k3}\,\eta_{l4} + \sum_{j,k,l=1}^{n-2}\big(c_{sjkl}\,\eta_{s1}+c_{tjkl}\,\eta_{t1}\big)\,\eta_{j2}\,\eta_{k3}\,\eta_{l4}\\ &= \frac{1}{[st]}\sum_{j,k,l=1}^{n-2}\sum_{i\neq s,t}^{n-2} c_{ijkl}\,m_{ist,1}\,\eta_{j2}\,\eta_{k3}\,\eta_{l4}\,,\end{aligned}\tag{A.7}$$

in which $m_{ist,1}$ is the first-order polynomial introduced in (1.4). We repeat this process and use the analogues of (A.6) for c_{iskl} and c_{itkl} to reexpress the sum over j in (A.7) in terms of $m_{jst,2}$. Repeating this for the k and l sums, we find

$$\mathcal{A}_n^{\rm NMHV} = \sum_{i,j,k,l\neq s,t}^{n-2} c_{ijkl}\,X_{ijkl}\,,\qquad X_{ijkl}\;\equiv\;\delta^{(8)}\big(\widetilde{Q}_a\big)\;\frac{m_{ist,1}\,m_{jst,2}\,m_{kst,3}\,m_{lst,4}}{[st]^4\langle n-1,n\rangle^4}\,.\tag{A.8}$$

The η-polynomial X_{ijkl} of degree 12 in (A.8) are manifestly invariant under both $\widetilde{Q}_a$ and Q^a supersymmetry. As in Sec. 4.1, it is convenient to set $s=n-3$ and $t=n-2$. Since the c-coefficients are fully symmetric we can symmetrize the X-polynomials and write

$$\mathcal{A}_n^{\rm NMHV} = \sum_{1\le i\le j\le k\le l\le n-4} c_{ijkl}\,X_{(ijkl)}\,,\qquad X_{(ijkl)}\equiv\sum_{\mathcal{P}(i,j,k,l)} X_{ijkl}\,.\tag{A.9}$$

The sum over permutations $\mathcal{P}(i,j,k,l)$ in the definition of $X_{(ijkl)}$ is explained below (4.6).

In the final **step 4** we identify the coefficients c_{ijkl} as on-shell amplitudes of the basis. Recall [6] that component amplitudes are obtained by applying Grassmann derivatives to the superamplitude. Consider amplitudes with negative-helicity gluons at positions $n-1$ and n. To extract such amplitudes from (A.9) we apply four $\eta_{n-1,a}$-derivatives and four η_{na}-derivatives to $\mathcal{A}_n^{\rm NMHV}$. These derivatives must hit the Grassmann δ-function and the result is simply a factor $\langle n-1,n\rangle^4$, which cancels the same factor in the denominator of $X_{(ijkl)}$. We must apply four more Grassmann derivatives $\frac{\partial}{\partial\eta_{i1}}\frac{\partial}{\partial\eta_{j2}}\frac{\partial}{\partial\eta_{k3}}\frac{\partial}{\partial\eta_{l4}}$ to $\mathcal{A}_n$ in order to extract an NMHV amplitude. These derivatives hit the product of $m_{i,n\text{-}3,n\text{-}2;a}$-polynomials and produce a factor of $[n-3,n-2]^4$ which cancels the remaining denominator factor of X_{ijkl}. As a result, the 12 η-derivatives just leave us with the coefficient c_{ijkl}. When $1\le i\le j\le k\le l\le n-4$, we have therefore identified c_{ijkl} as the amplitude $c_{ijkl}=A_n\big(\{i,j,k,l\}++--\big)$. As discussed in the main text, the notation means that line i carries the $SU(4)_R$ index 1, line j carries index 2 etc. With this identification of the c_{ijkl} coefficients we can now write the final result (4.8) in terms of the $(n-4)(n-3)(n-2)(n-1)/4! = \binom{n-1}{4}$ basis amplitudes $A_n\big(\{i,j,k,l\}++--\big)$ of the algebraic basis.

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