MODELING AND ESTIMATION OF SPACE-TIME STOCHASTIC PROCESSES

by

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ABSTRACT

This thesis examines the issues of modeling and estimation of space-time stochastic processes in the context of two physically motivated problems. These two problems are distinguished by the manner in which observations are made on the space-time processes involved.

The first problem involves a space-time process, called the signal field, being propagated by a time-invariant spatial process called the transmission field. Observations are made on the signal field via a spatially fixed sensor and these observations are processed to estimate the signal field and to infer the properties of the transmission field. Extensions of the problem to the case of multiple signal fields with a single sensor and the case of a single signal field with multiple sensors are also considered. Applications of these space-time models formulated here and the associated estimation and statistical inference results to various physical problems are pointed out wherever appropriate.

The second problem deals with the estimation of a time invariant spatial field via observations from a point sensor moving across it in space. A novel approach of modeling the field with a stochastic differential equation is proposed and the implications of this model for field estimation are examined. Results are derived for field estimation in both the cases of random and deterministic sensor motion. A novel problem of optimal field estimation via sensor motion control is introduced and solved explicitly in one special case.

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CHAPTER 1

INTRODUCTION

1.1 Motivation for our research

The theory of stochastic processes has played an important role in many branches of science, engineering, economics, management and even the social sciences. In the field of control and communication, its impact has been especially vital. Ever since the pioneering work of Wiener in the forties in applying statistical theory to this field [1]-[3], a tremendous amount of research has been done on it over the last three decades. The publication of Kalman and Bucy [4] in 1961 represents a major breakthrough in filtering theory and is the starting point for the present day research in recursive estimation via stochastic differential equations. At present, it can be said that, in principle, the nonlinear estimation problem has been solved in that it is well understood, and recursive and non-recursive (infinite dimensional) solutions have been developed. In actuality, of course, the development of implementable approximations remains an important research area. Recent publications such as [5]-[7] can be consulted for detailed expositions and solutions to the general nonlinear filtering problem.

The success of the statistical theory in control and communication problems has inevitably led researchers to explore similar applications for the theory in other fields. One field that has emerged recently in this exploration is the modeling and statistical analysis of random quantities which vary in space and time. Application to physical problems is no doubt the prime motivation for considering this class of problems and physical examples abound, as we shall point out, which demonstrate the importance of a statistical theory in this field. Another main reason, from the view-point of systems engineers, is to see if the ideas of modeling and statistical inference for stochastic processes in time can similarly be applied to processes in space and time. That is, since estimation theory concepts have been so thoroughly developed for processes in time, we feel that it is time to lay the foundations for a similar theory for processes which also vary in space. These issues provide the primary motivation for this research. Our goal is to understand the issues raised by space-time processes. It is, of course, impossible to answer all of the questions that can be raised and our intent, rather, is to take a fundamental step in extending stochastic analysis and estimation ideas to space-time processes.

A word of terminology here. We will use the terms space-time stochastic processes, random fields or fields interchangeably.

A space-time stochastic process can of course be viewed as a collection of random variables indexed by a vector parameter. Many workers in this field tend to view this vector parameter as multidimensional time. The early work in this field has mostly taken this point of view [8]-[10], but these authors were only concerned with certain mathematical analysis aspects of the problems, and their problem formulations and results were not motivated by any particular class of physical applications. However, their work does provide some fundamental understanding of the difficulties of multiparameter stochastic calculus. At present, Wong and Zakai [11]-[13] have also adopted such a point of view and their efforts have been directed at producing a multiparameter stochastic calculus. None of the authors who deal with the concept of multidimensional time have so far given a good explanation or an example of how the concept can be useful in practice. We feel that the results in the references on multiparameter processes cited above represent important but only very initial results in this area and do not form a usable calculus for dealing with practical problems yet. In particular, the artificial causality imposed on multidimensional time in these studies must be eliminated or at least its implication must be thoroughly understood before we can successfully apply their results to physically motivated random field problems.

Besides the work on multiparameter processes cited above, various researchers have dealt with different aspects of space-time processes through different approaches. First, there is the work on time-invariant spatial processes using the correlation function approach which is analogous to the traditional correlation function approach to temporal processes. The work of Chernov [14], Tatarski [27], Wong [46] and so on are all based on this approach. Using this approach, the authors above were able to characterize the properties of the fields of interest. Yaglom [64] has also treated rigorously the mathematical properties of random fields using the correlation approach. A few other isolated attempts in dealing with space-time processes can also be mentioned. The work of Woods [50] has dealt with Markovian random fields in discrete space and Ito [61] has attempted to establish a general theory for homogeneous or isotropic random vector fields. On the physical application side, the work of McGarty [62] on characterizing the structure of random fields generated by a scattering medium should be pointed out. All the work cited above has dealt mainly with characterizing the properties of random fields. In the area of estimation and statistical inference on random fields, a great deal still remains to be done. More recently, some progress has been made in this direction. Notable among the work accomplished is that of Fishman and Snyder [21]-[23] in the area of space-time point processes and that of Wong [24], [63] in recursive filtering of two-dimensional random fields employing some of his results in multiparameter stochastic calculus. In addition, the work of people in the field of statistical image processing should be mentioned. An image can be viewed as statistical data defined on a two-dimensional surface [65], [66] and this data is usually processed line-wise as a one-dimensional sequence of statistical data using the well developed theory of estimation for stochastic processes in time [67], [68]. Recently, however, Attasi [66] has developed some results for modeling and recursive estimation of statistical data defined on a two-dimensional discrete space with application to image processing in mind. The work of Wong and Zakai on multiparameter stochastic calculus cited earlier also has application to image processing as one possible motivation.

At present, it can still be said that the modeling and estimation of space-time stochastic processes is a very new topic and at this stage it is not clear how such problems should be approached in general, in contrast to estimation of stochastic processes in time. Therefore, we have taken the point of view that instead of trying to formulate general hypothetical space-time modeling and estimation problems, we should abstract some useful and physically meaningful examples and examine the issues of modeling and estimation of space-time processes in the context of these examples. This point of view is precisely the starting point of the research reported in this thesis. We note here that the work of Fishman and Snyder [21]-[23] in the area of space-time point processes is aligned with the point of view that we have adopted, while the work of Wong [24], [63] is more toward a general hypothetical formulation. The problems that we have considered in this thesis, although abstracted, are motivated by physical applications. This is unlike the cited work on multiparameter processes which may be useful eventually but are far away from physical reality in their present form. Of course, the solutions that we have provided to these problems do not indicate in general what space-time modeling and estimation problems are like. However, they do provide some insight into some aspects of space-time modeling and estimation and indicate some of the difficulties that arise in space-time estimation that never arise in estimation of stochastic processes in time. Our work here should be viewed as an initial attempt in taking a step toward developing a theory for estimation of space-time processes. As such, we cannot expect our problem formulations and our results to be very general. However, we feel that they are very basic, and many extensions are possible to handle more complicated space-time problems. All these will be pointed out in the remainder of the report.

We have restricted our work here to processes which vary in time and in only one spatial dimension and even for such processes we have considered the temporal and spatial variations separately. There are two basic reasons for this: (1) in order to consider the multidimensional case, one would need to use some type of multiparameter stochastic calculus; although a number of results (such as those of Wong and Zakai) have been obtained, it is not yet clear how to use such mathematical tools to formulate multidimensional space-time estimation problems along the lines we have considered here; (2) many of the important concepts of space-time estimation already arise in the one-dimensional case we do consider, and we feel that a thorough understanding of these is required before we introduce the further complication of several dimensions.

1.2 Structure of the thesis

This thesis deals with two main problems of space-time modeling and estimation. These two problems are distinguished by the method whereby observations are made on the space-time fields involved. In the first case, considered in Chapter 2, observations are made via a spatially fixed sensor and in the second case, considered in Chapter 3, observations are made via a sensor moving in space. We describe these two problems in more detail below.

In Chapter 2, we consider the problem modeled by a propagating space-time field, which we call the signal field, being transmitted by a time-invariant spatial field which we call the transmission medium. The signal field is assumed to be generated by a source located at a fixed point in space. Observations are made on the signal field as a function of time by a sensor located at a fixed spatial point in the transmission medium and the problems we are interested in are: (i) to estimate the signal field at the location of the sensor using the observations, and (ii) to infer the properties of the transmission field using the observations of the sensor and the estimates of the signal field. We give the complete solution to the signal estimation problem and examine it under various special cases, such as the case of linear time-invariant signal model in the steady state. To infer the properties of the transmission medium, we compute the a posteriori probability distribution of the travel time of the signal field from the source to the sensor. Since the delay effect is the only influence we assume the transmission field exerts on the signal field in propagating it from the source to the sensor, the delay time is the only quantity concerning the transmission field that we can estimate. It turns out that the signal estimation problem and the delay time estimation problem are coupled. We give the implementation of the overall problem of signal and delay time estimation and examine some suboptimal approximate implementations. Then we consider extensions of our problem, firstly to the case with multiple signal fields due to multiple signal sources and a single sensor, and secondly, to the case with a single signal field and an array of sensors. In both cases, we consider the problem of estimating the signal field(s) and the corresponding delay times involved. The signal estimation problem in both these cases turn out to be much more complicated than that in the basic one-source-one-sensor case and we draw upon some of the results in [37] on estimation for systems with delays. The signal estimation and delay time estimation problems are coupled in both cases and the complete solution to the overall problem of signal and delay time estimation requires an implementation exactly similar to the one-source-one-sensor case. We shall also illustrate our results with some examples involving deterministic signals.

In Chapter 3, we consider the problem of estimating a time-invariant spatial random field using observations from a point sensor moving across the field. The field is assumed to vary in one spatial dimension and we propose the novel approach of modeling the spatial variations of the field using a stochastic differential equation. Reasons are given to motivate the use of such a model which is really a spatial shaping filter, but we are mostly concerned with the implications of such a model for random field estimation. We derive the equations for estimating the field using the observations of the sensor for the case in which the velocity of the sensor is known perfectly and the case in which only noisy observations on the velocity and the position of the sensor are available. Our investigation of the problem of random field estimation using observations from a moving point sensor leads finally to a novel optimal control problem on the sensor. It is natural to conjecture that the velocity of the sensor affects the accuracy with which it can observe the field and hence the accuracy with which the field can be estimated from the observations of the sensor. Therefore, it is natural to consider the problem of designing an optimal velocity program for the sensor so that different parts of the field can be estimated with the desired accuracy. We solve this problem in the case in which the field and observation models are linear and in which the velocity of the sensor is known at each time.

Chapter 4 concludes the thesis with recapitulations of what we have done and what we think are good immediate extensions of our work. Finally, we will give some thoughts on possible future directions for research in this new area of random fields.

1.3 Contributions of our work

The major contribution of this thesis is the formulation of mathematical models for certain physically motivated space-time process problems and in indicating how the powerful techniques of stochastic analysis and estimation can be used in their solution.

The problems we formulate and solve in Chapter 2 are basic models which can have applications to such fields as wave propagation in a random medium [14], [27], statistical fluid mechanics [28]-[30], seismic signal processing [15], discrete multipath communication [16] and so on. Of course, in the present form, our models are rather inadequate to handle any of the above problems except in the simplest cases. However, we have taken the first step in building some very basic models on which extensions are possible to handle the more complicated cases. Before the more complicated problems can be faced, the issues raised by the basic models must be thoroughly understood. This is precisely where the significance of the work in Chapter 2 lies.

The models we have formulated and the problems we are interested in solving for the models in Chapter 2 are novel. They represent conceptualizations of some of the main problems in the areas of wave propagation in random media and statistical fluid mechanics, and the abstractions of these problems that we will study represent a new, initial attempt to apply many of the tools of stochastic analysis and recursive estimation to abstract space-time models. In the field of wave propagation in random media, Chernov [14] and Tatarski [27] have treated the general theory, including turbulence effects, for both sound and electromagnetic waves. They have dealt mainly with the physics of the problem and have analyzed some effects the randomness of the media can have on the propagating wave fields. The same remarks can be made of the work summarized in the survey papers of Frisch [51] and Dence and Spence [52]. Soong and Chuang [48] have also considered this problem using the approach of differential equations with random coefficients. However, none of the work cited so far has dealt with the problem of estimating the propagating wave field using noisy observations on the latter or of doing statistical inference on the random medium using such observations. The models we have proposed and the problems we have formulated are intended to answer some of these questions that have not been touched on by the previously cited workers. The work of people in the areas of radar and sonar communication [16], [31], [33], [53] and seismic signal processing [15], [25], [26], [34] has actually touched on the questions of estimating the wave field and of doing inference on the random medium. A good account of the theory currently used for estimating the signal field and for doing inference on the random medium can be found in Van Trees [16]. This theory employs the frequency domain approach and does not enable us to get on-line estimates of the wave field or to do on-line inference on the random medium. In other words, the observations have to be processed over a certain time interval before an estimate or an inference can be made. The work on radar and sonar communication and seismic signal processing in the references cited above is all based on this theory. We will see that the results in this thesis enable us to do on-line estimation and inference, i.e., an estimate or an inference can be made as the observations are being processed.

In the field of statistical fluid mechanics, problems of transport by a random fluid flow are even less well understood than the wave propagation problems above. The main difficulty is that random flows are usually turbulent, and turbulence is far from being well understood. The models we have proposed in this thesis only represent the case of a steady random flow. In [28], Monin and Yaglom have only discussed some basic statistical ideas for inferring the nature of a random turbulent flow but no concrete results are presented. References [29] and [30] also examine some statistical characterizations of turbulent flows using moment equations or spectral functions but do not deal with statistical inference on the flows. What we are trying to do in this thesis is to take a first step in abstracting a simplified model consisting only of a steady random flow and to understand thoroughly the signal processing and statistical inference problems associated with this model. The foundation that we lay in our work here will hopefully enable future researchers to build more complicated models for physically realistic types of flows and propagation effects.

The work in Chapter 3 is a mathematical formulation of a class of random field estimation problems which has already been carried out in practice. It is motivated by such applications as microwave sensing of atmospheric temperature and humidity fields using observations from satellites [17],[19]. We will elaborate on the Nimbus-5 system considered in [17] and [18] in Chapter 3. The measurement of the gravity field of the earth via instruments carried in a ship travelling horizontally [69] also falls into this class of random field estimation problems. The work on random field estimation currently being carried out in practice, such as [17]-[19], does not employ stochastic differential equation models for the fields of interest. Thus, our work appears to be the first to examine the issue of random field estimation via a dynamical model for the field, although the use of a dynamical model for various random fields has by now been proposed or investigated by other researchers [19], [20]. In [19], McGarty proposed the idea of fitting a state variable model to the power spectra of the data on the constituent densities of the upper atmosphere of the earth. In [20], workers at the Analytic Sciences Corporation, Reading, Mass., have examined the idea of fitting state variable models to gravity anomaly data for the earth. We derive the equations for estimating a random field modeled by a stochastic differential equation in both the cases of random and deterministic sensor motion, and consider some special cases of the dynamical field model in greater detail. The really novel contribution of our work is the introduction of optimal control theory into this area of random field estimation, calling attention for the first time to the idea of optimal field estimation via sensor motion control. Although we have only examined one case of sensor motion control in this thesis, we believe that we have proposed an idea which will lead to much further research in the future. Also, we feel that our proposed suboptimal field estimation system when sensor position is not known exactly has promise for future applications in this area.

We should remark here that our work in Chapter 3 is similar in spirit to some of the current work on image processing [65], [67], [68] in which an image is scanned line-wise and processed as a random process in time. The statistical information of the image is usually assumed to be given in terms of the mean and the two dimensional auto-correlation function. From this information, a dynamical model for the evolution of the signal (image) along each line is developed and estimation techniques for processes in time are applied to process the image. This is very similar to our work in Chapter 3 but we have considered many additional features of the problem of estimating a time invariant spatial field observed via a moving point sensor than is considered in the image processing literature. Of course, our problem formulations and our results in Chapter 3 do not have much applications to image processing but the analytical similarities with the latter are worth pointing out.

As stated earlier, the research in this thesis is based on the concept that with our present understanding of space-time problems, the modeling and estimation of space-time processes should be examined in the context of particular examples instead of in a general hypothetical framework. Our work therefore contributes to the understanding of space-time problems in the context of the examples we have studied. We believe that our concept of such an approach to space-time problems is a fruitful one and hopefully this concept is also one of the contributions of this thesis!

CHAPTER 2

SPACE-TIME ESTIMATION VIA OBSERVATIONS FROM SPATIALLY FIXED SENSORS

2.1 Motivation and the basic model

This class of problems is motivated by such topics as wave propagation in a random medium and transport of material by a steady fluid flow. Our basic formulation of this class of problems is as follows. We have a "propagating space-time field", which we call the signal field, being transmitted by a time-invariant spatial field called the transmission field. The signal field is generated by a signal source located at a fixed point in space and observations on the signal field are made as a function of time by a sensor located at a fixed spatial point in the transmission medium. The problems we are interested in are: (i) to estimate the signal field at the location of the sensor using the sensor observations, and (ii) to infer the properties of the transmission field by estimating its influence on the signal field.

The formulation proposed above models the following situations.

(1) The transport medium is a dielectric material with random time-invariant properties and the signal that propagates through it is an electromagnetic wave. Alternatively the transport medium is a random time-invariant material medium and the signal is a sound wave. In both these cases, it is of interest in practice to estimate the properties of the medium by processing the observations on the signal. The processing of seismic signals to estimate the structure of the subsurface of the earth [15],[25],[26] is a good example. In this type of applications it is important to obtain estimates of the signal itself because [26] the signal is the impulse response

of the earth and knowing the impulse response enables us to deduce the structure of the subsurface of the earth. This area of application is of great importance to geologic exploration such as oil prospecting. Similarly, the estimation of the random refractive index of the atmosphere using electromagnetic waves as signals [27] is also an important area of application.

(2) The transport medium is a steady fluid flow and the signal is some material transported by the flow. This problem is important in fluid mechanics where scientists have been trying to understand the nature of many types of flows, especially turbulent ones. Recently statistical techniques have been introduced into this area to help in characterizing the random nature of these flows [28]-[30]. In [28], Monin and Yaglom suggest introducing a dye into experimentally produced flows to help trace the structure of the turbulence and they discuss some statistical ideas for doing inference on the flow using observations on the dye. All these show that our problem formulation is in line with the ideas of people in the area of statistical fluid mechanics. Our model based on a time invariant random field might be too simple to deal with the many practically important types of flows, but we feel that understanding and solving the statistical inference problems for this abstracted model is a necessary first step before tackling the more complicated ones.

In the class of space-time problems discussed informally above, the signal estimation problem and the statistical inference problem on the transmission field are the two problems of interest. Depending on the application we have in mind for the basic model formulated above, the primary problem of interest might be either one or both of the problems. In the remainder of this chapter, whenever appropriate, we will point out specific possible applications of the results of either problem.

Since observations are made on the signal field, we can view the signal estimation problem as a problem of direct inference on the signal field and the inference problem on the transmission field as being indirect. We can look at the time invariant spatial transmission field as an information source on which no direct observations are possible. However, it exhibits itself through its influence on the signal field which is being transmitted. By processing our observations on the signal field, we want to "estimate the influence" of the transmission field on the signal field and thereby infer some properties of the former. Therefore, it is very important in building our basic model here to specify exactly what the influence of the transmission field on the signal field is. We shall assume here that the only influence of the transmission field on the signal field is a pure transport, i.e., the propagation of the signal from the source to the sensor involves only a pure time delay. We could, of course, assume more complicated models for the influence of the transmission field on the signal field, e.g., the case in which the transmission field modulates the amplitude of the signal field in addition to transporting it. However, we shall be more modest at this stage and consider only the case of a pure propagation time delay.

The basic model formulated above involves only one signal source and one sensor. In later sections of this chapter, we shall extend our model to the case of multiple signal sources and one sensor and the case of one signal source and multiple sensors and indicate possible applications for such extensions of our model.

We admit that the basic model and its various extensions we have given above may be too elementary to handle any real physical problems. However, our intent in this thesis is not to apply our models to any specific real application but rather to understand thoroughly the signal processing and statistical inference aspects of such models in order to assess their potential usefulness. With the groundwork that we lay in this thesis, future researchers would be able to build on our models and possibly apply them to actual situations. We shall only indicate, whenever appropriate, possible applications we have in mind for our models and our results.

2.2 Mathematical formulation of the basic model

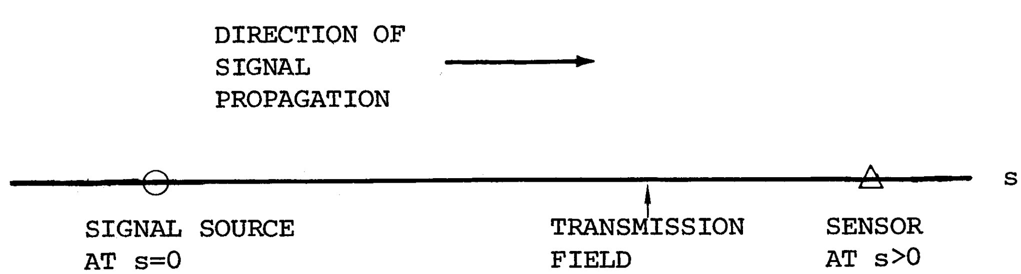

FIGURE 1: THE BASIC MODEL

We have chosen to consider here a time-invariant transmission medium which is a random field in one spatial dimension. As stated earlier, a usable multidimensional stochastic calculus would be necessary in order to deal with variations in more than one spatial dimension. The transmission field is characterized at each point s' by a unique random velocity v(s') which is the speed with which the signal propagates at that point in the medium. We assume that $v(s')>0$ for all $s'\geq 0$ so that signal propagation takes place constantly in the direction of increasing s. A sensor is fixed at some point $s>0$ in the field. The input to the transmission field is a signal field generated by a source located at $s=0$. The situation is depicted in Figure 1.

The signal generated by the source is modeled by an Ito diffusion process, i.e., the signal ϕ_t is given by

$$\left.\begin{aligned} d\phi_t &= \alpha(\phi_t,t)dt + \underline{\gamma}'(\phi_t,t)d\underline{\eta}_t \quad , \qquad t>0 \\ \phi_0 &= \text{random with given distribution,} \\ \phi_t &= 0 \qquad\qquad , \; t<0 \end{aligned}\right\} \tag{2.2.1}$$

Here, $\underline{\gamma}(\cdot,\cdot)$ is an n-vector and $\underline{\eta}_t$ is an n-vector of independent standard Wiener processes, i.e.,

$$E\{d\underline{\eta}_t d\underline{\eta}_t'\} = \underline{I}dt \tag{2.2.2}$$

The functions $\alpha(\cdot,\cdot)$ and $\underline{\gamma}(\cdot,\cdot)$ are assumed to satisfy conditions for the existence and uniquencess of ϕ.

The travel time t_s of the signal from the source at $s=0$ to the sensor at $s>0$ is given by

$$t_s = \int_0^s \frac{ds'}{v(s')} \tag{2.2.3}$$

and is a random variable. Assuming that we know the probability distributions of the random variables $v(s)$, for all s, we can, in principle, compute the a priori probability distribution of t_s. We suppose that the source starts generating the signal at time $t=0$, and thus the signal field first arrives at the sensor at time $t=t_s$. Since the transmission field only transports the signal field from the source to the sensor, the signal x_t at the location of the sensor is a delayed version of the signal from the source, i.e.,

$$x_t = \phi_{t-t_s} \tag{2.2.4}$$

The sensor makes noisy observations on the signal and these are modeled as

$$dz_t = h(\phi_{t-t_s},t)dt + dw_t \tag{2.2.5}$$

where $h(\cdot,\cdot)$ is jointly measurable with respect to both arguments and w_t is a standard Wiener process independent of $\underline{\eta}_t$ and of ϕ_0. Thus w_t is independent of ϕ_t. If we assume w_t and t_s to be independent also, then w_t and ϕ_{t-t_s} will be independent. This assumption will be made. We define here the cumulative observation σ-field:

$$Z_t = \sigma\{z_\tau, 0\leq\tau\leq t\} \tag{2.2.6}$$

The problems we are interested in are now:

(i) To estimate the signal x_t using the observations Z_t,

(ii) To infer the properties of the transmission field using the observations Z_t and the estimates of x_t.

We are interested in deriving on-time recursive solutions to the problems posed above. Our mathematical formulation of the problem is such that it is well suited for recursive solution. We shall see that our solution employs the now well established theory of estimation for continuous time processes via stochastic differential equations.

Note that some similar problems of processing space-time random processes have been considered by Baggeroer [31] via the frequency domain spectral function approach. The frequency domain approach was first originated by Wiener in the estimation of temporal processes in the forties and was not replaced by the present time domain approach until the breakthrough of Kalman and Bucy [4] in 1961. In the area of space-time signal processing, the frequency domain approach appears to be the only approach employed so far, as a sample, e.g. [31]-[34], of the vast literature will show us. Our work here therefore appears to be among the first to take a time domain approach.

To infer the properties of the transmission field, we have to estimate its influence on the signal field. Since the only influence of the transmission field on the signal field is a pure time delay, the quantity t_s is the only variable we can estimate concerning the transmission field. We shall see that we can recursively compute the a posteriori probability distribution of t_s, from which we can compute recursively the minimum mean square error estimate of t_s. Under various special situations the delay time estimate does enable us to estimate more about the transmission field. For instance, if the transmission velocity v(s) is a constant, then equation (2.2.3) reduces to

$$t_s = \frac{s}{v} \tag{2.2.7}$$

Assuming that s is known, we can then estimate the velocity v. Alternatively, v could be known and constant and we then can estimate the distance s between the source and the sensor. The latter case is important especially in radar/sonar communication problems [16] in which delay time estimates are used to estimate the distance from a target. Delay-time estimates are also very important in the processing of seismic signals [15].

We shall see that the delay time estimation problem and the signal estimation problem are coupled. In the next section, we shall present the complete solution in the case of a continuous range of values of t_s. Then, in Section 2.4, we examine the case in which t_s takes on only a finite number of possible values.

2.3 Solution for a continuous range of t_s

In this section, we present the solution to the signal estimation and delay time estimation problems formulated in the previous section for the basic model. We first deal with the signal estimation problem and present two solutions, one via a stochastic differential equation representation for the estimate and the other via a "multiple-model" approach. Then, we deal with the delay time estimation problem and discuss the implementation of the above results. Finally, we examine the behaviour of our results under special conditions, e.g., the case of linear time invariant signal model in the steady state, and we investigate suboptimal approximate implementations of our results using an assumed density approach. Some examples involving known signals will also be worked out. Throughout this section, t_s is assumed to take on a continuous range of values and the a priori probability distribution of t_s is absolutely continuous with respect to the Lebesque measure on the real line.

2.3a Dynamical Representation of the Signal Estimate

We are interested here in deriving the equation for generating the minimum mean square error estimate of the signal ϕ_{t-t_s} conditioned on the observations. We have defined the cumulative observations Z_t in equation (2.2.6) and the estimate $\hat{\phi}_{t-t_s}$ of the signal ϕ_{t-t_s} is given by

$$\hat{\phi}_{t-t_s} = E\{\phi_{t-t_s} \mid Z_t\} \qquad (2.3a.1)$$

We shall use the ^ notation for the minimum mean square error estimate of any random variable given the observations Z_t:

$$\hat{\cdot} \equiv E\{\cdot \mid Z_t\} \qquad (2.3a.2)$$

The following steps will be taken in the derivation of our result. We first derive a dynamical representation for the randomly delayed diffusion process ϕ_{t-t_s}. Then, a fundamental martingale representation theorem of Fujisaki etc. [5] is applied to derive the representation for the estimate $\hat{\phi}_{t-t_s}$.

A word of notation is appropriate here. We let $(\Omega,\mathcal{F},\mathcal{P})$ be the basic probability space on which all random variables are defined. Let

$\{\mathcal{F}_t\}_{t\geq 0}$ be an increasing family of sub-σ-fields of $\mathcal{F}$ that describe all events in time at the signal source at s=0. Thus, in particular, the processes ϕ_t and $\underline{\eta}_t$ are adapted to the family $\{\mathcal{F}_t\}_{t\geq 0}$. To describe events at the sensor, we construct the increasing family $\{\mathcal{B}_t\}_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$ such that

$$\mathcal{B}_t = \mathcal{G}_t \vee \sigma\{\phi_{\tau-t_s},\ 0\leq\tau\leq t\} \vee \sigma\{\{\omega : t_s(\omega)\leq\tau\}\,|\,\tau\leq t\} \tag{2.3a.3}$$

The notation $\mathcal{A} \vee \mathcal{M}$ denotes the smallest σ-field generated by the union of the σ-fields $\mathcal{A}$ and $\mathcal{M}$. The increasing family $\{\mathcal{G}_t\}_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$ describe events at the sensor which are not delayed versions of events at the source. We will define $\mathcal{G}_t$ to be the σ-field $\sigma\{w_\tau,\ 0\leq\tau\leq t\}$, where w_t is the observation noise in equation (2.2.5). With the above construction, both ϕ_{t-t_s} and w_t are adapted to $\{\mathcal{B}_t\}_{t\geq 0}$.

We now proceed to derive the semimartingale representation for ϕ_{t-t_s}. This representation is necessary in order for us to be able to make us of results in filtering theory. To derive this representation, we introduce a unit-jump process ψ_t defined by

$$\psi_t = \begin{cases} 0 & ,\ t<t_s \\ 1 & ,\ t\geq t_s \end{cases} \tag{2.3a.4}$$

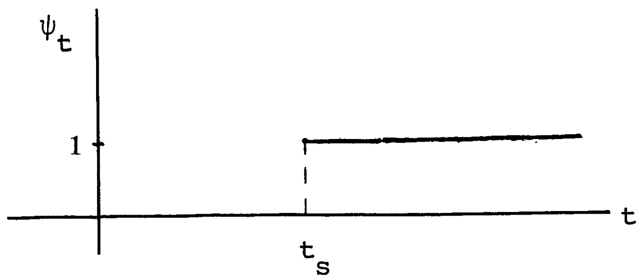

FIGURE 2: THE UNIT-JUMP PROCESS ψ_t

Then, we can write

$$\phi_{t-t_s} = \psi_t \phi_{t-t_s} \tag{2.3a.5}$$

Note that this is just purely a mathematical device. If we make use of only equation (2.2.1), the defining equation for ϕ_t, then we can only get the semimartingale representation for ϕ_{t-t_s} for $t>t_s$. However, we want the representation for ϕ_{t-t_s} for all $t\geq 0$. By writing ϕ_{t-t_s} as in equation (2.3a.5), we can derive the semimartingale representation for ϕ_{t-t_s} for all $t\geq 0$ if we have the semimartingale representation for ψ_t. Theorem 2.1 below gives us the required representation for ψ_t. Note that ψ_t is adapted to $\{\mathcal{B}_t\}_{t\geq 0}$.

Theorem 2.1: The unit-jump process ψ_t can be represented by the stochastic differential equation

$$d\psi_t = \lambda_t dt + dm_t \tag{2.3a.6}$$

where m_t is a martingale on $\{\mathcal{B}_t\}_{t\geq 0}$ and

$$\begin{aligned}\lambda_t &= \left[P_{t_s}(t) \Big/ \int_t^{\infty} P_{t_s}(\tau)d\tau \right] (1-\psi_t) \\ &\triangleq \rho_t(1-\psi_t)\end{aligned} \tag{2.3a.7}$$

$P_{t_s}(t)$ is the a priori probability density of t_s. ■

This result is not new [36],[54] and can be easily verified using the Doob-Meyer decomposition theorem [35]. The verification is carried out in Appendix 1 for the sake of completeness. With this representation for ψ_t and the defining equation (2.2.1) for ϕ_t, we can apply the Doleans-Dade, Meyer change of variables formula [36] to obtain the following representation for ϕ_{t-t_s} for all $t\geq 0$. The proof is given in Appendix 2.

<u>Theorem 2.2</u>: The signal ϕ_{t-t_s} is represented by

$$\begin{aligned}d\phi_{t-t_s} &= (\lambda_t\phi_0 + \psi_{t-}\alpha(\phi_{t-t_s}, t-t_s))dt \\ &+ [\phi_0 \quad , \quad \psi_{t-}\underline{\Upsilon}'(\phi_{t-t_s}, t-t_s)] \begin{bmatrix} dm_t \\ d\underline{\eta}_{t-t_s} \end{bmatrix}\end{aligned} \tag{2.3a.8}$$

ψ_{t-} denotes the left-continuous version of ψ_t. ■

Remark: Equation (2.3a.8) is nothing more than

$$d\phi_{t-t_s} = \phi_0 d\psi_t + \psi_{t-}(\alpha(\phi_{t-t_s}, t-t_s)dt + \underline{\gamma}'(\phi_{t-t_s}, t-t_s)d\underline{\eta}_{t-t_s}) \qquad (2.3a.9)$$

This is actually what we intuitively expect the representation (2.3a.8) to say. Since ϕ_{t-t_s} has a jump at $t=t_s$, the ϕ_0 term represents the contribution due to the fact that t_s might be the present instant.

We are now in a position to derive the filtering equation for generating the estimate $\hat{\phi}_{t-t_s}$. Before doing that, we will review some relevant results in filtering theory important for our derivation.

We will use the martingale approach in our derivation of the estimation equations. One of the main results that we need here is a martingale representation theorem for observation models of the form (2.2.5). The theorem was first proved by Fujisaki etc. [5] for the case of square integrable martingales and we shall state it below.

Theorem 2.3: Given the observation model

$$dz_t = h_t dt + dw_t \,, \qquad t \varepsilon [0,T] \qquad (2.3a.10)$$

where w_t is a standard Wiener process and $h_t(\omega)$ is a (t,ω)-measurable process such that $\int_0^T E|h_t|^2 dt < \infty$. Assume that for each $s \in [0,T]$, the σ-fields, $\sigma\{h_u, w_u, 0<u<s\}$ and $\sigma\{w_v - w_u, s<u<v<T\}$, are independent. Let

$$Z_t = \sigma\{z_\tau,\ 0\leq\tau\leq t\} \tag{2.3a.11}$$

Then, every separable square integrable martingale (μ_t, Z_t) is sample continuous and has the representation

$$\mu_t - E\{\mu_0\} = \int_0^t \Phi_s d\nu_s \tag{2.3a.12}$$

where (ν_t, Z_t) is the innovations process given by

$$d\nu_t = dz_t - \hat{h}_t dt \tag{2.3a.13}$$

and (Φ_t, Z_t) is a process satisfying

$$\int_0^T E|\Phi_t|^2\, dt \quad < \infty \tag{2.3a.14}$$

■

Using this result, our desired filtering equation can easily be derived. Many authors, for instance [6], have used the martingale representation theorem above to derive estimation equations for processes described by semimartingales. We need not go through the derivation again but will just state the result we need.

Theorem 2.4: Given the semimartingale (y_t, F_t) where

$$dy_t = f_t dt + dm_t\ , \qquad t \in [0,T] \tag{2.3a.15}$$

such that

(i) $(m_t, \mathcal{F}_t)$ is a martingale,

(ii) $(f_t, \mathcal{F}_t)$ is an adapted measurable process with

$$E|f_t|<\infty\ , \qquad E\int |f_t|^2\, dt < \infty$$

Assume observations z_t to be given by equation (2.3a.10) with the same assumptions as in Theorem 3. Then, the minimum mean square error estimate

$$\hat{y}_t = E\{y_t | Z_t\} \tag{2.3a.16}$$

is given by

$$d\hat{y}_t = \hat{f}_t dt + E\{(y_t - \hat{y}_t)(h_t - \hat{h}_t) + \frac{d}{dt} \langle m,w\rangle_t \Big| Z_t\} \cdot (dz_t - \hat{h}_t dt) \tag{2.3a.17}$$

■

Remark: The first step in the derivation of this result consists of showing that the process

$$\mu_t = \hat{y}_t - \int_0^t \hat{f}_\tau d\tau \tag{2.3a.18}$$

is a martingale on $\{Z_t\}_{t\geq 0}$ and therefore by Theorem 2.3 we have

$$\mu_t - E\{\mu_0\} = \int_0^t \Phi_s (dz_s - \hat{h}_s ds) \tag{2.3a.19}$$

The rest of the proof consists of showing that

$$\Phi_t = E\left\{(y_t-\hat{y}_t)(h_t-\hat{h}_t) + \frac{d}{dt}\langle m,w\rangle_t \middle| Z_t\right\} \tag{2.3a.20}$$

The result also assumes that $\langle m,w\rangle_t$ is differentiable in t. The notation $\langle\cdot,\cdot\rangle_t$ denotes the joint variance process of two martingales [36].

Application of the above results immediately leads to the recursive filtering equation for the signal ϕ_{t-t_s} and this result is given below.

<u>Theorem 2.5</u>: The estimate $\hat{\phi}_{t-t_s} = E\{\phi_{t-t_s}|Z_t\}$ of the signal ϕ_{t-t_s} given the observations Z_t is generated recursively by the following filter:

$$\begin{aligned} d\hat{\phi}_{t-t_s} = &\left(\rho_t E\{\phi_0\}(1-\hat{\psi}_{t|t}) + E\{\psi_{t-}\alpha(\phi_{t-t_s},t-t_s)|Z_t\}\right) dt \\ &+ E\left\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})(h(\phi_{t-t_s},t)-\hat{h}(\phi_{t-t_s},t))|Z_t\right\} \cdot \\ &\cdot d\nu_t \end{aligned} \tag{2.3a.21}$$

where ν_t is the innovations process given by

$$d\nu_t = dz_t - \hat{h}(\phi_{t-t_s},t)dt \tag{2.3a.22}$$

The initial condition is

$$\hat{\phi}_{t-t_s}\Big|_{t=0} = 0 \tag{2.3a.23}$$

■

Proof: It is easy to show, in spite of the dependence on the random variable t_s, that the process

$$\mu_t = \hat{\phi}_{t-t_s} - \int_0^t E\{\lambda_\tau \phi_0 + \psi_{\tau-}\ \alpha(\phi_{\tau-t_s}, \tau-t_s) | Z_\tau\} d\tau \tag{2.3a.24}$$

is a martingale on $\{Z_t\}_{t \geq 0}$. Thus, by Theorem 2.3, we have representation

$$\mu_t - E\{\mu_0\} = \int_0^t \Phi_s d\nu_s \tag{2.3a.25}$$

The rest of the proof goes through as the proof of Theorem 2.4 to give us

$$\Phi_t = E\{(\phi_{t-t_s} - \hat{\phi}_{t-t_s})(h(\phi_{t-t_s}, t) - \hat{h}(\phi_{t-t_s}, t)) + \frac{d}{dt} \langle v,w \rangle_t \Big| Z_t\} \tag{2.3a.26}$$

where v_t is the $\mathcal{B}_t$-martingale given by

$$dv_t = [\phi_0\ , \quad \psi_{t-}\underline{\gamma}'(\phi_{t-t_s}, t-t_s)] \begin{bmatrix} dm_t \\ d\underline{\eta}_{t-t_s} \end{bmatrix} \tag{2.3a.27}$$

We will now show that

$$\langle v,w \rangle_t = 0 \tag{2.3a.28}$$

We have

$$E\{\langle v,w\rangle_{t_2} - \langle v,w\rangle_{t_1} \Big| \mathcal{B}_{t_1}\} = E\{(v_{t_2} - v_{t_1})(w_{t_2} - w_{t_1}) \Big| \mathcal{B}_{t_1}\}$$

$$= E\{v_{t_2} w_{t_2} - v_{t_1} w_{t_1} \Big| \mathcal{B}_{t_1}\} \tag{2.3a.29}$$

for $t_1<t_2$. Since we have assumed that the observation Wiener noise w_t is independent of the signal ϕ_{t-t_s}, therefore w_t is independent of v_t. Thus,

$$E\{(v_{t_2}-v_{t_1})(w_{t_2}-w_{t_1})\Big|\mathcal{B}_{t_1}\} = E\{v_{t_2}-v_{t_1}\Big|\mathcal{B}_{t_1}\}E\{w_{t_2}-w_{t_1}\Big|\mathcal{B}_{t_1}\} = 0 \qquad (2.3a.30)$$

since $E\{w_{t_2}-w_{t_1}\Big|\mathcal{B}_{t_1}\} = 0$. Thus, by (2.3a.29),

$$E\{v_{t_2}w_{t_2}\Big|\mathcal{B}_{t_1}\} = v_{t_1}w_{t_1} \qquad (2.3a.31)$$

implying that v_tw_t is a $\mathcal{B}_t$-martingale and hence

$$v_tw_t = 0 \qquad \forall\, t \qquad (2.3a.32)$$

But this implies equation (2.3a.28) and so equations (2.3a.24), (2.3a.25) and (2.3a.26) give us the filter

$$\begin{aligned} d\hat{\phi}_{t-t_s} &= E\{\lambda_t\phi_0 + \psi_{t-}\alpha(\phi_{t-t_s},t-t_s)\Big|Z_t\}dt \\ &\quad + E\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})(h(\phi_{t-t_s},t) - \hat{h}(\phi_{t-t_s},t))\Big|Z_t\}. \\ &\quad \cdot d\nu_t \end{aligned} \qquad (2.3a.33)$$

The first term on the right can be simplified as

$$E\{\lambda_t \phi_0 \big| Z_t\} = E\{\rho_t (1-\psi_t)\phi_0 \big| Z_t\}$$

$$= \rho_t \int_0^\infty (1-\psi_t) E\{\phi_0 \big| Z_t,\ t_s=\tau\} \mathcal{P}(\tau<t_s\leq\tau+d\tau \big| Z_t)$$

$$= \rho_t \int_t^\infty (1-\psi_t) E\{\phi_0 \big| Z_t,\ t_s=\tau\} \mathcal{P}(\tau<t_s\leq\tau+d\tau \big| Z_t) \qquad (2.3a.34)$$

because for $t_s=\tau\leq t$, $\psi_t=1$ and so the integrand is zero. Now, for $t_s=\tau>t$, we have $\psi_t=0$ and note that

$$E\{\phi_0 \big| Z_t,\ t_s=\tau\} = E\{\phi_0\} \qquad (2.3a.35)$$

since Z_t contains no measurements on the signal ϕ_{t-t_s} given that $t_s>t$. Thus,

$$\begin{aligned} E\{\lambda_t\phi_0 \big| Z_t\} &= \rho_t E\{\phi_0\}\ \mathcal{P}(t_s>t \big| Z_t) \\ &= \rho_t E\{\phi_0\}\ (1-\hat{\psi}_{t|t}) \end{aligned} \qquad (2.3a.36)$$

where

$$\begin{aligned} \hat{\psi}_{t|t} &= E\{\psi_t \big| Z_t\} \\ &= \mathcal{P}(t_s\leq t \big| Z_t) \end{aligned} \qquad (2.3a.37)$$

is a quantity we will examine in Section 2.3c in connection with the estimation of t_s. Putting equation (2.3a.36) into (2.3a.33) gives us the result (2.3a.21). Equation (2.3a.23) for the initial condition is easily verified since the a priori probability measure for t_s is absolutely continuous with respect to the Lebesque measure on the real line. ■

It is interesting to note the first term $\rho_t E\{\phi_0\}\ (1-\hat{\psi}_{t|t})$ on the right of the filter (2.3a.21) of Theorem 2.5. The quantity ϕ_0 is the initial value of the signal, i.e., the "signal front". The first term thus shows that whenever $\rho_t(1-\hat{\psi}_{t|t})\neq 0$, i.e., whenever the a priori probability of arrival $\rho_t\neq 0$ and the a posteriori probability that the signal has arrived $\hat{\psi}_{t|t}<1$, the filter takes into account the possibility that the "signal front" is arriving at the present time t.

Equation (2.3a.21) of Theorem 2.5 gives us only a representation for the estimate $\hat{\phi}_{t-t_s}$. In general, the filter is non-implementable because the on-line computation of the terms on the right is infinite dimensional. Either we need the joint a posteriori probabilities $P(\phi\leq\phi_{t-t_s}<\phi+d\phi,\ \tau<t_s\leq\tau+d\tau\,|\,Z_t)$ or an infinite system of stochastic differential equations. We will see in a latar section on implementations that both of these are infinite dimensional problems.

Note that our signal estimation problem consists of only the filtering of a diffusion process observed with a fixed random time delay. It appears that this is the first time the filtering of a diffusion process observed with a fixed random time delay has been considered, although the case of a fixed known time delay has been considered before, e.g., [37]. We will see in the later sections that our work on delay time estimation, suboptimal approximations and so on are also novel applications of nonlinear filtering concepts and techniques.

2.3b Multiple-Model Solution to the Signal Estimation Problem

This approach is based on the following expression for the estimate of the signal:

$$E\{\phi_{t-t_s} \mid Z_t\} = \int_{-\infty}^{\infty} \phi \, p_{\phi_{t-t_s}}(\phi \mid Z_t) d\phi$$

$$= \int_{-\infty}^{\infty} \phi \int_{0}^{\infty} p_{\phi_{t-t_s}, t_s}(\phi, \tau \mid Z_t) d\tau \, d\phi$$

$$= \int_{0}^{\infty} \int_{-\infty}^{\infty} \phi \, p_{\phi_{t-t_s}}(\phi \mid Z_t, t_s=\tau) d\phi \, p_{t_s}(\tau \mid Z_t) d\tau$$

$$= \int_{0}^{\infty} E\{\phi_{t-\tau} \mid Z_t, t_s=\tau\} \, P(\tau < t_s \leq \tau + d\tau \mid Z_t)$$

$$= \int_{0}^{t} E\{\phi_{t-\tau} \mid Z_t, t_s=\tau\} \, P(\tau < t_s \leq \tau + d\tau \mid Z_t) \tag{2.3b.1}$$

The last step follows because for $\tau > t$, we know that $\phi_{t-\tau} = 0$.

Equation (2.3b.1) clearly exhibits the multiple-model nature of the solution [7]. For each value of $t_s = \tau \leq t$, we have one estimate of the signal ϕ_{t-t_s}. These estimates are weighted by the a posteriori probabilities of t_s and then summed. We will discuss the problem of computing the probabilities $P(\tau<t_s\leq\tau+d\tau \mid Z_t)$ in the next section in connection with the estimation of t_s. To generate the estimate $E\{\phi_{t-\tau} \mid Z_t,\ t_s=\tau\}$, we note that for each value of $t_s = \tau\leq t$, the signal $\phi_{t-\tau}$ satisfies the equation

$$d\phi_{t-\tau} = \alpha(\phi_{t-\tau},t-\tau)dt + \underline{\gamma}'(\phi_{t-\tau},t-\tau)d\underline{\eta}_{t-\tau}\ ,\quad t\geq\tau \tag{2.3b.2}$$

and the measurements are given by

$$dz_t = h(\phi_{t-\tau},t)dt + dw_t\quad ,\quad t\geq\tau \tag{2.3b.3}$$

so that the estimate

$$\hat{\phi}_{t-\tau} = E\{\phi_{t-\tau} \mid Z_t,\ t_s=\tau\},\qquad t\geq\tau \tag{2.3b.4}$$

is given by the filter [38]

$$\begin{aligned} d\hat{\phi}_{t-\tau} &= \hat{\alpha}(\phi_{t-\tau},t-\tau)dt \\ &\quad + E\{(\phi_{t-\tau}-\hat{\phi}_{t-\tau})(h(\phi_{t-\tau},t) - \hat{h}(\phi_{t-\tau},t)) \mid Z_t,t_s=\tau\}. \\ &\quad \cdot (dz_t-\hat{h}(\phi_{t-\tau},t)dt),\qquad t\geq\tau \end{aligned} \tag{2.3b.5}$$

with the initial condition, at $t=\tau$,

$$\hat{\phi}_{t-\tau}\Big|_{t=\tau} = \hat{\phi}_0 = E\{\phi_0\},\quad \text{given} \tag{2.3b.6}$$

Conceptually, our solution then consists of an infinite bank of filters at each time t, one filter for each value of $t_s = \tau \leq t$ and this bank grows with time t. Each filter is of the form given by equation (2.3b.5) and starts at time $t=\tau$ with initial condition given by equation (2.3b.6). In the general nonlinear case, the filter (2.3b.5) is non-implementable because to compute the terms on the right hand side, we need to carry along the density $p(\phi_{t-\tau} | Z_t)$ at each time t and this gives rise to an infinite dimensional problem. However, in the linear Gaussian case, the filter reduces to a readily implementable Kalman filter. The model for the signal $\phi_{t-\tau}$ reduces to

$$d\phi_{t-\tau} = \alpha_{t-\tau}\,\phi_{t-\tau}dt + \underline{\gamma}'_{t-\tau}d\underline{\eta}_{t-\tau} \quad , \qquad t \geq \tau \tag{2.3b.7}$$

and the measurements are given by

$$dz_t = h_t\phi_{t-\tau}dt + dw_t \quad , \qquad t \geq \tau \tag{2.3b.8}$$

Assuming that the initial value ϕ_0 is Gaussian, then the filter (2.3b.5) becomes the Kalman filter:

$$d\hat{\phi}_{t-\tau} = \alpha_{t-\tau}\hat{\phi}_{t-\tau}dt + h_t\sigma_\tau(t)(dz_t - h_t\hat{\phi}_{t-\tau}dt), \qquad t \geq \tau \tag{2.3b.9}$$

where the error covariance

$$\sigma_\tau(t) = E\{(\phi_{t-\tau} - \hat{\phi}_{t-\tau})^2 \Big| Z_t,\ t_s = \tau\}, \qquad t \geq \tau \tag{2.3b.10}$$

is given by the Riccati equation

$$\frac{d\sigma_\tau(t)}{dt} = 2\alpha_{t-\tau}\sigma_\tau(t) + \underline{\gamma}'_{t-\tau}\underline{\gamma}_{t-\tau} - h_t^2\sigma_\tau^2(t), \qquad t\geq\tau \tag{2.3b.11}$$

with the initial condition, at $t=\tau$,

$$\sigma_\tau(\tau) = E\{(\phi_0 - E\{\phi_0\})^2\} = \sigma_0 \quad , \text{ given} \tag{2.3b.12}$$

The Riccati equation can be solved a priori for values of $t\geq\tau$ to obtain $\sigma_\tau(t)$. From equation (2.3b.11), we see that we need to solve one Riccati equation for each value of the initial time τ. However, if the observation gain

$$h_t = h = \text{constant} \tag{2.3b.13}$$

then we need only solve one equation, namely

$$\frac{d\sigma_0(t)}{dt} = 2\alpha_t\sigma_0(t) + \underline{\gamma}'_t\underline{\gamma}_t - h^2\sigma_0^2(t), \qquad t\geq 0 \tag{2.3b.14}$$

$$\sigma_0(0) = \sigma_0 \quad \text{given} \tag{2.3b.15}$$

The solution $\sigma_\tau(t)$ to equation (2.3b.11) with $h_t=h$ is then given by

$$\sigma_\tau(t) = \sigma_0(t-\tau) \tag{2.3b.16}$$

In practice, it is of course impossible to implement the solution proposed here based on the multiple-model approach simply because it consists of an "uncountably infinite" number of filters at each time t and the bank of filters grows with time t. In addition, each filter is non-implementable in the general nonlinear case and is implementable only in the linear Gaussian case. However, the idea of this approach leads easily to an approximate suboptimal implementation. For instance, suppose we know that

$$t_i \leq t_s \leq t_f \tag{2.3b.17}$$

We can partition the interval $[t_i, t_f]$ as

$$t_i = t_0 < t_1 < \dots < t_n = t_f \tag{2.3b.18}$$

and implement the filters given by equation (2.3b.5) for the values of $t_s = t_0, t_1, \dots, t_n$. The partition (2.3b.18) can be chosen based on our knowledge of the a priori distribution of t_s. For instance, if the a priori density $p_{t_s}(t)$ has the form as in Figure 3, then we might choose the partition with more points around t^* and fewer points elsewhere because points nearer to t^* have a higher probability of occurrence. We will have more to say about implementations later.

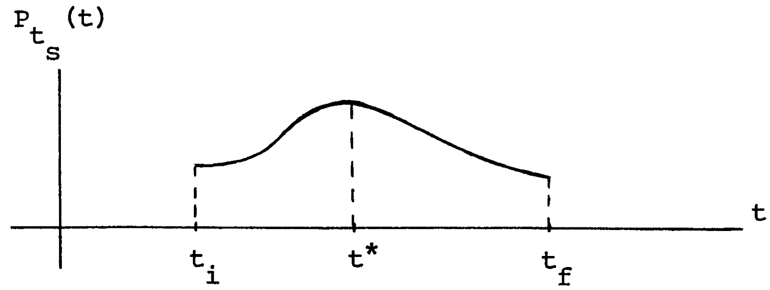

FIGURE 3: A POSSIBLE FORM OF $P_{t_s}(t)$

2.3c Estimation of Delay Time t_s

The delay time estimation problem is in principle solved by computing the probability distribution of t_s conditioned on the observations Z_t. We saw in Section 2.3b that this probability distribution is also used in the multiple-model approach to the signal estimation problem. The representation (2.3a.21) for the signal estimate $\hat{\phi}_{t-t_s}$ in Section 2.3a requires the conditional probability $\hat{\psi}_{t|t} = P(t_s \leq t | Z_t)$. The on-line computation of this probability distribution is accomplished by doing a filtering, a smoothing and a prediction problem on the unit-jump process ψ_t introduced in Section 2.3a, since at any time t, we have

$$E\{\psi_\tau | Z_t\} = P(\psi_\tau = 1 | Z_t) = P(t_s \leq \tau | Z_t) \tag{2.3c.1}$$

for any value of τ. The result is given in Theorem 2.6.

Theorem 2.6: The estimate

$$\hat{\psi}_{\tau|t} = E\{\psi_\tau | Z_t\} = P(t_s \leq \tau | Z_t) \tag{2.3c.2}$$

is generated recursively by the following equations:

$\tau=t$: $$d\hat{\psi}_{t|t} = \rho_t(1-\hat{\psi}_{t|t})dt + [E\{\psi_t h(\phi_{t-t_s},t)|Z_t\} - \hat{\psi}_{t|t}\,\hat{h}(\phi_{t-t_s},t)]d\nu_t \;,$$

$$\hat{\psi}_{0|0}=0 \tag{2.3c.3}$$

$\tau>t$: $$\hat{\psi}_{\tau|t} = 1-(1-\hat{\psi}_{t|t})\frac{P(t_s \geq \tau)}{P(t_s \geq t)} \tag{2.3c.4}$$

$\tau<t$: $$\hat{\psi}_{\tau|t} = \hat{\psi}_{\tau|\tau} + \int_\tau^t \Sigma(\tau,t')d\nu_{t'} \tag{2.3c.5}$$

Here,

$$d\nu_t = dz_t - \hat{h}(\phi_{t-t_s},t)dt \tag{2.3c.6}$$

is the innovations process and

$$\Sigma(\tau,t') = E\{\psi_\tau h(\phi_{t'-t_s},t')|Z_{t'}\} - \hat{\psi}_{\tau|t'}\,\hat{h}(\phi_{t'-t_s},t') \tag{2.3c.7}$$

■

Proof: Equation (2.3c.3) is obtained by considering the filtering problem on the system

$$d\psi_t = \rho_t(1-\psi_t)dt + dm_t \tag{2.3c.8}$$

with the observations

$$dz_t = h(\phi_{t-t_s},t)dt + dw_t \tag{2.3c.9}$$

and applying Theorem 2.4. Note that the term $\langle m,w\rangle_t=0$ since m_t has a jump at $t=t_s$. Thus we get equation (2.3c.3). Equation (2.3c.5) is obtained by considering a smoothing problem on the process ψ_t with the observations (2.3c.9) and the result is well known [6]. Similarly, application of prediction results [6] to the system (2.3c.8) with the observations (2.3c.9) results in

$$\hat{\psi}_{\tau|t} = \hat{\psi}_{t|t} + \int_t^{\tau} \rho_{t'}(1-\hat{\psi}_{t'|t})dt' \quad , \qquad \tau>t \tag{2.3c.10}$$

which will be simplified as follows to equation (2.3c.4). We have

$$d_\tau\hat{\psi}_{\tau|t} = \rho_\tau(1-\hat{\psi}_{\tau|t})d\tau \tag{2.3c.11}$$

where $d_\tau(\cdot)$ denotes the differential of the quantity for a differential change in τ. This can be rearranged as

$$\frac{d_\tau(\hat{\psi}_{\tau|t}-1)}{\hat{\psi}_{\tau|t}-1} = -\rho_\tau d\tau \tag{2.3c.12}$$

which is integrated to give

$$\ln\left|\frac{\hat{\psi}_{\tau|t}-1}{\hat{\psi}_{t|t}-1}\right| = -\int_t^\tau \rho_{t'}dt' \tag{2.3c.13}$$

and this simplifies to

$$\hat{\psi}_{\tau|t} = 1-(1-\hat{\psi}_{t|t})e^{-\int_t^\tau \rho_{t'}dt'} \tag{2.3c.14}$$

We now evaluate $\int_t^\tau \rho_{t'}dt'$. Since

$$\rho_t = P_{t_s}(t)\Big/\int_t^\infty P_{t_s}(t')dt'$$

$$= -\frac{d}{dt}\int_t^\infty P_{t_s}(t')dt'\Big/\int_t^\infty P_{t_s}(t')dt' \tag{2.3c.15}$$

then

$$\int_t^\tau \rho_{t'}dt' = -\left(\ln\int_\sigma^\infty P_{t_s}(t')dt'\right)\Bigg|_t^\tau$$

$$= \ln\frac{P(t_s \geq t)}{P(t_s \geq \tau)} \tag{2.3c.16}$$

and so $$e^{-\int_t^\tau \rho_{t'}dt'} = \frac{P(t_s \geq \tau)}{P(t_s \geq t)} \qquad (2.3c.17)$$

which gives equation (2.3c.4). Note that $\dfrac{P(t_s \geq \tau)}{P(t_s \geq t)}$ is precomputable.

$\blacksquare$

<u>Remark</u>: Although we have derived equation (2.3c.4) via a prediction approach, it can be done more easily as follows.

$$\begin{aligned}
\hat{\psi}_{\tau|t} = P(t_s \leq \tau | Z_t) &= P(t_s \leq t | Z_t) + P(t < t_s \leq \tau | Z_t) \\
&= \hat{\psi}_{t|t} + P(t_s \leq \tau | Z_t, t_s > t) P(t_s > t | Z_t) \\
&= \hat{\psi}_{t|t} + P(t_s \leq \tau | Z_t, t_s > t)(1 - \hat{\psi}_{t|t})
\end{aligned} \qquad (2.3c.18)$$

But given $t_s > t$, Z_t contains no information on t_s. Thus,

$$\begin{aligned}
P(t_s \leq \tau | Z_t, t_s > t) &= P(t_s \leq \tau | t_s > t) \\
&= \frac{P(t < t_s \leq \tau)}{P(t_s > t)}
\end{aligned} \qquad (2.3c.19)$$

and equation (2.3c.18) simplifies to (2.3c.4).

Computing the a posteriori probability distribution $P(t_s \leq \tau | Z_t)$ of t_s for all values of τ at each time t is an inherently infinite dimensional problem. This difficulty becomes more obvious when we consider implementations in a later section.

Finally, consider the problem of on-line estimation of the delay time t_s given the observations Z_t. If we are interested in the minimum mean square error estimate $E\{t_s|Z_t\}$, we can compute it as

$$E\{t_s|Z_t\} = \int_0^\infty \tau P(\tau<t_s\leq\tau+d\tau|Z_t) \tag{2.3c.20}$$

There is no way to compute this estimate on-line with finite dimensional computations. Another delay time estimate of common interest is the maximum a posteriori probability estimate [16] which is given by the value of t_s at which the a posteriori density of t_s given Z_t is a maximum. From Theorem 2.6, it is not easy to deduce that the density $P_{t_s}(\tau|Z_t)$ exists, especially from equation (2.3c.5) for $\tau<t$. However, by considering the system

$$dt_s = 0 \tag{2.3c.21}$$

and computing the estimate $E\{e^{jut_s}|Z_t\}$, we easily see that $P_{t_s}(\tau|Z_t)$ exists. Related results in Wozencraft [74] also indicate the existence of $P_{t_s}(\tau|Z_t)$.

2.3d Implementation of Results and Some Special Cases

In this section, we want to examine the implementation aspects of our signal and delay time estimation results and find cases in which the results admit finite dimensional implementations. We first note two points here. The first concerns the delay time estimation results and, as we have noted before, the on-line computation of the a posteriori distribution of t_s is an inherently infinite dimensional problem since we have to compute the whole distribution function at each time. We will see later in this section that the results in Theorem 2.6 for computing the a posteriori distribution of t_s require an infinite dimensional multiple-model type of implementation. There is no hope of finding a case in which the delay time estimation results admit a finite dimensional implementation except if we make a suboptimal approximation. We shall talk about one such approximation in the next section. The second point to note here is that the multiple-model solution to the signal estimation problem also requires inevitably an infinite dimensional implementation since we have assumed a continuum of values of t_s. Thus, the only result in which we can hope to find cases of finite dimensional implementation is the representation result for the signal estimate given in Theorem 2.5. We shall first investigate cases of finite dimensional implementation for this result, then examine the implementation of the signal and delay time estimation results in general and finally examine a combined implementation for the signal and delay time estimation results.

Special Cases of Optimal Finite Dimensional Implementation

We shall examine here the representation result for the signal estimate $\hat{\phi}_{t-t_s}$ given in Theorem 2.5 for cases of finite dimensional implementation. This equation is

$$
\begin{aligned}
d\hat{\phi}_{t-t_s} &= (\rho_t E\{\phi_0\}(1-\hat{\psi}_{t|t}) + E\{\psi_{t-}\alpha(\phi_{t-t_s},t-t_s)|Z_t\})dt \\
&\quad + E\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})(h(\phi_{t-t_s},t) - \hat{h}(\phi_{t-t_s},t))|Z_t\}\cdot \\
&\quad \cdot d\nu_t \quad ,
\end{aligned}
$$

$$
\hat{\phi}_{t-t_s}\Big|_{t=0} = 0 \qquad (2.3d.1)
$$

From our experience with filtering theory, we conjecture that a finite dimensional implementation should be possible in the linear Gaussian case. This is the case in which the signal and observation models are linear and given by

$$d\phi_t = \alpha_t \phi_t dt + \underline{\gamma}_t' d\underline{\eta}_t \quad , \qquad t>0$$

$$\phi_0 = \text{Gaussian random variable}$$

$$\phi_t = 0 \quad , \qquad t<0$$

$$dz_t = h_t \phi_{t-t_s} dt + dw_t \quad , \qquad t\geq 0$$

Equation (2.3d.1) now becomes

$$d\hat{\phi}_{t-t_s} = (\rho_t E\{\phi_0\}(1-\hat{\psi}_{t|t}) + E\{\psi_{t-}\alpha_{t-t_s}\phi_{t-t_s}|Z_t\})dt$$
$$+ h_t E\{(\phi_{t-t_s} - \hat{\phi}_{t-t_s})^2 | Z_t\} d\nu_t \quad , \tag{2.3d.2}$$

where here

$$d\nu_t = dz_t - h_t \hat{\phi}_{t-t_s} dt \tag{2.3d.3}$$

Equation (2.3d.2) shows that our conjecture is not quite correct yet for the following reasons.

Firstly, the term $E\{\psi_{t-}\alpha_{t-t_s}\phi_{t-t_s}|Z_t\}$, although linear in ϕ_{t-t_s}, involves the random gain $\psi_{t-}\alpha_{t-t_s}$, the randomness being due to t_s. To compute this term on-line requires the joint a posteriori probabilities $p_{\phi_{t-t_s}}(\phi|Z_t,t_s=\tau)d\phi P(\tau<t_s\leq\tau+d\tau|Z_t)$, or an infinite system of stochastic differential equations. Either method leads to an infinite dimensional implementation. The only case in which the computation of this term is finite dimensional is when $\alpha_t=\alpha$, a constant, in which case $E\{\psi_{t-}\alpha_{t-t_s}\phi_{t-t_s}\big|Z_t\} = \alpha E\{\psi_{t-}\phi_{t-t_s}\big|Z_t\}$ and we can easily show that

$$E\{\psi_{t-}\phi_{t-t_s}\big|Z_t\} = \hat{\phi}_{t-t_s} \tag{2.3d.4}$$

Equation (2.3d.4) is easily verified in Appendix 3.

Secondly, consider the term $E\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})^2\big|Z_t\}$. To compute this term on-line requires an infinite system of stochastic differential equations but this approach is not very interesting. The alternative way using the multiple-model approach turns out to be very appealing in this case although it is still infinite dimensional. We can write

$$\begin{aligned} E\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})^2\big|Z_t\} &= \int_{\tau\leq t} E\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})^2\big|Z_t,t_s=\tau\}P(\tau<t_s\leq\tau+d\tau\big|Z_t) \\ &= \int_{\tau\leq t} \sigma_\tau(t)P(\tau<t_s\leq\tau+d\tau\big|Z_t) \end{aligned} \tag{2.3d.5}$$

where the covariance $\sigma_\tau(t)$, for each $\tau\leq t$, can be precomputed by solving a Riccati equation. (See Section 2.3b, equations (2.3b.10) to (2.3b.12)). Since $\sigma_\tau(t)$ is precomputable, the problem reduces to the on-line computation of the probabilities $P(\tau<t_s\leq\tau+d\tau|Z_t)$ which is an infinite dimensional problem. We are interested in finding cases in which the computation in equation (2.3d.5) is finite dimensional. First recall from Section 2.3b that when the observation gain

$$h_t = h = \text{constant} \tag{2.3d.6}$$

we have

$$\sigma_\tau(t) = \sigma_0(t-\tau), \qquad t\geq\tau \tag{2.3d.7}$$

so that we only need to solve the Riccati equation for $\sigma_0(t)$ in order to compute $\sigma_\tau(t)$ for all τ. In this case, equation (2.3d.5) becomes

$$E\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})^2\Big|Z_t\} = \int_{\tau\leq t}\sigma_0(t-\tau)P(\tau<t_s\leq\tau+d\tau|Z_t) \tag{2.3d.8}$$

which resembles a convolution operation. This is illustrated in Figure 4. Next, note that in the time invariant case, since we have a scalar constant linear system with constant linear observations, the system is completely controllable and observable and therefore the Riccati equation (2.3b.14) for computing $\sigma_0(t)$ must reach a steady state as $t\to\infty$, i.e.,

$$\sigma_0(t) \to \sigma = \text{constant as } t\to\infty$$

where σ is given by

$$2\alpha\sigma + \underline{\gamma}'\underline{\gamma} - h^2\sigma^2 = 0 \tag{2.3d.9}$$

or

$$\sigma = \frac{\alpha}{h^2} + \sqrt{\left(\frac{\alpha}{h^2}\right)^2 + \underline{\gamma}'\underline{\gamma}/h^2} \tag{2.3d.10}$$

In this case, suppose we know in addition that

$$t_1 \leq t_s \leq t_2$$

and we are in the region $t >> t_2$. Then equation (2.3d.8) becomes

$$E\{(\phi_{t-t_s} - \hat{\phi}_{t-t_s})^2 \big| Z_t\} = \int_{t_1 \leq \tau \leq t_2} \sigma_0(t-\tau) P(\tau < t_s \leq \tau + d\tau \big| Z_t)$$

$$= \sigma \int_{t_1 \leq \tau \leq t_2} P(\tau < t_s \leq \tau + d\tau \big| Z_t)$$

$$= \sigma \tag{2.3d.11}$$

Thus, we have found one case in which $E\{(\phi_{t-t_s} - \hat{\phi}_{t-t_s})^2 \big| Z_t\}$ can be computed with a finite dimensional operation. Another case in which we expect finite dimensional computations for this term is when

$$\sigma_0(0) = \sigma \tag{2.3d.12}$$

i.e., the Riccati equation (2.3b.14) for computing $\sigma_0(t)$ starts with the steady state value σ. In this case, it is well known that

$$\sigma_0(t) = \sigma \quad , \qquad t \geq 0 \tag{2.3d.13}$$

and hence we have

$$E\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})^2\Big|Z_t\} = \sigma \int_{\tau\leq t} P(\tau<t_s\leq\tau+d\tau\Big|Z_t)$$

$$= \sigma\ \hat{\psi}_{t|t} \tag{2.3d.14}$$

Thus, the on-line computation of the covariance reduces to that of $\hat{\psi}_{t|t}$, which from Theorem 2.6, is given by

$$\begin{aligned} d\hat{\psi}_{t|t} &= \rho_t(1-\hat{\psi}_{t|t})dt \\ &\quad +[E\{\psi_t h(\phi_{t-t_s},t)\Big|Z_t\} - \hat{\psi}_{t|t}\hat{h}(\phi_{t-t_s},t)]d\nu_t \quad , \\ \hat{\psi}_{0|0} &=0 \end{aligned} \tag{2.3d.15}$$

In the linear case, this reduces to

$$\begin{aligned} d\hat{\psi}_{t|t} &= \rho_t(1-\hat{\psi}_{t|t})dt \\ &\quad + h_t\hat{\phi}_{t-t_s}(1-\hat{\psi}_{t|t})d\nu_t \end{aligned} \tag{2.3d.16}$$

and we see that $\hat{\psi}_{t|t}$ requires finite dimensional on-line computations provided we have $\hat{\phi}_{t-t_s}$.

Finally, coming to the first term $\rho_t E\{\phi_0\}(1-\hat{\psi}_{t|t})$ back in equation (2.3d.2), the only on-line computation to be done is in computing $\hat{\psi}_{t|t}$ which we have just discussed above.

Summarizing our discussion, we conclude that we have the following cases in which the representation for $\hat{\phi}_{t-t_s}$ given in Theorem 2.5 admits a finite dimensional implementation.

(i) We have a linear time-invariant signal and observation model:

$$\left.\begin{aligned} d\phi_t &= \alpha\phi_t dt + \underline{\gamma}' d\underline{\eta}_t , \qquad t>0 \\ \phi_0 &= \text{Gaussian random variable,} \\ \phi_t &= 0 , \qquad t<0 \end{aligned}\right\} \tag{2.3d.17}$$

$$dz_t = h\phi_{t-t_s} dt + dw_t \tag{2.3d.18}$$

Suppose that

$$\begin{aligned} \sigma_0 &= E\{(\phi_0 - E\{\phi_0\})^2\} \\ &= \sigma \end{aligned} \tag{2.3d.19}$$

where σ is the steady state value given in equation (2.3d.10). Then, $\hat{\phi}_{t-t_s}$ is generated by

$$\begin{aligned} d\hat{\phi}_{t-t_s} &= (\rho_t E\{\phi_0\}(1-\hat{\psi}_{t|t}) + \alpha\hat{\phi}_{t-t_s})dt \\ &\qquad + h\sigma\hat{\psi}_{t|t} d\nu_t , \\ &\hat{\phi}_{t-t_s}\big|_{t=0} = 0 \end{aligned} \tag{2.3d.20}$$

where

$$d\nu_t = dz_t - h\hat{\phi}_{t-t_s} dt \tag{2.3d.21}$$

and $\hat{\psi}_{t|t}$ is generated by

$$d\hat{\psi}_{t|t} = \rho_t(1-\hat{\psi}_{t|t})dt + h\hat{\phi}_{t-t_s}(1-\hat{\psi}_{t|t})d\nu_t \quad ,$$
$$\hat{\psi}_{0|0}=0 \tag{2.3d.22}$$

Note that if, in addition, we have $t_1 \leq t_s \leq t_2$, then for $t \geq t_2$, we have $\hat{\psi}_{t|t}=1$ in which case

$$d\hat{\phi}_{t-t_s} = \alpha\hat{\phi}_{t-t_s}dt + h\sigma d\nu_t \tag{2.3d.23}$$

This is just the steady state Kalman filter and the result is expected. Since we have linear time invariant signal and observation models and the covariance starts from the steady state, then for t greater than or equal to the largest possible value of t_s, the transients due to the unknown arrival time should have vanished.

(ii). We have the same signal and observation models in (i). Assume that

$$t_1 \leq t_s \leq t_2$$

Then, when $t >> t_2$, the estimate $\hat{\phi}_{t-t_s}$ is given by

$$d\hat{\phi}_{t-t_s} = \alpha\hat{\phi}_{t-t_s}dt + h\sigma d\nu_t \tag{2.3d.24}$$

which is again the steady state Kalman filter. This is expected since if $t_1 \leq t_s \leq t_2$ and $t >> t_2$, then the actual value of t_s does not matter because the transients that occur when the signal first arrives should have disappeared.

Note that although we have carried out our work above in the scalar signal model case, the results can be extended without difficulty to the case of the vector model:

$$d\underline{y}_t = \underline{A}\,\underline{y}_t dt + \underline{B}\,d\underline{\eta}_t \quad , \quad t>0 \tag{2.3d.25}$$

$$\phi_t = \underline{c}'\underline{y}_t \ , \qquad t>0 \tag{2.3d.26}$$

where $\underline{y}_t$ and $\underline{c}$ are m-vectors, $\underline{A}$ and $\underline{B}$ are mxm and mxn matrices respectively. Of course, we have to assume that $(\underline{A},\underline{B})$ is controllable and $(\underline{A},\underline{c}')$ is observable. This model generates a richer class of signals ϕ than the scalar model (2.3d.17). However, we shall not do this extension here.

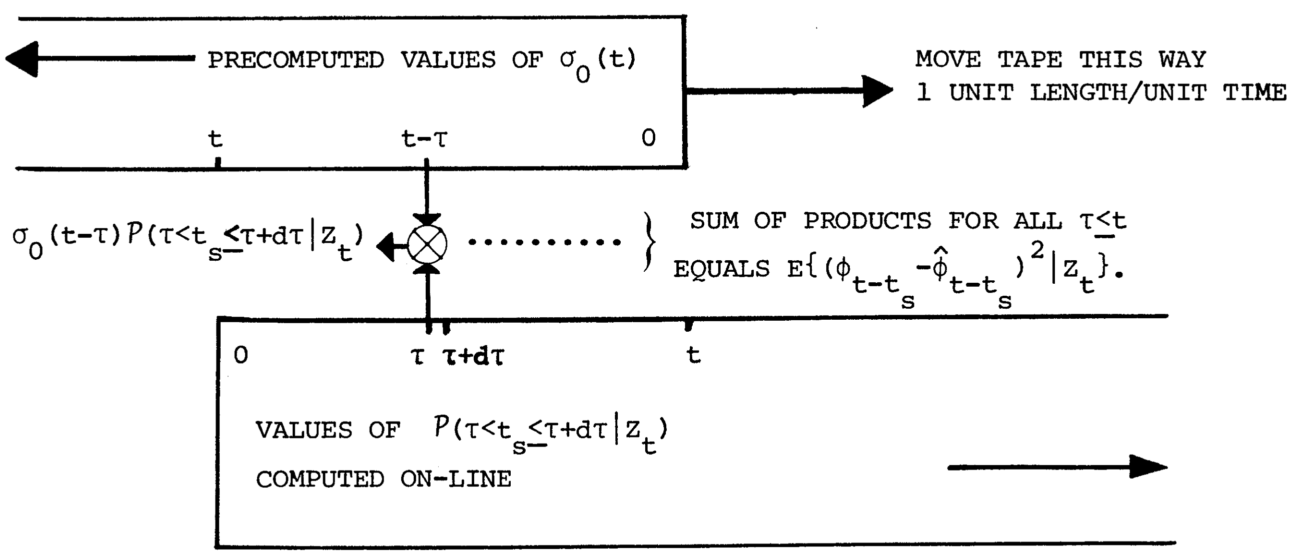

FIGURE 4: THE CONVOLUTION OPERATION OF EQUATION (2.3d.8)

Implementation of Signal and Delay Time Estimation Results in General

We discuss here the methods and the difficulties involved in the implementation of the signal and delay time estimation results in the general nonlinear case.

Consider first the representation result in Theorem 2.5 for the signal estimate $\hat{\phi}_{t-t_s}$:

$$d\hat{\phi}_{t-t_s} = (\rho_t E\{\phi_0\}(1-\hat{\psi}_{t|t}) + E\{\psi_{t-}\alpha(\phi_{t-t_s},t-t_s)\Big|Z_t\})dt$$
$$+ E\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})(h(\phi_{t-t_s},t)-\hat{h}(\phi_{t-t_s},t))\Big|Z_t\}d\nu_t \qquad (2.3d.27)$$

The first term on the right requires the on-line computation of $\hat{\psi}_{t|t}$ which, as we will see later, requires an infinite dimensional implementation in the general nonlinear case. To compute the second and the third terms on-line requires either an infinite system of stochastic differential equations or carrying along the joint a posteriori probabilities $P(\phi\leq\phi_{t-t_s}<\phi+d\phi,\ \tau<t_s\leq\tau+d\tau|Z_t)$. The manner in which an infinite system of stochastic differential equations arises is well known in nonlinear filtering theory [6], [38] and we shall not present the details any more. The on-line computation of the conditional probabilities $P(\phi\leq\phi_{t-t_s}<\phi+d\phi,\tau<t_s\leq\tau+d\tau|Z_t)$ has to be carried out as

$$P(\phi\leq\phi_{t-t_s}<\phi+d\phi,\ \tau<t_s\leq\tau+d\tau|Z_t)$$
$$= P_{\phi_{t-t_s}}(\phi|Z_t,t_s=\tau)d\phi\ P(\tau<t_s\leq\tau+d\tau|Z_t) \qquad (2.3d.28)$$

We will discuss later the implementation for computing the probabilities $P(\tau<t_s\leq\tau+d\tau|Z_t)$. To compute the density $p_{\phi_{t-t_s}}(\phi|Z_t,t_s=\tau)$ involves no new difficulties since there is no uncertainty in the arrival time given that $t_s=\tau$. We have the results

$$\tau>t: \quad p_{\phi_{t-t_s}}(\phi|Z_t,t_s=\tau) = \delta(\phi) \tag{2.3d.29}$$

since $\phi_t=0$, for $t<0$, by definition,

$$\tau=t: \quad p_{\phi_{t-t_s}}(\phi|Z_t,t_s=\tau) = p_{\phi_0}(\phi) \tag{2.3d.30}$$

which is the given a priori density of ϕ_0,

$$\tau<t: \quad dp = L(p)dt + (h-\hat{h})(dz_t-\hat{h}dt)p \tag{2.3d.31}$$

where

$$p \equiv p_{\phi_{t-t_s}}(\phi|Z_t,t_s=\tau) \tag{2.3d.32}$$

Equation (2.3d.31) is just the Kushner equation [38]. It is of course impossible to implement equation (2.3d.31) in the general nonlinear case. However, in the linear Gaussian case, the density p is Gaussian and so it is completely characterized by its mean and covariance which can be computed via equations (2.3b.9) and (2.3b.11). Equation (2.3d.28) calls for a point wise multiplication of the probabilities $p_{\phi_{t-t_s}}(\phi|Z_t,t_s=\tau)d\phi$ and $P(\tau<t_s\leq\tau+d\tau|Z_t)$ for different values of τ at each time t and therefore leads to an implementation essentially equivalent to the multiple-model approach of Section 2.3b. We have to compute the density $p_{\phi_{t-t_s}}(\phi|Z_t,t_s=\tau)$

for each possible value of $t_s=\tau$, thus giving rise to an infinite bank of filters. Again, we emphasize that these filters are non-implementable except in the linear Gaussian case.

Summarizing the above discussion, we conclude that the representation result (2.3d.27) for the signal estimate $\hat{\phi}_{t-t_s}$ is in general non-implementable because the on-line computation of each term on the right is infinite dimensional. However, the result is very useful because, as we have seen previously, in several linear time invariant cases, it does admit a finite dimensional optimal implementation.

The multiple-model solution to the signal estimation problem given in Section 2.3b is a conceptual implementation-oriented approach, giving rise to an infinite bank of filters. We have discussed this in Section 2.3b and will not go into any more details here. In Section 2.4, we will discuss it again when t_s takes on finitely many values.

Finally, we consider the implementation of the equations given in Theorem 2.6 for computing the a posteriori distribution $\hat{\psi}_{\tau|t}=P(t_s\leq\tau|Z_t)$ of t_s. These equations are reproduced here for convenience:

$$
\begin{aligned}
\tau=t:\quad d\hat{\psi}_{t|t} &= \rho_t(1-\hat{\psi}_{t|t})dt \\
&\quad + [E\{\psi_t h(\phi_{t-t_s},t)|Z_t\} - \hat{\psi}_{t|t}\hat{h}(\phi_{t-t_s},t)]d\nu_t\ , \\
\hat{\psi}_{0|0} &= 0
\end{aligned}
\tag{2.3d.33}
$$

$$\tau>t: \quad \hat{\psi}_{\tau|t} = 1- \quad (1-\hat{\psi}_{t|t})\frac{P(t_s \geq \tau)}{P(t_s \geq t)} \tag{2.3d.34}$$

$$\tau<t: \quad \hat{\psi}_{\tau|t} = \hat{\psi}_{\tau|\tau} + \int_{\tau}^{t} \Sigma(\tau,t')d\nu_{t'} \tag{2.3d.35}$$

where

$$\Sigma(\tau,t') = E\{\psi_\tau h(\phi_{t'-t_s},t')\Big|Z_{t'}\} - \hat{\psi}_{\tau|t'}\hat{h}(\phi_{t'-t_s},t') \tag{2.3d.36}$$

Note that in equation (2.3d.35), we only need to compute $\Sigma(\tau,t)$ at each time t. For each value of τ in equation (2.3d.35), $\hat{\psi}_{\tau|t}$ is computed recursively in time t as

$$d\hat{\psi}_{\tau|t} = \Sigma(\tau,t)d\nu_t \tag{2.3d.37}$$

starting with the initial condition $\hat{\psi}_{\tau|\tau}$ at time $t=\tau$. Thus, to implement equation (2.3d.35), we only need to compute $\Sigma(\tau,t)$ at each time t for all values of $\tau<t$.

The terms which we need to compute in order to implement equations (2.3d.33) to (2.3d.35) are $E\{\psi_t h(\phi_{t-t_s},t)|Z_t\}$, $\hat{h}(\phi_{t-t_s},t)$ and the first term $E\{\psi_\tau h(\phi_{t-t_s},t)|Z_t\}$ in $\Sigma(\tau,t)$. We have the following evaluation:

$$E\{\psi_\tau h(\phi_{t-t_s},t)|Z_t\}$$

$$= \int_{t'\leq\tau}\int_{\text{all}\,\phi} h(\phi,t)\,P(\phi<\phi_{t-t_s}\leq\phi+d\phi,t'<t_s\leq t'+dt'\Big|Z_t)$$

$$= \int_{t'\leq\tau} \int_{\text{all}\phi} h(\phi,t) P(\phi<\phi_{t-t_s}\leq\phi+d\phi \mid Z_t, t_s=t') \, P(t'<t_s\leq t'+dt' \mid Z_t)$$

$$= \int_{t'\leq\tau} E\{h(\phi_{t-t_s},t) \mid Z_t, t_s=t'\} \, P(t'<t_s\leq t'+dt' \mid Z_t) \tag{2.3d.38}$$

Setting $\tau=t$, we also have

$$E\{\psi_t h(\phi_{t-t_s},t) \mid Z_t\}$$

$$= \int_{t'\leq t} E\{h(\phi_{t-t_s},t) \mid Z_t, t_s=t'\} \, P(t'<t_s\leq t'+dt' \mid Z_t) \tag{2.3d.39}$$

Thus, both $E\{\psi_t h(\phi_{t-t_s},t) \mid Z_t\}$ and $E\{\psi_\tau h(\phi_{t-t_s},t) \mid Z_t\}$ can be computed via the multiple-model approach. If we also evaluate $\hat{h}(\phi_{t-t_s},t)$ as

$$\hat{h}(\phi_{t-t_s},t) = \int_{t'\leq t} E\{h(\phi_{t-t_s},t) \mid Z_t, t_s=t'\} \, P(t'<t_s\leq t'+dt' \mid Z_t)$$
$$+ h(0,t)(1-\hat{\psi}_{t|t}) \tag{2.3d.40}$$

then we immediately see that the whole set of equations for computing $P(t_s\leq\tau|Z_t)$, for all τ at each time t, can be implemented via the multiple-model approach which now involves an infinite bank of filters, each one for computing the estimate $E\{h(\phi_{t-t_s},t)|Z_t,t_s=t'\}$ and one for every value of $t_s=t'\leq t$. Again, this bank of filters grows with time t.

Since the estimate $E\{h(\phi_{t-t_s},t)\Big|Z_t,t_s=t'\}$, for each value of $t_s=t'<t$, is the primary quantity to be computed in the implementation of the delay time estimation results, we shall examine here how it is computed. Again, there are of course two ways to compute it. The first way is to carry along the density $P_{\phi_{t-t_s}}(\phi|Z_t,t_s=t')$ a method which we have discussed before (see equations (2.3d.29) to (2.3d.32)). The other way is to generate the stochastic differential equation representation for it. Given that $t_s=t'<t$, we know that

$$h(\phi_{t-t_s},t) = h(\phi_{t-t'},t) \tag{2.3d.41}$$

and an application of Ito's differential rule [40] gives us

$$\begin{aligned} dh(\phi_{t-t'},t) = {} & \left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial \phi}\,\alpha(\phi_{t-t'},t-t') + \frac{1}{2}\,\underline{\gamma}'(\phi_{t-t'},t-t')\underline{\gamma}(\phi_{t-t'},t-t')\,\frac{\partial^2 h}{\partial \phi^2}\right)dt \\ & + \frac{\partial h}{\partial \phi}\,\underline{\gamma}'(\phi_{t-t'},t-t')\,d\underline{\eta}_{t-t'} \end{aligned} \tag{2.3d.42}$$

Thus, the estimate

$$\hat{h}(\phi_{t-t'},t) = E\{h(\phi_{t-t_s},t)|Z_t,t_s=t'\} \tag{2.3d.43}$$

is given by

$$d\hat{h}(\phi_{t-t'},t) = E\left\{ \frac{\partial h}{\partial t} + \frac{\partial h}{\partial \phi}\alpha(\phi_{t-t'},t-t') + \frac{1}{2}\underline{\gamma}'(\phi_{t-t'},t-t')\underline{\gamma}(\phi_{t-t'},t-t')\frac{\partial^2 h}{\partial \phi^2} \middle| Z_t, t_s=t' \right\} dt$$

$$+ E\{(h(\phi_{t-t'},t) - \hat{h}(\phi_{t-t'},t))^2 | Z_t, t_s=t'\}.$$

$$\cdot (dz_t - \hat{h}(\phi_{t-t'},t)dt) \qquad (2.3d.44)$$

This is of course only a representation and we have to generate the stochastic differential equation for computing each term on the right hand side, ending up with an infinite system of equations. We shall not go any further into this problem here.

Note that in the linear Gaussian case, i.e., signal and observation models linear, initial signal value Gaussian, the basic quantity to be computed in the implementation of the delay time estimation results is the estimate $\hat{\phi}_{t-t'} = E\{\phi_{t-t_s} | Z_t, t_s=t'\}$. This estimate is readily computed by an implementable Kalman filter, as we have seen in Section 2.3b.

In the linear time invariant Gaussian case, we have seen earlier that if $\sigma_0=\sigma$, i.e., the initial covariance of the signal equals the steady state value, then $\hat{\psi}_{t|t}$ is generated by a finite dimensional filter. (See equation (2.3d.22)). Equation (2.3d.34) then shows that the computation of $\hat{\psi}_{\tau|t}$, for $\tau>t$, is also finite dimensional. However, even in this case, the computation of $\hat{\psi}_{\tau|t}$, $\tau<t$, is still not finite dimensional.

Combined Implementation of Signal Estimation and Delay Time Estimation Results

We have seen in the previous section that the multiple-model approach is the natural way to implement the equations for computing the a posteriori probability distribution of the delay time. The multiple-model approach is also one way of implementing the solution to the signal estimation problem. Thus, it appears to be possible to implement the solution to the entire problem of signal and delay time estimation via the multiple-model approach. Indeed, this is possible and we illustrate the overall implementation in Figure 5. The major component of the implementation is the growing infinite bank of filters, one for each value of $t_s = t' < t$, at each time t. Each filter generates the estimates $\hat{\phi}_{t-t'}$ and $\hat{h}(\phi_{t-t'},t)$. (Note that $\hat{h}(\phi_{t-t'},t)$ is used in computing $\hat{\phi}_{t-t'}$; see equation (2.3b.5)). In the linear Gaussian case, all the filters reduce to readily implementable Kalman filters.

Note the presence of the feedback loop around the box for computing the a posteriori probabilities $P(t' < t_s \leq t'+dt \mid Z_t)$. These probabilities together with the estimates $\hat{h}(\phi_{t-t'},t)$ are used to compute the updated a posteriori probabilities $P(t' < t_s \leq t'+dt' \mid Z_{t+dt})$ when the new observations dz_t are obtained. The new values of the a posteriori probabilities are fed back to be used in the next update.

The on-line computation of the a posteriori distribution of t_s, besides serving its role of providing on-line estimates of t_s, can also be viewed as a fine-tuning mechanism on the signal estimation algorithm. Because of the uncertainties in the delay time t_s, we do not know at each time t which point of the signal ϕ we are actually measuring and therefore such a fine-tuning based on updating our knowledge of t_s is necessary.

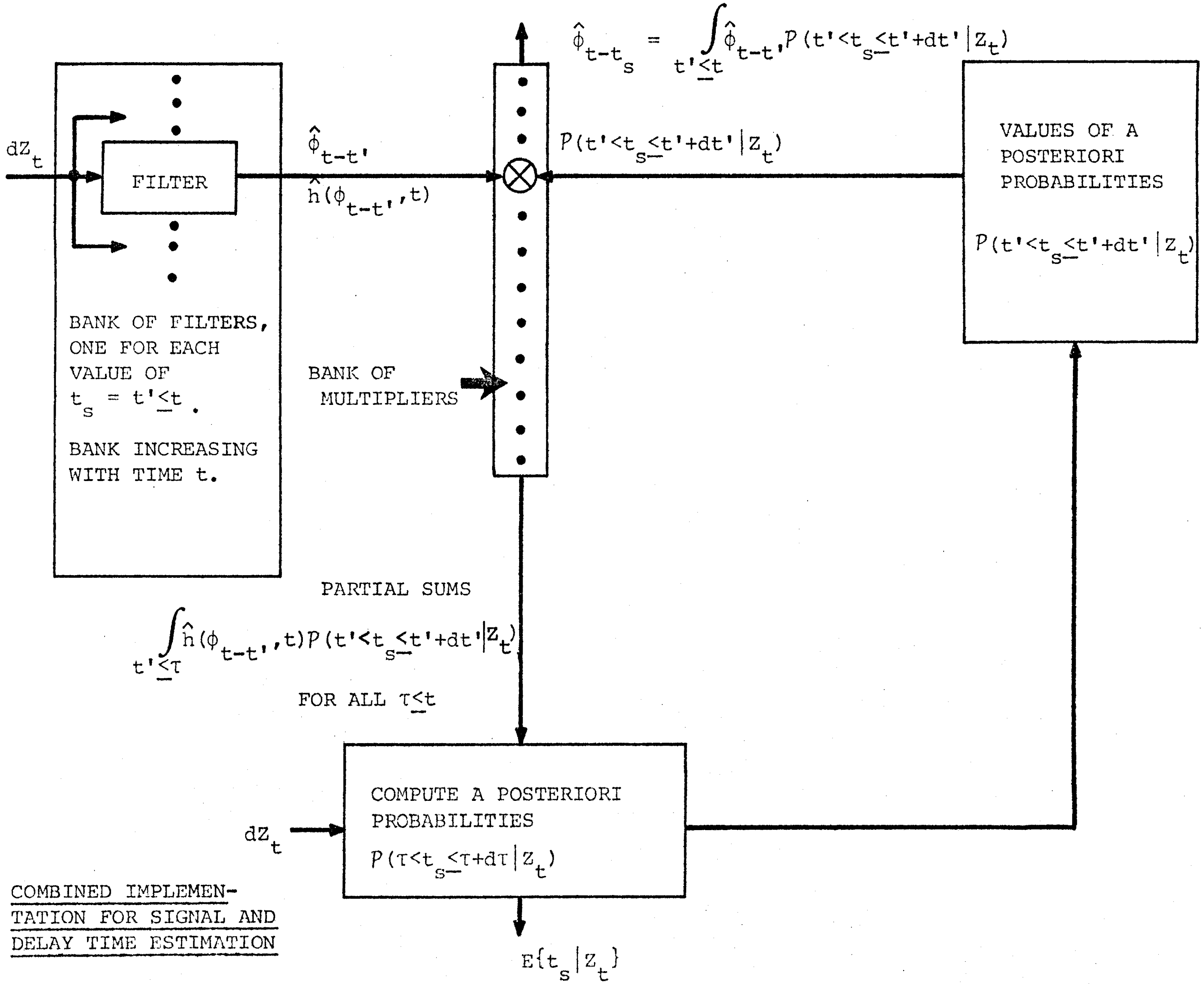

FIGURE 5: COMBINED IMPLEMENTATION FOR SIGNAL AND DELAY TIME ESTIMATION

2.3e Suboptimal Implementation via Assumed Density Approximation

We have seen that our signal and delay time estimation results are in general non-implementable and in the last section we found some special cases in which the representation result for the signal estimate is implementable. Those are the only cases in which our results admit a finite dimensional optimal implementation. In this section, we are interested in finding approximate approaches for deriving suboptimal finite dimensional implementations for our results. Such approximate implementations are important in practice because they provide the only means of actually implementing our solution. Many approaches for approximating optimal filters exist in the literature [41], [42]. We shall only make use of one of these approaches here.

The approach that we use here is based on an assumed density approximation to the a posteriori distribution of the delay time t_s. The idea is to assume that the conditional density of t_s given the observations Z_t has a known form which is characterized by a finite number of parameters and the problem of on-line computation of this conditional density then reduces to one of on-line determination of these parameters which hopefully is a finite dimensional problem. We have seen that the on-line computation of the conditional distribution of t_s is a crucial component of the solution to the entire problem of signal and delay time estimation and that it is an inherently infinite dimensional problem. Therefore, if we can find a finite dimensional approximate solution to this problem, then we can hopefully find more cases in which the entire signal and delay time estimation problem admits a finite dimensional solution.

In the following, we first consider the case in which the assumed conditional density of t_s is characterized by one unknown and nonrandom parameter and then extend it to the case of two unknown nonrandom parameters. Finally, we consider the case of two unknown parameters, one of which is nonrandom while the other is random. In all cases, we assume that the conditional density is exponential.

The One Unknown Nonrandom Parameter Case

We assume here that the a priori and a posteriori density of t_s is given by

$$p_{t_s}(\tau|Z_t) = \begin{cases} \beta e^{-\beta(\tau-T)} , & \tau \geq T \\ 0 , & \tau < T \end{cases} \qquad \text{for } t \geq 0 \tag{2.3e.1}$$

The parameter β is assumed to be unknown and nonrandom while T is assumed to be known. The value of T is the time before which the signal will not arrive with probability one. Thus, for values of t such that $0 \leq t \leq T$, the observations contain no measurements on the signal and hence on t_s and so

$$p_{t_s}(\tau|Z_t) = p_{t_s}(\tau) \quad , \qquad 0 \leq t \leq T \tag{2.3e.2}$$

We can see this easily since

$$\phi_{t-t_s} = 0, \qquad 0 \leq t \leq T \tag{2.3e.3}$$

and so

$$dz_t = h(0,t)dt + dw_t \ , \qquad 0\le t\le T \tag{2.3e.4}$$

which implies that

$$\sigma\{z_\tau,\ 0\le\tau\le t\} = \sigma\{w_\tau,\ 0\le\tau\le t\}, \qquad 0\le t\le T \tag{2.3e.5}$$

Now, since t_s and $\{w_\tau,\ 0\le\tau\}$ are independent, equation (2.3e.2) follows. We assume that the a priori density $P_{t_s}(\tau)$ is known, i.e.,

$$p_{t_s}(\tau) = \begin{cases} \beta_o e^{-\beta_o(\tau-T)} & , \quad \tau\ge T \\ 0 & , \quad \tau<T \end{cases} \tag{2.3e.6}$$

Thus, we know that

$$\beta = \beta_o \quad , \qquad 0\le t\le T \tag{2.3e.7}$$

For $t>T$, we want to determine β using the observations. We denote the value of β determined based on the observations Z_t by $\hat{\beta}_t$ and rewrite equation (2.3e.1) as

$$P_{t_s}(\tau|Z_t) = \begin{cases} \hat{\beta}_t e^{-\hat{\beta}_t(\tau-T)} & , \quad \tau\ge T \\ 0 & , \quad \tau<T \end{cases} \qquad \text{for} \quad t>T \tag{2.3e.8}$$

To generate $\hat{\beta}_t$, note that at each time $t>T$, we have

$$P(t_s \le t | Z_t) = \int_T^t \hat{\beta}_t e^{-\hat{\beta}_t(\tau-T)} d\tau$$

$$= 1-e^{-\hat{\beta}_t(t-T)} \tag{2.3e.9}$$

From Theorem 2.6., we know that

$$P(t_s \le t | Z_t) = \hat{\psi}_{t|t} \tag{2.3e.10}$$

can be generated, for t>T, by the filter

$$d\hat{\psi}_{t|t} = \rho_t(1-\hat{\psi}_{t|t})dt + [E\{\psi_t h(\phi_{t-t_s},t)|Z_t\} - \hat{\psi}_{t|t}\hat{h}(\phi_{t-t_s},t)]d\nu_t \tag{2.3e.11}$$

with initial condition

$$\hat{\psi}_{T|T} = 0 \tag{2.3e.12}$$

Note that ρ_t, $t \ge T$, is in this case defined in terms of the a priori density

$$\rho_t = p_{t_s}(t) \Big/ \int_t^\infty p_{t_s}(\tau) d\tau$$

$$= \beta_o e^{-\beta_o(t-T)} \Big/ \int_t^\infty \beta_o e^{-\beta_o(\tau-T)} d\tau$$

$$= \beta_o \tag{2.3e.13}$$

By equating equations (2.3e.9) and (2.3e.10), we have

$$1-e^{-\hat{\beta}_t(t-T)} = \hat{\psi}_{t|t} \tag{2.3e.14}$$

from which we get the equation for determining $\hat{\beta}_t$:

$$\hat{\beta}_t = \frac{1}{t-T} \ln \left(\frac{1}{1-\hat{\psi}_{t|t}}\right), \quad t>T \tag{2.3e.15}$$

Under the assumed exponential conditional density (2.3e.1), the conditional mean estimate of t_s is easily given by

$$E\{t_s|Z_t\} = T + \frac{1}{\hat{\beta}_t} = T + \frac{t-T}{\ln\left(\frac{1}{1-\hat{\psi}_{t|t}}\right)}, \quad t>T \tag{2.3e.16}$$

From the above development, we see that the conditional density of t_s can be computed on-line by computing $\hat{\beta}_t$ alone and the latter is derived from the estimate $\hat{\psi}_{t|t}$ which we have seen how to compute. The estimate $\hat{\psi}_{t|t}$ also gives the estimate $E\{t_s|Z_t\}$.

As we have discussed before, the filter (2.3e.11) for computing $\hat{\psi}_{t|t}$ requires an infinite dimensional implementation because of the terms $E\{\psi_t h(\phi_{t-t_s},t)|Z_t\}$ and $h(\phi_{t-t_s},t)$. However, in the case in which the observation model is linear, i.e.,

$$dz_t = h_t\phi_{t-t_s}dt + dw_t, \qquad t>0 \tag{2.3e.17}$$

then we have

$$E\{\psi_t h(\phi_{t-t_s},t)|Z_t\} = E\{\psi_t h_t \phi_{t-t_s}|Z_t\}$$

$$= h_t \hat{\phi}_{t-t_s} \qquad (2.3e.18)$$

so that the filter (2.3e.11) reduces to

$$d\hat{\psi}_{t|t} = \rho_t(1-\hat{\psi}_{t|t})dt + h_t\hat{\phi}_{t-t_s}(1-\hat{\psi}_{t|t})d\nu_t, \qquad t>T \qquad (2.3e.19)$$

which is finite dimensional provided that the signal estimate $\hat{\phi}_{t-t_s}$ can be computed with a finite dimensional filter. Referring back to the discussion in Section 2.3d, we conclude that the following case gives us a finite dimensional filter for computing $\hat{\phi}_{t-t_s}$. We have the linear signal model

$$d\phi_t = \alpha\phi_t dt + \underline{\gamma}_t' d\eta_t \ , \qquad t>0 \qquad (2.3e.20)$$

(note that it is only partly time-invariant) with the linear observation model

$$dz_t = h_t\phi_{t-t_s}dt + dw_t \ , \qquad t>0 \qquad (2.3e.21)$$

In addition, assume that the initial value ϕ_0 of the signal is Gaussian. Then, $\hat{\phi}_{t-t_s}$ is generated by the filter

$$d\hat{\phi}_{t-t_s} = (\rho_t(1-\hat{\psi}_{t|t})E\{\phi_0\} + \alpha\hat{\phi}_{t-t_s})dt + h_t E\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})^2|Z_t\}d\nu_t \ ,$$

$$t>T \qquad (2.3e.22)$$

The computation of the covariance $E\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})^2 \mid Z_t\}$ via the multiple-model approach is now finite dimensional because we have

$$E\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})^2 \mid Z_t\} = \int_{T\leq\tau\leq t} \sigma_\tau(t)\, P(\tau<t_s\leq\tau+d\tau \mid Z_t) \tag{2.3e.23}$$

and $\sigma_\tau(t)$ is precomputable while the computation of the probabilities $P(\tau<t_s\leq\tau+d\tau \mid Z_t)$ involves only computing the value of $\hat{\beta}_t$. Thus, the filter (2.3e.22) for $\hat{\phi}_{t-t_s}$ is finite dimensional implying that the filter (2.3e.19) for $\hat{\psi}_{t|t}$ is also finite dimensional. The overall implementation for signal and delay time estimation is illustrated for this case in Figure 6.

The finite dimensional implementation comes about mainly because the parameter of the assumed form of the conditional density $p_{t_s}(\tau \mid Z_t)$ can be computed on-line with a finite dimensional implementation. Note that previously we have discovered that in the linear time-invariant Gaussian case without the assumed exponential conditional density for t_s, the implementation is finite dimensional only if $t_1\leq t_s\leq t_2$ and if $t>>t_2$ in which case the solution $\sigma_0(t)$ of the Riccati equation reaches a steady state value σ or if $\sigma_0(0)$ is equal to the steady state value σ. Now with the assumed exponential conditional density of t_s, the implementation is finite dimensional whenever the initial signal value ϕ_0 is Gaussian and the signal and observation models are linear with only the first term α in the signal model being-time-invariant. The implementation is even simpler if the observation model is also time-invariant in addition to being linear, since, as we noted before, we now have $\sigma_\tau(t) = \sigma_0(t-\tau)$ so that we only need to solve one Riccati equation for $\sigma_0(t)$ and store these values instead of solving an infinite system of Riccati equations for $\sigma_\tau(t)$, one for each value of $t_s=\tau$, and storing all these values.

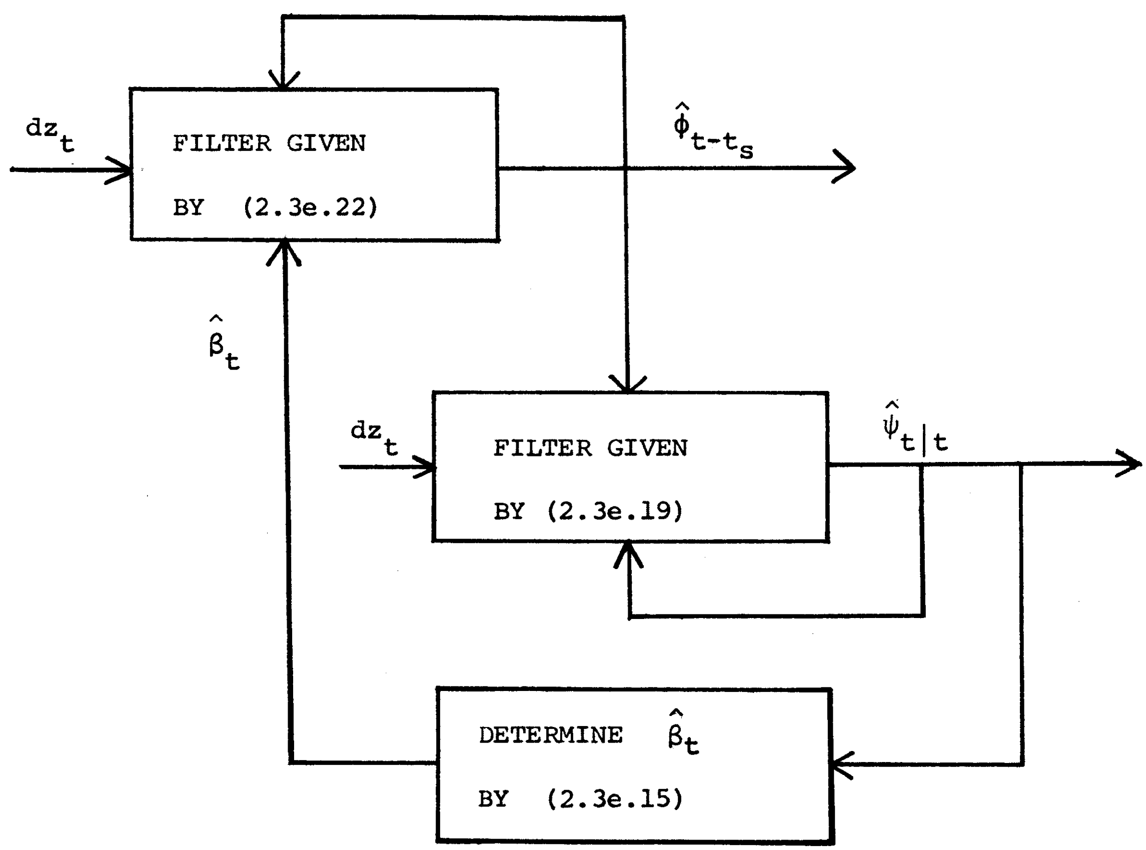

FIGURE 6: FINITE DIMENSIONAL IMPLEMENTATION FOR SIGNAL AND DELAY TIME ESTIMATION WITH ONE-PARAMETER ASSUMED EXPONENTIAL CONDITIONAL DENSITY FOR t_s

Our results above can be generalized without difficulty to the case of a vector signal model which produces a richer class of signals than the scalar model (2.3e.20). We shall not do this here.

Finally, note that equation (2.3e.23) for $E\{(\phi_{t-t_s} - \hat{\phi}_{t-t_s})^2 | Z_t\}$ need not be evaluated on-line. It can be evaluated off-line as a function of $\hat{\beta}_t$ and t and the results can be stored so that at each time t, when $\hat{\beta}_t$ is obtained, the value of $E\{(\phi_{t-t_s} - \hat{\phi}_{t-t_s})^2 | Z_t\}$ can be found from the stored values. We illustrate this briefly in the following example.

Example

Consider the signal model

$$\begin{aligned} d\phi_t &= -\phi_t dt + d\eta_t, \qquad t>0 \\ \phi_0 &= \text{known} \end{aligned} \tag{2.3e.24}$$

with the observation model

$$dz_t = \phi_{t-t_s} dt + dw_t \quad , \qquad t>0 \tag{2.3e.25}$$

In this case, $\sigma_0(t)$ is given by the Riccati equation

$$\frac{d\sigma_0(t)}{dt} = -2\sigma_0(t) - \sigma_0^2(t) + 1 \tag{2.3e.26}$$

Since ϕ_0 is known, we have

$$\sigma_0(0) = \sigma_0 = 0 \tag{2.3e.27}$$

The solution for $\sigma_0(t)$ is then given by

$$\sigma_0(t) = (\sqrt{2}-1) - \frac{1}{\left(1 + \frac{3}{4}\sqrt{2}\right) e^{2\sqrt{2}t} + \frac{1}{2\sqrt{2}}} \tag{2.3e.28}$$

Equation (2.3e.23) now becomes

$$E\{(\phi_{t-t_s} - \hat{\phi}_{t-t_s})^2 \big| Z_t\} = (\sqrt{2}-1) \int_T^t \hat{\beta}_t \, e^{-\hat{\beta}_t(\tau-T)} d\tau$$

$$- \int_T^t \frac{\hat{\beta}_t \, e^{-\hat{\beta}_t(\tau-T)}}{\left(1 + \frac{3}{4}\sqrt{2}\right) e^{2\sqrt{2}(t-\tau)} + \frac{1}{2\sqrt{2}}} d\tau \tag{2.3e.29}$$

The first integral above can be evaluated analytically while the second integral cannot be evaluated analytically unless $\hat{\beta}_t$ is an integer. However, we can in principle **evaluate** it numerically for all values of $\hat{\beta}_t$ and all t, creating a table of values which gives $E\{(\phi_{t-t_s} - \hat{\phi}_{t-t_s})^2 \big| Z_t\}$ as a function of $\hat{\beta}_t$ and t, i.e.,

$$E\{(\phi_{t-t_s} - \hat{\phi}_{t-t_s})^2 \big| Z_t\} = f(\hat{\beta}_t, t) \tag{2.3e.30}$$

At any time t, once the estimate $\hat{\beta}_t$ is obtained, the table immediately gives us $E\{(\phi_{t-t_s} - \hat{\phi}_{t-t_s})^2 \big| Z_t\}$.

Alternatively, the second integral above can be evaluated numerically for all values of $\hat{\beta}_t$ and t. Then for each t, the values of the integral can be approximated by a polynomial in $\hat{\beta}_t$. Having obtained the value of $\hat{\beta}_t$ at each time t, the value of $E\{(\phi_{t-t_s} - \hat{\phi}_{t-t_s})^2 | Z_t\}$ can thus be readily evaluated.

The Case of Two Unknown Nonrandom Parameters

In the previous case, we have assumed that T is known. Since the assumed conditional density of t_s is exponential, the most likely value of t_s is always T, i.e., T is the value of t_s that occurs with maximum probability although the minimum mean square error estimate of t_s conditioned on the observations Z_t is greater than T, according to equation (2.3e.16). It will be more interesting if the most likely value of t_s is allowed to vary as we obtain new observations. This is what we will do in this section. All we assume we know about T is that $T \geq t_0$, where t_0 is known and $t_0 \geq 0$. The work of the previous section carries over easily.

For values of t such that $0 \leq t \leq t_0$, the observations contain no measurements on the signal and hence on t_s. Then,

$$P_{t_s}(\tau | Z_t) = P_{t_s}(\tau)$$

$$= \begin{cases} \beta_0 e^{-\beta_0(\tau - T_0)} & , \quad \tau \geq T_0 \\ 0 & , \quad \tau < T_0 \end{cases} \qquad \text{for} \quad 0 \leq t \leq t_0 \tag{2.3e.31}$$

The a priori density $P_{t_s}(\tau)$ is assumed to be known and thus we know that

$$\beta = \beta_0 \quad , \qquad 0 \leq t \leq t_0 \tag{2.3e.32}$$

$$T = T_0 \quad , \qquad\qquad 0 \leq t \leq t_0 \tag{2.3e.33}$$

For $t>t_0$, we want to determine β and T using the observations. Denote the values of β and T determined based on the observations Z_t by $\hat{\beta}_t$ and $\hat{T}_t$ respectively. Following the method in the previous section of equating the equation

$$\mathcal{P}(t_s \leq t | Z_t) = 1 - e^{-\hat{\beta}_t (t - \hat{T}_t)} \tag{2.3e.34}$$

with the equation

$$\mathcal{P}(t_s \leq t | Z_t) = \hat{\psi}_{t|t} \tag{2.3e.35}$$

we get one equation for $\hat{\beta}_t$ and $\hat{T}_t$:

$$\hat{\beta}_t (t - \hat{T}_t) = \ln \left(\frac{1}{1 - \hat{\psi}_{t|t}} \right) \tag{2.3e.36}$$

To obtain another equation between $\hat{\beta}_t$ and $\hat{T}_t$ we note that the prediction estimate $\hat{\psi}_{t+\Delta|t}$, for fixed $\Delta>0$, is computable in terms of the estimate $\hat{\psi}_{t|t}$:

$$\hat{\psi}_{t+\Delta|t} = 1 - \left(1 - \hat{\psi}_{t|t}\right) \frac{\mathcal{P}(t_s \geq t+\Delta)}{\mathcal{P}(t_s \geq t)} \tag{2.3e.37}$$

(See Theorem 2.6).

Now, since

$$P(t_s \leq t+\Delta \,|\, Z_t) = 1 - e^{-\hat{\beta}_t (t+\Delta-\hat{T}_t)} \tag{2.3e.38}$$

and

$$\hat{\psi}_{t+\Delta|t} = P(t_s \leq t+\Delta \,|\, Z_t) \tag{2.3e.39}$$

we have

$$1 - e^{-\hat{\beta}_t (t+\Delta-\hat{T}_t)} = \hat{\psi}_{t+\Delta|t} \tag{2.3e.40}$$

or

$$\hat{\beta}_t (t+\Delta - \hat{T}_t) = \ln\left(\frac{1}{1-\hat{\psi}_{t+\Delta|t}}\right) \tag{2.3e.41}$$

Solving equations (2.3e.36) and (2.3e.41), we get

$$\hat{T}_t = t - \frac{f(t)}{1-f(t)}\,\Delta \tag{2.3e.42}$$

where

$$f(t) = \ln\left(\frac{1}{1-\hat{\psi}_{t|t}}\right) \Bigg/ \ln\left(\frac{1}{1-\hat{\psi}_{t+\Delta|t}}\right) \tag{2.3e.43}$$

Equation (2.3e.42) can be simplified as follows.

$$\frac{f(t)}{1-f(t)} = \ln\left(\frac{1}{1-\hat{\psi}_{t|t}}\right) \Bigg/ \ln\left(\frac{1-\hat{\psi}_{t|t}}{1-\hat{\psi}_{t+\Delta|t}}\right) \tag{2.3e.44}$$

But from equation (2.3e.37), we have

$$\frac{1-\hat{\psi}_{t|t}}{1-\hat{\psi}_{t+\Delta|t}} = \frac{P(t_s \geq t)}{P(t_s \geq t+\Delta)}$$

$$= \frac{e^{-\beta_0(t-T_0)}}{e^{-\beta_0(t+\Delta-T_0)}}$$

$$= e^{\beta_0 \Delta} \tag{2.3e.45}$$

Thus,

$$\hat{T}_t = t - \frac{1}{\beta_0} \ln\left(\frac{1}{1-\hat{\psi}_{t|t}}\right), \quad t>t_0 \tag{2.3e.46}$$

and using this in equation (2.3e.36) gives

$$\hat{\beta}_t = \beta_0, \qquad t>t_0 \tag{2.3e.47}$$

The work in this section is exactly the same as in the previous one parameter case except that $\hat{\psi}_{t|t}$ is used to determine two values, $\hat{\beta}_t$ and $\hat{T}_t$. We therefore arrive at the same case of finite dimensional suboptimal implementation as in the last section. (See equations (2.3e.17) to (2.3e.22)). The only difference is that the filters now start from time $t = t_0$ and the quantity ρ_t is now given by

$$\rho_t = \begin{cases} \beta_0, & t \geq T_0 \\ 0, & t<T_0 \end{cases} \tag{2.3e.48}$$

The Case of One Random and One Nonrandom Parameter

The work that we are trying to do in this section is similar to that in the previous case of two nonrandom parameters. The only difference is that in this case, the most likely value T of t_s is assumed to be random. The results and the approach, however, are similar to the previous two cases.

The a priori and a posteriori density of t_s is again assumed to be exponential, given by

$$P_{t_s}(\tau|Z_t) = \begin{cases} \beta e^{-\beta(\tau-T)} & , \quad \tau \geq T \\ 0 & , \quad \tau < T \end{cases} \qquad \text{for } t \geq 0 \qquad (2.3e.49)$$

The parameter β is again assumed to be unknown and nonrandom while T is assumed to be unknown and random. The a priori and a posteriori density of T is also assumed to be exponential, given by

$$P_T(t'|Z_t) = \begin{cases} \theta e^{-\theta(t'-t_0)} & , \quad t' \geq t_0 \\ 0 & , \quad t' < t_0 \end{cases} \qquad \text{for } t \geq 0 \qquad (2.3e.50)$$

Here, θ is unknown and nonrandom while t_0 is known and is the smallest possible value of T. Since T is random, equation (2.3e.49) is to be interpreted as

$$P_{t_s}(\tau|Z_t, T=t') = \begin{cases} \beta e^{-\beta(\tau-t')} & , \quad \tau \geq t' \\ 0 & , \quad \tau < t' \end{cases} \qquad \text{for } t \geq 0 \qquad (2.3e.51)$$

Thus, we have

$$P_{t_s}(\tau|Z_t) = \int_{t_0}^{\infty} P_{t_s}(\tau|Z_t, T=t')P_T(t'|Z_t)dt'$$

$$= \int_{t_0}^{\infty} \beta e^{-\beta(\tau-t')} u_{-1}(\tau-t')\theta e^{-\theta(t'-t_0)} u_{-1}(t'-t_0)dt'$$

$$= \beta\theta e^{-\beta\tau+\theta t_0} \int_{t_0}^{\tau} e^{(\beta-\theta)t'}dt' \; u_{-1}(\tau-t_0)$$

$$= \frac{\beta\theta}{\beta-\theta}\left(e^{-\theta(\tau-t_0)} - e^{-\beta(\tau-t_0)}\right) u_{-1}(\tau-t_0) \qquad (2.3e.52)$$

or

$$P_{t_s}(\tau|Z_t) = \begin{cases} \frac{\beta\theta}{\beta-\theta}\left(e^{-\theta(\tau-t_0)} - e^{-\beta(\tau-t_0)}\right), & \tau\geq t_0 \\ 0 & , \tau<t_0 \end{cases} \qquad \text{for } t\geq 0 \qquad (2.3e.53)$$

In the above, $u_{-1}(t')$ is the unit step function:

$$u_{-1}(t') = \begin{cases} 1 & , t'\geq 0 \\ 0 & , t<0 \end{cases} \qquad (2.3e.54)$$

In our present setting, since $t_s \geq T \geq t_0$, then for $t \leq t_0$, the signal ϕ_{t-t_s} will not arrive with probability one. Thus, as explained in the one parameter case, for $t \leq t_0$,

$$P_{t_s}(\tau|Z_t) = P_{t_s}(\tau) = \begin{cases} \dfrac{\beta_0\theta_0}{\beta_0-\theta_0}\left(e^{-\theta_0(\tau-t_0)} - e^{-\beta_0(\tau-t_0)}\right), & \tau\geq t_0 \\ 0 & \tau<t_0 \end{cases} \tag{2.3e.55}$$

and

$$P_T(t'|Z_t) = P_T(t') = \begin{cases} \theta_0 e^{-\theta_0(t'-t_0)} & , \quad t'\geq t_0 \\ 0 & , \quad t'<t_0 \end{cases} \tag{2.3e.56}$$

The a priori densities $P_{t_s}(\tau)$ and $P_T(t')$ are assumed known, i.e., β_0 and θ_0 are known. Thus, we know that

$$\beta = \beta_0 \tag{2.3e.57}$$

and

$$\theta = \theta_0 \tag{2.3e.58}$$

for $t\leq t_0$. For $t>t_0$, we want to determine β and θ using the observations Z_t. Denote the values of β and θ determined using the observations Z_t by $\hat{\beta}_t$ and $\hat{\theta}_t$. From equation (2.3e.53), we easily get, for each $t>t_0$,

$$P(t_s\leq t|Z_t) = \frac{\hat{\beta}_t\hat{\theta}_t}{\hat{\beta}_t-\hat{\theta}_t}\int_{t_0}^{t}[e^{-\hat{\theta}_t(\tau-t_0)} - e^{-\hat{\beta}_t(\tau-t_0)}]d\tau$$

$$= 1 + \frac{1}{\hat{\beta}_t-\hat{\theta}_t}\left[\hat{\theta}_t e^{-\hat{\beta}_t(t-t_0)} - \hat{\beta}_t e^{-\hat{\theta}_t(t-t_0)}\right] \tag{2.3e.59}$$

and from equation (2.3e.50), for $t>t_0$,

$$P(T\leq t|Z_t) = \int_{t_0}^{t} \hat{\theta}_t e^{-\hat{\theta}_t(t'-t_0)} dt'$$

$$= 1 - e^{-\hat{\theta}_t(t-t_0)} \qquad (2.3e.60)$$

However, we know that

$$P(t_s\leq t|Z_t) = \hat{\psi}_{t|t} \qquad (2.3e.61)$$

and similarly, defining the process π_t such that

$$\pi_t = \begin{cases} 1 , & t\geq T \\ 0 , & t<T \end{cases} \qquad (2.3e.62)$$

we have

$$P(T\leq t|Z_t) = E\{\pi_t|Z_t\} = \hat{\pi}_{t|t} \qquad (2.3e.63)$$

We shall shortly show how the estimates $\hat{\psi}_{t|t}$ and $\hat{\pi}_{t|t}$ can be generated in this case via a procedure similar to that in Theorem 2.6. Equating (2.3e.60) and (2.3e.63), we have

$$1 - e^{-\hat{\theta}_t(t-t_0)} = \hat{\pi}_{t|t} \qquad (2.3e.64)$$

or

$$\hat{\theta}_t = \frac{1}{t-T} \ln\left(\frac{1}{1-\hat{\pi}_{t|t}}\right) , \qquad t>t_0 \qquad (2.3e.65)$$

Similarly, equating (2.3e.59) and (2.3e.61), we have

$$1 + \frac{1}{\hat{\beta}_t - \hat{\theta}_t}\left[\hat{\theta}_t e^{-\hat{\beta}_t (t-t_0)} - \hat{\beta}_t e^{-\hat{\theta}_t (t-t_0)} \right] = \hat{\psi}_{t|t}\ , \quad t>t_0 \tag{2.3e.66}$$

We can, in principle, solve for $\hat{\beta}_t$ in terms of $\hat{\theta}_t$ and $\hat{\psi}_{t|t}$. A closed form expression for $\hat{\beta}_t$ in terms of $\hat{\theta}_t$ and $\hat{\psi}_{t|t}$ is not possible and we shall just leave equation (2.3e.66) as an implicit equation for $\hat{\beta}_t$.

Under the assumed density (2.3e.53), the conditional mean estimate of t_s is given by

$$E\{t_s | Z_t\} = t_0 + \frac{1}{\hat{\beta}_t} + \frac{1}{\hat{\theta}_t} \tag{2.3e.67}$$

while the most likely value, i.e., the maximum a posteriori probability estimate, of t_s is

$$\hat{t}^{\,map}_{s/t} = t_0 + \frac{\ln(\hat{\beta}_t / \hat{\theta}_t)}{\hat{\beta}_t - \hat{\theta}_t} \tag{2.3e.68}$$

Thus, both estimates of t_s can be computed on-line by computing $\hat{\beta}_t$ and $\hat{\theta}_t$.

Now, consider the computation of the estimates $\hat{\psi}_{t|t}$ and $\hat{\pi}_{t|t}$ on which everything else depends. We have to first enlarge the family of σ-fields $\{\mathcal{B}_t\}_{t\geq 0}$ defined by equation (2.3a.3), Section 2.3a, so that the process π_t is also adapted to $\{\mathcal{B}_t\}_{t\geq 0}$. Thus, we define $\mathcal{B}_t$ now as

$$\mathcal{B}_t = \mathcal{G}_t \vee \sigma\{\phi_{\tau-t_s}, 0\leq\tau\leq t\} \vee \sigma\{\{\omega:\ t_s(\omega)\leq\tau\} \mid \tau\leq t\}$$

$$\vee\ \sigma\{\{\omega:\ T(\omega)\leq\tau\} \mid \tau\leq t\} \tag{2.3e.69}$$

The process π_t is now clearly adapted to $\mathcal{B}_t$. By a proof similar to that in Appendix 1 for ψ_t, π_t can be shown to have the representation

$$d\pi_t = \bar{\rho}_t(1-\pi_t)dt + dn_t \tag{2.3e.70}$$

where

$$\bar{\rho}_t = P_T(t) \Big/ \int_t^\infty P_T(\tau)d\tau \tag{2.3e.71}$$

and n_t is a $\mathcal{B}_t$-martingale. Under the assumed a priori density $P_T(t)$ in (2.3e.56), we have

$$\bar{\rho}_t = \theta_0 \quad , \qquad t\geq t_0 \tag{2.3e.72}$$

From Theorem 2.1, we know that ψ_t has the representation

$$d\psi_t = \rho_t(1-\psi_t)dt + dm_t \tag{2.3e.73}$$

when t_s can take on any value $t\geq 0$. Now, we know in addition that $t_s \geq T$ where T is random. Under this new condition, we show in Appendix 4 that ψ_t now has the representation

$$d\psi_t = \rho_t(1-\psi_t)\pi_{t-}dt + \pi_{t-}dm_t \tag{2.3e.74}$$

With the assumed a priori density $P_{t_s}(\tau)$ in (2.3e.55), we have

$$\rho_t = \frac{e^{-\theta_0(t-t_0)} - e^{-\beta_0(t-t_0)}}{\frac{1}{\theta_0} e^{-\theta_0(t-t_0)} - \frac{1}{\beta_0} e^{-\beta_0(t-t_0)}} \quad , \qquad t \geq t_0 \tag{2.3e.75}$$

We can now derive the equations for the estimates $\hat{\psi}_{t|t}$ and $\hat{\pi}_{t|t}$, for $t \geq t_0$. By considering the filtering problem on the system (2.3e.70) and (2.3e.74) with the observations

$$dz_t = h(\phi_{t-t_s}, t)dt + dw_t \tag{2.3e.76}$$

and applying Theorem 2.4, we get

$$d\hat{\pi}_{t|t} = \bar{\rho}_t(1-\hat{\pi}_{t|t})dt + [E\{\pi_t h(\phi_{t-t_s}, t)|Z_t\} - \hat{\pi}_{t|t}\, \hat{h}(\phi_{t-t_s}, t)]d\nu_t ,$$

$$\hat{\pi}_{T|T} = 0 \tag{2.3e.77}$$

and

$$d\hat{\psi}_{t|t} = \rho_t(\hat{\pi}_{t-|t} - \widehat{\pi_{t-}\psi_t})dt + [E\{\psi_t h(\phi_{t-t_s}, t)|Z_t\} - \hat{\psi}_{t|t}\, \hat{h}(\phi_{t-t_s}, t)] \cdot$$

$$\cdot\, d\nu_t \quad , \qquad \hat{\psi}_{T|T} = 0 \tag{2.3e.78}$$

Note that $\langle n,w\rangle_t=0$ and $\langle m,w\rangle_t=0$ since n_t and m_t have jumps at T and t_s respectively. Equation (2.3e.78) can be further simplified since

$$\hat{\pi}_{t-|t} = \hat{\pi}_{t|t} \tag{2.3e.79}$$

which follows because, for each fixed t,

$$E\{\pi_t-\pi_{t-}|Z_t\} = P(T=t|Z_t)=0 \tag{2.3e.80}$$

For the same reason, we similarly have

$$\widehat{\pi_{t-}\psi_t} = \widehat{\pi_t\psi_t} = \hat{\psi}_{t|t} \tag{2.3e.81}$$

Thus, we have

$$d\hat{\psi}_{t|t} = \rho_t(\hat{\pi}_{t|t}-\hat{\psi}_{t|t})dt + [E\{\psi_t h(\phi_{t-t_s},t)|Z_t\} - \hat{\psi}_{t|t}\,\hat{h}(\phi_{t-t_s},t)]\cdot$$

$$\cdot\, d\nu_t \quad , \qquad \hat{\psi}_{T|T}=0 \tag{2.3e.82}$$

In general, there is no way to implement the filter (2.3e.77) because the first term $E\{\pi_t h(\phi_{t-t_s},t)|Z_t\}$ admits no implementation, not even a conceptual infinite dimensional multiple-model implementation, in the general nonlinear case. However, in the case when the observation model is linear, i.e.,

$$dz_t = h_t\phi_{t-t_s}dt + dw_t \tag{2.3e.83}$$

then, since we have

$$\pi_t\phi_{t-t_s} = \left\{\begin{array}{ll} \phi_{t-t_s} & , \quad t_s\le t \\ 0 & , \quad t_s>t \end{array}\right\} = \phi_{t-t_s} \tag{2.3e.84}$$

the filter (2.3e.77) reduces to

$$d\hat{\pi}_{t|t} = \bar{\rho}_t(1-\hat{\pi}_{t|t})dt + h_t\hat{\phi}_{t-t_s}(1-\hat{\pi}_{t|t})d\nu_t \,, \quad \hat{\pi}_{T|T}=0 \tag{2.3e.85}$$

while the filter (2.3e.82) reduces to

$$d\hat{\psi}_{t|t} = \rho_t(\hat{\pi}_{t|t}-\hat{\psi}_{t|t})dt + h_t\hat{\phi}_{t-t_s}(1-\hat{\psi}_{t|t})d\nu_t \quad , \ \hat{\psi}_{T|T}=0 \tag{2.3e.86}$$

Both the filters for $\hat{\pi}_{t|t}$ and $\hat{\psi}_{t|t}$ admit finite dimensional implementations in the linear observation model case if the filter for $\hat{\phi}_{t-t_s}$ is finite dimensional. This will be true for the signal model considered previously in the one-parameter case:

$$d\phi_t = \alpha\phi_t dt + \underline{\gamma}_t' d\underline{\eta}_t \,, \qquad t>0 \tag{2.3e.87}$$

$$\phi_0 = \text{Gaussian random variable}$$

For exactly the same reasons as in the one-parameter case, the filter for $\hat{\phi}_{t-t_s}$ is now finite dimensional:

$$d\hat{\phi}_{t-t_s} = (\rho_t(1-\hat{\psi}_{t|t})E\{\phi_0\} + \alpha\hat{\phi}_{t-t_s})dt + h_t E\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})^2|Z_t\}d\nu_t \tag{2.3e.88}$$

The overall implementation is illustrated in Figure 7.

2.3f Some Examples Involving Known Signals

In this section, we want to illustrate our signal and delay time estimation results via some examples in which the signal is known a priori. The

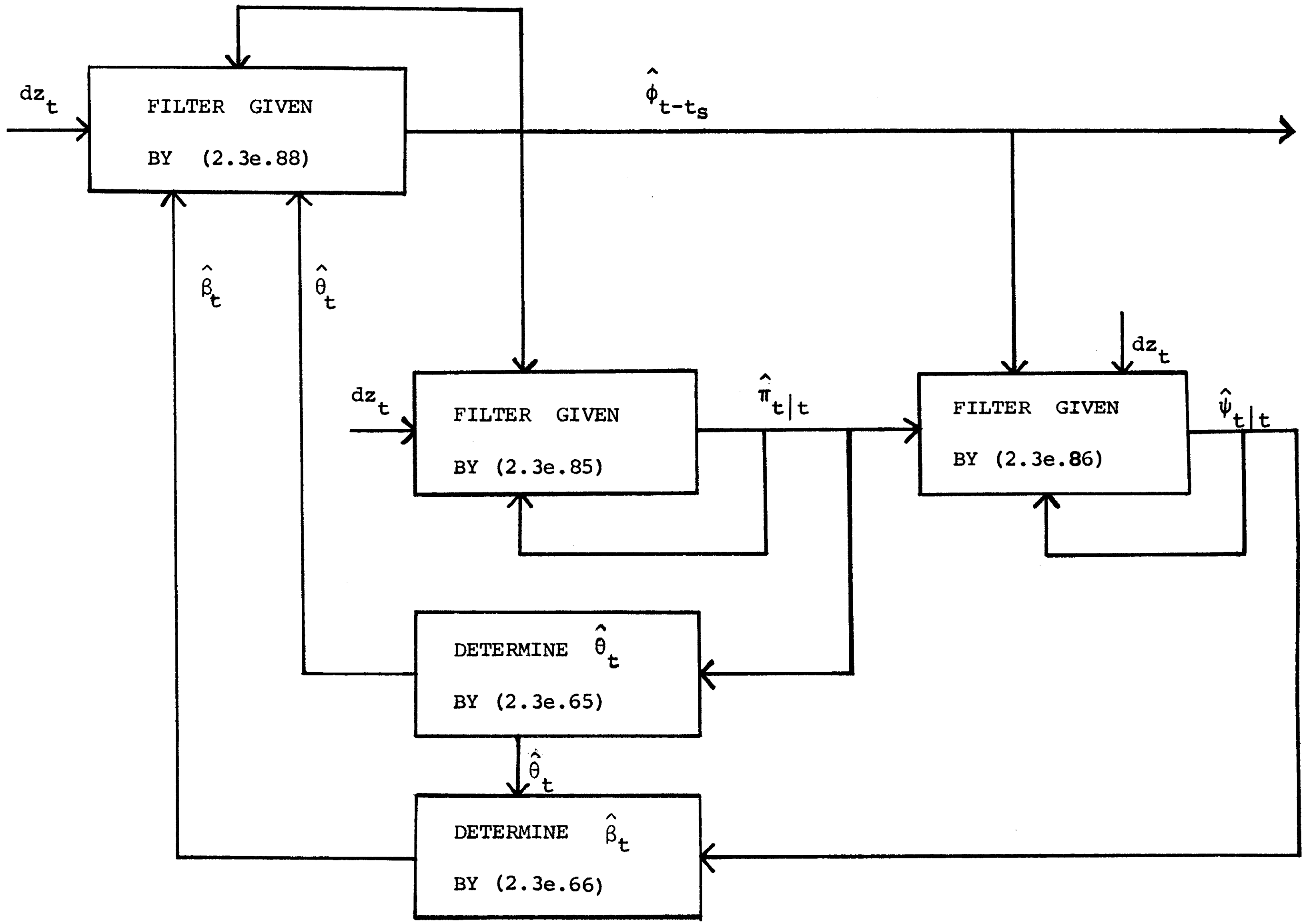

FIGURE 7: FINITE DIMENSIONAL IMPLEMENTATION FOR SIGNAL AND DELAY TIME ESTIMATION WITH TWO-PARAMETER ASSUMED EXPONENTIAL CONDITIONAL DENSITY FOR t_s

purpose of doing this is two fold. Firstly, we want to show that in the case of known signals, the solution to the optimal signal and delay time estimation problem can be implemented without a growing infinite bank of filters. Of course, the solution still requires an infinite amount of on-line computations but the implementation is so much simpler than an infinite bank of filters that it is appealing. Secondly, the case of known signals can be of great importance in practice. Suppose we are interested in inferring the properties of the time-invariant transmission field and the signal source is under our control. We can then send a known signal through the transmission field to the sensor and our delay time estimation results will enable us to infer the properties of the field. Other variants of this situation can also be mentioned. For instance, the velocity of the transmission field might be spatially constant and there is a reflector present in the transmission field. See Figure 8.

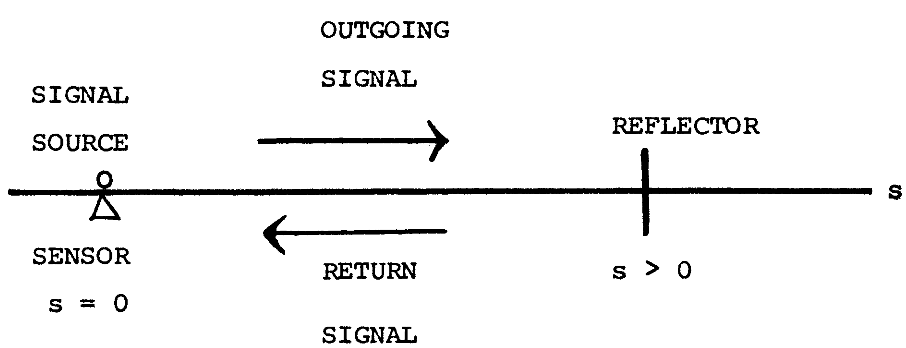

FIGURE 8: AN EXAMPLE INVOLVING ONE SOURCE, ONE SENSOR AND ONE REFLECTOR

By sending a known signal to the reflector and processing the return, we can estimate the travel time between the source and the reflector. If the distance of the reflector from the source is known, the delay time estimate enables us to estimate the velocity of the transmission field. Conversely, if the velocity of the transmission field is known, we can estimate the distance of the reflector from the source. This latter situation is very important in radar and sonar communication problems [16].

In what follows, we will analyze two examples, one involving an exponential signal and one involving a rectangular pulse.

<u>Example 1</u> <u>Exponential Signal</u>

We consider here the following signal model which is a special case of the model (2.2.1):

$$\left.\begin{aligned} d\phi_t &= -\alpha\phi_t dt \;, \qquad \alpha>0, \quad t>0 \\ \phi_0 &= \text{known}, \\ \phi_t &= 0 \qquad , \quad t<0 \end{aligned}\right\} \tag{2.3f.1}$$

The signal ϕ_t is then given by

$$\phi_t = \phi_0 e^{-\alpha t} u_{-1}(t) \tag{2.3f.2}$$

where $u_{-1}(t)$ is the unit step function. Since the signal model is time invariant, we will also assume a time invariant observation model:

$$dz_t = h\phi_{t-t_s} dt + dw_t \tag{2.3f.3}$$

By Theorem 2.5, the estimate $\hat{\phi}_{t-t_s}$ is given by

$$d\hat{\phi}_{t-t_s} = (\rho_t(1-\hat{\psi}_{t|t})\phi_0 - \alpha\hat{\phi}_{t-t_s})dt + hE\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})^2 \big| Z_t\}d\nu_t \ ,$$

$$\hat{\phi}_{t-t_s}\Big|_{t=0} = 0 \tag{2.3f.4}$$

where

$$d\nu_t = dz_t - h\hat{\phi}_{t-t_s}dt \tag{2.3f.5}$$

The second term in equation (2.3f.4) can be shown to be zero. We have

$$E\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})^2 \big| Z_t\} = \int_0^t \sigma_0(t-\tau)P(\tau<t_s\leq\tau+d\tau|Z_t) \tag{2.3f.6}$$

(See Section 2.3d, equation (2.3d.8)). The term $\sigma_0(t)$ is given by the Riccati equation

$$\frac{d\sigma_0(t)}{dt} = -2\alpha\sigma_0(t) - h^2\sigma_0^2(t) \ , \qquad t\geq 0 \tag{2.3f.7}$$

However, since ϕ_0 is known, the initial condition is

$$\sigma_0(0) = \sigma_0 = 0 \tag{2.3f.8}$$

The solution to equation (2.3f.7) with the initial condition (2.3f.8) is easily shown to be

$$\sigma_0(t) = 0 \,, \qquad t\geq 0 \tag{2.3f.9}$$

Thus,

$$E\{(\phi_{t-t_s}-\hat{\phi}_{t-t_s})^2 \big| Z_t\}=0, \qquad t\geq 0 \tag{2.3f.10}$$

and the filter (2.3f.4) reduces to

$$d\hat{\phi}_{t-t_s} = (\rho_t(1-\hat{\psi}_{t|t})\phi_0-\alpha\hat{\phi}_{t-t_s})dt \,, \quad \hat{\phi}_{t-t_s}\Big|_{t=0} =0 \tag{2.3f.11}$$

The on-line estimation of the signal ϕ_{t-t_s} therefore involves mainly the on-line computation of the estimate $\hat{\psi}_{t|t}$.

We turn now to the equations for computing the a posteriori distribution $\hat{\psi}_{\tau|t} = P(t_s\leq\tau|Z_t)$. (See Theorem 2.6). These are given by

$$\tau=t: \quad d\hat{\psi}_{t|t} = \rho_t(1-\hat{\psi}_{t|t})dt + h\hat{\phi}_{t-t_s}(1-\hat{\psi}_{t|t})d\nu_t \tag{2.3f.12}$$

$$\tau>t: \quad \hat{\psi}_{\tau|t} = 1 - (1-\hat{\psi}_{t|t}) \frac{P(t_s\geq\tau)}{P(t_s\geq t)} \tag{2.3f.13}$$

$$\tau<t: \quad \hat{\psi}_{\tau|t} = \hat{\psi}_{\tau|\tau} + \int_\tau^t \Sigma(\tau,t')d\nu_{t'} \tag{2.3f.14}$$

where

$$\Sigma(\tau,t) = hE\{\psi_\tau\phi_{t-t_s}\big|Z_t\} - h\hat{\psi}_{\tau|t}\,\hat{\phi}_{t-t_s} \tag{2.3f.15}$$

Note that equation (2.3f.14) is to be implemented as

$$d\hat{\psi}_{\tau|t} = \Sigma(\tau,t)d\nu_t \tag{2.3f.16}$$

i.e., one equation of the form (2.3f.16) for each $\tau<t$, starting with the initial condition $\hat{\psi}_{\tau|\tau}$ at time $t=\tau$. Thus, at each time t, we only need to compute $\Sigma(\tau,t)$ for all $\tau<t$.

In all the equations above for computing $\hat{\psi}_{\tau|t}$ the only equation that requires an infinite amount of on-line computations is (2.3f.14) because of the first term $E\{\psi_\tau\phi_{t-t_s}|Z_t\}$ in $\Sigma(\tau,t)$, since we have to compute it for all $\tau<t$. We have

$$\begin{aligned}
E\{\psi_\tau\phi_{t-t_s}\Big|Z_t\} &= \int_0^\tau E\{\phi_{t-t_s}\Big|Z_t\ ,t_s=t'\}P(t'<t_s\leq t'+dt'\Big|Z_t) \\
&= \int_0^\tau \phi_{t-t'}\ P(t'<t_s\leq t'+dt'\Big|Z_t) \\
&= \phi_0 e^{-\alpha t}\int_0^\tau e^{\alpha t'}P(t'<t_s\leq t'+dt'\Big|Z_t)
\end{aligned} \tag{2.3f.17}$$

The last equation shows us the kind of on-line computations that have to be carried out. At each time t, we have to evaluate the integral $\int_0^\tau e^{\alpha t'}\ P(t'<t_s\leq t'+dt'|Z_t)$ for all $\tau<t$. This can be done by performing

the integration forward in τ until $\tau=t$ and storing up all the intermediate results of the integration.

By performing the integration indicated above, $\Sigma(\tau,t)$ can thus be evaluated at each time t for all $\tau<t$ via equation (2.3f.15). When an incremental observation dz_t is obtained, the updated a posteriori distribution $\hat{\psi}_{\tau|t+dt}$ is computed via (2.3f.16) for all $\tau<t+dt$.

We now summarize the example as follows. For this case of a known exponential signal, the on-line computation of the signal estimate $\hat{\phi}_{t-t_s}$ involves mainly the computation of the estimate $\hat{\psi}_{t|t}$ and note from equations (2.3f.11) and (2.3f.12) that both these estimates are generated by finite dimensional filters. An infinite bank of filters is not required either for signal estimation or for delay time estimation. However, the delay time estimation results still require an infinite amount of on-line computations because at each time t, we have to evaluate the integral in equation (2.3f.17) for all $\tau<t$. In spite of this, the implementation is very much simpler than an infinite bank of filters.

It is interesting to note that for this case of an exponential signal, some "estimate" of t_s can be obtained with finite dimensional computations. Denote this "estimate" of t_s based on the observations Z_t by $\hat{t}_{s|t}$. Then, from equation (2.3f.2), for $t>\hat{t}_{s|t}$, we have

$$\hat{\phi}_{t-t_s} = \phi_0 e^{-\alpha(t-\hat{t}_{s|t})} \tag{2.3f.18}$$

which gives

$$\hat{t}_{s|t} = t - \frac{1}{\alpha} \ln\left(\frac{\phi_0}{\hat{\phi}_{t-t_s}}\right) \tag{2.3f.19}$$

Note that $\hat{t}_{s|t}$ is alway less than t. As we have pointed out before, the computation of $\hat{\phi}_{t-t_s}$ is finite dimensional and so equation (2.3f.19) is finite dimensional.

<u>Example 2</u> <u>Rectangular Pulse</u>

We assume here that the signal is a rectangular pulse given by

$$\phi_t = \begin{cases} 0 \ , & t<0 \\ 1 \ , & 0\leq t\leq T \\ 0 \ , & t>T \end{cases} \tag{2.3f.20}$$

Note that this model is not of the class (2.2.1) which we have analyzed in the main part of our work. Thus, the representation result for the signal estimate is no longer true but we shall see that the signal estimate can easily be generated. The delay time estimation results in Theorem 2.6, however, do not depend on the model for the signal ϕ_t and therefore are still true. We assume here a linear observation model:

$$dz_t = h_t \phi_{t-t_s} dt + dw_t \tag{2.3f.21}$$

The signal estimate is generated as follows:

$$\begin{aligned} \hat{\phi}_{t-t_s} &= P(t_s\leq t\leq T+t_s \mid Z_t) \\ &= P(t\leq T+t_s \mid Z_t) - P(t\leq t_s \mid Z_t) \end{aligned}$$

$$= P(t-T \leq t_s | Z_t) - P(t \leq t_s | Z_t)$$

$$= (1 - \hat{\psi}_{t-T|t}) - (1 - \hat{\psi}_{t|t})$$

i.e.
$$\hat{\phi}_{t-t_s} = \hat{\psi}_{t|t} - \hat{\psi}_{t-T|t} \tag{2.3f.22}$$

Thus, everything now boils down to the on-line computation of the a posteriori distribution $\hat{\psi}_{\tau|t}$. This is computed by the following equations:

$$\tau=t: \quad d\hat{\psi}_{t|t} = \rho_t(1-\hat{\psi}_{t|t})dt + h_t\hat{\phi}_{t-t_s}(1-\hat{\psi}_{t|t})d\nu_t \tag{2.3f.23}$$

or

$$d\hat{\psi}_{t|t} = \rho_t(1-\hat{\psi}_{t|t})dt + h_t(\hat{\psi}_{t|t}-\hat{\psi}_{t-T|t})(1-\hat{\psi}_{t|t})d\nu_t \tag{2.3f.24}$$

with initial condition

$$\hat{\psi}_{0|0} = 0 \tag{2.3f.25}$$

Here

$$d\nu_t = dz_t - h_t\hat{\phi}_{t-t_s}dt \tag{2.3f.26}$$

$$\tau>t: \quad \hat{\psi}_{\tau|t} = 1 - (1-\hat{\psi}_{t|t})\frac{P(t_s \geq \tau)}{P(t_s \geq t)} \tag{2.3f.27}$$

$$\tau<t: \quad \hat{\psi}_{\tau|t} = \hat{\psi}_{\tau|\tau} + \int_{\tau}^{t} \Sigma(\tau,t')d\nu_{t'} \tag{2.3f.28}$$

where

$$\Sigma(\tau,t) = h_t E\{\psi_\tau \phi_{t-t_s} | Z_t\} - h_t\hat{\psi}_{\tau|t}\,\hat{\phi}_{t-t_s} \tag{2.3f.29}$$

Of all the equations for computing $\hat{\psi}_{\tau|t}$, only equation (2.3f.28) requires an infinite amount of on-line computations. Again, equation (2.3f.28) is to be implemented as

$$d\hat{\psi}_{\tau|t} = \Sigma(\tau,t)d\nu_t \tag{2.3f.30}$$

i.e., one equation of the form (2.3f.30) for each $\tau<t$, starting with the initial condition $\hat{\psi}_{\tau|\tau}$ at time $t=\tau$. Thus, we only need to compute $\Sigma(\tau,t)$ for all $\tau<t$ at each time t. In equation (2.3f.29) for $\Sigma(\tau,t)$, the first term is evaluated as

$$E\{\psi_\tau \phi_{t-t_s} \Big| Z_t\} = \int_0^\tau E\{\psi_\tau \phi_{t-t_s} \Big| Z_t \text{ , } t_s=t'\} P(t'<t_s\leq t'+dt' \Big| Z_t)$$

$$= \int_0^\tau \phi_{t-t'} \, P(t'<t_s\leq t'+dt' \Big| Z_t)$$

$$= \begin{cases} \hat{\psi}_{\tau|t} \text{ ,} & 0\leq\tau<t\leq T \\ \hat{\psi}_{\tau|t} - \hat{\psi}_{t-T|t} \text{ ,} & 0\leq t-T<\tau<t \\ 0 \text{ ,} & 0\leq\tau\leq t-T \end{cases} \tag{2.3f.31}$$

Thus, the on-line computation of this term, at each time t, involves at most an infinite number of subtractions, one for each $\tau<t$. The term $\Sigma(\tau,t)$ is therefore readily computable at each time t for all $\tau<t$.

For this case of a rectangular pulse signal, the signal and delay time estimation problems both reduce to the on-line computation of the a posteriori distribution $\hat{\psi}_{\tau|t}$. The latter can be computed on-line by an infinite number of elementary algebraic operations and therefore the implementation is as simple as we can hope to get. Note that we can never hope to get away with a finite number of operations in computing $\hat{\psi}_{\tau|t}$ on-line, since we have to compute it for all τ at each time t.

Relation of our work to known results

Our results for processing known signals observed with a random time delay represent a new approach to the problem. However, we should point out that some existing results in the communications literature can also be applied to such signal processing problems. Essentially, in this case, we can use a correlation receiver or a matched filter receiver [74] to decide which value of the delay time is the true one based on the maximum a posteriori probability criterion. Once the true value of the delay time is decided on, the estimate of the signal is determined. We illustrate briefly here the structure of the correlation receiver.

Suppose the delay time t_s takes on the values $\{t_1, t_s, \ldots, t_n\}$, with the a priori probabilities $\{P_1, P_2, \ldots, P_n\}$ respectively, and the signal ϕ_t is known a priori. Then, the correlation receiver consists of a bank of n multipliers and n integrators. Let the observations be given by

$$r(t) = h_t \phi_{t-t_s} + n_w(t) \quad , \qquad t \geq 0 \tag{2.3f.32}$$

where $r(t)$ is the received signal and $n_w(t)$ is a zero-mean white Gaussian noise process. The correlation receiver first generates the set of numbers r_i , $i=1,\ldots,n$, where

$$r_i = \int_0^\infty r(t)\phi_{t-t_i} dt \tag{2.3f.33}$$

and from these the number ℓ_i , $i=1,\ldots,n$, where

$$\ell_i = r_i + c_i \tag{2.3f.34}$$

Here, c_i , $i=1,\ldots,n$, is precomputable and depends linearly on $\ln P_i$. Note that the ℓ_i's form a sufficient statistic for our decision problem. From each ℓ_i, the a posteriori probability $P(t_s=t_i \mid r(t),\ 0\leq t<\infty)$ can easily be obtain as

$$P(t_s=t_i \mid r(t),\ 0\leq t<\infty) = \frac{1}{\sqrt{2\pi N}}\, e^{-\left(\frac{R-2\ell_i}{N}\right)} \Big/ P(\underline{r}) \tag{2.3f.35}$$

where

$$\underline{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \tag{2.3f.36}$$

$$R = ||\underline{r}|| = \sum_{i=1}^{n} r_i^2 \tag{2.3f.37}$$

and N is the magnitude of the power spectrum of $n_w(t)$. It is shown by Wozencraft [74] that if the index k is such that $\ell_k \geq \ell_i$, for all i=1,...,n and i≠k, then the decision $t_s=t_k$ minimizes the probability of error. The last stage of the correlation receiver is a peak detector which selects the maximum of the numbers ℓ_i. Once the decision $t_s=t_k$ is made, the signal estimate is just ϕ_{t-t_k}.

We can point out now the similarities and the differences between the correlation receiver and the known signal results we derived above. Firstly, our results are recursive and enable us to do on-line estimation for both the signal and the delay time. The correlation receiver, however, is non-recursive and the estimates of the signal and the delay time can be obtained only after all the observations have been processed. Secondly, note that although we have discussed the correlation receiver for a finite number of possible values of t_s in the above, we should really interpret it as follows. Given that t_s takes on a continuum of values, we approximate this set of values by a finite subset only for implementation purposes and design a correlation receiver based on this finite set of values. This is similar to our results for processing known signals. If we were to actually implement our results for delay time estimation, we have to approximate the continuum of values of t_s by a finite subset and compute $\hat{\psi}_{\tau|t}$ for values of τ in this finite subset. When t_s actually takes on finitely many possible values, we also have results, presented in Section 2.4, for doing on-line signal and delay time estimation. Finally, note that all our results enable us to deal with random signals directly in contrast to the above results in the communications literature. However, the latter results provide us with a different insight into the problem than our formulation. There does not seem to be any way in which the implementation structure of our results is similar to the structure of a correlation receiver. The analogy to this is the relation between a Kalman filter starting with infinite initial covariance and a matched filter receiver generating a maximum-likelihood estimate.

The same remarks as above apply to the matched filter receiver which realizes the same decision rule as the correlation receiver.

In practice, it is not clear whether our recursive procedures are superior or inferior to the existing nonrecursive procedures. In any event, both solutions yield useful insights into the problem. Recursive solutions are definitely very appealing but for some applications, nonrecursive procedures are favored. For instance, in radar communication problems, a radar pulse might be only about 50ms in duration [69] and for such a signal, nonrecursive procedures employing, say, a matched filter receiver, implemented with the aid of the fast Fourier transform, is obviously preferrable to our recursive solution. We shall not try to advocate here one solution over another; the application in mind will decide the choice.

2.3g Summary Review of Solution

We have now completed the solution to the signal and delay time estimation problem in the case of a continuous range of values of the delay time t_s. We have assumed that the a posteriori probability distribution of t_s is absolutely continuous with respect to the Lebesque measure on the real line, i.e., t_s does not take on any value t with nonzero probability. The main results are as follows.

For the signal estimation problem, we have two solutions: a representation for the signal estimate $\hat{\phi}_{t-t_s}$ by means of a stochastic differential equation and a multiple-model solution. Both solutions are in the general nonlinear case non-implementable because they require an infinite amount of on-line computations. The multiple-model solution is inherently infinite dimensional since we have a continous range of values of t_s. Even in the linear Gaussian case, it involves an infinite bank of Kalman filters. The only way to get a finite dimensional implementation is to approximate the infinite bank of filters by a finite bank which is suboptimal. However, the representation result for the signal estimate leads to finite dimensional optimal implementations in several special cases. Moreover, using an assumed density for t_s, we have seen that the representation result easily leads to finite dimensional suboptimal implementations. In the case of known signals, we have seen one example in which the representation result admits a finite dimensional optimal implementation. Thus, the representation result appears to be a more interesting and more useful solution than the multiple-model approach.

For the delay time estimation problem, we can compute on-line the a posteriori distribution $\mathcal{P}(t_s \leq \tau | Z_t)$ of the delay time t_s given the observations Z_t. The solution is inherently infinite dimensional since we have to compute $\mathcal{P}(t_s \leq \tau | Z_t)$ for all τ at each time t. Only in the case where we assume that an a posteriori density $p_{t_s}(\tau | Z_t)$ exists and is of a known form characterized by a finite number of parameters is the delay time estimation solution finite dimensional. The equations for on-line computation of $\mathcal{P}(t_s \leq \tau | Z_t)$ require in general a multiple-model type of implementation involving an infinite bank of filters. However, in some examples involving known signals, an infinite bank of filters is not necessary. Although the on-line computations are still infinite dimensional, the required implementations are simpler and more appealing.

Throughout our results, it is apparent that the solutions to the signal and delay time estimation problems are coupled. This is to be expected. Because of uncertainties in t_s, the on-line computation of the conditional distribution of t_s is necessary to refine our estimate of ϕ_{t-t_s}. Conversely, since our measurements are on ϕ_{t-t_s}, the on-line computation of the conditional distribution of t_s inevitably involves estimates of the signal.

2.4 Solution for finitely many possible values of t_s

We assume here that t_s takes on finitely many possible values $\{t_1, t_2, \ldots, t_n\}$ with the nonzero a priori probabilities $\{P_1, P_2, \ldots, P_n\}$ respectively. This situation models the case in which the transmission medium can be one out of a finite number of possibilities. Alternatively, this situation models the case in which the random velocity characterizing the transport medium is spatially constant and takes on only one out of a finite number of values. Physically, this case of a finite set of values of t_s is not so interesting and important as the previous case of a continuous range of values of t_s. The results that we present here are not new and are included only for the sake of completeness. However, certain interesting applications of these results to the problem of statistical inference on the transmission field can be pointed out.

Since t_s takes on a finite set of values, it does not have a probability density function. The representation for the signal estimate $\hat{\phi}_{t-t_s}$ given in Theorem 2.5 is then no longer true. The only way to estimate the signal in this case is to use the multiple-model approach which is particularly easy in this case because we have

$$\hat{\phi}_{t-t_s} = \sum_{t_i \leq t} \hat{\phi}_{t-t_i} P_t^i \tag{2.4.1}$$

where

$$\hat{\phi}_{t-t_i} = E\{\phi_{t-t_s} \big| Z_t , t_s = t_i\} \tag{2.4.2}$$

and

$$P_t^i = P(t_s=t_i \mid Z_t) \tag{2.4.3}$$

Thus, at each time t, the signal estimate $\hat{\phi}_{t-t_s}$ is generated by a finite bank of filters which grows with time to at most n. Each of the estimates $\hat{\phi}_{t-t_i}$ is generated as in Section 2.3b. The equations for computing the a posteriori probabilities P_t^i are well known in this case, see for instance [7]. They are given by

$$dP_t^i = P_t^i \, (\hat{h}(\phi_{t-t_i},t) - \overline{h}(\phi_{t-t_s},t))(dz_t - \overline{h}(\phi_{t-t_s},t)dt)$$
$$P_0^i = P_i \ , \quad i=1,\ldots,n \tag{2.4.4}$$

where

$$\overline{h}(\phi_{t-t_s},t) = \sum_{i=1}^{n} \hat{h}(\phi_{t-t_i},t)P_t^i \tag{2.4.5}$$

and

$$\hat{h}(\phi_{t-t_i},t) = E\{h(\phi_{t-t_s},t) \mid Z_t,\ t_s=t_i\} \tag{2.4.6}$$

It is interesting to consider a problem of hypothesis testing on the transmission medium. Suppose our observations on the signal are limited to the time interval [0,T] where $T > \max_{1\leq i\leq n} t_i$ and based on these observations, we want to decide on the true value of t_s. This can be viewed as a problem of deciding what the transmission medium actually is given that it is one out of a finite number of possibilities. This hypothesis testing problem can be solved by computing the probabilities $\mathcal{P}_T^i$, $i=1,\ldots,n$ and making the decision that $t_s=t_i$, for that value of i for which $\mathcal{P}_T^i$ is maximum. This approach of solving the hypothesis testing problem is equivalent to the likelihood ratio approach [7].

2.5 The Multiple Source Problem

The aim of this section is to extend the results in the previous sections to the case of multiple signal sources. We shall only consider the case with two signal sources, as shown in Figure 9, with both sources generating the same signal and assume that the delay times involved take on a continuous range of values. The joint a priori distribution of the delay times is assumed to be absolutely continuous with respect to the Lebesque measure on the plane. The extention to an arbitrary number of sources is similar, while the finite hypothesis case can also be worked out in analogy with the signal source case.

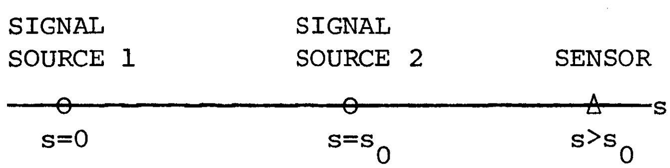

FIGURE 9: THE TWO SOURCE CASE

The motivation for considering the multiple source problem is to study the situation in which there are reflectors present in the transport medium to reflect the signal. Specifically, consider the situation in Figure 10. Because of the presence of the two reflectors, the

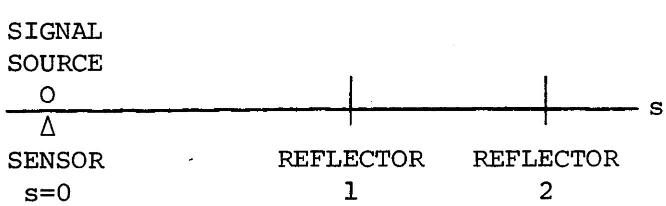

FIGURE 10: THE TWO REFLECTOR CASE

signal received at the sensor is made up of two signals, each one being the signal from the source but observed with a different random time delay. Conceptually, the reflectors in Figure 10 correspond to the sources in Figure 9. The situation with more than two reflectors is similar and we refer to this problem as the multiple reflection problem. The multiple reflection problem and the multiple source problem are conceptually equivalent and the two names will be used interchangeably. The case of only one reflector corresponds to the case of a single source which we have considered in detail before.

One possible application of the solution to the multiple reflection problem is to the discrete multipath communication problem [16]. Note that our problem formulation here applies to the nonresolvable case of the discrete multipath communication problem, i.e., the reflections of the signal received at the sensor overlap in time. Another possible application is to the problem of deducing the placement of the reflectors in the transmission medium. Assuming that the velocity of the transmission medium is spatially constant and is known, the delay time estimates enable us to estimate the distances of the reflectors from the sensor. This has possible applications to seismic signal processing [5], [25], [26].

2.5a Problem Formulation

We assume that both signal sources generate the same signal ϕ_t which is an Ito diffusion process given by the same model discussed before (see Section 2.2):

$$\left.\begin{aligned} d\phi_t &= \alpha(\phi_t,t)dt + \underline{\gamma}'(\phi_t,t)d\underline{\eta}_t, \quad t>0 \\ \phi_0 &= \text{random with known distribution}, \\ \phi_t &= 0, \qquad\qquad t<0 \end{aligned}\right\} \tag{2.5a.1}$$

Let t_{s1} and t_{s2} be the travel times of the signal from signal source 1 and signal source 2 respectively to the sensor. By the set-up of the model, we have:

$$t_{s1} > t_{s2} \tag{2.5a.2}$$

The sensor observes two signals, $\phi_{t-t_{s1}}$ due to signal source 1 and $\phi_{t-t_{s2}}$ due to signal source 2. Suppose the observations of the sensor are modeled as

$$dz_t = h_1(\phi_{t-t_{s1}}, t)dt + h_2(\phi_{t-t_{s2}}, t)dt + dw_t \tag{2.5a.3}$$

where $h_1(.,.)$ and $h_2(.,.)$ are jointly measurable with respect to both arguments and w_t is a standard Wiener process. We define the cumulative observation σ-field of the sensor as

$$Z_t = \sigma\{z_\tau, \ 0\leq\tau\leq t\} \tag{2.5a.4}$$

The problems we are interested in are now:

(i) To estimate the signals $\phi_{t-t_{s1}}$ and $\phi_{t-t_{s2}}$,

(ii) To infer the properties of the transmission field.

For the second problem, we shall see that we can compute on-line the joint a posteriori distribution $P(t_{s1}\leq\tau_1, \ t_{s2}\leq\tau_2 | Z_t)$ of the delay times t_{s1} and t_{s2}. This joint distribution is also used in computing the estimates of the signals. We present in the following sections the solution to the two problems posed above.

We note that the observation model (2.5a.3) can allow different amounts of reflection from the different reflectors. This is especially evident in the case when $h_1(.,.)$ and $h_2(.,.)$ are linear in $\phi_{t-t_{s1}}$ and $\phi_{t-t_{s2}}$ respectively, so that $h_1(t)$ and $h_2(t)$ can be regarded as the reflection coefficients from reflectors 1 and 2 respectively. In our model we take $h_1(t)$ and $h_2(t)$ to be deterministic, but one could consider them to be random (as in amplitude fading in a Rayleigh channel [16]).

We state here the statistical assumptions made in our analysis of the problem. As in the one source case, w_t is assumed independent of $\underline{\eta}_t$ and of ϕ_0 so that w_t is independent of ϕ_t. We will also assume w_t to be independent of t_{s1} and of t_{s2} and this implies that w_t is independent of $\phi_{t-t_{s1}}$ and of $\phi_{t-t_{s2}}$.

2.5b Signal Estimation

As in the one source case, we discuss here two solutions to the signal estimation problem, namely the solution via dynamical representations for the signal estimates and the multiple-model solution.

Dynamical Representations for Signal Estimates

We are interested in the stochastic differential equation representations for the estimates

$$\hat{\phi}_{t-t_{s1}} = E\{\phi_{t-t_{s1}} | Z_t\} \tag{2.5b.1}$$

and

$$\hat{\phi}_{t-t_{s2}} = E\{\phi_{t-t_{s2}} | Z_t\} \tag{2.5b.2}$$

It will be seen that such a representation is possible for $\hat{\phi}_{t-t_{s2}}$ and not for $\hat{\phi}_{t-t_{s1}}$.

The derivation of the representation for $\hat{\phi}_{t-t_{s2}}$ proceeds as in the one source case. To describe events at the sensor, we construct the increasing family $\{\mathcal{B}_t\}_{t\geq 0}$ of σ-fields such that

$$\begin{aligned}\mathcal{B}_t = \sigma\{w_\tau,\ 0\leq\tau\leq t\} \vee \sigma\{\phi_{\tau-t_{s1}},\ 0\leq\tau\leq t\} \vee \sigma\{\phi_{\tau-t_{s2}},\ 0\leq\tau\leq t\} \\ \vee\ \sigma\{\{\omega:\ t_{s1}\leq\tau\} | 0\leq\tau\leq t\} \\ \vee\ \sigma\{\{\omega:\ t_{s2}\leq\tau\} | 0\leq\tau\leq t\}\end{aligned} \tag{2.5b.3}$$

On the family $\{\mathcal{B}_t\}_{t\geq 0}$, the process $\phi_{t-t_{s2}}$ is a semimartingale and has the representation

$$\begin{aligned}d\phi_{t-t_{s2}} &= (\rho_{2t}(1-\psi_{2t})\phi_0 + \psi_{2t-}\alpha(\phi_{t-t_{s2}},\ t-t_{s2}))dt \\ &\quad + [\phi_0 \quad \psi_{2t-}\underline{\gamma}'(\phi_{t-t_{s2}},\ t-t_{s2})] \begin{bmatrix} dm_{2t} \\ d\underline{\eta}_{t-t_{s2}} \end{bmatrix}\end{aligned} \tag{2.5b.4}$$

where ψ_{2t} is the unit-jump process defined by

$$\psi_{2t} = \begin{cases} 1, & t\geq t_{s2} \\ 0, & t<t_{s2} \end{cases} \tag{2.5b.5}$$

and ρ_{2t} is given by

$$\rho_{2t} = p_{t_{s2}}(t) \Big/ \int_t^\infty p_{t_{s2}}(\tau)d\tau \tag{2.5b.6}$$

while m_{2t} is a $\mathcal{B}_t$-martingale (here we are using the fact that ψ_{2t} is also $\mathcal{B}_t$-adapted.) The derivation of this representation is the same as that of Theorem 2.2 in the one source case. Following the steps that lead to Theorem 2.5, we now have the following representation for $\hat{\phi}_{t-t_{s2}}$:

$$d\hat{\phi}_{t-t_{s2}} = (\rho_{2t}(1-\hat{\psi}_{2t|t})\ E\{\phi_0\} + E\{\psi_{2t-}\alpha(\phi_{t-t_{s2}},\ t-t_{s2})|Z_t\})\ dt$$
$$+ E\{\phi_{t-t_{s2}}(h_1(\phi_{t-t_{s1}},\ t) + h_2(\phi_{t-t_{s2}},\ t))$$
$$- \hat{\phi}_{t-t_{s2}}(\hat{h}_1(\phi_{t-t_{s1}},\ t) + \hat{h}_2(\phi_{t-t_{s2}},\ t))|Z_t\}d\nu_t\ ,$$
$$\hat{\phi}_{t-t_{s2}}\Big|_{t=0} = 0 \qquad (2.5b.7)$$

where

$$d\nu_t = dz_t - \hat{h}_1(\phi_{t-t_{s1}},\ t)\ dt - \hat{h}_2(\phi_{t-t_{s2}},\ t)dt \qquad (2.5b.8)$$

We will discuss later the implementation of this filter as well as its specialization to the linear case.

To generate the representation for the estimate $\hat{\phi}_{t-t_{s1}}$, note that $\phi_{t-t_{s1}}$ is a delayed version of $\phi_{t-t_{s2}}$ since $t_{s1} > t_{s2}$. Because the observation z_t contains a measurement on $\phi_{t-t_{s2}}$, the estimation of $\phi_{t-t_{s1}}$ should be viewed as a smoothing problem. We have found that it is impossible to write the stochastic differential equation representation for the estimate $\hat{\phi}_{t-t_{s1}}$ in all cases. This difficulty was noted by Kwong [37] in the case of a fixed known delay. A trivial extension of the argument by Kwong [37] easily leads us to this conclusion. The reader should refer to [37] for a thorough discussion of this point. In the case when t_{s1} and t_{s2} are known, say $t_{s1}=\tau_1$ and $t_{s2}=\tau_2$, a nonrecursive representation for $\hat{\phi}_{t-\tau_1}$ has been obtained by Kwong [37], where

$$\hat{\phi}_{t-\tau_1} = E\{\phi_{t-t_{s1}}|Z_t,\ t_{s1}=\tau_1,\ t_{s2}=\tau_2\} \tag{2.5b.9}$$

Using the estimate $\hat{\phi}_{t-\tau_1}$, the estimate $\hat{\phi}_{t-t_{s1}}$ can then be generated by using a multiple-model type of approach. This is discussed in detail in the following section. Note that at present, it is not known how to generalize the nonrecursive representation for $\hat{\phi}_{t-\tau_1}$ to one for $\hat{\phi}_{t-t_{s1}}$ because without conditioning on known values of t_{s1} and t_{s2}, the representation is not well defined.

<u>Multiple-Model Solution</u>

The estimates $\hat{\phi}_{t-t_{s1}}$ and $\hat{\phi}_{t-t_{s2}}$ can be generated as

$$\hat{\phi}_{t-t_{s1}} = \int_{\tau_2\leq\tau_1\leq t} E\{\phi_{t-t_{s1}}|Z_t,\ t_{s1}=\tau_1,\ t_{s2}=\tau_2\}\ \cdot$$
$$\cdot\ P(\tau_1<t_{s1}\leq\tau_1+d\tau_1,\ \tau_2<t_{s2}\leq\tau_2+d\tau_2|Z_t) \tag{2.5b.10}$$

$$\hat{\phi}_{t-t_{s2}} = \int_{\substack{\tau_2\leq t,\\ \tau_2\leq\tau_1}} E\{\phi_{t-t_{s2}}|Z_t,\ t_{s1}=\tau_1,\ t_{s2}=\tau_2\}\ \cdot$$
$$\cdot\ P(\tau_1<t_{s1}\leq\tau_1 + d\tau_1,\ \tau_2<t_{s2}\leq\tau_2+d\tau_2|Z_t) \tag{2.5b.11}$$

The computation of the probabilities $P(\tau_1<t_{s1}\leq\tau_1+d\tau_1,\ \tau_2<t_{s2}\leq\tau_2+d\tau_2|Z_t)$ will be discussed in a later section. We now discuss the generation of

the estimates

$$\hat{\phi}_{t-\tau_1} = E\{\phi_{t-t_{s1}} | Z_t,\ t_{s1}=\tau_1,\ t_{s2}=\tau_2\} \tag{2.5b.12}$$

and

$$\hat{\phi}_{t-\tau_2} = E\{\phi_{t-t_{s2}} | Z_t,\ t_{s1}=\tau_1,\ t_{s2}=\tau_2\} \tag{2.5b.13}$$

Given that $t_{s1}=\tau_1$ and $t_{s2}=\tau_2$, we have

$$d\phi_{t-\tau_1} = \alpha(\phi_{t-\tau_1},\ t-\tau_1)dt + \underline{\gamma}'(\phi_{t-\tau_1},\ t-\tau_1)\ d\underline{\eta}_{t-\tau_1}, \qquad t\geq\tau_1 \tag{2.5b.14}$$

$$d\phi_{t-\tau_2} = \alpha(\phi_{t-\tau_2},\ t-\tau_2)dt + \underline{\gamma}'(\phi_{t-\tau_2},\ t-\tau_2)d\underline{\eta}_{t-\tau_2}, \qquad t\geq\tau_2 \tag{2.5b.15}$$

and the observation equation is

$$dz_t = h_1(\phi_{t-\tau_1},t)dt + h_2(\phi_{t-\tau_2},t)dt + dw_t\ , \qquad t\geq 0 \tag{2.5b.16}$$

Thus, the estimation equation for $\hat{\phi}_{t-\tau_2}$ is [38]:

$$0\leq t<\tau_2 : \qquad \hat{\phi}_{t-\tau_2}=0 \tag{2.5b.17}$$

$$\begin{aligned}
\tau_2\leq t<\tau_1: \qquad d\hat{\phi}_{t-\tau_2} &= \hat{\alpha}(\phi_{t-\tau_2},\ t-\tau_2)dt \\
&+ E\{(\phi_{t-\tau_2}-\hat{\phi}_{t-\tau_2})(h_2(\phi_{t-\tau_2},t)-\hat{h}_2(\phi_{t-\tau_2},t)) \\
&\quad |Z_t,\ t_{s1}=\tau_1,\ t_{s2}=\tau_2\}(dz_t'-\hat{h}_2(\phi_{t-\tau_2},t)dt)\ , \\
\hat{\phi}_{t-\tau_2}\Big|_{t=\tau_2} &= E\{\phi_0\}
\end{aligned} \tag{2.5b.18}$$

where

$$dz'_t = dz_t - h_1(0,t)dt \tag{2.5b.19}$$

$$\begin{aligned}\tau_1 \leq t: \quad d\hat{\phi}_{t-\tau_2} &= \hat{\alpha}(\phi_{t-\tau_2}, t-\tau_2)dt \\ &+ E\{(\phi_{t-\tau_2} - \hat{\phi}_{t-\tau_2})\ (h_1(\phi_{t-\tau_1}, t) + h_2(\phi_{t-\tau_2}, t) \\ &- \hat{h}_1(\phi_{t-\tau_1}, t) - \hat{h}_2(\phi_{t-\tau_2}, t))\,|Z_t, t_{s1}=\tau_1, t_{s2}=\tau_2\}\cdot \\ &\cdot\ (dz_t - \hat{h}_1(\phi_{t-\tau_1}, t)dt - \hat{h}_2(\phi_{t-\tau_2}, t)dt)\end{aligned} \tag{2.5b.20}$$

In principle, to implement the filter (2.5b.18) in the general nonlinear case, we can write a stochastic differential equation for each term on the right hand side, ending up with an infinite system of equations. However, for the filter (2.5b.20), the same procedure is not possible because it is impossible in general to write a stochastic differential equation for the term $E\{\phi_{t-\tau_2} h_1(\phi_{t-\tau_1}, t)|Z_t, t_{s1}=\tau_1, t_{s2}=\tau_2\}$. This problem has been discussed in Kwong's thesis [37] and we refer the interested reader to this reference. In the general nonlinear case, it is impossible to compute this term recursively by any means. However, in the linear Gaussian case, this term reduces to $h_{1t}\ E\{\phi_{t-\tau_2}\phi_{t-\tau_1} |Z_t, t_{s1}=\tau_1, t_{s2}=\tau_2\}$ and is precomputable as we shall see later.

Now, consider the problem of finding the estimation equation for $\hat{\phi}_{t-\tau_1}$. We know that

$$\hat{\phi}_{t-\tau_1} = 0, \qquad t<\tau_1 \tag{2.5b.21}$$

For $t \geq \tau_1$, we encounter the same problem as before for $\hat{\phi}_{t-t_{s1}}$. The generation of this estimate is a smoothing problem since $\phi_{t-\tau_1}$ is a delayed version of $\phi_{t-\tau_2}$ and we have observations on $\phi_{t-\tau_2}$. In all cases, it is impossible to generate the dynamical representation for $\hat{\phi}_{t-\tau_1}$. See the discussion in [37].

However, a nonrecursive representation for $\hat{\phi}_{t-\tau_1}$ has been obtained by Kwong in [37] and is given by

$$\begin{aligned}
\hat{\phi}_{t-\tau_1} = \hat{\phi}_{t-(\tau_1-\tau_2)-\tau_2|t-(\tau_1-\tau_2)} + \int_{t-(\tau_1-\tau_2)}^{t} & E\{E_0[\phi_{t-\tau_1}|\phi_{s-\tau_1}] \cdot \\
& \cdot [h_1(\phi_{s-\tau_1}, s) + h_2(\phi_{s-\tau_2}, s) \\
& - \hat{h}_1(\phi_{s-\tau_1}, s) - \hat{h}_2(\phi_{s-\tau_2}, s)]|Z_s, t_{s1}=\tau_1, t_{s2}=\tau_2\} \cdot \\
& \cdot (dz_s - \hat{h}_1(\phi_{s-\tau_1}, s)ds - \hat{h}_2(\phi_{s-\tau_2}, s)ds)
\end{aligned} \tag{2.5b.22}$$

where $E_0\{\cdot\}$ denotes the expectation with respect to a probability measure $\mathcal{P}_0$ defined by

$$\begin{aligned}
\frac{d\mathcal{P}_0}{d\mathcal{P}} = \exp\Big[& -\int_0^T [h_1(\phi_{t-\tau_1}, t) + h_2(\phi_{t-\tau_2}, t)]dw_t \\
& - \frac{1}{2}\int_0^T [h_1(\phi_{t-\tau_1}, t) + h_2(\phi_{t-\tau_2}, t)]^2 dt\Big]
\end{aligned} \tag{2.5b.23}$$

Here [0, T] is the interval of time over which our problem is defined. Under the probability measure $\mathcal{P}_0$, the process z_t, the observations, is a standard Wiener process. Thus, intuitively, under the measure $\mathcal{P}_0$, no measurement on $\phi_{t-\tau_1}$ and $\phi_{t-\tau_2}$ is made. Equation (2.5b.22) is only a representation and is incomputable in practice. In the linear Gaussian case, this representation reduces to a readily implementable smoothing equation which is given in Section 2.5d.

2.5c Delay Time Estimation

We discuss here the on-line computation of the joint a posteriori distribution $\mathcal{P}(t_{s1}\leq\tau_1,\ t_{s2}\leq\tau_2|Z_t)$ which, besides being used for estimation of the delay times t_{s1} and t_{s2}, is also used in the multiple-model solution for computing the signal estimates $\hat{\phi}_{t-t_{s1}}$ and $\hat{\phi}_{t-t_{s2}}$ as illustrated in the previous section. This is done by first writing

$$\begin{aligned}&\mathcal{P}(\tau_1<t_{s1}\leq\tau_1+d\tau_1,\ \ \tau_2<t_{s2}\leq\tau_2+d\tau_2|Z_t)\\ &\quad= \mathcal{P}(\tau_2<t_{s2}\leq\tau_2+d\tau_2|Z_t,\ t_{s1}=\tau_1)\ \mathcal{P}(\tau_1<t_{s1}\leq\tau_1+d\tau_1|Z_t)\end{aligned} \tag{2.5c.1}$$

The first term on the right can be computed by considering an estimation problem on the process ψ_{2t} defined by equation (2.5b.5). Given that $t_{s1} = \tau_1$, we now have the new a priori density $P(t_{s2}|t_{s1}=\tau_1)$ for t_{s2}. In addition, given that $t_{s1}=\tau_1$, events at the sensor should now be described by the increasing family $\{\mathcal{B}'_t\}_{t\geq 0}$ of σ-fields such that

$$\begin{aligned}\mathcal{B}'_t = \sigma\{w_\tau,\ 0\leq\tau\leq t\}\ &\vee\ \sigma\{\phi_{\tau-\tau_1},\ 0\leq\tau\leq t\}\\ &\vee\ \sigma\{\phi_{\tau-t_{s2}},\ 0\leq\tau\leq t\}\\ &\vee\ \sigma\{\{\omega:\ \ t_{s2}\leq\tau\}|0\leq\tau\leq t\wedge\tau_1\}\end{aligned} \tag{2.5c.2}$$

The probabilities of these events should be assigned by the measure $\mathcal{P}(\cdot|t_{s1}=\tau_1)$. Under this new a priori probability measure (assuming it is absolutely continuous with respect to Lebesque measure), the process ψ_{2t} has the representation

$$d\psi_{2t} = \rho'_{2t}(1-\psi_{2t})dt + dm'_{2t} \tag{2.5c.3}$$

where

$$\rho'_{2t} = p_{t_{s2}}(t|t_{s1}=\tau_1) \Big/ \int_t^{\infty} p_{t_{s2}}(\tau|t_{s1}=\tau_1)d\tau \tag{2.5c.4}$$

and m'_{2t} is a martingale on $\{\mathcal{B}'_t\}_{t\geq 0}$.

The probability distribution $\hat{\psi}_{2\tau_2|t,\tau_1} = E\{\psi_{2\tau_2}|Z_t, t_{s1} = \tau_1\}$ $= \mathcal{P}(t_{s2}\leq\tau_2|Z_t, t_{s1}=\tau_1)$ is now computed by the following equations whose proofs are direct extensions of the one-source case:

$$\begin{aligned}
\tau_2=t: \quad d\hat{\psi}_{2t|t,\tau_1} &= \rho'_{2t}(1-\hat{\psi}_{2t|t,\tau_1})dt \\
&\quad + [E\{\psi_{2t}(h_1(\phi_{t-t_{s1}},t) + h_2(\phi_{t-t_{s2}}, t))|Z_t, t_{s1}=\tau_1\} \\
&\quad - \hat{\psi}_{2t|t,\tau_1}(\hat{h}_1(\phi_{t-t_{s1}}, t|\tau_1)+\hat{h}_2(\phi_{t-t_{s2}}, t|\tau_1))]. \\
&\quad \cdot d\nu_{t|\tau_1}, \\
\hat{\psi}_{20|0,\tau_1} &= 0
\end{aligned} \tag{2.5c.5}$$

where

$$\hat{h}_1(\phi_{t-t_{s1}}, t|\tau_1) = E\{h_1(\phi_{t-t_{s1}}, t)|Z_t, t_{s1}=\tau_1\} \tag{2.5c.6}$$

$$\hat{h}_2(\phi_{t-t_{s2}}, t|\tau_1) = E\{h_2(\phi_{t-t_{s2}}, t)|Z_t, t_{s1}=\tau_1\} \tag{2.5c.7}$$

and

$$d\nu_{t|\tau_1} = dz_t - \hat{h}_1(\phi_{t-t_{s1}}, t|\tau_1)dt - \hat{h}_2(\phi_{t-t_{s2}}, t|\tau_1)dt \tag{2.5c.8}$$

$$\tau_2>t: \quad \hat{\psi}_{2\tau_2|t,\tau_1} = 1-(1-\hat{\psi}_{2t|t,\tau_1})\frac{P(t_{s2}\geq\tau_2|t_{s1}=\tau_1)}{P(t_{s2}\geq t|t_{s1}=\tau_1)} \tag{2.5c.9}$$

$$\tau_2<t: \quad \hat{\psi}_{2\tau_2|t,\tau_1} = \hat{\psi}_{2\tau_2|\tau_2,\tau_1} + \int_{\tau_2}^{t} \Sigma_2(\tau_2,\tau|\tau_1)d\nu_{\tau|\tau_1} \tag{2.5c.10}$$

where

$$\begin{aligned}\Sigma_2(\tau_2, t|\tau_1) = E\{\psi_{2\tau_2}&(h_1(\phi_{t-t_{s1}}, t) + h_2(\phi_{t-t_{s2}}, t))\\ &|Z_t, t_{s1}=\tau_1\} - \hat{\psi}_{2\tau_2|t,\tau_1}\ \hat{h}_1(\phi_{t-t_{s1}}, t|\tau_1)\\ &- \hat{\psi}_{2\tau_2|t,\tau_1}\ \hat{h}_2(\phi_{t-t_{s2}}, t|\tau_1)\end{aligned} \tag{2.5c.11}$$

and $\hat{h}_1(\phi_{t-t_{s1}}, t|\tau_1)$ is defined by equation (2.5c.6) and $\hat{h}_2(\phi_{t-t_{s2}}, t|\tau_1)$ by equation (2.5c.7).

Next, the probability distribution $\hat{\psi}_{1\tau_1|t} = E\{\psi_{1\tau_1}|Z_t\} = P(t_{s1}\leq\tau_1|Z_t)$ is computed by the following equations which are obtained by considering an estimation problem on the process ψ_{1t} defined by

$$\psi_{1t} = \begin{cases} 1, & t\geq t_{s1} \\ 0, & t<t_{s1} \end{cases} \tag{2.5c.12}$$

and which has the representation

$$d\psi_{1t} = \rho_{1t}(1-\psi_{1t})dt + dm_{1t} \tag{2.5c.13}$$

where

$$\rho_{1t} = P_{t_{s1}}(t)\Big/\int_t^{\infty} P_{t_{s1}}(\tau)d\tau \tag{2.5c.14}$$

and m_{1t} is a $\mathcal{B}_t$-martingale.

$$\tau_1 = t: \quad d\hat{\psi}_{1t|t} = \rho_{1t}(1-\hat{\psi}_{1t|t})dt$$

$$+ [E\{\psi_{1t}(h_1(\phi_{t-t_{s1}},t) + h_2(\phi_{t-t_{s2}},t))|Z_t\}$$

$$- \hat{\psi}_{1t\ t}(\hat{h}_1(\phi_{t-t_{s1}},t) + \hat{h}_2(\phi_{t-t_{s2}},t))]d\nu_t,$$

$$\hat{\psi}_{10|0} = 0 \qquad (2.5c.15)$$

$$\tau_1 > t: \quad \hat{\psi}_{1\tau_1|t} = 1-(1-\hat{\psi}_{1t|t})\frac{\mathcal{P}(t_{s1} \geq \tau_1)}{\mathcal{P}(t_{s1} \geq t)} \qquad (2.5c.16)$$

$$\tau_1 < t: \quad \hat{\psi}_{1\tau_1|t} = \hat{\psi}_{1\tau_1|\tau_1} + \int_{\tau_1}^{t} \Sigma_1(\tau_1,\tau)d\nu_\tau \qquad (2.5c.17)$$

where

$$\Sigma_1(\tau_1,t) = E\{\psi_{1\tau_1}(h_1(\phi_{t-t_{s1}}, t) + h_2(\phi_{t-t_{s2}}, t))|Z_t\}$$

$$- \hat{\psi}_{1\tau_1|t}\ (\hat{h}_1(\phi_{t-t_{s1}}, t) + \hat{h}_2(\phi_{t-t_{s2}}, t)) \qquad (2.5c.18)$$

The derivation of these equations follows that of Theorem 2.6 directly.

With the equations for $\mathcal{P}(t_{s2} \leq \tau_2|Z_t, t_{s1}=\tau_1)$ and $\mathcal{P}(t_{s1} \leq \tau_1|Z_t)$, we can then generate the joint distribution $\mathcal{P}(t_{s1} \leq \tau_1, t_{s2} \leq \tau_2|Z_t)$ on-line using equation (2.5c.1).

As in the one source case, the above equations for computing $\mathcal{P}(t_{s2} \leq \tau_2|Z_t, t_{s1}=\tau_1)$ and $\mathcal{P}(t_{s1} \leq \tau_1|Z_t)$ have to be implemented via the multiple-model approach. For instance, in equation (2.5c.18), the term $E\{\psi_{1\tau_1}h_2(\phi_{t-t_{s2}},t)|Z_t\}$ is evaluated as

$$E\{\psi_{1\tau_1} h_2(\phi_{t-t_{s2}},t)|Z_t\}$$

$$= \int_{0\le t_2'\le t_1'\le \tau_1} E\{h_2(\phi_{t-t_{s2}},t)|Z_t,\ t_{s1}=t_1',\ t_{s2}=t_2'\}\cdot$$

$$\cdot\ P(t_1'<t_{s1}\le t_1' + dt_1',\quad t_2'<t_{s2}\le t_2' + dt_2'|Z_t) \tag{2.5c.19}$$

It is easy to see that the basic quantities to be computed in implementing the delay time estimation equations are the estimates

$$\hat{h}_1(\phi_{t-\tau_1},\ t) = E\{h_1(\phi_{t-t_{s1}},\ t)|Z_t,\ t_{s1}=\tau_1,\ t_{s2}=\tau_2\} \tag{2.5c.20}$$

and

$$\hat{h}_2(\phi_{t-\tau_2},\ t) = E\{h_2(\phi_{t-t_{s2}},\ t)|Z_t,\ t_{s1}=\tau_1,\ t_{s2}=\tau_2\} \tag{2.5c.21}$$

However, to generate the equations for computing these estimates involves the same difficulties as those encountered before in deriving the equations for $\hat{\phi}_{t-\tau_1}$ and $\hat{\phi}_{t-\tau_2}$. In the next section, we examine the linear Gaussian case in which all these estimates are computable.

2.5d The Linear Gaussian Case

We saw in the previous sections that one way to generate the complete solution to the signal and delay time estimation problems is the multiple-model approach. We shall first discuss the multiple-model solution here and then examine the representation results for the signal estimates.

Multiple-Model Solution

In the linear Gaussian case, the signal model (2.5a.1) specializes to

$$\left.\begin{aligned} d\phi_t &= \alpha_t \phi_t dt + \underline{\gamma}_t' d\underline{\eta}_t, \quad t>0 \\ \phi_0 &= \text{Gaussian random variable} \\ \phi_t &= 0 \qquad\qquad , \quad t<0 \end{aligned}\right\} \tag{2.5d.1}$$

and the observation model (2.5a.3) becomes

$$dz_t = h_{1t}\phi_{t-t_{s1}}dt + h_{2t}\phi_{t-t_{s2}}dt + dw_t \tag{2.5d.2}$$

The basic quantities called for by the multiple-model approach are the estimates

$$\hat{\phi}_{t-\tau_1} = E\{\phi_{t-t_{s1}} | Z_t, \ \ t_{s1}=\tau_1, \ t_{s2}=\tau_2\} \tag{2.5d.3}$$

and

$$\hat{\phi}_{t-\tau_2} = E\{\phi_{t-t_{s2}} | Z_t, \ t_{s1}=\tau_1, \ t_{s2}=\tau_2\} \tag{2.5d.4}$$

The solution for computing these estimates has been derived rigorously by Kwong [37] and we present it here for the special case of our model. Given that $t_{s1}=\tau_1$ and $t_{s2}=\tau_2$ the observation model becomes

$$dz_t = h_{1t}\phi_{t-\tau_1}dt + h_{2t}\phi_{t-\tau_2}dt + dw_t \tag{2.5d.5}$$

It is well known that ϕ_t is a Gaussian process and its distribution conditioned on the observations Z_t is also Gaussian. In fact, the same is true for $\phi_{t+\theta}$ for any θ such that $-t\leq\theta\leq 0$ [37]. Thus, its a posteriori density given Z_t is completely characterized by the conditional mean and the conditional covariance.

The conditional means $\hat{\phi}_{t-\tau_1}$ and $\hat{\phi}_{t-\tau_2}$ are given by the following equations:

$$t<\tau_2<\tau_1: \qquad \hat{\phi}_{t-\tau_1} = \hat{\phi}_{t-\tau_2} = 0 \tag{2.5d.6}$$

$$\tau_2 \leq t<\tau_1: \qquad \hat{\phi}_{t-\tau_1} = 0 \tag{2.5d.7}$$

$$d\hat{\phi}_{t-\tau_2} = \alpha_{t-\tau_2}\hat{\phi}_{t-\tau_2}dt + P_0(t|\tau_1,\tau_2)h_{2t}(dz_t - h_{2t}\hat{\phi}_{t-\tau_2}dt),$$

$$\hat{\phi}_{t-\tau_2}\Big|_{t=\tau_2} = \hat{\phi}_0 = E\{\phi_0\} \tag{2.5d.8}$$

where the error covariance

$$P_0(t|\tau_1,\tau_2) = E\{(\phi_{t-\tau_2} - \hat{\phi}_{t-\tau_2})^2 \, |Z_t, \; t_{s1}=\tau_1, \; t_{s2}=\tau_2\} \tag{2.5d.9}$$

is given by the Riccati equation

$$\frac{dP_0(t|\tau_1,\tau_2)}{dt} = 2\alpha_{t-\tau_2} P_0(t|\tau_1, \tau_2) + \underline{\gamma}'_{t-\tau_2}\underline{\gamma}_{t-\tau_2} - h^2_{2t}P^2_0(t|\tau_1, \tau_2),$$

$$P_0(t=\tau_2|\tau_1, \tau_2) = E\{(\phi_0 - E\{\phi_0\})^2\} = \Sigma_0, \text{ given} \tag{2.5d.10}$$

$\tau_2<\tau_1\leq t$:

$$d\hat{\phi}_{t-\tau_2} = \alpha_{t-\tau_2}\hat{\phi}_{t-\tau_2} dt + [P_0(t|\tau_1,\tau_2)h_{2t} + P_1(t, \tau_2-\tau_1|\tau_1,\tau_2)h_{1t}] \cdot$$

$$\cdot \, [dz_t - h_{1t}\hat{\phi}_{t-\tau_1}dt - h_{2t}\hat{\phi}_{t-\tau_2}dt] \tag{2.5d.11}$$

$$\begin{aligned}\hat{\phi}_{t-\tau_1} = \; & \hat{\phi}_{t-\tau_1|t-(\tau_1-\tau_2)} \\ & + \int_{t-(\tau_1-\tau_2)}^{t} [P_2(s, t-(\tau_1-\tau_2)-s, 0|\tau_1,\tau_2)h_{2s} \\ & + P_2(s, t-(\tau_1-\tau_2)-s, -(\tau_1-\tau_2)|\tau_1,\tau_2)h_{1s}] \\ & \cdot \, [dz_s - (h_{1s}\hat{\phi}_{s-\tau_1} + h_{2s}\hat{\phi}_{s-\tau_2})ds]\end{aligned} \tag{2.5d.12}$$

Note that the first term $\hat{\phi}_{t-\tau_1 | t-(\tau_1 - \tau_2)}$ in equation (2.5d.12) is the estimate generated by the filter (2.5d.11):

$$\hat{\phi}_{t-\tau_1|t-(\tau_1-\tau_2)} = \hat{\phi}_{t-\tau_2-(\tau_1-\tau_2)|t-(\tau_1-\tau_2)}$$

$$= E\{\phi_{t-(\tau_1-\tau_2)-\tau_2} \Big| Z_{t-(\tau_1-\tau_2)},\ t_{s1}=\tau_1,\ \ t_{s2}=\tau_2\} \tag{2.5d.13}$$

Equations (2.5d.11) and (2.5d.12) can be understood more easily if we write $x_t = \phi_{t-\tau_2}$ and $x_{t-(\tau_1-\tau_2)} = \phi_{t-\tau_1}$.

The covariances P_i, $i=0, 1, 2$ are defined by

$$P_2(t,\theta,\xi|\tau_1,\tau_2) = E\{(\phi_{t-\tau_2+\theta} - \hat{\phi}_{t-\tau_2+\theta})(\phi_{t-\tau_2+\xi} - \hat{\phi}_{t-\tau_2+\xi}) \Big| Z_t,$$

$$t_{s1} = \tau_1,\ t_{s2}=\tau_2\},$$

$$\theta \leq 0, \quad \xi \leq 0 \tag{2.5d.14}$$

$$P_1(t,\theta|\tau_1,\tau_2) = P_2(t,\theta,0|\tau_1,\ \tau_2) \tag{2.5d.15}$$

$$P_0(t|\tau_1,\tau_2) = P_2(t,0,0|\tau_1,\tau_2) \tag{2.5d.16}$$

Note that the definitions (2.5d.16) and (2.5d.9) are the same. The above covariances are precomputable by the following equations which are obtained by direct substitution into Kwong's results [37]:

$$\frac{d}{dt} P_0(t|\tau_1,\tau_2) = 2\alpha_{t-\tau_2} P_0(t|\tau_1,\tau_2) - h_{2t}^2 P_0^2(t|\tau_1,\tau_2)$$

$$- h_{1t}^2 P_1^2(t,\ -(\tau_1-\tau_2)|\tau_1,\tau_2)$$

$$- 2h_{1t}h_{2t}P_0(t|\tau_1,\tau_2)P_1(t,\ -(\tau_1-\tau_2)|\tau_1,\tau_2)$$

$$+ \underline{\gamma}'_{t-\tau_2}\underline{\gamma}_{t-\tau_2} \tag{2.5d.17}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta}\right) P_1(t,\theta|\tau_1,\tau_2) = P_1(t,\theta|\tau_1,\tau_2)\alpha_{t-\tau_2}$$

$$- P_1(t,\theta|\tau_1,\tau_2)\ h_{2t}^2 P_0(t|\tau_1,\tau_2)$$

$$- P_2(t,\theta,\ -(\tau_1-\tau_2)|\tau_1,\tau_2)h_{1t}^2 P_1(t,-(\tau_1-\tau_2)|\tau_1,\tau_2)$$

$$- P_1(t,\theta|\tau_1,\tau_2)h_{1t}h_{2t}\ P_1(t,\ -(\tau_1-\tau_2)|\tau_1,\tau_2)$$

$$- P_2(t,\theta,\ -(\tau_1-\tau_2)|\tau_1,\tau_2)h_{1t}h_{2t}P_0(t|\tau_1,\tau_2)$$

(2.5d.18)

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \xi}\right) P_2(t,\theta,\xi|\tau_1,\tau_2) = -P_1(t,\theta|\tau_1,\tau_2)h_{2t}^2 P_1(t,\xi|\tau_1,\tau_2)$$

$$- P_2(t,\theta,-(\tau_1-\tau_2)|\tau_1,\tau_2)h_{1t}^2 P_2(t,-(\tau_1-\tau_2),\xi|\tau_1,\tau_2)$$

$$- P_1(t,\theta|\tau_1,\tau_2)h_{1t}h_{2t}P_2(t,-(\tau_1-\tau_2),\xi|\tau_1,\tau_2)$$

$$- P_2(t,\theta,-(\tau_1-\tau_2)|\tau_1,\tau_2)h_{1t}h_{2t}P_1(t,\xi|\tau_1,\tau_2)$$

(2.5d.19)

The initial conditions are

$$P_0(\tau_2|\tau_1,\tau_2) = \Sigma_0,\ P_1(\tau_2,\theta|\tau_1,\tau_2) = 0, \qquad P_2(\tau_2,\theta,\xi|\tau_1,\tau_2) = 0$$

for $\theta<0$, $\xi<0$ (2.5d.20)

$$P_1(t,0|\tau_1,\tau_2) = P_0(t|\tau_1,\tau_2) \tag{2.5d.21}$$

$$P_2(t,\theta,0|\tau_1,\tau_2) = P_1(t,\theta|\tau_1,\tau_2) \tag{2.5d.22}$$

Representations for the Signal Estimates

We have seen that it is impossible to generate the representation, recursive or nonrecursive, for $\hat{\phi}_{t-t_{s1}}$ in Section 2.5b. The only way to generate the estimate $\hat{\phi}_{t-t_{s1}}$ is by means of the multiple-model approach even in the linear Gaussian case. Thus, the computation of the estimate $\hat{\phi}_{t-t_{s1}}$ is infinite dimensional in all cases.

Consider now the representation (2.5b.7) for the estimate $\hat{\phi}_{t-t_{s2}}$ in the linear Gaussian case. This representation becomes

$$d\hat{\phi}_{t-t_{s2}} = (\rho_{2t}(1-\hat{\psi}_{2t|t})\ E\{\phi_0\} + E\{\psi_{2t-}\alpha_{t-t_{s2}}\phi_{t-t_{s2}}|Z_t\})dt$$
$$+ E\{h_{1t}\phi_{t-t_{s2}}\phi_{t-t_{s1}} + h_{2t}\phi^2_{t-t_{s2}}$$
$$- h_{1t}\hat{\phi}_{t-t_{s2}}\hat{\phi}_{t-t_{s1}} - h_{2t}\hat{\phi}^2_{t-t_{s2}}|Z_t\}d\nu_t \tag{2.5d.23}$$

where

$$d\nu_t = dz_t - h_{1t}\hat{\phi}_{t-t_{s1}}\ dt - h_{2t}\hat{\phi}_{t-t_{s2}}\ dt \tag{2.5d.24}$$

Evidently, the filter (2.5d.23) is always infinite dimensional because the innovations process ν_t involves the estimate $\hat{\phi}_{t-t_{s1}}$. However, we are interested in finding cases in which the remaining terms on the right of the filter (2.5d.23) is implementable.

When $\alpha_t = \alpha$, a constant, the second term becomes

$$E\{\psi_{2t-}\ \alpha_{t-t_{s2}}\phi_{t-t_{s2}}|Z_t\} = \alpha E\{\psi_{2t-}\phi_{t-t_{s2}}|Z_t\}$$
$$= \alpha\hat{\phi}_{t-t_{s2}} \tag{2.5d.25}$$

The last step follows as in the one-course case. The second term is now finite dimensional.

For the third term, we write

$$E\{h_{1t}\phi_{t-t_{s2}}\phi_{t-t_{s1}} + h_{2t}\phi^2_{t-t_{s2}}$$

$$-h_{1t}\hat{\phi}_{t-t_{s2}}\hat{\phi}_{t-t_{s1}} - h_{2t}\hat{\phi}^2_{t-t_{s2}}|Z_t\}$$

$$= h_{1t}E\{(\phi_{t-t_{s2}}-\hat{\phi}_{t-t_{s2}})\ (\phi_{t-t_{s1}}-\hat{\phi}_{t-t_{s1}})|Z_t\}$$

$$+ h_{2t}E\{(\phi_{t-t_{s2}}-\hat{\phi}_{t-t_{s2}})^2|Z_t\} \qquad (2.5d.26)$$

We now compute each of the two terms on the right-hand side of (2.5d.26) using the multiple-model philosophy:

$$h_{1t}\ E\{(\phi_{t-t_{s2}}-\hat{\phi}_{t-t_{s2}})(\phi_{t-t_{s1}}-\hat{\phi}_{t-t_{s1}})|Z_t\}$$

$$= h_{1t}\int_{\tau_1=0}^{t}\int_{\tau_2=0}^{\tau_1} P_1(t,-(\tau_1-\tau_2)|\tau_1,\tau_2)\mathcal{P}(\tau_1<t_{s1}\leq\tau_1+d\tau_1,$$

$$\tau_2<t_{s2}\leq\tau_2+d\tau_2|Z_t) \qquad (2.5d.27)$$

where $P_1(.,.|\tau_1,\tau_2)$ is defined by equation (2.5d.15) and is precomputable by equation (2.5d.18). Similarly,

$$h_{2t}\ E\{(\phi_{t-t_{s2}} - \hat{\phi}_{t-t_{s2}})^2|Z_t\}$$

$$= h_{2t}\int_{\tau_1=0}^{\infty}\int_{\tau_2=0}^{\tau_1\wedge t} P_0(t|\tau_1,\tau_2)\mathcal{P}(\tau_1<t_{s1}\leq\tau_1 + d\tau_1,$$

$$\tau_2<t_{s2}\leq\tau_2+d\tau_2|Z_t) \qquad (2.5d.28)$$

where $P_0(t|\tau_1,\tau_2)$ is defined by equation

(2.5d.16) and is precomputable by equation (2.5d.17). Equations (2.5d.27) and (2.5d.28) are of course infinite dimensional and there does not yet seem to be any case in which they could become finite dimensional. Based on our experience with the one-source case, we expect them to become finite dimensional when $P_1(t, -(\tau_1-\tau_2)|\tau_1,\tau_2)$ and $P_0(t|\tau_1,\tau_2)$ are equal to their steady state values. However, these steady state values, although independent of t, would still be functions of τ_1 and τ_2 and so we would still have to evaluate the integrals in (2.5d.27) and (2.5d.28) over the appropriate ranges of τ_1 and τ_2 on-line. In fact, we expect P_1 and P_0 in the steady state to be functions of $\tau_1-\tau_2$ only. In spite of this difficulty, we can still hope to evaluate these two equations with finite dimensional computations suboptimally. The evaluation of the integrals in (2.5d.27) and (2.5d.28) in the steady state appears to be an interesting open problem.

The way in which equations (2.5d.27) and (2.5d.28) could be finite dimensional is when $P_1(t,-(\tau_1,\tau_2)|\tau_1,\tau_2)$ and $P_0(t|\tau_1,\tau_2)$ are equal to their steady state values, say $\overline{P}_1(\tau_1,\tau_2)$ and $\overline{P}_0(\tau_1,\tau_2)$ respectively, and we use an assumed joint density, characterized by a finite number of parameters, for t_{s1} and t_{s2}. Then, since $\overline{P}_1(\tau_1,\tau_2)$ and $\overline{P}_0(\tau_1,\tau_2)$ are precomputable and the on-line computation of the joint density of t_{s1} and t_{s2} involves only the on-line determination of the parameters characterizing the density, equations (2.5d.27) and (2.5d.28) are easily seen to be finite dimensional. In fact, they can be evaluated off-line in terms of the parameters of the assumed density and once the parameters are determined, the values of the two covariances are obtained. Of course, this is only a suboptimal approach based on what we have done in the one-source case. We expect that the on-line determination of the parameters of the assumed joint density of t_{s1} and t_{s2} would involve the estimates $\hat{\psi}_{1t|t}$ and $\hat{\psi}_{2t|t}$ and we will examine these estimates shortly.

The conditions under which $P_0(t|\tau_1,\tau_2)$ and $P_1(t,-(\tau_1-\tau_2)|\tau_1,\tau_2)$ are equal to their steady state values are discussed by Kwong [72]. We shall not discuss these conditions here and the interested reader should consult [72].

Finally, let us consider the computation of $\hat{\psi}_{1t|t}$ and $\hat{\psi}_{2t|t}$ which are given by the filters:

$$\begin{aligned} d\hat{\psi}_{1t|t} = {} & \rho_{1t}(1-\hat{\psi}_{1t|t})dt \\ & + [E\{\psi_{1t}(h_1(\phi_{t-t_{s1}},t) + h_2(\phi_{t-t_{s2}}, t))|Z_t\} \\ & \quad - \hat{\psi}_{1t\,t}(\hat{h}_1(\phi_{t-t_{s1}}, t) + \hat{h}_2(\phi_{t-t_{s2}},t))]d\nu_t , \\ & \hat{\psi}_{10|0} = 0 \end{aligned} \tag{2.5d.29}$$

$$\begin{aligned} d\hat{\psi}_{2t|t} = {} & \rho_{2t}(1-\hat{\psi}_{2t|t})dt \\ & + [E\{\psi_{2t}(h_1(\phi_{t-t_{s1}}, t) + h_2(\phi_{t-t_{s2}}, t))|Z_t\} \\ & - \hat{\psi}_{2t|t}(\hat{h}_1(\phi_{t-t_{s1}}, t) + \hat{h}_2(\phi_{t-t_{s2}}, t))]d\nu_t , \\ & \hat{\psi}_{20|0} = 0 \end{aligned} \tag{2.5d.30}$$

Equation (2.5d.29) has been presented before (see equation (2.5c.15)). The derivation of equation (2.5d.30) is similar. In the linear Gaussian case these reduce to:

$$d\hat{\psi}_{1t|t} = \rho_{1t}(1 - \hat{\psi}_{1t|t})dt$$

$$+ [h_{1t}\hat{\phi}_{t-t_{s1}}(1 - \hat{\psi}_{1t|t}) + h_{2t}(E\{\psi_{1t}\phi_{t-t_{s2}}|Z_t\}$$

$$- \hat{\psi}_{1t|t}\hat{\phi}_{t-t_{s2}})]d\nu_t ,$$

$$\hat{\psi}_{10|0} = 0 \qquad (2.5d.31)$$

$$d\hat{\psi}_{2t|t} = \rho_{2t}(1 - \hat{\psi}_{2t|t})dt$$

$$+ [h_{1t}\hat{\phi}_{t-t_{s1}}(1 - \hat{\psi}_{2t|t})$$

$$+ h_{2t}\hat{\phi}_{t-t_{s2}}(1 - \hat{\psi}_{2t|t})]d\nu_t ,$$

$$\hat{\psi}_{20|0} = 0 \qquad (2.5d.32)$$

The last two equations show that even in the linear Gaussian case, the computation of $\hat{\psi}_{1t|t}$ and $\hat{\psi}_{2t|t}$ are infinite dimensional. The reason is that the estimate $\hat{\phi}_{t-t_{s1}}$ is involved, which also occurs in the innovations process ν_t. There is an additional term involved in equation (2.5d.31) for $\hat{\psi}_{1t|t}$, namely $E\{\psi_{1t}\phi_{t-t_{s2}}|Z_t\}$. We will here examine what the computation of this term involves.

The derivation of the filter for $\widehat{\psi_{1t}\phi_{t-t_{s2}}}$ is straightforward along the lines we have derived other filters in this chapter and therefore is omitted. In the linear Gaussian case, this filter is given by

$$
\begin{aligned}
d\,\widehat{\psi_{1t}\phi_{t-t_{s2}}} &= [\rho_{1t}(\hat{\phi}_{t-t_{s2}} - \widehat{\psi_{1t}\phi_{t-t_{s2}}}) + \widehat{\psi_{1t-}\alpha_{t-t_{s2}}\phi_{t-t_{s2}}}]dt \\
&\quad + [h_{1t}\, E\{\phi_{t-t_{s1}}\phi_{t-t_{s2}} - \hat{\phi}_{t-t_{s1}}\widehat{\psi_{1t}\phi_{t-t_{s2}}}|Z_t\} \\
&\quad + h_{2t}\, E\{\psi_{1t}\phi^2_{t-t_{s2}} - \hat{\phi}_{t-t_{s2}}\widehat{\psi_{1t}\phi_{t-t_{s2}}}|Z_t\}]d\nu_t
\end{aligned}
\tag{2.5d.33}
$$

This filter is of course infinite dimensional and the terms involved in this filter are very similar to those in the filter (2.5d.23) for $\hat{\phi}_{t-t_{s2}}$. The second term is finite dimensional only when $\alpha_t = \alpha$ so that

$$
\widehat{\psi_{1t-}\alpha_{t-t_{s2}}\phi_{t-t_{s2}}} = \alpha\,\widehat{\psi_{1t}\phi_{t-t_{s2}}}
\tag{2.5d.34}
$$

The only term which is new and requires consideration is:

$$
\begin{aligned}
h_{2t}\, E\{\psi_{1t}\phi^2_{t-t_{s2}}|Z_t\} &= h_{2t}\, E\{\psi_{1t}(\phi^2_{t-t_{s2}} - \hat{\phi}^2_{t-t_{s2}})|Z_t\} \\
&\quad + h_{2t}E\{\psi_{1t}\hat{\phi}^2_{t-t_{s2}}|Z_t\} \\
&= h_{2t}E\{\psi_{1t}(\phi^2_{t-t_{s2}} - \hat{\phi}^2_{t-t_{s2}})|Z_t\} \\
&\quad + h_{2t}\hat{\psi}_{1t|t}\hat{\phi}^2_{t-t_{s2}}
\end{aligned}
\tag{2.5d.35}
$$

Now,

$$
\begin{aligned}
&E\{\psi_{1t}(\phi^2_{t-t_{s2}} - \hat{\phi}^2_{t-t_{s2}})|Z_t\} \\
&= \int_{\tau_1=0}^{t} \int_{\tau_2=0}^{\tau_1} P_0(t|\tau_1,\tau_2)\mathcal{P}(\tau_1 < t_{s1} \le \tau_1 + d\tau_1,\ \tau_2 < t_{s2} \le \tau_2 + d\tau_2 | Z_t)
\end{aligned}
\tag{2.5d.36}
$$

Equation (2.5d.36) is essentially the same as equation (2.5d.28) and we shall avoid further discussion. Notice that the filter (2.5d.33) does not introduce any new term which requires another filter for its computation. This is fortunate because otherwise the new filter might introduce terms which again lead to new filters. We have ended up here with a finite bank of filters.

We summarize our discussion here as follows. The representation for $\hat{\phi}_{t-t_{s2}}$ in the linear Gaussian case is infinite dimensional in all cases because the estimate $\hat{\phi}_{t-t_{s1}}$ is required. The terms involved in this filter lead to a finite bank of filters for their computation. Even when the covariances $P_1(t, -(\tau_1-\tau_2)|\tau_1,\tau_2)$ and $P_0(t|\tau_1,\tau_2)$ are equal to their steady state values, the covariances on the right of the filter for $\hat{\phi}_{t-t_{s2}}$ still require an infinite amount of on-time computations. The computation of the estimates $\hat{\psi}_{1t|t}$ and $\hat{\psi}_{2t|t}$ is also always infinite dimensional. The main reason why the estimation of $\phi_{t-t_{s2}}$ is always infinite dimensional is that the estimate $\hat{\phi}_{t-t_{s1}}$ is needed. When the delay times t_{s1} and t_{s2} are known, we have seen earlier in this section that the estimates of $\phi_{t-t_{s1}}$ and $\phi_{t-t_{s2}}$ are readily computed by a smoother and a linear filter respectively. (See equations (2.5d.11) and (2.5d.12). In our case in which t_{s1} and t_{s2} are unknown, it is impossible to generalize the smoother (2.5d.12) to one for $\hat{\phi}_{t-t_{s1}}$. At present, the only way to generate $\hat{\phi}_{t-t_{s1}}$ is the infinite dimensional multiple model approach employing a growing infinite bank of smoothers. This is the problem that makes our estimation problem much more difficult than when the delay times are known. The problem of finding a representation for $\hat{\phi}_{t-t_{s1}}$, whether recursive or non-recursive, remains unsolved at present. The only way we can hope to find cases in which the computation of $\hat{\phi}_{t-t_{s1}}$ is finite dimensional is by first deriving a representation for $\hat{\phi}_{t-t_{s1}}$. Otherwise, using the multiple model approach, we can only hope to get a suboptimal estimate of $\phi_{t-t_{s1}}$ by implementing a finite subset of the infinite bank of smoothers.

We conclude this section by listing the ways in which our problem here with random delay times is more complex compared to the case of known delays. (1) We need a growing infinite bank of smoothers to compute $\hat{\phi}_{t-t_{s1}}$. In the case of known delays, the computation of this term involves a single smoother which is still infinite dimensional. (2) We have to compute $\hat{\psi}_{2t|t}$ and in some cases $\hat{\psi}_{1t|t}$ also. These estimates do not arise in the known delay case but are analogous to the computation of $\hat{\psi}_{t|t}$ in the single source case. They take into account uncertainty in our knowledge of the delay times. (3) We have to compute $\psi_{1t}\phi_{t-t_{s2}}$. This is a new term not seen in the single source case. It adds an additional equation but does not complexify the system nearly as much as (1) above. (4) Even in the steady state, the covariances in the filter for $\hat{\phi}_{t-t_{s2}}$ require an infinite amount of on-line computations. This is unlike the one-source case in which the covariance is finite-dimensional in the steady state.

2.5e Implementation of Results in General

In this section, we will examine the requirements for implementing our results in general.

Consider first the representation (2.5b.7) for the estimate $\hat{\phi}_{t-t_{s2}}$:

$$d\hat{\phi}_{t-t_{s2}} = (\rho_{2t}(1 - \hat{\psi}_{2t|t})\, E\{\phi_0\} + E\{\psi_{2t-}\alpha(\phi_{t-t_{s2}},\, t-t_{s2})|Z_t\})dt$$

$$+ E\{\phi_{t-t_{s2}}\,(h_1(\phi_{t-t_{s1}},t) + h_2(\phi_{t-t_{s2}},t))$$

$$- \hat{\phi}_{t-t_{s2}}(\hat{h}_1(\phi_{t-t_{s1}},t) + \hat{h}_2(\phi_{t-t_{s2}},\, t))|Z_t\}.$$

$$.\ d\nu_t$$

$$\hat{\phi}_{t-t_{s2}}\Big|_{t=0} = 0 \qquad (2.5e.1)$$

Neglecting for the moment the first term, we see that to compute the remaining terms on the right requires either an infinite system of stochastic differential equations or carrying along the joint conditional probabilities $P(\phi_1 \leq \phi_{t-t_{s1}} < \phi_1 + d\phi_1,\ \phi_2 \leq \phi_{t-t_{s2}} < \phi_2 + d\phi_2, \tau_1 < t_{s1} \leq \tau_1 + d\tau_1,\ \tau_2 < t_{s2} \leq \tau_2 + d\tau_2 | Z_t)$. It is possible to write the stochastic differential equations for those terms which involve only $\phi_{t-t_{s2}}$. For those terms which involve both $\phi_{t-t_{s1}}$ and $\phi_{t-t_{s2}}$, this is impossible and the reason is given in [37]. The joint conditional probabilities are computed as

$$P(\phi_1 \leq \phi_{t-t_{s1}} < \phi_1 + d\phi_1,\ \ \phi_2 \leq \phi_{t-t_{s2}} < \phi_2 + d\phi_2,$$

$$\tau_1 < t_{s1} \leq \tau_1 + d\tau_1\ ,\ \tau_2 < t_{s2} \leq \tau_2 + d\tau_2 | Z_t)$$

$$= P_{\phi_{t-t_{s1}},\ \phi_{t-t_{s2}}}(\phi_1,\ \phi_2 \Big| Z_t,\ t_{s1}=\tau_1,\ t_{s2}=\tau_2)\, d\phi_1 d\phi_2\ .$$

$$.\ P(\tau_1 < t_{s1} \leq \tau_1 + d\tau_1,\ \tau_2 < t_{s2} \leq \tau_2 + d\tau_2 | Z_t) \qquad (2.5e.2)$$

We have seen in Section 2.5c how the joint conditional distribution of t_{s1} and t_{s2} is computed. To compute the joint conditional density $p_{\phi_{t-t_{s1}}, \phi_{t-t_{s2}}}(\phi_1, \phi_2 | Z_t, t_{s1}=\tau_1, t_{s2}=\tau_2)$ recursively is, however, impossible mainly because the conditional density $p_{\phi_{t-t_{s1}}}(\phi_1 | Z_t, t_{s1}=\tau_1, t_{s2}=\tau_2)$ cannot be computed recursively. The reason is the same as in Section 2.5b where we explained why the estimate $\hat{\phi}_{t-\tau_1} = E\{\phi_{t-t_{s1}} | Z_t, t_{s1}=\tau_1, t_{s2}=\tau_2\}$ cannot be computed recursively. It might still be possible to compute this joint conditional density by some other nonrecursive procedure which we do not have at present. In the linear Gaussian case, the joint conditional density above is Gaussian and is completely characterized by the mean and the covariance. The computation of the latter two quantities has been illustrated in Section 2.5d.

Finally, consider the first term on the right of equation (2.5e.1). The only quantity we have to compute on-line is the estimate $\hat{\psi}_{2t|t}$ which is given by

$$
\begin{aligned}
d\hat{\psi}_{2t|t} &= \rho_{2t}(1 - \hat{\psi}_{2t|t})dt \\
&\quad + [E\{\psi_{2t}(h_1(\phi_{t-t_{s1}}, t) + h_2(\phi_{t-t_{s2}}, t)) | Z_t\} \\
&\quad - \hat{\psi}_{2t|t}(\hat{h}_1(\phi_{t-t_{s1}}, t) + \hat{h}_2(\phi_{t-t_{s2}}, t))]d\nu_t, \\
\hat{\psi}_{20|0} &= 0
\end{aligned}
\tag{2.5e.3}
$$

where

$$
\rho_{2t} = p_{t_{s2}}(t) \Big/ \int_t^\infty p_{t_{s2}}(\tau)d\tau \tag{2.5e.4}
$$

This equation is derived in the same way as equation (2.5c.15) for $\hat{\psi}_{1t|t}$. As pointed out in Section 2.5c, the way to implement this equation is the infinite dimensional multiple-model approach in the general nonlinear case. Even in the linear time-invariant Gaussian case, we have seen that the filter for $\hat{\psi}_{2t|t}$ is still infinite dimensional.

Next, for the estimate $\hat{\phi}_{t-t_{s1}}$, we have seen in Section 2.5b that a representation, recursive or nonrecursive, is not possible in all cases. The only way to generate this estimate is by the infinite dimensional multiple model approach.

The remainder of the results on signal estimation via the multiple-model appraoch and on delay time estimation all require the same implementation - the multiple-model implementation using an infinite bank of filters. We have seen this in Section 2.5b and 2.5c. Thus, as in the one-source case, the complete solution to the overall problem of signal and delay time estimation can conceptually be considered to be given by a growing infinite bank of filters, one for each possible pair of values of the delay times t_{s1} and t_{s2}. See Figure 5 for the illustration in the one-source case. In the linear Gaussian case, the estimates $\hat{\phi}_{t-\tau_1}$, and $\hat{\phi}_{t-\tau_2}$ are the only quantities to be generated by the bank of filters and we have presented the estimation equations in Section 2.5d.

2.5f An Example Involving a Known Signal

In this section, we illustrate the results of the multiple source problem by an example involving a rectangular pulse signal. For the primary motivation for considering known signal examples, refer to Section 2.3f. Here, we can add that in the two reflector case of Figure

10, sending a known signal from the source and processing the reflections at the sensor enable us to deduce the placement of the reflectors provided the velocity of the transmission field is known.

The signal ϕ_t is a rectangular pulse given by

$$\phi_t = \begin{cases} 0 \;, & t<0 \\ 1 \;, & 0\leq t\leq T \\ 0 \;, & t>T \end{cases} \tag{2.5f.1}$$

and assume a linear observation model

$$dz_t = h_{1t}\phi_{t-t_{s1}} dt + h_{2t}\phi_{t-t_{s2}} dt + dw_t \tag{2.5f.2}$$

The signal estimates are given by

$$\hat{\phi}_{t-t_{s1}} = \hat{\psi}_{1t|t} - \hat{\psi}_{1t-T|t} \tag{2.5f.3}$$

and

$$\hat{\phi}_{t-t_{s2}} = \hat{\psi}_{2t|t} - \hat{\psi}_{2t-T|t} \tag{2.5f.4}$$

Thus, as in the one-source case both the signal and delay time estimation problems are solved by computing the joint conditional distribution of t_{s1} and t_{s2}. From this joint conditional distribution, we can obtain the marginal conditional distributions $\hat{\psi}_{1\tau_1|t}$ and $\hat{\psi}_{2\tau_2|t}$ by which the signal estimates $\hat{\phi}_{t-t_{s1}}$ and $\hat{\phi}_{t-t_{s2}}$ are computed. In the rest of this section, we will discuss the computation of the joint conditional distribution of t_{s1} and t_{s2} for which the estimates $\hat{\psi}_{1\tau_1|t}$ and $\hat{\psi}_{2\tau_2|t,\tau_1}$ are required. See Section 2.5c.

The equations for computing the estimates $\hat{\psi}_{1\tau_1|t}$ are now given as follows. See equations (2.5c.15) through (2.5c.18).

$$\begin{aligned}
\tau_1 = t: \quad d\hat{\psi}_{1t|t} &= \rho_{1t}(1 - \hat{\psi}_{1t|t})dt \\
&\quad + [h_{1t}(\hat{\psi}_{1t|t} - \hat{\psi}_{1t-T|t})(1 - \hat{\psi}_{1t|t}) \\
&\quad + h_{2t}\, E\{\psi_{1t}\phi_{t-t_{s2}}|Z_t\} \\
&\quad - h_{2t}\hat{\psi}_{1t|t}(\hat{\psi}_{2t|t} - \hat{\psi}_{2t-T|t})]d\nu_t, \\
\hat{\psi}_{10|0} &= 0
\end{aligned} \tag{2.5f.5}$$

$$\tau_1 > t: \quad \hat{\psi}_{1\tau_1|t} = 1 - (1 - \hat{\psi}_{1t|t})\frac{\mathcal{P}(t_{s1} \geq \tau_1)}{\mathcal{P}(t_{s1} \geq t)} \tag{2.5f.6}$$

$$\tau_1 < t: \quad \hat{\psi}_{1\tau_1|t} = \hat{\psi}_{1\tau_1|\tau_1} + \int_{\tau_1}^{t} \Sigma_1(\tau_1,\tau)d\nu_\tau \tag{2.5f.7}$$

where

$$\begin{aligned}
\Sigma_1(\tau_1, t) &= h_{1t}(E\{\psi_{1\tau_1}\phi_{t-t_{s1}}|Z_t\} \\
&\qquad - \hat{\psi}_{1\tau_1|t}(\hat{\psi}_{1t|t} - \hat{\psi}_{1t-T|t})) \\
&\quad + h_{2t}(E\{\psi_{1\tau_1}\phi_{t-t_{s2}}|Z_t\} \\
&\qquad - \hat{\psi}_{1\tau_1|t}(\hat{\psi}_{2t|t} - \hat{\psi}_{2t-T|t}))
\end{aligned} \tag{2.5f.8}$$

The terms that require infinite dimensional on-line computations are the following:

$$E\{\psi_{1\tau_1}\phi_{t-t_{s2}}|Z_t\}$$

$$=\int_{0\le t_2'\le t_1'\le\tau_1} E\{\phi_{t-t_{s2}}|Z_t, t_{s1}=t_1', t_{s2}=t_2'\}\cdot$$

$$\cdot\ \mathcal{P}(t_1'<t_{s1}\le t_1'+dt_1',\ t_2'<t_{s2}\le t_2'+dt_2'|Z_t)$$

$$=\int_{0\le t_2'\le t_1'\le\tau_1} \phi_{t-t_2'}\mathcal{P}(t_1'<t_{s1}\le t_1'+dt_1',\ t_2'<t_{s2}\le t_2'+dt_2'|Z_t)$$

$$=\begin{cases}\mathcal{P}(t_{s1}\le\tau_1,\ t_{s2}\le\tau_1|Z_t), & 0\le\tau_1<t\le T\\ \mathcal{P}(t_{s1}\le\tau_1,\ t-T<t_{s2}\le\tau_1|Z_t), & 0\le t-T\le\tau_1<t\\ 0, & 0\le\tau_1\le t-T\end{cases}\tag{2.5f.9}$$

(in equation (2.5f.8)),

$$E\{\psi_{1t}\phi_{t-t_{s2}}|Z_t\}=\begin{cases}\mathcal{P}(t_{s1}\le t,\ t_{s2}\le t|Z_t), & 0\le t\le T\\ \mathcal{P}(t_{s1}\le t,\ t-T<t_{s2}\le t|Z_t), & T<t\end{cases}\tag{2.5f.10}$$

(in equation (2.5f.5)).

$$E\{\psi_{1\tau_1}\phi_{t-t_{s1}}|Z_t\}$$

$$=\int_{0\le t_2'\le t_1'\le\tau_1} E\{\phi_{t-t_{s1}}|Z_t,\ t_{s1}=t_1',\ t_{s2}=t_2'\}\cdot$$

$$\cdot\ \mathcal{P}(t_1'<t_{s1}\le t_1'+dt_1',\ t_2'<t_{s2}\le t_2'+dt_2'|Z_t)$$

$$=\int_{0\le t_2'\le t_1'\le\tau_1} \phi_{t-t_1'}\ \mathcal{P}(t_1'<t_{s1}\le t_1'+dt_1',\ t_2'<t_{s2}\le t_2'+dt_2'|Z_t)$$

$$=\begin{cases}\mathcal{P}(t_{s1}\le\tau_1,\ t_{s2}\le\tau_1|Z_t), & 0\le\tau_1<t\le T\\ \mathcal{P}(t-T<t_{s1}\le\tau_1,\ t_{s2}\le\tau_1|Z_t), & 0\le t-T\le\tau_1<t\\ 0, & 0\le\tau_1\le t-T\end{cases}\tag{2.5f.11}$$

(in equation (2.5f.8)).

Thus, the joint conditional distribution of t_{s1} and t_{s2} is the only quantity needed in implementing the equations for computing the estimates $\hat{\psi}_{1\tau_1|t}$ at each time t for all values of τ_1. To compute this joint conditionaldis-tribution, we need the estimates $\hat{\psi}_{2\tau_2|t,\tau_1}$ in addition to the estimates $\hat{\psi}_{1\tau_1|t}$ given above. The equations for computing the estimates $\hat{\psi}_{2\tau_2|t,\tau_1}$ are now given as follows. See equations (2.5c.5) through (2.5c.11).

$$\tau_2 = t: \quad d\hat{\psi}_{2t|t,\tau_1} = \rho'_{2t}(1 - \hat{\psi}_{2t|t,\tau_1})dt$$
$$+ [h_{1t}E\{\psi_{2t}\phi_{t-t_{s1}}|Z_t, t_{s1}=\tau_1\}$$
$$+ h_{2t}E\{\psi_{2t}\phi_{t-t_{s2}}|Z_t, t_{s1}=\tau_1\}$$
$$- h_{1t}\hat{\psi}_{2t|t,\tau_1}E\{\phi_{t-t_{s1}}|Z_t, t_{s1}=\tau_1\}$$
$$- h_{2t}\hat{\psi}_{2t|t,\tau_1}E\{\phi_{t-t_{s2}}|Z_t, t_{s1}=\tau_1\}] \cdot d\nu_{t|\tau_1},$$
$$\hat{\psi}_{20|0,\tau_1} = 0 \tag{2.5f.12}$$

$$\tau_2 > t: \quad \hat{\psi}_{2\tau_2|t,\tau_1} = 1 - (1 - \hat{\psi}_{2t|t,\tau_1}) \frac{P(t_{s2} \geq \tau_2 | t_{s1}=\tau_1)}{P(t_{s2} \geq t | t_{s1}=\tau_1)} \tag{2.5f.13}$$

$$\tau_2 < t: \quad \hat{\psi}_{2\tau_2|t,\tau_1} = \hat{\psi}_{2\tau_2|\tau_2,\tau_1} + \int_{\tau_2}^{t} \Sigma_2(\tau_2, \tau|\tau_1)d\nu_{\tau|\tau_1} \tag{2.5f.14}$$

where

$$\Sigma_2(\tau_2, t|\tau_1) = h_{1t}\, E\{\psi_{2\tau_2}\phi_{t-t_{s1}}|Z_t, t_{s1}=\tau_1\}$$
$$+ h_{2t}\, E\{\psi_{2\tau_2}\phi_{t-t_{s2}}|Z_t, t_{s1}=\tau_1\}$$
$$- h_{1t}\hat{\psi}_{2\tau_2|t,\tau_1}\, E\{\phi_{t-t_{s1}}|Z_t, t_{s1}=\tau_1\}$$
$$- h_{2t}\hat{\psi}_{2\tau_2|t,\tau_1}\, E\{\phi_{t-t_{s2}}|Z_t, t_{s1}=\tau_1\} \tag{2.5f.15}$$

These equations can be further simplified. Equation (2.5f.12) simplifies to

$$\tau_2 = t : \quad d\hat{\psi}_{2t|t,\,\tau_1} = \rho'_{2t}(1 - \hat{\psi}_{2t|t,\,\tau_1})dt$$
$$+ h_{2t}\, E\{\phi_{t-t_{s2}} | Z_t \;,\; t_{s1}=\tau_1\} \cdot$$
$$\cdot\; (1 - \hat{\psi}_{2t|t,\,\tau_1})\; d\nu_{t|\tau_1} \;,$$
$$\hat{\psi}_{20|0,\,\tau_1} = 0 \qquad (2.5f.16)$$

since in equation (2.5f.12), the second and fourth terms are equal:

$$h_{1t}\, E\{\psi_{2t}\phi_{t-t_{s1}} | Z_t \;,\; t_{s1} = \tau_1\}$$
$$= h_{1t}\phi_{t-\tau_1}\, E\{\psi_{2t} | Z_t \;,\; t_{s1}=\tau_1\}$$
$$= h_{1t}\, \hat{\psi}_{2t|t,\,\tau_1} E\{\phi_{t-t_{s1}} | Z_t \;,\; t_{s1}=\tau_1\} \qquad (2.5f.17)$$

Similarly, equation (2.5f.15) simplifies to

$$\Sigma_2(\tau_2 \;,\; t|\tau_1) = h_{2t}E\{\psi_{2\tau_2}\phi_{t-t_{s2}} | Z_t \;,\; t_{s1}=\tau_1\}$$
$$- h_{2t}\, \hat{\psi}_{2\tau_2|t,\,\tau_1} E\{\phi_{t-t_{s2}} | Z_t,\; t_{s1}=\tau_1\} \qquad (2.5f.18)$$

The terms in the above equations which require infinite dimensional on-line computations are:

$$E\{\psi_{2\tau_2}\phi_{t-t_{s2}} | Z_t,\; t_{s1}=\tau_1\}$$
$$= \int_{0\le t'_2 \le \min(t,\, \tau_1,\, \tau_2)} \phi_{t-t'_2}\; P(t'_2 < t_{s2} \le t'_2 + dt'_2 \,|\, Z_t)$$
$$= \begin{cases} P(t_{s2} \le \tau_m | Z_t) \;, & 0 \le t \le T \\ P(t-T < t_{s2} \le \tau_m | Z_t), & 0 < T \le t \end{cases} \qquad (2.5f.19)$$

where

$$\tau_m = \min\ (t,\ \tau_1,\ \tau_2) \tag{2.5f.20}$$

(in equation (2.5f.18)),

$$E\{\phi_{t-t_{s2}} | Z_t\ ,\ t_{s1} = \tau_1\}$$

$$= \begin{cases} P(t_{s2} \le \tau'_m | Z_t)\ , & 0 \le t \le T \\ P(t-T < t_{s2} \le \tau'_m | Z_t), & 0 < T \le t \end{cases} \tag{2.5f.21}$$

where

$$\tau'_m = \min\ (t,\ \tau_1) \tag{2.5f.22}$$

(in equations (2.5f.16) and (2.5f.18)).

With the estimates $\hat{\psi}_{1\tau_1|t}$ and $\hat{\psi}_{2\tau_2|t,\ \tau_1}$ computed above, we can compute the joint conditional distribution of t_{s1} and t_{s2} as in Section 2.5c. Note again that this joint conditional distribution is the only quantity needed in implementing the whole set of equations above for $\hat{\psi}_{1\tau_1|t}$ and $\hat{\psi}_{2\tau_2|t,\tau_1}$. The marginal conditional distribution $\hat{\psi}_{2\tau_2|t}$ of t_{s2} can be obtained from the joint conditional distribution and it is used in computing the signal estimate $\hat{\phi}_{t-t_{s2}}$ by equation (2.5f.4).

As in the one-source case, the example here involving a known rectangular pulse signal does not require an infinite bank of filters for implementation. The implementation of the solution to the overall problem of signal and delay time estimation is still infinite dimensional but involves only an infinite amount of subtractions on-line. Both the signal estimation and delay time estimation problems are solved by computing on-line the joint conditional distribution of t_{s1} and t_{s2}.

2.5g Concluding Remarks on The Multiple Source Problem

We have now completed the analysis of the multiple source problem along the lines of the basic one-source-one-sensor case. A few points deserve mentioning before we move on to the next problem.

Estimation problems for systems with time delays, even if the delays are known, are inherently infinite dimensional [37]. If the delays are unknown, the problem becomes even more complicated. In our problem, the whole difficulty starts from the estimation of $\phi_{t-t_{s1}}$. A representation for $\hat{\phi}_{t-t_{s1}}$, recursive or nonrecursive, is at present impossible. The only way to compute this estimate is the infinite dimensional multiple model approach.

The main reason why a representation for $\hat{\phi}_{t-t_{s1}}$ is impossible is because the delay times are unknown. We have seen that conditioned on known values of the delay times, a nonrecursive representation is possible in the general nonlinear case although it is non-implementable. In the linear Gaussian case, this representation reduces to an implementable smoothing equation. However, it is impossible to generalize this representation to the case of unknown delays because without conditioning on known delays, the representation is not well defined. At present, it is not clear how unknown delays can be taken into account in the representation.

The manner in which our problem here with random delays is more difficult than the case of known delays can be summarized as follows. (1) We need a growing infinite bank of smoothers to compute $\hat{\phi}_{t-t_{s1}}$. In the case of known delays, this estimate is computed by a single smoother which is still infinite dimensional. Since the estimate $\hat{\phi}_{t-t_{s1}}$ is infinite dimensional, our solution to the entire problem of signal and delay time estimation is infinite dimensional in all cases because the estimate of $\phi_{t-t_{s1}}$ or $h(\phi_{t-t_{s1}}, t)$ is needed in generating the innovations. (2) Because of uncertainty in our knowledge of the delay times, we have to compute $\hat{\psi}_{1t|t}$ and $\hat{\psi}_{2t|t}$ which do not arise in the case of known delays. This is analogous to the computation of $\hat{\psi}_{t|t}$ in the one-source case. (3) We have to compute the estimate $\widehat{\psi_{1t}\phi_{t-t_{s2}}}$. This is a new term not seen in the single source case. It adds an additional equation but does not complexify the system nearly as much as (1) above. (4) The covariances in the filter for $\hat{\phi}_{t-t_{s2}}$ requires an infinite amount of on-line computations even in the steady state in the linear Gaussian case. This contrasts with the one source case in which the covariance is finite dimensional in the linear Gaussian case in the steady-state. In the case of known delays, the covariance in the linear Gaussian case is precomputable.

The only way we can suggest at present to obtain a finite dimensional suboptimal approximation for computing $\hat{\phi}_{t-t_{s1}}$ is to approximate the infinite bank of smoothers by a finite bank. Each of these smoothers is infinite dimensional and a finite dimensional approximation in this case is not known. Even if we use an assumed density characterized by a finite number of parameters for the delay times t_{s1} and t_{s2}, it is not possible to come up with a finite dimensional suboptimal implementation. The main reason is that the on-line computation of the parameters of this assumed density requires the estimates $\hat{\psi}_{1t|t}$ and $\hat{\psi}_{2t|t}$ and both these estimates require infinite dimensional on-line computations. At present, it is not clear if there is any other way to come up with a finite dimensional approximation to our results.

We have not worked out the case in which t_{s1} and t_{s2} take on a finite set of possible values. However, the results are very similar to the one-source case and we will just mention them here. For the signal estimation problem, the only solution is the multiple-model approach which in this case involves only a finite bank of estimators. The equations describing these estimators are the same as those presented in the previous sections on the multiple-model solution when t_{s1} and t_{s2} take on a continuum of values. For the delay time estimation problem, we compute the a posteriori probabilities $\mathcal{P}(t_{s1}=\tau_1^i, t_{s2}=\tau_2^i | Z_t)$ where (τ_1^i, τ_2^i) is one possible pair of values for t_{s1} and t_{s2}. The stochastic differential equations describing the evolution of these probabilities are similar to those presented in the one-source case.

2.6 The Multiple Sensor Problem

This section extends the analysis of our basic model to the case of a single signal source and multiple sensors. We shall only analyze the case of two sensors as shown in Figure 11; the extension to an arbitrary number of sensors is similar. The delay times involved are assumed to take on a continuous range of values and their joint a priori distribution is assumed to be absolutely continuous with respect to the Lebesque measure on the plane. The case of only a finite possible set of values of the delay times can also be worked out as in the one-sensor case.

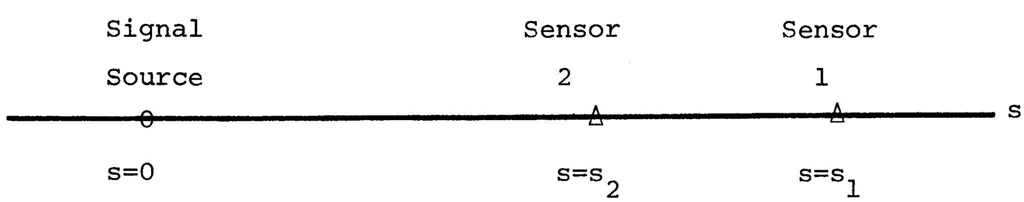

FIGURE 11: THE TWO SENSOR CASE

Sensor arrays are common in practice, e.g., seismometer arrays, antenna arrays and so on. We are interested in estimating the signal at the location of the sensors and the travel times of the signal from the source to each of the sensors. The latter estimates will enable us to estimate the travel time of the signal between the sensors. In many practical applications of sensor arrays, this estimate is very important. For instance, it is used in the resolution, i.e., bearing estimation, of propagating signal fields [33], [43]. Consider the situation in Figure 12 which is conceptually the same as our model in Figure 11.

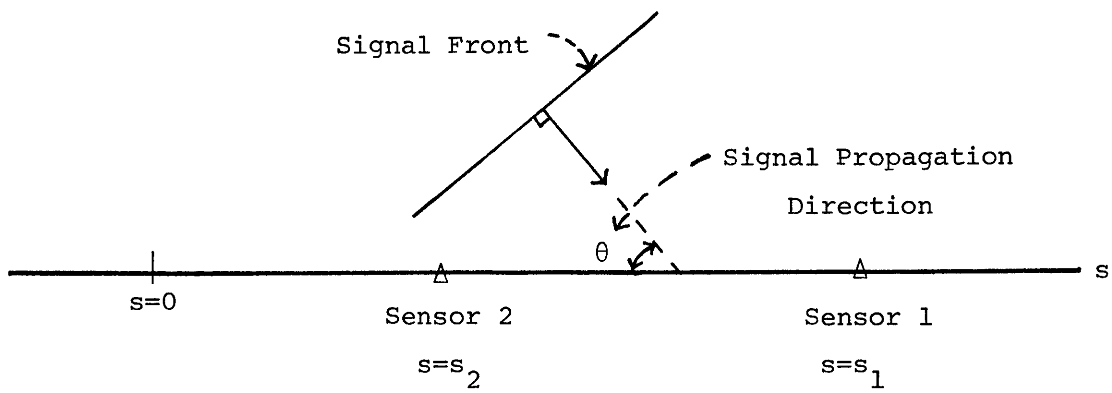

FIGURE 12: BEARING ESTIMATION FOR PROPAGATING SIGNAL FIELD

Assuming that the propagation velocity v of the transmission field is spatially constant and is known and assuming that the separation s_1-s_2 of the two sensors is known, we see that the travel time T of the signal field between the sensors and the propagation direction are related by

$$\frac{s_1-s_2}{v\cos\theta} = T \tag{2.6.1}$$

From this equation, we see that an estimate of the travel time T will give us an estimate of the signal propagation direction θ. Bearing estimation of propagating signal fields is in fact one of the main applications of sensor arrays.

We shall see that the analysis of the multiple sensor problem is very similar to that of the multiple source problem discussed in the last section.

2.6a Problem Formulation

Assume again the following Ito diffusion process model for the signal from the source:

$$\left.\begin{aligned} d\phi_t &= \alpha(\phi_t, t)dt + \underline{\gamma}'(\phi_t, t)d\underline{\eta}_t \quad, \; t>0 \\ \phi_0 &= \text{random with given distribution,} \\ \phi_t &= 0 \;, \qquad t<0 \end{aligned}\right\} \tag{2.6a.1}$$

Let t_{s1} and t_{s2} be the travel times of the signal from the source to sensor 1 and sensor 2 respectively. By the set-up of the model, we have

$$t_{s1} > t_{s2} \tag{2.6a.2}$$

The signals observed at sensors 1 and 2 are $\phi_{t-t_{s1}}$ and $\phi_{t-t_{s2}}$ respectively. We assume the following observation models:

$$\text{Sensor 1:} \quad dz_{1t} = h_1(\phi_{t-t_{s1}}, t)dt + dw_{1t} \tag{2.6a.3}$$

$$\text{Sensor 2:} \quad dz_{2t} = h_2(\phi_{t-t_{s2}}, t)dt + dw_{2t} \tag{2.6a.4}$$

The functions $h_1(\cdot,\cdot)$ and $h_2(\cdot,\cdot)$ are assumed to be jointly measurable with respect to both arguments. The processes w_{1t} and w_{2t} are independent standard Wiener processes, both independent of $\underline{\eta}_t$ and of ϕ_0. Thus, both w_{1t} and w_{2t} are independent of ϕ_t. It is also assumed that both w_{1t} and w_{2t} are independent of t_{s1} and of t_{s2}. Hence, both w_{1t} and w_{2t} are independent of $\phi_{t-t_{s1}}$ and $\phi_{t-t_{s2}}$. Since we are interested in collective processing of the measurements of the sensors, we define the cumulative observations

$$Z_t = \sigma\{z_{1\tau}, z_{2\tau}, 0\leq\tau\leq t\} \tag{2.6a.5}$$

The problems we are interested in are now:

(i) To estimate the signals $\phi_{t-t_{s1}}$ and $\phi_{t-t_{s2}}$,

(ii) To infer the properties of the transmission field.

As in the multiple source problem, we shall see that for the second problem above, we can compute on-line the joint a posteriori distribution $\mathcal{P}(t_{s1}\leq\tau_1, t_{s2}\leq\tau_2 | Z_t)$ of the delay times t_{s1} and t_{s2}.

2.6b Signal Estimation

We present here the representation results and the multiple-model solution to the signal estimation problem.

Dynamical Representations for Signal Estimates

Since the observations of the sensors are processed collectively, we define the increasing family $\{\mathcal{B}_t\}_{t\geq 0}$ of σ-fields to describe events at both sensors:

$$\mathcal{B}_t = \sigma\{w_{1\tau}, w_{2\tau}, 0\leq\tau\leq t\}\vee\sigma\{\phi_{\tau-t_{s1}}, \phi_{\tau-t_{s2}}, 0\leq\tau\leq t\}$$
$$\vee\sigma\{\{\omega: t_{s1}\leq\tau_1, t_{s2}\leq\tau_2\} \big| 0\leq\tau_1\leq t, 0\leq\tau_2\leq t\} \tag{2.6b.1}$$

This is similar to the definition in the multiple source problem. We are interested in the dynamical representations for the signal estimates

$$\hat{\phi}_{t-t_{s1}} = E\{\phi_{t-t_{s1}} | Z_t\} \tag{2.6b.2}$$

and

$$\hat{\phi}_{t-t_{s2}} = E\{\phi_{t-t_{s2}} | Z_t\} \tag{2.6b.3}$$

Following the same steps as in the multiple source problem, we have the representation for $\hat{\phi}_{t-t_{s2}}$:

$$\begin{aligned} d\hat{\phi}_{t-t_{s2}} &= (\rho_{2t}(1-\hat{\psi}_{2t|t})E\{\phi_0\} + E\{\psi_{2t}-\alpha(\phi_{t-t_{s2}},t-t_{s2}) | Z_t\})dt \\ &+ E\{(\phi_{t-t_{s2}}-\hat{\phi}_{t-t_{s2}})(h_1(\phi_{t-t_{s1}},t)-\hat{h}_1(\phi_{t-t_{s1}},t)) | Z_t\}d\nu_{1t} \\ &+ E\{(\phi_{t-t_{s2}}-\hat{\phi}_{t-t_{s2}})(h_2(\phi_{t-t_{s2}},t)-\hat{h}_2(\phi_{t-t_{s2}},t)) | Z_t\}d\nu_{2t} , \\ \hat{\phi}_{t-t_{s2}}\Big|_{t=0} &= 0 \end{aligned} \tag{2.6b.4}$$

where

$$d\nu_{1t} = dz_{1t} - \hat{h}_1(\phi_{t-t_{s1}},t)dt \tag{2.6b.5}$$

and

$$d\nu_{2t} = dz_{2t} - \hat{h}_2(\phi_{t-t_{s2}},t)dt \tag{2.6b.6}$$

are the innovations processes. The process ψ_{2t} is given by

$$\psi_{2t} = \begin{cases} 1 , & t \geq t_{s2} \\ 0 , & t < t_{s2} \end{cases} \tag{2.6b.7}$$

While

$$\rho_{2t} = P_{t_{s2}}(t) \Big/ \int_t^{\infty} P_{t_{s2}}(\tau)d\tau \tag{2.6b.8}$$

Note that the structure of this representation (2.6b.4) is very much the same as that of the representation (2.5b.7) for $\hat{\phi}_{t-t_{s2}}$ in the multiple source problem. We will discuss later the implementation of this filter in general and its specialization to the linear case.

To generate the representation for $\hat{\phi}_{t-t_{s1}}$ involves the same difficulty as in the multiple-source problem and the problem is not solvable in all cases. See Section 2.5b of the multiple source problem for the explanation. When t_{s1} and t_{s2} take on known values τ_1 and τ_2 respectively, a nonrecursive representation for $\hat{\phi}_{t-\tau_1}$ has been derived in [37], where

$$\hat{\phi}_{t-\tau_1} = E\{\phi_{t-t_{s1}} \Big| Z_t , t_{s1}=\tau_1 , t_{s2}=\tau_2\} \tag{2.6b.9}$$

We shall present this representation in the next section on the multiple-model solution. It is impossible to generalize this representation to the case of unknown random values of t_{s1} and t_{s2} because without conditioning on known values of t_{s1} and t_{s2}, the representation is not well defined.

Multiple-Model Solution

In the multiple-model approach, the signal estimates are given by

$$\hat{\phi}_{t-t_{s1}} = \int_{\tau_2\le\tau_1\le t} \hat{\phi}_{t-\tau_1} P(\tau_1<t_{s1}\le\tau_1+d\tau_1 , \tau_2<t_{s2}\le\tau_2+d\tau_2 \Big| Z_t) \tag{2.6b.10}$$

$$\hat{\phi}_{t-t_{s2}} = \int_{\substack{\tau_2\le t,\\ \tau_2\le\tau_1}} \hat{\phi}_{t-\tau_2} P(\tau_1<t_{s1}\le\tau_1+d\tau_1 ,\tau_2<t_{s2}\le\tau_2+d\tau_2 \Big| Z_t) \tag{2.6b.11}$$

where

$$\hat{\phi}_{t-\tau_1} = E\{\phi_{t-t_{s1}} \Big| Z_t\ ,\ t_{s1}=\tau_1\ ,t_{s2}=\tau_2\} \tag{2.6b.12}$$

and

$$\hat{\phi}_{t-\tau_2} = E\{\phi_{t-t_{s2}} \Big| Z_t\ ,\ t_{s1}=\tau_1\ ,t_{s2}=\tau_2\} \tag{2.6b.13}$$

The computation of the probabilities $P(\tau_1<t_{s1}\leq\tau_1+d\tau_1\ ,\tau_2<t_{s2}\leq\tau_2+d\tau_2 \big| Z_t)$ is discussed in the next section. The estimates $\hat{\phi}_{t-\tau_1}$ and $\hat{\phi}_{t-\tau_2}$ can be generated as follows. Given that $t_{s1}=\tau_1$ and $t_{s2}=\tau_2$, we have

$$d\phi_{t-\tau_1} = \alpha(\phi_{t-\tau_1},t-\tau_1)dt + \underline{\gamma}'(\phi_{t-\tau_1},t-\tau_1)d\underline{\eta}_{t-\tau_1}\ ,t\geq\tau_1 \tag{2.6b.14}$$

and

$$d\phi_{t-\tau_2} = \alpha(\phi_{t-\tau_2}\ ,t-\tau_2)dt + \underline{\gamma}'(\phi_{t-\tau_2}\ ,t-\tau_2)d\underline{\eta}_{t-\tau_2}\ ,t\geq\tau_2 \tag{2.6b.15}$$

and the observation equations are

$$dz_{1t} = h_1(\phi_{t-\tau_1},t)dt + dw_{1t} \tag{2.6b.16}$$

and

$$dz_{2t} = h_2(\phi_{t-\tau_2},t)dt + dw_{2t} \tag{2.6b.17}$$

Thus, the estimation equation for $\hat{\phi}_{t-\tau_2}$ is [38]:

$$0\leq t<\tau_2:\quad \hat{\phi}_{t-\tau_2} = 0 \tag{2.6b.18}$$

$$\tau_2 \le t < \tau_1: \quad d\hat{\phi}_{t-\tau_2} = \hat{\alpha}(\phi_{t-\tau_2}, t-\tau_2)dt$$
$$+ E\{(\phi_{t-\tau_2} - \hat{\phi}_{t-\tau_2})(h_2(\phi_{t-\tau_2}, t) - \hat{h}_2(\phi_{t-\tau_2}, t)) \Big| Z_t, t_{s1}=\tau_1, t_{s2}=\tau_2\} \cdot$$
$$\cdot (dz_{2t} - \hat{h}_2(\phi_{t-\tau_2}, t)dt),$$
$$\hat{\phi}_{t-\tau_2}\Big|_{t=\tau_2} = E\{\phi_0\} \qquad (2.6b.19)$$

$$\tau_1 \le t: \quad d\hat{\phi}_{t-\tau_2} = \hat{\alpha}(\phi_{t-\tau_2}, t-\tau_2)dt$$
$$+ E\{(\phi_{t-\tau_2} - \hat{\phi}_{t-\tau_2})(h_1(\phi_{t-\tau_1}, t) - \hat{h}_1(\phi_{t-\tau_1}, t)) \Big| Z_t, t_{s1}=\tau_1, t_{s2}=\tau_2\} \cdot$$
$$\cdot (dz_{1t} - \hat{h}_1(\phi_{t-\tau_1}, t)dt)$$
$$+ E\{(\phi_{t-\tau_2} - \hat{\phi}_{t-\tau_2})(h_2(\phi_{t-\tau_2}, t) - \hat{h}_2(\phi_{t-\tau_2} t)) \Big| Z_t, t_{s1}=\tau_1, t_{s2}=\tau_2\} \cdot$$
$$\cdot (dz_{2t} - \hat{h}_2(\phi_{t-\tau_2}, t)dt) \qquad (2.6b.20)$$

To implement the filter (2.6b.19) in the general nonlinear case, we can write a stochastic differential equation for each term on the right hand side, ending up with an infinite system of equations. However, for the filter (2.6b.20), the same procedure is not possible because it is impossible to write a stochastic differential equation for the term $E\{\phi_{t-\tau_2} h_1(\phi_{t-\tau_1}, t) \Big| Z_t, t_{s1}=\tau_1, t_{s2}=\tau_2\}$. This problem has been encountered before in the multiple source problem (see Section 2.5b) and we refer the reader to Kwong's thesis [37] where the problem is discussed. It is impossible to compute this term recursively by any means in the general nonlinear case. However, we shall see that in the linear Gaussian case, this term is precomputable.

Next, we turn to the problem of generating the estimate $\hat{\phi}_{t-\tau_1}$. We know that

$$\hat{\phi}_{t-\tau_1} = 0 , \qquad t<\tau_1 \tag{2.6b.21}$$

For $t \geq \tau_1$, since $\phi_{t-\tau_1}$ is a delayed version of $\phi_{t-\tau_2}$, the estimation of $\phi_{t-\tau_1}$ is a smoothing problem because we also have observations on $\phi_{t-\tau_2}$. In any case, it is impossible to generate the dynamical representation for $\hat{\phi}_{t-\tau_1}$. See the discussion in [37]. However, a nonrecursive representation has been derived in [37] and is given by

$$\begin{aligned}
\hat{\phi}_{t-\tau_1} &= \hat{\phi}_{t-(\tau_1-\tau_2)-\tau_2 \,|\, t-(\tau_1-\tau_2)} \\
&+ \int_{t-(\tau_1-\tau_2)}^{t} E\{E_0[\phi_{t-\tau_1} \,|\, \phi_{s-\tau_1}][h_1(\phi_{s-\tau_1},s)-\hat{h}_1(\phi_{s-\tau_1},s)] \,|\, Z_s, t_{s1}=\tau_1 , t_{s2}=\tau_2\} \cdot \\
&\qquad \cdot (dz_{1s}-\hat{h}_1(\phi_{s-\tau_1},s)ds) \\
&+ \int_{t-(\tau_1-\tau_2)}^{t} E\{E_0[\phi_{t-\tau_1} \,|\, \phi_{s-\tau_1}][h_2(\phi_{s-\tau_2},s)-\hat{h}_2(\phi_{s-\tau_2},s)] \,|\, Z_s , t_{s1}=\tau_1, t_{s2}=\tau_2\} \cdot \\
&\qquad \cdot (dz_{2s}-\hat{h}_2(\phi_{s-\tau_2} ,s)ds)
\end{aligned} \tag{2.6b.22}$$

Here $E_0\{\cdot\}$ denotes the expectation with respect to a probability measure $\mathcal{P}_0$ defined by

$$\begin{aligned}
\frac{d\mathcal{P}_0}{d\mathcal{P}} = \exp[&- \int_0^T h_1(\phi_{t-\tau_1},t)dw_{1t} - \int_0^T h_2(\phi_{t-\tau_2},t)dw_{2t} \\
&- \frac{1}{2} \int_0^T (h_1^2(\phi_{t-\tau_1},t) + h_2^2(\phi_{t-\tau_2},t))dt]
\end{aligned} \tag{2.6b.23}$$

where [0,T] is the interval of time over which our problem is defined. Under the probability measure P_0, the observation processes z_{1t} and z_{2t} are independent standard Wiener processes and thus intuitively, under the measure P_0, no measurements on $\phi_{t-\tau_1}$ and $\phi_{t-\tau_2}$ are made. Equation (2.6b.22) is only a representation which is incomputable in practice. In the linear Gaussian case, this representation reduces to a readily implementable smoothing equation which is given in Section 2.6d.

2.6c Delay Time Estimation

The on-line computation of the joint conditional distribution $P(t_{s1}\leq\tau_1, t_{s2}\leq\tau_2 | Z_t)$ is carried out as in Section 2.5c for the multiple source problem. We write

$$P(\tau_1<t_{s1}\leq\tau_1+d\tau_1\ ,\ \tau_2<t_{s2}\leq\tau_2+d\tau_2 | Z_t)$$

$$= P(\tau_2<t_{s2}\leq\tau_2+d\tau_2 | Z_t\ , t_{s1}=\tau_1)P(\tau_1<t_{s1}\leq\tau_1+d\tau_1 | Z_t) \qquad (2.6c.1)$$

The conditional distribution $P(t_{s2}\leq\tau_2 | Z_t\ , t_{s1}=\tau_1)$ is computed by considering an estimation problem on the process ψ_{2t} defined by equation (2.6b.7). Given that $t_{s1}=\tau_1$, events at the two sensors should be described by the increasing family $\{\mathcal{B}'_t\}_{t\geq 0}$ of σ-fields such that

$$\mathcal{B}'_t = \sigma\{w_{1\tau}, w_{2\tau}, 0\leq\tau\leq t\}\vee\sigma\{\phi_{\tau-\tau_1}, 0\leq\tau\leq t\}\vee\sigma\{\phi_{\tau-t_{s2}}, 0\leq\tau\leq t\}$$

$$\vee\sigma\{\{\omega: t_{s2}(\omega)\leq\tau\} | 0\leq\tau\leq t\wedge\tau_1\} \qquad (2.6c.2)$$

and the probabilities of these events should be assigned by the measure $P(\cdot | t_{s1}=\tau_1)$. The process ψ_{2t} now has the representation

$$d\psi_{2t} = \rho_{2t}'(1-\psi_{2t})dt + dm_{2t}' \tag{2.6c.3}$$

where

$$\rho_{2t}' = P_{t_{s2}}(t \mid t_{s1}=\tau_1) \Big/ \int_t^\infty P_{t_{s2}}(\tau \mid t_{s1}=\tau_1)d\tau \tag{2.6c.4}$$

and m_{2t}' is a martingale on $\{\mathcal{B}_t'\}_{t\geq 0}$. The conditional distribution $\hat{\psi}_{2\tau_2|t,\tau_1} = E\{\psi_{2\tau_2} \mid Z_t, t_{s1}=\tau_1\} = P(t_{s2}\leq\tau_2 \mid Z_t, t_{s1}=\tau_1)$ is now computed by the following equations whose derivations are direct extensions of those in Theorem 2.6:

$$\begin{aligned}
\tau_2=t: \quad d\hat{\psi}_{2t|t,\tau_1} &= \rho_{2t}'(1-\hat{\psi}_{2t|t,\tau_1})dt \\
&+ [E\{\psi_{2t}h_1(\phi_{t-t_{s1}},t) \mid Z_t, t_{s1}=\tau_1\} \\
&- \hat{\psi}_{2t|t,\tau_1}\hat{h}_1(\phi_{t-t_{s1}},t \mid \tau_1)]d\nu_{1t|\tau_1} \\
&+ [E\{\psi_{2t}h_2(\phi_{t-t_{s2}},t) \mid Z_t, t_{s1}=\tau_1\} \\
&- \hat{\psi}_{2t|t,\tau_1}\hat{h}_2(\phi_{t-t_{s2}},t \mid \tau_1)]d\nu_{2t|\tau_1}, \\
\hat{\psi}_{20|0,\tau_1} &= 0
\end{aligned} \tag{2.6c.5}$$

where

$$\hat{h}_1(\phi_{t-t_{s1}},t \mid \tau_1) = E\{h_1(\phi_{t-t_{s1}},t) \mid Z_t, t_{s1}=\tau_1\} \tag{2.6c.6}$$

$$\hat{h}_2(\phi_{t-t_{s2}},t|\tau_1) = E\{h_2(\phi_{t-t_{s2}},t)\Big|Z_t\ ,t_{s1}=\tau_1\} \tag{2.6c.7}$$

$$d\nu_{1t|\tau_1} = dz_{1t} - \hat{h}_1(\phi_{t-t_{s1}},t|\tau_1)dt \tag{2.6c.8}$$

and

$$d\nu_{2t|\tau_1} = dz_{2t} - \hat{h}_2(\phi_{t-t_{s2}},t|\tau_1)dt \tag{2.6c.9}$$

$$\tau_2>t:\quad \hat{\psi}_{2\tau_2|t,\tau_1} = 1 - (1-\hat{\psi}_{2t|t,\tau_1})\ \frac{P(t_{s2}\geq\tau_2\big|t_{s1}=\tau_1)}{P(t_{s2}\geq t\big|t_{s1}=\tau_1)} \tag{2.6c.10}$$

$$\tau_2<t:\quad \hat{\psi}_{2\tau_2|t,\tau_1} = \hat{\psi}_{2\tau_2|\tau_2,\tau_1}$$

$$+\int_{\tau_2}^{t} \Sigma_1(\tau_2\ ,\tau|\tau_1)d\nu_{1\tau|\tau_1}$$

$$+\int_{\tau_2}^{t} \Sigma_2(\tau_2\ ,\tau|\tau_1)d\nu_{2\tau|\tau_1} \tag{2.6c.11}$$

where

$$\Sigma_1(\tau_2\ ,t|\tau_1) = E\{\psi_{2\tau_2}h_1(\phi_{t-t_{s1}},t)\Big|Z_t\ ,t_{s1}=\tau_1\}$$

$$- \hat{\psi}_{2\tau_2|t,\tau_1}\hat{h}_1(\phi_{t-t_{s1}},t\Big|\tau_1) \tag{2.6c.12}$$

and

$$\Sigma_2(\tau_2\ ,t|\tau_1) = E\{\psi_{2\tau_2}h_2(\phi_{t-t_{s2}}\ ,t)\Big|Z_t\ ,t_{s1}=\tau_1\}$$

$$-\hat{\psi}_{2\tau_2|t,\tau_1}\hat{h}_2(\phi_{t-t_{s2}},t\Big|\tau_1) \tag{2.6c.13}$$

Next, the conditional distribution $P(t_{s1} \leq \tau_1 | Z_t)$ is obtained by considering an estimation problem on the process ψ_{1t} defined by

$$\psi_{1t} = \begin{cases} 1 , & t \geq t_{s1} \\ 0 , & t < t_{s1} \end{cases} \tag{2.6c.14}$$

This process has the representation

$$d\psi_{1t} = \rho_{1t}(1-\psi_{1t})dt + dm_{1t} \tag{2.6c.15}$$

where

$$\rho_{1t} = p_{t_{s1}}(t) \Bigg/ \int_t^\infty p_{t_{s1}}(\tau)d\tau \tag{2.6c.16}$$

and m_{1t} is a martingale on $\{\mathcal{B}_t\}_{t \geq 0}$. The conditional distribution $\hat{\psi}_{1\tau_1|t} = E\{\psi_{1\tau_1} | Z_t\} = P(t_{s1} \leq \tau \, | Z_t)$ is now computed by the following equations which are extensions of those in Theorem 2.6:

$$\begin{aligned} \tau_1 = t: \quad d\hat{\psi}_{1t|t} = {} & \rho_{1t}(1-\hat{\psi}_{1t|t})dt \\ & + [E\{\psi_{1t}h_1(\phi_{t-t_{s1}}, t) | Z_t\} \\ & \quad - \hat{\psi}_{1t|t} \hat{h}_1(\phi_{t-t_{s1}}, t)]d\nu_{1t} \\ & + [E\{\psi_{1t}h_2(\phi_{t-t_{s2}}, t) | Z_t\} \\ & \quad - \hat{\psi}_{1t|t} \hat{h}_2(\phi_{t-t_{s2}}, t)]d\nu_{2t} , \end{aligned}$$

$$\hat{\psi}_{10|0} = 0 \tag{2.6c.17}$$

where

$$d\nu_{1t} = dz_{1t} - \hat{h}_1(\phi_{t-t_{s1}}, t)dt \tag{2.6c.18}$$

and

$$d\nu_{2t} = dz_{2t} - \hat{h}_2(\phi_{t-t_{s2}}, t)dt \tag{2.6c.19}$$

$$\tau_1 > t: \quad \hat{\psi}_{1\tau_1}\Big|t = 1 - (1-\hat{\psi}_{1t|t}) \frac{P(t_{s1} \geq \tau_1)}{P(t_{s1} \geq t)} \tag{2.6c.20}$$

$$\tau_1 < t: \quad \hat{\psi}_{1\tau_1}\Big|t = \hat{\psi}_{1\tau_1}\Big|\tau_1 + \int_{\tau_1}^{t} \Sigma_1(\tau_1, \tau)d\nu_{1\tau}$$

$$+ \int_{\tau_1}^{t} \Sigma_2(\tau_1, \tau)d\nu_{2\tau} \tag{2.6c.21}$$

where

$$\Sigma_1(\tau_1, t) = E\{\psi_{1\tau_1} h_1(\phi_{t-t_{s1}}, t)\Big|Z_t\}$$
$$- \hat{\psi}_{1\tau_1}|t\ \hat{h}_1(\phi_{t-t_{s1}}, t) \tag{2.6c.22}$$

and

$$\Sigma_2(\tau_1, t) = E\{\psi_{1\tau_1} h_2(\phi_{t-t_{s2}}, t)\Big|Z_t\}$$
$$- \hat{\psi}_{1\tau_1}\Big|t\ \hat{h}_2(\phi_{t-t_{s2}}, t) \tag{2.6c.23}$$

With the equations for $P(t_{s2}\leq\tau_2 | Z_t, t_{s1}=\tau_1)$ and $P(t_{s1}\leq\tau_1 | Z_t)$, we can then generate the joint conditional distribution $P(t_{s1}\leq\tau_1, t_{s2}\leq\tau_2 | Z_t)$ on-line using equation (2.6c.1). All the above equations for on-line computation of $P(t_{s2}\leq\tau_2 | Z_t, t_{s1}=\tau_1)$ and $P(t_{s1}\leq\tau_1 | Z_t)$ have to be implemented via the multiple-model approach and the basic quantities to be computed are the estimates

$$\hat{h}_1(\phi_{t-\tau_1}, t) = E\{h_1(\phi_{t-t_{s1}}, t) \big| Z_t, t_{s1}=\tau_1, t_{s2}=\tau_2\} \tag{2.6c.24}$$

and

$$\hat{h}_2(\phi_{t-\tau_2}, t) = E\{h_2(\phi_{t-t_{s2}}, t) \big| Z_t, t_{s1}=\tau_1, t_{s2}=\tau_2\} \tag{2.6c.25}$$

To generate the equations for computing these estimates involves the same difficulties for $\hat{\phi}_{t-\tau_1}$ and $\hat{\phi}_{t-\tau_2}$. We shall examine in the next section the linear Gaussian case in which all these estimates are computable.

2.6d <u>The Linear Gaussian Case</u>

<u>The Multiple-Model Solution</u>

The solution here is again derived from Kwong's results [37]. The signal model (2.6a.1) becomes

$$\left.\begin{aligned} d\phi_t &= \alpha_t\phi_t dt + \underline{\gamma}_t' d\underline{\eta}_t \quad , \ t>0 \\ \phi_0 &= \text{random with given Gaussian distribution} \\ \phi_t &= 0 \quad , \ t<0 \end{aligned}\right\} \tag{2.6d.1}$$

and the observations at the sensors are

$$\text{Sensor 1:} \quad dz_{1t} = h_{1t}\phi_{t-t_{s1}}dt + dw_{1t} \tag{2.6d.2}$$

$$\text{Sensor 2:} \quad dz_{2t} = h_{2t}\phi_{t-t_{s2}}dt + dw_{2t} \tag{2.6d.3}$$

We have seen in the previous sections that the complete solution to the signal and delay time estimation problems can be given by the multiple model approach. In this case, the basic quantities required in the multiple model solution are

$$\hat{\phi}_{t-\tau_1} = E\{\phi_{t-t_{s1}} \mid Z_t, t_{s1}=\tau_1, t_{s2}=\tau_2\} \tag{2.6d.4}$$

and

$$\hat{\phi}_{t-\tau_2} = E\{\phi_{t-t_{s2}} \mid Z_t, t_{s1}=\tau_1, t_{s2}=\tau_2\} \tag{2.6d.5}$$

We shall just deal with the generation of these two estimates in the rest of this section. Given that $t_{s1}=\tau_1$ and $t_{s2}=\tau_2$, the observation models become

$$\text{Sensor 1:} \quad dz_{1t} = h_{1t}\phi_{t-\tau_1}dt + dw_{1t} \tag{2.6d.6}$$

$$\text{Sensor 2:} \quad dz_{2t} = h_{2t}\phi_{t-\tau_2}dt + dw_{2t} \tag{2.6d.7}$$

It is easy to see [37] that conditioned on the observations Z_t, the distribution of $\phi_{t+\theta}$, for any θ such that $-t\leq\theta\leq 0$, is Gaussian. This distribution is completely characterized by its mean and covariance. We will show below the equations for computing the conditional means $\hat{\phi}_{t-\tau_1}$ and $\hat{\phi}_{t-\tau_2}$ and their associated covariances.

$$t<\tau_2<\tau_1: \quad \hat{\phi}_{t-\tau_2} = \hat{\phi}_{t-\tau_1} = 0 \tag{2.6d.8}$$

$$\tau_2\leq t<\tau_1: \quad \hat{\phi}_{t-\tau_1} = 0 \tag{2.6d.9}$$

$$d\hat{\phi}_{t-\tau_2} = \alpha_{t-\tau_2}\hat{\phi}_{t-\tau_2}dt + P_0(t|\tau_1,\tau_2)h_{2t}(dz_{2t}-h_{2t}\hat{\phi}_{t-\tau_2}dt),$$

$$\hat{\phi}_{t-\tau_2}\Big|_{t=\tau_2} = E\{\phi_0\} \tag{2.6d.10}$$

where

$$P_0(t|\tau_1,\tau_2) = E\{(\phi_{t-\tau_2}-\hat{\phi}_{t-\tau_2})^2\Big|Z_t\ ,t_{s1}=\tau_1\ ,t_{s2}=\tau_2\} \tag{2.6d.11}$$

is given by the Riccati equation

$$\frac{dP_0(t|\tau_1,\tau_2)}{dt} = 2\alpha_{t-\tau_2}P_0(t|\tau_1,\tau_2) + \underline{\gamma}'_{t-\tau_2}\underline{\gamma}_{t-\tau_2}-h^2_{2t}P^2_0(t|\tau_1,\tau_2),$$

$$P_0(\tau_2|\tau_1,\tau_2) = E\{(\phi_0- E\{\phi_0\})^2\} = \Sigma_0 \quad \text{given} \tag{2.6d.12}$$

$$\begin{aligned}\tau_2<\tau_1\leq t: \quad d\hat{\phi}_{t-\tau_2} &= \alpha_{t-\tau_2}\hat{\phi}_{t-\tau_2}dt \\ &+ h_{1t}P_1(t,-(\tau_1-\tau_2)|\tau_1,\tau_2)(dz_{1t}-h_{1t}\hat{\phi}_{t-\tau_1}dt) \\ &+ h_{2t}P_0(t|\tau_1,\tau_2)(dz_{2t}-h_{2t}\hat{\phi}_{t-\tau_2}dt)\end{aligned} \tag{2.6d.13}$$

$$\hat{\phi}_{t-\tau_1} = \hat{\phi}_{t-\tau_1|t-(\tau_1-\tau_2)}$$

$$+ \int_{t-(\tau_1-\tau_2)}^{t} h_{1s} P_2(s,t-(\tau_1-\tau_2)-s,-(\tau_1-\tau_2)|\tau_1,\tau_2)(dz_{1s}-h_{1s}\hat{\phi}_{s-\tau_1}ds)$$

$$+ \int_{t-(\tau_1-\tau_2)}^{t} h_{2s} P_2(s,t-(\tau_1-\tau_2)-s,0|\tau_1,\tau_2)(dz_{2s}-h_{2s}\hat{\phi}_{s-\tau_2}ds) \qquad (2.6d.14)$$

The term $\hat{\phi}_{t-\tau_1|t-(\tau_1-\tau_2)}$ in the last equation is the filter estimate generated by equation (2.6d.13):

$$\hat{\phi}_{t-\tau_1|t-(\tau_1-\tau_2)} = \hat{\phi}_{t-(\tau_1-\tau_2)-\tau_2|t-(\tau_1-\tau_2)}$$

$$= E\{\phi_{t-(\tau_1-\tau_2)-\tau_2}|Z_{t-(\tau_1-\tau_2)}, t_{s1}=\tau_1, t_{s2}=\tau_2\} \qquad (2.6d.15)$$

All the above equations are better understood if we write $x(t) = \phi_{t-\tau_2}$ and $x(t-(\tau_1-\tau_2)) = \phi_{t-\tau_1}$. The definitions of the covariances P_i, $i=0,1,2$ are the same as in the multiple-source problem:

$$P_2(t,\theta,\xi|\tau_1,\tau_2) = E\{(\phi_{t-\tau_2+\theta} - \hat{\phi}_{t-\tau_2+\theta})(\phi_{t-\tau_2+\xi} - \hat{\phi}_{t-\tau_2+\xi})$$
$$|Z_t, t_{s1}=\tau_1, t_{s2}=\tau_2\} \qquad (2.6d.16)$$

$$P_1(t,\theta|\tau_1,\tau_2) = P_2(t,\theta,0|\tau_1,\tau_2) \qquad (2.6d.17)$$

$$P_0(t|\tau_1,\tau_2) = P_2(t,0,0|\tau_1,\tau_2) \qquad (2.6d.18)$$

where $\theta \leq 0$, $\xi \leq 0$. These covariances can be precomputed by the following equations which are derived from Kwong's results [37]:

$$\frac{d}{dt} P_0(t|\tau_1,\tau_2) = 2\alpha_{t-\tau_2} P_0(t|\tau_1,\tau_2) - h_{2t}^2 P_0^2(t|\tau_1,\tau_2) + \underline{\gamma}'_{t-\tau_2}\underline{\gamma}_{t-\tau_2}$$

$$- h_{1t}^2 P_1^2(t,-(\tau_1-\tau_2)|\tau_1,\tau_2) \tag{2.6d.19}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta}\right) P_1(t,\theta|\tau_1,\tau_2) = \alpha_{t-\tau_2} P_1(t,\theta|\tau_1,\tau_2) - h_{2t}^2 P_0(t|\tau_1,\tau_2) P_1(t,\theta|\tau_1,\tau_2)$$

$$- h_{1t}^2 P_1(t,-(\tau_1-\tau_2)|\tau_1,\tau_2) P_2(t,\theta,-(\tau_1-\tau_2)|\tau_1,\tau_2) \tag{2.6d.20}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \xi}\right) P_2(t,\theta,\xi|\tau_1,\tau_2) = - h_{2t}^2 P_1(t,\theta|\tau_1,\tau_2) P_1(t,\xi|\tau_1,\tau_2)$$

$$- h_{1t}^2 P_2(t,\theta,-(\tau_1-\tau_2)|\tau_1,\tau_2) P_2(t,\xi,-(\tau_1-\tau_2)|\tau_1,\tau_2) \tag{2.6d.21}$$

The initial conditions are

$$P_0(\tau_2|\tau_1,\tau_2) = \Sigma_0 \text{ , } P_1(\tau_2,\theta|\tau_1,\tau_2)=0, \; P_2(\tau_2,\theta,\xi|\tau_1,\tau_2)=0 \text{ ,}$$

for $\theta<0$ and $\xi<0$. (2.6d.22)

$$P_1(t,0|\tau_1,\tau_2) = P_0(t|\tau_1,\tau_2) \tag{2.6d.23}$$

$$P_2(t,\theta,0|\tau_1,\tau_2) = P_1(t,\theta|\tau_1,\tau_2) \tag{2.6d.24}$$

Representations for the Signal Estimates

We have seen that it is impossible to generate the representation for $\hat{\phi}_{t-t_{s1}}$, recursive or nonrecursive, and the only way to compute this estimate is the infinite dimensional multiple model approach even in the linear Gaussian case. Thus, we shall only discuss the representation for the estimate $\hat{\phi}_{t-t_{s2}}$.

In the linear Gaussian case, the representation (2.6b.4) for $\hat{\phi}_{t-t_{s2}}$ reduces to

$$
\begin{aligned}
d\hat{\phi}_{t-t_{s2}} &= (\rho_{2t}(1-\hat{\psi}_{2t|t})E\{\phi_0\} + E\{\psi_{2t}-\alpha_{t-t_{s2}}\phi_{t-t_{s2}}\Big| Z_t\})dt \\
&\quad + h_{1t}E\{(\phi_{t-t_{s2}}-\hat{\phi}_{t-t_{s2}})(\phi_{t-t_{s1}}-\hat{\phi}_{t-t_{s1}})\Big|Z_t\}d\nu_{1t} \\
&\quad + h_{2t}E\{(\phi_{t-t_{s2}}-\hat{\phi}_{t-t_{s2}})^2\Big|Z_t\}d\nu_{2t}\,, \\
&\hat{\phi}_{t-t_{s2}}\Big|_{t=0} = 0
\end{aligned}
\tag{2.6d.25}
$$

The structure of each term in this filter is exactly the same as that of the filter (2.5d.23) for $\hat{\phi}_{t-t_{s2}}$ in the linear Gaussian case of the multiple source problem. We shall therefore avoid the details and just state the results concerning the conditions under which each term becomes finite dimensional.

For the first term on the right of the filter (2.6d.25), we only need to comput $\hat{\psi}_{2t|t}$ on-line. This estimate is given by the filter

$$d\hat{\psi}_{2t|t} = \rho_{2t}(1-\hat{\psi}_{2t|t})dt$$

$$+ h_{1t}\hat{\phi}_{t-t_{s1}}(1-\hat{\psi}_{2t|t})d\nu_{1t}$$

$$+ h_{2t}\hat{\phi}_{t-t_{s2}}(1-\hat{\psi}_{2t|t})d\nu_{2t} \quad ,$$

$$\hat{\psi}_{20|0} = 0 \tag{2.6d.26}$$

which is derived in a manner similar to equation (2.6c.17) for $\hat{\psi}_{1t|t}$. It is immediately obvious that this filter is infinite dimensional because it requires the estimate $\hat{\phi}_{t-t_{s1}}$. The only way we can avoid computing the estimate $\hat{\psi}_{2t|t}$ is when $t_1 \le t_{s2} \le t_2$ and we are in the region $t \ge t_2$ so that $\hat{\psi}_{2t|t} = 1$.

For the second term on the right of the filter (2.6d.25), the on-line computation is finite dimensional if $\alpha_t = \alpha$, a constant, so that

$$E\{\psi_{2t-}\alpha_{t-t_{s2}}\phi_{t-t_{s2}}\Big|Z_t\} = \alpha E\{\psi_{2t-}\phi_{t-t_{s2}}\Big|Z_t\}$$

$$= \alpha\hat{\phi}_{t-t_{s2}} \tag{2.6d.27}$$

For the last two terms, we have

$$h_{1t}\, E\{(\phi_{t-t_{s2}}-\hat{\phi}_{t-t_{s2}})(\phi_{t-t_{s1}}-\hat{\phi}_{t-t_{s1}})\Big|Z_t\}$$

$$= h_{1t}\int_{\tau_1=0}^{t}\int_{\tau_2=0}^{\tau_1} P_1(t,-(\tau_1-\tau_2)\Big|\tau_1,\tau_2)P(\tau_1<t_{s1}\le\tau_1+d\tau_1,\tau_2<t_{s2}\le\tau_2+d\tau_2\Big|Z_t) \tag{2.6d.28}$$

$$h_{2t}\, E\{(\phi_{t-t_{s2}}-\hat{\phi}_{t-t_{s2}})^2 \big| Z_t\}$$

$$= h_{2t}\int_{\tau_1=0}^{\infty}\int_{\tau_2=0}^{t\wedge\tau_1} P_0(t|\tau_1,\tau_2)\,P(\tau_1<t_{s1}\le\tau_1+d\tau_1,\tau_2<t_{s2}\le\tau_2+d\tau_2 \big| Z_t) \qquad (2.6d.29)$$

where $P_0(t|\tau_1,\tau_2)$ and $P_1(t,-(\tau_1-\tau_2)|\tau_1,\tau_2)$ are defined by equations (2.6d.18) and (2.6d.17) and precomputable by equations (2.6d.19) and (2.6d.20) respectively. These two equations are of course infinite dimensional and there does not seem to be any case in which they could become finite dimensional. Even in the steady state, $P_0(t|\tau_1,\tau_2)$ and $P_1(t,-(\tau_1-\tau_2)|\tau_1,\tau_2)$ are still functions of τ_1 and τ_2. This is exactly the same as in the multiple source problem and we shall avoid further discussion.

The conditions under which $P_0(t|\tau_1,\tau_2)$ and $P_1(t,-(\tau_1-\tau_2)|\tau_1,\tau_2)$ will reach their steady state values are discussed in Kwong [72]. We refer the reader to this reference for a complete discussion.

Finally, we discuss here the computation of the estimate $\hat{\psi}_{1t|t}$. In the linear Gaussian case, this estimate is generated by the filter (see equation (2.6c.17)):

$$\begin{aligned} d\hat{\psi}_{1t|t} &= \rho_{1t}(1-\hat{\psi}_{1t|t})dt \\ &\quad + h_{1t}\hat{\phi}_{t-t_{s1}}(1-\hat{\psi}_{1t|t})d\nu_{1t} \\ &\quad + h_{2t}[E\{\psi_{1t}\phi_{t-t_{s2}} \big| Z_t\} - \hat{\psi}_{1t|t}\hat{\phi}_{t-t_{s2}}]d\nu_{2t}\ , \\ &\quad \hat{\psi}_{10|0} = 0 \end{aligned} \qquad (2.6d.30)$$

All the terms involved in this filter have been encountered before except $E\{\psi_{1t}\phi_{t-t_{s2}} | Z_t\}$. This filter is of course infinite dimensional since the estimate $\hat{\phi}_{t-t_{s1}}$ is involved. We now examine the computation of $E\{\psi_{1t}\phi_{t-t_{s2}} | Z_t\}$.

In the linear Gaussian case, this estimate is given by the filter:

$$d\widehat{\psi_{1t}\phi_{t-t_{s2}}} = \left[\rho_{1t}(\hat{\phi}_{t-t_{s2}} - \widehat{\psi_{1t}\phi_{t-t_{s2}}}) + \widehat{\psi_{1t-}\alpha_{t-t_{s2}}\phi_{t-t_{s2}}}\right]dt$$

$$+ h_{1t}E\{\phi_{t-t_{s2}}\phi_{t-t_{s1}} - \widehat{\psi_{1t}\phi_{t-t_{s2}}}\ \hat{\phi}_{t-t_{s1}} \Big| Z_t\}d\nu_{1t}$$

$$+ h_{2t}\ E\{\psi_{1t}\phi^2_{t-t_{s2}} - \widehat{\psi_{1t}\phi_{t-t_{s2}}}\ \hat{\phi}_{t-t_{s2}} \Big| Z_t\}d\nu_{2t} \qquad (2.6d.31)$$

Again, the second term is finite dimensional when $\alpha_t=\alpha$, a constant, so that

$$\widehat{\psi_{1t-}\alpha_{t-t_{s2}}\phi_{t-t_{s2}}} = \alpha\ \widehat{\psi_{1t}\phi_{t-t_{s2}}} \qquad (2.6d.32)$$

Of all the remaining terms, we only need to consider the estimate $E\{\psi_{1t}\phi^2_{t-t_{s2}} | Z_t\}$ which we have not encountered yet in this section. We have

$$E\{\psi_{1t}\phi^2_{t-t_{s2}}\big|Z_t\} = E\{\psi_{1t}(\phi^2_{t-t_{s2}} - \hat{\phi}^2_{t-t_{s2}})\big|Z_t\}$$

$$+ E\{\psi_{1t}\hat{\phi}^2_{t-t_{s2}}\big|Z_t\}$$

$$= \int_{\tau_1=0}^{t} \int_{\tau_2=0}^{\tau_1} P_0(t|\tau_1,\tau_2)P(\tau_1<t_{s1}\leq\tau_1+d\tau_1,\tau_2<t_{s2}\leq\tau_2+d\tau_2\big|Z_t)$$

$$+ \hat{\psi}_{1t|t}\,\hat{\phi}^2_{t-t_{s2}} \tag{2.6d.33}$$

The first term on the right of equation (2.6d.33) is very much the same as the term on the right of equation (2.6d.29) and we shall avoid further discussion. Note that the filter for $\widehat{\psi_{1t}\phi_{t-t_{s2}}}$ does not introduce any term which requires a new filter for its computation. This is fortunate because we now end up here with a finite bank of filters for computing the terms required in implementing the filter (2.6d.25) for $\hat{\phi}_{t-t_{s2}}$.

We summarize our discussion as follows. In the linear Gaussian case, the filter (2.6d.25) for $\hat{\phi}_{t-t_{s2}}$ is still infinite dimensional in all cases because the estimate $\hat{\phi}_{t-t_{s1}}$ is involved. The remaining terms on the right require in general an infinite amount of on-line computations but they can be represented by a finite bank of filters. Even in the steady state, the covariance terms require an infinite dimensional on-line computation. The strong similarities between our results here and the corresponding results in the multiple source problem should be noted. The ways in which our problem here with random delays is more difficult than the case of known delays are the same as in the multiple source problem.

Because the structure of the terms involved in the solution of this multiple sensor problem is exactly the same as those in the multiple source problem, we shall avoid discussing the results again here. It should be evident by now that the discussion of the multiple source problem carries over directly. In particular, we shall avoid discussing the computational requirements in the general nonlinear case of the multiple sensor problem.

2.6e An Example Involving A Known Signal

We will again consider the results of the multiple sensor problem in the case of a known rectangular pulse signal. Since the results are almost identical to those in the multiple source problem, we will avoid the derivations and only present the results.

The signal ϕ_t is given by

$$\phi_t = \begin{cases} 0 , & t<0 \\ 1 , & 0\leq t\leq T \\ 0 , & t>T \end{cases} \tag{2.6e.1}$$

and the observations are

$$dz_{1t} = h_{1t}\phi_{t-t_{s1}} dt + dw_{1t} \tag{2.6e.2}$$

and

$$dz_{2t} = h_{2t}\phi_{t-t_{s2}} dt + dw_{2t} \tag{2.6e.3}$$

The signal estimates are given by

$$\hat{\phi}_{t-t_{s1}} = \hat{\psi}_{1t|t} - \hat{\psi}_{1t-T|t} \tag{2.6e.4}$$

$$\hat{\phi}_{t-t_{s2}} = \hat{\psi}_{2t|t} - \hat{\psi}_{2t-T|t} \tag{2.6e.5}$$

Thus, the signal and delay time estimation problems are both solved by computing the joint conditional distribution of t_{s1} and t_{s2}. From this joint conditional distribution, we can obtain the marginal distributions $\hat{\psi}_{1\tau|t}$ and $\hat{\psi}_{2\tau|t}$ by which we can compute the signal estimates $\hat{\phi}_{t-t_{s1}}$ and $\hat{\phi}_{t-t_{s2}}$. We will discuss the computation of this joint conditional distribution in the rest of this section. As discussed in Section 2.6c, we have to compute the estimates $\hat{\psi}_{1\tau_1|t}$ and $\hat{\psi}_{2\tau_2|t,\tau_1}$.

The equations for computing $\hat{\psi}_{1\tau_1|t}$ are given as follows. See equations (2.6c.17) to (2.6c.23).

$$\begin{aligned}\tau_1=t: \quad d\hat{\psi}_{1t|t} &= \rho_{1t}(1-\hat{\psi}_{1t|t})dt \\ &\quad + [h_{1t}(\hat{\psi}_{1t|t}-\hat{\psi}_{1t-T|t})(1-\hat{\psi}_{1t|t})]d\nu_{1t} \\ &\quad + h_{2t}[E\{\psi_{1t}\phi_{t-t_{s2}}\big|Z_t\} - \hat{\psi}_{1t|t}(\hat{\psi}_{2t|t}-\hat{\psi}_{2t-T|t})]d\nu_{2t}\,, \\ \hat{\psi}_{10|0} &= 0 \end{aligned} \tag{2.6e.6}$$

$$\tau_1>t: \quad \hat{\psi}_{1\tau_1|t} = 1 - (1-\hat{\psi}_{1t|t})\,\frac{P(t_{s1}\geq\tau_1)}{P(t_{s1}\geq t)} \tag{2.6e.7}$$

$$\begin{aligned}\tau_1<t: \quad \hat{\psi}_{1\tau_1|t} &= \hat{\psi}_{1\tau_1|\tau_1} + \int_{\tau_1}^{t} \Sigma_1(\tau_1,\tau)d\nu_{1\tau} \\ &\quad + \int_{\tau_1}^{t} \Sigma_2(\tau_1,\tau)d\nu_{2\tau}\end{aligned} \tag{2.6e.8}$$

where

$$\Sigma_1(\tau_1,t) = h_{1t}[E\{\psi_{1\tau_1}\phi_{t-t_{s1}}\Big|Z_t\} - \hat{\psi}_{1\tau_1|t}(\hat{\psi}_{1t|t}-\hat{\psi}_{1t-T|t})] \tag{2.6e.9}$$

and

$$\Sigma_2(\tau_1,t) = h_{2t}[E\{\psi_{1\tau_1}\phi_{t-t_{s2}}\Big|Z_t\} - \hat{\psi}_{1\tau_1|t}(\hat{\psi}_{2t|t}-\hat{\psi}_{2t-T|t})] \tag{2.6e.10}$$

The terms that require infinite dimensional on-line computations are evaluated as follows:

$$E\{\psi_{1\tau_1}\phi_{t-t_{s2}}\Big|Z_t\} = \begin{cases} P(t_{s1}\le\tau_1, t_{s2}\le\tau_1\big|Z_t), & 0\le\tau_1<t\le T \\ P(t_{s1}\le\tau_1, t-T<t_{s2}\le\tau_1\big|Z_t), & 0\le t-T\le\tau_1<t \\ 0 & , 0\le\tau_1\le t-T \end{cases} \tag{2.6e.11}$$

(in equation (2.6e.10)),

$$E\{\psi_{1t}\phi_{t-t_{s2}}\Big|Z_t\} = \begin{cases} P(t_{s1}\le t, t_{s2}\le t\big|Z_t) & , 0\le t\le T \\ P(t_{s1}\le t, t-T<t_{s2}\le t\big|Z_t) & , T<t \end{cases} \tag{2.6e.12}$$

(in equation (2.6e.6)),

$$E\{\psi_{1\tau_1}\phi_{t-t_{s1}}\Big|Z_t\} = \begin{cases} P(t_{s1}\le\tau_1, t_{s2}\le\tau_1\big|Z_t), & 0\le\tau_1<t\le T \\ P(t-T<t_{s1}\le\tau_1, t_{s2}\le\tau_1\big|Z_t), & 0\le t-T\le\tau_1<t \\ 0 & , 0\le\tau_1\le t-T \end{cases} \tag{2.6e.13}$$

(in equation (2.6e.9)).

Next, the equations for computing $\hat{\psi}_{2\tau_2|t,\tau_1}$ are as follows. See equations (2.6c.5) to (2.6c.13).

$$\tau_2=t: \quad d\hat{\psi}_{2t|t,\tau_1} = \rho_{2t}'(1-\hat{\psi}_{2t|t,\tau_1})dt + h_{2t}E\{\phi_{t-t_{s2}}\Big|Z_t, t_{s1}=\tau_1\}(1-\hat{\psi}_{2t|t,\tau_1})d\nu_{2t|\tau_1},$$
$$\hat{\psi}_{20|0,\tau_1} = 0 \tag{2.6e.14}$$

$$\tau_2>t: \quad \hat{\psi}_{2\tau_2|t,\tau_1} = 1 - (1-\hat{\psi}_{2t|t,\tau_1})\frac{P(t_{s2}\geq\tau_2|t_{s1}=\tau_1)}{P(t_{s2}\geq t|t_{s1}=\tau_1)} \tag{2.6e.15}$$

$$\tau_2<t: \quad \hat{\psi}_{2\tau_2|t,\tau_1} = \hat{\psi}_{2\tau_2|\tau_2,\tau_1} + \int_{\tau_2}^{t} \Sigma_1(\tau_2,\tau|\tau_1)d\nu_{1\tau|\tau_1} + \int_{\tau_2}^{t} \Sigma_2(\tau_2,\tau|\tau_1)d\nu_{2\tau|\tau_1} \tag{2.6e.16}$$

where

$$\Sigma_1(\tau_2,t|\tau_1)=0 \tag{2.6e.17}$$

and

$$\Sigma_2(\tau_2,t|\tau_1) = h_{2t}E\{\psi_{2\tau_2}\phi_{t-t_{s2}}\Big|Z_t, t_{s1}=\tau_1\} - h_{2t}\hat{\psi}_{2\tau_2|t,\tau_1}E\{\phi_{t-t_{s2}}\Big|Z_t, t_{s1}=\tau_1\} \tag{2.6e.18}$$

The terms which require infinite dimensional on-line computations are evaluated as follows:

$$E\{\psi_{2\tau_2}\phi_{t-t_{s2}}\Big|Z_t,t_{s1}=\tau_1\}$$

$$=\begin{cases}P(t_{s2}\leq\tau_m\big|Z_t) & , \quad 0\leq t\leq T\\ P(t-T<t_{s2}\leq\tau_m\big|Z_t) & , \quad 0<T\leq t\end{cases} \tag{2.6e.19}$$

where

$$\tau_m = \min(t,\tau_1,\tau_2) \tag{2.6e.20}$$

(in equation (2.6e.18)),

$$E\{\phi_{t-t_{s2}}\Big|Z_t,t_{s1}=\tau_1\}$$

$$=\begin{cases}P(t_{s2}\leq\tau'_m\big|Z_t) & , \quad 0\leq t\leq T\\ P(t-T<t_{s2}\leq\tau'_m\big|Z_t) & , \quad 0<T\leq t\end{cases} \tag{2.6e.21}$$

where

$$\tau'_m = \min(t,\tau_1) \tag{2.6e.22}$$

(in equations (2.6e.14) and (2.6e.18)).

With the estimates $\hat{\psi}_{1\tau_1|t}$ and $\hat{\psi}_{2\tau_2|t,\tau_1}$, we can compute the joint conditional distribution of t_{s1} and t_{s2}. In all the equations above for $\hat{\psi}_{1\tau_1|t}$ and $\hat{\psi}_{2\tau_2|t,\tau_1}$, this joint conditional distribution is the only quantity needed in performing the on-line computations. The on-line computations, though infinite dimensional, involve only an infinite amount of algebraic operations. The overall solution to the problem of signal and delay time estimation is given in the joint conditional distribution of t_{s1} and t_{s2}.

2.6f Concluding Remarks on the Multiple Sensor Problem

We have now completed the analysis of the multiple sensor problem and noted the remarkable similarities between this and the multiple source problem. A few concluding remarks here are in order.

Firstly, all the estimation results in this problem are infinite dimensional. This is not surprising since estimation problems for systems with time delays are inherently infinite dimensional even if the delays are known [37]. Because of the close similarities between this and the multiple source problem, we have exactly the same remarks here as in Section 2.5g concerning the additional difficulties compared to the case of known delays and so on.

Secondly, the case in which the delay times take on a finite number of possible values can also be worked out in a manner similar to the basic one-source-one-sensor case. We shall not work out the details but just mention the results. For the signal estimation problem, the only solution is the multiple-model approach which now involves only a finite bank of estimators. These estimators are described by the same equations presented in the previous sections on the multiple-model solution when t_{s1} and t_{s2} take on a continuum of values. For the delay time estimation problem, we compute the a posteriori probabilities $\mathcal{P}(t_{s1}=\tau_1^i\ ,\ t_{s2}=\tau_2^i \mid Z_t)$, where (τ_1^i,τ_2^i) is one possible pair of values for t_{s1} and t_{s2}. The evolution of these probabilities is described by stochastic differential equations similar to those presented in the one-source-one-sensor case.

CHAPTER 3

ESTIMATION OF TIME-INVARIANT RANDOM FIELDS OBSERVED VIA A MOVING POINT SENSOR

3.1 Introduction and Motivation

The basic idea for the problems we consider here is as follows. Given a time-invariant spatially varying random quantity, we want to estimate it when it is observed via a point sensor moving along it in space. There are many physical examples that motivate the consideration of such problems.

Space-time random processes which are observed and estimated via a moving point sensor abound in nature, good examples being the gravitational field of a planet [20], [44], the atmospheric pressure, temperature or humidity fields of the earth, [17], [18], a distribution of pollutants or gas in the atmosphere [19] and so on. Of all the above examples, only the gravitational field is time invariant. However, the other examples can also be considered time invariant for the purpose of observation and estimation via a moving point sensor because they usually vary so slowly compared with the motion of the sensor that they can be considered time-invariant during the time that the sensor is moving across them. Such an assumption was made in [19] for the purpose of estimating the constituent densities of the upper atmosphere.

We shall only be concerned with time invariant spatial random processes which vary in one spatial dimension. The reasons are as follows. Firstly, many time invariant fields exist in nature and, as we have pointed out above, many of the time varying fields in practice are "quasi-time-invariant". Secondly, in order to consider variations in more than one dimension simultaneously, we need a usable multidimensional stochastic calculus to handle the problem. As pointed out in Chapter 1, such a tool is not yet available and therefore we are forced to restrict consideration to variations in one dimension. However, we shall see that our work here for time invariant fields in one spatial dimension points out many of the features common to estimation of time invariant fields in more than one spatial dimension.

In all the work that follows, we will refer to the process of estimating a time invariant spatial field observed with a moving point sensor as spatial mapping.

A good example of spatial mapping in practice is the microwave sensing of atmospheric temperature and humidity fields via satellites [17], [18]. The system referred to in [17] and [18] is the Nimbus-5 satellite which carries a five-channel microwave spectrometer that has operated since December 1972. It is making the first satellite-based microwave measurements of global atmospheric temperature profiles, as well as measurements of atmospheric water content and other geophysical parameters. The measurements are stored and processed off-line. We shall not be concerned in our work here with applying our results to any example such as the Nimbus-5 system above. The Nimbus-5 example is pointed out here only as a motivation for our work. However, in the remainder of this chapter, whenever appropriate, we will point out potential applications of our work and our ideas to this Nimbus-5 example.

3.2 Modeling time invariant spatial random fields

In order to handle spatial mapping problems, we find it necessary and useful to model the spatial variations of the time invariant random field. This will enable us to describe the variations in time of the process as observed by the sensor. Modeling a time-invariant random field is not a new problem. Many authors have tried to do this before via the classical approach of using correlation functions as was done for random processes in time. For instance, in [14], Chernov modeled the random refractive index of the atmosphere using analytical correlation functions which approximate those measured experimentally. By imposing a structure on the correlation function, various types of random fields have been defined, e.g., homogeneous random fields, homogeneous isotropic random fields and so on [45], [46]. In this research, we have taken a somewhat different approach motivated by a desire to use many of the powerful results of estimation theory. Since we are only considering time-invariant fields that vary in one spatial dimension, we have taken the approach of describing the variations of the field in this spatial dimension by a stochastic differential equation in space.

A stochastic differential equation model allows one to perform certain types of analysis that one cannot do just with a correlation function model. As we will see, this type of model enables us to derive readily recursive estimation procedures. It is interesting to note that the main difference between the Kalman filtering theory [4] and the Wiener filtering theory [2] also lies in such a difference of a model for the process. The former theory has now replaced the latter in most applications because it employs a dynamical model for the process and yields recursive filtering algorithms which are readily implementable. A dynamical model for a random process has the additional advantage that if there are unknown parameters in the model, they can be identified on-line using observations on the process. Note that a stochastic differential equation can be viewed as a shaping filter and, given correlation information about a random process, we can find a shaping filter whose output will possess that correlation function.

Of course, the use of such a model employing a stochastic differential equation in space implies that we know a great deal about the field and thus there are questions that remain concerning its utility. On the other hand, one can take the point of view that, given correlation data, we can fit a spatial shaping filter to the data. Such a point of view has also been proposed by other investigators. In [19], McGarty mentioned such ideas for modeling the constituent densities of the atmosphere. In discrete space, such an approach has been proposed in [20]. The question of the utility of such differential models is a difficult one and a study of this issue is beyond the scope of this research. Rather, we wish to understand the implications of such a model for the problem of random field estimation.

3.3 Problem Formulation

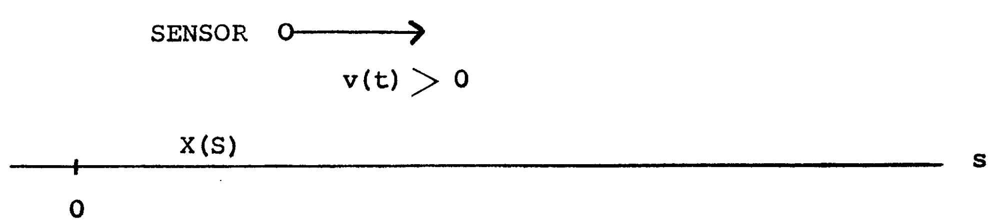

FIGURE 13: SPATIAL MAPPING

Our spatial mapping problem can be set up mathematically as follows. See Figure 13. We have a time invariant spatial random field which varies in one spatial dimension. We propose to use the following spatial shaping filter model for the field of interest:

$$dx(s) = f(x(s),s)ds + \underline{g}'(x(s),s)d\underline{w}(s), \qquad s\geq 0 \tag{3.3.1}$$

where $x(0)$ is a given random variable. Here, $x(\cdot)$ is the random quantity of interest and s is the spatial coordinate. The process $\underline{w}(s)$ is an n-vector of independent Wiener processes with $E[d\underline{w}(s)d\underline{w}'(s)] = \tilde{\underline{Q}}(s)ds$. The functions $f(\cdot,\cdot)$ and $\underline{g}(\cdot,\cdot)$ are assumed to satisfy conditions for the existence and uniqueness of solutions $x(\cdot)$.

The field is observed via a point sensor which moves in the direction of increasing s with a positive velocity $v(t)$. The equation of motion of the sensor is given by

$$ds(t) = v(t)dt \quad , \quad s(0)=0 \tag{3.3.2}$$

where $s(t)$ is the coordinate of the sensor at time t. The velocity $v(t)$ is either known a priori or is unknown a priori and has to be computed on-line using noisy or perfect observations. The value of the field measured by the sensor at any time t is $x(s(t))$ and this value is a function of t, i.e.,

$$x(s(t)) = \tilde{x}(t) \tag{3.3.3}$$

Assume that the sensor makes noisy observations on the field and that these observations are modeled as:

$$dz_1(t) = c(\tilde{x}(t),t)dt + d\beta_1(t) \tag{3.3.4}$$

where $\beta_1(t)$ is a standard Wiener process. Depending on our knowledge of the motion of the sensor, estimation of the field can be done in two ways which we will describe in the following sections.

We state here the assumptions concerning the processes involved. We assume that $\{\beta_1(\tau_1) - \beta_1(\tau_2),\ \tau_1>\tau_2\geq t\}$ is independent of $\{s(\tau),\ v(\tau),\ 0\leq\tau\leq t\}$. By assuming that $\{\beta_1(\tau_1) - \beta_1(\tau_2),\ \tau_1>\tau_2\geq t\}$ is independent of $\{w(s(\tau)),\ 0\leq\tau\leq t\}$ and $x(0)$, then $\{\beta_1(\tau_1) - \beta_1(\tau_2),\ \tau_1>\tau_2\geq t\}$ is independent of $\{\tilde{x}(\tau),\ 0\leq\tau\leq t\}$.

Before we can estimate the field using the observations (3.3.4), we have to be able to describe the evolution of $\tilde{x}(t)$ as a function of t in terms of the motion of the sensor. To do this, we need the following result.

<u>Theorem 3.1</u>: Let $\{w(s),\ \mathcal{F}_s,\ s\geq 0\}$ be a Wiener process with $E\{dw^2(s)\} = \tilde{Q}(s)ds$ with respect to the parameter s, where

$$\mathcal{F}_s = \sigma\{w(s'),\ 0\leq s'\leq s\} \tag{3.3.5}$$

Assume that the process s(t) satisfies

$$ds(t) = v(t)dt\ , \qquad s(0)=0 \tag{3.3.6}$$

where $v(t)>0$ is a given continuous random process. Let t(s) denote the inverse of s(t). Further assume that the increments $w(s_1) - w(s_2)$, $s_1>s_2>s$, are independent of $\{s(\tau)\wedge s,\ \forall\ \tau\geq 0\}$ and $\{v(t(s')),\ 0\leq s'\leq s\}$. Define the increasing family of σ-fields

$$\mathcal{G}_s = \mathcal{F}_s \vee \sigma\{s(\tau)\wedge s, \forall\ \tau\geq 0\} \vee \sigma\{v(t(s')),\ 0\leq s'\leq s\} \tag{3.3.7}$$

Then for each t, s(t) is a stopping time with respect to $\mathcal{G}_s$, and on the family

$\{\tilde{G}_t\}_{t\geq 0}$, where

$$\tilde{G}_t = G_{s(t)} \tag{3.3.8}$$

the process

$$\tilde{w}(t) = w(s(t)) \tag{3.3.9}$$

is a martingale with respect to time t and is given by

$$d\tilde{w}(t) = v^{1/2}(t)\ d\eta(t) \tag{3.3.10}$$

where $\{\eta_t\ ,\tilde{G}_t\}$ is a Wiener process with respect to time t with

$E(d\eta^2(t)) = Q(t)dt = \tilde{Q}(s(t))dt.$

■

This theorem is not new and is available in [47]. Our proof in Appendix 5 using martingale theory is much easier than that in [47] and is included for the sake of completeness.

Using this result, we can easily obtain from equation (3.3.1) the following equation for $\tilde{x}(t)$:

$$d\tilde{x}(t) = \tilde{f}(\tilde{x}(t),t)v(t)dt + \underline{\tilde{g}}'(\tilde{x}(t),t)v^{1/2}(t)d\underline{\eta}(t) \tag{3.3.11}$$

where

$$\tilde{f}(\cdot,t) = f(\cdot,s(t)) \tag{3.3.12}$$

and

$$\underline{\tilde{g}}(\cdot,t) = \underline{g}(\cdot,s(t)) \tag{3.3.13}$$

In the following sections, we will consider the problem of estimating the process $\tilde{x}(t)$ and thereby the process x(s).

Remark: In the spatial mapping problem we have formulated here, we have assumed that the sensor makes direct observations on the field and these observations are to be processed to estimate the field. This problem formulation applies to such problems as microwave sensing of atmospheric fields using satellites like the Nimbus 5. It does not apply to such problems as gravity field mapping via spacecraft tracking data [44]. In this latter type of problems, the time invariant spatial random field is a force field which affects the motion of the sensor and the field is to be estimated by processing observations on the motion of the sensor. However, we shall not be concerned with the latter type of problems here although we will briefly mention it in Chapter IV.

3.4 Field Estimation with Deterministic Sensor Motion

We assume here that the velocity v(t) of the sensor at each time t is known a priori or is observed perfectly. Then since v(t) is known, we have a simple nonlinear filtering problem involving the system (3.3.12) with the observations (3.3.4). The minimum mean square error estimate

$$\hat{\tilde{x}}(t) = E\{\tilde{x}(t)|Z_{1t}\} \tag{3.4.1}$$

where

$$Z_{1t} = \sigma\{z_1(\tau),\ 0\leq\tau\leq t\} \tag{3.4.2}$$

is given by [38]

$$\begin{aligned} d\hat{\tilde{x}}(t) = {} & \hat{\tilde{f}}(\tilde{x}(t),t)v(t)dt \\ & + E\{(\tilde{x}(t) - \hat{\tilde{x}}(t))\ (c(\tilde{x}(t),t) - \hat{c}(\tilde{x}(t),t))|Z_{1t}\}\cdot \\ & \cdot(dz_1(t) - \hat{c}(\tilde{x}(t),t)dt) \end{aligned} \tag{3.4.3}$$

This is a nonlinear filter which is in general non-implementable. We will not go into the questions of implementation here.

Since $v(t)$ is known for each t, so is $s(t)$. Thus, in this case, we can associate our estimate $\hat{\tilde{x}}(t)$ at each time t with a point s in space. This completes the spatial mapping problem.

In the case of linear field and observation models, the filter (3.4.3) reduces to the Kalman filter which is implementable. The field model becomes

$$dx(s) = f(s)x(s)ds + \underline{g}'(s)d\underline{w}(s) \tag{3.4.4}$$

and the observation model becomes

$$dz_1(t) = c(t)\tilde{x}(t)dt + d\beta_1(t) \tag{3.4.5}$$

The evolution of $x(t)$ is given by

$$d\tilde{x}(t) = \tilde{f}(t)\tilde{x}(t)v(t)dt + \tilde{\underline{g}}'(t)v^{1/2}(t)d\underline{\eta}(t) \tag{3.4.6}$$

where

$$\tilde{f}(t) = f(s(t)) \tag{3.4.7}$$

and

$$\tilde{\underline{g}}(t) = \underline{g}(s(t)) \tag{3.4.8}$$

The estimate $\hat{\tilde{x}}(t)$ is now given by

$$d\hat{\tilde{x}}(t) = \tilde{f}(t)v(t)\hat{\tilde{x}}(t)dt + c(t)\sigma(t)(dz_1(t) - c(t)\hat{\tilde{x}}(t)dt) \tag{3.4.9}$$

which is the readily implementable Kalman filter. The covariance

$$\sigma(t) = E\{(\tilde{x}(t) - \hat{\tilde{x}}(t))^2 \,|\, Z_{1t}\} \tag{3.4.10}$$

is precomputable by solving the Riccati equation

$$\frac{d\sigma(t)}{dt} = 2\tilde{f}(t)v(t)\sigma(t) + v(t)\underline{\tilde{g}}'(t)\underline{Q}(t)\underline{\tilde{g}}(t)$$
$$-\sigma^2(t)c^2(t)R^{-1}(t) \quad ,$$
$$\sigma(0) = \sigma_0 \quad , \text{ given} \tag{3.4.11}$$

where

$$R(t)dt = E\{d\beta_1^2(t)\} \tag{3.4.12}$$

and

$$\underline{Q}(t) = \underline{\tilde{Q}}(s(t)) \tag{3.4.13}$$

We shall encounter this linear case again in a later section when we consider the problem of sensor velocity control for optimal field estimation.

Note that since $v(t)$ enters in the second term on the right of the Riccati equation (3.4.11), it essentially scales the intensity, i.e., the covariance $\underline{Q}(t)$, of the noise $\underline{\eta}(t)$ that drives the process $\tilde{x}(t)$. This is also evident from equation (3.4.6) for $\tilde{x}(t)$ in which there is a gain of $v^{1/2}(t)$ on the input $d\underline{\eta}(t)$. The intuitive interpretation is that the higher the velocity of the sensor, the higher the intensity of fluctuation in the process $\tilde{x}(t)$ it observes and consequently the worse its observations. This corresponds to our intuition that the faster the sensor moves, the less information it can get from the field. The presence of $v(t)$ in the first term on the right of the Riccati equation represents the fact that the spatial correlation of the field is reflected as a correlation in time through the velocity $v(t)$. Hence the faster sensor moves, the less correlation we expect to see in the observed time process.

The result that we arrive at above in the linear case is of greater importance in practice than the nonlinear filter (3.4.3). This is true not only because it is an implementable filter but also because in practice linear models will be the first to be tried out because of their simplicity. We shall always examine our results in the special case of linear models in the following sections.

3.5 Field Estimation with Random Sensor Motion

3.5a The Problem and Its Basic Difficulty

We assume in this section that the velocity $v(t)$ of the sensor is not known a priori and is observed in the presence of noise, i.e., we have an observation on the sensor of the form

$$dz_2(t) = v(t)dt + d\beta_2(t) \tag{3.5a.1}$$

where $\beta_2(t)$ is a standard Wiener process independent of $\beta_1(t)$. In addition, assume that we also have an observation on the position of the form

$$dz_3(t) = s(t)dt + d\beta_3(t) \tag{3.5a.2}$$

where $\beta_3(t)$ is a standard Wiener process independent of $\beta_1(t)$ and $\beta_2(t)$.

As with $\beta_1(t)$, we will assume that $\{\beta_2(\tau_1)-\beta_2(\tau_2),\ \beta_3(\tau_1)-\beta_3(\tau_2),\ \tau_1>\tau_2\geq t\}$ is independent of $\{s(\tau),\ v(\tau),\ 0\leq\tau\leq t\}$, $\{w(s(\tau)),\ 0\leq\tau\leq t\}$ and $x(0)$. Then, the former is also independent of $\{\tilde{x}(\tau),\ 0\leq\tau\leq t\}$. The measurements $z_1(t)$ on the field and the measurements $z_2(t)$ and $z_3(t)$ on the sensor are now to be processed collectively to estimate the field and the motion of the sensor.

In order to deal with the motion of the sensor, we need to first model its velocity. Suppose its velocity $v(t)$ is modeled by the dynamical system

$$dv(t) = u(t)dt + \underline{k}'(v(t),t)d\underline{\xi}(t) \tag{3.5a.3}$$

Here $u(t)$ is the control input or the acceleration of the sensor. It is either predetermined and applied open loop or determined on-line based on the measurements, i.e., $u(t)$ is measurable with respect to the measurements Z_t, where

$$Z_t = \sigma\{z_1(\tau),\ z_2(\tau),\ z_3(\tau),\ 0\leq\tau\leq t\} \tag{3.5a.4}$$

In either case, $u(t)$ is known at each time t. The process $\underline{\xi}(t)$ is a vector of independent standard Wiener processes which is independent of $\underline{\eta}(t)$, $\beta_1(t)$, $\beta_2(t)$ and $\beta_3(t)$. The term $\underline{k}'(v(t),t)d\underline{\xi}(t)$ models random perturbations on the motion of the sensor. Since we assume that $v(t)>0$ for all $t\geq 0$, we have to place some constraints on equation (3.5a.3). These conditions require that $k(\cdot,\cdot)$ be random and in fact it must depend on $v(\cdot)$. We can assume that

$$v(t) = \phi(\underline{y}(t)) \tag{3.5a.5}$$

where $\phi(\cdot)$ is a positive function and $\underline{y}(t)$ is an Ito diffusion process. Then, Ito's differential rule shows that $v(t)$ satisfies equation (3.5a.3) where

$\underline{k}(\cdot,\cdot)$ now depends on $v(\cdot)$.

An alternative model for the velocity which guarantees that $v(t)>0$ for all $t\geq 0$ is the lognormal bilinear model

$$dv(t) = (u(t)dt + \underline{b}'(t)d\underline{\xi}(t))v(t) \qquad (3.5a.6)$$

Here $u(t)$ is the same as in equation (3.5a.3) while $\underline{b}(t)$ is deterministic, i.e., known a priori. It is easy to see that $\ln v(t)$ is an Ito diffusion process and has a Gaussian distribution. Thus, $v(t)>0$ for all $t\geq 0$.

Finally, we also have the equation of motion

$$ds(t) = v(t)dt \qquad (3.5a.7)$$

We can now view the problem of estimating $\tilde{x}(t)$ as a nonlinear filtering problem involving the system made up of equations (3.3.12) for $\tilde{x}(t)$, (3.5a.3) or (3.5a.6) for $v(t)$ and (3.5a.7) for $s(t)$, with the observations $z_1(t)$, $z_2(t)$ and $z_3(t)$ given by equations (3.3.4), (3.5a.1) and (3.5a.2) respectively. The filtering equations can be written down at once by [38] to yield the estimate

$$\hat{\tilde{x}}(t) = E\{\tilde{x}(t)|Z_t\} \qquad (3.5a.8)$$

where Z_t is defined in equation (3.5a.4). The filter is given as follows:

$$\begin{aligned} d\hat{\tilde{x}}(t) = {} & \widehat{\tilde{f}(\tilde{x}(t),t)v(t)}dt \\ & + E\{(\tilde{x}(t)-\hat{\tilde{x}}(t))(c(\tilde{x}(t),t)-\hat{c}(\tilde{x}(t),t))|Z_t\} \cdot \\ & \quad \cdot (dz_1(t)-\hat{c}(\tilde{x}(t),t)dt) \\ & + E\{(\tilde{x}(t)-\hat{\tilde{x}}(t))(v(t)-\hat{v}(t))|Z_t\}(dz_2(t)-\hat{v}(t)dt) \\ & + E\{(\tilde{x}(t)-\hat{\tilde{x}}(t))(s(t)-\hat{s}(t))|Z_t\}(dz_3(t)-\hat{s}(t)dt) \end{aligned} \qquad (3.5a.9)$$

where

$$\hat{\cdot} = E\{\cdot|Z_t\} \tag{3.5a.10}$$

This filter is of course non-implementable because it is infinite dimensional. In the linear case in which the field model is linear, i.e.,

$$d\tilde{x}(t) = \tilde{f}(t)\tilde{x}(t)v(t)dt + \underline{\tilde{g}}'(t)v^{1/2}(t)d\underline{\eta}(t) \tag{3.5a.11}$$

(see equation (3.4.6)) and the observation model on the field is linear, i.e.,

$$dz_1(t) = c(t)\tilde{x}(t)dt + d\beta_1(t) \tag{3.5a.12}$$

(see equation (3.4.5)), the filter above becomes

$$\begin{aligned} d\hat{\tilde{x}}(t) = {} & \widehat{\tilde{f}(t)\tilde{x}(t)v(t)}dt \\ & + c(t)\ E\{(\tilde{x}(t)-\hat{\tilde{x}}(t))^2\Big|Z_t\}\ (dz_1(t)-c(t)\hat{\tilde{x}}(t)dt) \\ & + E\{(\tilde{x}(t)-\hat{\tilde{x}}(t))(v(t)-\hat{v}(t))|Z_t\}\ (dz_2(t)-\hat{v}(t)dt) \\ & + E\{(\tilde{x}(t)-\hat{\tilde{x}}(t))(s(t)-\hat{s}(t))|Z_t\}\ (dz_3(t)-\hat{s}(t)dt) \end{aligned} \tag{3.5a.13}$$

This filter is still infinite dimensional and therefore non-implementable. The main reason is that equation (3.5a.11) for $\tilde{x}(t)$, although linear in $\tilde{x}(t)$, contains the random gain $\tilde{f}(t)v(t)$. In the filter (3.5a.13), the term $E\{\tilde{x}(t)v(t)|Z_t\}$, for instance, requires an infinite number of equations to be computed on-line. We illustrate this briefly as follows in the case when $f(t)=f=\text{constant}$. To compute $E\{\tilde{x}(t)v(t)|Z_t\}$, we first write the stochastic differential equation for $\tilde{x}(t)v(t)$. Using Ito's rule, we see that $d(\tilde{x}(t)v(t))$ contains the term $v(t)d\tilde{x}(t)$ which in turn contains the term $fv^2(t)\tilde{x}(t)dt$. Thus, to compute $E\{\tilde{x}(t)v(t)|Z_t\}$ requires computing $E\{v^2(t)\tilde{x}(t)|Z_t\}$ which in turn requires computing $E\{v^3(t)\tilde{x}(t)|Z_t\}$, and so on resulting in an infinite system of equations.

Although the filter (3.5a.13) is infinite dimensional, there exists a possibility of computing some terms on the right hand side by finite dimensional approximations. For instance, we can replace $v(t)$ by $\hat{v}(t)$ and $\tilde{f}(t)=f(s(t))$ by $f(\hat{s}(t))$ in equation (3.5a.11). Then, in the filter (3.5a.13), the first term on the right becomes

$$\widehat{\tilde{f}(t)\tilde{x}(t)v(t)}dt = f(\hat{s}(t))\hat{v}(t)\hat{\tilde{x}}(t)dt \tag{3.5a.14}$$

We will discuss this type of approximations in more detail later in this section.

Although we do not present them here, the filters for computing the estimates $\hat{v}(t)$ and $\hat{s}(t)$ are of the same form as (3.5a.9) and involve the measurements $z_1(t)$, $z_2(t)$ and $z_3(t)$.

In the paragraphs above, we have examined the problem of estimating $\tilde{x}(t) = x(s(t))$ in some detail. However, estimating $\tilde{x}(t)$ does not solve the field estimation problem completely in this case of random sensor motion. Since $v(t)$ is not known perfectly at each time t, neither is $s(t)$. Thus, at each time t, we do not know with which point s in space to associate our estimate $\hat{\tilde{x}}(t)$. In fact, since $s(t)$ can take on different values, $\tilde{x}(t)$ cannot be associated with any fixed spatial point s. It is now evident that since $s(t)$ is unknown at each time t, computing the estimate $\hat{\tilde{x}}(t)$ is not the optimal way to solve the spatial mapping problem. In the following section we examine methods for avoiding this difficulty.

One suboptimal approximate method of spatial mapping we would like to point out here is to associate the estimate $\hat{\tilde{x}}(t)$ with the point $\hat{s}(t)$ provided we can estimate the position $s(t)$ accurately. Although in theory there is no guarantee that $\hat{s}(t)$ will increase monotonically in t, in practice we can often be sure it is very likely to do so.

3.5b Some Methods of Field Estimation

The discussion of the previous section has led us to investigate the problem of estimating the field $x(s)$ at known positions s at each time t. Since we can compute the estimate $\hat{s}(t)$ of the position of the sensor at each time t, where

$$\hat{s}(t) = E\{s(t)|Z_t\} \tag{3.5b.1}$$

we can therefore try to estimate $x(\hat{s}(t))$, i.e., the value of the field at the estimated position of the sensor. This is what we will attempt to do next.

The estimate $\hat{s}(t)$ is given by the filter

$$\begin{aligned} d\hat{s}(t) = \hat{v}(t)dt &+ E\{s(t)c(\tilde{x}(t),t)-\hat{s}(t)\hat{c}(\tilde{x}(t),t)|Z_t\}\cdot \\ &\cdot (dz_1(t)-\hat{c}(\tilde{x}(t),t)dt) \\ &+ E\{s(t)v(t)-\hat{s}(t)\hat{v}(t)|Z_t\}\ (dz_2(t)-\hat{v}(t)dt) \\ &+ E\{s^2(t)-\hat{s}^2(t)|Z_t\}\ (dz_3(t)-\hat{s}(t)dt) \end{aligned} \tag{3.5b.2}$$

Note that although the field does not affect the motion of the sensor, the field value $\tilde{x}(t)$ is still correlated with the velocity $v(t)$ and hence the position $s(t)$. This is because $x(t)$ is correlated with $\{v(\tau),\ 0\leq\tau<t\}$. Thus, the second term on the right of equation (3.5b.2) is nonzero.

In order to estimate $x^*(t) = x(\hat{s}(t))$ recursively on-line, we first need to find the stochastic differential equation representation for $x^*(t)$ so as to be able to apply the well-known results of filtering theory. However at this point, we find that a stochastic differential equation for $x^*(t)$ is impossible. The main reason is that we need a result like Theorem 3.1 to characterize the process $w(\hat{s}(t))$ in order to transform equation (3.3.1) for $x(s)$ to an equation for $x^*(t)$. Such a result is not possible. Theorem 3.1 only holds when $s(t)$ is monotonically increasing in t. However, $\hat{s}(t)$ given by equation (3.5b.2) fluctuates in t because the equation is driven by Wiener processes. The difficulty that prevents us from characterizing $w(\hat{s}(t))$ basically stems from the fact that we are trying to define a process of the form $\mu(\rho(t))$ where $\mu(\rho)$ is a Wiener process in the parameter ρ and $\rho(t)$ is a Wiener process in the parameter t. Such a process at this time defies detailed analysis. Therefore, at present, we do not see any way of characterizing $x^*(t)$ by which we can derive filtering formulas for it.

Since we cannot estimate $x^*(t) = x(\hat{s}(t))$, we will try to solve the field estimation problem by estimating $x(\tilde{s}(t))$, where $\tilde{s}(t)$ is the position defined by

$$d\tilde{s}(t) = \hat{v}(t)dt \tag{3.5b.3}$$

$$\tilde{s}(0) = 0 \tag{3.5b.4}$$

This position is known at each time t given the observations. Note that it is in some sense an "open loop" measurement of $s(t)$. Since we have measurements $z_1(t)$, $z_2(t)$ and $z_3(t)$, the velocity estimate $\hat{v}(t)$ is computed by the filter

$$d\hat{v}(t) = u(t)dt + E\{(v(t)-\hat{v}(t))(c(\tilde{x}(t),t)-\hat{c}(\tilde{x}(t),t))|Z_t\} \cdot$$
$$\cdot\ (dz_1(t)-\hat{c}(\tilde{x}(t),t)dt)$$
$$+ E\{(v(t)-\hat{v}(t))^2 \Big| Z_t\}\ (dz_2(t)-\hat{v}(t)dt)$$
$$+ E\{(v(t)-\hat{v}(t))(s(t)-\hat{s}(t))|Z_t\} \cdot$$
$$\cdot\ (dz_3(t)-\hat{s}(t)dt) \tag{3.5b.5}$$

where we assume the model (3.5a.3) for $v(t)$. We now need to characterize the process $x(\tilde{s}(t))$ and here again the main problem is to characterize $w(\tilde{s}(t))$. Since the computation of the estimate $\hat{v}(t)$ involves the measurement $z_1(t)$ on $\tilde{x}(t) = x(s(t))$, the following problem arises in characterizing $w(\tilde{s}(t))$. Suppose at some time t, the position $\tilde{s}(t)$ is less than $s(t)$. Then, $\tilde{s}(t)$ is correlated with $x(s)$, for $\tilde{s}(t) \leq s \leq s(t)$ and hence with $w(s)$, for $\tilde{s}(t) \leq s \leq s(t)$. We cannot then straightforwardly apply a version of Theorem 3.1 defined for $\hat{v}(t)$ and $\tilde{s}(t)$ because $w(s_1)-w(s_2)$, $s_1 > s_2 \geq \tilde{s}(t)$, is <u>not</u> independent of $\{\tilde{s}(\tau),\ 0 \leq \tau \leq t\}$ and $\{\hat{v}(t(s')),\ 0 \leq s' \leq s(t)\}$. In short, the process $w(s)$ may no longer be a Wiener process on the family $\{\mathcal{G}'_s\}_{s \geq 0}$ where

$$\mathcal{G}'_s = \sigma\{w(s'),\ 0 \leq s' \leq s\} \vee \sigma\{\tilde{s}(\tau) \wedge s,\ \tau \geq 0\}$$
$$\vee\ \sigma\{\hat{v}(t(s')),\ 0 \leq s' \leq s\} \tag{3.5b.6}$$

and without this property, we cannot derive a representation for $w(\tilde{s}(t))$ similar to that in Theorem 3.1. If, however, $\tilde{s}(t)$ is greater than $s(t)$, then $w(s_1)-w(s_2)$, $s_1 > s_2 \geq \tilde{s}(t)$, is indeed independent of $\{\tilde{s}(\tau),\ o \leq \tau \leq t\}$ and $\{\hat{v}(t(s')),\ 0 \leq s' \leq s(t)\}$ and we can apply a version of Theorem 3.1 defined for

$\hat{v}(t)$ and $\tilde{s}(t)$ to obtain the representation for $w(\tilde{s}(t))$. Since at each time t, we do not know if $s(t)$ is less than or greater than $\tilde{s}(t)$, the representation for $w(\tilde{s}(t))$ cannot be obtained in this manner.

The only way to get around the problem above is to use the estimate of $v(t)$ computed using only the measurements $z_2(t)$ and $z_3(t)$ on $v(t)$ and $s(t)$, i.e.,

$$d\hat{v}^*(t) = u(t)dt + E\{(v(t)-\hat{v}^*(t))^2 \,|\, Z_{23t}\}\ (dz_2(t)-\hat{v}^*(t)dt)$$
$$+ E\{(v(t)-\hat{v}^*(t))(s(t)-\hat{s}^*(t)) \,|\, Z_{23t}\}\ (dz_3(t)-\hat{s}^*(t)dt) \qquad (3.5b.7)$$

(assuming the model (3.5a.3) for $v(t)$)

where

$$Z_{23t} = \sigma\{z_2(\tau),\ z_3(\tau),\ 0\leq\tau\leq t\} \qquad (3.5b.8)$$

and

$$d\hat{s}^*(t) = \hat{v}^*(t)dt + E\{(s(t)-\hat{s}^*(t))(v(t)-\hat{v}^*(t)) \,|\, Z_{23t}\}\cdot$$
$$\cdot\ (dz_2(t)-\hat{v}^*(t)dt) + E\{(s(t)-\hat{s}^*(t))^2 \,|\, Z_{23t}\}\ (dz_3(t)-\hat{s}^*(t)dt) \qquad (3.5b.9)$$

Note that these are only suboptimal approximations to the conditional means of $v(t)$ and $s(t)$ given Z_t since

$$\hat{v}^*(t) = E\{v(t)\,|\,Z_{23t}\} \qquad (3.5b.10)$$

and

$$\hat{s}^*(t) = E\{s(t)\,|\,Z_{23t}\} \qquad (3.5b.11)$$

We restrict $u(t)$ to be Z_{23t}-measurable in this case. Now, define the position

$\tilde{s}^*(t)$ by

$$d\tilde{s}^*(t) = \hat{v}^*(t)dt \tag{3.5b.12}$$

The process $\hat{v}^*(t)$ is not correlated with $\{x(\sigma+\tilde{s}^*(t))-x(\tilde{s}^*(t)), \forall\sigma>0\}$ and neither is $\tilde{s}^*(t)$. Thus, it is now true that $\{w(\sigma+\tilde{s}^*(t))-w(\tilde{s}^*(t)), \forall\sigma>0\}$ is independent of $\{\tilde{s}^*(\tau), 0\leq\tau\leq t\}$ and $\{\hat{v}^*(t(s')), 0\leq s'\leq\tilde{s}^*(t)\}$. Applying a version of Theorem 3.1 for $\tilde{s}^*(t)$ and $\hat{v}^*(t)$, we have

$$d\underline{w}(\tilde{s}^*(t)) = \hat{v}^{*1/2}(t)d\underline{\eta}^*(t) \tag{3.5b.13}$$

where $\underline{\eta}^*(t)$ is a vector of independent standard Wiener processes defined with respect to the velocity $\hat{v}^*(t)$. We can now characterize the process

$$\tilde{x}^*(t) = x(\tilde{s}^*(t)) \tag{3.5b.14}$$

by the stochastic differential equation

$$\begin{aligned} d\tilde{x}^*(t) &= f(\tilde{x}^*(t), \tilde{s}^*(t))d\tilde{s}^*(t) + \underline{g}'(\tilde{x}^*(t), \tilde{s}^*(t))d\underline{w}(\tilde{s}^*(t)) \\ &= \tilde{f}^*(\tilde{x}^*(t),t)\hat{v}^*(t)dt + \underline{\tilde{g}}^{*\prime}(\tilde{x}^*(t),t)\hat{v}^{*1/2}(t)d\underline{\eta}^*(t) \end{aligned} \tag{3.5b.15}$$

where

$$\tilde{f}^*(\cdot,t) = f(\cdot,\tilde{s}^*(t)) \tag{3.5b.16}$$

and

$$\underline{\tilde{g}}^*(\cdot,t) = \underline{g}(\cdot,\tilde{s}^*(t)) \tag{3.5b.17}$$

The spatial mapping problem is now accomplished by estimating $\tilde{x}^*(t)$. The minimum mean square error estimate

$$\hat{\tilde{x}}^*(t) = E\{\tilde{x}^*(t)|Z_t\} \tag{3.5b.18}$$

is given by [38] (here we use the fact that $\hat{v}^*$ and $\tilde{s}^*$ are Z_t -measurable):

$$\begin{aligned} d\hat{\tilde{x}}^*(t) = {} & \hat{\tilde{f}}^*(\tilde{x}^*(t),t)\hat{v}^*(t)dt \\ & + E\{\tilde{x}^*(t)c(\tilde{x}(t),t)-\hat{\tilde{x}}^*(t)\hat{c}(\tilde{x}(t),t)|Z_t\}\cdot \\ & \cdot\ (dz_1(t)-\hat{c}(\tilde{x}(t),t)dt) \\ & + E\{\tilde{x}^*(t)v(t)-\hat{\tilde{x}}^*(t)\hat{v}(t)|Z_t\}\cdot \\ & \cdot\ (dz_2(t)-\hat{v}(t)dt) \\ & + E\{\tilde{x}^*(t)s(t)-\hat{\tilde{x}}^*(t)\hat{s}(t)|Z_t\}\ (dz_3(t)-\hat{s}(t)dt) \end{aligned} \tag{3.5b.19}$$

This filter is nonlinear and is in general non-implementable because the terms on the right are incomputable.

The above approach of mapping the field by estimating $x(\tilde{s}^*(t))$ has some obvious drawbacks. As time goes on, the difference between $\tilde{s}^*(t)$ and $s(t)$ may get large and hence the point $s(t)$ of the field that the sensor is measuring may be getting further and further away from the point $\tilde{s}^*(t)$ that we are trying to estimate. If the field is correlated over short distances only, the performance of our estimation scheme would deteriorate because the field at the point being observed and the field at the point being estimated are weakly correlated.

In the case of linear field and observation models, the filter (3.5b.19) reduces to

$$
\begin{aligned}
d\hat{\tilde{x}}^*(t) = {} & \tilde{f}^*(t)\hat{v}^*(t)\hat{\tilde{x}}^*(t)dt \\
& + c(t)E\{\tilde{x}^*(t)\tilde{x}(t)-\hat{\tilde{x}}^*(t)\hat{\tilde{x}}(t)\,|Z_t\} \;\; (dz_1(t)-c(t)\hat{\tilde{x}}(t)dt) \\
& + E\{\tilde{x}^*(t)v(t)-\hat{\tilde{x}}^*(t)\hat{v}(t)\,|Z_t\} \;\; (dz_2(t)-\hat{v}(t)dt) \\
& + E\{\tilde{x}^*(t)s(t)-\hat{\tilde{x}}^*(t)\hat{s}(t)\,|Z_t\} \;\; (dz_3(t)-\hat{s}(t)dt)
\end{aligned} \tag{3.5b.20}
$$

where

$$\tilde{f}^*(t) = f(\tilde{s}^*(t)) \tag{3.5b.21}$$

This filter is still non-implementable because some terms on the right are incomputable. We have illustrated in Section 3.5a that the estimate $\hat{\tilde{x}}(t)$ has to be computed by an infinite number of equations even in the case of linear field and observation models. In addition, there are added complications in the computation of terms like $E\{\tilde{x}^*(t)\tilde{x}(t)\,|Z_t\}$ which is the conditional second order moment of a process sampled at two different rates. There does not seem to be any case in which the filter (3.5b.19) for estimating $\tilde{x}^*(t) = x(\tilde{s}^*(t))$ is implementable. As such, we have only produced the representations for the estimate $\hat{\tilde{x}}^*(t)$. Of course, these may be useful in devising suboptimal approximations.

In the paragraphs above, we have examined some optimal methods of spatial mapping which consist of estimating the field x(s) at known positions s at each time t. The first two methods of estimating $x(\hat{s}(t))$ and $x(\tilde{s}(t))$ are not achievable since a semimartingale characterization of $x(\hat{s}(t))$ and of $x(\tilde{s}(t))$ is impossible. The third method of estimating $x(\tilde{s}^*(t))$ leads to non-implementable infinite dimensional filters even in the linear case. We now turn to some suboptimal spatial mapping schemes which are readily achievable and hopefully lead to finite dimensional implementable filters.

Note that in Section 3.5a, we have mentioned one method of suboptimal approximate spatial mapping. This method consists of associating the estimate $\hat{\tilde{x}}(t) = \hat{x}(s(t))$ with the point $\hat{s}(t)$ provided we can estimate the position $s(t)$ accurately. We have discussed the problems of computing the estimate $\hat{\tilde{x}}(t)$ in Section 3.5a and will not go into this any more.

Another method of suboptimal approximate spatial mapping which is especially useful in the linear case is the following. We essentially decouple the estimation of the motion of the sensor from the estimation of the field. We have seen in the case of deterministic sensor motion that with the position $s(t)$ and the velocity $v(t)$ of the sensor known at each time t, we only use the observation $z_1(t)$ on $x(s(t))$ to estimate the field. Suppose now we want to use a similar procedure to estimate the field when the sensor motion is random and $s(t)$ and $v(t)$ are not known precisely at any time t. The idea is to compute the estimates $\hat{v}^*(t)$ and $\hat{s}^*(t)$ of the velocity and position using only the measurements $z_2(t)$ and $z_3(t)$ on $v(t)$ and $s(t)$ respectively. See equations (3.5b.7) and (3.5b.9). As noted before, these are only suboptimal estimates since the measurement $z_1(t)$ on the field gives us additional information on the motion of the sensor. We have only decoupled the estimation of the motion of the sensor from the estimation of the field in order to get implementable suboptimal filters. These estimates $\hat{v}^*(t)$ and $\hat{s}^*(t)$ are now substituted for $v(t)$ and $s(t)$ whenever the latter quantities appear as if they were the actual velocity and position. Thus, equation (3.3.12) for $\tilde{x}(t) = x(s(t))$ becomes:

$$d\tilde{x}(t) = \bar{f}(\tilde{x}(t),t)\hat{v}^*(t)dt + \underline{\bar{g}}'(\tilde{x}(t),t)\hat{v}^{*1/2}(t)d\underline{\eta}(t) \qquad (3.5b.22)$$

where

$$\overline{f}(\cdot,t) = f(\cdot,\hat{s}^*(t)) \tag{3.5b.23}$$

and

$$\overline{\underline{g}}(\cdot,t) = \underline{g}(\cdot,\hat{s}^*(t)) \tag{3.5b.24}$$

Note that at each time t, given $\hat{s}^*(t)$ and $\hat{v}^*(t)$, the randomness in $\overline{f}(\cdot,\cdot)$ and $\overline{\underline{g}}(\cdot,\cdot)$ is only due to $\tilde{x}(t)$. We now generate the estimate of $\tilde{x}(t)$ using only the measurements $z_1(t)$. The estimate $\hat{\tilde{x}}(t)$ is given by the filter

$$\begin{aligned} d\hat{\tilde{x}}(t) &= \hat{\overline{f}}(\tilde{x}(t),t)\hat{v}^*(t)dt \\ &+ E\{(\tilde{x}(t)-\hat{\tilde{x}}(t))(c(\tilde{x}(t),t)-\hat{c}(\tilde{x}(t),t))\big|Z_{1t}\}. \\ &\cdot (dz_1(t)-\hat{c}(\tilde{x}(t),t)dt) \end{aligned} \tag{3.5b.25}$$

The estimate $\hat{\tilde{x}}(t)$ is now associated with the point $\hat{s}^*(t)$ and the spatial mapping problem is solved. As mentioned before, in theory, there is no guarantee that $\hat{s}^*(t)$ will be monotonically increasing in t but in practice it is very likely to be so, since position estimates can often be made very accurately.

In the nonlinear case, the filter (3.5b.25) of course will not be implementable. However, in the linear case, the result becomes very interesting. Equation (3.5b.22) reduces to

$$d\tilde{x}(t) = \overline{f}(t)\hat{v}^*(t)\tilde{x}(t)dt + \overline{\underline{g}}'(t)\hat{v}^{*1/2}(t)d\underline{\eta}(t) \tag{3.5b.26}$$

where

$$\overline{f}(t) = f(\hat{s}^*(t)) \tag{3.5b.27}$$

and

$$\underline{\overline{g}}(t) = \underline{g}(\hat{s}^*(t)) \tag{3.5b.28}$$

All the gains in equation (3.5b.26) are now measurable with respect to

$$Z_{23t} = \sigma\{z_2(\tau), z_3(\tau),\ 0 \leq \tau \leq t\} \tag{3.5b.29}$$

and thus can be considered known at each time t. With a linear observation model on the field,

$$dz_1(t) = c(t)\tilde{x}(t)dt + d\beta_1(t) \tag{3.5b.30}$$

the estimate $\hat{\tilde{x}}(t)$ is now easily seen to be given by the Kalman filter

$$d\hat{\tilde{x}}(t) = \overline{f}(t)\hat{v}^*(t)\hat{\tilde{x}}(t)dt + c(t)\tilde{\sigma}(t)(dz_1(t) - c(t)\hat{\tilde{x}}(t)dt) \tag{3.5b.31}$$

where the covariance

$$\tilde{\sigma}(t) = E\{(\tilde{x}(t) - \hat{\tilde{x}}(t))^2 \big| Z_{1t}\} \tag{3.5b.32}$$

is computed <u>on-line</u> by the Riccati equation

$$\begin{aligned} \frac{d\tilde{\sigma}(t)}{dt} &= 2\overline{f}(t)\hat{v}^*(t)\tilde{\sigma}(t) + \hat{v}^*(t)\underline{\overline{g}}'(t)\underline{Q}(t)\underline{\overline{g}}(t) \\ &\quad - \tilde{\sigma}^2(t)c^2(t)R^{-1}(t), \\ \tilde{\sigma}(0) &= \sigma_0\ ,\ \text{given} \end{aligned} \tag{3.5b.33}$$

Equations (3.5b.31) and (3.5b.33) for computing the estimate $\hat{\tilde{x}}(t)$ are readily implementable. The interesting feature here is that the Riccati equation

(3.5b.33) has to be solved on-line to yield the covariance $\tilde{\sigma}(t)$ because the gains in the equation depend on $\hat{v}^*(t)$ and $\hat{s}^*(t)$. The covariance $\tilde{\sigma}(t)$ can be computed only as $\hat{v}^*(t)$ and $\hat{s}^*(t)$ become available.

Now, let us consider the computation of the estimates $\hat{v}^*(t)$ and $\hat{s}^*(t)$ which are given by equations (3.5b.7) and (3.5b.9):

$$d\begin{bmatrix} \hat{v}^*(t) \\ \hat{s}^*(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{v}^*(t) \\ \hat{s}^*(t) \end{bmatrix} dt + \begin{bmatrix} u(t) \\ 0 \end{bmatrix} dt + \underline{\Sigma}(t) \begin{bmatrix} dz_2(t) - \hat{v}^*(t)dt \\ dz_3(t) - \hat{s}^*(t)dt \end{bmatrix} \tag{3.5b.34}$$

where

$$\underline{\Sigma}(t) = E\left\{ \begin{bmatrix} v(t)-\hat{v}^*(t) \\ s(t)-\hat{s}^*(t) \end{bmatrix} [v(t)-\hat{v}^*(t) \;\; s(t)-\hat{s}^*(t)] \middle| Z_{23t} \right\} \tag{3.5b.35}$$

The filter (3.5b.34) is infinite dimensional because $\underline{\Sigma}(t)$ has to be computed on-line with an infinite dimensional implementation. The reason is that the velocity $v(t)$ is given by the nonlinear stochastic differential equation

$$dv(t) = u(t)dt + \underline{k}'(v(t),t)d\underline{\xi}(t) \tag{3.5b.36}$$

which guarantees that $v(t)$ is positive for all $t \geq 0$. (See discussion following equation (3.5a.3)). There are however two ways of divising suboptimal implementable approximations to the filter (3.5b.34).

In the first method, we assume that $\underline{k}(v(t),t)$ can be replaced by $\underline{k}(\hat{v}^*(t),t)$ so that equation (3.5b.36) becomes linear in $v(t)$. The filter (3.5b.34) becomes a Kalman filter and the covariance $\underline{\Sigma}(t)$ is computed on-line by the Riccati equation

$$\dot{\underline{\Sigma}}(t) = \underline{A}\,\underline{\Sigma}(t) + \underline{\Sigma}(t)\underline{A}' + \underline{B}(t)\,\underline{\Xi}(t)\underline{B}'(t) - \underline{\Sigma}(t)\underline{\Theta}^{-1}(t)\underline{\Sigma}(t),$$

$$\underline{\Sigma}(0) = \underline{0} \qquad (3.5b.37)$$

where

$$\underline{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad (3.5b.38)$$

$$\underline{B}(t) = \begin{bmatrix} \underline{k}'(\hat{v}^*(t),t) \\ \underline{0}' \end{bmatrix} \qquad (3.5b.39)$$

$$\underline{\Xi}(t)dt = E\{d\underline{\xi}(t)d\underline{\xi}'(t)\} \qquad (3.5b.40)$$

and

$$\underline{\Theta}(t)dt = E\left\{\begin{bmatrix} d\beta_2(t) \\ d\beta_3(t) \end{bmatrix}[d\beta_2(t) \quad d\beta_3(t)]\right\} \qquad (3.5b.41)$$

The initial condition $\underline{\Sigma}(0)=\underline{0}$ arises from the fact that we assume s(0) and v(0) are known. The Riccati equation (3.5b.37) is solved on-line because the gain $\underline{B}(t)$ depends on the estimate $\hat{v}^*(t)$. The result we have arrived at is the extended Kalman filter.

In the second method, we assume that u(t) is sufficiently large that we can use a linear model for v(t) :

$$dv(t) = u(t)dt + \underline{k}'(t)d\underline{\xi}(t) \qquad (3.5b.42)$$

where $\underline{k}(t)$ is known a priori. With such a model, we cannot guarantee that $v(t) > 0$ at every t with probability one but we can only be sure that $v(t) < 0$ with very small probability. Physically, this corresponds to the situation in which the control input we apply on the sensor is so large compared to the random perturbations that we can be sure that $v(t) < 0$ with very small probability. With the model (3.5b.42) for v(t), the filter (3.5b.34) again becomes a Kalman filter and $\underline{\Sigma}(t)$ is computed by the Riccati equation (3.5b.37) where now

$$\underline{B}(t) = \begin{bmatrix} \underline{k}'(t) \\ \\ \underline{0}' \end{bmatrix} \tag{3.5b.43}$$

Since the gains in the Riccati equation are now known a priori, $\underline{\Sigma}(t)$ is precomputable.

3.6 Some Special Cases

A Special Case of the Field Model

In the last section, we have seen the difficulties of spatial mapping when the motion of the sensor is random. We consider in this section a special case of the field model (3.3.1) in which we can hope to avoid the difficulties of the last section. This model is given by

$$dx(s) = f(x(s), s, \underline{r})ds \tag{3.6.1}$$

In this case, the randomness in x(s) is due to the random parameters $\underline{r}$ which may include the random initial condition x(0). Reference [48] examines models of this type extensively. The parameters $\underline{r}$ can be identified on-line by augmenting equation (3.6.1) with the equation

$$d\underline{r} = \underline{0} \qquad (3.6.2)$$

and then applying filtering theory to this augmented system. We shall not deal with this problem of estimating $\underline{r}$ any further.

Consider the field estimation problem when the sensor motion is deterministic. In this case, the field is mapped by estimating $\tilde{x}(t) = x(s(t))$. Since $\tilde{x}(t)$ satisfies the equation

$$\begin{aligned} d\tilde{x}(t) &= f(\tilde{x}(t), s(t), \underline{r})v(t)dt \\ &= \tilde{f}(\tilde{x}(t), t, \underline{r})v(t)dt \end{aligned} \qquad (3.6.3)$$

the estimate

$$\hat{\tilde{x}}(t) = E\{\tilde{x}(t)|Z_{1t}\} \qquad (3.6.4)$$

is given by

$$\begin{aligned} d\hat{\tilde{x}}(t) &= \hat{\tilde{f}}(\tilde{x}(t), t, \underline{r})v(t)dt \\ &\quad + E\{(\tilde{x}(t)-\hat{\tilde{x}}(t))(c(\tilde{x}(t),t)-\hat{c}(\tilde{x}(t),t))|Z_{1t}\} \cdot \\ &\quad \cdot (dz_1(t)-\hat{c}(\tilde{x}(t),t)dt) \end{aligned} \qquad (3.6.5)$$

In the general nonlinear case, this filter is of course non-implementable. In the case of linear field model

$$dx(s) = f(s,\underline{r})x(s)ds \qquad (3.6.6)$$

and assuming a linear observation model, the filter (3.6.5) becomes

$$d\hat{\tilde{x}}(t) = E\{\tilde{f}(t,\underline{r})\tilde{x}(t)\big|Z_t\}v(t)dt$$

$$+ c(t)E\{(\tilde{x}(t)-\hat{\tilde{x}}(t))^2\big|Z_t\}\ (dz_1(t)-c(t)\hat{\tilde{x}}(t)dt) \tag{3.6.7}$$

This filter is still non-implementable because we have to carry along the joint conditional density of $x(t)$ and $\underline{r}$. Actually, although the field model (3.6.6) is linear in $x(s)$, it should be considered nonlinear because $\underline{r}$ should be considered as additional state variables that satisfy equation (3.6.2). If the model (3.6.6) does not depend on $\underline{r}$, the randomness in $x(s)$ being due to $x(0)$ alone, then the filter (3.6.7) can be easily seen to be a readily implementable Kalman filter. But this is a <u>very</u> special case of what we have done earlier.

Next, consider the field estimation problem when the sensor motion is random. Since the field model (3.6.1) is not driven by any Wiener processes, the field can be mapped by estimating $x^*(t) = x(\hat{s}(t))$. The estimate $\hat{s}(t)$ of the position of the sensor is given by

$$d\hat{s}(t) = \hat{v}(t)dt + \sigma_1(t)d\nu_1(t) + \sigma_2(t)d\nu_2(t) + \sigma_3(t)d\nu_3(t) \tag{3.6.8}$$

where

$$\sigma_1(t) = E\{s(t)c(\tilde{x}(t),t)-\hat{s}(t)\hat{c}(\tilde{x}(t),t)\big|Z_t\} \tag{3.6.9}$$

$$\sigma_2(t) = E\{s(t)v(t)-\hat{s}(t)\hat{v}(t)\big|Z_t\} \tag{3.6.10}$$

$$\sigma_3(t) = E\{s^2(t)-\hat{s}^2(t)\big|Z_t\} \tag{3.6.11}$$

$$d\nu_1(t) = dz_1(t)-\hat{c}(\tilde{x}(t),t)dt \tag{3.6.12}$$

$$d\nu_2(t) = dz_2(t)-\hat{v}(t)dt \tag{3.6.13}$$

and

$$d\nu_3(t) = dz_3(t)-\hat{s}(t)dt \tag{3.6.14}$$

Thus, the process $\mathbf{x}^*(t) = x(\hat{s}(t))$ satisfies the equation

$$\begin{aligned} dx^*(t) &= f(x^*(t),\hat{s}(t),\underline{r})(\hat{v}(t)dt + \sigma_1(t)d\nu_1(t) + \sigma_2(t)d\nu_2(t) + \sigma_3(t)d\nu_3(t)) \\ &= f^*(x^*(t),t,\underline{r})(\hat{v}(t)dt + \sigma_1(t)d\nu_1(t) + \sigma_2(t)d\nu_2(t) + \sigma_3(t)d\nu_3(t)) \end{aligned} \tag{3.6.15}$$

Applying results from [6], the estimate

$$\hat{\mathbf{x}}^*(t) = E\{x^*(t)|Z_t\} \tag{3.6.16}$$

is generated by the filter

$$\begin{aligned} d\hat{x}^*(t) = {} & \hat{f}^*(\mathbf{x}^*(t),t,\underline{r})\hat{v}(t)dt \\ & + E\{x^*(t)c(\tilde{x}(t),t)-\hat{x}^*(t)\hat{c}(\tilde{x}(t),t) + \frac{d}{dt}\langle m,\beta_1\rangle_t \Big| Z_t\}d\nu_1(t) \\ & + E\{x^*(t)v(t)-\hat{x}^*(t)\hat{v}(t) + \frac{d}{dt}\langle m,\beta_2\rangle_t \Big| Z_t\}d\nu_2(t) \\ & + E\{x^*(t)s(t)-\hat{x}^*(t)\hat{s}(t) + \frac{d}{dt}\langle m,\beta_3\rangle_t \Big| Z_t\}d\nu_3(t) \end{aligned} \tag{3.6.17}$$

where m_t is the martingale defined by

$$dm_t = f^*(x^*(t),t,\underline{r})[\sigma_1(t),\sigma_2(t),\sigma_3(t)]\begin{bmatrix} d\nu_1(t) \\ d\nu_2(t) \\ d\nu_3(t) \end{bmatrix} \tag{3.6.18}$$

We have assumed that the joint variance processes $\langle m,\beta_1\rangle_t$, $\langle m,\beta_2\rangle_t$ and

$\langle m,\beta_3\rangle_t$ are differentiable, an assumption required in [6]. This assumption is actually true because we have, by definition [49], [36],

$$\langle m,\beta_1\rangle_t = \int_0^t f^*(x^*(\tau),\tau,\underline{r})[\sigma_1(\tau),\sigma_2(\tau),\sigma_3(\tau)] \begin{bmatrix} d\langle\nu_1,\beta_1\rangle_\tau \\ d\langle\nu_2,\beta_1\rangle_\tau \\ d\langle\nu_3,\beta_1\rangle_\tau \end{bmatrix} \tag{3.6.19}$$

$$\langle m,\beta_2\rangle_t = \int_0^t f^*(x^*(\tau),\tau,\underline{r})[\sigma_1(\tau),\sigma_2(\tau),\sigma_3(\tau)] \begin{bmatrix} d\langle\nu_1,\beta_2\rangle_\tau \\ d\langle\nu_2,\beta_2\rangle_\tau \\ d\langle\nu_3,\beta_2\rangle_\tau \end{bmatrix} \tag{3.6.20}$$

$$\langle m,\beta_3\rangle_t = \int_0^t f^*(x^*(\tau),\tau,\underline{r})[\sigma_1(\tau),\sigma_2(\tau),\sigma_3(\tau)] \begin{bmatrix} d\langle\nu_1,\beta_3\rangle_\tau \\ d\langle\nu_2,\beta_3\rangle_\tau \\ d\langle\nu_3,\beta_3\rangle_\tau \end{bmatrix} \tag{3.6.21}$$

which clearly show that they are differentiable. The evaluation of the joint variance processes $\langle\nu_2,\beta_1\rangle_t$, etc., is carried out in Appendix 6. The results are as follows:

$$\langle\nu_1,\beta_1\rangle_t = t \tag{3.6.22}$$

$$\langle\nu_2,\beta_1\rangle_t = 0 \tag{3.6.23}$$

$$\langle\nu_3,\beta_1\rangle_t = 0 \tag{3.6.24}$$

$$\langle\nu_1,\beta_2\rangle_t = 0 \tag{3.6.25}$$

$$\langle\nu_2,\beta_2\rangle_t = t \tag{3.6.26}$$

$$\langle\nu_3,\beta_2\rangle_t = 0 \tag{3.6.27}$$

$$\langle\nu_1,\beta_3\rangle_t = 0 \tag{3.6.28}$$

$$\langle\nu_2,\beta_3\rangle_t = 0 \tag{3.6.29}$$

$$\langle \nu_3 , \beta_3 \rangle_t = t \tag{3.6.30}$$

Thus,

$$\frac{d}{dt} \langle m, \beta_1 \rangle_t = f^*(x^*(t), t, \underline{r}) \sigma_1(t) \tag{3.6.31}$$

$$\frac{d}{dt} \langle m, \beta_2 \rangle_t = f^*(x^*(t), t, \underline{r}) \sigma_2(t) \tag{3.6.32}$$

and

$$\frac{d}{dt} \langle m, \beta_3 \rangle_t = f^*(x^*(t), t, \underline{r}) \sigma_3(t) \tag{3.6.33}$$

The filter (3.6.17) now reduces to

$$\begin{aligned} d\hat{x}^*(t) &= \widehat{f^*(x^*(t), t, \underline{r})}\hat{v}(t)dt \\ &\quad + [\sigma_4(t) + \sigma_1(t)\widehat{f^*}(x^*(t), t, \underline{r})] d\nu_1(t) \\ &\quad + [\sigma_5(t) + \sigma_2(t)\widehat{f^*}(x^*(t), t, \underline{r})] d\nu_2(t) \\ &\quad + [\sigma_6(t) + \sigma_3(t)\widehat{f^*}(x^*(t), t, \underline{r})] d\nu_3(t) \end{aligned} \tag{3.6.34}$$

where

$$\sigma_4(t) = E\{x^*(t)c(\tilde{x}(t), t) - \hat{x}^*(t)\hat{c}(\tilde{x}(t), t) \,|\, Z_t\} \tag{3.6.35}$$

$$\sigma_5(t) = E\{x^*(t)v(t) - \hat{x}^*(t)\hat{v}(t) \,|\, Z_t\} \tag{3.6.36}$$

and

$$\sigma_6(t) = E\{x^*(t)s(t) - \hat{x}^*(t)\hat{s}(t) \,|\, Z_t\} \tag{3.6.37}$$

The result (3.6.34) is only a representation and is not implementable in all cases. Besides other difficulties, the estimate $\hat{c}(\tilde{x}(t),t)$ is never computable except via an infinite system of equations. Even in the linear case when $\hat{c}(\tilde{x}(t),t) = c(t)\hat{\tilde{x}}(t)$, we have seen in Section 3.5a that the computation of the estimate $\hat{\tilde{x}}(t) = E\{x(s(t))\big|Z_t\}$ is infinite dimensional. The manner in which infinite dimensional problems arise in the computation of the other terms in the filter (3.6.34) is similar. We shall not go into the discussion any further.

Estimation of the field at a fixed spatial point

We consider in this section the estimation of the field at a fixed spatial point $s_0>0$ and use the field model (3.3.1). This problem has some possible practical value and is certainly of theoretical interest. Again, we consider the cases of deterministic and random sensor motion.

When the sensor motion is deterministic, the estimation problem is straightforward. If the position of the sensor $s(t)$ is less than s_0, then the estimate

$$\hat{x}(s_0\big|t) = E\{x(s_0)\big|Z_{1t}\} \tag{3.6.38}$$

is given by the prediction equation [6]

$$\hat{x}(s_0\big|t) = \hat{\tilde{x}}(t) + \int_{s(t)}^{s_0} E\{\mathbf{f}(x(s),s)\big|Z_{1t}\}\mathbf{ds} \tag{3.6.39}$$

where

$$\hat{\tilde{x}}(t) = E\{x(s(t))\big|Z_{1t}\} \tag{3.6.40}$$

is the filtered estimate we have considered before in Section 3.4.

If $s(t) = s_0$, then

$$\hat{x}(s_0 \mid t) = \hat{\tilde{x}}(t) \tag{3.6.41}$$

and finally, if $s(t)$ is greater than s_0 , the estimate $\hat{x}(s_0|t)$ is given by the smoothing equation [6]

$$\hat{x}(s_0|t) = \hat{\tilde{x}}(t_0) + \int_{t_0}^{t} E\{x(s_0)(c(\tilde{x}(\tau),\tau)-\hat{c}(\tilde{x}(\tau),\tau)) \Big| Z_{1\tau}\}\cdot \\ \cdot(dz_1(\tau)-\hat{c}(\tilde{x}(\tau),\tau)d\tau) \tag{3.6.42}$$

where we assume that

$$s(t_0) = s_0 \tag{3.6.43}$$

The results above are only representations which are not implementable in the general non-linear case. In the case of linear field model so that

$$f(x(s),s) = f(s)x(s) \tag{3.6.44}$$

the prediction result (3.6.39) is implementable because we can now write

$$d_s\hat{x}(s|t) = f(s)\hat{x}(s|t)ds, \qquad \text{for} \quad s>s(t) \tag{3.6.45}$$

where

$$\hat{x}(s|t) = E\{x(s) \Big| Z_{1t}\} \tag{3.6.46}$$

The estimate $\hat{x}(s_0|t)$ is then obtained by integrating equation (3.6.45) forward in s from $s = s(t)$ to $s = s_0$. However, the smoothing result (3.6.42) is still not implementable even in the case of linear field and observation models.

Consider now the case of random sensor motion. In this case, the estimation problem is not so straightforward as before. Because the position $s(t)$ of the sensor is not known precisely at each time t, a multiple-model type of approach is necessary. We can write the estimate

$$\hat{x}(s_0|t) = E\{x(s_0)|Z_t\} \tag{3.6.47}$$

as follows:

$$\hat{x}(s_0\Big|t) = \int_0^\infty E\{x(s_0)\Big|Z_t\ ,\ s(t)=s'\}P_{s(t)}(s'\Big|Z_t)ds' \tag{3.6.48}$$

However, the generation of the estimate $E\{x(s_0)\Big|Z_t\ ,s(t)=s'\}$ is a problem which cannot be solved except in the simple case of a random constant velocity. The reason is as follows. If $s'=s_0$, the estimate $E\{x(s_0)\Big|Z_t\ ,s(t)=s'\}$ is equal to the estimate $E\{x(s(t))\Big|Z_t\ ,s(t)=s'\}$. If $s'<s_0$, the estimate $E\{x(s(t))\Big|Z_t\ ,s(t)=s'\}$ is required in computing the estimate $E\{x(s_0)\Big|Z_t\ ,s(t)=s'\}$ by means of a prediction equation of the form of (3.6.39). However, the estimate $E\{x(s(t))\Big|Z_t\ ,s(t)=s'\}$, which is not equal to the estimate $E\{x(s(t))\Big|Z_t\} = \hat{x}(t)$, cannot be generated by any estimator at the present time. Finally, if $s'>s_0$, the estimation equation for $E\{x(s_0)\Big|Z_t\ ,s(t)=s'\}$ cannot be written down at all. Although we are given $s(t)=s'$, the time t_0 at which the sensor was at s_0 is unknown since the sensor motion is random. Thus a smoothing estimator of the form of equation (3.6.42) cannot be written.

In the special case in which the velocity of the sensor is an unknown constant random variable v, the difficulties mentioned above can be overcome.

Consider first the generation of the estimate $E\{x(s(t))\big|Z_t\ ,s(t)=s'\}$. Since $s(t) = vt$, conditioning on $s(t)=s'$ is the same as conditioning on $v=s'/t$ and this determines the entire motion of the sensor. Thus, the estimate

$$\hat{\tilde{x}}(t\big|s') = E\{x(s(t))\big|Z_t\ ,s(t)=s'\} \tag{3.6.49}$$

is given by

$$\begin{aligned} d\hat{\tilde{x}}(t\big|s') &= \hat{\tilde{f}}(\tilde{x}(t),t)v\,dt \\ &\quad + E\{(\tilde{x}(t)-\hat{\tilde{x}}(t))(c(\tilde{x}(t),t)-\hat{c}(\tilde{x}(t),t))\big|Z_t\ ,s(t)=s'\}\cdot \\ &\quad \cdot\ (dz_1(t)-\hat{c}(\tilde{x}(t),t)dt) \end{aligned} \tag{3.6.50}$$

where here

$$\hat{\cdot} = E\{\cdot\,|\,Z_t\ ,s(t)=s'\}$$

Note that we need one filter of the form (3.6.50) for each possible value of $v = s'/t$ because for each $t>0$, $s(t)$ can take on any value $s'>0$. The measurements $z_2(t)$ and $z_3(t)$ are not used in the filter (3.6.50) because they are measurements on $v(t)$ and $s(t)$ respectively and the latter quantities are now assumed known. We can now generate the estimate $E\{x(s_0)\big|Z_t\ ,s(t)=s'\}$, for $s'\leq s_0$, as follows. If $s'=s_0$, then

$$E\{x(s_0)\big|Z_t\ ,s(t)=s'\} = E\{x(s(t))\big|Z_t\ ,s(t)=s'\} = \hat{\tilde{x}}(t|s') \tag{3.6.51}$$

If $s'<s_0$, then

$$E\{x(s_0)\Big|Z_t\ ,s(t)=s'\} = \hat{\tilde{x}}(t|s') + \int_{s'}^{s_0} E\{f(x(s),s)\Big|Z_t\ ,s(t)=s'\}ds \quad (3.6.52)$$

Note that when $f(\cdot,\cdot)$ is linear in $x(s)$, equation (3.6.52) is implementable as with equation (3.6.39). Finally, if $s'>s_0$, then the time t_0 at which the sensor is at the point s_0 is

$$t_0 = \frac{s_0 t}{s'} \quad (3.6.53)$$

Thus,

$$E\{x(s_0)\Big|Z_t\ ,s(t)=s'\} = \hat{\tilde{x}}(t_0)$$
$$+ \int_{t_0}^{t} E\{x(s_0)(c(\tilde{x}(\tau),\tau)-\hat{c}(\tilde{x}(\tau),\tau))\Big|Z_\tau\ ,s(t)=s'\} \cdot$$
$$\cdot\ (dz_1(\tau)-\hat{c}(\tilde{x}(\tau),\tau)d\tau) \quad (3.6.54)$$

The generation of the estimate $\hat{\tilde{x}}(t_0)$ has been considered before in Section 3.5a.

We have now considered in detail the generation of the estimate $E\{x(s_0)|Z_t\ ,s(t)=s'\}$. Returning to equation (3.6.48), the density of $s(t)$ conditioned on Z_t can be generated by considering the filtering problem on the system consisting of the states $v(t)$, $s(t)$ and $\tilde{x}(t) = x(s(t))$ with the observations $z_1(t)$, $z_2(t)$ and $z_3(t)$. The joint conditional density of $v(t)$, $s(t)$ and $\tilde{x}(t)$ is immediately given by Kushner's equation [38] and from this we obtain the conditional density of $s(t)$. We now see that in the case of random sensor motion, the estimation of $x(s_0)$ is always infinite dimensional because the unknown random position of the sensor gives rise to an infinite dimensional multiple-model approach for estimating $x(s_0)$.

3.7 Optimal Field Estimation via Sensor Motion Control

One of the most interesting features of the problem of estimating a random field using observations from a moving point sensor seems to be in the problem of sensor speed versus estimation accuracy. It is conceivable that the speed with which the sensor moves across the field affects the quality of its observations on the field and hence the accuracy of the estimates of the field. We first illustrate our conjecture concretely in mathematical terms and then formulate and solve an optimal control problem on the motion of the sensor so that the field is observed and estimated in a manner optimal with respect to some criterion involving the estimation error covariance. Our work here will be carried out only for the case of linear field and observation models with deterministic sensor motion, i.e., the velocity of the sensor is computed a priori and there are no random inputs driving the sensor. In this case, we have seen in Section 3.4 that the field is readily estimated by means of a Kalman filter and the estimation error covariance is precomputable. The problem of finding the optimal velocity of the sensor so as to minimize a cost functional involving the estimation error covariance can now be formulated and solved as a deterministic optimal control problem and we expect to be able to find explicit expressions for the optimal velocity in some special cases. If we consider the case of a nonlinear field or observation model or that of random sensor motion, the estimation error covariance has to be computed on line using the observations, and the sensor motion control problem becomes a complex stochastic control problem. We will not consider the latter case here mainly because it is not as analytically tractable as the linear case with deterministic sensor motion. In addition, since this is the first time such sensor control problems for optimal field estimation are considered, we feel that we should not involve ourselves with highly complicated mathematical problems. Rather, we should limit ourselves at present to a problem which is as simple as possible mathematically so that we can develop an insight and an intuitive feel for the sensor control problem in general. It is our hope that the following analysis will provide the desired insight and also a foundation for future work.

Consider here the linear case with deterministic sensor motion, encountered before in Section 3.4. The field model is given by

$$dx(s) = f(s)\,x(s)\,ds + \underline{g}'(s)\,d\underline{w}(s) \tag{3.7.1}$$

The process $\tilde{x}(t) = x(s(t))$ measured by the sensor is given by

$$d\tilde{x}(t) = \tilde{f}(t)v(t)\tilde{x}(t)dt + \underline{\tilde{g}}'(t)v^{\frac{1}{2}}(t)d\underline{\eta}(t) \tag{3.7.2}$$

where

$$\tilde{f}(t) = f(s(t)) \tag{3.7.3}$$

$$\underline{\tilde{g}}(t) = \underline{g}(s(t)) \tag{3.7.4}$$

The observations on the field are given by

$$dz_1(t) = c(t)\tilde{x}(t)dt + d\beta_1(t) \tag{3.7.5}$$

The velocity $v(t)$ of the sensor is assumed known at each time t, i.e., it is observed perfectly at each time t or it is known a priori and there are no random inputs perturbing the motion. The field is then mapped by computing the estimate

$$\hat{\tilde{x}}(t) = E\{x(s(t))|Z_{1t}\} \tag{3.7.6}$$

which is given by the Kalman filter:

$$d\hat{\tilde{x}}(t) = \tilde{f}(t)v(t)\hat{\tilde{x}}(t)dt + c(t)\sigma(t)R^{-1}(t)(dz_1(t)-c(t)\hat{\tilde{x}}(t)dt) \tag{3.7.7}$$

where

$$R(t)dt = E\{d\beta_1^2(t)\} \tag{3.7.8}$$

and

$$\sigma(t) = E\{(\tilde{x}(t) - \hat{\tilde{x}}(t))^2|Z_{1t}\} \tag{3.7.9}$$

is the estimation error covariance given by the Riccati equation:

$$\frac{d\sigma(t)}{dt} = 2\tilde{f}(t)v(t)\sigma(t) + v(t)\underline{\tilde{g}}'(t)\underline{Q}(t)\underline{\tilde{g}}(t) - \sigma^2(t)c^2(t)R^{-1}(t)$$

$$\sigma(0) = \sigma_0, \text{ given} \tag{3.7.10}$$

Here

$$\begin{aligned} \underline{Q}(t)dt &= E\{d\underline{\eta}(t)d\underline{\eta}'(t)\} \\ &= \underline{\tilde{Q}}(s(t))dt \end{aligned} \tag{3.7.11}$$

where $\underline{\tilde{Q}}(s)ds = E\{d\underline{w}(s)d\underline{w}(s')\}$. It is obvious from equation (3.7.10) that the velocity v(t) of the sensor affects the quality of the estimates as measured by the error covariance $\sigma(t)$. This is also true in the case of nonlinear field and observation models. We consider below the formulation and solution of the sensor control problem for optimal field estimation in the linear case with deterministic sensor motion.

<u>Formulation of the Sensor Control Problem</u>

The problem we are considering here is completely new and admits formulations as control problems on the sensor at several levels of complexity. The formulation that we present here is only one possible formulation, and it has been chosen both because of its potential usefulness and for its analytical tractability.

We want to find the optimal velocity program $v^*(t)$, $t\varepsilon[0,T]$, for the sensor so as to minimize a cost functional which involves the estimation error covariance $\sigma(t)$, $t\varepsilon[0,T]$. First, we place a constraint on the motion of the sensor:

$$\int_0^T v(t)dt = s_o \tag{3.7.12}$$

This means that the sensor has to cover a distance of s_o in a length of time T. We can suppose that s_o is the length of the section of the field that we want to estimate. In addition, we impose the constraint

$$v(t) > 0, \qquad t \ \varepsilon \ [0,T] \tag{3.7.13}$$

This constraint prevents the sensor from sweeping over any point of field more than once. Since the motion of the sensor is deterministic, the constraint (3.7.13) implies a one-to-one relation between the spatial coordinate s and the time t because these two variables are related as

$$s(t) = \int_0^t v(\tau)d\tau \qquad (3.7.14)$$

where s(t) is the position of the sensor at time t. We denote by t(s) the inverse of the function s(t), i.e., t(s) is the time at which the sensor is at the point s. The one-to-one relation between s and t enables us to denote the dependence of any variable on s or on t interchangeably. Thus we define

$$\tilde{v}(s) = v(t(s)) \qquad (3.7.15)$$

We shall now formulate the criterion of optimality. This criterion should contain a cost that measures the accuracy with which the field is estimated. We prepare to use the term

$$J' = \int_0^{s_0} q(s)\tilde{\sigma}(s)ds \qquad (3.7.16)$$

where $\tilde{\sigma}(s) = \sigma(t(s))$ is the estimation error covariance. The function q(s) is a positive function which we determine a priori, and it reflects our judgment of the relative accuracy with which different parts of the field have to be estimated. Since we will solve the control problem in time, we transform the cost J' in equation (3.7.16) as

$$J' = \int_0^T \tilde{q}(t)\sigma(t)v(t)dt \qquad (3.7.17)$$

where

$$\tilde{q}(t) = q(s(t)) \tag{3.7.18}$$

The velocity v(t) is the control in this sensor control problem. In order to make the control problem well posed and to penalize large magnitudes of the velocity, we will include in the criterion of optimality, a weighted integral of the square of the velocity. (Alternatively, we could require $v(t) \leq M$ for some bound M.) Thus, we finally write the criterion of optimality as

$$J = \int_0^T [\tilde{q}(t)\sigma(t)v(t) + r(t)v^2(t)]dt \tag{3.7.19}$$

where r(t) is a positive function. From the discussion earlier in this section, the estimation error covariance $\sigma(t)$ is given by the Riccati equation

$$\frac{d\sigma(t)}{dt} = 2\tilde{f}(t)v(t)\sigma(t) + v(t)\underline{\tilde{g}}'(t)\underline{Q}(t)\underline{\tilde{g}}(t) - \sigma^2(t)c^2(t)R^{-1}(t),$$

$$\sigma(0) = \sigma_0, \text{ given} \tag{3.7.20}$$

Note that if v(t) is determined a priori, $\sigma(t)$ can be precomputed. We can now state our optimal control problem as follows:

Find the optimal velocity $v^*(t)$, $t\varepsilon[0,T]$, subject to the constraints (3.7.12), (3.7.13), and (3.7.20) so that the cost functional J in equation (3.7.19) is minimized.

Solution to the Sensor Control Problem

The optimal control problem formulated above can be solved by a direct application of the minimum principle [55]-[57]. The control in this problem is v and the state variables are σ and s. Thus, we have the state equations

$$\frac{d\sigma(t)}{dt} = 2f(s(t))v(t)\sigma(t) + v(t)\underline{g}'(s(t))\underline{Q}(s(t))\underline{g}(s(t)) - \sigma^2(t)c^2(t)R^{-1}(t) \tag{3.7.21}$$

$$\frac{ds(t)}{dt} = v(t) \tag{3.7.22}$$

The initial values of these states are known:

$$\sigma(0) = \sigma_0 \tag{3.7.23}$$

$$s(0) = 0 \tag{3.7.24}$$

The constraint (3.7.12) is now transformed into the terminal condition

$$s(T) = s_0 \tag{3.7.25}$$

It is convenient for computational reasons to modify the problem by incorporating another constraint:

$$\sigma(T) = \bar{\sigma} \tag{3.7.26}$$

Without such a constraint, the two-point boundary value problem that must be solved is more difficult. The solution to that problem can be obtained, but we have explicitly considered only this simpler case for demonstration purposes. The constraint (3.7.26) means that we are interested in eventually achieving the value $\bar{\sigma}$ for the estimation error covariance. In practice, we usually would want $\bar{\sigma}$ to be less than σ_0. Next, in order to have a closed constraint set for the controls, we modify constraint (3.7.13) to

$$v(t) \geq \varepsilon, \qquad \forall t \in [0,T] \tag{3.7.27}$$

where $\varepsilon > 0$ is an arbitrarily small but fixed constant. Finally, the Hamiltonian H is given by

$$\begin{aligned} H = {} & P_0[q(s(t))\sigma(t)v(t) + r(t)v^2(t)] \\ & + P_1(t)[2f(s(t))v(t)\sigma(t) + v(t)\underline{g}'(s(t))\underline{Q}(s(t))\underline{g}(s(t)) \\ & \qquad - \sigma^2(t)c^2(t)R^{-1}(t)] \\ & + P_2(t)v(t) + \mu(t)[\varepsilon - v(t)] \end{aligned} \tag{3.7.28}$$

where

$$\mu(t) \begin{cases} \geq 0 & \varepsilon - v(t) = 0 \\ = 0 & \varepsilon - v(t) < 0 \end{cases} \tag{3.7.29}$$

(See [57]). The variables P_0, $P_1(t)$, $P_2(t)$ and $\mu(t)$ are costate variables. Now, we apply the minimum principle given in Theorem 5-10 of [56]. In order that $v^*(t)$ be optimal, it is necessary that the following conditions be satisfied (* denotes optimal):

(a) $$\frac{d\sigma^*(t)}{dt} = 2f(s^*(t))v^*(t)\sigma^*(t) + v^*(t)\underline{g}'(s^*(t))\underline{Q}(s^*(t))\underline{g}(s^*(t)) - \sigma^{*2}(t)c^2(t)R^{-1}(t) \qquad (3.7.30)$$

$$\frac{ds^*(t)}{dt} = v^*(t) \qquad (3.7.31)$$

$$P_0^* \geq 0 \qquad (3.7.32)$$

$$\frac{dp_1^*(t)}{dt} = -\left.\frac{\partial H}{\partial \sigma}\right|_* = -P_0^* q(s^*(t))v^*(t) - 2p_1^*(t)f(s^*(t))v^*(t) + 2\sigma^*(t)c^2(t)R^{-1}(t)p_1^*(t) \qquad (3.7.33)$$

$$\frac{dp_2^*(t)}{dt} = -\left.\frac{\partial H}{\partial s}\right|_* = -P_0^*\sigma^*(t)v^*(t)\frac{\partial q}{\partial s}(s^*(t)) - 2p_1^*(t)v^*(t)\sigma^*(t)\frac{\partial f}{\partial s}(s^*(t)) -2p_1^*(t)v^*(t)\frac{\partial \underline{g}'}{\partial s}(s^*(t))\underline{Q}(s^*(t))\underline{g}(s^*(t)) -P_1^*(t)v^*(t)\underline{g}'(s^*(t))\frac{\partial \underline{Q}}{\partial s}(s^*(t))\underline{g}(s^*(t)) \qquad (3.7.34)$$

$$\sigma^*(0) = \sigma_0 \qquad (3.7.35)$$

$$\sigma^*(T) = \bar{\sigma} \qquad (3.7.36)$$

$$s^*(0) = 0 \qquad (3.7.37)$$

$$s^*(T) = s_0 \qquad (3.7.38)$$

(b) Minimization of H with respect to v.

$$\left.\frac{\partial H}{\partial v}\right|_* = 0 = P_0^* q(s^*(t))\sigma^*(t) + 2r(t)v^*(t)p_0^*$$
$$+ 2p_1^*(t)f(s^*(t))\sigma^*(t)$$
$$+ P_1^*(t)\underline{g}'(s^*(t))\underline{Q}(s^*(t))\underline{g}(s^*(t)) + P_2^*(t) - \mu^*(t) \tag{3.7.39}$$

Since

$$\left.\frac{\partial^2 H}{\partial v^2}\right|_* = 2r(t)p_0^* \geq 0 \tag{3.7.40}$$

we conclude that v* obtained from equation (3.7.39) must necessarily minimize H. Equation (3.7.39) gives us only one solution for v* and so this must necessarily be a global minimum. If we are considering the region $v(t) > \varepsilon$, we have $\mu^*(t) = 0$ so that equation (3.7.39) gives us

$$P_0^* v^*(t) = -\tfrac{1}{2}r^{-1}(t)[P_0^* q(s^*(t))\sigma^*(t)$$
$$+ 2p_1^*(t)f(s^*(t))\sigma^*(t)$$
$$+ p_1^*(t)\underline{g}'(s^*(t))\underline{Q}(s^*(t))\underline{g}(s^*(t)) + p_2^*(t)] \tag{3.7.41}$$

Note also that the initial and terminal conditions on the costates are free.

We have now obtained all the conditions that characterize the optimal velocity $v^*(t)$ and the optimal estimation error covariance $\sigma^*(t)$. In principle, the optimal control problem has been solved. However, the solution as such does not give us much insight into the sensor control problem. It is evidently impossible to obtain any algebraic simplification on the set of necessary conditions above. Usually, in practice, the necessary conditions in an optimal control problem have to be solved numerically on a computer. In what follows, we shall consider a special case in which we can obtain an explicit solution to the optimal velocity $v^*(t)$ and the optimal estimation error covariance $\sigma^*(t)$.

<u>The Case of Spatially Invariant Field Model and Time Invariant Observation Model</u>

We consider here the case in which the field model is spatially invariant and given by

$$dx(s) = fx(s)ds + \underline{g}'d\underline{w}(s) \tag{3.7.42}$$

so that $\tilde{x}(t) = x(s(t))$ is given by

$$d\tilde{x}(t) = fv(t)\tilde{x}(t)dt + \underline{g}'v^{\frac{1}{2}}(t)d\underline{\eta}(t) \tag{3.7.43}$$

and the observation model on the field is time-invariant and given by

$$dz_1(t) = c\tilde{x}(t)dt + d\beta_1(t) \tag{3.7.44}$$

In addition, assume that $r(t)$ in the criterion of optimality reduces to

$$r(t) = r > 0 \tag{3.7.45}$$

Then the necessary conditions above reduce to:

a) $$\frac{d\sigma^*(t)}{dt} = 2fv^*(t)\sigma^*(t) + v^*(t)\underline{g}'\underline{Q}\,\underline{g} - \sigma^{*2}(t)c^2/R \tag{3.7.46}$$

$$\frac{ds^*(t)}{dt} = v^*(t) \tag{3.7.47}$$

$$P_0^* \geq 0 \tag{3.7.48}$$

$$\frac{dP_1^*(t)}{dt} = -P_0^* q(s^*(t))v^*(t) - 2P_1^*(t)fv^*(t)$$

$$+ 2\sigma^*(t)P_1^*(t)c^2/R \tag{3.7.49}$$

$$\frac{dP_2^*(t)}{dt} = -P_0^*\sigma^*(t)v^*(t)\frac{\partial q}{\partial s}(s^*(t)) \tag{3.7.50}$$

$$\sigma^*(0) = \sigma_0 \tag{3.7.51}$$

$$\sigma^*(T) = \bar{\sigma} \tag{3.7.52}$$

$$s^*(0) = 0 \tag{3.7.53}$$

$$s^*(T) = s_0 \tag{3.7.54}$$

b) Minimization of H with respect to v:

$$\left.\frac{\partial H}{\partial v}\right|_* = 0 = P_0^* q(s^*(t))\sigma^*(t) + 2rv^*(t)P_0^*$$

$$+ 2P_1^*(t)f\,\sigma^*(t) + P_1^*(t)\underline{g}'\underline{Q}\,\underline{g} + P_2^*(t) - \mu^*(t) \tag{3.7.55}$$

A Special Case

The set of necessary conditions above is still not solvable in closed form. We now consider the following special case in which a closed form solution is possible:

$$f = 0 \tag{3.7.56}$$

$$\underline{g}'\underline{Q}\,\underline{g} = 1 \tag{3.7.57}$$

$$r = 1/2 \tag{3.7.58}$$

$$c^2/R = 1/2 \tag{3.7.59}$$

$$q(s) = q = 1 \tag{3.7.60}$$

Since f=0, the field x(s) consists of a weighted sum of independent standard Wiener processes with total intensity $\underline{g}'\underline{Q}\,\underline{g} = 1$. Of course, this case does not constitute a realistic example of a field but we are picking this example so as to obtain explicit solutions. Hopefully, the explicit solution will give us some insight into the general problem, which must be solved using numerical methods. The choice of $q(s) = 1$ means that the accuracy of all parts of the field is of equal weight. The necessary conditions now reduce to:

a) $$\frac{d\sigma^*(t)}{dt} = v^*(t) - \frac{1}{2}\sigma^{*2}(t) \tag{3.7.61}$$

$$\frac{ds^*(t)}{dt} = v^*(t) \tag{3.7.62}$$

$$P_0^* \geq 0 \tag{3.7.63}$$

$$\frac{dP_1^*(t)}{dt} = -P_0^*\, v^*(t) + \sigma^*(t) P_1^*(t) \tag{3.7.64}$$

$$\frac{dP_2^*(t)}{dt} = 0 \tag{3.7.65}$$

$$\sigma^*(0) = \sigma_0 \tag{3.7.66}$$

$$\sigma^*(T) = \bar{\sigma} \tag{3.7.67}$$

$$s^*(0) = 0 \tag{3.7.68}$$

$$s^*(T) = s_0 \tag{3.7.69}$$

b) Minimization of H with respect to v:

$$\left.\frac{\partial H}{\partial v}\right|_* = 0 = P_0^*\sigma^*(t) + P_0^*v^*(t) + P_1^*(t) + P_2^*(t) - \mu^*(t) \tag{3.7.70}$$

The above equations can now be simplified as follows. We can set $P_0^*=1$ if $P_0^* > 0$. The only case in which $P_0^* = 0$ is when the terminal conditions $\sigma^*(T) = \bar{\sigma}$ and $s^*(T) = s_0$ are so difficult to meet that optimization becomes irrelevant (i.e. either we cannot achieve these conditions or there is only one possible trajectory); that is, all we want is to find the velocity $v^*(t)$ that will meet these terminal conditions [55]. This is also evident from equation (3.7.70) since when $P_0^* = 0$, we cannot determine $v^*(t)$ from this H-minimization condition. We will here assume that the terminal conditions on σ and s are so given that they can be met with more than one velocity profile $v(t)$, $0 \leq t \leq T$, and therefore set

$$P_0^* = 1 \tag{3.7.71}$$

Consider now the case when $v(t) > \varepsilon$ so that $\mu^*(t)=0$. Then, equation (3.7.70) gives

$$v^*(t) = -\sigma^*(t) - P_1^*(t) - P_2^*(t) \tag{3.7.72}$$

From the available equations we can easily get a differential equation for $\sigma^*(t)$. Differentiating equation (3.7.72) to obtain

$$\frac{dv^*(t)}{dt} = -\frac{d\sigma^*(t)}{dt} - \frac{dP_1^*(t)}{dt} - \frac{dP_2^*(t)}{dt}$$

and substituting from equations (3.7.61), (3.7.64) and (3.7.65) we arrive at the equation

$$\frac{dv^*(t)}{dt} = \frac{1}{2}\sigma^{*2}(t) - \sigma^*(t)P_1^*(t)$$

Using $P_1^*(t)$ from (3.7.72) and noting that, by (3.7.65),

$$P_2^*(t) = P_2^*(0) \tag{3.7.73}$$

we get

$$\frac{dv^*(t)}{dt} = \frac{1}{2}\sigma^{*2}(t) + \sigma^*(t)(v^*(t) + \sigma^*(t) + P_2^*(0))$$

$$= \frac{3}{2}\sigma^{*2}(t) + \sigma^*(t)(v^*(t) + P_2^*(0))$$

Next, use $v^*(t)$ from (3.7.61) to get

$$\frac{dv^*(t)}{dt} = \frac{3}{2}\sigma^{*2}(t) + \sigma^*(t)\left(\frac{d\sigma^*(t)}{dt} + \frac{1}{2}\sigma^{*2}(t) + P_2^*(0)\right)$$

$$= \sigma^*(t)\left(\frac{d\sigma^*(t)}{dt} + P_2^*(0)\right) + \frac{3}{2}\sigma^{*2}(t) + \frac{1}{2}\sigma^{*3}(t) \tag{3.7.74}$$

Finally, differentiate (3.7.61) and substitute from (3.7.74) to obtain

$$\frac{d^2\sigma^*(t)}{dt^2} = \frac{dv^*(t)}{dt} - \sigma^*(t)\,\frac{d\sigma^*(t)}{dt}$$

$$= P_2^*(0)\sigma^*(t) + \frac{3}{2}\,\sigma^{*2}(t) + \frac{1}{2}\,\sigma^{*3}(t) \qquad (3.7.75)$$

Solving this equation then gives us the optimal estimation error covariance $\sigma^*(t)$ assuming that $v^*(t)>\varepsilon\ \forall\ t$. To solve this equation, multiply the left side by $2\,\frac{d\sigma^*}{dt}$ dt and the right side by $2\,d\sigma^*$:

$$2\,\frac{d\sigma^*}{dt}\,\frac{d}{dt}\left(\frac{d\sigma^*}{dt}\right)dt = 2(P_2^*(0)\sigma^* + \frac{3}{2}\,\sigma^{*2} + \frac{1}{2}\,\sigma^{*3})d\sigma^*$$

which gives

$$\frac{d}{dt}\left(\frac{d\sigma^*}{dt}\right)^2 dt = 2(P_2^*(0)\sigma^* + \frac{3}{2}\,\sigma^{*2} + \frac{1}{2}\,\sigma^{*3})d\sigma^*$$

An integration gives

$$\left(\frac{d\sigma^*}{dt}\right)^2 = P_2^*(0)\sigma^{*2} + \sigma^{*3} + \frac{1}{4}\,\sigma^{*4} + c \qquad (3.7.76)$$

where

$$c = \left(\frac{d\sigma^*}{dt}\bigg|_{t=0}\right)^2 - (P_2^*(0)\sigma^{*2}(0) + \sigma^{*3}(0) + \frac{1}{4}\,\sigma^{*4}(0)) \qquad (3.7.77)$$

The solution to equation (3.7.76) is found in terms of elliptic functions [58]. By writing equation (3.7.76) as

$$\left(\frac{d\sigma^*}{dt}\right)^2 = h^2(\sigma^*-\alpha)(\sigma^*-\beta)(\sigma^*-\gamma)(\sigma^*-\delta),\ h^2=1/4 \tag{3.7.78}$$

then the solution is given by [58]

$$\sigma^*(t) = (\beta Y^2 - A\alpha)\Big/(Y^2-A) + (\sigma^*(0)-\alpha) \tag{3.7.79}$$

where

$$Y = \mathrm{sn}\{hMt,\ k\} \tag{3.7.80}$$

$$A = \frac{\beta-\delta}{\alpha-\delta} \tag{3.7.81}$$

$$k^2 = \frac{(\beta-\gamma)(\alpha-\delta)}{(\alpha-\gamma)(\beta-\delta)} \tag{3.7.82}$$

$$M^2 = (\beta-\delta)(\alpha-\gamma)/4 \tag{3.7.83}$$

The function $\mathrm{sn}\{\cdot,\cdot\}$ is an elliptic function known as the sinus amplitudinus function [58], [59] and it is tabulated in [60]. The solution (3.7.79) for $\sigma^*(t)$ does not depend on the sign we take for $\frac{d\sigma^*}{dt}$ in equation (3.7.78) since we can absorb the + or - sign in the factor h which also occurs in the first argument of the solution for Y in equation (3.7.80). The function, $\mathrm{sn}\{u,k\}$, is an odd function in u and the solution for $\sigma^*(t)$ involves only the square of this function. We have now obtained a closed form solution for the optimal estimation error covariance $\sigma^*(t)$, and using this in the Riccati equation (3.7.61) will give us the optimal velocity $v^*(t)$ of the sensor. The initial values $P_1^*(0)$ and $P_2^*(0)$ of the costate variables are free and can be selected so that the terminal conditions on σ^* and s^* can be met.

Since elliptic functions are not common functions it is difficult to visualize how the solution above behaves. We therefore work out below a numerical example to enable us to see how the solution behaves in one particular case.

In the example we have chosen, the terminal time is $T = 0.5$. The initial conditions are $\sigma^*(0)=1$ and $s^*(0)=0$ and the terminal conditions are $\sigma^*(0.5)= 0.41$ and $s^*(0.5)=3.94$. It turns out that to meet these terminal conditions, we have to choose $P_1^*(0)=2.75$ and $P_2^*(0)= -10.25$. Then, in equation (3.7.78), the roots on the right hand side are given by $\alpha=3$, $\beta= -2$, $\gamma= -8.25$ and $\delta=3.52$. We now have $A=10.62$, $k=0.231$ and $M=j\ 3.99$. To evaluate the $\mathrm{sn}\{\cdot,\cdot\}$ function when the first argument is imaginary, we use the relation [59]

$$\begin{aligned}\mathrm{sn}\{ju,k\} &= j\ \mathrm{sc}\{u,k'\}\\ &= j\ \frac{\mathrm{sn}\{u,k'\}}{\mathrm{cn}\{u,k'\}}\end{aligned} \tag{3.7.84}$$

where

$$k' = (1-k^2)^{1/2} \tag{3.7.85}$$

and

$$\mathrm{cn}^2\{u,k'\} = 1 - \mathrm{sn}^2\{u,k'\} \tag{3.7.86}$$

The values of $\sigma^*(t)$ can be evaluated directly from equation (3.7.79) with the aid of elliptic function tables, while $v^*(t)$ is evaluated from the

Riccati equation (3.7.61), using equation (3.7.78) for $\frac{d\sigma^*}{dt}$. Table 3.1 below shows the values of $\sigma^*(t)$ and $v^*(t)$ for t=0 to t=0.5 at increments of 0.1 for t.

TABLE 3.1

t	$\sigma^*(t)$	$v^*(t)$
0	1	6.50
0.1	0.98	6.51
0.2	0.92	6.53
0.3	0.81	6.57
0.4	0.65	6.61
0.5	0.41	6.67

Actually, this numerical example was obtained by solving the problem backwards. We first pick the initial conditions $\sigma^*(0)$, $s^*(0)$, $P_1^*(0)$ and $P_2^*(0)$ so that $v^*(0)$ from equation (3.7.72) is positive. With these values selected, we evaluate $\left.\frac{d\sigma^*}{dt}\right|_{t=0}$ from equation (3.7.61) and hence c from equation (3.7.77). The coefficients in equation (3.7.76) for $\left(\frac{d\sigma^*}{dt}\right)^2$ are now all known. Then, we find the roots α, β, γ and δ for equation (3.7.78) so that equation (3.7.76) is satisfied. We can now evaluate $\sigma^*(t)$ and $v^*(t)$ via the method mentioned before. The terminal conditions $\sigma^*(0.5)$ and $s^*(0.5)$ are not picked a priori but just result from the values of $\sigma^*(0)$, $P_1^*(0)$ and so on that we pick. We will have more to say about the numerical solution of such optimal control problems in general in a later section.

Note that in the example, $v^*(t) > 0$ for all $0 \leq t \leq 0.5$. Thus, we do not have to invoke the condition $\mu^*(t) \geq 0$ which happens when $v(t)=\varepsilon$, for some cannot be expected. We note that the necessary conditions we have derived are highly nonlinear, involving products of the state and costate variables and the control. In the case of spatially invariant field and time invariant observation models, the nonlinearities in the system of equations are somewhat simpler, and one might hope to find an explicit solution in this case, as we have for one special case.

In practice, however, the solution to an optimal control problem usually cannot be found explicitly and it has to be determined numerically on a computer. In the case of the sensor motion control problem that we have formulated, the initial and terminal conditions on the state variables are specified. Thus, the initial and terminal conditions on the costate variables are **free**. In a numerical solution of such an optimal control problem, we need to use a numerical technique such as Newton's method [70], [71] to solve for the optimal state and costate trajectories and thereby the optimal control. We illustrate here how this method works for our sensor control problem. For notational simplicity, we define

$$\underline{y}(t) = \begin{bmatrix} \sigma(t) \\ s(t) \end{bmatrix} \tag{3.7.87}$$

as the vector of state variables and define

$$\underline{P}(t) = \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} \tag{3.7.88}$$

as the vector of costate variables. We assume $P_0=1$. Write the set of cannonical equations for the state and costate variables in the necessary conditions of optimality (equations (3.7.30) to (3.7.34)) for simplicity as

This is a linear system of the form

$$\begin{bmatrix} \dot{\underline{y}}_{k+1}(t) \\ \dot{\underline{p}}_{k+1}(t) \end{bmatrix} = \underline{A}_k(t) \begin{bmatrix} \underline{y}_{k+1}(t) \\ \underline{p}_{k+1}(t) \end{bmatrix} + \underline{m}_k(t) \tag{3.7.100}$$

where

$$\underline{m}_k(t) = \begin{bmatrix} \tilde{\underline{a}}(\underline{y}_k(t),\ \underline{p}_k(t),t) \\ \tilde{\underline{b}}(\underline{y}_k(t),\ \underline{p}_k(t),t) \end{bmatrix} - \underline{A}_k(t) \begin{bmatrix} \underline{y}_k(t) \\ \underline{p}_k(t) \end{bmatrix} \tag{3.7.101}$$

and its solution at t=T is given by [73]

$$\begin{bmatrix} \underline{y}_{k+1}(T) \\ \underline{p}_{k+1}(T) \end{bmatrix} = \underline{\Phi}_k(T,0) \begin{bmatrix} \underline{y}_{k+1}(0) \\ \underline{p}_{k+1}(0) \end{bmatrix} + \underline{n}_k(T,0) \tag{3.7.102}$$

where

$$\underline{n}_k(T,0) = \int_0^T \underline{\Phi}_k(T,\tau)\underline{m}_k(\tau)d\tau \tag{3.7.103}$$

and $\underline{\Phi}_k(\cdot,\cdot)$ is the transition matrix associated with the system

$$\begin{bmatrix} \dot{\underline{y}}(t) \\ \dot{\underline{p}}(t) \end{bmatrix} = \underline{A}_k(t) \begin{bmatrix} \underline{y}(t) \\ \underline{p}(t) \end{bmatrix} \tag{3.7.104}$$

$\varepsilon > 0$ arbitrarily small (as long as we take $\varepsilon < 6.5$). Note also that $\sigma^*(t)$ decreases with t although $v^*(t)$ increases with t. The fact that $v^*(t)$ increases with t is because we have to meet the terminal condition on s^*. That $\sigma^*(t)$ decreases with t although $v^*(t)$ increases with t is not surprising since we are taking observations as we move along.

We should remark that although we have obtained an explicit solution in one special case, we have not obtained a great deal of insight into the nature of problems of sensor motion control for optimal field estimation. It is clear that more work is needed in order to understand the problem thoroughly. Numerical techniques are needed to obtain the solution in general since we cannot always hope to find explicit solutions. The application of one numerical technique to our sensor control problem is explained in the following section.

Summary of the Sensor Control Problem

We have now solved the problem of optimal field estimation via sensor motion control in the case of linear field and observation models with deterministic sensor motion. We have derived the necessary conditions for optimality in the general case of a spatially varying linear field model and time varying linear observation model. In a special case of a spatially invariant linear field model and a time invariant linear observation model, we have been able to derive an explicit solution for the optimal estimation error covariance. In the general case of a spatially invariant linear field model and time invariant linear observation model, it may still be possible, in principle, to derive an explicit solution for the optimal estimation error covariance or the optimal velocity. However, for the case of a spatially varying linear field model and a time varying linear observation model, such explicit solutions

$$\dot{\underline{y}}^*(t) = \underline{a}(\underline{y}^*(t),\ \underline{P}^*(t),\ v^*(t),t) \tag{3.7.89}$$

$$\dot{\underline{P}}^*(t) = \underline{b}(\underline{y}^*(t),\ \underline{P}^*(t),\ v^*(t),t) \tag{3.7.90}$$

with the given boundary conditions

$$\underline{y}^*(0) = \begin{bmatrix} \sigma^*(0) \\ s^*(0) \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ 0 \end{bmatrix} \equiv \underline{y}_0 \tag{3.7.91}$$

and

$$\underline{y}^*(T) = \begin{bmatrix} \sigma^*(T) \\ s^*(T) \end{bmatrix} = \begin{bmatrix} \bar{\sigma} \\ s_0 \end{bmatrix} \equiv \bar{\underline{y}} \tag{3.7.92}$$

From the H-minimal condition of equation (3.7.39), we can solve for the optimal control $v^*(t)$ in terms of the optimal state and costate variables $\underline{y}^*(t)$ and $\underline{P}^*(t)$, assuming that $v^*(t) > \varepsilon$ so that $\mu^*(t)=0$. Suppose

$$v^*(t) = h(\underline{y}^*(t),\ \underline{P}^*(t),t) \tag{3.7.93}$$

Then we can substitute this relation into equations (3.7.89) and (3.7.90) and eliminate $v^*(t)$:

$$\dot{\underline{y}}^*(t) = \tilde{\underline{a}}(\underline{y}^*(t),\ \underline{P}^*(t),t) \tag{3.7.94}$$

$$\dot{\underline{p}}^*(t) = \tilde{\underline{b}}(\underline{y}^*(t),\ \underline{P}^*(t),t) \tag{3.7.95}$$

We now have to solve this set of coupled equations (3.7.94) and (3.7.95) with the boundary conditions (3.7.91) and (3.7.92). Newton's method is used to determine, in an iterative manner, the functions $\underline{y}^*(t)$ and $\underline{p}^*(t)$, for all $t \in [0,T]$. The method generates a sequence of time functions $\{\underline{y}_k(t)\}$ and $\{\underline{p}_k(t)\}$, $t \in [0,T]$, $k=0,1,2,...,$ which always meet the boundary conditions (3.7.91) and (3.7.92) but do not necessarily satisfy the differential equations (3.7.94) and (3.7.95). The idea is to choose this sequence of functions so that they approach the solutions of the differential equations (3.7.94) and (3.7.95). An initial guess for $\underline{y}_0(t)$ and $\underline{p}_0(t)$ is first made over the entire interval $[0,T]$. Suppose we have arrived at the k-th guess $\underline{y}_k(t)$ and $\underline{p}_k(t)$, $t \in [0,T]$. Then, the (k+1)th guess is generated as follows. If the (k+1)-th guess is the true solution, then

$$\dot{\underline{y}}_{k+1}(t) = \underline{\tilde{a}}(\underline{y}_{k+1}(t), \underline{p}_{k+1}(t),t) \tag{3.7.96}$$

$$\dot{\underline{p}}_{k+1}(t) = \underline{\tilde{b}}(\underline{y}_{k+1}(t), \underline{p}_{k+1}(t),t) \tag{3.7.97}$$

We now linearize this system about the k-th guess:

$$\begin{bmatrix} \dot{\underline{y}}_{k+1}(t) \\ \dot{\underline{p}}_{k+1}(t) \end{bmatrix} = \begin{bmatrix} \underline{\tilde{a}}(\underline{y}_k(t), \underline{p}_k(t),t) \\ \underline{\tilde{b}}(\underline{y}_k(t), \underline{p}_k(t),t) \end{bmatrix} + \underline{A}_k(t) \begin{bmatrix} \underline{y}_{k+1}(t)-\underline{y}_k(t) \\ \underline{p}_{k+1}(t)-\underline{p}_k(t) \end{bmatrix} \tag{3.7.98}$$

where

$$\underline{A}_k(t) \equiv \left. \begin{bmatrix} \dfrac{\partial \underline{\tilde{a}}}{\partial \underline{y}} & \dfrac{\partial \underline{\tilde{a}}}{\partial \underline{p}} \\ \dfrac{\partial \underline{\tilde{b}}}{\partial \underline{y}} & \dfrac{\partial \underline{\tilde{b}}}{\partial \underline{p}} \end{bmatrix} \right|_{\underline{y}=\underline{y}_k(t),\ \underline{p}=\underline{p}_k(t)} \tag{3.7.99}$$

Note that $\underline{A}_k(t)$, $\underline{m}_k(t)$ and $\underline{n}_k(T,0)$ are known given the k-th guess. Equation (3.7.102) now gives us a relation between $\underline{y}_{k+1}(T)$, $\underline{y}_{k+1}(0)$ and $\underline{P}_{k+1}(0)$. But we require that the boundary conditions

$$\underline{y}_{k+1}(0) = \underline{y}_0 \ , \ \underline{y}_{k+1}(T) = \overline{\underline{y}} \tag{3.7.105}$$

be satisfied. Thus, we can find the required value for $\underline{P}_{k+1}(0)$. We now have the initial condition for the (k+1)-th guess, i.e., $\underline{y}_{k+1}(0)$ and $\underline{P}_{k+1}(0)$ and using this in equation (3.7.100) gives us the (k+1)-th trajectory $\underline{y}_{k+1}(t)$, $\underline{P}_{k+1}(t)$, $t \ \varepsilon \ [0,T]$. The iteration is continued likewise and usually the true solution is considered to have been obtained when

$$\max_{t \ \varepsilon [0,T]} \left\| \begin{bmatrix} \underline{y}_{k+1}(t) - \underline{y}_k(t) \\ \underline{P}_{k+1}(t) - \underline{P}_k(t) \end{bmatrix} \right\| < \delta \tag{3.7.106}$$

where δ is a predetermined arbitrarily small positive number. The optimal control is then obtained from equation (3.7.93).

Note that we have assumed $v^*(t) > \varepsilon$ in the discussion of Newton's method above. Whenever $v^*(t)=\varepsilon$, the condition $\mu^*(t) \geq 0$ would have to be invoked. In this case, we cannot just apply Newton's method above directly. It is not clear at present what numerical technique can be used to handle the situation.

In the sensor control problem that we have considered, if, instead of specifying the terminal condition $\sigma^*(T)$, we had let it be free, the resulting numerical problem would be even more complex. In this case, we would have the terminal condition $P_1^*(T)=0$ and we would have to solve a split two-point boundary value problem. A discussion of such two-point boundary value problems can be found in [70], [71] and we shall not detail them any more. Note that Newton's method as described above cannot be applied for split two-point boundary value problems. Other methods, such as the method of steepest descent [70], [71], have to be used instead.

Considering extensions to the special case of the sensor control problem we have considered, it appears to be very likely that we can still obtain a differential equation for $\sigma^*(t)$ alone if we change the values of $\underline{g}'\underline{Q}\,\underline{g}$, r and c^2/R provided we keep f=0 and q(s)=constant. If f is nonzero and q is non-constant, several nonlinear terms arise in the necessary conditions, and it is not clear if we can perform a similar analysis.

Note that although we have considered the sensor motion control problem using the velocity as the control, we could also have formulated the problem using the acceleration as the control and the velocity as an additional state variable. In this case, the positivity constraint on the velocity becomes a state-variable inequality constraint and this would require an approach slightly different than the approach we used in the previous sections. For a discussion on control problems with state-variable constraints, see [57]. Actually, it is more meaninful physically to consider the acceleration as the control in the sensor control problem. However, we shall not solve this problem here.

Finally, note that in the sensor control problem in this section and the spatial mapping problem in the previous sections, we very often encounter the case of the spatially invariant field model

$$dx(s) = fx(s)ds + \underline{g}'d\underline{w}(s) \tag{3.7.107}$$

Because f is a scalar constant, the class of processes generated by this model is quite special. We can extend our work without difficulty to the case of the vector model

$$d\underline{y}(s) = \underline{A}\ \underline{y}(s)ds + \underline{B}d\underline{w}(s) \tag{3.7.108}$$

$$x(s) = \underline{c}'\underline{y}(s) \tag{3.7.109}$$

This model of course generates a richer class of processes. We shall however not do this extension here.

CHAPTER 4

CONCLUSION AND SUGGESTIONS FOR FUTURE RESEARCH

In this thesis, we have examined the issues of space-time modeling and estimation in the context of two particular physically motivated examples. While we have been motivated by potential applications to several problems of practical importance, the major emphasis of our work has been in the study of the modeling of space-time processes and of the consequences of such models for problems of optimal estimation. For the problem in Chapter 2, we pointed out such potential applications as wave propagation in random media, statistical fluid mechanics, discrete multipath communication and seismic signal processing. For the problem in Chapter 3, we foresee applications such as microwave sensing of atmospheric temperature and humidity fields using satellite observations and gravity field mapping via instruments carried in a moving vehicle. For the models formulated in both chapters, we have analyzed in detail the estimation and statistical inference problems involved. We will point out in the rest of this chapter several problem areas which are immediate extensions of the work we have done but first we list the contributions of our work.

The main contribution of this thesis is the abstraction of mathematical models for certain physically motivated space-time process problems and in indicating how the powerful techniques of stochastic analysis and estimation can be used in their solution. As we have pointed out before, many previous researchers have investigated into random field problems via various different approaches and with different applications in mind. However, their work has mostly been directed at characterizing the properties of random fields and very little has been done in the area of space-time modeling and estimation.

Our work here is among the first attempts in conceptualizing real physical problems involving random fields and developing a theory of estimation and statistical inference for the models we formulated for such problems. We have laid a foundation in this thesis upon which future researchers can build in order to handle more complicated and realistic space-time problems. A number of new ideas have been introduced, such as sensor control for optimal field estimation, which are certain to lead to much further research in the future. Moreover, we have laid down some concrete concepts and mathematical results for the classes of space-time problems that we have formulated. Although our problem formulations are admittedly narrow, we have nevertheless achieved something more solid for these problems than many other workers have for their random field problems. For instance, in [28], Monin and Yaglom have only discussed some basic statistical ideas for inferring the nature of a random turbulent fluid flow. The discussion, though general, is vague and gives no indication at all of how such statistical ideas can be implemented in practice. In contrast, although our problem formulations deal with very simple cases, we have derived concrete mathematical results and discussed how in practice they can be implemented. Our work further indicates the degree of complexity that the solution of a random field estimation problem can involve. This could never have been seen in such a general qualitative discussion as in [28]. We believe that many random field problems in practice can be solved if appropriate mathematical models are abstracted for such problems. This is one of the main reasons for our formulation of the models in this thesis. Hopefully, our work here can help to inspire future researchers to create models for other space-time problems.

We list here the specific contributions we have made to the topic of random fields in the context of the space-time problems we have considered.

(1) We considered the problem of estimating a random signal field propagated by a random transmission medium. In the process, we provided the first solution to the problem of estimating a diffusion process observed with a random time delay.

(2) We discovered several cases of optimal finite dimensional implementation to the signal estimation results in (1).

(3) One way of deriving finite dimensional suboptimal approximate implementations to the signal estimation results was demonstrated.

(4) An entirely novel on-line procedure for estimating random delay times in propagating signals was presented.

(5) Some results were derived for inferring the properties of random time invariant transmission fields using known propagating signals.

(6) Similar contributions, though more restricted, were made in the case of multiple signal fields and in the case of multiple sensors.

(7) We derived the results for estimating a time invariant spatial field modeled by a stochastic differential equation and observed by a moving point sensor. The implications of using such a model for the problem of random field estimation are uncovered by the results.

(8) A novel problem of optimal field estimation via sensor motion control is formulated and solved explicitly in one special case. Although the special case that we have solved might not be important in practice, the ideas introduced have much further implications for future research and practical applications.

In the following paragraphs, we will contribute some ideas for future research which are direct extensions of the work we have done.

In Chapter 2, we have seen in both the multiple source and multiple sensor problems that the estimation of $\phi_{t-t_{s1}}$ can at present be accomplished only by the infinite dimensional multiple model approach consisting of a growing infinite bank of smoothers. A representation for the estimate $\hat{\phi}_{t-t_{s1}}$ is impossible at the moment, whether recursive or nonrecursive. This makes both the multiple source and multiple sensor problems more difficult than the basic one-source-one-sensor problem or the estimation problems in which the delay times are known. Thus, deriving a representation for the estimate $\hat{\phi}_{t-t_{s1}}$ is a very important problem for future research. Only by examining the representation for $\hat{\phi}_{t-t_{s1}}$, whether recursive or nonrecursive, can we hope to find cases in which the computation of $\hat{\phi}_{t-t_{s1}}$ is finite dimensional, if they exist. At present, the only way we can hope to compute $\hat{\phi}_{t-t_{s1}}$ with a finite dimensional implementation is to approximate the infinite bank of smoothers by a finite bank. However, each of these smoothers is infinite dimensional [37] and a finite dimensional approximation is not known in this case. Estimation problems for systems with time delays, even if the delays are known, are infinite dimensional [37]. Thus, it is hard to expect, and even appears to be impossible, that our problems here with random delays can admit finite dimensional optimal implementations. However, a representation for $\hat{\phi}_{t-t_{s1}}$ can still give us additional insight into how finite dimensional suboptimal approximations can be made.

For the work of Chapter 2, it will be interesting to consider, as a first extension, the case in which the observation model contains an unknown random parameter, i.e., the observation model is given by

$$dz_t = h(\phi_{t-t_s}, r, t)dt + dw_t \tag{4.1}$$

Here, $h(\cdot,\cdot,\cdot)$ is jointly measurable with respect to all three arguments and w_t is a standard Wiener process satisfying the same assumptions made in Section 2.2. In addition, it is assumed to be independent of r. The parameter r is a random variable with a known a priori distribution. The problems we are interested in are now:

(i) To estimate the signal ϕ_{t-t_s} using the observations Z_t ,

(ii) To estimate the delay time t_s ,

(iii) To estimate the parameter r.

One of the main reasons for examining the above problems with the observation model (4.1) is to study the situation in which the transmission medium modulates the amplitude of the signal field in addition to causing random transmission delays from the source to the sensor. In general, since both r and t_s are determined by the random transmission field, one would want to consider the case in which they are statistically correlated. In the linear case the parameter r has a direct interpretation as an amplitude attenuation factor:

$$dz_t = h_t r\, \phi_{t-t_s} dt + dw_t \tag{4.2}$$

One approach for solving the above problems which we can suggest here immediately is the multiple model approach. By conditioning on each known value of the parameter r, say $r=r'$, we can apply the results of Chapter 2 to obtain an estimate of the signal ϕ_{t-t_s} as well as the conditional distribution of the delay time t_s. The only problem that remains now is to find the equations for computing on-line the conditional distribution $P(r \leq r' | Z_t)$ of the parameter r given the observations Z_t. Once this distribution is available, the solution to the above problems is obtained as follows. Define

$$\hat{\phi}_{t-t_s}|r' = E\{\phi_{t-t_s} | Z_t , r=r'\} \tag{4.3}$$

Then, the estimate of the signal ϕ_{t-t_s} is given by

$$\hat{\phi}_{t-t_s} = \int_{r'} \hat{\phi}_{t-t_s}|r' \; P(r'<r\leq r'+dr' | Z_t) \tag{4.4}$$

Similarly, the conditional distribution of the delay time t_s is given by

$$P(t_s \leq t' | Z_t) = \int_{r'} P(t_s \leq t' | Z_t , r=r') \, P(r'<r\leq r'+dr' | Z_t) \tag{4.5}$$

The problem of deriving the equations for computing on-line the conditional distribution $P(r\leq r' | Z_t)$ of the parameter r given the observations Z_t remains to be solved.

When we further extend our problem to the case of multiple signal sources, i.e., when the observation model becomes

$$dz_t = h_1(\phi_{t-t_{s1}}, r_1, t)dt + h_2(\phi_{t-t_{s2}}, r_2, t)dt + dw_t \tag{4.6}$$

it becomes even more interesting. Here, r_1 and r_2 are (possibly correlated) random variables, both independent of w_t. When we consider the multiple source problem as the multiple reflection problem, the parameters r_1 and r_2 can be used to model the random reflection coefficients from the two reflectors. This becomes more apparent in the linear case, when the observation model (4.6) reduces to

$$dz_t = h_{1t} r_1 \phi_{t-t_{s1}} dt + h_{2t} r_2 \phi_{t-t_{s2}} dt + dw_t \tag{4.7}$$

Again, one approach that we can immediately suggest for solving the multiple source problem is the multiple model approach. By conditioning on known values of r_1 and r_2, the results of Chapter 2 can be applied to estimate the signals $\phi_{t-t_{s1}}$ and $\phi_{t-t_{s2}}$ as well as the delay times t_{s1} and t_{s2}. The only problem that remains to be solved is to derive the equations for on-line computation of the joint conditional distribution of r_1 and r_2. The case in which the gains r_1 and r_2 are random in equation (4.6) or (4.7) is important in practice because when we apply the results for the multiple reflection problem to, say, the discrete multipath communication problem, r_1 and r_2 can be used to model the random amplitude fading in a Rayleigh communication channel [16].

A more complicated version of the multiple reflection problem above can also be considered. This is the case in which reflection and transmission of the signal field take place on both sides of each reflector, giving rise to reverberations which happen all the time in most wave propagation problems, especially seismic signal processing [15]. Considering the two-reflector situation in Figure 14, we see that since the signal field is continually being reflected between the two reflectors, more and more return signals are received by the sensor, each successive return being weaker than the previous one. The observation model for this problem is of the form

$$dz_t = h_1(\phi_{t-t_{s1}}, r_1, t)dt + h_2(\phi_{t-t_{s2}}, r_2, t)dt + \ldots + dw_t \tag{4.8}$$

which contains an infinite number of terms because theoretically there is an infinite number of return signals generated by the reverberations. It is also easy to see that there is a simple relation among the delay times, i.e.,

$$t_{si+1} - t_{si} = 2T \tag{4.9}$$

where T is the travel time between the two reflectors. It is not clear at this point how the problem of estimating the signals $\phi_{t-t_{s1}}$, $\phi_{t-t_{s2}}$, etc., can be solved since the observation model (4.8) contains an infinite number of terms.

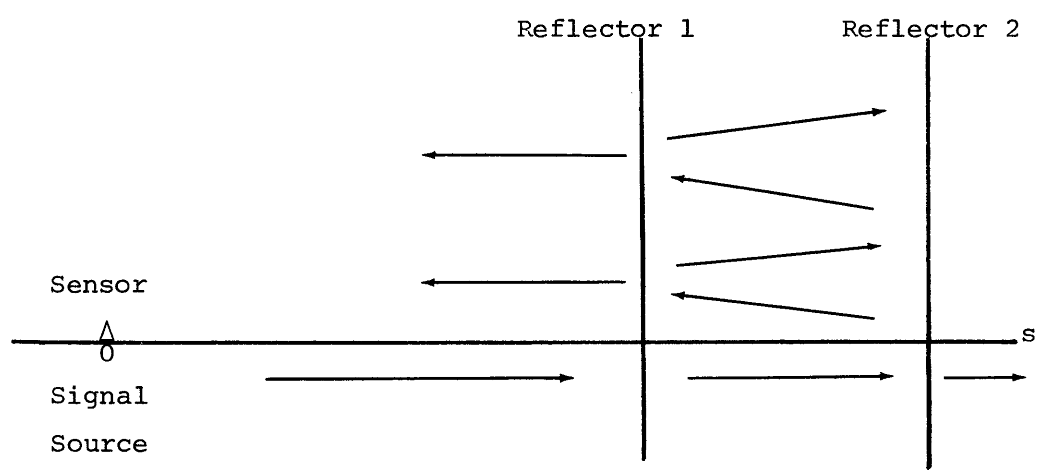

FIGURE 14: A TWO-REFLECTOR SITUATION INVOLVING REVERBERATIONS

Finally, we remark that when applying our results of Chapter 2 to any specific problem, one would need to evaluate the performance of the estimators by simulation. Such performance analyses would also be useful for comparing our results with those derived via the classical frequency domain approach. This is clearly a crucial next step in determining the utility of the framework we have developed.

In Chapter 3, we considered the problem of estimating a spatial random field using observations from a moving point sensor. We assumed that the field did not affect the motion of the sensor, and we have pointed out that such a problem formulation does not apply to such problems as gravity field mapping using spacecraft tracking data [44]. However, we feel that a problem formulation applicable to the latter type of problems is not only theoretically interesting but also useful in practice. We shall present here a problem formulation for the case in which the field affects the motion of the sensor; the solution of the problem is suggested for future research.

Assume that the random field is an acceleration (or force) field, i.e., that $x(s)$ is the random acceleration (or force) experienced by the sensor at point s. The equations of motion are

$$ds(t) = v(t)dt \tag{4.10}$$

$$dv(t) = x(s(t))dt = \tilde{x}(t)dt \tag{4.11}$$

The field $x(s)$ is still assumed to be given by the spatial shaping filter model (3.3.1). The equation for the acceleration $x(s(t)) = \tilde{x}(t)$ can be obtained from the field model (3.3.1) using the same space-time transformation as in Theorem 3.1. The result is similar to equation (3.3.11):

$$d\tilde{x}(t) = \tilde{f}(\tilde{x}(t),t)v(t)dt + \underline{\tilde{g}}'(\tilde{x}(t),t)\left|v(t)\right|^{1/2}d\underline{\xi}(t) \tag{4.12}$$

where $\tilde{f}(\cdot,\cdot)$ and $\underline{\tilde{g}}(\cdot,\cdot)$ are as defined in equations (3.3.12) and (3.3.13). The process $\underline{\xi}(t)$ is periodic, with period 2T (defined later), and over each period is defined by

$$\underline{\xi}(2nT+t) = \begin{cases} \underline{\eta}(t)\ , & 0\leq t\leq T \\ \underline{\eta}(2T-t), & T\leq t\leq 2T \end{cases} \qquad n = \text{positive integer} \tag{4.13}$$

where $\underline{\eta}(t)$ is a vector of independent standard Wiener processes. The interval of time T is length of time during which the sensor can move under the influence of the acceleration field before coming to a stop and reversing its direction of motion. The nature of the motion of the sensor in the acceleration field can be briefly described as follows. Since x(s) is an acceleration, its integral

$$P(s) = \int_0^s x(s')ds' \tag{4.14}$$

is a potential. The sensor moving in the acceleration field can be viewed as a particle rolling in the potential field. If the particle starts from rest from a point s_0 , it will roll off in the direction of lower potential until it reaches a point s_1 , such that

$$P(s_0) = P(s_1) \tag{4.15}$$

It stops at s_1 and reverses its direction of motion until it comes back to rest at s_0. Thereafter, the motion is repeated. The length of time T is the time it takes the particle to go from s_0 to s_1 or from s_1 to s_0. Note that we have assumed the motion to be conservative, i.e., there is no dissipation of energy. The motion is periodic with period 2T.

Now, assume that the motion of the sensor is observed through its velocity and position, i.e., we have observations

$$dz_1(t) = v(t)dt + dw_1(t) \tag{4.16}$$

$$dz_2(t) = s(t)dt + dw_2(t) \tag{4.17}$$

where $w_1(t)$ and $w_2(t)$ are independent standard Wiener processes such that $\{w_1(\tau_1)-w_1(\tau_2),\ \tau_1>\tau_2\geq t\}$ and $\{w_2(\tau_1)-w_2(\tau_2),\ \tau_1>\tau_2\geq t\}$ are independent of $\{\underline{\xi}(\tau),\ 0\leq\tau\leq t\}$. Using these observations, we want to estimate the section of the field traversed by the sensor, i.e., x(s) for $s \in [s_0,s_1]$. Considering the problem over an interval of time $nT\leq t\leq(n+1)T$, n a positive integer, the process $\underline{\xi}(t)$ is a vector of independent standard Wiener processes. We can therefore view the problem as a standard nonlinear filtering problem consisting of the system (4.10) through (4.12) and the observations (4.16) and (4.17) and write down the filtering equation for $E\{\tilde{x}(t)\,|\,Z_t\} = E\{x(s(t))\,|\,Z_t\}$. However, the same problems will arise here as in Chapter 3 due to the position s(t) not being known exactly. The same discussion as in Section 3, Chapter 3, on such problems applies here and we shall not repeat it. A more interesting feature of the field estimation problem here lies in the fact that since the sensor will reverse its direction of motion after every interval of time T, we will be re-estimating the old values of the field after every interval of time T. In addition, the value of T is random and we do not know <u>when</u> we start re-estimating the old values. At present, it is not clear how this estimation problem can be done. It seems that some combination of filter and smoother is needed and that T has to be estimated on line so that at the appropriate time we can switch from the filter to the smoother when we are re-estimating the old values of the field.

The problem we have formulated above is important because it models the problem of gravity field mapping using spacecraft tracking data. The phenomenon of the sensor reversing its direction of motion and re-experiencing the old values of the acceleration field is analogous to a spacecraft re-experiencing the old values of the gravity field after making a revolution round the planet.

We can also extend the problem formulated above to the case in which the motion of the sensor is dissipative. This is the case in which the velocity of the sensor is given, for instance, by

$$dv(t) = \tilde{x}(t)dt - D(v(t))dt \tag{4.18}$$

where $D(v(t))$ is a velocity-dependent dissipative force. In this case, it is well known that the sensor travels for shorter and shorter intervals of time between stops and the distance traveled between stops also decreases. The field estimation problem is now more complicated than before when sensor motion is conservative because the time it travels before reversing its direction of motion changes with each stop, making it more difficult than before to predict when we will start re-estimating old values of the field. Additional complications arise due to the fact that a smaller portion of the field is re-estimated after each stop. At present, no definite solution can be foreseen for this problem and we suggest it for future research.

The work in Chapter 3 has all been carried out for the case of a continuous field $x(s)$. The same is true of the gravity field mapping problem that we have just formulated. However, there are many random fields in practice which naturally involve jump discontinuities, especially in the area of image processing [65], [67], [68]. Thus, the modeling of random fields with jump discontinuities and their estimation using observations from a moving point sensor is also an important area for future research. At this stage, we feel that fields with jump discontinuities can be modeled using an analog of equation (3.3.1), our spatial shaping filter model for a continuous field. This model is given by

$$dx(s) = f(x(s),s)ds + \underline{g}'(x(s),s)d\underline{N}(s) \tag{4.19}$$

where $\underline{N}(s)$ is a vector of independent standard Poisson processes. The theory of equations of the form (4.19) has been treated extensively, e.g., [6]. Since $\underline{N}(s)$ is a jump process, it gives rise to jump discontinuities in $x(s)$. We conjecture that the estimation of such discontinuous fields can be dealt with along the lines by which we deal with continuous fields.

Finally, an extension of the problem of optimal field estimation via sensor motion control to the case of nonlinear field and observation models should be considered. It is well known that in this case the estimation error covariance has to be computed on-line using the measurements [38]. The problem of controlling the velocity of the sensor to minimize some functional of the error covariance will now become an on-line stochastic control problem. We shall not attempt to formulate the problem here but suggest it for future researchers in this field.

The problems we have considered in Chapter 3 as well as the gravity field mapping type of problems we just formulated above are all based on a one-dimensional model for the fields of interest. For more realistic applications to such problems or to problems in other areas such as image processing [65], [67], [68], a model in multidimensional space would be necessary. Before such models can be made, a usable multidimensional stochastic calculus would be necessary and we have cited such work before in Chapter 1. However, in order to make use of such multidimensional stochastic calculus, the artificial causality imposed in its definition must be understood and its implications clarified before realistic models can be built.

It should also be pointed out that all of our work in this thesis has been done for the case of a continuous parameter, i.e., continuous time or continuous space. Perhaps some insight could also be gained if we carry out our work here for the case of a discrete parameter, i.e., discrete time or discrete space. An appropriate model for a field in one-dimensional discrete space could be the following:

$$x(k+1) = f(x(k),k) + \underline{g}'(x(k),k)\underline{w}(k) \tag{4.20}$$

Here, $x(\cdot)$ is the random quantity of interest, k is a nonnegative integer and $\underline{w}(k)$ is a vector of independent white Gaussian sequences. One simplification that we certainly can get in the case of a discrete parameter is that the need for a spatial stochastic calculus may be eliminated. This simplification could prove to be helpful when we try to extend our work to the case of a multidimensional discrete parameter. The study of such discrete parameter fields is very interesting theoretically and might prove to be of great help in understanding random fields with a multidimensional continuous parameter.

Although we have studied in detail two problems of modeling and estimation of space-time stochastic processes, we have by no means covered great territory in the area of random fields. Rather, our contributions in this thesis have only shed some light on one corner of a vast unexplored research area which still harbors lots of open issues and unanswered questions for both theoreticians and applied scientists. We feel that we have made significant progress in this thesis, but we are still at the threshold of a new and exciting area for future research.

APPENDIX 1

Verification of Theorem 2.1

Theorem 2.1: The unit-jump process ψ_t can be represented by the stochastic differential equation

$$d\psi_t = \lambda_t dt + dm_t$$

where m_t is a martingale on $\{\mathcal{B}_t\}_{t\geq 0}$ and

$$\lambda_t = \left[p_{t_s}(t) \Big/ \int_t^\infty p_{t_s}(\tau)d\tau \right] (1-\psi_t) \triangleq \rho_t(1-\psi_t)$$

$p_{t_s}(t)$ is the a priori probability density of t_s.

■

This is intended to be a non-rigorous verification of the result. For a rigorous proof, see [54].

The process ψ_t is a submartingale on $\{\mathcal{B}_t\}_{t\geq 0}$. It is easy to check that it is of class DL [35]. Thus, by the Doob-Meyer decomposition theorem [35], we can write

$$\psi_t = a_t + m_t \tag{A1.1}$$

where $\{a_t, \mathcal{B}_t\}$ is an increasing process and $\{m_t, \mathcal{B}_t\}$ is a martingale. The decomposition (A1.1) is unique. We now have to show that

$$a_t = \int_0^t \lambda_\tau d\tau \tag{A1.2}$$

and this can be done by showing that $\psi_t - \int_0^t \lambda_\tau d\tau$ is a martingale on $\{\mathcal{B}_t\}_{t\geq 0}$. Now,

$$E\left\{\left(\psi_{t+\Delta t} - \int_0^{t+\Delta t} \lambda_\tau \, d\tau\right) - \left(\psi_t - \int_0^t \lambda_\tau d\tau\right) \Big| \mathcal{B}_t\right\}$$

$$= E\left\{ \psi_{t+\Delta t} - \psi_t - \int_t^{t+\Delta t} \lambda_\tau d\tau \Big| \mathcal{B}_t\right\}$$

$$= E\left\{ \psi_{t+\Delta t} - \psi_t \Big| \mathcal{B}_t\right\} - \lambda_t \Delta t \tag{A1.3}$$

But we have

$$E\{\psi_{t+\Delta t} - \psi_t \big| \mathcal{B}_t\} = \mathcal{P}(\psi_{t+\Delta t} - \psi_t = 1 \big| \mathcal{B}_t)$$

$$= \mathcal{P}(\psi_{t+\Delta t} - \psi_t = 1 \big| \mathcal{B}_t)\Big|_{\psi_{t=1}} + \mathcal{P}(\psi_{t+\Delta t} - \psi_t = 1 \big| \mathcal{B}_t)\Big|_{\psi_{t=0}}$$

$$= \mathcal{P}(\psi_{t+\Delta t} - \psi_t = 1 \big| \mathcal{B}_t)\Big|_{\psi_{t=0}} (1-\psi_t)$$

$$= \mathcal{P}(t_s \leq t+\Delta t \,|\, t_s > t) \;\; (1-\psi_t)$$

$$= \left[\mathcal{P}(t < t_s \leq t+\Delta t) \middle/ \mathcal{P}(t_s > t)\right](1-\psi_t)$$

$$\approx \left[p_{t_s}(t)\Delta t \middle/ \int_t^\infty p_{t_s}(\tau) d\tau\right] (1-\psi_t)$$

$$= \lambda_t \; \Delta t \tag{A1.4}$$

Thus,

$$E\left\{\left(\psi_{t+\Delta t}-\int_0^{t+\Delta t}\lambda_\tau d\tau\right)-\left(\psi_t-\int_0^t\lambda_\tau d\tau\right)\middle|\mathcal{B}_t\right\}=0 \tag{A1.5}$$

From this, we can show that for any $t'\geq t$,

$$E\left\{\left(\psi_{t'}-\int_0^{t'}\lambda_\tau d\tau\right)-\left(\psi_t-\int_0^t\lambda_\tau d\tau\right)\middle|\mathcal{B}_t\right\}$$

$$=E\left\{\psi_{t'}-\psi_t-\int_t^{t'}\lambda_\tau d\tau\ \middle|\ \mathcal{B}_t\right\}=0 \tag{A1.6}$$

To do this, partition the interval $[t,t']$ as:

$$t=t_0<t_1<\ldots<t_n=t'$$

so that $t_{i+1}-t_i=\Delta t_i$ is small. Then,

$$E\left\{\psi_{t'}-\psi_t-\int_t^{t'}\lambda_\tau d\tau\middle|\mathcal{B}_t\right\}$$

$$=E\left\{\psi_{t_n}-\psi_{t_{n-1}}+\psi_{t_{n-1}}-\ldots+\psi_{t_{i+1}}-\psi_{t_i}+\ldots-\psi_{t_0}\right.$$

$$\left.-\left(\int_{t_{n-1}}^{t_n}\lambda_\tau d\tau+\ldots+\int_{t_i}^{t_{i+1}}\lambda_\tau d\tau+\ldots+\int_{t_0}^{t_1}\lambda_\tau d\tau\right)\middle|\mathcal{B}_t\right., \tag{A1.7}$$

The typical term is

$$E\left\{\psi_{t_{i+1}} - \psi_{t_i} - \int_{t_i}^{t_{i+1}} \lambda_\tau d\tau \,\middle|\, \mathcal{B}_t\right\}$$

$$= E\left\{E\left\{\psi_{t_{i+1}} - \psi_{t_i} - \int_{t_i}^{t_{i+1}} \lambda_\tau d\tau \,\middle|\, \mathcal{B}_{t_i}\right\} \,\middle|\, \mathcal{B}_t\right\}$$

$$= 0$$

by (A1.5). Thus, (A1.6) follows and $\psi_t - \int_0^t \lambda_\tau d\tau$ is a martingale on $\{\mathcal{B}_t\}_{t\geq 0}$.

APPENDIX 2

Proof of Theorem 2.2

Theorem 2.2: The signal ϕ_{t-t_s} is represented by

$$d\phi_{t-t_s} = (\lambda_t\phi_0 + \psi_{t-}\alpha(\phi_{t-t_s}, t-t_s))dt$$

$$+ [\phi_0 \, , \, \psi_{t-} \underline{\gamma}'(\phi_{t-t_s}, t-t_s)] \begin{bmatrix} dm_t \\ d\underline{\eta}_{t-t_s} \end{bmatrix}$$

ψ_{t-} denotes the left-continuous version of ψ_t. ■

Using the Doleans-Dade Meyer change of variables formula [36], we have:

$$\phi_{t-t_s} = \psi_t\phi_{t-t_s}$$

$$= \psi_0\phi_{-t_s} + \int_0^t \phi_{(\tau-t_s)-}d\psi_\tau + \int_0^t \psi_{\tau-}d\phi_{\tau-t_s}$$

$$+ \int_0^t d\langle\psi^c \, , \, \phi^c_{-t_s}\rangle_\tau$$

$$+ \sum_{\tau\leq t} (\psi_\tau\phi_{\tau-t_s} - \psi_{\tau-}\,\phi_{(\tau-t_s)-})$$

$$- \sum_{\tau\leq t} [\phi_{(\tau-t_s)-}(\psi_\tau-\psi_{\tau-}) + \psi_{\tau-}(\phi_{\tau-t_s}-\phi_{(\tau-t_s)-})] \qquad \text{(A2.1)}$$

The notation ψ^c_t and $\phi^c_{t-t_s}$ denotes the continuous part of the processes ψ_t

and ϕ_{t-t_s}. Since ψ_t is purely discontinuous, we have $\psi_t^c=0$ and so the fourth term in the right side of (A2.1) is zero. The second term $\int_0^t \phi_{(\tau-t_s)-}d\psi_\tau$ is also zero since $d\psi_\tau\neq 0$ only at $\tau=t_s$ and $\phi_{(\tau-t_s)-}=0$ at $\tau=t_s$. Thus, by rearrangement,

$$\phi_{t-t_s} = \psi_0\phi_{-t_s} + \int_0^t \psi_{\tau-}d\phi_{\tau-t_s}$$

$$+ \sum_{\tau\leq t} [\phi_{\tau-t_s}(\psi_\tau-\psi_{\tau-}) - \phi_{(\tau-t_s)-}(\psi_\tau-\psi_{\tau-})] \tag{A2.2}$$

The last term on the right is zero since $\psi_\tau-\psi_{\tau-}\neq 0$ only at $\tau=t_s$ and $\phi_{(\tau-t_s)-}=0$ at $\tau=t_s$. It can easily be seen that the term $\sum_{\tau\leq t} \phi_{\tau-t_s}(\psi_\tau-\psi_{\tau-})$ is equal to $\phi_0\psi_t$. Hence,

$$\phi_{t-t_s} = \psi_0\phi_{-t_s} + \int_0^t \psi_{\tau-}d\phi_{\tau-t_s} + \phi_0\psi_t \tag{A2.3}$$

or

$$
\begin{aligned}
d\phi_{t-t_s} &= \psi_{t-} d\phi_{t-t_s} + \phi_0 d\psi_t \\
&= \psi_{t-} \left(\alpha(\phi_{t-t_s}, t-t_s)dt + \underline{\gamma}'(\phi_{t-t_s}, t-t_s) d\underline{\eta}_{t-t_s}\right) \\
&\quad + \phi_0(\lambda_t dt + dm_t) \\
&= (\lambda_t \phi_0 + \psi_{t-} \alpha(\phi_{t-t_s}, t-t_s))dt \\
&\quad + [\phi_0 \quad \psi_{t-} \underline{\gamma}'(\phi_{t-t_s}, t-t_s)] \begin{bmatrix} dm_t \\ d\underline{\eta}_{t-t_s} \end{bmatrix}
\end{aligned}
\tag{A2.4}
$$

APPENDIX 3

Verification of equation (2.3d.4)

We have, at each fixed time t,

$$E\{\phi_{t-t_s} - \psi_{t-}\phi_{t-t_s} \mid Z_t\}$$

$$= E\{(\psi_t - \psi_{t-})\phi_{t-t_s} \mid Z_t\}$$

$$= \int_{\tau \leq t} E\{(\psi_t - \psi_{t-})\phi_{t-t_s} \mid Z_t, t_s = \tau\} P(\tau < t_s \leq \tau + d\tau \mid Z_t) \tag{A3.1}$$

But $\psi_t - \psi_{t-} \neq 0$ only at $t_s = \tau = t$ and $P(t_s = t \mid Z_t) = 0$.

Therefore,

$$E\{\psi_{t-}\phi_{t-t_s} \mid Z_t\} = \hat{\phi}_{t-t_s} \tag{A3.2}$$

APPENDIX 4

Verification of equation (2.3e.74)

We can write, since $t_s \geq T$,

$$\psi_t = \pi_t \psi_t \tag{A4.1}$$

Using the Doleans-Dade Meyer change of variables formula,

$$\psi_t = \pi_0 \psi_0 + \int_0^t \psi_{\tau-} d\pi_\tau + \int_0^t \pi_{\tau-} d\psi_\tau + \int_0^t d\langle \pi^c , \psi^c \rangle_\tau$$

$$+ \sum_{\tau \leq t} \left[\pi_\tau \psi_\tau - \pi_{\tau-} \psi_{\tau-} \right] - \sum_{\tau \leq t} \left[\psi_{\tau-} (\pi_\tau - \pi_{\tau-}) + \pi_{\tau-} (\psi_\tau - \psi_{\tau-}) \right] \tag{A4.2}$$

We now note the following:

(1) Since $\pi^c = 0$ and $\psi^c = 0$, the fourth term is zero.

(2) Since $d\pi_\tau \neq 0$ only at $\tau = T$ and since $t_s \geq T$ implies that $\psi_{T-} = 0$, thus the second term is zero.

(3) Since $\pi_\tau - \pi_{\tau-} \neq 0$ only at $\tau = T$ and $\psi_{T-} = 0$, thus the sixth term is zero.

These reduce (A4.2) to

$$\psi_t = \pi_0 \psi_0 + \int_0^t \pi_{\tau-} d\psi_\tau + \sum_{\tau \leq t} \left[\pi_\tau \psi_\tau - \pi_{\tau-} \psi_{\tau-} \right] - \sum_{\tau \leq t} \pi_{\tau-} (\psi_\tau - \psi_{\tau-})$$

$$= \pi_0 \psi_0 + \int_0^t \pi_{\tau-} d\psi_\tau + \sum_{\tau \leq t} \psi_\tau (\pi_\tau - \pi_{\tau-}) \tag{A4.3}$$

But

$$\sum_{\tau \leq t} \psi_\tau (\pi_\tau - \pi_{\tau-}) = \psi_T \pi_t \tag{A4.4}$$

and

$$\psi_T = 0 \qquad\qquad \text{w.p.1} \tag{A4.5}$$

because

$$\mathcal{P}(\psi_T = 1) = \mathcal{P}(t_s \leq T) = \mathcal{P}(t_s = T) = 0 \tag{A4.6}$$

Thus, we finally have

$$\psi_t = \pi_0 \psi_0 + \int_0^t \pi_{\tau-} d\psi_\tau \tag{A4.7}$$

or

$$d\psi_t = \pi_{t-}(\lambda_t dt + dm_t) \tag{A4.8}$$

where we make use of the representation

$$d\psi_t = \lambda_t dt + dm_t \quad , \qquad \text{for} \quad t \geq T \quad . \tag{A4.9}$$

APPENDIX 5

Proof of Theorem 3.1

Theorem 3.1: Let $\{w(s), \mathcal{F}_s, s \geq 0\}$ be a Wiener process with $E\{dw^2(s)\} = \tilde{Q}(s)ds$ with respect to the parameter s, where

$$\mathcal{F}_s = \sigma\{w(s'), 0 \leq s' \leq s\} \tag{A5.1}$$

Assume that the process s(t) satisfies

$$ds(t) = v(t)dt, \qquad s(0)=0 \tag{A5.2}$$

where $v(t) > 0$ is a given continuous random process. Let t(s) denote the inverse of s(t). Further, assume that the increments $w(s_1)-w(s_2)$, for $s_1 > s_2 \geq s$, are independent of $\{s(\tau)\wedge s, \forall \tau \geq 0\}$ and $\{v(t(s')), 0 \leq s' \leq s\}$. Define the increasing family of σ-fields

$$\mathcal{G}_s = \mathcal{F}_s \vee \sigma\{s(\tau)\wedge s, \forall \tau \geq 0\} \vee \sigma\{v(t(s')), 0 \leq s' \leq s\}, \tag{A5.3}$$

Then, for each t, s(t) is a stopping time with respect to $\mathcal{G}_s$ and on the family $\{\tilde{\mathcal{G}}_t\}_{t \geq 0}$ where

$$\tilde{\mathcal{G}}_t = \mathcal{G}_{s(t)} \tag{A5.4}$$

the process

$$\tilde{w}(t) = w(s(t)) \tag{A5.5}$$

is a martingale with respect to time t and is given by

$$d\tilde{w}(t) = v^{1/2}(t)d\eta(t) \tag{A5.6}$$

where $\{\eta_t, \tilde{G}_t\}$ is a Wiener process with respect to time t with

$E\{d\eta^2(t)\} = Q(t)dt = \tilde{Q}(s(t))dt$

■

Proof: We first show that s(t), for every t, is a G_s stopping time. First we note

$$\{s(t)<s\} = \{s(t)\Lambda s<s\} \in G_s \tag{A5.7}$$

Furthermore, by the continuity of s(t)

$$\{s(t)=s\} = \bigcap_m \bigcap_{n>m} \bigcap_{\substack{\tau \text{ rational} \\ \tau<t}} \{s-\frac{1}{m} \leq s(\tau)\Lambda s \leq s-\frac{1}{n}\} \in G_s \tag{A5.8}$$

Hence s(t) is a stopping time.

By the assumption that the increments $w(s_1)-w(s_2)$, $s_1>s_2\geq s$, are independent of $\{s(\tau)\Lambda s, \forall \tau\geq 0\}$ and $\{v(t(s')), 0\leq s'\leq s\}$, the process w(s) is a Wiener process on $\{G_s\}_{s\geq 0}$. Thus, $\tilde{w}(t) = w(s(t))$ is a continuous L^2-martingale on $\{\tilde{G}_t\}_{t\geq 0}$. Note that s(t) and v(t) are adapted to $\tilde{G}_t$ and v(t) is predictable on $\{\tilde{G}_t\}_{t\geq 0}$ since it is continuous. Now, define the process $\eta(t)$ as the stochastic integral

$$\eta(t) = \int_0^t v^{-1/2}(\tau)d\tilde{w}(\tau) \tag{A5.9}$$

We can easily verify that $\eta(t)$ is a standard Wiener process on $\{\tilde{G}_t\}_{t\geq 0}$ as follows:

(i) $\eta(t)$ is a continuous $\tilde{G}_t$ -martingale since it is a stochastic integral with respect to the continuous martingale $\{\tilde{w}(t),\tilde{G}_t\}$.

(ii) $\eta(0)=0$

(iii) Since

$$\eta(t) = \int_0^t v^{-1/2}(\tau)d\tilde{w}(\tau)$$

$$= \int_0^{s(t)} v^{-1/2}(t(s'))dw(s') \tag{A5.10}$$

where

$$s' = s(\tau) \tag{A5.11}$$

then, for $t_1>t_2$,

$$E\{\left|\eta(t_1)-\eta(t_2)\right|^2\Big|\tilde{G}_{t_2}\}$$

$$= E\{\left|\int_{s(t_2)}^{s(t_1)} v^{-1/2}(s^{-1}(s'))dw(s')\right|^2\Big|G_{s(t_2)}\}$$

$$= E\{\int_{s(t_2)}^{s(t_1)} v^{-1}(s^{-1}(s'))d\langle w\rangle_{s'}\Big|G_{s(t_2)}\}$$

$$= E\{\int_{s(t_2)}^{s(t_1)} v^{-1}(s^{-1}(s'))ds'\Big|G_{s(t_2)}\}$$

$$= t_1 - t_2 \tag{A5.12}$$

because

$$v^{-1}(t(s'))ds' = d\tau \tag{A5.13}$$

Equation (A5.9) now gives us the desired result:

$$d\tilde{w}(t) = v^{1/2}(t)d\eta(t) \tag{A5.14}$$

APPENDIX 6

Derivation of equations (3.6.22) through (3.6.30)

To derive these equations, we first construct the σ-fields with respect to which the processes are adapted. As in Appendix 5, we first construct the family $\{\mathcal{G}_s\}_{s\geq 0}$ by

$$\mathcal{G}_s = \sigma\{w(s'),\ 0\leq s'\leq s\} \vee \sigma\{s(\tau)\wedge s,\ \forall \tau\geq 0\} \vee \sigma\{v(t(s')),\ 0\leq s'\leq s\} \tag{A6.1}$$

and assume $\sigma\{x(0)\}\subset\mathcal{G}_0$. It is still assumed that the increments $w(s_1)-w(s_2)$, $s_1>s_2\geq s$, are independent of $\{s(\tau)\wedge s, \forall\tau\geq 0\}$ and $\{v(t(s')),\ 0\leq s'\leq s\}$. Now, define the family $\{\mathcal{B}_t\}_{t\geq 0}$ by

$$\mathcal{B}_t = \mathcal{G}_{s(t)} \vee \sigma\{\beta_1(\tau),\beta_2(\tau),\beta_3(\tau),\ 0\leq\tau\leq t\} \tag{A6.2}$$

It is easy to see that all processes of interest which vary with t, including spatial processes sampled in time, are adapted to $\{\mathcal{B}_t\}_{t\geq 0}$. By the assumption we made earlier that $\{\beta_1(\tau_1)-\beta_1(\tau_2),\beta_2(\tau_1)-\beta_2(\tau_2),\ \beta_3(\tau_1)-\beta_3(\tau_2),\ \tau_1>\tau_2\geq t\}$ is independent of $\{s(\tau),\ v(\tau),\ 0\leq\tau\leq t\}$ and $\{w(s(\tau)),\ 0\leq\tau\leq t\}$, the processes $\beta_1(t)$, $\beta_2(t)$ and $\beta_3(t)$ are Wiener processes on $\{\mathcal{B}_t\}_{t\geq 0}$. We did not consider these σ-fields in Chapter 3 because they were not needed except in the derivations in this appendix.

Consider the derivation of equation (3.6.22) for $\langle\nu_1, \beta_1\rangle_t$. For any

$t>t_1$,with t_1 fixed, we have

$$E\{\langle\nu_1, \beta_1\rangle_{t+\Delta t} - \langle\nu_1, \beta_1\rangle_t \Big| \mathcal{B}_{t_1}\}$$

$$= E\{(\nu_1(t+\Delta t) - \nu_1(t))(\beta_1(t+\Delta t) - \beta_1(t)) \Big| \mathcal{B}_{t_1}\} \qquad \text{(A6.3)}$$

But

$$\nu_1(t+\Delta t) - \nu_1(t) = (c(\tilde{x}(t),t) - \hat{c}(\tilde{x}(t),t))\Delta t$$
$$+ (\beta_1(t+\Delta t) - \beta_1(t)) \qquad \text{(A6.4)}$$

and we have

$$E\{[c(\tilde{x}(t),t) - \hat{c}(\tilde{x}(t),t)]\Delta t[\beta_1(t+\Delta t) - \beta_1(t)] \Big| \mathcal{B}_{t_1}\}=0 \qquad \text{(A6.5)}$$

since both $c(\tilde{x}(t),t)$ and $\hat{c}(\tilde{x}(t),t)$ are independent of the future increment $\beta_1(t+\Delta t) - \beta_1(t)$ of the observation noise. Thus, (A6.3) becomes

$$E\{\langle\nu_1, \beta_1\rangle_{t+\Delta t} - \langle\nu_1, \beta_1\rangle_t \Big| \mathcal{B}_{t_1}\}$$

$$= E\{(\beta_1(t+\Delta t) - \beta_1(t))^2 \Big| \mathcal{B}_{t_1}\} = \Delta t \qquad \text{(A6.6)}$$

From (A6.6), it is easy to show that for any $t>t_1$, with t_1 fixed, we have

$$E\{\langle\nu_1, \beta_1\rangle_t - \langle\nu_1, \beta_1\rangle_{t_1} \Big| \mathcal{B}_{t_1}\} = t-t_1 \qquad \text{(A6.7)}$$

which implies that

$$\langle\nu_1, \beta_1\rangle_t = t \qquad \text{(A6.8)}$$

Using the same technique for $<\nu_2, \beta_1>_t$, we have

$$E\{<\nu_2, \beta_1>_{t+\Delta t} - <\nu_2, \beta_1>_t \Big| \mathcal{B}_{t_1}\}$$

$$= E\{(\nu_2(t+\Delta t) - \nu_2(t))(\beta_1(t+\Delta t) - \beta_1(t)) \Big| \mathcal{B}_{t_1}\}, \quad t>t_1 \tag{A6.9}$$

But

$$\nu_2(t+\Delta t) - \nu_2(t) = (v(t) - \hat{v}(t))\Delta t + (\beta_2(t+\Delta t) - \beta_2(t)) \tag{A6.10}$$

and we have

$$E\{(v(t) - \hat{v}(t))\Delta t(\beta_1(t+\Delta t) - \beta_1(t)) \Big| \mathcal{B}_{t_1}\} = 0 \tag{A6.11}$$

$$E\{(\beta_2(t+\Delta t) - \beta_2(t))(\beta_1(t+\Delta t) - \beta_1(t)) \Big| \mathcal{B}_{t_1}\} = 0 \tag{A6.12}$$

Equation (A6.11) follows since $v(t)$ and $\hat{v}(t)$ are independent of the future increment $\beta_1(t+\Delta t) - \beta_1(t)$ of the observation noise $\beta_1(t)$ while (A6.12) follows by independence of $\beta_1(t)$ and $\beta_2(t)$. Thus,

$$E\{<\nu_2, \beta_1>_{t+\Delta t} - <\nu_2, \beta_1>_t \Big| \mathcal{B}_{t_1}\} = 0 \tag{A6.13}$$

and from this it follows that

$$<\nu_2, \beta_1>_t = 0 \tag{A6.14}$$

The derivation of the other equations is similar.

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BIOGRAPHICAL NOTE

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Mr. Kam believes in the training of the mind as well as the body. Since a teenager, he likes practicing the Chinese Martial Arts, or Kung Fu, which he believes is the ultimate form of exercise for the human body.

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