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On relativistic quantum mechanics of the Majorana particle: quaternions, paired plane waves, and orthogonal representations of the Poincaré group

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Abstract

The standard momentum operator $-i\nabla$ has the trivial domain (the null vector) if the L^2 Hilbert space consists of only real-valued functions. In consequence, it is useless in quantum mechanics of the relativistic Majorana particle which is formulated in such a Hilbert space. Instead, one can consider the axial momentum operator introduced in (2019) *Phys. Lett. A* **383** 1242. In the present paper we report several new results which elucidate usability of the axial momentum observable. First, a new motivation for the axial momentum is given, and the Heisenberg uncertainty relation checked. Next, we show that the general solution of time evolution equation written in the axial momentum basis has a connection with quaternions. Furthermore, it turns out that in the case of massive Majorana particles, single traveling monochromatic plane waves are not possible, but there exist solutions which have the form of two plane waves traveling in opposite directions. Another issue discussed here in detail is relativistic invariance. A single real, orthogonal and irreducible representation of the Poincaré group—consistent with the lack of antiparticle—is unveiled.

Keywords: relativistic quantum mechanics, Dirac equation, Majorana particle, real bispinors

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1. Introduction

The discovery of a nonvanishing mass of neutrinos has led to many conjectures about the nature of these particles. In particular, it is possible that they are relativistic massive fermions of the Majorana type. While state of the art description of fundamental particles is provided by quantum field theory (notwithstanding its well-known problems), the slightly older framework of relativistic quantum mechanics is also useful, especially in the case of single particles. Relativistic quantum mechanics has many important applications e.g. in atomic physics, theory of elementary particles, and even in condensed matter physics, see, e.g. [1–3]. Among relativistic wave equations, the most popular is of course the one proposed by P A M Dirac, but other equations are interesting as well, in particular the Proca and the Salpeter equations recently discussed in [4, 5], respectively. There is no doubt that relativistic quantum mechanics is the source of important insights. Quantum mechanics of the Majorana particle is not an exception in this respect.

Relativistic quantum mechanics of the Majorana particle significantly differs from quantum mechanics of the Dirac particle. It has several unusual features. Certain aspects of it have already been considered in [6–10]. In particular, there is a rather intriguing problem concerning the momentum observable (see below) which to the best of our knowledge was first considered in [7], and recently readdressed in [10], where the axial momentum operator has been proposed as a momentum-like observable. In the present paper we utilize expansions into eigenfunctions of the axial momentum operator in order to discuss the general solution of the time evolution equation, as well as the relativistic invariance.

The time evolution equation for the Majorana particle coincides with the Dirac equation[3],

$$i\gamma^{\mu}\partial_{\mu}\psi - m\psi = 0, \tag{1}$$

in which certain Majorana-type representation for Dirac matrices γ^{μ} is assumed, that is these matrices are purely imaginary, and m is a non-negative real number. The crucial difference with the Dirac particle is that all four components of the bispinor ψ are real numbers. This assumption is consistent with equation (1). Thus, the relevant Hilbert space $\mathcal{H}$ consists of all real normalizable bispinors, and the pertinent algebraic number field is that of real numbers $\mathbb{R}$, not the more common in quantum mechanics algebraic field of complex numbers $\mathbb{C}$. There exist other formulations of Majorana quantum mechanics,, e.g. [6, 9], but they are equivalent to the one adopted here, and of course the problem with the momentum operator is present there too.

Quantum mechanics with real numbers or even quaternions in place of the algebraic field of complex numbers is not very popular, but it has been thoroughly discussed in literature [11–13]. In particular, it is known that in the real and quaternionic cases the discrete symmetries P, T, and C are represented by unitary operators, while in complex quantum mechanics anti-unitary symmetry operators can also appear.

The standard momentum operator $\hat{\mathbf{p}} = -i\nabla$ turns real bispinors into imaginary ones, therefore its domain in $\mathcal{H}$ is trivial—it consists of the null vector only. Thus, a new momentum-like operator is needed. Such operator, called the axial momentum and denoted by $\hat{\mathbf{p}}_5$, has been proposed in [10], namely

$$\hat{\mathbf{p}}_5 = -i\gamma_5\nabla$$

[3] We use the natural units, $c = \hbar = 1$.

in the Schroedinger picture. It is Hermitean, and its spectrum is continuous. Moreover, it can be regarded as the generator of spatial translations. On the other hand, it hasrather peculiar features. First, it does not commute with Hamiltonian in the case of the massive Majorana particle ($m > 0$).[4] In consequence, its direction is not constant in time in the Heisenberg picture—the axial momentum contains a rotating component of the magnitude m/E_{p}, where $E_{\mathrm{p}} = \sqrt{m^2 + \mathbf{p}_5^2}$ is the energy of the particle [10]. This component is negligibly small at high energies, but it cannot be neglected at the energies comparable with m. The eigenfunctions of the axial momentum can be used as a basis for Fourier-type expansion of time-dependent wave functions of the Majorana particle [10]. It turns out that in place of simple time-dependent $U(1)$ phase factors known from the case of Dirac particles there are certain cumbersome time-dependent $SO(4)$ matrices.

In the present paper we continue the investigations initiated in [10]. We begin by providing a new motivation for the axial momentum operator. It is based on a mapping between the Majorana and Weyl bispinors. Next, we show that the classic position vs momentum uncertainty relation remains unchanged when the standard momentum operator is replaced by the axial one. After these introductory remarks, we examine the general solution of the wave equation in the form of expansion in the basis of eigenfunctions of the axial momentum $\psi_{\mathbf{p}}(\mathbf{x})$. We notice that the solution can be regarded as position and time-dependent quaternion. Next, we transform the solution to a more convenient form without the cumbersome $SO(4)$ matrices, namely we rewrite it as a superposition of traveling plane waves, see formula (13). That this is at all possible is a surprise because the direction of the axial momentum is not constant in time if $m > 0$. Interestingly, it turns out that in the massive case the plane waves necessarily come in pairs. The paired plane waves have the opposite wave vectors $\mathbf{p}$ and $-\mathbf{p}$, hence they travel in opposite directions. Their amplitudes are not equal, the ratio is $1 : m/E_q$.

Finally, we elaborate on the relativistic invariance of the model using the amplitudes defined in the basis of eigenfunctions of the axial momentum. It turns out that the realization of the relativistic invariance in the space of these amplitudes is similar to the picture known from the case of the Dirac particle, which is a rather encouraging result. On one hand, this finding is perhaps surprising because the axial momentum is not conserved if $m \neq 0$. On the other hand, the γ_5 matrix is a scalar with respect to the Poincaré transformations, hence one may expect that its presence does not destroy the relativistic invariance. The representation of the Poincaré group we have obtained in the massive case is orthogonal, irreducible, and equivalent to a real version of the well-known spin 1/2 unitary irreducible representation. Recall that in the case of the Dirac particle one obtains a reducible representation composed of two spin 1/2 irreducible representations. Such representations, here discussed within the framework of relativistic quantum mechanics, usually reappear unchanged when one considers single particle sectors for the related quantum field. In the Dirac case, the two spin 1/2 representations correspond to the particle and its antiparticle. In the Majorana case we expect the particle only.

Our overall conclusion is that the axial momentum operator is a reasonable replacement for the ordinary momentum one (which should not be used in the Majorana case anyway). Certain peculiar features present in the case of the massive Majorana particle, like the mixing of modes with opposite axial momenta (discussed in section 3), are negligible at energies $E_{\mathrm{p}} \gg m$. At lower energies however, they cannot be neglected. We regard them as intrinsic physical features of the relativistic massive Majorana particle.

[4] This fact is explained by a relation of the γ_5 matrix with the well-known axial $U(1)$ charge, which is not conserved in the massive case.

The paper is organized as follows. In the next section we introduce the axial momentum operator using the mapping between the Weyl and Majorana bispinors, and we derive the uncertainty relation. In section 3, after a brief recap of necessary results from [10], we point out the connection with quaternions, and we discuss the traveling plane waves. Section 4 is devoted to analysis of the representation of the Poincaré group that exists in the space of solutions of the evolution equation.

2. The axial momentum: new motivation, and the Heisenberg uncertainty relation

Throughout this paper we work with the Dirac matrices γ^μ in a Majorana-type representation, i.e. the matrices are purely imaginary. Then also the matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is purely imaginary. Furthermore, γ_5 is Hermitian, hence also anti-symmetric: $\gamma_5^{\mathrm{T}} = -\gamma_5$, and $\gamma_5^2 = I$, where I is the 4 by 4 unit matrix. We work with the following set of the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad \gamma^1 = i\begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \qquad \gamma^2 = i\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix},$$

$$\gamma^3 = -i\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad \text{and} \quad \gamma_5 = i\begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}.$$

Here σ_k are the Pauli matrices, and σ_0 is the 2 by 2 unit matrix.

In all Majorana-type representations charge conjugation C is represented just by the complex conjugation. Therefore, the Majorana bispinors, which by definition are invariant under C, have only real components in such representations. The operator $\hat{\mathbf{p}}_5$ commutes with C, in contradistinction to $\hat{\mathbf{p}}$.

The motivation for the axial momentum given in [10] refers to a Lagrangian in classical field theory and to the Noether theorem. Moreover, it applies to the massless case ($m = 0$) only. Below we give an independent, simple and more general motivation.

There is a simple one-to-one mapping M between linear spaces of the Majorana bispinors and right-handed (or left-handed) Weyl bispinors. From arbitrary right-handed Weyl bispinor ϕ, which by definition has the property $\gamma_5\phi = \phi$, we form $\psi = \phi + \phi^* \equiv M(\phi)$, which is real, hence Majorana, bispinor. The asterisk denotes the complex conjugation. Now, because the matrix γ_5 is purely imaginary, ϕ^* is a left-handed bispinor, $\gamma_5\phi^* = -\phi^*$. It follows that $\gamma_5\psi = \phi - \phi^*$, and $\phi = (I + \gamma_5)\psi/2 \equiv \psi_R$, $\phi^* = (I - \gamma_5)\psi/2 \equiv \psi_L$. This shows that the mapping M is invertible. Notice that it preserves linear combinations only if their coefficients are real. The Weyl bispinors are complex, hence the standard momentum operator $\hat{\mathbf{p}} = -i\nabla$ is well-defined for them. In particular, it commutes with the γ_5 matrix, therefore also $\hat{\mathbf{p}}\phi$ is right-handed Weyl bispinor. Let us find the Majorana bispinor that corresponds to $\hat{\mathbf{p}}\phi$:

$$M(\hat{\mathbf{p}}\phi) = \hat{\mathbf{p}}\phi + (\hat{\mathbf{p}}\phi)^* = \hat{\mathbf{p}}(\psi_R - \psi_L) = -i\gamma_5\nabla\psi = \hat{\mathbf{p}}_5\psi.$$

We see that the standard momentum operator in the space of right-handed Weyl bispinors gives rise to the axial momentum operator in the space of Majorana bispinors.

The axial momentum commutes with γ_5, therefore it can be used also in the space of right-handed Weyl bispinors. However, in this space it coincides with $\hat{\mathbf{p}}$ because $\gamma_5\phi = \phi$.

The presence of the one-to-one mapping M might suggest that the two quantum mechanics, Majorana and Weyl, are equivalent to each other. For the equivalence, the mapping M should preserve scalar product. It turns out that it is not the case. Let us take $\psi_1 = M(\phi_1)$, $\psi_2 = M(\phi_2)$, and compare the scalar product of the Majorana bispinors $\int d^3x\ \psi_1^T\psi_2$ with the scalar product $\int d^3x\ \phi_1^\dagger\phi_2$ of the corresponding Weyl bispinors. We have

$$\int d^3x\ \psi_1^T\psi_2 = \int d^3x\,(\phi_1^T + \phi_1^\dagger)(\phi_2 + \phi_2^*) = \int d^3x\,(\phi_1^\dagger\phi_2 + (\phi_1^\dagger\phi_2)^*).$$

Here we have used the identity $\phi_1^T\phi_2 \equiv 0$, which follows from the antisymmetry of γ_5: $\phi_1^T\phi_2 = \phi_1^T\gamma_5\phi_2 = -\phi_1^T\gamma_5^T\phi_2 = -(\gamma_5\phi_1)^T\phi_2 = -\phi_1^T\phi_2$. Thus we see that in general the scalar product is not preserved by M,

$$\int d^3x\ \psi_1^T\psi_2 \neq \int d^3x\ \phi_1^\dagger\phi_2.$$

Note also the differences in evolution equations. In the Weyl case, the evolution equation has the form (1) with $m = 0$, namely $i\gamma^\mu\partial_\mu\phi = 0$, while in the Majorana case $m \neq 0$ is allowed. Using the mapping inverse to M one can of course transform equation (1) for the Majorana bispinor ψ to the space of right-handed Weyl bispinors—we obtain $i\gamma^\mu\partial_\mu\phi - m\phi^* = 0$, which is known as the Dirac equation for ϕ with the Majorana mass term (recall that ϕ^* is the charge conjugation of ϕ). This last equation cannot be accepted as a quantum mechanical evolution equation for the Weyl bispinor ϕ because it is not linear over $\mathbb{C}$—it is linear only over $\mathbb{R}$. The point is that the Hilbert space of the right-handed Weyl bispinors[5] is linear over $\mathbb{C}$, therefore also quantum mechanical evolution equation for these bispinors should be linear over $\mathbb{C}$, otherwise the superposition principle is broken.

Commutator of the axial momentum with position operator $\hat{x}^j = x^jI$, where I is the four by four unit matrix, has the form

$$[\hat{x}^j,\ \hat{p}_5^k] = i\delta_{jk}\gamma_5, \tag{2}$$

which differs by γ_5 from the commutator $[\hat{x}^j,\ \hat{p}^k]$. In spite of the difference, the implied uncertainty relation has the usual form

$$\langle\psi|(\Delta\hat{x}^j)^2|\psi\rangle\langle\psi|(\Delta\hat{p}_5^k)^2|\psi\rangle \geqslant \frac{1}{4}\delta_{jk},$$

where $\Delta\hat{x}^j = \hat{x}^j - \langle\psi|\hat{x}^j|\psi\rangle$, $\Delta\hat{p}_5^k = \hat{p}_5^k - \langle\psi|\hat{p}_5^k|\psi\rangle$. The uncertainty relation is obtained in the standard manner. Let us consider

$$I(\alpha) = \langle\psi|(\alpha\Delta\hat{p}_5^k + i\gamma_5\Delta\hat{x}^j)(\alpha\Delta\hat{p}_5^k - i\gamma_5\Delta\hat{x}^j)|\psi\rangle,$$

where α is a real variable. It is clear that $I(\alpha) \geqslant 0$. On the other hand, using commutator (2) we have

$$I(\alpha) = \alpha^2\langle\psi|(\Delta\hat{p}_5^k)^2|\psi\rangle - \alpha\delta_{jk} + \langle\psi|(\Delta\hat{x}^j)^2|\psi\rangle.$$

We know that this quadratic polynomial in α does not have two distinct real roots. The uncertainty relation follows as the necessary and sufficient condition for this.

In the massive case ($m > 0$) the axial momentum has nontrivial time evolution in the Heisenberg picture, because it does not commute with the Hamiltonian $\hat{h}$ shown below. This aspect is discussed in detail in [10].

[5] By definition, it includes all bispinors which obey the condition $\gamma_5\phi = \phi$. Arbitrary complex linear combination of such bispinors also is right-handed. The complex conjugate bispinor ϕ^* is left-handed.

3. Time evolution of the axial momentum amplitudes and quaternions

The general solution of equation (1) in the basis of eigenfunctions of the axial momentum was found in [10]. It is complete from a theoretical viewpoint, but rather clumsy if one thinks about concrete applications. Below we transform that solution to a much simpler form. Furthermore, we point out that the general solution can be described in terms of quaternions. Such a link of the Majorana quantum mechanics with the algebra of quaternions is yet another intriguing feature of it, in addition to the non Hermitian Hamiltonian $\hat{h}$ and non conservation of the axial momentum in the case of free massive Majorana particle.

Let us begin by recalling necessary facts from the paper [10]. The Dirac equation (1) is rewritten as

$$\partial_t \psi = \hat{h}\psi, \tag{3}$$

where the Hamiltonian

$$\hat{h} = -\gamma^0\gamma^k\partial_k - im\gamma^0$$

is real and anti-symmetric, but it is not Hermitian if $m \neq 0$. Nevertheless, the scalar product $\langle\psi_1(t)|\psi_2(t)\rangle = \int \mathrm{d}^3x\ \psi_1^{\mathrm{T}}(\mathbf{x},t)\,\psi_2(\mathbf{x},t)$ is constant in time because $\hat{h}$ is anti-symmetric. The time evolution operator is orthogonal one.

The normalized eigenfunctions of the axial momentum have the form

$$\psi_{\mathbf{p}}(\mathbf{x}) = (2\pi)^{-3/2}\ \exp(\mathrm{i}\gamma_5\mathbf{p}\mathbf{x})\,v, \tag{4}$$

They obey the conditions

$$\hat{\mathbf{p}}_5\psi_{\mathbf{p}}(\mathbf{x}) = \mathbf{p}\ \psi_{\mathbf{p}}(\mathbf{x}), \qquad \int \mathrm{d}^3x\ \psi_{\mathbf{p}}^{\mathrm{T}}(\mathbf{x})\,\psi_{\mathbf{q}}(\mathbf{x}) = \delta(\mathbf{p}-\mathbf{q}).$$

where v is an arbitrary constant, normalized ($v^{\mathrm{T}}v = 1$) and real bispinor. We call the functions $\psi_{\mathbf{p}}(\mathbf{x})$ the axial plane waves[6]. Note that

$$\exp(\mathrm{i}\gamma_5\mathbf{p}\mathbf{x}) = \cos(\mathbf{p}\mathbf{x})I + i\gamma_5\ \sin(\mathbf{p}\mathbf{x}).$$

The expansion of $\psi(\mathbf{x},t)$ into the axial plane waves has the form

$$\psi(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\sum_{\alpha=1}^{2}\int \mathrm{d}^3p\,\mathrm{e}^{\mathrm{i}\gamma_5\mathbf{p}\mathbf{x}}\left(v_\alpha^{(+)}(\mathbf{p})c_\alpha(\mathbf{p},t) + v_\alpha^{(-)}(\mathbf{p})d_\alpha(\mathbf{p},t)\right), \tag{5}$$

where the basis bispinors $v_\alpha^{(\pm)}$ obey the conditions

$$\gamma^0\gamma^k p^k\ v_\alpha^{(\pm)} = \pm|\mathbf{p}|\ v_\alpha^{(\pm)}.$$

The eigenvalues $\pm|\mathbf{p}|$ correspond to helicities $\pm 1/2$, respectively, [10]. They are double degenerate ($\alpha = 1, 2$). Thus each single mode in (5) is common normalized eigenstate of $\hat{\mathbf{p}}_5$ and of the helicity. The index $\alpha = 1, 2$ reflects the degeneracy of the common eigenstates which is an artifact of the reality of our Hilbert space.

[6] Note that the time dependence is not included—it is represented by a time-dependent $SO(4)$ matrix, see formula (7) below.

The basis bispinors have the following form

$$v_1^{(+)}(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| - p^2)}} \begin{pmatrix} -p^3 \\ p^2 - |\mathbf{p}| \\ p^1 \\ 0 \end{pmatrix}, \qquad v_2^{(+)}(\mathbf{p}) = i\gamma_5\, v_1^{(+)}(\mathbf{p}),$$

$$v_1^{(-)}(\mathbf{p}) = i\gamma^0\, v_1^{(+)}(\mathbf{p}), \qquad v_2^{(-)}(\mathbf{p}) = i\gamma_5\, v_1^{(-)}(\mathbf{p}) = -\gamma_5\gamma^0 v_1^{(+)}(\mathbf{p}). \tag{6}$$

They are orthonormal,

$$(v_\alpha^{(\epsilon)})^{\mathrm{T}}(\mathbf{p})\, v_\beta^{(\epsilon')}(\mathbf{p}) = \delta_{\epsilon\epsilon'}\delta_{\alpha\beta},$$

where $\epsilon, \epsilon' = +, -$ refer to the helicity, and $\alpha, \beta = 1, 2$. The basis (6) has quite remarkable properties: it is real; generated from $v_1^{(+)}$ by the quaternions which are introduced below; and it does not depend on the mass m—it is scale invariant.

Time dependence of the real amplitudes $c_\alpha(\mathbf{p}, t), d_\alpha(\mathbf{p}, t)$ in expansion (5) is found by solving equation (3). To this end, the amplitudes are split into the even and odd parts,

$$c_\alpha(\mathbf{p}, t) = c_\alpha^{'}(\mathbf{p}, t) + c_\alpha^{''}(\mathbf{p}, t), \qquad d_\alpha(\mathbf{p}, t) = d_\alpha^{'}(\mathbf{p}, t) + d_\alpha^{''}(\mathbf{p}, t),$$

where $c_\alpha^{'}(-\mathbf{p}, t) = c_\alpha^{'}(\mathbf{p}, t),\ c_\alpha^{''}(-\mathbf{p}, t) = -c_\alpha^{''}(\mathbf{p}, t)$, and analogously for d', d''. Furthermore, we introduce the notation

$$\vec{c}(\mathbf{p}, t) = \begin{pmatrix} c_1' \\ c_1'' \\ c_2' \\ c_2'' \end{pmatrix}, \qquad \vec{d}(\mathbf{p}, t) = \begin{pmatrix} d_1' \\ d_1'' \\ d_2' \\ d_2'' \end{pmatrix}, \qquad K_\pm(\mathbf{p}) = \begin{pmatrix} 0 & -n^1 & \pm n^2 & \pm n^3 \\ n^1 & 0 & \mp n^3 & \pm n^2 \\ \mp n^2 & \pm n^3 & 0 & n^1 \\ \mp n^3 & \mp n^2 & -n^1 & 0 \end{pmatrix},$$

where

$$n^1 = \frac{m\, p^1}{E_{\mathrm{p}}\,\sqrt{(p^1)^2 + (p^3)^2}}, \qquad n^2 = \frac{|\mathbf{p}|}{E_{\mathrm{p}}}, \qquad n^3 = \frac{m\, p^3}{E_{\mathrm{p}}\,\sqrt{(p^1)^2 + (p^3)^2}},$$

and $E_{\mathrm{p}} = \sqrt{m^2 + \mathbf{p}^2}$. The time dependence of the amplitudes is given by following formula [10]

$$\vec{c}(\mathbf{p}, t) = \exp(tE_{\mathrm{p}}\, K_+(\mathbf{p}))\, \vec{c}(\mathbf{p}, 0), \qquad \vec{d}(\mathbf{p}, t) = \exp(tE_{\mathrm{p}}\, K_-(\mathbf{p}))\, \vec{d}(\mathbf{p}, 0). \tag{7}$$

The matrices $K_\pm(\mathbf{p})$ are anti-symmetric, hence the matrices $\exp(tE_{\mathrm{p}}K_\pm(\mathbf{p}))$ belong to the $SO(4)$ group. Because $K_\pm^2 = -I$, we have the formula

$$\exp(tE_{\mathrm{p}}\, K_\pm(\mathbf{p})) = \cos(tE_{\mathrm{p}})I + \sin(tE_{\mathrm{p}})K_\pm(\mathbf{p}). \tag{8}$$

Here we end the recapitulation of the relevant for this work facts from [10].

Using expansion (5) we immediately obtain the Plancherel formula

$$\langle\psi_1|\psi_2\rangle = \int \mathrm{d}^3 p \sum_{\alpha=1}^{2} \left(c_\alpha^{(1)}(\vec{p}, t)c_\alpha^{(2)}(\vec{p}, t) + d_\alpha^{(1)}(\vec{p}, t)d_\alpha^{(2)}(\vec{p}, t)\right),$$

where the amplitudes $c_\alpha^{(1)}$, $d_\alpha^{(1)}$ correspond to ψ_1, and $c_\alpha^{(2)}$, $d_\alpha^{(2)}$ to ψ_2. Let us remind that the integration variable **p** is the eigenvalue of the axial, not the ordinary, momentum. Splitting the amplitudes into the even and odd parts, we may write

$$\langle\psi_1|\psi_2\rangle = \int \mathrm{d}^3p \left((\vec{c}^{\,(1)})^{\mathrm{T}}\vec{c}^{\,(2)} + (\vec{d}^{\,(1)})^T\vec{d}^{\,(2)} \right). \tag{9}$$

Because the time evolution is given by the orthogonal matrices, as shown in formula (7), we again see that the scalar product is constant in time.

In the general solution of equation (3) quoted above the amplitudes c_α, d_α are split into the even and odd components which mix during the time evolution, see formula (7). It turns out that the solution can be rewritten in a more transparent form. To this end, we use formula (7) and (8), as well as the concrete form (6) of the basis bispinors. After straightforward and somewhat lengthy calculation the general solution is transformed to the following form

$$\begin{aligned}\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}}\sum_{\alpha=1}^{2}\int \mathrm{d}^3p\ \Big[&\big(\cos(E_{\mathrm{p}}t)I - \hat{L}_+\ \sin(E_{\mathrm{p}}t)\big)\,\mathrm{e}^{\mathrm{i}\gamma_5\mathbf{px}}v_\alpha^{(+)}(\mathbf{p})c_\alpha(\mathbf{p},0)\\ &+ \big(\cos(E_{\mathrm{p}}t)I + \hat{L}_-\ \sin(E_{\mathrm{p}}t)\big)\,\mathrm{e}^{\mathrm{i}\gamma_5\mathbf{px}}v_\alpha^{(-)}(\mathbf{p})d_\alpha(\mathbf{p},0)\Big],\end{aligned} \tag{10}$$

where

$$\hat{L}_\pm = i\gamma_5\frac{|\mathbf{p}|}{E_{\mathrm{p}}} \pm i\gamma^0\frac{m}{E_{\mathrm{p}}}.$$

In formula (10) we have the initial values of the full amplitudes c_α, d_α, while in (7),the even and odd parts appear separately.

Notice that

$$\cos(E_{\mathrm{p}}t) \mp \hat{L}_\pm\ \sin(E_{\mathrm{p}}t) = \exp(\mp\hat{L}_\pm E_{\mathrm{p}}t),$$

because $\hat{L}_\pm^2 = -I$. Therefore, in the massless case, i.e. $m = 0$, formula (10) acquires a very simple form, namely

$$\begin{aligned}\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}}\sum_{\alpha=1}^{2}\int \mathrm{d}^3p\ \Big[&\mathrm{e}^{\mathrm{i}\gamma_5(\mathbf{px}-|\mathbf{p}|t)}v_\alpha^{(+)}(\mathbf{p})c_\alpha(\mathbf{p},0)\\ &+ \mathrm{e}^{\mathrm{i}\gamma_5(\mathbf{px}+|\mathbf{p}|t)}v_\alpha^{(-)}(\mathbf{p})d_\alpha(\mathbf{p},0)\Big]\end{aligned} \tag{11}$$

Thus, in this case the modes with different helicity do not mix during time evolution, in accordance with the theory of irreducible representations of the Poincaré group.

Intriguingly, the general solution (7) and its equivalent form (10) can be rewritten in terms of quaternions. The quaternionic units $\hat{i}, \hat{j}, \hat{k}$ are introduced as follows:

$$\hat{i} = i\gamma_5, \qquad \hat{j} = i\gamma^0, \qquad \hat{k} = -\gamma_5\gamma^0 = i\gamma^1\gamma^2\gamma^3.$$

They obey the usual conditions

$$\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = -I, \qquad \hat{i}\hat{j} = \hat{k}, \qquad \hat{k}\hat{i} = \hat{j}, \qquad \hat{j}\hat{k} = \hat{i}.$$

The bispinor basis $v_\alpha^{(\pm)}(\mathbf{p})$ is generated from $v_1^{(+)}(\mathbf{p})$ by acting with $\hat{i}, \hat{j}, \hat{k}$, see formula (6). Moreover, all matrices present in formulas (7) and (10) can be expressed by $I, \hat{i}, \hat{j}, \hat{k}$. In particular, $K_{\pm}(\mathbf{p}) = \mp n^2\hat{i} \pm n^3\hat{j} + n^1\hat{k}$. Therefore, the time evolution of the amplitudes $\vec{c}$, $\vec{d}$ at each fixed value of the axial momentum **p** is given by a time dependent quaternion.

Solution (10) can be written in a Fourier form, in which no matrices are present, only trigonometric functions and the basis bispinors (6). This is possible because the quaternions acting on the basis bispinors do not yield any new bispinors, but only permute them. This form of solution (10) reads

$$\psi(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 p \, \left[\cos(\mathbf{px})\cos(E_\mathrm{p}t)V_\mathrm{cc}(\mathbf{p}) + \cos(\mathbf{px})\sin(E_\mathrm{p}t)V_\mathrm{cs}(\mathbf{p}) \right.$$
$$\left. + \sin(\mathbf{px})\cos(E_\mathrm{p}t)V_\mathrm{sc}(\mathbf{p}) + \sin(\mathbf{px})\sin(E_\mathrm{p}t)V_\mathrm{ss}(\mathbf{p})\right], \tag{12}$$

where the V_cc stand for linear combinations of the basis bispinors, namely

$$V_\mathrm{cc}(\mathbf{p}) = c_1(\mathbf{p},0)v_1^{(+)}(\mathbf{p}) + c_2(\mathbf{p},0)v_2^{(+)}(\mathbf{p}) + d_1(\mathbf{p},0)v_1^{(-)}(\mathbf{p}) + d_2(\mathbf{p},0)v_2^{(-)}(\mathbf{p}),$$

$$V_\mathrm{cs}(\mathbf{p}) = \frac{1}{E_\mathrm{p}}\left[\left(m\, d_1(\mathbf{p},0) + |\mathbf{p}|c_2(\mathbf{p},0)\right) v_1^{(+)}(\mathbf{p}) \right.$$
$$- \left(m\, d_2(\mathbf{p},0) + |\mathbf{p}|c_1(\mathbf{p},0)\right) v_2^{(+)}(\mathbf{p}) - \left(m\, c_1(\mathbf{p},0) + |\mathbf{p}|d_2(\mathbf{p},0)\right) v_1^{(-)}(\mathbf{p})$$
$$\left. + \left(m\, c_2(\mathbf{p},0) + |\mathbf{p}|d_1(\mathbf{p},0)\right) v_2^{(-)}(\mathbf{p})\right],$$

$$V_\mathrm{sc}(\mathbf{p}) = -c_2(\mathbf{p},0)v_1^{(+)}(\mathbf{p}) + c_1(\mathbf{p},0)v_2^{(+)}(\mathbf{p}) - d_2(\mathbf{p},0)v_1^{(-)}(\mathbf{p}) + d_1(\mathbf{p},0)v_2^{(-)}(\mathbf{p}),$$

$$V_\mathrm{ss}(\mathbf{p}) = \frac{1}{E_\mathrm{p}}\left[-\left(m\, d_2(\mathbf{p},0) - |\mathbf{p}|c_1(\mathbf{p},0)\right) v_1^{(+)}(\mathbf{p}) \right.$$
$$- \left(m\, d_1(\mathbf{p},0) - |\mathbf{p}|c_2(\mathbf{p},0)\right) v_2^{(+)}(\mathbf{p}) + \left(m\, c_2(\mathbf{p},0) - |\mathbf{p}|d_1(\mathbf{p},0)\right) v_1^{(-)}(\mathbf{p})$$
$$\left. + \left(m\, c_1(\mathbf{p},0) - |\mathbf{p}|d_2(\mathbf{p},0)\right) v_2^{(-)}(\mathbf{p})\right].$$

Formula (12) is a convenient starting point for analysis of concrete examples of solutions.

Solution (12) is a superposition of standing plane waves. In order to rewrite it in terms of traveling plane waves we use trigonometric formulas such as $\cos(\mathbf{px})\cos(E_\mathrm{p}t) = \frac{1}{2}(\cos(\mathbf{px} - E_\mathrm{p}t) + \cos(\mathbf{px} + E_\mathrm{p}t))$, etc. We obtain

$$\psi(\mathbf{x},t) = \frac{1}{2(2\pi)^{3/2}} \int \mathrm{d}^3 p \, \left[\cos(\mathbf{px} - E_\mathrm{p}t)\, A_+(\mathbf{p}) + \cos(\mathbf{px} + E_\mathrm{p}t)\, A_-(\mathbf{p}) \right. \tag{13}$$
$$\left. + \sin(\mathbf{px} - E_\mathrm{p}t)\, B_+(\mathbf{p}) + \sin(\mathbf{px} + E_\mathrm{p}t)\, B_-(\mathbf{p})\right],$$

where

$$A_\pm(\mathbf{p}) = v_1^{(+)}(\mathbf{p})A_\pm^1(\mathbf{p}) + v_2^{(+)}(\mathbf{p})A_\pm^2(\mathbf{p}) + v_1^{(-)}(\mathbf{p})A_\pm^3(\mathbf{p}) + v_2^{(-)}(\mathbf{p})A_\pm^4(\mathbf{p}),$$

$$B_\pm(\mathbf{p}) = v_1^{(+)}(\mathbf{p})B_\pm^1(\mathbf{p}) + v_2^{(+)}(\mathbf{p})B_\pm^2(\mathbf{p}) + v_1^{(-)}(\mathbf{p})B_\pm^3(\mathbf{p}) + v_2^{(-)}(\mathbf{p})B_\pm^4(\mathbf{p}),$$

and

$$A_\pm^1 = (1 \pm \frac{p}{E_\mathrm{p}})c_1 \mp \frac{m}{E_\mathrm{p}}d_2, \qquad A_\pm^2 = (1 \pm \frac{p}{E_\mathrm{p}})c_2 \mp \frac{m}{E_\mathrm{p}}d_1,$$
$$A_\pm^3 = \left(1 \mp \frac{p}{E_\mathrm{p}}\right)d_1 \pm \frac{m}{E_\mathrm{p}}c_2, \qquad A_\pm^4 = \left(1 \mp \frac{p}{E_\mathrm{p}}\right)d_2 \pm \frac{m}{E_\mathrm{p}}c_1,$$

$$B^1_\pm = -(1 \pm \frac{p}{E_\mathrm{p}})c_2 \mp \frac{m}{E_\mathrm{p}}d_1, \qquad B^2_\pm = (1 \pm \frac{p}{E_\mathrm{p}})c_1 \pm \frac{m}{E_\mathrm{p}}d_2,$$

$$B^3_\pm = -\left(1 \mp \frac{p}{E_\mathrm{p}}\right)d_2 \pm \frac{m}{E_\mathrm{p}}c_1, \qquad B^4_\pm = \left(1 \mp \frac{p}{E_\mathrm{p}}\right)d_1 \mp \frac{m}{E_\mathrm{p}}c_2.$$

In these formulas $p \equiv |\mathbf{p}|$, $E_\mathrm{p} = \sqrt{\mathbf{p}^2 + m^2}$, and the amplitudes c_1, c_2, d_1, d_2 are the ones present in formula (10) (the arguments $(\mathbf{p}, 0)$ have been omitted for brevity). Let us remind again that $\mathbf{p}$ is the eigenvalue of the axial momentum.

Let us consider now a single mode with fixed value $\mathbf{q}$ of the axial momentum, i.e. we put in the formulas above $c_\alpha(\mathbf{p}, 0) = c_\alpha\delta(\mathbf{p} - \mathbf{q}), d_\alpha(\mathbf{p}, 0) = d_\alpha\delta(\mathbf{p} - \mathbf{q})$, where c_α, d_α, $\alpha = 1, 2$, are constants now. In the massless case,

$$A^1_+ = 2c_1, \quad A^2_+ = 2c_2, \quad A^3_+ = A^4_+ = A^1_- = A^2_- = 0, \quad A^3_- = 2d_1, \quad A^4_- = 2d_2,$$

$$B^1_+ = -2c_2, \quad B^2_+ = 2c_1, \quad B^3_+ = B^4_+ = B^1_- = B^2_- = 0, \quad B^3_- = -2d_2, \quad B^4_- = 2d_1.$$

We see that in this case the A_+, B_+ part on the rhs of formula (13) is independent of the A_-, B_- part. In particular, we can put one of them to zero in order to obtain a plane wave propagating in the direction of $\mathbf{q}$ or $-\mathbf{q}$.

The massive case is different—the plane wave always has the two components propagating in the opposite directions, $\mathbf{q}$ and $-\mathbf{q}$. If we assume that $A_- = 0$, simple calculation shows that also $B_- = A_+ = B_+ = 0$; if we put $B_- = 0$ then also $A_- = A_+ = B_+ = 0$.

Let us put the constants $d_1 = d_2 = 0$. In the massless case this assumption gives the plane wave moving in the direction $\mathbf{q}$,

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \cos(\mathbf{qx} - E_q t)\,(c_1 v_1^{(+)}(\mathbf{q}) + c_2 v_2^{(+)}(\mathbf{q}))$$
$$+ \frac{1}{(2\pi)^{3/2}} \sin(\mathbf{qx} - E_q t)\,(-c_2 v_1^{(+)}(\mathbf{q}) + c_1 v_2^{(+)}(\mathbf{q})).$$

In the massive case all four components in (13) do not vanish. However, the amplitudes of the $-\mathbf{q}$ components, i.e. A_-, B_-, are negligibly small in the high frequency limit, i.e. when $m/E_q \ll 1$. In this limit

$$A^1_+ \approx 2c_1, \qquad A^2_+ \approx 2c_2, \qquad A^3_+ = \frac{m}{E_q}c_2, \qquad A^4_+ = \frac{m}{E_q}c_1,$$
$$B^1_+ \approx -2c_2, \qquad B^2_+ \approx 2c_1, \qquad B^3_+ = \frac{m}{E_q}c_1, \qquad B^4_+ = -\frac{m}{E_q}c_2,$$

and

$$A^1_- \approx \frac{m^2}{2E_q^2}c_1, \qquad A^2_- \approx \frac{m^2}{2E_q^2}c_2, \qquad A^3_- = -\frac{m}{E_q}c_2, \qquad A^4_- = -\frac{m}{E_q}c_1,$$
$$B^1_- \approx -\frac{m^2}{2E_q^2}c_2, \qquad B^2_- \approx \frac{m^2}{2E_q^2}c_1, \qquad B^3_- = -\frac{m}{E_q}c_1, \qquad B^4_- = \frac{m}{E_q}c_2.$$

On the other hand, in the limit of long waves, i.e. $q \ll m$,

$$A^1_\pm \approx c_1, \qquad A^2_\pm \approx c_2, \qquad A^3_\pm \approx \pm c_2, \qquad A^4_\pm \approx \pm c_1,$$

and

$$B^1_\pm \approx -c_2, \qquad B^2_\pm \approx c_1, \qquad B^3_\pm \approx \pm c_1, \qquad B^4_\pm \approx \mp c_2.$$

In this case the $\mathbf{q}$ and $-\mathbf{q}$ components have approximately equal magnitudes.

4. Relation with irreducible representations of the Poincaré group

The Poincaré transformations of the real bispinor $\psi(x)$ have the standard form,

$$\psi'(x) = S(L)\psi(L^{-1}(x-a)), \tag{14}$$

with $S(L) = \exp(\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]/8)$, where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ parameterize the proper orthochronous Lorentz group, $L = \exp(\omega^\mu{}_\nu)$, in a vicinity of the unit element. Below we show that in the massive case these transformations imply transformations of the axial momentum dependent amplitudes which coincide with the real form of a single standard unitary Wigner's representation with spin 1/2. This representation being real and unitary is in fact orthogonal one. The conclusion is that as far as the relativistic transformations is the issue, the expansion into the axial plane waves of the Majorana bispinor has the properties expected for a single massive spin 1/2 particle.

We do not discuss here representations of the Poincaré group pertaining to the massless Majorana particle ($m = 0$). The massless case is simpler because now the operator $\hat{\mathbf{p}}_5$ commutes with the Hamiltonian. We find two independent irreducible representations with the helicities $\pm 1/2$. The appearance of a gauge structure with implied gauge equivalence classes of real bispinors is noteworthy. Detailed analysis of the massless case is presented in lecture notes [14].

We start from the following expansion into the eigenstates of the axial momentum

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 p}{E_\mathrm{p}}\, \mathrm{e}^{\mathrm{i}\gamma_5 \mathbf{p}\mathbf{x}}\, v(\mathbf{p}, t), \tag{15}$$

where $v(\mathbf{p}, t)$ is a real bispinor, and $E_\mathrm{p} = \sqrt{m^2 + \mathbf{p}^2}$. Equation (3) gives time evolution equation for v

$$\dot{v}(\mathbf{p}, t) = -i\gamma^0\gamma^k\gamma_5 p^k v(\mathbf{p}, t) - im\gamma^0 v(-\mathbf{p}, t). \tag{16}$$

The reason for $v(-\mathbf{p}, t)$ in the last term on the rhs is that γ^0 anti-commutes with γ_5 and therefore $\gamma^0 \exp(\mathrm{i}\gamma_5 \mathbf{p}\mathbf{x}) = \exp(-\mathrm{i}\gamma_5 \mathbf{p}\mathbf{x})\gamma^0$. Taking time derivative of equation (16) we obtain the following equation

$$\ddot{v}(\mathbf{p}, t) = -E_\mathrm{p}^2 v(\mathbf{p}, t).$$

Let us write its general solution in the form

$$v(\mathbf{p}, t) = \exp(-\mathrm{i}\gamma_5 E_\mathrm{p} t) v_+(\mathbf{p}) + \exp(\mathrm{i}\gamma_5 E_\mathrm{p} t) v_-(-\mathbf{p}), \tag{17}$$

where the argument of v_- is $-\mathbf{p}$ for later convenience. Then $\psi(\mathbf{x}, t)$ can be written as

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 p}{E_\mathrm{p}} \left(\mathrm{e}^{\mathrm{i}\gamma_5(\mathbf{p}\mathbf{x} - E_\mathrm{p} t)}\, v_+(\mathbf{p}) + \mathrm{e}^{-\mathrm{i}\gamma_5(\mathbf{p}\mathbf{x} - E_\mathrm{p} t)}\, v_-(\mathbf{p})\right), \tag{18}$$

where in the last term we have changed the integration variable to $-\mathbf{p}$. Furthermore, equation (16) is satisfied by $v(\mathbf{p}, t)$ of the form (17) only if $v_{\pm}(\mathbf{p})$ obey the following conditions

$$E_{\mathrm{p}}\gamma_5 v_{\pm}(\mathbf{p}) = \gamma^0\gamma^k p^k \gamma_5 v_{\pm}(\mathbf{p}) \pm m\gamma^0 v_{\mp}(\mathbf{p}). \tag{19}$$

Applying the transformation law (14) with $a = 0$ to solution (18), we obtain Lorentz transformation of the bispinors $v_{\pm}(\mathbf{p})$,

$$v'_{\pm}(p) = S(L)\, v_{\pm}(L^{-1}p), \tag{20}$$

where now we use the four-vector p instead of $\mathbf{p}$ for convenience in notation: $v_+(p) \equiv v_+(\mathbf{p})$ and $p^0 = E_{\mathrm{p}}$. The spacetime translations $x' = x + a$ are represented by $SO(4)$ factor

$$v'_{\pm}(\mathbf{p}) = \mathrm{e}^{\pm \mathrm{i}\gamma_5 pa}\, v_{\pm}(\mathbf{p}). \tag{21}$$

In the massive case, $v_-(\mathbf{p})$ can be expressed by $v_+(\mathbf{p})$, see (19). The scalar product $\langle\psi_1|\psi_2\rangle = \int \mathrm{d}^3x\ \psi_1^{\mathrm{T}}(\mathbf{x}, t)\psi_2(\mathbf{x}, t)$ acquires explicitly Poincaré invariant (and time independent) form

$$\langle\psi_1|\psi_2\rangle = \frac{2}{m^2}\int \frac{\mathrm{d}^3p}{E_{\mathrm{p}}}\, \overline{v_{1+}(\mathbf{p})}\, (\gamma^0 E_{\mathrm{p}} - \gamma^k p^k)\, v_{2+}(\mathbf{p}), \tag{22}$$

where $\overline{v_{1+}(\mathbf{p})} = v_{1+}^{\mathrm{T}}(\mathbf{p})\gamma^0$, and $v_{1+}(v_{2+})$ corresponds to $\psi_1(\psi_2)$ by formula (18).

Transformations (20) and (21) are unitary with respect to this scalar product. Thus, we have here real unitary, i.e. orthogonal, representation of the Poincaré group. It turns out that it is irreducible and equivalent to a real version of the standard spin 1/2 unitary representation. Detailed analysis of the representation is given below. Let us recall that in the case of massive Dirac particle one finds a reducible representation which is a direct sum of two spin 1/2 representations.

Representation (20) can be cast in the standard form which involves the Wigner rotations and a representation of $SU(2)$ group [15]. To this end, we introduce the standard momentum $\overset{(0)}{p} = (m, 0, 0, 0)$, where $m > 0$, as well as a Lorentz boost $H(p)$ such that $H(p)\overset{(0)}{p} = p$. Furthermore, at each p we introduce the basis of real bispinors,

$$v_i(p) = S(H(p))v_i(\overset{(0)}{p}), \tag{23}$$

where $i = 1, 2, 3, 4$, and $v_i(\overset{(0)}{p})$ is a basis at $\overset{(0)}{p}$ such that $mv_i^{\mathrm{T}}(\overset{(0)}{p})v_k(\overset{(0)}{p}) = \delta_{ik}$ (the factor m is included for dimensional reason). Here again we use the four momentum in the notation as in (20). The bispinor $v_+(p)$ is decomposed in this basis,

$$v_+(p) = a^i(p)v_i(p).$$

The scalar product (22) is equal to

$$\langle\psi_1|\psi_2\rangle = \frac{2}{m^2}\int \frac{\mathrm{d}^3p}{E_{\mathrm{p}}}\, a_1^k(p)a_2^k(p), \tag{24}$$

where $k = 1, 2, 3, 4$. The real dimensionless amplitudes a_1^k, a_2^k correspond to ψ_1, ψ_2, respectively.

Let us find the relativistic transformation law of the amplitudes $a^k(p)$. In the case of Lorentz transformations, using (20) we have

$$v'_+(p) = a'^k(p)v_k(p) = S(L)a^i(L^{-1}p)v_i(L^{-1}p) \quad ;$$
$$= a^i(L^{-1}p)S(H(p))S(H^{-1}(p)LH(L^{-1}p))v_i(\overset{(0)}{p}).$$

The Lorentz transformation $\mathcal{R}(L,p)) = H^{-1}(p)LH(L^{-1}p)$ leaves $\overset{(0)}{p}$ invariant—it is a rotation, known as the Wigner rotation. Therefore, we may write

$$S(\mathcal{R}(L,p))v_i(\overset{(0)}{p}) = D_{ki}(\mathcal{R}(L,p))v_k(\overset{(0)}{p}). \tag{25}$$

In consequence,

$$v'_+(p) = a^i(L^{-1}p)D_{ki}(\mathcal{R}(L,p))v_k(p),$$

and finally

$$a'^k(p) = D_{ki}(\mathcal{R}(L,p))a^i(L^{-1}p). \tag{26}$$

The invariance of the scalar product (24) implies orthogonality of the 4 by 4 real matrix with the elements $D_{ki}(\mathcal{R}(L,p))$.

The space-time translation $\psi'(x) = \psi(x-a)$ results in a change of the amplitudes[7], $a^i(p) \rightarrow \underline{a}^i(p)$. Let us compute $\underline{a}^i(p)$. Using formula (21) we have

$$v'_+(p) = \underline{a}^k(p)v_k(p) = \mathrm{e}^{\mathrm{i}\gamma_5 pa}a^k(p)v_k(p) = a^k(p)S(H(p))\mathrm{e}^{\mathrm{i}\gamma_5 pa}\, v_k(\overset{(0)}{p})$$
$$= ma^k(p)S(H(p))\left(v_l^{\mathrm{T}}(\overset{(0)}{p})\mathrm{e}^{\mathrm{i}\gamma_5 pa}\, v_k(\overset{(0)}{p})\right) v_l(\overset{(0)}{p}).$$

At this point it is convenient to choose the basis $v_k(\overset{(0)}{p})$ in the Kronecker form, in which the ith component of the bispinor $v_k(\overset{(0)}{p})$ is equal to $\delta_{ik}/\sqrt{m}$. In this basis

$$mv_l^{\mathrm{T}}(\overset{(0)}{p})\mathrm{e}^{\mathrm{i}\gamma_5 pa}v_k(\overset{(0)}{p}) = (\mathrm{e}^{\mathrm{i}\gamma_5 pa})_{lk}.$$

In consequence,

$$\underline{a}^l(p) = (\mathrm{e}^{\mathrm{i}\gamma_5 pa})_{lk}a^k(p). \tag{27}$$

The matrix $\mathrm{e}^{\mathrm{i}\gamma_5 pa}$ is orthogonal, and the scalar product (24) is of course invariant with respect to the transformations (27).

Formula (25) opens the way to identification of the pertinent orthogonal representation of the Poincaré group. This representation is uniquely characterized by representation (26) of the Wigner rotations [15]. In order to identify this last representation it suffices to take in formula (25) $p = \overset{(0)}{p}$ and $L = R$, where R is arbitrary rotation. Then $\mathcal{R}(R, \overset{(0)}{p}) = R$. Let us again use the Kronecker basis introduced above. On this basis, formula (25) can now be rewritten as $S(R) = D(R)$, where the matrix elements of $D(R)$ are equal to $D_{ki}(R)$. Therefore we now turn to the matrices $S(R)$.

[7] We use the notation $\underline{a}$ because a' is already used.

The matrices $S(R)$ have the form

$$S(R) = \exp(\frac{1}{2}(\omega_{12}\gamma^1\gamma^2 + \omega_{31}\gamma^3\gamma^1 + \omega_{23}\gamma^2\gamma^3)).$$

They form a subgroup of the SO(4) group. There exist real orthogonal matrices $\mathcal{O}$ such that

$$\mathcal{O}\gamma^1\gamma^2\mathcal{O}^{-1} = \hat{i}, \qquad \mathcal{O}\gamma^2\gamma^3\mathcal{O}^{-1} = \hat{j}, \qquad \mathcal{O}\gamma^3\gamma^1\mathcal{O}^{-1} = \hat{k},$$

where $\hat{i}, \hat{j}, \hat{k}$ are the quaternions introduced in the previous section. For example, one may take the matrix

$$\mathcal{O} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma_0 & -\sigma_1 \\ -\sigma_0 & -\sigma_1 \end{pmatrix}.$$

Thus, the matrices $\mathcal{O}S(R)\mathcal{O}^{-1}$ are elements of the algebra of quaternions, and as such they can be written in the form

$$\mathcal{O}S(R)\mathcal{O}^{-1} = s_0 I_4 + s_1\hat{i} + s_2\hat{j} + s_3\hat{k}, \tag{28}$$

where s_0, s_k are real functions of the parameters ω_{ik}. These matrices also belong to the $SO(4)$ group. Furthermore, because

$$(\mathcal{O}S(R)\mathcal{O}^{-1})^{\mathrm{T}} = s_0 I_4 - s_1\hat{i} - s_2\hat{j} - s_3\hat{k}$$

and $\mathcal{O}S(R)\mathcal{O}^{-1}(\mathcal{O}S(R)\mathcal{O}^{-1})^{\mathrm{T}} = I_4$, we obtain the relation $(s_0)^2 + (s_1)^2 + (s_2)^2 + (s_3)^2 = 1$.

On the other hand, let us consider the spin 1/2 representation $T(u)$ of $SU(2)$ group, $T(u)\xi = u\xi$, where $u \in SU(2)$ and ξ is a two-component spinor (in general complex). This representation can be rewritten in real form simply by using the real and imaginary parts. Thus we write

$$u = \begin{pmatrix} \alpha & -\beta \\ \beta^* & \alpha^* \end{pmatrix}, \qquad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

where $\alpha = \alpha' + i\alpha'', \beta = \beta' + i\beta'', \xi_1 = \xi_1' + i\xi_1'', \xi_2 = \xi_2' + i\xi_2''$, and $\alpha\alpha^* + \beta\beta^* = (\alpha')^2 + (\alpha'')^2 + (\beta')^2 + (\beta'')^2 = 1$. Next, we form the four-component real vector $\vec{\xi}$ and the 4 by 4 real matrix $\hat{T}(u)$:

$$\vec{\xi} = \begin{pmatrix} \xi_1' \\ \xi_1'' \\ \xi_2' \\ \xi_2'' \end{pmatrix}, \qquad \hat{T}(u) = \begin{pmatrix} \alpha' & -\alpha'' & -\beta' & \beta'' \\ \alpha'' & \alpha' & -\beta'' & -\beta' \\ \beta' & \beta'' & \alpha' & \alpha'' \\ -\beta'' & \beta' & -\alpha'' & \alpha' \end{pmatrix}.$$

It turns out that $\vec{\xi_u} = \hat{T}(u)\vec{\xi}$, where $\xi_u \equiv T(u)\xi$. The matrix $\hat{T}(u)$ can be rewritten in terms of the quaternions,

$$\hat{T}(u) = \alpha'\, I_4 + \beta'\,\hat{i} + \beta''\,\hat{j} + \alpha''\,\hat{k}. \tag{29}$$

The rhs of this formula coincides with the rhs of formula (28) if $\alpha' = s_0, \beta' = s_1, \beta'' = s_2$, and $\alpha'' = s_3$.

In conclusion, the representation of the Wigner rotations given by the matrices $S(\mathcal{R}(L, p))$ is equivalent to the real form of the spin 1/2 representation $T(u)$ of $SU(2)$ group. Thus, we have found that the representation of the Poincaré group, given by formulas (21) and (26), is the spin 1/2, $m > 0$ representation. Notice that we have obtained just one such spin 1/2 representation. In the case of Dirac particle a direct sum of two spin 1/2 representations appears, one for particle and the other for antiparticle.

5. Summary and remarks

1. Let us summarize our main results. We have shown that the axial momentum operator for the Majorana particle is related to the ordinary momentum for the Weyl particle by the one-to-one mapping between the two models, and that it obeys the Heisenberg uncertainty relation. Next, using the eigenfunctions of the axial momentum operator, we have written the general solution of the Dirac equation for the real bispinor in the form of superposition of traveling plane waves, with the eigenvalues $\mathbf{p}$ of the axial momentum playing the role of wave vectors, i.e. giving the wave length and the direction of propagation. In the massive case this superposition has the special feature that the plane waves come in pairs with the opposite axial momenta, $\mathbf{p}$ and $-\mathbf{p}$. This is due to the fact that in the massive case the axial momentum does not commute with the Hamiltonian $\hat{h}$. Therefore, the eigenvectors of $\hat{\mathbf{p}}_5$ are not stationary states—the minimal stationary subspace in the Hilbert space is spanned by the two modes $\mathbf{p}, -\mathbf{p}$. The presence of such paired plane waves could perhaps serve as a signature of the massive Majorana particle. This effect is relatively small at high energies, but quite sizable at energies close to the rest mass of the particle. Last but not least, we have shown how one can unveil the irreducible spin 1/2 representation of the Poincaré group working with the axial momentum basis.

Apart from the results listed above, there are quite interesting purely theoretical aspects, namely the reformulation in terms of quaternions, and fully-fledged relativistic quantum mechanics over the algebraic field of real numbers $\mathbb{R}$ in place of complex numbers.

We conclude that the axial momentum $\hat{\mathbf{p}}_5 = -i\gamma_5\nabla$ can be accepted as the replacement for the ordinary momentum $\hat{\mathbf{p}} = -i\nabla$. This latter operator is not an observable for the Majorana particle because it does not commute with the charge conjugation C, in contradistinction to $\hat{\mathbf{p}}_5$. We think that the axial momentum is the proper observable to be used in theoretical analysis of experimental data for relativistic Majorana particles, when they are available.

2. The investigations of the axial momentum can be continued in several directions. In our opinion, two are especially interesting. First, we would like to check time evolution of wave packets with certain fixed initial profile of the axial momentum. Formulas (12) and (13) seem to be a good starting point for work in this direction. We believe that such a basic knowledge about evolution of normalizable wave functions can be helpful in experimental searches for the Majorana particles.

The second very interesting and important topic is the application of the axial plane waves in quantum theory of the Majorana field. The amplitudes $a^i(p)$ introduced right above formula (24), where $p = (p^0, \mathbf{p})$ with $\mathbf{p}$ being eigenvalue of the axial momentum and $p^0 = \sqrt{m^2 + \mathbf{p}^2}$, have clear transformation law with respect to the Poincaré group. This fact suggests that precisely these amplitudes should be replaced by creation and annihilation operators of the Majorana particle when quantizing the real Majorana field.

Finally, one may use the axial momentum instead of the ordinary momentum in quantum mechanics of the Dirac particle. Here the ordinary momentum has the advantage: it commutes with the Dirac Hamiltonian in the case of free particle—but the use of the axial momentum, which is after all a legitimate observable, can lead to new insights.

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