Mathematische Annalen (2021) 380:789–823
https://doi.org/10.1007/s00208-021-02151-4
Mathematische Annalen

Poincaré duality for L^p cohomology on subanalytic singular spaces

Guillaume Valette[1]

Received: 14 September 2020 / Revised: 4 December 2020 / Accepted: 8 January 2021 /
Published online: 21 January 2021

Abstract

We investigate the problem of Poincaré duality for L^p differential forms on bounded subanalytic submanifolds of $\mathbb{R}^n$ (not necessarily compact). We show that, when p is sufficiently close to 1 then the L^p cohomology of such a submanifold is isomorphic to its singular homology. In the case where p is large, we show that L^p cohomology is dual to intersection homology. As a consequence, we can deduce that the L^p cohomology is Poincaré dual to L^q cohomology, if p and q are Hölder conjugate to each other and p is sufficiently large.

Mathematics Subject Classification 14F40 · 58A10 · 55N33 · 57P10 · 32B20

1 Introduction

The history of L^p forms on singular varieties began when Cheeger computed the L^2 cohomology groups for varieties with metrically conical singularities and started constructing a Hodge theory for singular compact varieties [5–8]. This enabled him to derive Poincaré duality results for singular varieties. These groups turned out to be related to intersection cohomology [9], which clarified the interplay between Poincaré duality for L^2 cohomology and the geometry of the underlying variety. A significant achievement was then made by W. C. Hsiang and V. Pati who proved that the L^2 cohomology of complex normal algebraic surfaces is isomorphic to intersection cohomology[19].

Communicated by Thomas Schick.

Research partially supported by the NCN grant 2014/13/B/ST1/00543.

✉ Guillaume Valette
gvalette@impan.pl

1 Institute of Mathematics, Jagiellonian University, ul. S. Łojasiewicza 6, 30-348 Kraków, Poland

Since Cheeger's work on L^2 forms, many other authors have investigated L^p forms on singular varieties focusing on various classes of Riemmanian manifolds, with different restrictions on the metric near the singularities, like in the case of the so-called f-horns [2,14,23,24,32,33]. In the present paper, assuming only that the given set is subanalytic (possibly singular) we investigate the problem of Poincaré duality for L^p forms for p sufficiently large or close to 1. Our approach relies on the precise description of the Lipschitz geometry initiated by the author in [27–29].

In order to describe the achievements of this article, let us recall the de Rham theorem recently proved in [29] which motivated the present paper.

Theorem 1.1 *[29] Let X be a compact subanalytic pseudomanifold. For any j, we have:*

$$H^j_\infty(X_{reg}) \simeq I^t H^j(X).$$

Furthermore, the isomorphism is induced by the natural map provided by integration on allowable simplices.

Here, H^j_∞ denotes the L^∞ cohomology while $I^{\mathbf{t}}H^j(X)$ stands for the intersection cohomology of X in the maximal perversity. The definitions of these cohomology theories are recalled in Sects. 2.3 and 2.4 below. We write X_{reg} for the nonsingular part of X, i.e. the set of points at which X is a smooth manifold.

Intersection homology was introduced by M. Goresky and R. MacPherson in order to investigate the topology of singular sets. What makes it very attractive is that they showed in their fundamental paper [17] that it satisfies Poincaré duality for a quite large class of sets (recalled in Theorem 2.8 below) enclosing all the complex analytic sets (see also [18]). The above theorem thus raises the very natural question whether we can hope for Poincaré duality for L^∞ cohomology of subanalytic pseudomanifolds, or more generally for L^p cohomology $p \in [1, \infty]$.

The natural candidate for being dual to L^p cohomology is L^q cohomology with $\frac{1}{p} + \frac{1}{q} = 1$. We start by proving a de Rham theorem for the L^p cohomology of a subanalytic submanifold $M \subset \mathbb{R}^n$ in the case where p is close to 1 (Theorem 2.9). We also generalize Theorem 1.1 by proving a De Rham theorem for L^p cohomology for p sufficiently large (Theorem 2.10). These results can be regarded as subanalytic versions of Cheeger's theorems.

This enables us to establish some Poincaré duality results for L^p cohomology (Corollaries 2.11 and 2.12). Intersection homology turns out to be very useful to assess the lack of duality between L^p and L^q cohomology. In particular, we see that the obstruction for this duality to hold is of purely topological nature. Although the L^p condition is closely related to the metric structure of the singularities, the theorems below show that the knowledge of the topology of the singularities is enough to ensure Poincaré duality. It is worthy of notice that the only data of the topology of X_{reg} is not enough.

Organization of the article.

In Sect. 2, we set-up our framework, state our de Rham theorems for L^p cohomology, and derive two corollaries about Poincaré duality. The proof of these de Rham theorems is postponed to section 5.

The strategy used to establish them in Sect. 5 is classical: we first establish some Poincaré Lemmas for L^p cohomology (Lemmas 5.2 and 5.5) and then conclude by a sheaf theoretic argument. Our Poincaré Lemmas for L^p cohomology require to define some homotopy operators on L^p forms. The construction of these operators (see (3.20) and (3.27)) as well as the study of their properties is carried out in Sect. 4.

Because of the of metric nature of the L^p condition, this requires a delicate study of the Lipschitz properties of subanalytic singularities, which is the subject matter of Sect. 3. Using the techniques developed in [27–29], we show that the conical structure of subanalytic set-germs may be required to have nice Lipschitz properties (Theorem 3.5). This theorem, which is of its own interest, improves significantly the results of [29] where it was shown that every subanalytic germ may be retracted in a Lipschitz way. Since the homeomorphism of the conical structure provided by Theorem 3.5 is not smooth but just subanalytic and Lipschitz (unlike in [7,33]), we have problems to pull-back smooth differential forms to smooth ones and we shall also require stratification theory (in Sects. 3.2, 3.3, and 3.5) to overcome these difficulties (the subanalytic character of the homeomorphism of Theorem 3.5 is therefore essential). We therefore work with nonsmooth forms in Sects. 3, 4, and 5, that we differentiate as distributions (see Definition 3.12). This kind of problem is actually classical and already arose in Cheeger's original paper [5] as well as in other settings [33], when working locally at a singular point of the closure of the given manifold. It is actually well-known since de Rham's works on currents and the study of regularization of L^p forms carried out in [15], that weakly smooth forms give rise to the same cohomology groups as the smooth ones (see Corollary 3.16), and we rely on these techniques to overcome these difficulties (see Sect. 3.4).

2 Framework and main results

2.1 Some notations

Throughout this article, m, n, j, and k will stand for integers. By "smooth", we will mean C^∞.

We denote by $|.|$ the Euclidean norm of $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$ and $\varepsilon > 0$, we respectively denote by $S(x, \varepsilon)$ and $B(x, \varepsilon)$ the sphere and the open ball of radius ε that are centered at x (for the Euclidean distance). We also write $\overline{B}(x, \varepsilon)$ for the corresponding closed ball. Given a subset A of $\mathbb{R}^n$, we denote the closure of A by $cl(A)$ and set $\delta A = cl(A) \setminus A$.

Given two functions ξ and ζ defined on a subset A of $\mathbb{R}^n$ and a subset B of A, we write "$\xi \lesssim \zeta$ on B" if there is a constant C such that $\xi(x) \le C\zeta(x)$, for all $x \in B$. We write "$\xi \sim \zeta$ on B" if we have both $\xi \lesssim \zeta$ and $\zeta \lesssim \xi$ on B.

The graph of a mapping $f : A \to B$ will be denoted Γ_f. A mapping $\xi : A \to \mathbb{R}^k$, $A \subset \mathbb{R}^n$, is said to be *Lipschitz* if it is Lipschitz with respect to the metric $|.|$, i.e., if there is a constant C such that $|\xi(x) - \xi(x')| \leq C|x - x'|$, for all x and x' in A. We will say *C-Lipschitz* if we wish to specify the constant.

Given a manifold M, we denote by $\Lambda_0^j(M)$ the set of C^∞ differential j-forms on M with compact support. We write $\operatorname{supp}\varphi$ for the support of a form φ on M, and $\Lambda_{or}^j(M)$ for the space of forms $\varphi \in \Lambda_0^j(M)$ for which $\operatorname{supp}\varphi$ fits in an oriented open subset of M.

2.2 The subanalytic category

We now recall some basic facts about subanalytic sets and functions.

Definition 2.1 Let N be an analytic manifold. A subset $E \subset N$ is called (locally) *semi-analytic* if it is locally defined by finitely many real analytic equalities and inequalities. More precisely, for each $a \in N$, there is a neighborhood U of a, and real analytic functions f_i, g_{ij} on U, where $i = 1, \dots, r$, $j = 1, \dots, s$, such that

$$E \cap U = \bigcup_{i=1}^{r} \bigcap_{j=1}^{s} \{x \in U : g_{ij}(x) > 0 \text{ and } f_i(x) = 0\}. \tag{2.1}$$

We denote by $\mathbb{P}_1$ the 1-dimensional real projective space. Let $\mathcal{V} : \mathbb{R} \to \mathbb{P}_1$ be the mapping defined by $\mathcal{V}(x) = [1 : x] \in \mathbb{P}_1$ for every $x \in \mathbb{R}$. Every subset of $\mathbb{R}^n$ may be regarded as a subset of $\mathbb{P}_1^n$ via the homeomorphism (onto its image)

$$\begin{aligned} \mathcal{V}^n &: \mathbb{R}^n \to \mathbb{P}_1^n, \\ (y_1, \dots, y_n) &\mapsto (\mathcal{V}(y_1), \dots, \mathcal{V}(y_n)). \end{aligned}$$

A subset Z of $\mathbb{R}^n$ is *globally semi-analytic* if $\mathcal{V}^n(Z)$ is a semi-analytic subset of $\mathbb{P}_1^n$. Of course, globally semi-analytic sets are semi-analytic. Clearly, a bounded subset of $\mathbb{R}^n$ is semi-analytic if and only if it is globally semi-analytic.

Working with *globally* semi-analytic sets will make it possible to avoid some pathological situations at infinity. In particular, it will enable us to work without any properness assumption. The function $\sin x$ is a typical example of a function which is semi-analytic but not globally semi-analytic.

Definition 2.2 A subset $E \subset \mathbb{R}^n$ is *subanalytic* (resp. *globally subanalytic*) if it can be represented as the projection of a semi-analytic (resp. globally semi-analytic) set; more precisely, if there exists a semi-analytic (resp. globally semi-analytic) set $Z \subset \mathbb{R}^{n+p}$, $p \in \mathbb{N}$, such that $E = \pi(Z)$, where $\pi : \mathbb{R}^{n+p} \to \mathbb{R}^n$ is the projection omitting the p last coordinates. In particular, globally semi-analytic sets are globally subanalytic.

We say that *a mapping* $f : A \to B$ *is subanalytic* (resp. *globally subanalytic*), $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ subanalytic, if its graph is a subanalytic (resp. globally subanalytic) subset of $\mathbb{R}^{n+m}$. In the case $B = \mathbb{R}$, we say that f is a *subanalytic* (resp. *globally subanalytic*) *function*.

The advantage of the globally subanalytic category is that, unlike the globally semi-analytic category, it is stable under linear proje ction. Globally subanalytic sets constitute a nice category to study the geometry of semi-analytic sets: it is also stable under union, intersection, complement, and Cartesian product. Moreover, these sets enjoy many finiteness properties. For instance, they always have finitely many connected components, each of them being globally subanalytic.

If X is a subanalytic set then X_{reg}, which is the set of points at which X is a C^∞ manifold (of dimension dim X or smaller), is an open dense subanalytic subset of X. Another feature of the subanalytic category which will be important for our purpose is the famous Łojasiewicz's inequality. We shall use it in the following form.

Proposition 2.3 *(Łojasiewicz inequality) Let f and g be two globally subanalytic functions on a globally subanalytic set A. Assume that f is bounded and that* $\lim_{t\to 0} f(\gamma(t)) = 0$, *for every globally subanalytic arc $\gamma : (0, \varepsilon) \to A$ such that* $\lim_{t\to 0} g(\gamma(t)) = 0$. *Then there exist $N \in \mathbb{N}$ and $C \in \mathbb{R}$ such that for any $x \in A$:*

$$|f(x)|^N \leq C|g(x)|. \tag{2.2}$$

This inequality originates in [21]. Several improvements were then obtained. This form is due to [26] (Proposition 1.1). We refer to [3,12] for more about subanalytic sets.

2.3 L^p cohomology.

Let M be a C^∞ submanifold of $\mathbb{R}^n$. We equip M with the Riemannian metric inherited from the ambient space, this set being endowed with the Euclidean inner product. This metric gives rise to a measure vol_M.

Given a measurable function $f : M \to \mathbb{R}$ (the word *measurable* will always refer to this measure), we will denote by $\int_{x\in M} f(x)$ (or sometimes $\int_M f$) the integral of f with respect to vol_M (in other words, we will not match the measure in the notation).

For $p \in [1, \infty)$, we then say that *the function f is L^p* if it is L^p with respect to the measure vol_M, i.e., if $\int_{x\in M} |f(x)|^p < \infty$. We will write $|f|_p$ for the *L^p norm of f* (possibly infinite), i.e., $|f|_p := \left(\int_{x\in M} |f(x)|^p\right)^{\frac{1}{p}}$.

We say that f is L^∞ if there is a constant C such that $|f(x)| \leq C$ for almost every $x \in M$. The L^∞ norm of f will be denoted $|f|_\infty$ and will be, as usual, the *essential supremum of f on M*, i.e.,

$$|f|_\infty := \text{ess sup}_{x\in M}|f(x)| := \inf\{a \in \mathbb{R} : vol_M(|f|^{-1}([a, \infty)) = 0\},$$

with the convention that this infimum is infinite if the considered set is empty.

Differentiable forms will always be assumed to be at least measurable (i.e., giving rise to a measurable function when composed with a smooth section of multivectors). Given a differential j-form ω on M, we will denote by $|\omega(x)|$ the norm of the linear mapping $\omega(x) : \otimes^j T_x M \to \mathbb{R}$ with respect to the metric of M. As usual, we will denote by d the exterior differentiation of forms (for all manifolds and all j).

Definition 2.4 Given $p \in [1, +\infty]$, we say that *a differential j-form ω on M is L^p* if the function $f(x) := |\omega(x)|$ is L^p. In the case where p is finite, this means that

$$|\omega|_p := \left(\int_{x \in M} |\omega(x)|^p \right)^{\frac{1}{p}} < \infty.$$

In the case $p = \infty$ this means that there exists a constant C such that:

$$|\omega|_\infty := \text{ess sup}_{x \in M} |\omega(x)| < \infty.$$

We denote by $\Omega_p^j(M)$ the real vector space constituted by the smooth L^p differential j-forms ω on M for which $d\omega$ is also L^p. The cohomology groups of the cochain complex $(\Omega_p^j(M), d)_{j \in \mathbb{N}}$ are called the *L^p cohomology groups of M* and will be denoted by $H_p^j(M)$.

2.4 Intersection homology

We recall the definition of intersection homology introduced by Goresky and Macpherson [17,18]. We do it in the subanalytic category.

Definition 2.5 Let $X \subset \mathbb{R}^n$ be a subanalytic subset. A *stratification of X* is a finite partition of this set into subanalytic C^∞ submanifolds of $\mathbb{R}^n$, called *strata*.

We now are going to define inductively on the dimension of X the locally topologically trivial stratifications of X. For $\dim X = 0$, every stratification is locally topologically trivial.

We denote by cL the open cone over the space $L \subset \mathbb{R}^n$ of vertex at the origin, $c\emptyset$ being reduced to the origin. Observe that if L is stratified by Σ then cL is stratified by $cS \setminus \{0\}$, $S \in \Sigma$, and the origin.

A stratification Σ of X is said to be *locally topologically trivial* if for every $x \in S \in \Sigma$, there is a subanalytic homeomorphism

$$h : U_x \to B(0_{\mathbb{R}^i}, 1) \times cL,$$

(where $i = \dim S$) with U_x neighborhood of x in X and $L \subset X \setminus \{x\}$ compact subanalytic subset having a locally topologically trivial stratification such that h maps the strata of U_x (induced stratification) onto the strata of $B(0_{\mathbb{R}^i}, 1) \times cL$ (product stratification).

Definition 2.6 A subanalytic subset $X \subset \mathbb{R}^n$ is an *m-dimensional subanalytic pseudomanifold* if X_{reg} is an m-dimensional manifold and $\dim X \setminus X_{reg} < m - 1$.

A *stratified pseudomanifold* (of dimension m) is the data of an m-dimensional subanalytic pseudomanifold X together with a locally topologically trivial stratification Σ of X having no stratum of dimension $(m - 1)$.

Given $X \subset \mathbb{R}^n$, the *singular k-simplices of X* will be the continuous globally subanalytic mappings $c : \Delta_j \to \mathbb{R}^n$ such that $|c| \subset X$, Δ_j being the j-simplex spanned by $0, e_1, \dots, e_j$, where $e_1, \dots, e_j$ is the canonical basis of $\mathbb{R}^j$, and $|c|$ the support of the chain c. We denote by $C_j(X)$ the $\mathbb{R}$-vector spaces of singular subanalytic sets, i.e. finite combinations (with real coefficients) of singular subanalytic simplices, and we will write ∂c for the boundary of c.

Definition 2.7 Let (X, Σ) be an m-dimensional stratified pseudomanifold and let X_i denote the union of all the strata of Σ of dimension less than or equal to i. A *perversity* is a sequence of integers $p = (p_2, p_3, \dots, p_m)$ such that $p_2 = 0$ and $p_{k+1} = p_k$ or $p_k + 1$. Given a perversity p, a subset $Y \subset X$ is called (p, i)-*allowable* if for all k

$$\dim Y \cap X_{m-k} \le p_k + i - k.$$

Define $I^p C_i(X)$ as the subgroup of $C_i(X)$ consisting of those chains σ such that $|\sigma|$ is (p, i)-allowable and $|\partial\sigma|$ is $(p, i-1)$-allowable. The i^{th} *intersection homology group of perversity* p, denoted $I^p H_i(X)$, is the i^{th} homology group of the chain complex $I^p C_\bullet(X)$. The i^{th} *intersection cohomology group of perversity* p, denoted $I^p H^i(X)$, is defined as $\mathbf{Hom}(I^p H_i(X), \mathbb{R})$.

In [17,18] Goresky and MacPherson have proved that these homology groups are finitely generated and independent of the (locally topologically trivial) stratification. Since topologically trivial stratifications exist for all subanalytic pseudomanifolds [13] (Whitney (b)-regular stratifications do have this property), we will not always specify the chosen stratification.

Moreover, Goresky and MacPherson also proved that their theory satisfy a generalized version of Poincaré duality. We set $\mathbf{t} := (0, 1, \dots, m-2)$.

Theorem 2.8 (Generalized Poincaré duality [17,18]) *Let X be a compact oriented stratified pseudomanifold and let p and q be perversities with $p + q = \mathbf{t}$. For all j, we have:*

$$I^p H^j(X) \simeq I^q H^{m-j}(X).$$

In particular, in the case of the perversities $p = 0 = (0, \dots, 0)$ and $q = \mathbf{t}$, we get

$$I^0 H^j(X) \simeq I^{\mathbf{t}} H^{m-j}(X). \tag{2.3}$$

2.5 The de Rham theorems

In this section we state our de Rham theorems. The proofs require technical preliminaries and will appear in Sect. 5.

Theorem 2.9 *Given a bounded subanalytic submanifold M of $\mathbb{R}^n$, we have for each $p \in [1, \infty)$ sufficiently close to 1 and each integer j:*

$$H_p^j(M) \simeq H^j(M).$$

Such a theorem is of course no longer true without the subanalycity assumption. We will also show that the isomorphism is given by integration of forms on simplices. By "p sufficiently close to 1", we mean that there is $p_0 \in (1, \infty]$ such that this statement holds for all $1 \leq p < p_0$. The bound for p will be provided by the famous Łojasiewicz's inequality (see Proposition 2.3).

We then will improve Theorem 1.1 by showing:

Theorem 2.10 *Given a bounded subanalytic m-dimensional submanifold M of $\mathbb{R}^n$ such that* $\dim \delta M \leq m-2$, *we have for all $p \in [1, \infty]$ large enough and each integer j:*

$$H_p^j(M) \simeq I^{\mathbf{t}} H^j(X),$$

where X denotes the closure of M.

Again, by "p large enough", we mean that there is $p_0 \in [1, \infty)$ such that this statement holds for all $p > p_0$.

2.6 L^q as a Poincaré dual for L^p

Given $p \in [1, \infty]$, we call the number $q \in [1, \infty]$ that satisfies $\frac{1}{p} + \frac{1}{q} = 1$, the *Hölder conjugate of p*. Thanks to Goresky and MacPherson's generalized Poincaré duality, we can derive explicit topological criteria on the singularity to determine whether L^p cohomology is Poincaré dual to L^q cohomology.

Corollary 2.11 *Let X be a compact oriented m-dimensional subanalytic pseudomanifold. Take $p \in [1, \infty]$ and denote by q its Hölder conjugate. If $H^j(X_{reg}) \simeq I^0 H^j(X)$ then, for p sufficiently close to 1, L^p cohomology is Poincaré dual to L^q cohomology in dimension j, in the sense that*

$$H_p^j(X_{reg}) \simeq H_q^{m-j}(X_{reg}).$$

Proof This is a consequence of Theorems 2.9, 2.10, and Goresky and MacPherson's generalized Poincaré duality (2.3). □

Corollary 2.12 *Let $M \subset \mathbb{R}^n$ be a bounded subanalytic oriented m-dimensional C^∞ submanifold. Take $p \in [1, \infty]$ and denote by q its Hölder conjugate. If* $\dim \delta M = k$ *then, for p sufficiently close to 1, L^p cohomology is Poincaré dual to L^q cohomology in dimension j, for each $j < m - k - 1$, i.e., for each such j:*

$$H_p^j(M) \simeq H_q^{m-j}(M).$$

Proof We may assume $k < m - 1$ since otherwise the statement is vacuous. Observe that $X := cl(M)$ is then a compact subanalytic pseudomanifold. Fix a stratification that makes of X a stratified pseudomanifold. We can choose this stratification compatible with δM and such that there is no stratum S satisfying $k < \dim S < m$ (all the strata of positive codimension may be assumed to be included in δM). By definition of 0-allowable chains (see Sect. 2.4), the support of a singular chain $\sigma \in I^0C_j(X)$ may not intersect the strata of the singular locus of dimension less than $(m-j)$. If $j < m-k$ (or equivalently $k < m-j$) $|\sigma|$ thus must lie entirely in M, which entails

$$I^0C_j(X) = C_j(M).$$

If $j < m-k-1$ then the same applies to $(j+1)$ and consequently

$$I^0H_j(X) = H_j(M).$$

The result is therefore again a consequence of Theorems 2.9, 2.10, and Goresky and MacPherson's generalized Poincaré duality (2.3). □

Remark 2.13 It could be seen that this duality is provided by the natural pairing given by integration. Considering L^p cohomology with bounded support, it is possible to generalize this duality to the case of unbounded manifolds.

3 Lipschitz properties of subanalytic sets and mappings

3.1 Lipschitz conic structure of subanalytic set-germs

The study of the metric geometry of singularities is more challenging than the study of their topology. For instance it is well-known that subanalytic sets can be triangulated and hence are locally homeomorphic to cones. The situation is more complicated if one is interested in the description of the aspect of singularities from the metric point of view. We however are going to prove that this conic structure may be required to have some nice metric properties (Theorem 3.5) that will make it possible to establish our de Rham theorems later on.

Definition 3.1 A *cell decomposition of* $\mathbb{R}^n$ is a finite partition of $\mathbb{R}^n$ into globally subanalytic sets $(C_i)_{i\in I}$, called *cells*, satisfying certain properties explained below.

$n = 1$: A *cell decomposition of* $\mathbb{R}$ is given by a finite subdivision $a_1 < \cdots < a_l$ of $\mathbb{R}$. The cells of $\mathbb{R}$ are the singletons $\{a_i\}$, $1 \le i \le l$, and the intervals (a_i, a_{i+1}), $0 \le i \le l$, where $a_0 = -\infty$ and $a_{l+1} = +\infty$.

$n > 1$: A *cell decomposition of* $\mathbb{R}^n$ is the data of a cell decomposition of $\mathbb{R}^{n-1}$ and, for each cell D of $\mathbb{R}^{n-1}$, some C^∞ globally subanalytic functions on D (which, by induction, is a C^∞ manifold):

$$\zeta_{D,1} < \cdots < \zeta_{D,l(D)} : D \to \mathbb{R}.$$

The *cells of* $\mathbb{R}^n$ are the *graphs*

$$\{(x, \zeta_{D,i}(x)) : x \in D\},$$

and the *bands*

$$(\zeta_{D,i}, \zeta_{D,i+1}) := \{(x, y) : x \in D \text{ and } \zeta_{D,i}(x) < y < \zeta_{D,i+1}(x)\},$$

for $0 \le i \le l(D)$, where $\zeta_{D,0}(x) = -\infty$ and $\zeta_{D,l(D)+1}(x) = +\infty$.

A cell decomposition is said to be *compatible with finitely many sets* $A_1, \dots, A_k$ if the A_i's are unions of cells.

It is well-known that given some globally subanalytic sets $A_1, \dots, A_k$, it is always possible to find a cell decomposition compatible with this family of sets. This fact is true on every o-minimal structure. A detailed proof in this framework can be found in [10].

Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection omitting the last coordinate. If D is a cell of $\mathbb{R}^n$, we call $E := \pi(D)$, the *basis* of D. Observe that if $\mathcal{D}$ is a cell decomposition of $\mathbb{R}^n$, then $\pi(\mathcal{D}) := \{\pi(D) : D \in \mathcal{D}\}$ is a cell decomposition of $\mathbb{R}^{n-1}$.

Definition 3.2 Let $A, B \subset \mathbb{R}^n$. A globally subanalytic map $h : A \to B$ is x_1-*preserving* if it preserves the first coordinate in the canonical basis of $\mathbb{R}^n$, i.e., if for any $x = (x_1, \dots, x_n) \in A$, $\mu(h(x)) = x_1$, where $\mu : \mathbb{R}^n \to \mathbb{R}$ is the orthogonal projection onto the first coordinate.

If R is a positive real number and n a positive integer, we set

$$\mathcal{C}_n(R) := \{x = (x_1, \dots, x_n) \in [0, \infty) \times \mathbb{R}^{n-1} : |x| \le R x_1\}.$$

We also set $\mathcal{C}_1(R) := [0, +\infty)$.

We shall need the following lemma which was proved in [29] (Lemma 2.2.3) to compute L^∞ cohomology. In this lemma all the germs are germs at the origin.

Lemma 3.3 *Let $A_1, \dots, A_\mu$ be germs of subanalytic subsets of $\mathcal{C}_n(R)$, $R > 0$, and $\eta_1, \dots, \eta_l$ be germs of nonnegative globally subanalytic functions on $\mathcal{C}_n(R)$. There exist a germ of subanalytic x_1-preserving bi-Lipschitz homeomorphism (onto its image) $\Phi : (\mathcal{C}_n(R), 0) \to (\mathcal{C}_n(R), 0)$ and a cell decomposition $\mathcal{D}$ of $\mathbb{R}^n$ such that:*

- *(i) $\mathcal{D}$ is compatible with (some representatives of the germs) $\Phi(A_1), \dots, \Phi(A_\mu)$.*
- *(ii) Every cell of $\mathcal{D}$ which is a graph (i.e., not a band, see Definition 3.1) is the graph of a function that has bounded derivative.*
- *(iii) On each $D \in \mathcal{D}$, every germ $\eta_i \circ \Phi^{-1}(x)$ is $\sim$ to the germ of a function of the form:*

$$|x_n - \theta(\tilde{x})|^r a(\tilde{x}) \tag{2.4}$$

(for $x = (\tilde{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$) where $a, \theta : E \to \mathbb{R}$ are globally subanalytic functions on the basis E of D with θ Lipschitz and $r \in \mathbb{Q}$.

Remark 3.4 In (ii), it is required that the function defining the cells have bounded derivative. Such functions are not necessarily Lipschitz (with respect to the Euclidean distance). They are nevertheless Lipschitz with respect to the so-called inner metric (given by the shortest path joining two points). It follows from the existence of L-reqular cell decompositions [20] that there is a partition of the basis of each cell in such a way that the two metric be equivalent on each element of this partition. This means that in (ii), we could require the functions defining the cells to be Lipschitz (with respect to the Euclidean distance).

Given $A \subset \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$, we denote by $x_0 * A$ the cone over the space A of vertex x_0, i.e., the set of points of type $tx_0 + (1-t)y$ with $y \in A$ and $t \in [0, 1]$ (by convention, $x_0 * \emptyset$ will be reduced to the point x_0).

Theorem 3.5 *Let $X \subset \mathbb{R}^n$ be subanalytic and $x_0 \in X$. For $\varepsilon > 0$ small enough, there exists a Lipschitz subanalytic homeomorphism*

$$H : x_0 * (S(x_0, \varepsilon) \cap X) \to \overline{B}(x_0, \varepsilon) \cap X,$$

satisfying $H_{|S(x_0,\varepsilon)\cap X} = Id$, preserving the distance to x_0, and having the following metric properties:

(i) The natural retraction by deformation onto x_0

$$r : [0, 1] \times \overline{B}(x_0, \varepsilon) \cap X \to \overline{B}(x_0, \varepsilon) \cap X,$$

defined by

$$r(s, x) := H(sH^{-1}(x) + (1 - s)x_0),$$

is Lipschitz. Indeed, there is a constant C such that for every fixed $s \in [0, 1]$, the mapping r_s defined by $x \mapsto r_s(x) := r(s, x)$, is Cs-Lipschitz.

(ii) For each $\delta > 0$, the restriction of H^{-1} to $\{x \in X : \delta \leq |x - x_0| \leq \varepsilon\}$ is Lipschitz and, for each $s \in (0, 1]$, the map $r_s^{-1} : \overline{B}(x_0, s\varepsilon) \cap X \to \overline{B}(x_0, \varepsilon) \cap X$ is Lipschitz.

Proof For $\varepsilon > 0$, let us set

$$\mathcal{C}_n(R, \varepsilon) := \{x = (x_1, \dots, x_n) \in \mathcal{C}_n(R) : 0 \leq x_1 \leq \varepsilon\}.$$

The idea is to replace the distance to x_0 with the function given by the projection onto the first coordinate. We will prove by induction on n the following statements.

($\mathbf{A}_n$) Let R be a positive real number, $X_1, \dots, X_s$ finitely many subanalytic subsets of $\mathcal{C}_n(R)$, and $\xi_1, \dots, \xi_l$ some bounded subanalytic nonnegative functions on $\mathcal{C}_n(R)$.

For every positive small enough real number ε, there exists a Lipschitz x_1-preserving subanalytic homeomorphism $h : \mathcal{C}_n(R, \varepsilon) \to \mathcal{C}_n(R, \varepsilon)$ such that $h(\varepsilon, x) = (\varepsilon, x)$ for all $x \in B(0_{\mathbb{R}^{n-1}}, R\varepsilon)$, and satisfying

(1) $h(0 * X_{j,\varepsilon}) = X_j \cap \{x \in \mathbb{R}^n : 0 < x_1 \leq \varepsilon\}$, for all $j = 1, \dots s$, where $X_{j,\varepsilon} = X_j \cap \{x \in \mathbb{R}^n : x_1 = \varepsilon\}$.

(2) The natural retraction by deformation onto the origin

$$r : [0, 1] \times \mathcal{C}_n(R, \varepsilon) \to \mathcal{C}_n(R, \varepsilon)$$

defined by

$$r(s, x) := h(sh^{-1}(x)),$$

is Lipschitz. Indeed, there is a constant C such that for every fixed $s \in [0, 1]$, the retraction r_s, defined by

$$x \mapsto r_s(x) := r(s, x),$$

is Cs-Lipschitz.

(3) For each $\eta > 0$, the restriction of h^{-1} to $\{x \in X : \eta \leq |x| \leq \varepsilon\}$ is Lipschitz and, for each $s \in (0, 1]$, the map

$$r_s^{-1} : \mathcal{C}_n(R, s\varepsilon) \to \mathcal{C}_n(R, \varepsilon)$$

is Lipschitz.

(4) There is a constant C such that for all $x \in \mathcal{C}_n(R, \varepsilon)$, $s \in (0, 1)$, and all $k \leq l$ we have:

$$\xi_k \circ h(sx) \leq C\, \xi_k \circ h(x). \tag{2.5}$$

(5) For each $\delta > 0$ there is a positive constant c_δ such that we have for all $x \in \mathcal{C}_n(R, \varepsilon)$, $s \in (\delta, 1)$, and all $k \leq l$:

$$c_\delta\, \xi_k \circ h(x) \leq \xi_k \circ h(sx). \tag{2.6}$$

Before proving these statements, let us make it clear that these yield the theorem. Let $X \subset \mathbb{R}^n$ be a subanalytic set. We can assume that $0 \in cl(X)$ and work nearby the origin. The set

$$\check{X} := \{(t, x) \in \mathbb{R} \times X : t = |x|\},$$

is a subset of $\mathcal{C}_{n+1}(R)$ (for $R > 1$) to which we can apply $(\mathbf{A}_{n+1})$. This provides a Lipschitz x_1-preserving homeomorphism

$$h : \mathcal{C}_{n+1}(R, \varepsilon) \to \mathcal{C}_{n+1}(R, \varepsilon),$$

which, thanks to (1) of $(\mathbf{A}_{n+1})$, gives rise to a homeomorphism

$$H : 0 * (S(0, \varepsilon) \cap X) \to X \cap B(0, \varepsilon)$$

(since $\check{X}_\varepsilon = S(0, \varepsilon) \cap X$). Because the projection defined by $P(t, x) := x$ induces a bi-Lipschitz homeomorphism between $\check{X}$ and X, properties (i) and (ii) of the theorem come down from (2) and (3) of $(\mathbf{A}_{n+1})$.

The assertions (4) and (5) are not necessary to derive statement of the theorem. They are required so as to perform the proofs of (2) and (3) during the induction step of the proof of $(\mathbf{A}_n)$.

As $(\mathbf{A}_1)$ is trivial (h being the identity map), we fix some $n > 1$. We also fix some globally subanalytic subsets $X_1, \dots, X_s$ of $\mathcal{C}_n(R)$, $R > 0$, as well as some globally subanalytic bounded functions $\xi_1, \dots, \xi_l : \mathcal{C}_n(R) \to \mathbb{R}$.

The induction hypothesis will be applied to the elements of a suitable cell decomposition of $\mathbb{R}^{n-1}$. This requires some preliminaries.

Apply Lemma 3.3 to the collection of globally subanalytic sets constituted by the (germs of) the X_i's, the set $\mathcal{C}_n(R)$ itself, and the union of the zero loci of the ξ_i's together with the finite family of functions $\xi_1, \dots, \xi_l$. We get a (germ of) x_1-preserving globally subanalytic bi-Lipschitz map $\Phi : \mathcal{C}_n(R) \to \mathcal{C}_n(R)$ and a cell decomposition $\mathcal{D}$ of $\mathbb{R}^n$ such that (i), (ii), and (iii) of the latter lemma hold.

Let Θ be a cell of $\mathcal{D}$ which lies in $\Phi(\mathcal{C}_n(R))$ and which is a graph, say of a function $\eta : \Theta' \to \mathbb{R}$, Θ' standing for the basis of Θ. By (ii) of Lemma 3.3 (and Remark 3.4), η is a Lipschitz function. It thus may be extended to a globally subanalytic Lipschitz function on the whole of $\mathbb{R}^{n-1}$. Repeating this for all the cells Θ of $\mathcal{D}$ lying in $\Phi(\mathcal{C}_n(R))$ which are graphs (i.e., not bands), we get finitely many globally subanalytic Lipschitz functions $\eta_1, \dots, \eta_v$. Using the min and max operators if necessary, we may transform this family into a family satisfying $\eta_1 \le \cdots \le \eta_v$.

As we may work up to a x_1-preserving bi-Lipschitz map, in what follows we will identify Φ with the identity map. Hence, thanks to (iii) of Lemma 3.3, we may assume that on each cell every function ξ_i is $\sim$ to a function of the form which appears in (2.4).

We are now going to introduce some bounded $(n-1)$-variable functions $\sigma_1, \dots, \sigma_p$, to which we will apply (4) and (5) of the induction hypothesis. This will be of service to establish (2) and (3) and to show that the ξ_k's satisfy (4) and (5). These $(n-1)$-variable functions will be provided by the estimate (2.4) and the η_j's.

Fix an integer $1 \le j < v$ and a cell Λ of $\mathcal{D}$. Set for simplicity $D := \Lambda \cap (\eta_j, \eta_{j+1})$.

For each $k \le l$, the function ξ_k is $\sim$ to a function like in (2.4) on D, i.e., there exist $(n-1)$-variable functions on $\pi_{e_n}(D)$ (π_{e_n} denoting the orthogonal projection along e_n, the last vector of the canonical basis), say θ_k and a_k, and $\alpha_k \in \mathbb{Q}$ such that:

$$\xi_k(\tilde{x}, x_n) \sim |x_n - \theta_k(\tilde{x})|^{\alpha_k} a_k(\tilde{x}),$$

for $(\tilde{x}, x_n) \in D \subset \mathbb{R}^{n-1} \times \mathbb{R}$.

As the zero loci of the ξ_k's are included in the graphs of the η_i's, we have on $\pi_{e_n}(D)$ for every k, either $\theta_k \le \eta_j$ or $\theta_k \ge \eta_{j+1}$. We will assume for simplicity that $\theta_k \le \eta_j$.

This means that for $(\tilde{x}, x_n) \in D \subset \mathbb{R}^{n-1} \times \mathbb{R}$:

$$\xi_k(\tilde{x}, x_n) \sim \min((x_n - \eta_j(\tilde{x}))^{\alpha_k} a_k(\tilde{x}), (\eta_j(\tilde{x}) - \theta_k(\tilde{x}))^{\alpha_k} a_k(\tilde{x})), \tag{2.7}$$

if α_k is negative and

$$\xi_k(\tilde{x}, x_n) \sim \max((x_n - \eta_j(\tilde{x}))^{\alpha_k} a_k(\tilde{x}), (\eta_j(\tilde{x}) - \theta_k(\tilde{x}))^{\alpha_k} a_k(\tilde{x})), \tag{2.8}$$

in the case where α_k is nonnegative.

First, consider the following functions:

$$\kappa_k(\tilde{x}) := (\eta_j(\tilde{x}) - \theta_k(\tilde{x}))^{\alpha_k} a_k(\tilde{x}), \qquad k = 1, \dots, l. \tag{2.9}$$

For every k, the function κ_k is bounded on D since it is equivalent to the function $\tilde{x} \mapsto \lim_{x_n \to \eta_j(\tilde{x})} \xi_k(\tilde{x}, x_n)$ which is bounded. As the function $(\eta_{j+1} - \eta_j)$ is Lipschitz and vanishes at the origin (since it extends a function whose graph lies in $\mathcal{C}_n(R)$) the function $x = (x_1, \dots, x_{n-1}) \mapsto \frac{(\eta_{j+1} - \eta_j)(x)}{x_1}$ is bounded on $\mathcal{C}_{n-1}(R)$. We thus can complete the family κ by adding the function $\frac{(\eta_{j+1} - \eta_j)(x)}{x_1}$ as well as the functions $\min((\eta_{j+1} - \eta_j)^{\alpha_k} a_k, 1)$, $1 \le k \le l$. This family κ of course depends on the fixed set D (intersection of some cell Λ of $\mathcal{D}$ with (η_j, η_{j+1}), for some j). The union of all these families obtained for every such set D eventually provides us the desired collection of functions $\sigma_1, \dots, \sigma_p$.

We now turn to the construction of the desired homeomorphism. Refine the cell decomposition $\pi_{e_n}(\mathcal{D})$ into a cell decomposition $\mathcal{D}'$ of $\mathbb{R}^{n-1}$ compatible with the zero loci of the functions $(\eta_j - \eta_{j+1})$. Apply the induction hypothesis to the family constituted by the cells of $\mathcal{D}'$. This provides a x_1-preserving homeomorphism $\tilde{h} : \mathcal{C}_{n-1}(R, \varepsilon) \to \mathcal{C}_{n-1}(R, \varepsilon)$, satisfying $\tilde{h}(\varepsilon, x) = x$, for all $x \in B(0_{\mathbb{R}^{n-2}}, R\varepsilon)$. In addition, thanks to the induction hypothesis, we may assume that the functions $\sigma_1, \dots, \sigma_p$ as well as all the functions $\xi_i(x, \eta_j(x))$ satisfy (2.5) and (2.6) (with respect to $\tilde{h}$).

For $q \in (\eta_j, \eta_{j+1})$, decomposed as $(\tilde{q}, q_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we set

$$\nu(q) := \frac{q_n - \eta_j(\tilde{q})}{\eta_{j+1}(\tilde{q}) - \eta_j(\tilde{q})}. \tag{2.10}$$

In order to define the desired homeomorphism h, take now an element $x \in \mathcal{C}_n(R, \varepsilon)$. If the point $q := \frac{\varepsilon}{x_1} x$ belongs to (η_j, η_{j+1}), for some $j < v$, we set:

$$h(x) = h(\tilde{x}, x_n) := (\tilde{h}(\tilde{x}), \nu(q)(\eta_{j+1} - \eta_j) \circ \tilde{h}(\tilde{x}) + \eta_j \circ \tilde{h}(\tilde{x})).$$

If the point q belongs to the graph of η_j, for some $j \le v$, we set

$$h(x) := (\tilde{h}(\tilde{x}), \eta_j(\tilde{h}(\tilde{x})).$$

Since $\mathcal{D}'$ is compatible with the sets $\{\eta_j = \eta_{j+1}\}$ and since (1) holds for $\tilde{h}$ for each cell of $\mathcal{D}'$, the mapping h is a homeomorphism on $\mathcal{C}_n(R, \varepsilon)$. Moreover, as by construction h satisfies (1) for the Γ_{η_j}'s, this property holds true for all the cells of $\mathcal{D}$. As $\mathcal{D}$ is compatible with the X_j's, this yields (1) for h.

Let us check that (2) and (3) hold for $\tilde{h}$. Fix for this purpose $j < v$ and set for simplicity $\overline{\eta}_j := \eta_{j+1} - \eta_j$. In virtue of the induction hypothesis, inequality (2.5) is fulfilled by the functions $\frac{\overline{\eta}_j(\tilde{x})}{\tilde{x}_1}$, where $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_{n-1}) \in \mathcal{C}_{n-1}(R, \varepsilon)$. It means that (applying (2.5) with $s = \frac{\tilde{x}_1}{\varepsilon}$) there is a constant C such that for all such $\tilde{x}$:

$$\overline{\eta}_j \circ \tilde{h}(\tilde{x}) \leq \frac{C\tilde{x}_1}{\varepsilon} \cdot \overline{\eta}_j \circ \tilde{h}\left(\frac{\varepsilon}{\tilde{x}_1}\tilde{x}\right). \tag{2.11}$$

Therefore, as $\tilde{h}$ is Lipschitz, the mapping h, which maps linearly the vertical segment $[(\tilde{x}, \frac{\tilde{x}_1}{\varepsilon} \cdot \eta_j \circ \tilde{h}(\frac{\varepsilon}{x_1}\tilde{x})), (\tilde{x}, \frac{\tilde{x}_1}{\varepsilon} \cdot \eta_{j+1} \circ \tilde{h}(\frac{\varepsilon}{x_1}\tilde{x}))]$ onto the segment $[(\tilde{h}(\tilde{x}), \eta_j \circ \tilde{h}(\tilde{x})), (\tilde{h}(\tilde{x}), \eta_{j+1} \circ \tilde{h}(\tilde{x}))]$, must be Lipschitz as well. Moreover, for the same reason, (2.6) entails that h^{-1} is Lipschitz on $\mathcal{C}_n(R, \varepsilon) \setminus \mathcal{C}_n(R, \delta)$ for every $\delta > 0$.

Furthermore, as (2.5) holds for $\frac{\overline{\eta}_j}{x_1}$, composing with $\tilde{h}^{-1}$ in this inequality, we get for $x \in \mathcal{C}_n(R, \varepsilon)$ and $s \in [0, 1]$:

$$\overline{\eta}_j(r_s(x)) \leq Cs \cdot \overline{\eta}_j(x), \tag{2.12}$$

which yields that r_s is Cs Lipschitz for each s (that r is Lipschitz with respect to s is clear since h is itself Lipschitz). Finally, using (2.6) in exactly the same way, we can show that r_s^{-1} is Lipschitz for every positive s. This yields (2) and (3).

It remains to establish (4) and (5). The claimed estimates are clear on the graphs of the η_j's for we have required the functions $\xi_k(x, \eta_j(x))$ to satisfy such inequalities when applying the induction hypothesis. Hence, let us fix $k \leq l$, $j < v$, as well as $D \in \mathcal{D}'$, and show that (2.5) and (2.6) hold for ξ_k on the set $(\eta_{j|D}, \eta_{j+1|D})$.

Observe that, as ξ_k is bounded, it is enough to prove this for the function $\min(\xi_k, 1)$, and, by (2.7) and (2.8), it actually suffices to show that the functions $\min((x_n - \eta_j(\tilde{x}))^{\alpha_k} a_k(\tilde{x}), 1)$ and $\min(|\theta_k - \eta_j|(\tilde{x})^{\alpha_k} a_k(\tilde{x}), 1)$ both admit such estimates. For the latter function, this follows from the induction hypothesis since we have required the κ_i's (see (2.9)) and $\tilde{h}$ to have this property. We thus only need to deal with the function $\min((x_n - \eta_j(\tilde{x}))^{\alpha_k} a_k(\tilde{x}), 1)$.

For simplicity, we set

$$F(\tilde{x}, x_n) := (x_n - \eta_j(\tilde{x}))^{\alpha_k} \cdot a_k(\tilde{x}),$$

and

$$G(\tilde{x}) := (\eta_{j+1} - \eta_j)(\tilde{x})^{\alpha_k} \cdot a_k(\tilde{x}).$$

We have to show the desired inequalities for $\min(F, 1)$. By definition of ν (see (2.10)) we have for $x = (\tilde{x}, x_n) \in (\eta_{j|D}, \eta_{j+1|D})$:

$$F(x) = \nu(x)^{\alpha_k} \cdot G(\tilde{x}). \tag{2.13}$$

Remark that the function $\nu(h(sx))$ is constant with respect to $s \in [0, 1]$, which implies that for $x = (\tilde{x}, x_n) \in \mathcal{C}_n(R, \varepsilon)$ and $s \in [0, 1]$ we have:

$$F(h(sx)) = \nu(h(x))^{\alpha_k} \cdot G(\tilde{h}(s\tilde{x})). \tag{2.14}$$

We assume first that α_k is negative. Thanks to the induction hypothesis, we know that for $0 < \delta \leq s \leq 1$:

$$c_\delta \min(G \circ \tilde{h}(x), 1) \leq \min(G \circ \tilde{h}(s\tilde{x}), 1) \leq C \min(G \circ \tilde{h}(\tilde{x}), 1),$$

for some positive constants c_δ, C (with C independent of δ). Multiplying by ν^{α_k} and applying (2.14), this implies that:

$$\begin{aligned} c_\delta \min(F \circ h(x), 1, \nu(h(x))^{\alpha_k}) &\leq \min(F \circ h(sx), 1, \nu(h(x))^{\alpha_k}) \\ &\leq C \min(F \circ h(x), 1, \nu(h(x))^{\alpha_k}) \end{aligned}$$

But, as α_k is negative, $\min(F, \nu^{\alpha_k}, 1) = \min(F, 1)$ and we are done.

We now assume that α_k is nonnegative. This implies that F is bounded (since, by (2.8), $F \leq \xi_k$), which entails that G is bounded as well. Moreover, thanks to (2.14), it actually suffices to show the desired inequality for G and $\tilde{h}$. As G is bounded, $G \sim \min(G, 1)$, which satisfies (2.5) and (2.6), in virtue of the induction hypothesis ($\min(G, 1)$ is one of the σ_i's). This establishes (4) and (5). □

Remark 3.6 We have proved that, given finitely many subanalytic set germs $X_1, \ldots, X_s$ at $x_0 \in \mathbb{R}^n$, the respective homeomorphisms of the Lipschitz conic structure of the X_i's can be required to be induced by the same Lipschitz homeomorphism $H : x_0 * S(x_0, \varepsilon) \to \overline{B}(x_0, \varepsilon)$.

3.2 Horizontally C^1 mappings

We wish to prove that globally subanalytic bi-Lipschitz homeomorphisms pull-back smooth differentiable forms to weakly differentiable forms (see Sect. 3.5). This requires stratification theory and we shall make use of the notion of horizontally C^1 mappings introduced in [22]. In this section, we give basic definitions and prove a preliminary lemma.

Definition 3.7 A *stratified mapping* is the data of a mapping $h : X \to Y$, $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^k$, together with some stratifications Σ and Σ' of X and Y respectively, such that h smoothly maps every stratum of Σ into a stratum of Σ'.

A stratified mapping $h : (X, \Sigma) \to (Y, \Sigma')$ is said to be *horizontally* C^1 if, for any sequence $(x_l)_{l \in \mathbb{N}}$ in a stratum S of Σ tending to some point x in a stratum $S' \in \Sigma$ and for any sequence $u_l \in T_{x_l} S$ tending to a vector u in $T_x S'$, we have

$$\lim d_{x_l} h_{|S}(u_l) = d_x h_{|S'}(u).$$

If h is horizontally C^1 then the derivative of $h_{|S}$ is bounded away from infinity on every bounded subset of S for every $S \in \Sigma$. The lemma below can be considered as a converse of this observation. It relies on the existence of Whitney (a) regular stratifications which is well-known for subanalytic mappings [13,25]. We recall that a stratification is *Whitney* (a) *regular* if for every sequence x_i in a stratum S converging to a point y in a stratum S', in such a way that $T_{x_i} S$ has a limit in the Grassmannian, we have $\lim T_{x_i} S \supset T_y S'$.

Lemma 3.8 *Let $h : X \to Y$ be a subanalytic continuous mapping. If $|d_x h|$ (which exists almost everywhere) is bounded on every bounded subset of X then there exist two stratifications Σ and Σ', of X and Y respectively, making of $h : (X, \Sigma) \to (Y, \Sigma')$ a horizontally C^1 stratified mapping.*

Proof Let $\pi_1 : \Gamma_h \to X$ (resp. $\pi_2 : \Gamma_h \to Y$) be the projection onto the source (resp. target) space of h. Take Whitney (a) regular stratifications Σ_h and Σ' of Γ_h and Y respectively such that $\pi_2 : (\Gamma_h, \Sigma_h) \to (Y, \Sigma')$ is a stratified mapping.

Let Σ be the stratification of X constituted by the respective images of the strata of Σ_h under the mapping π_1. Observe that $h : (X, \Sigma) \to (Y, \Sigma')$ is a stratified mapping.

In order to show that h is horizontally C^1 (with respect to Σ), fix a stratum S of Σ, a sequence $x_l \in S$ tending to a point x belonging to a stratum $S' \in \Sigma$, as well as a sequence $u_l \in T_{x_l} S$ tending to some $u \in T_x S'$.

Let Z be the stratum of Σ_h that projects onto S via π_1 and set (extracting a subsequence if necessary, we may assume that this sequence is convergent)

$$\tau := \lim T_{(x_l, h(x_l))} Z.$$

Claim. The restriction of π_1 to τ is one-to-one.

To see this, observe that, as $(\Gamma_h)_{reg}$ is dense in Γ_h, for every l we can find an element $y_l \in (\Gamma_h)_{reg}$ close to $(x_l, h(x_l))$. For every l, let Z^l be the stratum of Σ_h containing y_l (choosing y_l sufficiently generic, we may assume that Z^l is open in Γ_h). By the Whitney (a) condition, the angle between $T_{(x_l, h(x_l))} Z$ and $T_{y_l} Z^l$ is small if y_l is chosen sufficiently close from $(x_l, h(x_l))$. It means that we can assume that $T_{y_l} Z^l$ tends to a limit τ' which contains τ. As h has locally bounded derivative $\pi_{1|\tau'}$ must be one-to-one (the graph of a mapping having a bounded first order derivative may not have a vertical limit tangent vector), yielding the claim.

For every l, there is a unique vector $v_l \in T_{(x_l, h(x_l))} Z$ which projects onto u_l. The above claim implies that the norm of v_l is bounded (since otherwise we would have $\lim \pi_1(\frac{v_l}{|v_l|}) = 0$) and we may assume that v_l is converging to a vector v. The vector v then necessarily projects onto u.

Let Z' be the stratum of Σ_h that projects onto S' and let w be the vector tangent to Z' at $(x, h(x))$ which projects onto u. By the Whitney (a) condition $(v - w) \in \tau$. Therefore, since $(w - v)$ lies in the kernel of $\pi_{1|\tau}$, it must be zero (by the above claim). Hence, v is tangent to Z', which entails that:

$$\lim d_{x_l} h_{|S}(u_l) = \lim \pi_2(v_l) = \pi_2(v) = d_x h_{|S'}(u).$$

□

Remark 3.9 In the situation of the above lemma, take in addition finitely many subanalytic subsets $A_1, \dots, A_k$ of X. Since the stratification of X given by the above lemma is provided by a Whitney stratification of the graph of h, we see that we can require the A_i's to be unions of strata of this stratification.

3.3 Stratified and weakly differentiable forms

We give the definition of stratified forms and then prove that these forms naturally give rise to weakly differentiable forms.

Definition 3.10 Let $X \subset \mathbb{R}^n$ be subanalytic and let Σ be a stratification of X.

A *stratified differential* 0*-form on* (X, Σ) is a collection of functions $\omega_S : S \to \mathbb{R}$, $S \in \Sigma$, that glue together into a continuous function on X.

A *stratified differential* j*-form on* (X, Σ), $j > 0$, is a collection $(\omega_S)_{S\in\Sigma}$ where, for every S, ω_S is a continuous differential j-form on S such that for any sequence (x_i, ξ_i), $i \in \mathbb{N}$, with x_i tending to some $x \in S' \in \Sigma$ and ξ_i tending to some $\xi \in \otimes^j T_x S'$ we have

$$\lim \omega_S(x_i, \xi_i) = \omega_{S'}(x, \xi).$$

We say that $\omega = (\omega_S)_{S\in\Sigma}$ is *differentiable* if ω_S is C^1 for every $S \in \Sigma$ and if $d\omega := (d\omega_S)_{S\in\Sigma}$ is a stratified form.

The integral of a stratified form on X is then defined as the sum of the respective integrals of the corresponding forms on the top dimensional strata (see [30,31] for details). An interesting feature of stratified forms is to admit a Stokes' formula, which we recall now because it will be useful for our purpose.

Proposition 3.11 *[30,31] Let $X \subset \mathbb{R}^n$ be a subanalytic C^0 compact oriented m-dimensional manifold with boundary ∂X and let Σ be a stratification of X. For any differentiable stratified m-form $\omega := (\omega_S)_{S\in\Sigma}$ on X we have:*

$$\int_X d\omega = \int_{\partial X} \omega.$$

We now are going to introduce the weakly differentiable forms. Let us, for the remaining part of this section, fix a subanalytic C^∞ submanifold M of $\mathbb{R}^n$. This manifold is not assumed to be orientable. For each k, $\mathbb{R}^k$ will be assumed to be endowed with the orientation given by the canonical basis.

We say that a differential form ω on M is L^1_{loc} if the restriction of ω to every compact subset of M is L^1.

Definition 3.12 Let U be an open subset of $\mathbb{R}^m$. An L^1_{loc} differential j-form α on U is called *weakly differentiable* if there exists an L^1_{loc} $(j+1)$-form ω on U such that for any form $\varphi \in \Lambda_0^{m-j-1}(U)$:

$$\int_U \alpha \wedge d\varphi = (-1)^{j+1} \int_U \omega \wedge \varphi.$$

The form ω is then called *the weak exterior differential of* α and we write $\omega = \overline{d}\alpha$. A form ω on M is *weakly differentiable* if it gives rise to such a form via the coordinate systems of M (and the weak exterior differential is then obtained by pulling-back the corresponding weak exterior differential).

Lemma 3.13 *Let $\alpha := (\alpha_S)_{S\in\Sigma}$ be a stratified j-form, where Σ is a stratification of M. If α is differentiable then it is weakly differentiable in the sense that the form α' defined by $\alpha'(x) := \alpha_S(x)$, for $S \in \Sigma$, $\dim S = \dim M$, and $x \in S$ (this form is defined almost everywhere) is weakly differentiable.*

Proof As the problem is local (we can use a partition of unity), we can assume that M is a small open ball $B(0_{\mathbb{R}^m}, \varepsilon)$, and consequently that $cl(M)$ is a manifold with boundary $S(0_{\mathbb{R}^m}, \varepsilon)$. We are going to show that α' is weakly differentiable and that its weak exterior differential is the form α'' defined (almost everywhere) by $\alpha''(x) := d\alpha_S(x)$, for $x \in S \in \Sigma$ and $\dim S = m := \dim M$. Let $\varphi \in \Lambda^{m-j-1}_{or}(M)$ and observe that the form $\beta := (\beta_S)_{S\in\Sigma}$ defined by $\beta_S(x) := \alpha_S(x)\wedge\varphi(x)$ is a differentiable stratified form and that its exterior differential is the stratified form $(d\alpha_S \wedge \varphi + (-1)^j \alpha_S \wedge d\varphi)_{S\in\Sigma}$. It thus suffices to establish that $\int_M d\beta = 0$.

Take a stratification Σ' of $cl(M)$ such that all the strata of Σ are unions of strata of Σ' and such that $S(0_{\mathbb{R}^m}, \varepsilon)$ is a union of strata. As φ has compact support in M, the form β gives rise to a stratified form on Σ' which is identically zero on the strata of $S(0_{\mathbb{R}^m}, \varepsilon)$. The required equality then follows from Proposition 3.11. ◻

3.4 Regularizing operators

Since the pull-back of a smooth form under a subanalytic bi-Lipschitz mapping is a stratified form, it is not necessarily smooth but just weakly differentiable. We will thus have to regularize the nonsmooth L^p forms that will appear. In this section, we recall the properties of the de Rham regularizing operators together with the L^p estimates obtained in [15].

The letter M still stands for a smooth subanalytic submanifold of $\mathbb{R}^n$ (although the subanalytic character of the underlying manifold is actually not needed in this section). We denote by $\overline{\Omega}^j(M)$ the space of weakly differentiable j-forms on M. We also set $\overline{\Omega}(M) = \bigoplus_{j\le m} \overline{\Omega}^j(M)$.

For every $p \in [1, \infty)$, we denote by $\overline{\Omega}^j_p(M)$ the set of weakly differentiable j-forms which are L^p and which have an L^p weak exterior differential. Together with $\overline{d}$, these $\mathbb{R}$-vector spaces constitute a cochain complex. We denote by $\overline{H}^j_p(M)$ the resulting cohomology groups.

Theorem 3.14 *There exist sequences of linear operators R_i and A_i, $i \in \mathbb{N}$, on $\overline{\Omega}(M)$ satisfying the following properties:*

(1) *If $\omega \in \overline{\Omega}^j(M)$ then $R_i\omega$ and $A_i\omega$ respectively belong to $\Omega^j(M)$ and $\overline{\Omega}^{j-1}(M)$ and satisfy:*

$$R_i\omega - \omega = \overline{d}A_i\omega + A_i\overline{d}\omega. \tag{2.15}$$

(2) *If ω is L^p, $A_i\omega$ and $R_i\omega$ are L^p as well, $1 \le p \le \infty$.*
(3) *For each L^p form ω, $1 \le p < \infty$, $R_i\omega \to \omega$ and $A_i\omega \to 0$ for the L^p norm, as $i \to \infty$.*
(4) *If W is a neighborhood of the support of $\omega \in \overline{\Omega}(M)$ then the supports of $R_i\omega$ and $A_i\omega$ are included in W for all i sufficiently large.*

This theorem comes down from the main theorem of [15] (see also [16], section 12). The construction of R_i and A_i is however due to de Rham [11].

Remark 3.15 The following observations will be useful. Let $\omega \in \overline{\Omega}(M)$.

(i) Applying (2.15) to $\overline{d}\omega$ we get

$$R_i\overline{d}\omega - \overline{d}\omega = \overline{d}A_i\overline{d}\omega.$$

Moreover, applying $\overline{d}$ to (2.15), we see that $dR_i\omega - \overline{d}\omega = \overline{d}A_i\overline{d}\omega$. Together with the preceding equality, this entails:

$$R_i\overline{d}\omega = dR_i\omega. \tag{2.16}$$

(ii) If $\overline{d}\omega$ is L^p, for some $p \in [1, \infty]$, then, by (2.16) and (3.14) of Theorem 3.14, $dR_i\omega = R_i\overline{d}\omega$ is L^p. Moreover, in the case where ω is L^p as well, by (2.15), we can then conclude that $\overline{d}A_i\omega$ is L^p.

The following consequence of the existence of regularizing operators will be needed to establish our de Rham theorems for L^p cohomology:

Corollary 3.16 *[15,16] For all $p \in [1, \infty]$, the inclusions $\Omega^j_p(M) \hookrightarrow \overline{\Omega}^j_p(M)$, $j \in \mathbb{N}$, induce isomorphisms in cohomology.*

Proof Fix $p \in [1, \infty]$ and, for $j \in \mathbb{N}$, denote by $\Lambda^j : H^j_p(M) \to \overline{H}^j_p(M)$ the mapping induced by the inclusion between the two cochain complexes. Let $R_i : \overline{\Omega}(M) \to \Omega(M)$ be the regularizing operator provided by Theorem 3.14. By Remark 3.15, R_i induces for every j a mapping $R^j_i : \overline{H}^j_p(M) \to H^j_p(M)$, which, due to (2.15), is nothing but the inverse of Λ^j. □

3.5 Weakly differentiable forms and subanalytic Lipschitz mappings

We now show that subanalytic Lipschitz mappings (not necessarily smooth) induce natural mappings on differential forms, which will entail that bi-Lipschitz subanalytic mappings naturally induce isomorphisms in L^p cohomology (Proposition 3.18). Let us here emphasize that, although differentiable forms are not required to be subanalytic, the subanalytic character of the mappings is essential. The letter M still stands for a subanalytic smooth submanifold of $\mathbb{R}^n$.

Proposition 3.17 *Let $h : M \to M'$ be a Lipschitz subanalytic mapping, with $M' \subset \mathbb{R}^k$ smooth submanifold. For every smooth form ω on M', the form $h^*\omega$ (which is* *well defined almost everywhere) is weakly differentiable and satisfies* $\overline{d}h^*\omega = h^*d\omega$, *almost everywhere.*

Moreover, if h *is locally bi-Lipschitz then the same conclusion holds for every weakly differentiable form* ω *on* M'.

Proof By Lemma 3.8, h is horizontally C^1 with respect to some stratifications of M and M'. For every $\omega \in \Omega^j(M')$, $h^*\omega$ thus gives rise to a differentiable stratified form on this stratification. By Lemma 3.13, this means that $h^*\omega$ is weakly differentiable, and thanks to Stokes' formula for stratified forms (Proposition 3.11), the formula $\overline{d}h^*\omega = h^*d\omega$ easily follows by integration by parts.

To prove the last statement, fix a form $\omega \in \overline{\Omega}^j(M')$, $j \in \mathbb{N}$ as well as an open subset U of M' on which ω is L^1. As $\Omega_1^j(U)$ is dense in $\overline{\Omega}_1^j(U)$, we can find a sequence $\omega_i \in \Omega_1^j(U)$ such that $\omega_i \to \omega$ and $d\omega_i \to \overline{d}\omega$ for the L^1 norm. If h is bi-Lipschitz, this implies that $h^*\omega_i$ tends to $h^*\omega$ and that $h^*d\omega_i$ tends to $h^*\overline{d}\omega$ for this norm. Moreover, since we have proved that the result holds true in the smooth case, we also know that for every $\varphi \in \Lambda_{or}^{m-j-1}(U)$

$$\int_U h^*\omega_i \wedge d\varphi = (-1)^{j+1} \int_U d\omega_i \wedge \varphi.$$

Passing to the limit as $i \to \infty$, we get the desired result. □

This leads us to the subanalytic bi-Lipschitz invariance of L^p cohomology:

Proposition 3.18 *Let* $j \in \mathbb{N}$ *and* $p \in [1, \infty]$. *If* $h : M \to M'$ *is a subanalytic bi-Lipschitz mapping, with* $M' \subset \mathbb{R}^{n'}$ *smooth submanifold, then* $H_p^j(M) \simeq H_p^j(M')$.

Proof By Proposition 3.17 and Corollary 3.16, h induces an isomorphism between $H_p^j(M)$ and $H_p^j(M')$ for all j and all p. □

Similarly, subanalytic Lipschitz homotopies (not necessarily differentiable) induce operators on smooth forms, like in the case of smooth homotopies.

Proposition 3.19 *Let* $h : [0, 1] \times M \to M$ *be a subanalytic Lipschitz homotopy and let* ∂_t *denote the constant vector field* $(1, 0)$ *on* $[0, 1] \times M$. *If we set for* $x \in M$ *and* $\omega \in \Omega^j(M)$*:*

$$\mathcal{H}\omega(x) := \int_0^1 (h^*\omega)_{\partial_t}(t, x)dt,$$

then we have

$$\overline{d}\mathcal{H}\omega + \mathcal{H}d\omega = h_1^*\omega - h_0^*\omega, \tag{2.17}$$

where $h_i : M \to M$, $i = 0, 1$, *is defined by* $h_i(x) = h(i, x)$.

Proof Proposition 3.17 implies that $h^*\omega$ is weakly differentiable and that $\overline{d}h^*\omega = h^*d\omega$. Moreover, thanks to Lemma 3.8, $h^*\omega$ gives rise to a stratified form.

Thanks to our Stokes' formula for stratified forms, we now can end the proof with an integration by parts. Namely, if $\varphi \in \Lambda^{m-j}_{or}(M)$, regarding it as a form on $[0,1] \times M$ constant with respect to t, we can write (for relevant orientations):

$$\begin{aligned} \int_M \mathcal{K}\omega \wedge d\varphi &= \int_{[0,1]\times M} h^*\omega \wedge d\varphi \\ &= (-1)^j \int_{[0,1]\times M} \overline{d}(h^*\omega \wedge \varphi) \\ &\quad -(-1)^j \int_{[0,1]\times M} \overline{d}h^*\omega \wedge \varphi. \end{aligned} \tag{2.18}$$

By Stokes' formula for stratified forms, we also have:

$$\int_{[0,1]\times M} \overline{d}(h^*\omega \wedge \varphi) = \int_M h_1^*\omega \wedge \varphi - \int_M h_0^*\omega \wedge \varphi.$$

Together with (2.18), this yields the desired equality. □

4 Some operators on L^p forms

We are going to define some operators on L^p forms on subanalytic varieties which will be useful to establish our Poincaré Lemma for L^p cohomology (Lemma 5.5). The usual Poincaré Lemma is devoted to smooth forms on an open ball or more generally on the so called star-shaped domains. On this kind of domains, it is well-known that some retractions by deformation give rise to differential operators on forms. The local conic structure given in Theorem 3.5 will make it possible to define some operators by the same process as on the star-shaped domains. We start by defining them and then study their properties.

We fix for all this section an m-dimensional subanalytic submanifold M of $\mathbb{R}^n$. Set $X := cl(M)$, fix $x_0 \in X$, and apply Theorem 3.5 to the germ of X at x_0. This provides a positive real number ε as well as a Lipschitz subanalytic homeomorphism

$$H : x_0 * (S(x_0,\varepsilon) \cap X) \to \overline{B}(x_0,\varepsilon) \cap X,$$

preserving the distance to x_0 and satisfying conditions (i) and (ii) of the latter theorem.

For simplicity, we then set for this section

$$N_{x_0} = S(x_0,\varepsilon) \cap M$$

and

$$U_{x_0} = B(x_0,\varepsilon) \cap M.$$

We can assume (see Remark 3.6) that H maps the open cone $(x_0 * N_{x_0}) \setminus N_{x_0} \cup \{x_0\}$ onto U_{x_0}. In particular, H gives rise to a globally subanalytic homeomorphism:

$$h : (0,1) \times N_{x_0} \to U_{x_0}, \qquad (t,x) \mapsto h(t,x) := H(tx). \tag{3.19}$$

For simplicity, we also set

$$Z_{x_0} = (0,1) \times N_{x_0}.$$

4.1 The operator $\mathcal{K}_\nu$, $\nu > 0$

Given a vector field ξ and a j-form ω on a manifold P, we denote by ω_ξ the differential $(j-1)$-form defined by $\omega_\xi(x)(\zeta) := \omega(x)(\xi(x) \otimes \zeta)$ for $x \in P$ and $\zeta \in \otimes^{j-1} T_x P$.

Denote by ∂_t the constant vector field $(1,0)$ on $(0,1) \times N_{x_0}$ and fix an L^1 differential j-form ω on U_{x_0} with $j \geq 1$. We first define a differential form $\mathcal{H}_\nu \omega$ by setting for almost every $(t,y) \in (0,1) \times N_{x_0}$ and $0 < \nu \leq 1$:

$$\mathcal{H}_\nu \omega(t,y) = \int_\nu^t (h^*\omega)_{\partial_t}(s,y)ds. \tag{3.20}$$

The desired operator is then defined by pushing forward this differential form by means of h:

$$\mathcal{K}_\nu \omega = h^{-1*} \mathcal{H}_\nu \omega. \tag{3.21}$$

This defines an operator $\mathcal{K}_\nu$ on L^1 differential forms for every $\nu \in (0,1]$.

4.2 The operator $\mathcal{K}_0$

The case $\nu = 0$ is more delicate since we are not sure that the mapping $t \mapsto (h^*\omega)_{\partial_t}(t,y)$ is L^1 on $[0,1]$. This fact is however clearly true if ω is an L^∞ form. The proposition below shows that we actually can define $\mathcal{K}_0 \omega$ analogously when ω is an L^p form with p sufficiently big.

Proposition and Definition 4.1 *For $p \in [1,\infty]$ sufficiently big, the form $(h^*\omega)_{\partial_t}$ is L^1 on $(0,1) \times N_{x_0}$ for every L^p differential j-form ω, $j \geq 1$.*

For such p and ω, the differential form

$$\mathcal{H}_0 \omega(t,y) := \int_0^t (h^*\omega)_{\partial_t}(s,y)ds \tag{3.22}$$

is thus (almost everywhere on $(0,1) \times N_{x_0}$) well-defined and we can set

$$\mathcal{K}_0 \omega = h^{-1*} \mathcal{H}_0 \omega.$$

Proof The function $(s, x) \mapsto \operatorname{jac} h(s, x)$ (this Jacobian is well defined on a subanalytic dense subset of Z_{x_0}) is globally subanalytic. As h is bi-Lipschitz above the complement of every neighborhood of the origin, $\operatorname{jac} h(s, x)$ can only tend to zero when s goes to zero. Therefore, by Łojasiewicz's inequality (see (2.2)), there is a positive integer k and a constant C such that for $(s, x) \in (0, 1) \times N_{x_0}$

$$s^k \leq C \operatorname{jac} h(s, x). \tag{3.23}$$

We are going to prove that $(h^*\omega)_{\partial_t}$ is L^1 for all L^p forms ω when $p > k + 1$. Fix such a form ω and such a real number p.

Since h has bounded first derivative, it is enough to show that $\omega \circ h$ is L^1. For this purpose, let us notice that since ω is L^p, so is $|\omega \circ h| \cdot (\operatorname{jac} h)^{\frac{1}{p}}$. It thus suffices to show that $(\operatorname{jac} h)^{-\frac{1}{p}}$ is L^q, where $q \geq 1$ is the Hölder conjugate of p. To prove this, write

$$\int_{Z_{x_0}} (\operatorname{jac} h)^{-\frac{q}{p}} \overset{(3.23)}{\lesssim} \int_{N_{x_0}} \int_0^1 s^{-\frac{kq}{p}} ds \lesssim \int_0^1 s^{-\frac{kq}{p}} ds = \int_0^1 s^{-\frac{k}{p-1}} ds < \infty,$$

since $k < p - 1$. This establishes that $\omega \circ h$ is L^1, which yields that so is $(h^*\omega)_{\partial_t}$. □

4.3 The homotopy ρ_ν

Given $\nu \in [0, 1]$, we can define a homotopy $\rho_\nu : (0, 1] \times U_{x_0} \to U_{x_0}$ as follows. Let for $(t, s, y) \in (0, 1) \times (0, 1) \times N_{x_0}$

$$\theta_\nu(t, s, y) := h(ts + (1 - t)\nu, y). \tag{3.24}$$

We then push-forward θ_ν by means of h^{-1} by setting for $(t, x) \in (0, 1] \times U_{x_0}$,

$$\rho_\nu(t, x) := \theta_\nu(t, h^{-1}(x)).$$

As the homeomorphism H (used at the beginning of this section to define $\mathcal{K}_\nu$) was assumed to send the open cone $(x_0 * N_{x_0}) \setminus (N_{x_0} \cup \{x_0\})$ onto U_{x_0}, we see that this homotopy stays in U_{x_0} for all $t \in (0, 1]$. Notice also that it follows from Theorem 3.5 that this mapping is locally Lipschitz near every point of $(0, 1] \times U_{x_0}$.

Remark also that for every $x \in U_{x_0}$, $\rho_0(t, x)$ coincides with $r_t(x)$, where r is the mapping given in the latter theorem (although the mapping r is defined on X, we will regard it in the sequel as a mapping from $(0, 1] \times U_{x_0}$ into U_{x_0} and r_t as a mapping from U_{x_0} to itself for all $t \in (0, 1]$).

Let us here stress the fact that Theorem 3.5 ensures that there is a constant C such that for all $t \in (0, 1]$, the mapping $r_t : U_{x_0} \to U_{x_0}$ is Ct-Lipschitz. Moreover, for every $t \in (0, 1]$, the mapping r_t is bi-Lipschitz.

Note also that since H preserves the distance to x_0, we have for all $(t, x) \in (0, 1] \times U_{x_0}$:

$$|r_t(x) - x_0| = t|x - x_0|. \tag{3.25}$$

The next proposition provides an alternative definition of the operator $\mathcal{K}_0$ using r. This kind of computation is of course very classical. As r is Lipschitz, this characterization will be helpful to estimate the L^p norm of $\mathcal{K}_0\omega$ in Sect. 4.5.

Proposition 4.2 *For every L^1 form ω on U_{x_0}, we have for each $\nu \in (0,1]$:*

$$\mathcal{K}_\nu\omega(x) = \int_0^1 (\rho_\nu^*\omega)_{\partial_t}(t,x)dt, \tag{3.26}$$

where ∂_t is the constant vector field $(1,0)$ on $[0,1]\times U_{x_0}$.

Moreover, if $r:(0,1)\times U_{x_0}\to U_{x_0}$ is the just above defined mapping, we have for each $p\in[1,\infty]$ large enough, each L^p form ω, and each $x\in U_{x_0}$:

$$\mathcal{K}_0\omega(x) = \int_0^1 (r^*\omega)_{\partial_t}(t,x)dt, \tag{3.27}$$

Proof Fix $(t,y)\in Z_{x_0}$ and $\nu\in(0,1]$. Making the substitution $s=ut+(1-u)\nu$, $u\in[0,1]$, in the integral defining $\mathcal{H}_\nu\omega(t,y)$ (equality (3.20)) we obtain

$$\mathcal{H}_\nu\omega(t,y) = t\int_0^1 (h^*\omega)_{\partial_t}(ut+(1-u)\nu,y)du = \int_0^1 (\theta_\nu^*\omega)_{\partial_u}(u,t,y)du,$$

where ∂_u is the constant vector field $(1,0,0)$ on $[0,1]^2\times U_{x_0}$ (and θ_ν is as in (3.24)). As $\mathcal{K}_\nu = h^{-1*}\mathcal{H}_\nu$ and $\rho_\nu(u,x)=\theta_\nu(u,h^{-1}(x))$ for all u, after a pull-back of by means of h^{-1}, we get (3.26).

Observe that if the necessary integrability conditions are satisfied then the above computation applies in the case where $\nu=0$ as well. Hence, since $\rho_0(t,x)=r(t,x)$, this argument yields (3.27) for all $p\in[1,\infty]$ sufficiently large for the conclusion of Proposition 4.1 to hold (i.e., for the form which is integrated in (3.22) to be L^1). □

4.4 $\mathcal{K}_\nu$ and weakly differentiable forms

Given $\nu\in(0,1)$, let $\pi_\nu := h\circ P_\nu\circ h^{-1}$, where $P_\nu(t,x):=(\nu,x)$.

Proposition 4.3 *For all $\omega\in\Omega_1^j(U_{x_0})$, $j\ge 1$, and all $\nu\in(0,1)$, the form $\mathcal{K}_\nu\omega$ is weakly differentiable and we have:*

$$\overline{d}\mathcal{K}_\nu\omega + \mathcal{K}_\nu d\omega = \omega - \pi_\nu^*\omega. \tag{3.28}$$

In particular, if ω is equal to zero in the vicinity of N_{x_0} then we have:

$$\overline{d}\mathcal{K}_1\omega + \mathcal{K}_1 d\omega = \omega. \tag{3.29}$$

Proof Equality (3.28) follows from (2.17) and (3.26). The second statement follows from the fact that, if ω is equal to zero in the vicinity of N_{x_0} then $\pi_\nu^*\omega$ vanishes and $\mathcal{K}_\nu\omega=\mathcal{K}_1\omega$, for all ν close to 1. □

We wish to establish an analogous result in the case $\nu = 0$ for p sufficiently large (Proposition 4.5). This is a bit more delicate since the forms are not defined at x_0. For simplicity, we set for $\omega \in \overline{\Omega}^j(M)$ and $\varphi \in \Lambda_{or}^{m-j}(M)$

$$< \omega, \varphi >:= \int_M \omega \wedge \varphi.$$

We shall need the following fact.

Proposition 4.4 *For p large enough we have for each L^p j-form ω on U_{x_0}, $j \geq 1$, and each $\varphi \in \Lambda_{or}^{m-j+1}(U_{x_0})$*

$$\lim_{\nu \to 0} < \mathcal{K}_\nu \omega, \varphi >=< \mathcal{K}_0 \omega, \varphi > .$$

Proof Take p large enough for the conclusion of Proposition 4.1 to hold and fix an L^p j-form ω on U_{x_0}, $j \geq 1$. Since h is bi-Lipschitz on the preimage of the support of any compactly supported form φ, it suffices to establish that $\mathcal{H}_\nu \omega$ tends to $\mathcal{H}_0 \omega$ for the L^1 norm. Remark that $\mathcal{H}_\nu \omega$ tends to $\mathcal{H}_0 \omega$ pointwise. As a matter of fact, since

$$|\mathcal{H}_\nu \omega(t, x)| = |\int_\nu^t (h^*\omega)_{\partial_t}(s, x) ds| \leq \int_0^1 |(h^*\omega)_{\partial_t}(s, x)| ds$$

which is L^1 on Z_{x_0} (and constant with respect to t), the result follows from Lebesgue's dominated convergence theorem. □

Proposition 4.5 *For p large enough and $j \geq 1$, we have for all $\omega \in \Omega_p^j(U_{x_0})$:*

$$\overline{d} \mathcal{K}_0 \omega + \mathcal{K}_0 d\omega = \omega. \tag{3.30}$$

Proof For $\nu \in (0, 1)$, define a mapping $h_\nu : N_{x_0} \to N_{x_0}$ by $h_\nu(x) := h(\nu, x)$. If ν remains bounded below away from zero then so does the function $(\nu, x) \mapsto \operatorname{jac} h_\nu(x)$. Consequently, by Łojasiewicz's inequality (see 2.2), there exists a rational number k such that (almost everywhere) on $(0, 1) \times N_{x_0}$ we have:

$$\nu^k \lesssim \operatorname{jac} h_\nu(x). \tag{3.31}$$

Fix $p \geq k + 1$ sufficiently large for the conclusion of Proposition 4.4 to hold and take a differential form $\omega \in \Omega_p^j(U_{x_0})$. We have to prove that for all $\varphi \in \Lambda_{or}^{m-j}(U_{x_0})$:

$$< \mathcal{K}_0 \omega, d\varphi >=< \omega, \varphi > - < \mathcal{K}_0 d\omega, \varphi > . \tag{3.32}$$

Fix such a differential form φ. As ω and $\overline{d}\omega$ are L^1, by Proposition 4.3, we know that for all $\nu \in (0, 1)$

$$< \mathcal{K}_\nu \omega, d\varphi >=< \omega, \varphi > - < \pi_\nu^* \omega, \varphi > - < \mathcal{K}_\nu d\omega, \varphi > .$$

Moreover, applying Proposition 4.4 to both ω and $d\omega$, we see that

$$\lim_{\nu\to 0} < \mathcal{K}_\nu\omega, d\varphi >=< \mathcal{K}_0\omega, d\varphi > \quad \text{and} \quad \lim_{\nu\to 0} < \mathcal{K}_\nu d\omega, \varphi >=< \mathcal{K}_0 d\omega, \varphi > .$$

As a matter of fact (3.32), reduces to show that there is a sequence ν_i tending to zero such that

$$\lim_{i\to+\infty} < \pi_{\nu_i}^*\omega, \varphi >= 0. \tag{3.33}$$

For simplicity, set

$$\theta(\nu) := \int_{z\in N_{x_0}} |\omega(h_\nu(z))|^p.$$

Observe first that by definition of θ we have

$$\int_0^1 \nu^k\theta(\nu)d\nu \overset{(3.31)}{\lesssim} \int_0^1\int_{z\in N_{x_0}} |\omega(h_\nu(z))|^p \operatorname{jac} h_\nu(z)d\nu = |\omega|_p^p < \infty,$$

which means that the function $\nu^k\theta(\nu)$ belongs to $L^1((0,1))$. Since $p \geq k+1$, this implies that there exists a sequence of positive numbers ν_i tending to zero such that

$$\lim_{\nu\to 0} \nu_i^p\theta(\nu_i) = 0 \tag{3.34}$$

(for if we had $\nu^k\theta(\nu) \geq \frac{\eta}{\nu}$, for some $\eta > 0$ and all $\nu > 0$ small, then $\nu^k\theta(\nu)$ could not be L^1). Denote by K the support of φ. We claim that

$$\lim_{i\to\infty} |\pi_{\nu_i}^*\omega_{|K}|_1 = 0. \tag{3.35}$$

Proving this claim will yield (3.33).

Since K is compact, there is a positive real number s such that $K \subset h([s,1)\times N_{x_0})$. By definition of r, for every $\nu \in (0,1)$ and $x \in K$ we have $\pi_\nu(x) = r_\mu(x)$, where $\mu = \frac{\nu}{|x-x_0|}$. Thanks to (i) of Theorem 3.5, we deduce that π_ν is $C\nu$-Lipschitz on K for some constant C independent of $\nu \in (0,1)$. Hence, for $x \in K$ and $\nu \in (0,1)$ we have (since $j \geq 1$)

$$|\pi_\nu^*\omega(x)| \lesssim \nu^j|\omega(\pi_\nu(x))| \leq \nu|\omega(\pi_\nu(x))|. \tag{3.36}$$

We thus get for $\nu \in (0,s)$:

$$|\pi_\nu^*\omega_{|K}|_1 \lesssim \int_{x\in K} \nu|\omega(\pi_\nu(x))| \leq \left(\int_{x\in K} \nu^p|\omega(\pi_\nu(x))|^p\right)^{\frac{1}{p}} \mathcal{H}^m(K)^{\frac{1}{q}},$$

by Hölder's inequality. Making the substitution $y = h^{-1}(x)$ in the last integral, this entails (since $h_\nu = \pi_\nu \circ h$)

$$|\pi_\nu^* \omega_{|K}|_1 \lesssim \Big(\int_{y \in h^{-1}(K)} \nu^p |\omega(h_\nu(y))|^p \mathrm{jac}\, h(y) \Big)^{\frac{1}{p}} \lesssim \Big(\int_{y \in h^{-1}(K)} \nu^p |\omega(h_\nu(y))|^p \Big)^{\frac{1}{p}}.$$

We therefore can conclude that for $i \in \mathbb{N}$

$$|\pi_{\nu_i}^* \omega_{|K}|_1 \lesssim \Big(\int_s^1 \int_{z \in N_{x_0}} \nu_i^p |\omega(h_{\nu_i}(z))|^p \, dt \Big)^{\frac{1}{p}} = \Big((1-s) \nu_i^p \theta(\nu_i) \Big)^{\frac{1}{p}},$$

which tends to zero (by choice of the sequence ν_i, see (3.34)). This establishes (3.35), which yields in turn (3.33). □

4.5 L^p bounds

Proposition 4.6 *There is a constant C such that for any large enough p we have for each L^p j-form ω, $j \geq 1$, on U_{x_0}:*

$$|\mathcal{K}_0 \omega|_p \leq C |\omega|_p. \tag{3.37}$$

Proof Since r_s is bi-Lipschitz for each $s > 0$ (see section 4.3 for r_s), the function $(s, x) \mapsto \mathrm{jac}\, r_s(x)$ (defined on a subanalytic dense subset of $[0, 1] \times U_{x_0}$) can only tend to zero when s goes to zero. Consequently, by Łojasiewicz's inequality (see Proposition 2.3), there is a positive integer k and a constant C such that for almost all $(s, x) \in (0, 1) \times U_{x_0}$

$$s^k \leq C \ \mathrm{jac}\, r_s(x). \tag{3.38}$$

We shall establish (3.37) for all $p \in (k+1, \infty]$.

Let p be a real number greater than $(k+1)$ (we postpone the case $p = \infty$) and let ω be an L^p j-form on U_{x_0}, $j \geq 1$. We shall estimate $|\mathcal{K}_0 \omega|_p$ using (3.27). For this purpose, we first estimate the L^p norm of $\omega \circ r_s$. Indeed, setting $y = r_s(x)$, we see that

$$\begin{aligned} |\omega \circ r_s|_p &= \Big(\int_{x \in U_{x_0}} |\omega(r_s(x))|^p \Big)^{\frac{1}{p}} \\ &= \Big(\int_{y \in r_s(U_{x_0})} |\omega(y)|^p \cdot \mathrm{jac}\, r_s^{-1}(y) \Big)^{\frac{1}{p}}, \end{aligned} \tag{3.39}$$

which, by (3.38), yields that

$$|\omega \circ r_s|_p \leq C^{\frac{1}{p}} \cdot s^{\frac{-k}{p}} \cdot |\omega|_p. \tag{3.40}$$

Now, as r_s^* has bounded derivative (by a constant independent of s) we have $|r_s^*\omega(x)| \le C'|\omega(r_s(x))|$, for some constant C' independent of x and s. By (3.27), we deduce

$$|\mathcal{K}_0\omega(x)| \le C' \int_0^1 |\omega(r_s(x))| ds, \tag{3.41}$$

which, thanks to Minkowski's inequality, entails that

$$|\mathcal{K}_0\omega|_p \le C' \int_0^1 |\omega \circ r_s|_p ds \overset{(3.40)}{\le} C^{\frac{1}{p}} C' |\omega|_p \int_0^1 s^{-\frac{k}{p}} ds,$$

showing that $\mathcal{K}_0$ is bounded for the L^p norm independently of p (since $p > k+1$).

In the case $p = \infty$, it immediately follows from (3.41) that

$$|\mathcal{K}_0\omega|_\infty \le C' \int_0^1 |\omega|_\infty \, ds = C'|\omega|_\infty,$$

for each L^∞ j-form ω on U_{x_0}. Hence, the result is clear in the case $p = \infty$ as well. □

Proposition 4.7 *There is a constant C such that for any p sufficiently close to 1 we have for each L^p form ω on U_{x_0}:*

$$|\mathcal{K}_1\omega|_p \le C|\omega|_p. \tag{3.42}$$

Proof As we can cover M by finitely many orientable manifolds and use a partition of unity, we may assume that U_{x_0} is oriented. Take $\varphi \in \Lambda_0^{m-j+1}(U_{x_0})$ as well as an L^1 j-form ω on U_{x_0}, and observe that we have (for the relevant orientation on N_{x_0})

$$\begin{aligned}
< \omega, \mathcal{K}_0\varphi > &= \int_{U_{x_0}} \omega \wedge \mathcal{K}_0\varphi \\
&= \int_{(0,1)\times N_{x_0}} h^*\omega \wedge \mathcal{H}_0\varphi \qquad \text{(pulling back via } h\text{)} \\
&= \int_{x\in N_{x_0}} \int_0^1 (h^*\omega)_{\partial_t}(t,x) \wedge \mathcal{H}_0\varphi(t,x) dt \\
&= \int_{x\in N_{x_0}} \int_0^1 \left(\int_0^t (h^*\omega)_{\partial_t}(x,t) \wedge (h^*\varphi)_{\partial_s}(s,x) ds \right) dt \quad \text{(by (3.22))} \\
&= \int_{x\in N_{x_0}} \int_{0<s\le t<1} (h^*\omega)_{\partial_t}(t,x) \wedge (h^*\varphi)_{\partial_s}(s,x) ds\, dt.
\end{aligned}$$

Making the same computation for $< \mathcal{K}_1\omega, \varphi >$ and applying Fubini's Theorem, we see that

$$< \omega, \mathcal{K}_0\varphi >= (-1)^j < \mathcal{K}_1\omega, \varphi > .$$

Let now q be a real number sufficiently big for the conclusion of Proposition 4.6 to hold (for every L^q form) and denote by p its Hölder conjugate (which is close to 1). By the above, for every L^p j-form ω on U_{x_0} and each $\varphi \in \Lambda_0^{m-j+1}(U_{x_0})$ we have:

$$| < \mathcal{K}_1\omega, \varphi > | =< \omega, \mathcal{K}_0\varphi > \leq |\mathcal{K}_0\varphi|_q \cdot |\omega|_p \overset{(3.37)}{\leq} C|\varphi|_q \cdot |\omega|_p,$$

which yields (3.42). □

5 Proof of the de Rham theorems

Throughout this section, the letter M will stand for a bounded subanalytic submanifold of $\mathbb{R}^n$ and X for its closure.

5.1 The sheaves

We will conclude by means of a sheaf theoretic argument. The problem is that Ω_p^j is not a sheaf on M. We thus shall work with the sheaf on X of locally L^p forms which has the same global sections (recall that M is not compact).

For $p \in [1, \infty)$ and $U \subset X$ open, let $\mathcal{F}_p^j(U)$ be the $\mathbb{R}$-vector space of the C^∞ j-forms ω on $U \cap M$ for which ω and $d\omega$ are both locally L^p (locally in U, not in $U \cap M$), i.e., those that satisfy for every $x_0 \in U$ and $\varepsilon > 0$ small enough $\int_{B(x_0,\varepsilon)\cap M} |\omega|^p + |d\omega|^p < \infty$ (if $p < \infty$) or (in the case $p = \infty$) $\sup_{x \in B(x_0,\varepsilon)\cap M} |\omega(x)| + |d\omega(x)| < \infty$.

Clearly, $(\mathcal{F}_p^j)_{j\in\mathbb{N}}$ is a complex of sheaves on X for every $p \in [1, \infty]$. Observe that as X is compact, locally L^p is equivalent to L^p, which entails that $\mathcal{F}_p^j(X) = \Omega_p^j(M)$ for all $p \in [1, \infty]$. Note also that all these sheaves are soft and therefore acyclic.

Given an open subset U of X, we will denote by $\mathcal{F}_{p,c}^j(U)$ the sections of $\mathcal{F}_p^j(U)$ that are compactly supported.

Here, it is worthwhile stressing the fact that for $x_0 \in \delta M$ the elements of $\mathcal{F}_{p,c}^j(B(x_0, \varepsilon) \cap X)$ are forms on $B(x_0, \varepsilon) \cap M$ which do not need to be zero near the points of δM. Such forms just have to be zero near $S(x_0, \varepsilon)$. In particular, they are not necessarily compactly supported as forms on $B(x_0, \varepsilon) \cap M$.

Similarly, given an open subset U of X, we will write $\overline{\mathcal{F}}_p^j(U)$ for the space of weakly differentiable locally L^p j-forms on $U \cap M$ that have an L^p weak exterior differential, and $\overline{\mathcal{F}}_{p,c}^j$ for the compactly supported sections of this sheaf. Observe that the elements of $\mathcal{F}_{p,c}^j(U)$ and $\overline{\mathcal{F}}_{p,c}^j$ are L^p forms on $U \cap M$. We have:

Lemma 5.1 *For all $p \in [1, \infty]$ and every open subset U of X, the inclusions $\mathcal{F}_{p,c}^j(U) \hookrightarrow \overline{\mathcal{F}}_{p,c}^j(U)$, $j \in \mathbb{N}$, induce isomorphisms in cohomology.*

Proof If $\omega \in \overline{\mathcal{F}}_{p,c}^j(U)$ then, by Theorem 3.14 (and Remark 3.15), $R_i\omega \in \mathcal{F}_{p,c}^j(U)$ for all i large enough. By (2.15), this implies that the mapping $R : H^j(\overline{\mathcal{F}}_{p,c}^\bullet(U)) \to$

$H^j(\mathcal{F}^\bullet_{p,c}(U))$, defined by $R(\omega) := R_i(\omega)$, for i large enough, is the inverse of the mapping induced by the inclusion between the two cochain complexes. □

We also need to introduce a complex $\mathcal{D}^j_c$ of compactly supported singular oriented cochains in a similar way. Given an open subset V of M and $j \in \mathbb{N}$, let $C^j(V)$ denote the cochain complex of the singular cochains of V. It is a consequence of a well-known subdivision argument that although these presheaves are not sheaves on M, the respective associated sheaves $\mathcal{C}^j$ give rise to the same cohomology groups.

Given now an open subset U of X, we let $\mathcal{D}^j(U) := \mathcal{C}^j(U \cap M)$ and we will denote by $\mathcal{D}^j_c(U)$ the subspace of compactly supported sections.

5.2 The case *p* close to 1

The first step is to prove a Poincaré Lemma for L^p forms with compact support. We show:

Lemma 5.2 *Given x_0 in δM, there is $\varepsilon > 0$ such that for all $p \geq 1$ sufficiently close to 1 and each $j \in \mathbb{N}$, we have:*

$$H^j(\mathcal{F}^\bullet_{p,c}(B(x_0, \varepsilon) \cap X)) \simeq 0.$$

Proof Let ε be some positive real number satisfying the conclusion of Theorem 3.5 (which enables us to define the homotopy operators $\mathcal{K}_\nu$ of Sect. 4). A closed 0-form with compact support being identically zero, the result is clear if $j = 0$. Let us thus fix a closed form $\omega \in \mathcal{F}^j_{p,c}(B(x_0, \varepsilon) \cap X)$ with $j > 0$. As ω is a compactly supported section, by Proposition 4.3, $\mathcal{K}_1\omega$ is a weakly differentiable $(j-1)$-form satisfying $\overline{d}\mathcal{K}_1\omega = \omega$. Furthermore, by Proposition 4.7, it is L^p if p is sufficiently close to 1. By Lemma 5.1, this entails that ω is the derivative of a compactly supported section on $B(x_0, \varepsilon) \cap X$. □

The just above lemma holds *for p sufficiently close to 1* in the sense that there is $p_0 \in (1, \infty]$ such that its statement holds for all $p \in [1, p_0)$. If we define $p_M(x_0)$ as the biggest such real number p_0, this number of course depends on the geometry of X near x_0 and may vary on this set. However, we have:

Lemma 5.3 $\inf_{x_0 \in X} p_M(x_0) > 1$.

Proof If M_{x_0} denotes the germ of M at x_0, the family $(M_{x_0})_{x_0 \in X}$ is globally subanalytic. As a matter of fact, by generic subanalytic bi-Lipschitz triviality (see Theorem 2.2 of [27]), we know that there is a finite partition of X, such that given any two points x_0 and x_0' in the same element of this partition, there is a (germ of) globally subanalytic bi-Lipschitz homeomorphism that maps M_{x_0} onto $M_{x_0'}$. Hence, by Proposition 3.18, p_M can take only finitely many values. □

Given $k \in \mathbb{N}$ and an open subset W of M, let

$$\begin{aligned} \phi^k_W : \Omega^k(W) &\to \mathcal{C}^k(W) \\ \omega &\mapsto [\phi^j_W(\omega) : \sigma \mapsto \int_\sigma \omega]. \end{aligned}$$

Theorem 5.4 *For each $p \geq 1$ sufficiently close to 1 and each $j \in \mathbb{N}$, the mapping ϕ_M^j induces an isomorphism between $H_p^j(M)$ and $H^j(M)$.*

Proof Given an open subset U of X and $p \in [1, \infty)$, we denote by $\lambda_U^j : \mathcal{F}_{p,c}^j(U) \to \mathcal{D}_c^j(U)$ the mapping induced by $\phi_{U \cap M}^j$. It is easily checked from the definitions that for all $p \in [1, \infty)$

$$\mathcal{E}^j(U) := \mathbf{Hom}(\mathcal{D}_c^{m-j}(U), \mathbb{R}) \text{ and } \mathcal{G}_p^j(U) := \mathbf{Hom}(\mathcal{F}_{p,c}^{m-j}(U), \mathbb{R})$$

are complexes of flabby sheaves ($\mathcal{D}_c^j$ and $\mathcal{F}_{p,c}^j$ are sometimes called *cosheaves* in the literature, see for instance [4] Propositions V.1.6 and V.1.10). Moreover, the mappings $\mu_U^j : \mathcal{E}^j(U) \to \mathcal{G}_p^j(U)$, $j \in \mathbb{N}$, defined as the respective adjoints of the λ_U^{m-j}, constitute a morphism of complexes of sheaves.

It thus easily follows from sheaf theory (see for instance [4], section IV, Theorem 2.2) that it is enough to show that for every $x_0 \in X$ and every $\varepsilon > 0$ small enough, the mapping $\lambda_{B(x_0,\varepsilon)\cap X}^j$ is an isomorphism for every j (since the morphisms $\mu_{B(x_0,\varepsilon)\cap X}^j$ are then isomorphisms as well).

If x_0 is a point of M, this is a direct consequence of the usual Poincaré Lemma. We thus can assume that $x_0 \in \delta M$, in which case, it easily comes down from the conic structure of X at x_0 (see Theorem 3.5) that

$$H^j(\mathcal{D}_c^\bullet(B(x_0, \varepsilon) \cap X)) \simeq 0, \quad \text{for all } j,$$

so that the desired result follows from Lemma 5.2 (for all p close to 1, see Lemma 5.3). □

5.3 The case where *p* is large

Fix $x_0 \in X$ and set $U_{x_0} := M \cap B(x_0, \varepsilon)$, where $\varepsilon > 0$ is provided by Theorem 3.5. We now have:

Lemma 5.5 *(Poincaré Lemma for p large) For $p \in [1, \infty]$ large enough, we have for all $j > 0$:*

$$H_p^j(U_{x_0}) \simeq 0.$$

Proof By Corollary 3.16, it is enough to show that if p is sufficiently large then for every closed form $\omega \in \Omega_p^j(U_{x_0})$, $j > 0$, there is $\alpha \in \overline{\Omega}_p^{j-1}(U_{x_0})$ such that $\omega = \overline{d}\alpha$. But, by Propositions 4.1, 4.5 and 4.6, if p is sufficiently large and if ω is such a form then $\alpha := \mathcal{K}_0\omega$ has all the required properties. □

We may here make an observation analogous to the one we made in Lemma 5.3. If we define $q_M(x_0)$ as the smallest real number q_0 such that Lemma 5.5 holds for all $p \in (q_0, \infty]$, the same argument as in the proof of Lemma 5.3 then establishes:

$$\sup_{x_0 \in X} q_M(x_0) < \infty. \tag{4.43}$$

proof of Theorem 2.10 In virtue of Theorem 1.1, it is enough to show that the inclusion of complexes $\mathcal{F}_\infty^j \to \mathcal{F}_p^j$ induces isomorphisms between the respective cohomology groups of the global sections. Since these sheaves are acyclic (these are fine sheaves), it suffices to prove that $(\mathcal{F}_\infty^j)_{j\in\mathbb{N}}$ and $(\mathcal{F}_p^j)_{j\in\mathbb{N}}$ constitute resolutions of the same sheaf $\mathcal{A}$ for each p sufficiently large (see for instance [4], section II-4.2). But, if we define $\mathcal{A}(V)$ as the set of 0-forms $\omega : V \cap M \to \mathbb{R}$ which are locally constant (at every point of $V \cap M$), then, by Lemma 5.5, the sequence

$$0 \hookrightarrow \mathcal{A} \hookrightarrow \mathcal{F}_p^0 \xrightarrow{d} \mathcal{F}_p^1 \xrightarrow{d} \mathcal{F}_p^2 \xrightarrow{d} \dots$$

is exact for all $p \in [1, \infty]$ sufficiently large (see (4.43)). □

5.4 An example

The definition of intersection homology is somewhat more technical than the definition of the usual homology and it may be unclear for the reader who is not well acquainted with this theory to figure out the extent to which the de Rham theorems presented in this article make it possible to compute the L^p cohomology groups. For this reason, we end this paper with an example on which we discuss the results of this paper. This is also a way to show more concretely the interplay between the vanishing cycles, the lack of duality of singular spaces, and L^p cohomology classes.

Let X be the suspension of the torus. It is the set constituted by two cones over a torus that are attached along this torus. It is the most basic example on which Poincaré duality fails for singular homology but holds for intersection homology [17]. Let x_0 and x_1 be the two isolated singular points.

This example, which has very simple singularities (metrically conical), is already enough to illustrate how the singularities affect Poincaré duality for L^p cohomology. As shown by the results of this article, the cohomology groups actually only depend on the topology of the underlying singular space. It is however possible to produce homeomorphic algebraic examples that are not metrically conical by making the cycles that generate the torus vanish at different rates at a singular point (like in [1] for instance). Of course, the real number p_0 from which Theorem 2.10 is valid then depends on the rate of vanishing of the cycles.

Take p in $[1, \infty]$ sufficiently close to 1 for Theorem 2.9 to hold for X_{reg}. If p is sufficiently close to 1 then, by Theorem 2.10, we also have $H_q^j(X_{reg}) \simeq I^{\mathbf{t}}H^j(X)$, where q is the Hölder conjugate of p. The cohomology groups involved in these theorems are gathered in the table below.

Cohomology groups $j =$	0	1	2	3
$I^{\mathbf{t}}H^j(X)$ and $H_q^j(X_{reg})$	$\mathbb{R}$	0	$\mathbb{R}^2$	$\mathbb{R}$
$I^0H^j(X)$	$\mathbb{R}$	$\mathbb{R}^2$	0	$\mathbb{R}$
$H^j(X_{reg})$ and $H_p^j(X_{reg})$	$\mathbb{R}$	$\mathbb{R}^2$	$\mathbb{R}$	0

All these results may be obtained from the isomorphisms given in Sect. 2 and a triangulation. Below, we interpret them geometrically.

Let $T \subset X$ be the original torus and let σ and τ be the two generators of $H_2(X)$ supported by the respective suspensions of the two circles generating the torus T. For $\varepsilon > 0$ set:

$$\sigma^\varepsilon := \{x \in |\sigma| : d(x, \{x_0, x_1\}) = \varepsilon\},$$

where $|\sigma|$ stands for the support of the cycle σ.

If ω is an L^q 2-form that is equal to zero near the singular points and that satisfies

$$\int_\sigma \omega = 1, \tag{4.44}$$

and if $\omega = d\alpha$, for some 1-form α, then $\int_{\sigma^\varepsilon} \alpha \equiv 1$ (by Stokes' formula). As the volume of σ^ε tends to zero, α cannot be an L^q form if q is big. Consequently, if ω is an L^q closed 2-form, zero near the singularities and satisfying (4.44), it must represent a nontrivial class. This accounts for the fact that $H^2_q(X_{reg}) \simeq \mathbb{R}^2$. In fact, every nontrivial class may be represented by a shadow form [2].

The nontrivial L^p classes of 1-forms are dual to the generators of the torus T (while, as we have seen in the preceding paragraph, the nontrivial L^q classes are dual to their respective suspensions σ and τ). We see that L^p cohomology is dual to L^q cohomology in dimension 0 and 1 (as it is established by Corollary 2.12).

However, the above form α may be L^p and this accounts for the fact that the L^p cohomology of the 2-forms is not isomorphic to $\mathbb{R}^2$. The only nontrivial L^p class of 2-forms is actually provided by the forms whose integral on T is nonzero. We see in particular that this collapsing torus induces a gap between $H^2_p(X_{reg})$ and $H^1_q(X_{reg})$. This is a typical example of the way the singularities affect the duality between L^p and L^q cohomology.

References

1. Birbrair, L., Gol'dshtein, V.M.: An example of noncoincidence of L_p-cohomology and intersection homology for real algebraic varieties. Int. Maths. Res. Notes **1994**(6), 261–271 (1994)
2. Brasselet, J.-P., Goresky, M., MacPherson, R.: Simplicial differential forms with poles. Am. J. Math. **113**(6), 1019–1052 (1991)
3. Bierstone, E., Milman, P.D.: Semianalytic and subanalytic sets. Inst. Hautes Études Sci. Publ. Math. No. 67, 5–42 (1988)

4. Bredon, G.E.: Sheaf theory, 2nd edn. Graduate Texts in Mathematics, 170. Springer-Verlag, New York, xii+502 pp (1997)
5. Cheeger, J.: On the spectral geometry of spaces with cone-like singularities. Proc. Nat. Acad. Sci. U.S.A. **76**(5), 2103–2106 (1979)
6. Cheeger, J.: On the Hodge theory of Riemannian pseudomanifolds, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), pp. 91–146, Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I. (1980)
7. Cheeger, J.: Hodge theory of complex cones. Analysis and topology on singular spaces, II, III (Luminy, 1981), 118–134, Astérisque, 101-102, Soc. Math. France, Paris (1983)
8. Cheeger, J.: Spectral geometry of singular Riemannian spaces. J. Differ. Geom. **1983**, 575–657 (1984)
9. Cheeger, J., Goresky, M., MacPherson, R.: L^2-Cohomology and Intersection Homology of Singular Algebraic Varieties, Seminar in Differential Geometry, S. T. Yau, ed. Princeton University Press, Princeton N. J. (1981)
10. Coste, M.: An Introduction to O-minimal Geometry, Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica, Istituti Editoriali e Poligrafici Internazionali, Pisa (2000)
11. de Rham, G.: Differential Manifolds, English Translation of "Variétés Différentiables". Springer, New York (1980)
12. Denef, J., van den Dries, L.: p-adic and real subanalytic sets. Ann. Math. (2) **128**(1), 79–138 (1988)
13. Denkowska, Z., Stasica, J.: Ensembles Sous-analytiques à la Polonaise. Editions Hermann, Paris (2007)
14. Dodziuk, J.: Sobolev spaces of differential forms and de Rham Hodge isomorphism. J. Differ. Geom. **16**(1), 63–73 (1981)
15. Gol'dshtein, V.M., Kuz'minov, V.I., Shvedov, I.A.: A property of de Rham regularization operators. Siberian Math. J. **25**(2) (1984)
16. Gol'dshtein, V.M., Troyanov, M.: Sobolev inequalities for differential forms and $L_{p,q}$-cohomology. Journal of Geom. Anal. **16**(4), 597–631 (2006)
17. Goresky, M., MacPherson, R.: Intersection homology theory. Topology **19**(2), 135–162 (1980)
18. Goresky, M., MacPherson, R.: Intersection homology. II. Invent. Math. **72**(1), 77–129 (1983)
19. Hsiang, W.C., Pati, V.: L^2-cohomology of normal algebraic surfaces. I. Invent. Math. **81**(3), 395–412 (1985)
20. Kurdyka, K.: On a subanalytic stratification satisfying a Whitney property with exponent 1, Real algebraic geometry (Rennes, 1991), 316–322, Lecture Notes in Math., 1524. Springer, Berlin (1992)
21. Łojasiewicz, S.: Ensembles semi-analytiques, Inst. Hautes Etudes Sci., Bures-sur-Yvette (1965) available on-line: https://perso.univ-rennes1.fr/michel.coste/Lojasiewicz.pdf
22. Murolo, C., Trotman, D.: Horizontally-C^1 controlled stratified maps and Thom's first isotopy theorem. C. R. Acad. Sci. Paris Sér. I Math. **330**(8), 707–712 (2000)
23. Saper, L.: L_2-cohomology and intersection homology of certain algebraic varieties with isolated singularities. Invent. Math. **82**(2), 207–255 (1985)
24. Saper, L.: L_2-cohomology of Kähler varieties with isolated singularities. J. Differ. Geom. **36**(1), 89–161 (1992)
25. Shiota, M.: Geometry of Subanalytic and Semialgebraic Sets, Progress in Mathematics, 150. Birkhäuser Boston Inc, Boston, MA (1997)
26. Valette, A.: Łojasiewicz inequality at singular points. Proc. Am. Math. Soc. **147**(3), 1109–1117 (2019)
27. Valette, G.: Lipschitz triangulations. Illinois J. Math. **49**(3), 953–979 (2005)
28. Valette, G.: The link of the germ of a semi-algebraic metric space. Proc. Am. Math. Soc. **135**, 3083–3090 (2007)
29. Valette, G.: L^∞ cohomology is intersection cohomology. Adv. Math. **231**(3–4), 1818–1842 (2012)
30. Valette, G.: Stokes formula for stratified forms. Ann. Polon. Math. **114**(3), 197–206 (2015)
31. Valette, G.: On subanalytic geometry, survey, available at http://www2.im.uj.edu.pl/gkw/sub.pdf
32. Weber, A.: An isomorphism from intersection homology to L_p-cohomology. Forum Math. **7**(4), 489–512 (1995)
33. Youssin, B.: L^p cohomology of cones and horns. J. Differ. Geom. **39**(3), 559–603 (1994)