Strategic Dynamic Vehicle Routing with Spatio-Temporal Dependent Demands

by

Diego Feijer

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Abstract

Dynamic vehicle routing problems address the issue of determining optimal routes for a set of vehicles, to serve a given set of demands that arrive sequentially in time. Traditionally, demands are assumed to be generated over time by an exogenous stochastic process. This thesis is concerned with the study of dynamic vehicle routing problems where demands are strategically placed in the space by an agent with selfish interests and physical constraints. In particular, we focus on the following problem: a team of vehicles seek to device dynamic routing policies that minimize the average waiting time of a typical demand, from the moment it is placed in the space until its location is visited; while an adversarial agent operating from a central depot with limited capacity aims at the opposite, strategically choosing the spatio-temporal point process according to which place demands.

We model the above problem and its inherent pure conflict of interests as a zero-sum game, and characterize equilibria under heavy load regime. For the analysis we discriminate between two cases: bounded and unbounded domains. In both cases we show that a routing policy based on performing successive TSP tours through outstanding demands and a power-law spatial distribution of demands are optimal, saddle point of the utility function of the game. The latter emerges as the unique solution of maximizing a non-convex nowhere differentiable functional over the infinite-dimensional space of probability densities; the non-convexity is the result of the spatio-temporal dependence induced by the physical constraints imposed on the behavior of the agent, and the non-differentiability is due to the emptiness of the interior of the positive cone of integrable functions. We solve this problem applying Fenchel conjugate duality for partially finite programming in the case of bounded domains; and a direct duality approach exploiting the structure of a concave integral functional part of the objective and the linear equality constraints, for unbounded domains. Remarkably, all the results obtained hold for any domain with a sufficiently smooth boundary, clossedness or connectedness is not needed. We provide numerical simulations to validate the theory.

Thesis Supervisor: Emilio Frazzoli
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Chapter 1

Introduction

1.1 Background

Dynamic vehicle routing problems address the issue of determining optimal routes for a set of vehicles, to serve a given set of demands that arrive sequentially in time. Usually, at a given instant in time, only the location of the current demands is known; future demand is uncertain. It is therefore convenient to think of the arrival of demands as governed by an underlying stochastic process. There exists many real-world setting in which problems of this nature arise: taxicabs picking up passengers, trucks delivering supplies to factories, aircraft with sensors visiting locations with suspicious activities.

Vehicle routing problems were initially analyzed for the case where demands are *static*, in the sense that no new demands arrive over time [37]. In these models a team of vehicles is required to visit the location of the demands, and spend a certain amount of on-site service time. Typically, the goal in these problems is to device scheduling policies that minimize the total distance traveled by the vehicles. Later, Psaraftis [30] contemplated the scenario where demands are *dynamic*, and are revealed as time goes by. In contrast to its static counterpart, dynamic vehicle routing would differ mainly in three aspects: first, the wait for service would often be more important than the travel cost; second, routing policies would not consist of pre-planned routes but rather indicate how routes should dynamically evolve in response to demands; third and most importantly, they would present queuing phenomena.

The addition of queuing considerations is a very important characteristic of dynamic vehicle routing problems. Just like in a queuing system, if the arrival rate of demands exceeds certain level, the routing system may become congested and will be unable to provide service without excessive delays. This issue of stability in terms of the number of waiting demands was early recognized by Psaraftis, but was not formally address until Bertsimas and Van Ryzin [5], [6]. In their seminal paper, they analyzed a version of a dynamic vehicle routing problem in which demands arrive at random locations within an Euclidean region according to a spatio-temporal renewal process, and a vehicle serving a demand spends some random on-site service time. The goal was to find policies that minimized the average time elapsed from the issuance of a demand and the completion of its service. Integrating ideas from combinatorial optimization, queuing theory and geometrical probability, they were able to provide fundamental lower bounds on the system time under different traffic regimes, and they constructed optimal and near-optimal routing policies.

Since then the study of dynamic vehicle routing problems has become a very active research area, primarily motivated by the extensive developments in the fields of autonomy, networking and robotics. A wide range of different directions have been pursued, the vast majority of them sharing the same methodology: (i) a queuing model for the system in question, and analysis of its structure; (ii) derivation of fundamental limitations on performance metrics, independent of routing policies; (iii) design of algorithms or heuristics that are optimal or optimal within a constant factor in specific regimes; (iv) validation of the algorithms through numerical simulations. An excellent summary of many contributions with the above features is [15].

Throughout the existing literature on dynamic vehicle routing, demands are assumed to be generated over time by an exogenous process. A recurrent theme is that demands are either customers that need to be picked up, raw material or merchandise to be delivered, failures that must be serviced by a mobile repair person, sites of suspicious activity that must be inspected. Thus far, to the best of our knowledge, vehicle routing problems where demands are strategically placed in the space by an agent with selfish interests and physical constraints, have not yet been considered.

In this thesis we focus on the following problem: a team of vehicles seek to device dynamic routing policies that minimize the average waiting time of a typical demand, from the moment it is placed in the space until its location is visited; while a malicious agent operating from a home base (or central depot) with limited capacity, aims at the opposite strategically choosing the spatio-temporal point process according to which place targets. Practical settings with this structure include: a criminal robbing banks, where the longer the police takes to get to the crime scene the more time he has to scape; war like situations, where an enemy places bombs in a region and it is imperative for the soldiers to disable them as quickly as possible.

We model the above problem and its inherent pure conflict of interests as a zero-sum game [18] with two opponents, each making the best possible decisions aware of the fact that his antagonist is behaving the same. We show that the game has a finite value, and we characterize an equilibrium (saddle point of the utility function). The proposed game is geometric in nature, thus similar in spirit to [1] where game theory is used to design motion coordination strategies among a team of vehicles to service demands; and the pioneering work of Isaacs [20] on differential games, where he rigorously studied pursuit, evasion and warfare problems from a game-theoretic perspective.

The equilibrium of the game, which emerges as the joint optimization of the average waiting time of a typical demand over the strategy space, consists of two components: a routing policy for the team of vehicles, and a spatio-temporal stochastic process for the agent. For the characterization of the optimal strategy for the team of vehicles, we adopt the work by Xu [39] regarding tight asymptotic lower bounds on the system time under heavy load regimes. On the other hand, in order to determine the optimal point process for the agent we rely on tools from convex analysis and partially finite optimization (optimization of a functional subject to a finite number of constraints).

The fact that the agent has finite capacity, and therefore needs to return to the depot between successive rounds of target placements, induces a dependence between the temporal rate and the spatial distribution of targets. Through a logarithmic transformation the problem is decoupled and essentially reduce to a minimization of a convex integral functional, as originally studied by Rockafellar in [34, 35], over the space of probability densities with support over the two dimensional Euclidean space. Such objects naturally fit within the context of spectral estimation, where the objective function is entropy-like. For entropy optimization problems, which are pervasive in engineering fields [26] and have recently found application in finance [14, 12], the goal is to describe the properties of a stochastic process based on the knowledge of its moments. In an effort to rigorously formalize the arguments used in the published literature on this subject to derive optimal solutions, Borwein and Lewis [8, 10, 11] developed a general mathematical framework based on duality under which to study these kind of problem. They provided necessary and sufficient conditions for the existence and attainment of optimal solutions, issues that as they showed, cannot be taken for granted when dealing with these problems. These results will be of paramount importance in our analysis.

1.2 Organization of the Thesis

Apart from the present introductory Chapter 1, and Chapter 6 which contains the concluding remarks, the rest of the thesis is organized as follows:

- Chapter 2 contains general background material on classic convex theory, partially finite optimization and functional analysis.

- In Chapter 3 we study the problem of strategic dynamic vehicle routing with spatio-temporal dependent demands over a bounded domain. We analyze the game under light and heavy load regimes, and characterize the equilibria. The heavy load case is the most interesting, for which we show that a routing policy based on the solution to the Traveling Salesman Problem and a power-law spatial distribution are optimal. The latter emerges as the unique optimum of the problem of minimizing a nowhere differentiable convex integral functional subject to linear constraints over the positive cone of the space of integrable functions, which is solved using Fenchel conjugate duality and results by Borwein and Lewis.

- Chapter 4 deals with the same problem over the entire two dimensional Euclidean space. In this case, the same routing policy remains optimal, provided a mild regularity condition is imposed on the behavior of the tail of the distribution of demands. However, even with the addition of this condition, conjugate integral functionals are not well-defined thus the previous framework cannot be used to yield the optimal spatial density. Remarkably, exploiting the structure of the objective function and the linear constraints defining the optimization problem, we prove a similar duality theorem which guarantees the existence and attainment of the optimal solution.

- Finally, Chapter 5 presents numerical simulations that validate and shed light on the theoretical results developed in the previous chapters.

Chapter 2

Preliminaries

In this chapter we provide the mathematical background on which this thesis will rely upon. It is divided in three section: the first section is devoted to convex theory, were the correspondence between sets and functions is the prevailing idea; the second section contains some basic notions from functional analysis which are ubiquitous in this work; the third and last section is of key importance since it presents the duality framework under which we will cast most of the analysis developed in this thesis.

2.1 Convex Analysis

We begin by introducing some concepts and properties from convex analysis, as developed in [4] and [32].

Basic Concepts and Notation

A point $\mathbf{x}$ in the n-dimensional Euclidean space $\mathbb{R}^n$ will be conceived as a column vector, where x_i denotes its i-th component. The inner product of two vectors in $\mathbb{R}^n$ will be written as $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T\mathbf{y} = \sum_i x_i y_i$. Thus, the usual Euclidean norm will be given by $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$. The non-negative orthant is the set $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0\}$, where $\geq$ is to be understood component-wise.

Definition 2.1. *A set $X \subseteq \mathbb{R}^n$ is convex if for any $\mathbf{x}, \mathbf{y} \in C$ and any $\alpha \in [0, 1]$, we have*

$$\alpha\mathbf{x} + (1 - \alpha)\mathbf{y} \in X.$$

In other words, a set X is convex if the line segment between $\mathbf{x}$ and $\mathbf{y}$ lies in X.

Generally, it is preferable to work with functions that are real-valued. However, on occasions it will be convenient to define *extended real-valued functions* that can take infinite values at some points. Henceforth, all the concepts we introduce will refer to this class of functions.

Definition 2.2. *A function $f : X \subseteq \mathbb{R}^n \to (-\infty, +\infty]$ is convex if X is a convex set and for all $\mathbf{x}, \mathbf{y} \in X$ and $\alpha \in [0, 1]$, we have*

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}). \tag{2.1}$$

Geometrically, this inequality means that the line segment between $(\mathbf{x}, f(\mathbf{x}))$ and $(\mathbf{y}, f(\mathbf{y}))$ lies above the graph of f. If the inequality in (2.1) holds in the strict sense whenever $\mathbf{x} \neq \mathbf{y}$ and $\alpha \in (0, 1)$, then f is said to be *strictly convex.*

Definition 2.3. *A function f is concave if $-f$ is convex.*

From the above definition it is easy to see that for concave functions the inequality in (2.1) is reversed. In fact, this transformation is not restricted to this property alone, but holds for most of the properties valid for convex functions.

Epigraph and Effective Domain

The *epigraph* of a function $f : X \subseteq \mathbb{R}^n \to [-\infty, +\infty]$ is the set defined as

$$\text{epi } f = \{(\mathbf{x}, w) \in \mathbb{R}^{n+1} : \mathbf{x} \in X, w \in \mathbb{R}, f(\mathbf{x}) \leq w\}.$$

The *effective domain* of f is defined to be the set

$$\text{dom } f = \{\mathbf{x} \in X : f(\mathbf{x}) < \infty\},$$

which is the projection of epi f on $\mathbb{R}^n$. Note that if we restrict f to its effective domain, or if we enlarge the domain of f by defining $f(\mathbf{x}) = \infty$ for every $\mathbf{x} \notin X$, the epigraph and the effective domain remain the same. For extended real-valued functions, its convexity (concavity) can be characterized through their epigraphs.

Definition 2.4. *Let X be a convex subset of $\mathbb{R}^n$. A function $f : X \to [-\infty, +\infty]$ is convex if* epi f *is a convex subset of $\mathbb{R}^{n+1}$.*

It can be readily verified that the above definition is consistent with the earlier definition of convexity given for real-valued functions. Thus we can use properties of sets to infer corresponding properties of functions. It turns out that the reverse also holds, through the notion of indicator function of a set. Given a set $X \subseteq \mathbb{R}^n$, we define its *indicator function* $\delta : \mathbb{R}^n \to (-\infty, +\infty]$ as

$$\delta(\mathbf{x}|X) = \begin{cases} 0 & \text{if } \mathbf{x} \in X, \\ \infty & \text{otherwise.} \end{cases} \tag{2.2}$$

Then, a set is convex if and only if its indicator function is convex. This interplay of geometry and analysis, made possible through this fundamental idea of identifying functions with sets and sets with functions, lies at the heart of convex analysis, and will be used extensively throughout this thesis.

When faced to the problem of minimizing a convex function, it is often important to exclude the degenerate case where f is identically equal to ∞, and the case where f takes the value $-\infty$ at some point. Therefore, we say that f is *proper* if $f(\mathbf{x}) < +\infty$ for at least one $\mathbf{x} \in X$ and $f(\mathbf{x}) > -\infty$ for all $\mathbf{x} \in X$. In other words, a function is proper if its epigraph is non-empty and does not contain a vertical line.

Definition 2.5. *A function $f : X \subseteq \mathbb{R}^n \to [-\infty, +\infty]$ is said to be closed, if its epigraph is a closed set.*

Relative Interior

We now introduce one of the most important topological concepts of convex sets, that of relative interior. This notion is motivated by the fact that a line segment embedded in $\mathbb{R}^2$ does have a natural interior when regarded in the appropriate dimension, which is not an interior in the sense of the whole space.

The *affine hull* of a subset X of $\mathbb{R}^n$, denoted aff X, is the smallest affine set containing X (namely, the intersection of all affine sets containing X). The *relative interior* of X, denoted ri X, is defined as the interior which results when X is considered as a subset of its affine hull. Hence, ri X consists of all the points $\mathbf{x} \in X$ for which there exists and open sphere S centered at x such that $S \cap \text{aff } X \subset X$.

The key property of relative interiors is that if X is a nonempty convex set, then ri X is nonempty and convex as well (in contrast to the interior of X, which is certainly convex but might be empty). Often, finding the relative interior of a set based on its definition might be cumbersome. The next lemma, as stated in [4], provides us with an equivalent characterization for convex sets.

Lemma 2.1. *Let X be a nonempty convex set. Then, $\mathbf{x} \in \text{ri } X$ if and only if, for every $\mathbf{y} \in X$ there exists a scalar $\alpha > 0$ such that $\mathbf{x} + \alpha(\mathbf{x} - \mathbf{y}) \in X$. In other words, $\mathbf{x} \in \text{ri } X$ if and only if, every line segment in X having $\mathbf{x}$ as one of the endpoints can be prolonged beyond $\mathbf{x}$ without leaving X.*

Weierstrass' Theorem

In optimization one of the main concerns is the existence of optimal solutions. The set of minima of a real-valued function can be written as

$$\bigcap_{k=1}^{\infty} V_{\eta_k}^f, \quad \text{where} \quad V_{\eta_k}^f = \{\mathbf{x} \in \text{dom } f : f(\mathbf{x}) \leq \eta_k\},$$

and $\{\eta_k\}$ is any sequence of scalars, such that $\eta_k \downarrow \inf_{\mathbf{x}} f(\mathbf{x})$. It follows that if the level sets of f are nonempty and compact, then the set of minima of f will be nonempty and compact as well (because it is the intersection of a decreasing sequence of nonempty and compact sets in $\mathbb{R}^n$). This is the classical theorem of Weierstrass, which we now generalize to extended real-valued functions (see [4] for more details).

Theorem 2.1. *Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a closed proper function. If either* dom f *is bounded, or there exists $\eta \in \mathbb{R}$ such that the level set V^f_η is nonempty and bounded, then the set of minima of f over $\mathbb{R}^n$ is nonempty and compact.*

Conjugate Functions

We now introduce the concept of conjugacy, due to Fenchel, and further develop it in a more general context later in the chapter. Given an extended real-valued function $f : \mathbb{R}^n \to [-\infty, +\infty]$, we define its *convex conjugate* as the function $f^\star : \mathbb{R}^n \to [-\infty, +\infty]$ given by

$$f^\star(\mathbf{y}) = \sup_{\mathbf{x}\in\mathbb{R}^n} \{\langle \mathbf{x}, \mathbf{y}\rangle - f(\mathbf{x})\}.$$

Since $f^\star$ is the pointwise supremum of a collection of affine functions, it is always closed and convex regardless of the structure of f. Furthermore, if dom f is non-empty, then $f^\star(\mathbf{x}^\star) > -\infty$ for all $\mathbf{x}^\star \in V^\star$.

Subgradients

Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a proper convex function. A vector $\mathbf{d} \in \mathbb{R}^n$ is called a *subgradient* of f at a point $\mathbf{x} \in$ dom f if,

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{d}, \mathbf{y} - \mathbf{x}\rangle, \quad \text{for all} \quad \mathbf{y} \in \mathbb{R}^n. \tag{2.3}$$

If f is a real-valued differentiable function, then $\mathbf{d}$ is the usual gradient of f at $\mathbf{x}$.

The set of all subgradients of f at $\mathbf{x}$ is the *subdifferential of f at* $\mathbf{x}$, and is denoted by $\partial f(\mathbf{x})$. As the following proposition shows, it is possible to define a subgradient at every relative interior point of the effective domain of f.

Proposition 2.1. *If* $\mathbf{x} \in$ ri (dom f) *then $\partial f(\mathbf{x}) = S^\perp + G$, where S is the subspace that is parallel to* aff (dom f) *and G is a nonempty compact set; in particular, $\partial f(\mathbf{x}) \neq \emptyset$.*

The next definitions involve the strict concavity of an extended-real valued function when confined to its effective domain (see [32]).

Definition 2.6. *A proper convex function $f : \mathbb{R}^n \to (-\infty, +\infty]$ is said to be essentially strictly convex, if f is strictly convex on every convex subset of $\{\mathbf{x} : \partial f(\mathbf{x}) \neq \emptyset\}$. In particular, when $n = 1$ this is equivalent to f being strictly convex on* dom f.

Definition 2.7. *A proper convex function* $f : \mathbb{R}^n \to (-\infty, +\infty]$ *is essentially smooth if* f *is differentiable on the interior of* $\operatorname{dom} f$*, and for any sequence* $\{\mathbf{x}_k\}$ *in the interior of* $\operatorname{dom} f$ *such that* $\mathbf{x}_k \to \mathbf{x}$ *with* $\mathbf{x}$ *in the boundary of* $\operatorname{dom} f$*, we have* $\|\nabla f(\mathbf{x}_k)\| \to \infty$.

As the following result shows, both of the above definitions are related through the conjugation operation.

Proposition 2.2. *If* $f : \mathbb{R}^n \to (-\infty, +\infty]$ *is closed, proper and convex, then* f *is essentially strictly convex if and only if* $f^\star$ *is essentially smooth.*

2.2 Functional Analysis and $\mathcal{L}_p$ Spaces

We present here some basic notions regarding functional analysis: linear functionals, dual spaces and adjoint operators, and $\mathcal{L}_p$ spaces. For omitted definitions and more details the reader is referred to [17] or [36].

Linear Functionals and Dual Spaces

Definition 2.8. *Let* V *be a vector space, and let* $\mathbb{K}$ *denote either* $\mathbb{R}$ *or* $\mathbb{C}$*. A linear map* $\mathbf{A} : V \to \mathbb{K}$ *is called a linear functional. We will write* $\mathbf{A}(\mathbf{v}) = \mathbf{A}\mathbf{v}$*, for every* $\mathbf{v} \in V$.

Definition 2.9. *Let* V *be a vector space equipped with a norm* $\|\cdot\|$*. A real linear functional* $\mathbf{A} : V \to \mathbb{R}$ *is called bounded, if there exists* $M \geq 0$ *such that* $\|\mathbf{A}\mathbf{v}\| \leq M\|\mathbf{v}\|$*, for all* $\mathbf{v} \in V$.

Note that this is different from the usual notion of boundedness for functions defined over sets, for which $\mathbf{A}$ would be bounded if $\|\mathbf{A}\mathbf{v}\| \leq C$ for all $\mathbf{v}$. Evidently, no non-zero linear map can satisfy that condition since $\mathbf{A}(\eta\mathbf{v}) = \eta\mathbf{A}\mathbf{v}$ for every scalar η. Therefore, the above definition is to be interpreted as $\mathbf{A}$ being bounded on bounded subsets of V.

Of particular importance is the case of continuous linear functionals over normed vector spaces. The next result characterizes this class of maps.

Proposition 2.3. *Let* V *be a normed vector space, with norm* $\|\cdot\|$*, and let* $\mathbf{A} : V \to \mathbb{R}$ *be a real linear functional. Then* $\mathbf{A}$ *is continuous if and only if,* $\mathbf{A}$ *is bounded.*

The modern theory of optimization over normed vector spaces revolves around the connections between a normed vector space V and its dual $V^\star$, defined as the space consisting of all continuous linear functionals on the original space. It can be shown that $V^\star$ is a Banach space (normed vector space which is complete with respect to the norm metric). In the resolution of optimization problems with constraints described by linear operators, an associated operator is usually involved in all duality relations. It is called the adjoint or transpose, and it is formally defined as follows.

Definition 2.10. *Let V and W be normed vector space, and let $\mathbf{A} : V \to W$ be a bounded linear operator. We define the adjoint or transpose of $\mathbf{A}$ as the operator $\mathbf{A}^T : W^\star \to V^\star$ defined by $\mathbf{A}^T f = f \circ \mathbf{A}$.*

$\mathcal{L}_p$ Spaces

$\mathcal{L}_p$ spaces are Banach spaces of measurable functions defined on a fixed measure space, whose norms are defined in terms of integrals. In this thesis we are primarily interested in real $\mathcal{L}_p$ spaces defined over $\mathbb{R}^n$ equipped with the usual Lebesgue measure, γ. Formally, for $1 \leq p < \infty$, we define

$$\mathcal{L}_p(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} : \|f\|_p < \infty\}, \quad \text{where} \quad \|f\|_p := \int_{\mathbb{R}^n} |f(\mathbf{x})| d\mathbf{x}.$$

For the limiting case $p = \infty$, we define $\mathcal{L}_\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} : \|f\|_\infty < \infty\}$ with

$$\|f\|_\infty := \inf \{\alpha \geq 0 : \mu(\{\mathbf{x} : |f(\mathbf{x})| > \alpha\}) = 0\},$$

and the convention that $\inf \emptyset = \infty$. Hence, $\mathcal{L}_\infty(\mathbb{R}^n)$ is the set of real bounded functions defined on $\mathbb{R}^n$; $\|f\|_\infty$ is often called the *essential supremum* of $|f|$ and is sometimes written as $\|f\|_\infty = \operatorname{ess\,sup}_{\mathbf{x} \in \mathbb{R}^n} |f(\mathbf{x})|$.

2.3 Optimization of Integral Functionals

We now introduce the framework developed in [8] and [11] by Borwein & Lewis, for the minimization of a convex integral functional over the positive cone of integrable functions subject to a finite number of linear equality constraints. The results gathered in this section will play a central role in the subsequent development of the thesis.

Conjugate Functions and Fenchel Duality

The concept of conjugacy previously introduced can be formulated in a general way as follows. Let V and $V^\star$ be vector spaces, equipped with a bilinear product $\langle \cdot, \cdot \rangle$ on the product space $V \times V^\star$, and consider a convex function $f : V \to [-\infty, +\infty]$. The *(Fenchel) conjugate function* of f with respect to $\langle \cdot, \cdot \rangle$, is a function $f^\star : V^\star \to [-\infty, +\infty]$ defined as

$$f^\star(\mathbf{x}^\star) := \sup\{\langle \mathbf{x}, \mathbf{x}^\star \rangle - f(\mathbf{x}) : \mathbf{x} \in V\}. \tag{2.4}$$

Fenchel's duality theory is concern with the problem of minimizing the difference of two proper functions, $f - g$, convex and concave respectively. This problem is very general and includes the minimization of a convex function over a convex set X, in which case we let $g = -\delta(\cdot|X)$. The following duality theorem resides in the connection between minimizing $f - g$ (convex) and maximizing $g^\star - f^\star$ (concave).

Theorem 2.2. *Let V and $V^\star$ be vector spaces paired by a bilinear product $\langle\cdot,\cdot\rangle$ on $V \times V^\star$. Let $\mathbf{A} : V \to \mathbb{R}^n$ be a linear map with adjoint $\mathbf{A}^T$, let $f : V \to (-\infty, +\infty]$ be a proper convex function, and let $g : \mathbb{R}^n \to [-\infty, +\infty)$ be a proper concave function. Then, if*

$$\text{ri}\,(\mathbf{A}\text{dom}\,f) \cap \text{ri}\,(\text{dom}\,g) \neq \emptyset,$$

we have,

$$\inf\{f(\mathbf{x}) - g(\mathbf{A}\mathbf{x}) : \mathbf{x} \in V\} = \sup\{g^\star(\boldsymbol{\xi}) - f^\star(\mathbf{A}^T\boldsymbol{\xi}) : \boldsymbol{\xi} \in \mathbb{R}^n\},$$

with the supremum on the right being attained when finite.

The reader is referred to either [32] or [4] for the proof in the case where V has finite dimension, and to [10] when V is infinite-dimensional. The latter case is often called *partially finite* because the linear operator $\mathbf{A}$ maps V into $\mathbb{R}^n$.

Partially Finite Convex Programming in $\mathcal{L}_1$

Let $\mathcal{S} \in \mathbb{R}^n$ be a finite measure set, and let $h : \mathbb{R} \to \mathbb{R}$ be a closed proper convex function. Consider the functional $\mathcal{I} : \mathcal{L}_1(\mathcal{S}) \to [-\infty, +\infty]$ defined by

$$\mathcal{I}(\varphi) = \int_{\mathcal{S}} h(\varphi(\mathbf{x}))d\mathbf{x}. \tag{2.5}$$

If we interpret the value of this integral as in [33], then $\mathcal{I}$ is a well-defined convex operator. The object of study is the following optimization problem:

$$\inf\ \mathcal{I}(\varphi) \quad \text{subject to} \quad \mathbf{A}\varphi = \mathbf{b},\ \varphi \in \mathcal{L}_1(\mathcal{S}), \tag{2.6}$$

where $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{A} : \mathcal{L}_1(\mathcal{S}) \to \mathbb{R}^n$ is a continuous linear operator with components $A_i \in \mathcal{L}_\infty(\mathcal{S})$, defined for all $i = 1, \ldots, n$ by

$$(\mathbf{A}\varphi)_i = \int_{\mathcal{S}} A_i(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}. \tag{2.7}$$

The most widely encountered instance of (2.6) is the problem of entropy optimization (see [26] and references therein), where the goal is to describe the statistical properties of an underlying stochastic process from a finite set of measurements of its moments. Applications range from the estimation of the power spectrum of a signal [3], to the more recent inference of the probability distribution for the price of an asset from option prices [14], [12].

We seek to solve problem (2.6) using Fenchel duality. The motivation is that through Theorem 2.2 we can transform a complicated minimization problem over the infinite-dimensional space of integrable functions, into an arguably easier maximization problem over the usual n-dimensional Euclidean space. To that end, let $V = \mathcal{L}_1(\mathcal{S})$ and $V^\star = \mathcal{L}_\infty(\mathcal{S})$. Then, it is possible to define a bilinear product on $V \times V^\star$ by,

$$(\varphi, \varphi^\star) \longmapsto \langle \varphi, \varphi^\star \rangle := \int_S \varphi(\mathbf{x})\varphi^\star(\mathbf{x})d\mathbf{x}. \tag{2.8}$$

As the following result shows [34], to compute the convex conjugate of the integral functional $\mathcal{I}$ with respect to (2.8), we may just conjugate the integrand h.

Proposition 2.4. *Let $\mathcal{S}$ be a finite measure set in $\mathbb{R}^n$, and let V and $V^\star$ be as above with bilinear product given by (2.8). Then, for any $\varphi^\star \in V^\star$, we have*

$$\mathcal{I}^\star(\varphi^\star) = \int_S h^\star(\varphi^\star(\mathbf{x}))d\mathbf{x}. \tag{2.9}$$

Theorem 2.2 with $f := \mathcal{I}$, $g(\mathbf{A}\varphi) := -\delta(\mathbf{A}\varphi - \mathbf{b}|\mathbf{0})$ and Proposition 2.4, furnishes the next result.

Corollary 2.1. *Consider the problem defined by (2.6) and (2.7), and assume that the constraint qualification*

$$\mathbf{b} \in \text{ri}\,(\mathbf{A}\text{dom}\,\mathcal{I}), \tag{2.10}$$

holds. Then, we have

$$\inf\{\mathcal{I}(\varphi) : \mathbf{A}\varphi = \mathbf{b},\ \varphi \in \mathcal{L}_1(\mathcal{S})\} = \sup\{\langle \boldsymbol{\xi}, \mathbf{b}\rangle - \mathcal{I}^\star(\mathbf{A}^T\boldsymbol{\xi}) : \boldsymbol{\xi} \in \mathbb{R}^n\}, \tag{2.11}$$

where $\mathbf{A}^T : \mathbb{R}^n \to \mathcal{L}_\infty(\mathcal{S})$ is the adjoint map, given by $\mathbf{A}^T\boldsymbol{\xi} := \sum_{i=1}^n \xi_i A_i$. Moreover, the supremum on the right-hand side of (2.11) is attained by some $\boldsymbol{\xi}^*$ whenever finite.

The left-hand side in (2.11) will be referred to as the *primal problem*, and $\varphi \in \mathcal{L}_1(\mathcal{S})$ the *primal variable*; the right-hand side will be referred to as the *dual problem*, and the vector $\boldsymbol{\xi} \in \mathbb{R}^n$ the *dual variable* or simply *multiplier*. The dual is always a convex problem, regardless of the structure of the primal. The following proposition gives sufficient conditions for the uniqueness of $\boldsymbol{\xi}^*$.

Proposition 2.5. *Consider the dual problem* $\sup\{\langle \boldsymbol{\xi}, \mathbf{b}\rangle - \mathcal{I}^\star(\mathbf{A}^T\boldsymbol{\xi}) : \boldsymbol{\xi} \in \mathbb{R}^n\}$. *If the set of constraint functions $\{A_i\}_{i=1}^n$ is linearly independent and $h^\star$ is essentially strictly convex (as in Definition 2.6), then any optimal solution is unique.*

We now have all the ingredients required to state the chief result, which yields the existence, uniqueness and characterization of the primal optimal solution $\varphi^*(\mathbf{x}) \in \mathcal{L}_1(\mathcal{S})$ in terms of $(h^\star)'$, the optimal dual solution $\boldsymbol{\xi}^*$ and the linear operator of constraints $\mathbf{A}$.

Theorem 2.3. *Consider the primal-dual pair (2.11) of Corollary 2.1. Assume that h is an essentially strictly convex and essentially smooth function, and suppose the following condition is satisfied:*

$$\Delta := \lim_{x\to\infty} \frac{h(x)}{x} > \operatorname*{ess\,sup}_{\mathbf{x}\in\mathcal{S}} \mathbf{A}^T\boldsymbol{\xi}^*(\mathbf{x}). \tag{2.12}$$

Then, the primal optimal solution to problem (2.6) is given by,

$$\varphi^*(\mathbf{x}) := (h^\star)'(\mathbf{A}^T\boldsymbol{\xi}^*(\mathbf{x})) = (h^\star)'\left(\sum_{i=1}^{n}\xi_i^* A_i(\mathbf{x})\right), \tag{2.13}$$

where $\boldsymbol{\xi}^* \in \mathbb{R}^n$ is the dual solution.

The proof of Theorem 2.3, which can be found in [8], builds on results derived in [35] by Rockafellar regarding the subgradients of convex integral functionals, and is mainly based on differentiating the dual objective function at the optimum. It is worth mentioning that the condition in (2.12) is concerned with the rate of growth of the integrand h, and is sufficient to assure the continuity of $\mathcal{I}^\star$ relative to the $\|\cdot\|_\infty$ metric at the dual optimum.

Chapter 3

Strategic Dynamic Vehicle Routing over a Bounded Domain

In this chapter we present one of the main theoretical results of the thesis. We begin with a mathematical description of the problem and a formulation of a zero-sum game, followed by an equilibrium analysis which relies heavily on notions borrowed from convex theory and the conjugate duality framework introduced in Chapter 2.

3.1 Problem Description

Consider a bounded set $\mathcal{S} \subseteq \mathbb{R}^2$ with $\mu(\mathcal{S}) > 0$, let $\overline{\mathcal{S}}$ be its closure and assume that for every $\mathbf{x} \in \overline{\mathcal{S}}$ there exists a ball $\mathcal{B}$ centered at $\mathbf{x}$, such that $\mu(\mathcal{B} \cap \mathcal{S}) > 0$. Targets stored in a depot located at $\mathbf{s} \in \mathcal{S}$ are carried and placed in $\mathcal{S}$ by an agent according to a spatio-temporal stochastic process, with spatial density $\varphi : \mathcal{S} \to \mathbb{R}_+$ and temporal rate $\lambda \geq 0$. Let $\mathbf{X}_i \sim \varphi$ represent the coordinates in $\mathbb{R}^2$ of the location of the i-th target, and assume that the locations are independent across targets. Without loss of generality, let us assume that the agent has both unitary capacity and speed, and let $\tau > 0$ denote the average time spent at the depot between two successive target placements. Then, the rate at which targets are placed in $\mathcal{S}$ is given by

$$\lambda = \frac{1}{2u + \tau}, \quad \text{with} \quad u = \mathbb{E}_{\varphi}[\|\mathbf{X} - \mathbf{s}\|] := \int_{\mathcal{S}} \|\mathbf{x} - \mathbf{s}\| \varphi(\mathbf{x}) d\mathbf{x}, \tag{3.1}$$

where $\mathbf{X}$ denotes the location of a randomly chosen target. A natural dependence between the temporal rate λ and the spatial density φ of targets is thus induced: in order to sustain a higher rate, the density has to be more concentrated around $\mathbf{s}$; or equivalently, as targets are distributed further away from the depot, the smaller the rate at which they can be placed becomes. We henceforth write λ_φ to explicitly show this dependence.

A vehicle operating from a base station and moving at speed v must service the targets. The service is done according to policies that consist on choosing either one of the outstanding targets whose location is to visit next; or alternatively choosing to go back to the base, at which the vehicle has the option of remaining indefinitely. Within this type of policies we are further interested in *stable* and *spatially unbiased* policies, which we now define.

Definition 3.1. *A policy is said to be stable if the expected number of outstanding targets is bounded almost surely at all times. A policy is said to be spatially unbiased if*

$$\mathbb{E}[T|\mathbf{X} \subseteq \mathcal{S}_1] = \mathbb{E}[T|\mathbf{X} \subseteq \mathcal{S}_2], \quad \textit{for every pair of sets} \quad \mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{S}, \tag{3.2}$$

where $\mathbf{X}$ *is the location of a given random target, and* T *represents its waiting time.*

Let Π denote the class of policies π with the aforementioned characteristics, and let $T_i(\pi, \varphi)$ represent the time elapsed from the issuance of the i-th target until a vehicle reaches its location. Define the *system time* $\overline{T} : \Pi \times \mathcal{F}$ by

$$\overline{T}(\pi, \varphi) := \limsup_{i \to \infty} \mathbb{E}_{\varphi}[T_i(\pi, \varphi)], \tag{3.3}$$

where

$$\mathcal{F} = \left\{ \varphi : \mathcal{S} \to \mathbb{R}_+ \text{ s.t. } \int_S \varphi(\mathbf{x}) d\mathbf{x} = 1 \right\}. \tag{3.4}$$

In the context described above, we consider a two-person zero-sum game between the agent and the vehicle, with the system time defined in (3.3) as payoff function. In other words, in this strictly competitive setting the agent will seek to maximize the system time, while the goal of the vehicle will be exactly the opposite. A solution, or equilibrium, of the game will be a pair $(\pi^*, \varphi^*) \in \Pi \times \mathcal{F}$ for which

$$\sup_{\varphi \in \mathcal{F}} \inf_{\pi \in \Pi} \overline{T}(\pi, \varphi) = \overline{T}(\pi^*, \varphi^*) = \inf_{\pi \in \Pi} \sup_{\varphi \in \mathcal{F}} \overline{T}(\pi, \varphi). \tag{3.5}$$

A point satisfying condition (3.5) is called a *saddle point* for the function $\overline{T}$. Finding such a point will be the focus of the subsequent development.

3.2 Load Regimes and System Time

In the now very extensive literature on dynamic vehicle routing, closed form expressions for the system time are usually only available under two limiting regimes: *light load*, where demands arrive at a rate very close to zero; and *heavy load*, characterized by having arrivals at a rate tending to infinity. The reason for this lies at the interconnection between these problems and queuing theory, and the lack of expressions for the average waiting time corresponding to G/G/m queues as a function of the load.

For the problem at hand, as described by (3.1) and the explanation that followed, the rate of arrivals of targets is intertwined with the spatial density. Hence, it is not possible to discriminate a priori between light or heavy load based on λ_φ approaching zero or infinity. However, these traffic regimes are as experienced by the vehicle, and thus only make sense in the time scale defined by the speed v. Therefore, if the vehicle moves with $v \to \infty$ it will measure a rate of arrivals $\lambda_\varphi \to 0$, light load; whereas if it moves with $v \to 0$ it will see targets appearing at a rate $\lambda_\varphi \to \infty$, heavy load.

Light Load vs. Heavy Load

Under the light load regime, with high probability (i.e. with probability approaching 1 as $v \to \infty$), at any instant in time there will be at most one outstanding target. This is due to the fact that with high probability the vehicle will reach the target before the agent can return to the depot. Hence, the problem in this case is of no interest, since both the notions of optimal service policy and spatio-temporal dependence between the location of the targets and the rate at which they are placed become completely insignificant.

In the heavy load regime, for all $\pi \in \Pi$ and all $\varphi \in \mathcal{F}$ the system time is asymptotically lower bounded by,

$$\overline{T}(\pi, \varphi) \geq \frac{\kappa^2}{2v^2}\lambda_\varphi \left(\int_S \sqrt{\varphi(\mathbf{x})} d\mathbf{x} \right)^2 \quad \text{as} \quad v \to 0. \tag{3.6}$$

Note that the inequality (3.6) should formally be written with $\overline{T}(\pi, \varphi)v^2$ on the left-hand side, so that the right-hand side becomes finite; but in order to make the results more transparent, we henceforth prefer to adopt the less formal style. This bound was originally proved by [7], and it was conjectured that κ was actually equal to the constant β that appears in the asymptotic result for the length of the shortest path in the Traveling Salesman Problem (TSP) over the Euclidean plane [2] (for more details on the constant β and its value, the reader is referred to [29]). This conjecture was later proved in [39].

Contrary to the light load case, a solution to problem (3.5) when the utility function is the system time given by (3.6), is not immediate. We study this problem in the remaining of the chapter.

3.3 Equilibrium Analysis

Consider the following routing policy introduced by Bertsimas and van Ryzin in [7], which we will refer to as π^*:

Unbiased TSP-based Policy. *Let r be a large enough positive integer. From a central point in $\mathcal{S}$ partition the space into r sets $\mathcal{S}_1, \dots, \mathcal{S}_r$, such that $\int_{\mathcal{S}_k} \varphi(\mathbf{x})d\mathbf{x} = 1/r$. Within each set of the partition, form sets of targets with size n/r, and as these sets are constructed, deposit them in a queue and service them in a "first come, first served" fashion. The service of each set is achieved by construction a TSP tour and following it in an arbitrary direction. Finally, optimize over n.*

It was shown in [39] that in heavy traffic

$$\overline{T}(\pi^*, \varphi) = \frac{\beta^2}{2v^2}\lambda_\varphi \left(\int_{\mathcal{S}} \sqrt{\varphi(\mathbf{x})}d\mathbf{x} \right)^2 = \inf_{\pi \in \Pi} \overline{T}(\pi, \varphi), \tag{3.7}$$

for any $\varphi \in \mathcal{F}$. Hence,

$$\sup_{\varphi \in \mathcal{F}} \overline{T}(\pi^*, \varphi) = \sup_{\varphi \in \mathcal{F}} \inf_{\pi \in \Pi} \overline{T}(\pi, \varphi) \leq \inf_{\pi \in \Pi} \sup_{\varphi \in \mathcal{F}} \overline{T}(\pi, \varphi),$$

where the last inequality follows from the min-max inequality, which holds true on any product space (see [4], for example). By definition of infimum we get,

$$\inf_{\pi \in \Pi} \sup_{\varphi \in \mathcal{F}} \overline{T}(\pi, \varphi) \leq \sup_{\varphi \in \mathcal{F}} \overline{T}(\pi^*, \varphi),$$

and we arrive at

$$\sup_{\varphi \in \mathcal{F}} \inf_{\pi \in \Pi} \overline{T}(\pi, \varphi) = \sup_{\varphi \in \mathcal{F}} \overline{T}(\pi^*, \varphi) = \inf_{\pi \in \Pi} \sup_{\varphi \in \mathcal{F}} \overline{T}(\pi, \varphi).$$

Therefore, if a solution $\varphi^* \in \mathcal{F}$ to the maximization problem between the equality signs above exists, it would constitute together with π^* a saddle point of $\overline{T}$ as $v \to 0$, the equilibrium of the game (3.5).

3.4 The Optimal Spatial Density

The optimal spatial density that will maximize the system time as $v \to 0$ will emerge as the solution to the following optimization problem:

$$\sup_{\varphi} \ \lambda_\varphi \left(\int_{\mathcal{S}} \sqrt{\varphi(\mathbf{x})}d\mathbf{x} \right)^2 \quad \text{subject to} \quad \varphi \in \mathcal{F}. \tag{3.8}$$

In its original form, problem (3.8) is the product between a convex and a concave function. Hence, it is not convex thus hard to tackle. However, applying a logarithmic transformation to the objective function and introducing a new variable $\gamma := \log \lambda_\varphi$ yields the following equivalent formulation

$$\sup_{\gamma,\varphi} \ \gamma + 2\log \int_{\mathcal{S}} \sqrt{\varphi(\mathbf{x})}d\mathbf{x} \quad \text{subject to} \quad \gamma \in \Gamma,\ \varphi \in \mathcal{F}. \tag{3.9}$$

Since $0 \leq \mathbb{E}_\varphi[\|\mathbf{x}-\mathbf{s}\|] \leq \max_{\mathbf{x}\in\overline{\mathcal{S}}} \|\mathbf{x}-\mathbf{s}\|$, from (3.1) it can be easily seen that

$$\Gamma = \left[-\log\left(2\max_{\mathbf{x}\in\overline{\mathcal{S}}}\|\mathbf{x}-\mathbf{s}\| + \tau\right), -\log\tau\right] \subset \mathbb{R}. \tag{3.10}$$

Expressing the dependence of φ on the real variable γ,

$$\int_{\mathcal{S}} \|\mathbf{x}-\mathbf{s}\|\varphi(\mathbf{x})d\mathbf{x} = \frac{e^{-\gamma}-\tau}{2}, \tag{3.11}$$

allows us to rewrite (3.9) as

$$\sup_{\gamma\in\Gamma}\left\{\gamma + 2\sup_{\varphi\in\mathcal{F}_\gamma}\log\int_{\mathcal{S}}\sqrt{\varphi(\mathbf{x})}d\mathbf{x}\right\}, \tag{3.12}$$

where

$$\mathcal{F}_\gamma = \left\{\varphi : \mathcal{S} \to \mathbb{R}_+ \text{ s.t. } \varphi \in \mathcal{F}, \int_{\mathcal{S}}\|\mathbf{x}-\mathbf{s}\|\varphi(\mathbf{x})d\mathbf{x} = \frac{e^{-\gamma}-\tau}{2}\right\}. \tag{3.13}$$

Problem (3.12) decouples the spatio-temporal dependence of the stochastic process of target locations, splitting (3.8) into two connected sub-problems: one for the spatial component, and another one for the temporal component. The former entails a maximization over an infinite-dimensional space to determine an optimal parametric family of spatial probability densities, parametrized by γ; the latter is a scalar maximization which yields the optimal rate, therefore completely identifying the optimal density from the previously found parametric family, rendering the solution to (3.8).

The Optimal Parametric Family

Given $\gamma \in \Gamma$, we wish to solve

$$\sup\left\{\log\int_{\mathcal{S}}\sqrt{\varphi(\mathbf{x})}d\mathbf{x} : \varphi \in \mathcal{F}_\gamma\right\},$$

or equivalently,

$$\inf\ \mathcal{I}(\varphi) := \int_{\mathcal{S}} -\sqrt{\varphi(\mathbf{x})}d\mathbf{x} \quad \text{subject to} \quad \varphi \in \mathcal{F}_\gamma. \tag{3.14}$$

First let us note that, as stated by the following lemma, problem (3.14) is feasible for every $\gamma \in \Gamma$ and has a value of zero when γ is either of the extremes of the interval Γ in (3.10).

Lemma 3.1. *For every $\gamma \in \operatorname{int} \Gamma$, there exists a density $\varphi \in \mathcal{D}_\gamma$. Moreover, when γ belongs to the boundary of Γ the value of (3.14) is zero.*

Proof. Let $\mathbf{y} \in \mathcal{A} = \{\mathbf{x} \in \overline{\mathcal{S}} : \mathbf{x} \in \arg\max_{\mathbf{x}} \|\mathbf{x} - \mathbf{s}\|\}$, and note that $\mu(\mathcal{A}) = 0$. Because of the smoothness assumption imposed on the set $\mathcal{S}$, there exists balls $\mathcal{B}_{\mathbf{s},r_1}$ and $\mathcal{B}_{\mathbf{y},r_2}$ with radii r_1 and r_2 centered at $\mathbf{s}$ and $\mathbf{y}$, respectively, such that $\mu(\mathcal{B}_{\mathbf{s},r_1} \cap \mathcal{S}) > 0$ and $\mu(\mathcal{B}_{\mathbf{y},r_2} \cap \mathcal{S}) > 0$. Clearly, we can reduce the radii and still maintain the same property.

Let φ_1 and φ_2 denote the densities associated with uniform distributions defined over $\mathcal{B}_{\mathbf{s},r_1}$ and $\mathcal{B}_{\mathbf{y},r_2}$, then $\lim_{r_1 \to 0} \mathbb{E}_{\varphi_1}[\|\mathbf{x}-\mathbf{s}\|] = 0$ and $\lim_{r_2 \to 0} \mathbb{E}_{\varphi_2}[\|\mathbf{x}-\mathbf{s}\|] = \max_{\mathbf{x}\in\overline{\mathcal{S}}} \|\mathbf{x}-\mathbf{s}\|$. Furthermore, these limiting values will only be achieved by singular distributions with supports over $\{\mathbf{s}\}$ and $\mathcal{A}$, respectively; therefore, the integral in (3.14) is zero and we get,

$$\inf \{\mathcal{I}(\varphi) : \varphi \in \mathcal{F}_\gamma\} = 0, \quad \text{for} \quad \gamma \in \left\{ -\log\left(2 \max_{\mathbf{x}\in\overline{\mathcal{S}}} \|\mathbf{x} - \mathbf{s}\| + \tau\right), -\log \tau \right\}.$$

Now consider a density φ defined as a linear combination of φ_1 and φ_2, with support over $\mathcal{B}_{\mathbf{s},r_1} \cup \mathcal{B}_{\mathbf{y},r_2}$. Then, by the linearity of the expectation we get,

$$\mathbb{E}_\varphi[\|\mathbf{x} - \mathbf{s}\|] = \alpha \mathbb{E}_{\varphi_1}[\|\mathbf{x} - \mathbf{s}\|] + (1 - \alpha)\mathbb{E}_{\varphi_2}[\|\mathbf{x} - \mathbf{s}\|], \quad \text{with} \quad \alpha \in [0, 1].$$

Thus, given $\gamma \in \Gamma$ we can always construct (with appropriate choices of r_1, r_2 and α) a density φ, such that

$$\mathbb{E}_\varphi[\|\mathbf{x} - \mathbf{s}\|] = \frac{e^{-\gamma} - \tau}{2},$$

which shows the desired result. ■

Since the objective function is convex in φ and the equality constraints defining $\mathcal{F}_\gamma$ are linear, problem (3.14) is convex thus amenable to solve through Lagrange duality. The Lagrangian for this problem is the function $L : \mathcal{L}_1(\mathcal{S}) \times \mathbb{R}^2 \to \mathbb{R}$ defined as

$$L(\varphi, \boldsymbol{\xi}) := \mathcal{I}(\varphi) + \xi_1 \left(1 - \int_{\mathcal{S}} \varphi(\mathbf{x}) d\mathbf{x}\right) + \xi_2 \left(\frac{e^{-\gamma} - \tau}{2} - \int_{\mathcal{S}} \|\mathbf{x} - \mathbf{s}\| \varphi(\mathbf{x}) d\mathbf{x}\right).$$

If some constraint qualifications are met, assuring that strong duality holds, then the solution to (3.14) can be obtained by solving

$$\sup_{\boldsymbol{\xi}\in\mathbb{R}^2} \inf_{\varphi\geq 0} L(\varphi, \boldsymbol{\xi}).$$

For the minimization over φ one might be tempted to differentiate the Lagrangian (in the Fréchet sense [27]); however, the Lagrangian is nowhere differentiable since the positive cone $\{\varphi \in \mathcal{L}_1(\mathcal{S}) : \varphi \geq 0\}$ has empty interior and its complement is dense in $\mathcal{L}_1(\mathcal{S})$. As a result, this approach cannot be rigorously justified (see [8] for further discussions). To bypass this technical difficulty we will cast problem (3.14) under the conjugate duality framework presented in Section 2.3.

Define the continuous linear operator with linearly independent components

$$\mathbf{A} : \mathcal{L}_1(\mathcal{S}) \to \mathbb{R}^2, \quad \text{such that} \quad \mathbf{A}\varphi = \begin{pmatrix} \int_{\mathcal{S}} \varphi(\mathbf{x}) d\mathbf{x} \\ \int_{\mathcal{S}} \|\mathbf{x} - \mathbf{s}\| \varphi(\mathbf{x}) d\mathbf{x} \end{pmatrix}, \tag{3.15}$$

and let $\mathbf{b}_\gamma \in \mathbb{R}^2$ be the column vector $\left(1, \frac{e^{-\gamma - \tau}}{2}\right)$. Then, expressing the constraints that define $\mathcal{F}_\gamma$ in (3.13) in terms of $\mathbf{A}$ and $\mathbf{b}_\gamma$, we can rewrite problem (3.14) as

$$\inf \ \mathcal{I}(\varphi) \quad \text{subject to} \quad \mathbf{A}\varphi = \mathbf{b}_\gamma,\ \varphi \geq 0,\ \varphi \in \mathcal{L}_1(\mathcal{S}). \tag{3.16}$$

Defining the functional

$$\mathcal{I}_+(\varphi) := \int_{\mathcal{S}} h(\varphi(\mathbf{x})) d\mathbf{x}, \quad \text{with} \quad h(x) := -\sqrt{x} + \delta(x|\mathbb{R}_+), \tag{3.17}$$

we can further write (3.16) as

$$\inf \ \mathcal{I}_+(\varphi) \quad \text{subject to} \quad \mathbf{A}\varphi = \mathbf{b}_\gamma,\ \varphi \in \mathcal{L}_1(\mathcal{S}). \tag{3.18}$$

Note that the integrand h is proper and convex. Moreover, we claim that it is also closed. Indeed, consider a sequence $\{x_k, w_k\} \subseteq \text{epi } h$ such that $(x_k, w_k) \to (x, w)$ as $k \to \infty$. We can assume that $\{x_k\} \subseteq \text{dom } h$ since, as discussed in Chapter 2, restricting a function to its effective domain does not change its epigraph. Then, since dom $h = [0, \infty)$ is a closed set, it follows that $x \in \text{dom } h$, thus $w \geq \lim_{k\to\infty} -\sqrt{x_k} = -\sqrt{x}$. This implies that $(x, w) \in \text{epi } h$, which shows the closedness of h. Consequently, formulation (3.18) exhibits the same structure as (2.6); therefore the results in Section 2.3 are applicable.

The next conjugate duality theorem will be of great significance in the subsequent analysis. It determines the dual of (3.18) and states that the duality gap is zero. Before we formally state and prove the theorem, we need the following key lemma.

Lemma 3.2. *For every* $\gamma \in \text{int } \Gamma$, *we have* $\mathbf{b}_\gamma \in \text{ri } (\mathbf{A}\text{dom } \mathcal{I}_+)$.

Proof. By definition,

$$\text{ri } (\mathbf{A}\text{dom } \mathcal{I}_+) = \text{ri } \{\mathbf{d} \in \mathbb{R}^2 : \exists\, \varphi \in \mathcal{L}_1(\mathcal{S}), \text{ with } \mathcal{I}_+(\varphi) < \infty \text{ and } \mathbf{A}\varphi = \mathbf{d}\},$$

and because $\{\varphi \in \mathcal{L}_1(\mathcal{S}) : \varphi \geq 0\} \subset \text{dom } \mathcal{I}_+$, it readily follows from Lemma 2.1 that ri $(\mathbf{A}\text{dom } \mathcal{I}_+) = \{\mathbf{d} \in \mathbb{R}^2 : \mathbf{d} > 0\}$. Thus, due to the fact that $\mathbf{b}_\gamma > 0$ for every $\gamma \in \text{int } \Gamma$, we have $\mathbf{b}_\gamma \in \text{ri } (\mathbf{A}\text{dom } \mathcal{I}_+)$. ■

Theorem 3.1. *Let* Γ *be defined as in (3.10), and let* int Γ *denote its interior. Then, for every* $\gamma \in \text{int } \Gamma$ *the dual of problem (3.18) is given by*

$$D(\gamma) := \sup_{\boldsymbol{\xi}\in\mathbb{R}^2} \left\{ \langle \boldsymbol{\xi}, \mathbf{b}_\gamma \rangle + \int_S \frac{d\mathbf{x}}{4\mathbf{A}^T\boldsymbol{\xi}(\mathbf{x})} : \mathbf{A}^T\boldsymbol{\xi} < 0 \right\}, \tag{3.19}$$

where $\mathbf{A}^T : \mathcal{S} \to \mathcal{L}_\infty(\mathcal{S})$ *is the adjoint map,* $\mathbf{A}^T\boldsymbol{\xi}(\mathbf{x}) = \xi_1 + \xi_2\|\mathbf{x} - \mathbf{s}\|$. *Furthermore, (3.19) admits a unique solution* $\boldsymbol{\xi}^*(\gamma)$, *and the optimal value achieved is finite and equal to the infimum in (3.18).*

Proof. The dual of problem (3.18) is given by,

$$\sup_{\boldsymbol{\xi}\in\mathbb{R}^2} \left\{ \langle \boldsymbol{\xi}, \mathbf{b}_\gamma \rangle - \mathcal{I}_+^*(\mathbf{A}^T\boldsymbol{\xi}) \right\},$$

and since the conjugate of the integrand function h is,

$$h^*(y) := \sup_{x\in\mathbb{R}}\{xy - h(x)\} = \sup_{x\geq 0}\{xy + \sqrt{x}\} = \begin{cases} -\frac{1}{4y} & y < 0, \\ \infty & \text{otherwise.} \end{cases} \tag{3.20}$$

by Proposition 2.4 it must be equal to

$$\sup_{\boldsymbol{\xi}\in\mathbb{R}^2} \left\{ \langle \boldsymbol{\xi}, \mathbf{b}_\gamma \rangle + \int_S \frac{d\mathbf{x}}{4\mathbf{A}^T\boldsymbol{\xi}(\mathbf{x})} : \mathbf{A}^T\boldsymbol{\xi} < 0 \right\}.$$

From Lemma 3.2 it follows that for every $\gamma \in \text{int } \Gamma$ the constraint qualification (2.10) is satisfied, and Corollary 2.1 implies that (3.19) is equal to (3.18) (thus equal to (3.14)).

Over the set

$$\mathcal{M} = \{\boldsymbol{\xi} \in \mathbb{R}^2 : \mathbf{A}^T\boldsymbol{\xi}(\mathbf{x}) < 0, \text{ for all } \mathbf{x} \in \mathcal{S}\}, \tag{3.21}$$

that describes the maximization in the dual problem, we must have $\langle \boldsymbol{\xi}, \mathbf{b}_\gamma \rangle < 0$. To see this, consider an arbitrary $\gamma \in \text{int } \Gamma$ and let φ be a density with support over $\mathcal{S}$ such that $\mathbf{A}\varphi = \mathbf{b}_\gamma$, whose existence is guaranteed by Lemma 3.1. Then, for every $\mathbf{x} \in \mathcal{S}$ we have $\varphi(\mathbf{x})(\xi_1 + \xi_2\|\mathbf{x} - \mathbf{s}\|) < 0$; hence,

$$\langle \boldsymbol{\xi}, \mathbf{b}_\gamma \rangle = \xi_1 \int_S \varphi(\mathbf{x})d\mathbf{x} + \xi_2 \int_S \|\mathbf{x} - \mathbf{s}\|\varphi(\mathbf{x})d\mathbf{x} < 0.$$

Therefore, the dual optimal value is bounded above by zero, thus must be achieved at some $\boldsymbol{\xi}^*(\gamma)$. Finally, the fact that h^* is essentially strictly convex (it is strictly convex over its effective domain) and the set of functions $\{1, \|\mathbf{x} - \mathbf{s}\|\}$ that define $\mathbf{A}$ is linearly independent, implies through Proposition 2.5, that $\boldsymbol{\xi}^*(\gamma)$ is unique for every $\gamma \in \text{int } \Gamma$. ∎

Corollary 3.1. *The optimal dual solution has $\xi_1^*(\gamma) < 0$, for every $\gamma \in \operatorname{int} \Gamma$.*

Proof. By definition, $\mathbf{A}^T\boldsymbol{\xi}^*(\gamma)(\mathbf{x}) = \xi_1^*(\gamma) + \xi_2^*(\gamma)\|\mathbf{x} - \mathbf{s}\| < 0$ for all $\mathbf{x} \in \mathcal{S}$. Hence, letting $\mathbf{x} = \mathbf{s}$ renders the result. ■

Based on the preceding theorem, the following proposition characterizes the unique optimal parametric family of spatial densities.

Proposition 3.1. *Consider the optimization problem defined by (3.14) and (3.13). Then, for every $\gamma \in \operatorname{int} \Gamma$ the unique optimal solution is given by,*

$$\varphi_\gamma^*(\mathbf{x}) = \frac{1}{4(\xi_1^*(\gamma) + \xi_2^*(\gamma)\|\mathbf{x} - \mathbf{s}\|)^2}, \quad \text{for all} \quad \mathbf{x} \in \mathcal{S}. \tag{3.22}$$

Proof. The function h defined in (3.17) is both essentially strictly convex and essentially smooth. Indeed, it is strictly convex and differentiable when restricted to its effective domain $[0, \infty)$, and $|h'(x)| = x^{-3/2}$ which tends to ∞ as $x \to 0$. Moreover, h satisfies the growth condition (2.12), since $\Delta = 0 > \operatorname{ess\,sup}_{\mathbf{x} \in \mathcal{S}} \mathbf{A}^T\boldsymbol{\xi}^*(\mathbf{x})$. Then, invoking Theorem 2.3 we conclude that for every $\gamma \in \operatorname{int} \Gamma$, the optimal solution to (3.14) is given by $\varphi_\gamma^*(\mathbf{x}) = (h^*)'(\mathbf{A}^T\boldsymbol{\xi}^*(\gamma)(\mathbf{x}))$ for all $\mathbf{x} \in \mathcal{S}$, where $\boldsymbol{\xi}^*(\gamma)$ is the unique dual solution determined by Theorem 3.1. Finally, from (3.20) we have that $(h^*)'(x) = 1/4x^2$ for all $x < 0$, and we arrive at (3.22). ■

Remarks:

- Through the use of conjugate duality, Theorem 3.1, we have transformed the infinite-dimensional optimization problem (3.14) into a maximization of a strictly concave function over a convex set in $\mathbb{R}^2$, and although the unique solution to (3.19) cannot be expressed in closed form it can be efficiently found numerically.

- The solution $\varphi_\gamma^*(\mathbf{x})$ obtained in Proposition 3.1 belongs to $\mathcal{C}(\mathcal{S})$, the set of continuous functions with support over $\mathcal{S}$ which is dense in $\mathcal{L}_1(\mathcal{S})$ and has a positive cone with non-empty interior. We could have chosen $\mathcal{C}(\mathcal{S})$ as the underlying working space and solve (3.14) through differentiation of the Lagrangian; however, the uniqueness result obtained for $\mathcal{L}_1(\mathcal{S})$ is much stronger.

The Optimal Parameter

We now study the optimization over γ in (3.12), and show that there exists a unique solution γ^*. Since for every $\gamma \in \operatorname{int} \Gamma$ the dual optimum $\boldsymbol{\xi}^*(\gamma)$ is unique, γ^* will determine the unique spatial density from the family described in (3.22) that attains the maximum in (3.8).

We start by providing some results concerning the behavior of $\boldsymbol{\xi}^*$ as a function of $\gamma \in \text{int } \Gamma$, that will play a key role in establishing the existence and uniqueness of the solution to (3.12). Specifically,

Proposition 3.2. *Consider the dual problem defined in (3.19). Then, the function* $\boldsymbol{\xi}^* : \text{int } \Gamma \to \mathcal{M}$ *is differentiable, and* $(\xi_2^*)'(\gamma) < 0$. *Also,* $D'(\gamma) = \langle \boldsymbol{\xi}^*(\gamma), \mathbf{b}'_\gamma \rangle$ *for all* $\gamma \in \text{int } \Gamma$.

Proof. The set $\mathcal{M}$ defined in (3.21) is open, therefore the following first order condition must be satisfied at $\boldsymbol{\xi}^*$:

$$G(\gamma, \boldsymbol{\xi}^*) = \mathbf{b}_\gamma - \frac{\partial \mathcal{I}_+^*}{\partial \boldsymbol{\xi}}(\mathbf{A}^T \boldsymbol{\xi}^*) = 0. \tag{3.23}$$

This equation implicitly defines $\boldsymbol{\xi}^*(\gamma)$ with a Jacobian

$$\frac{\partial G}{\partial \boldsymbol{\xi}} = -\frac{\partial^2 \mathcal{I}_+^*}{\partial \boldsymbol{\xi}^2},$$

which is negative definite for every $\boldsymbol{\xi} \in \mathcal{M}$ because of the strict convexity of $\mathcal{I}_+^*$ (strictly convex function composed with a linear function). Thus, it is nonsingular and the implicit theorem function furnishes the differentiability of $\boldsymbol{\xi}^*(\gamma)$. Moreover,

$$(\boldsymbol{\xi}^*)' = \left(\frac{\partial^2 \mathcal{I}_+^*}{\partial \boldsymbol{\xi}^2} \right)^{-1} \frac{\partial G}{\partial \gamma}. \tag{3.24}$$

The inverse of the Hessian of $\mathcal{I}_+^*$ is positive definite, and $\frac{\partial G}{\partial \gamma}$ is the column vector of entries $(0, -\frac{1}{2}e^{-\gamma})$. Hence, left-multiplying (3.24) by the transpose of $\frac{\partial G}{\partial \gamma}$ yields $-\frac{1}{2}e^{-\gamma}(\xi_2^*)' > 0$, and so $(\xi_2^*)' < 0$.

Finally, since for every $\gamma \in \text{int } \Gamma$ we have,

$$D(\gamma) = \langle \boldsymbol{\xi}^*(\gamma), \mathbf{b}_\gamma \rangle - \mathcal{I}_+^*(\mathbf{A}^T \boldsymbol{\xi}^*(\gamma)),$$

it follows that D is differentiable and

$$D'(\gamma) = \langle \boldsymbol{\xi}^*(\gamma), \mathbf{b}'_\gamma \rangle + \left\langle (\boldsymbol{\xi}^*)'(\gamma), \mathbf{b}_\gamma - \frac{\partial \mathcal{I}_+^*}{\partial \boldsymbol{\xi}}(\mathbf{A}^T \boldsymbol{\xi}^*(\gamma)) \right\rangle;$$

the second term vanishes due to (3.23). ■

The next theorem in conjunction with Proposition 3.1 completely characterizes the unique optimal spatial density, solution to problem (3.8).

Theorem 3.2. *The optimization problem defined by (3.12) and (3.10) admits a unique optimal solution* $\gamma^* \in \text{int}\,\Gamma$.

Proof. For all $\gamma \in \Gamma$, let $\Psi(\gamma)$ denote the objective function in (3.12), i.e.,

$$\Psi(\gamma) = \gamma + 2\log F(\gamma), \quad \text{with} \quad F(\gamma) := \sup_{\varphi \in \mathcal{F}_\gamma} \int_S \sqrt{\varphi(\mathbf{x})}d\mathbf{x}. \tag{3.25}$$

From Theorem 3.1 we know that $F(\gamma) = -D(\gamma)$ over int Γ, and Lemma 3.1 implies that $\Psi(\gamma) = -\infty$ when γ is at the boundary of the interval Γ. Thus, dom Ψ = int Γ, and Ψ is proper. The function Ψ is also closed; indeed consider any sequence $\{\gamma_k, w_k\} \subset$ epi Ψ such that $(\gamma_k, w_k) \to (\gamma, w)$. Recall that restricting a function to its effective domain does not affect the epigraph; hence, we can assume that $\{\gamma_k\} \subset$ dom Ψ. Then, by Proposition 3.2 we know Ψ is continuous over its effective domain, and $\Psi(\gamma) = \lim_{k\to\infty} \Psi(\gamma_k) \geq \lim_{k\to\infty} w_k = w$, which shows that $(\gamma, w) \in$ epi Ψ. Now, since Ψ tends to $-\infty$ at the boundary of its effective domain, we can find a scalar η such that the upper level set $\{\gamma \in \text{dom}\,\Psi : \Psi(\gamma) \geq \eta\}$ is nonempty and bounded. Therefore we can invoke Theorem 2.1 (generalized version of Weierstrass' theorem) to conclude that the set of maxima Γ^* is nonempty and compact; moreover, $\Gamma^* \subseteq \text{int}\,\Gamma$.

For every $\gamma^* \in \Gamma^*$ note that since γ^* is an interior point of Γ, the following first order condition must be satisfied:

$$\Psi'(\gamma^*) = 1 + 2\frac{F'(\gamma^*)}{F(\gamma^*)} = 0. \tag{3.26}$$

Combining Theorem 3.1 with Proposition 3.2, we get

$$F'(\gamma^*) = -\langle \boldsymbol{\xi}^*(\gamma^*), \mathbf{b}'_{\gamma^*}\rangle = \frac{1}{2}\xi_2^*(\gamma^*)e^{-\gamma^*}.$$

Also, Proposition 3.1 leads to

$$D(\gamma) = \int_S \frac{d\mathbf{x}}{2\mathbf{A}^T\boldsymbol{\xi}^*(\gamma)(\mathbf{x})}, \quad \text{for all} \quad \gamma \in \text{int}\,\Gamma,$$

which implies that $D(\gamma) = 2\langle \boldsymbol{\xi}^*(\gamma), \mathbf{b}_\gamma\rangle$. Thus, returning to (3.26) and after some simple algebra we conclude that every $\gamma^* \in \Gamma^*$ must satisfy

$$2\xi_1^*(\gamma^*) = \tau\xi_2^*(\gamma^*), \tag{3.27}$$

where $\tau > 0$. Let $\tilde{\Gamma} = \{\gamma \in \text{int}\,\Gamma : \xi_2^*(\gamma) < 0\}$, and note from Corollary 3.1 that $\Gamma^* \subseteq \tilde{\Gamma}$. From Proposition 3.2 it follows that ξ_2^* is continuous, thus $\tilde{\Gamma}$ is an open set. Inside this set, $(\xi_2^*)'(\gamma) < 0$ and by Proposition 3.1 it is clear that ξ_1^* should be increasing so that the density defined in (3.22) integrates to unity over $\mathcal{S}$. Hence, returning to (3.27) we conclude that the maximizer γ^* has to be unique. ■

Corollary 3.2. *The optimal spatial density solution to (3.8) can be written as*

$$\varphi^*(\mathbf{x}) = \frac{K}{(\tau + 2\|\mathbf{x} - \mathbf{s}\|)^2}, \quad \textit{for all} \quad \mathbf{x} \in \mathcal{S}, \tag{3.28}$$

where $K > 0$ *is a normalization constant.*

Proof. Let $K = (\tau/2\xi_1^*(\gamma^*))^2$ and plug (3.27) back in (3.22). ∎

Remark: If a target is placed at location $\mathbf{x}$, then from (3.1) we note that $\tau + 2\|\mathbf{x} - \mathbf{s}\|$ is the average time the agent has to wait before he can place another target in $\mathcal{S}$. As explained at the beginning of the chapter, this is the source of the spatio-temporal dependence between the location and the rate of targets, and not surprisingly, it is reflected on the shape of the optimal spatial density φ^*.

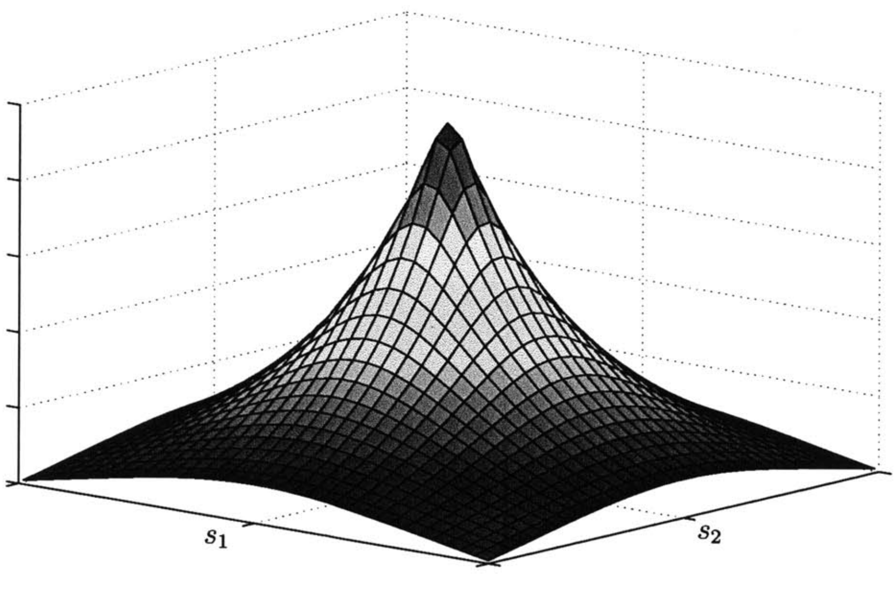

Figure 3-1: Spatial density φ^* defined in (3.28), unique maximizer of (3.8) in $\mathcal{L}_1(\mathcal{S})$.

It is important to highlight the fact that all of the results obtained are valid for any bounded set $\mathcal{S}$ with a smooth boundary, as defined at the beginning of the chapter. Neither clossedness nor connectedness is required for $\mathcal{S}$, which allows for any support constraint to be incorporated into the problem.

Chapter 4

Analysis of the Problem over Unbounded Regions

In this chapter we analyze the problem of dynamic vehicle routing with strategic spatio-temporal dependent demands over an unbounded region. Although the results obtained here are almost identical to those in Chapter 3, the approach is different: rather than relying on Fenchel duality, we exploit the geometric structure of the optimization problem.

4.1 Mathematical Formulation

An interesting question is whether the results obtained in the previous chapter can be extended to spatial densities with unbounded support in $\mathbb{R}^2$. Again, the spatio-temporal dependence between the location and rate of targets suggests that the game should still have a finite value. Mathematically, however, we must proceed with caution. In order to avoid the degenerate case where the optimal spatial density is singular with support $\{\infty\}$, we need to impose a mild restriction on the tail behavior of the densities the agent is allow to choose. Namely, we will work within the family

$$\mathcal{H} = \left\{ \varphi : \mathbb{R}^2 \to \mathbb{R}_+ \text{ s.t. } \int_{\mathbb{R}^2} \varphi(\mathbf{x}) d\mathbf{x} = 1, \int_{\mathbb{R}^2} \|\mathbf{x} - \mathbf{s}\|^q \varphi(\mathbf{x}) d\mathbf{x} = M \right\}, \tag{4.1}$$

where $q > 2$ and $M < \infty$. Since any constraint on the support can be taken into account through the introduction of indicator functions without changing the nature of the problem, we will assume without loss of generality, that supp $\varphi = \mathbb{R}^2$ for every $\varphi \in \mathcal{H}$.

Note that for every $\varphi \in \mathcal{H}$, the optimality of the TSP-based policy π^* stated by (3.7) remains valid (the proof given in [39] relies mainly on the classic TSP asymptotic result for the length of the shortest tour, which still holds [31]). Therefore, as $v \to 0$ the value of the zero-sum game is given by

$$\sup_{\varphi\in\mathcal{H}} \inf_{\pi\in\Pi} \overline{T}(\pi,\varphi) = \sup_{\varphi\in\mathcal{H}} \overline{T}(\pi^*,\varphi) = \inf_{\pi\in\Pi} \sup_{\varphi\in\mathcal{H}} \overline{T}(\pi,\varphi),$$

where

$$\overline{T}(\pi^*,\varphi) = \frac{\beta^2}{2v^2}\lambda_\varphi \left(\int_{\mathbb{R}^2} \sqrt{\varphi(\mathbf{x})}d\mathbf{x}\right)^2, \quad \text{for all} \quad \varphi \in \mathcal{H}.$$

Consequently, in order to find the optimal spatial density we face an optimization problem similar to (3.8); the difference is that the integration is carried over the entire $\mathbb{R}^2$, and $\varphi \in \mathcal{H}$. Applying the same logarithmic transformation and change of variables leads to the equivalent "decoupled" formulation

$$\sup_{\gamma\in\Gamma} \left\{ \gamma + 2 \sup_{\varphi\in\mathcal{H}_\gamma} \log \int_{\mathbb{R}^2} \sqrt{\varphi(\mathbf{x})}d\mathbf{x} \right\}, \tag{4.2}$$

where

$$\mathcal{H}_\gamma = \left\{ \varphi : \mathcal{S} \to \mathbb{R}_+ \text{ s.t. } \varphi \in \mathcal{H}, \int_{\mathbb{R}^2} \|\mathbf{x}-\mathbf{s}\|\varphi(\mathbf{x})d\mathbf{x} = \frac{e^{-\gamma}-\tau}{2} \right\}. \tag{4.3}$$

In this case Jensen's inequality gives, $0 \le \mathbb{E}_\varphi[\|\mathbf{x}-\mathbf{s}\|] \le (\mathbb{E}_\varphi[\|\mathbf{x}-\mathbf{s}\|^q])^{1/q} = M^{1/q}$, which defines

$$\Gamma = \left[-\log\left(2M^{1/q}+\tau\right), -\log\tau\right) \subset \mathbb{R}, \tag{4.4}$$

Thus the agenda will be the same as the one followed in Chapter 3: start by finding the optimal parametric family of spatial densities, and then determine the optimal rate.

4.2 Duality Approach

Consider the optimization problem

$$\inf\ \mathcal{I}(\varphi) := \int_{\mathbb{R}^2} -\sqrt{\varphi(\mathbf{x})}d\mathbf{x} \quad \text{subject to} \quad \varphi \in \mathcal{H}_\gamma, \tag{4.5}$$

equivalent to

$$\sup \left\{ \log \int_{\mathbb{R}^2} \sqrt{\varphi(\mathbf{x})}d\mathbf{x} : \varphi \in \mathcal{H}_\gamma \right\},$$

and note that (4.5) is feasible for every $\gamma \in \operatorname{int}\Gamma$, since we can always construct a power-law distribution whose density belongs to $\mathcal{H}_\gamma$. Let $\mathbf{A}$ represent a linear map defined over $\mathcal{L}_1(\mathbb{R}^2)$ with linearly independent components $\{A_i\} = \{1, \|\mathbf{x}-\mathbf{s}\|, \|\mathbf{x}-\mathbf{s}\|^q\}$, such that

$$(\mathbf{A}\varphi)_i = \int_{\mathbb{R}^2} A_i(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}. \tag{4.6}$$

Then, as shown in Chapter 3, defining $h(x) := -\sqrt{x} + \delta(x|\mathbb{R}_+)$ for all $x \in \mathbb{R}$ and letting $\mathbf{b}_\gamma \in \mathbb{R}^3$ be the column vector $\left(1, \frac{e^{-\gamma}-\tau}{2}, M\right)$, we can express problem (4.5) as

$$\inf \mathcal{I}_+(\varphi) := \int_{\mathbb{R}^2} h(\varphi(\mathbf{x}))d\mathbf{x} \quad \text{subject to} \quad \mathbf{A}\varphi = \mathbf{b}_\gamma,\ \varphi \in \mathcal{L}_1(\mathbb{R}^2). \tag{4.7}$$

Although problem (4.7) is very similar to (3.18), the approach used in Chapter 3 cannot be applied to find an optimal solution. Since for every $\varphi^\star \in \mathcal{L}_\infty(\mathbb{R}^2)$ we have $\lim_{\|x\|\to\infty} \varphi^\star(\mathbf{x}) < \infty$, it follows that $\lim_{\|x\|\to\infty} h^\star(\varphi^\star(\mathbf{x})) \neq 0$. Hence, the integral in (2.9) diverges for all $\varphi^\star \in \mathcal{L}_\infty(\mathbb{R}^2)$, which is the key result upon which the whole conjugate duality framework is built. In fact, general results establishing existence and characterization of solutions for problems like (4.7) seem to be restricted to the case where the domain of integration is of finite measure, where the issue of integrability does not pose any difficulties. The few results that are available for unbounded domains rely heavily on the structure of the problem, and are thus very specific. For example, in [21] a similar problem is solved; namely, (4.7) with $h(x) = -x\log(x) + \delta(x|\mathbb{R}_+)$, the problem of maximum entropy subject to moment constraints; however, the existence of an optimal solution is based on a particular property [23] of Shannon's entropy: that if $\mathcal{I}_+(\varphi_k) \to \mathcal{I}_+(\varphi)$ then $\varphi_k \to \varphi$ in $\mathcal{L}_1$. This property is not exhibited in our case. We will therefore solve (4.7) directly, without using Fenchel duality. As in [14], we first prove a partially finite duality result which will then allow us to show primal attainment.

A Partially Finite Duality Result

Lemma 4.1. *The function defined by*

$$V(\mathbf{b}) := \inf \left\{\mathcal{I}_+(\varphi) : \mathbf{A}\varphi = \mathbf{b}, \varphi \in \mathcal{L}_1(\mathbb{R}^2)\right\}, \tag{4.8}$$

is proper and convex over its effective domain.

Proof. Properness is evident because $V(\mathbf{b}) \leq 0$. To show convexity, consider two vectors $\mathbf{b}_1$,$\mathbf{b}_2 \in \operatorname{dom} V$, and $\alpha \in [0,1]$. For an arbitrary $\epsilon > 0$, let φ_1 and φ_2 be such that $\varphi_i \in \mathcal{L}_1(\mathbb{R}^2)$, $\mathbf{A}\varphi_i = \mathbf{b}_i$ and $\mathcal{I}_+(\varphi_i) \leq V(\mathbf{b}_i) + \epsilon$. Then, the fact that $\alpha\varphi_1 + (1-\alpha)\varphi_2$ is feasible implies that

$$V(\alpha\mathbf{b}_1 + (1-\alpha)\mathbf{b}_2) \leq \mathcal{I}_+(\alpha\varphi_1 + (1-\alpha)\varphi_2).$$

On the other hand, the convexity of $\mathcal{I}_+$ leads to,

$$\begin{aligned}\mathcal{I}_+(\alpha\varphi_1 + (1-\alpha)\varphi_2) &\leq \alpha\mathcal{I}_+(\varphi_1) + (1-\alpha)\mathcal{I}_+(\varphi_2)\\ &\leq \alpha V(\mathbf{b}_1) + (1-\alpha)V(\mathbf{b}_2) + \epsilon,\end{aligned}$$

and given that ϵ is arbitrary, the result follows. ■

Theorem 4.1. *Let Γ be defined as in (4.4), and V as in (4.8). Then, for every $\gamma \in \operatorname{int}\Gamma$ we have,*

$$V(\mathbf{b}_\gamma) = D(\gamma) := \max_{\boldsymbol{\xi}\in\mathbb{R}^2}\left\{\langle \boldsymbol{\xi}, \mathbf{b}_\gamma\rangle + \int_{\mathbb{R}^2} \frac{d\mathbf{x}}{4\mathbf{A}^T\boldsymbol{\xi}(\mathbf{x})} : \mathbf{A}^T\boldsymbol{\xi} < 0\right\}, \tag{4.9}$$

where $\mathbf{A}^T$ is the adjoint map, $\mathbf{A}^T\boldsymbol{\xi}(\mathbf{x}) = \xi_1 + \xi_2\|\mathbf{x}-\mathbf{s}\| + \xi_3\|\mathbf{x}-\mathbf{s}\|^q$. Moreover, the maximizer of the right-hand side of (4.9) is unique.

Proof. Consider an arbitrary $\gamma \in \operatorname{int}\Gamma$. By definition of conjugate function (3.20),

$$h^\star(y) \geq xy - h(x), \quad \text{for all} \quad x, y \in \mathbb{R}.$$

For any $\varphi \in \mathcal{H}_\gamma$ and $\boldsymbol{\xi} \in \mathbb{R}^3$, we have

$$\begin{aligned}\int_{\mathbb{R}^2} h^\star(\mathbf{A}^T\boldsymbol{\xi}(\mathbf{x}))d\mathbf{x} &\geq \int_{\mathbb{R}^2} \varphi(\mathbf{x})\mathbf{A}^T\boldsymbol{\xi}(\mathbf{x})d\mathbf{x} - \int_{\mathbb{R}^2} h(\varphi(\mathbf{x}))d\mathbf{x}\\ &= \langle \boldsymbol{\xi}, \mathbf{b}_\gamma\rangle - \int_{\mathbb{R}^2} h(\varphi(\mathbf{x}))d\mathbf{x}.\end{aligned} \tag{4.10}$$

Moreover, for every $\boldsymbol{\xi}$ in the set

$$\mathcal{M} = \{\boldsymbol{\xi} \in \mathbb{R}^3 : \mathbf{A}^T\boldsymbol{\xi}(\mathbf{x}) < 0, \text{ for all } \mathbf{x} \in \mathbb{R}^2\}, \tag{4.11}$$

the integral on the left-hand side of (4.10) is finite, hence

$$\mathcal{I}_+(\varphi) \geq \langle \boldsymbol{\xi}, \mathbf{b}_\gamma\rangle - \int_{\mathbb{R}^2} h^\star(\mathbf{A}^T\boldsymbol{\xi}(\mathbf{x}))d\mathbf{x} = \langle \boldsymbol{\xi}, \mathbf{b}_\gamma\rangle + \int_{\mathbb{R}^2} \frac{d\mathbf{x}}{4\mathbf{A}^T\boldsymbol{\xi}(\mathbf{x})}.$$

Therefore, taking the infimum over $\varphi \in \mathcal{H}_\gamma$ and then the supremum over $\boldsymbol{\xi} \in \mathcal{M}$ renders $V(\mathbf{b}_\gamma) \geq D(\gamma)$, for any $\gamma \in \operatorname{int}\Gamma$.

We now prove the other inequality together with the attainment of the supremum on the right-hand side of (4.9). A similar proof as the one given for Lemma 3.2 establishes that $\mathbf{b}_\gamma \in \operatorname{ri}(\mathbf{A}\operatorname{dom}\mathcal{I}_+)$, which clearly implies that $\mathbf{b}_\gamma \in \operatorname{ri}(\operatorname{dom} V)$. By Lemma 4.1, V is proper and convex, hence the subdifferential of V at $\mathbf{b}_\gamma$ is non-empty, as stated in Proposition 2.1. Let $\boldsymbol{\xi}^*_\gamma \in \partial V(\mathbf{b}_\gamma)$; we have by definition (2.3),

$$V(\mathbf{y}) \geq V(\mathbf{b}_\gamma) + \left\langle \boldsymbol{\xi}^*_\gamma, \mathbf{y} - \mathbf{b}_\gamma\right\rangle, \quad \text{for all} \quad \mathbf{y} \in \mathbb{R}^3.$$

For some $\rho > 0$, define the set $\mathcal{S}_\rho = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}-\mathbf{s}\| \leq \rho\}$. Then, for every $\varphi \in \mathcal{L}_1(\mathcal{S}_\rho)$ such that $\mathbf{A}\varphi = \mathbf{y}$, we get

$$\left\langle \boldsymbol{\xi}^*_\gamma, \mathbf{b}_\gamma\right\rangle - V(\mathbf{b}_\gamma) \geq \left\langle \boldsymbol{\xi}^*_\gamma, \mathbf{A}\varphi\right\rangle - \mathcal{I}_+(\varphi).$$

Taking the supremum over $\varphi \in \mathcal{L}_1(\mathcal{S}_\rho)$ yields,

$$\langle \boldsymbol{\xi}_\gamma^*, \mathbf{b}_\gamma \rangle - V(\mathbf{b}_\gamma) \geq \mathcal{I}_+^*(\mathbf{A}^T \boldsymbol{\xi}_\gamma^*). \tag{4.12}$$

Note that for every $\boldsymbol{\xi} \in \mathcal{M}$ and any φ feasible for (4.7), we have $\varphi \mathbf{A}^T \boldsymbol{\xi} < 0$ which, through integration, leads to $\langle \boldsymbol{\xi}, \mathbf{b}_\gamma \rangle < 0$. Hence $D(\gamma) \leq 0$, and since $D(\gamma) \leq V(\mathbf{b}_\gamma) \leq 0$, it follows that $V(\mathbf{b}_\gamma)$ is finite. Therefore, $\mathcal{I}_+^*(\mathbf{A}^T \boldsymbol{\xi}_\gamma^*)$ is finite and given the fact that $\mu(\mathcal{S}_\rho) < \infty$, we can invoke Proposition 2.4 with (3.20) to arrive at,

$$\mathcal{I}_+^*(\mathbf{A}^T \boldsymbol{\xi}_\gamma^*) = -\int_{\mathcal{S}_\rho} \frac{d\mathbf{x}}{4\mathbf{A}^T \boldsymbol{\xi}_\gamma^*(\mathbf{x})}, \quad \text{with} \quad \mathbf{A}^T \boldsymbol{\xi}_\gamma^*(\mathbf{x}) < 0 \text{ for all } \mathbf{x} \in \mathcal{S}_\rho, \tag{4.13}$$

which holds for any arbitrary $\rho > 0$. Finally, the fact that $\mathbf{A}^T \boldsymbol{\xi}_\gamma^*(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathcal{S}_\rho$ and all $\rho > 0$ implies that $\mathbf{A}^T \boldsymbol{\xi}_\gamma^*(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathbb{R}^2$, and because of the circular symmetry of the integrand in (4.13) we can let $\rho \to \infty$ in (4.12) and apply the monotone convergence theorem (see [17]) to conclude that,

$$V(\mathbf{b}_\gamma) \leq \langle \boldsymbol{\xi}_\gamma^*, \mathbf{b}_\gamma \rangle + \int_{\mathbb{R}^2} \frac{d\mathbf{x}}{4\mathbf{A}^T \boldsymbol{\xi}_\gamma^*(\mathbf{x})}.$$

The uniqueness of $\boldsymbol{\xi}_\gamma^*$ follows from the strict concavity of the objective function and the fact that the components of $\mathbf{A}$ are linearly independent. ■

Corollary 4.1. *For every $\gamma \in \text{int } \Gamma$, $\xi_1^*(\gamma) < 0$ and $\xi_3^*(\gamma) \leq 0$.*

Proof. By definition, $\mathbf{A}^T \boldsymbol{\xi}^*(\gamma)(\mathbf{x}) = \xi_1^*(\gamma) + \xi_2^*(\gamma)\|\mathbf{x} - \mathbf{s}\| + \xi_3^*(\gamma)\|\mathbf{x} - \mathbf{s}\|^q < 0$ for all $\mathbf{x} \in \mathbb{R}^2$. Hence, setting $\mathbf{x} = \mathbf{s}$ shows that $\xi_1^*(\gamma) < 0$; on the other hand, if $\xi_3^*(\gamma) > 0$, then $\mathbf{A}^T \boldsymbol{\xi}^*(\gamma)(\mathbf{x})$ would become positive for large enough $\|\mathbf{x} - \mathbf{s}\|$, but this cannot happen. ■

As stated in [12], for problems with the structure of (4.7) partially finite duality is usually only available when the integrand function h is defined over a finite measure set (like in Chapter 3). Besides the Kullback-Liebler divergence (see [16] for definition and details), this provides another example for which a duality result also holds.

Primal Attainment and Uniqueness

Based on the preceding duality theorem, the next result characterizes the unique family of spatial densities solution to (4.7).

Proposition 4.1. *For every $\gamma \in \text{int } \Gamma$, the unique optimal solution to (4.7) and (4.3) is given by,*

$$\varphi_\gamma^*(\mathbf{x}) = \frac{1}{4(\mathbf{A}^T \boldsymbol{\xi}^*(\gamma)(\mathbf{x}))^2}, \quad \textit{for all} \quad \mathbf{x} \in \mathbb{R}^2, \tag{4.14}$$

where $\boldsymbol{\xi}^* \in \mathbb{R}^3$ is the dual optimum (maximizer of (4.9)).

Proof. Fix $\gamma \in \operatorname{int} \Gamma$ and let $\boldsymbol{\xi}^*$ be the maximizer of the right-hand side of (4.9) for γ, whose existence was shown in Theorem 4.1. Since the set $\mathcal{M}$ defined in (4.11) is open, $\boldsymbol{\xi}^*$ must satisfy the first order condition: $\mathbf{b}_\gamma = \mathbf{A}\varphi^*$, where φ^* is given by (4.14). Then, φ^* is feasible for problem (4.7), and

$$\mathcal{I}_+(\varphi^*) = \int_{\mathbb{R}^2} \frac{\mathbf{A}^T\boldsymbol{\xi}^*(\mathbf{x})d\mathbf{x}}{4(\mathbf{A}^T\boldsymbol{\xi}^*(\mathbf{x}))^2} + \int_{\mathbb{R}^2} \frac{d\mathbf{x}}{4\mathbf{A}^T\boldsymbol{\xi}^*(\mathbf{x})} = \langle \boldsymbol{\xi}^*, \mathbf{b}_\gamma \rangle + \int_{\mathbb{R}^2} \frac{d\mathbf{x}}{4\mathbf{A}^T\boldsymbol{\xi}^*(\mathbf{x})}.$$

Returning to (4.9) we observe that $\mathcal{I}_+(\varphi^*) = D(\gamma)$, which implies through Theorem 4.1, that φ^* is indeed optimal. The uniqueness is given by the uniqueness of $\boldsymbol{\xi}^*$, in agreement with the fact that $\mathcal{I}_+$ is strictly convex. ■

Finally, we show the existence of a spatial density φ^*, optimal for (4.2). The uniqueness of φ^*, as established for the case in Chapter 3, is harder to prove in this case and is left as an open question.

Proposition 4.2. *The set of optimal solutions Γ^* for the problem defined by (4.2) and (4.4) is nonempty and compact. Moreover, $\Gamma^* \subseteq \operatorname{int} \Gamma$ and $\boldsymbol{\xi}^*(\gamma^*) \leq 0$ for all $\gamma^* \in \Gamma^*$.*

Proof. For all $\gamma \in \Gamma$, let $\Psi(\gamma)$ denote the objective function in (4.2), i.e.,

$$\Psi(\gamma) = \gamma + 2 \log F(\gamma), \quad \text{with} \quad F(\gamma) := \sup_{\varphi \in \mathcal{H}_\gamma} \int_{\mathbb{R}^2} \sqrt{\varphi(\mathbf{x})} d\mathbf{x}.$$

The function Ψ is closed (the proof being identical to that in Theorem 3.2), and given that $F(\gamma) = -D(\gamma)$ over $\operatorname{int} \Gamma$ as implied by Theorem 4.1, it is also proper. Hence, since $\operatorname{dom} \Psi \subseteq \Gamma$ and Γ is bounded we can invoke Weierstrass' theorem, cf. Theorem 2.1, to conclude that Γ^* is nonempty and compact.

From Jensen's inequality we know that $\mathbb{E}_\varphi[\|\mathbf{x} - \mathbf{s}\|] \leq (\mathbb{E}_\varphi[\|\mathbf{x} - \mathbf{s}\|^q])^{1/q}$, where the inequality is strict for every continuous distribution. Therefore, for γ at the left endpoint of the interval Γ in (4.4) the only feasible distribution is singular with support over the set $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x} - \mathbf{s}\|^q = M\}$. Then, for $\gamma = -\log(2M^{1/q} + \tau)$ we have $F(\gamma) = 0$, thus $\Psi(\gamma) = -\infty$, and $\Gamma^* \subseteq \operatorname{int} \Gamma$. This in turn implies that every $\gamma^* \in \Gamma^*$ must satisfy the first order optimality condition $\Psi'(\gamma^*) = 0$ (the differentiability of Ψ is due to that of $\boldsymbol{\xi}^*$, which is based on the implicit function theorem as in Proposition 3.2). After some simple algebraic manipulations (see Theorem 3.2), we arrive at the equivalent condition:

$$2(\xi_1^*(\gamma^*) + M\xi_3^*(\gamma^*)) = \tau\xi_2^*(\gamma^*), \quad \text{for all} \quad \gamma^* \in \Gamma^*,$$

and from Corollary 4.1, we conclude that $\xi_2^*(\gamma^*) < 0$ for all $\gamma^* \in \Gamma^*$. ■

Chapter 5

Numerical Explorations

This chapter contains numerical experiments that shed light on the theoretical results derived in the preceding sections. We will analyze the system time achieved by the optimal spatial density, and quantify its change when the vehicle's routing policy discriminates targets based on their location on the space.

5.1 Simulations

Throughout, let $\mathcal{S}_\rho = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x} - \mathbf{s}\| \leq \rho\}$ be the support of densities, and $\tau = 0.1$.

Finding the Optimal Density

In order to find φ^*, we proceed as in Chapter 3: (i) first solve the dual problem (3.19) using an interior point algorithm to find $\boldsymbol{\xi}^*(\gamma)$, and then (ii) find γ^*, the solution to problem (3.12). For $\rho = 1$, (i) and (ii) are illustrated in Figure 5-1 and Figure 5-2, respectively. Note that the behavior of $\boldsymbol{\xi}^*(\gamma)$ and $\Psi(\gamma)$ coincides with the analytical description presented in Chapter 3.

The Optimal System Time

If the physical constraint imposed by the agent carrying and placing the targets on $\mathcal{S}$ were removed and the rate were fixed, then the distribution that attains the maximum system time would be uniform; this was proved in [7] using a Hardy-Littlewood-Pólya inequality [19]. However, when the spatio-temporal dependence is introduced, a uniform distribution will induce a rate that is always smaller than λ_{φ^*}. Indeed, this is shown in Figure 5-3 and it is due to the fact that all locations are equally likely regardless their distance from the depot. The overall effect is a degradation on the system time, which is observed in Figure 5-4 to be higher that 20% already for $\rho = 5$.

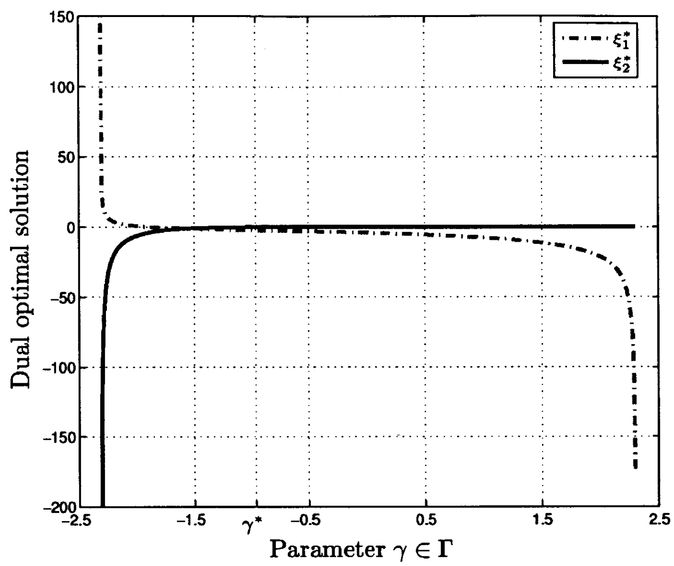

Figure 5-1: Evolution of dual optimal solution $\boldsymbol{\xi}^*(\gamma)$.

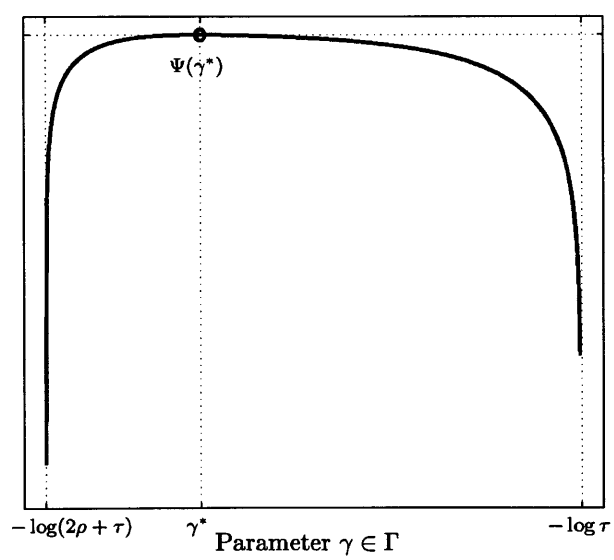

Figure 5-2: Objective function $\Psi(\gamma)$ in (3.25).

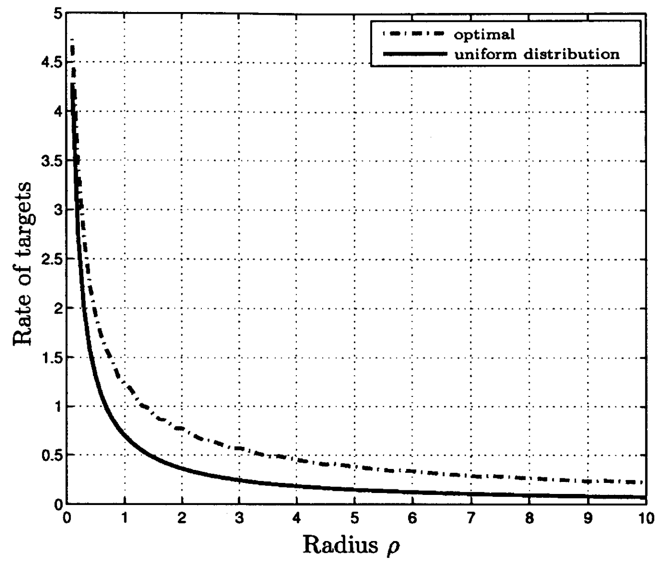

Figure 5-3: Rate of targets attained by uniform distribution.

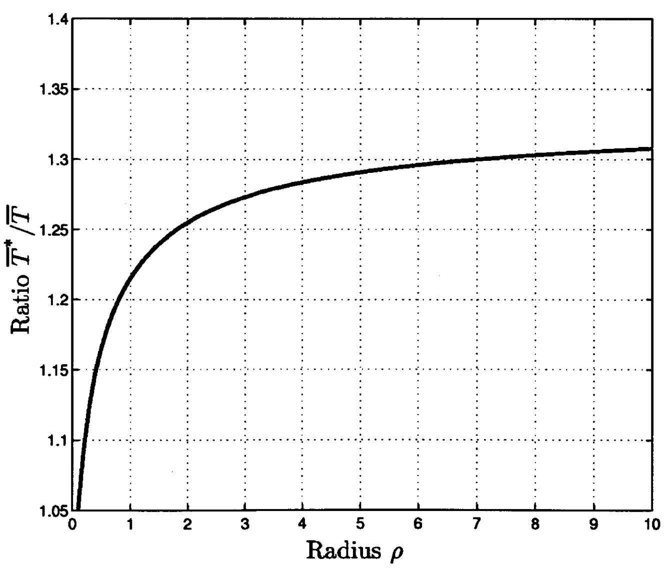

Figure 5-4: System time $\overline{T}$ with uniform distribution and the optimum $\overline{T}^*$.

Effect of the Speed of the Vehicle

Recall that the pair (π^*, φ^*) constitutes an equilibrium for the game in the limit as $v \to 0$. Therefore, understanding how the relative error between $\overline{T}(\pi^*, \varphi^*)$ and the measured optimal system time $\overline{T}^*$ decreases as v becomes closer to zero is an issue of practical significance.

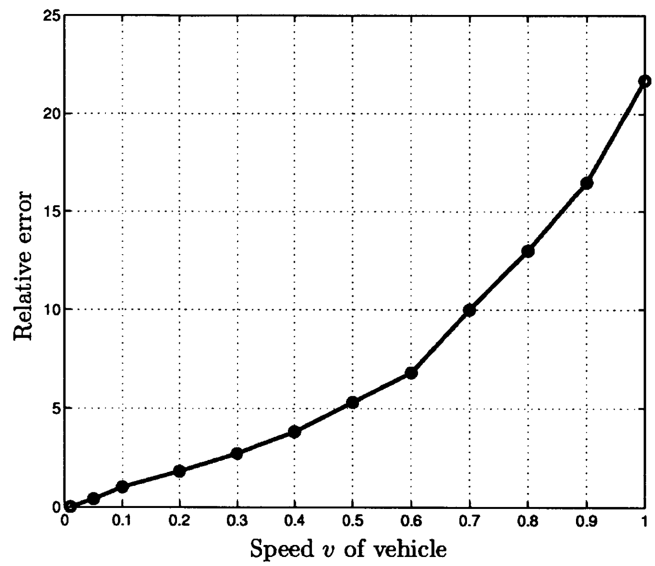

Figure 5-5: Relative error of optimal system time $\overline{T}^*$ as a function of the speed v.

To that end, we implemented in Matlab the TSP-based routing policy described in Section 3.3 based on the Lin & Kernighan's algorithm [25]. The results obtained are gathered in Figure 5-5, where we note that for $v = 0.01$ the relative error is already less than 5%. This observation is actually not surprising, since as implied in [24], the expression for the system time (3.6) in heavy load is usually a fairly good approximation for the system time under "intermediate" load regimes.

5.2 Spatial Characteristics of Routing Policies

Thus far we have consider the case where the vehicle services target locations according to spatially unbiased routing policies (cf. Definition 3.1). Assume now that we remove this hypothesis, and allow the vehicle to visit targets following policies for which the mean waiting time vary depending on the location of the demand; let Σ be the class of spatially biased policies. Since (3.2) imposes a constraint on the service policy, the system time of the optimal spatially biased policy should be smaller than $\overline{T}(\pi^*, \varphi)$ for all $\varphi \in \mathcal{F}$.

As established in [7] and [39], for all $\sigma \in \Sigma$ and all $\varphi \in \mathcal{F}$ the system time is asymptotically lower bounded by,

$$\overline{T}(\sigma, \varphi) \geq \frac{\beta^2}{2v^2}\lambda_\varphi \left(\int_{\mathcal{S}} \varphi^{2/3}(\mathbf{x}) d\mathbf{x} \right)^3 \quad \text{as} \quad v \to 0.$$

Moreover, there exists a routing policy $\sigma^* \in \Sigma$ based on performing TSP tours through outstanding target locations (such that locations where the density of targets is higher are given more priority), that attains the minimum

$$\overline{T}(\sigma^*, \varphi) = \frac{\beta^2}{2v^2}\lambda_\varphi \left(\int_{\mathcal{S}} \varphi^{2/3}(\mathbf{x}) d\mathbf{x} \right)^3, \quad \text{for all} \quad \varphi \in \mathcal{F}.$$

Hence, we get

$$\sup_{\varphi \in \mathcal{F}} \inf_{\sigma \in \Sigma} \overline{T}(\sigma, \varphi) = \sup_{\varphi \in \mathcal{F}} \overline{T}(\sigma^*, \varphi) = \inf_{\sigma \in \Sigma} \sup_{\varphi \in \mathcal{F}} \overline{T}(\sigma, \varphi).$$

In order to characterize an equilibrium for the game over $\Sigma \times \mathcal{F}$, we need only find the maximizer φ^*. Proceeding in a similar way as in Chapter 3, it can be shown that the problem

$$\sup_{\varphi} \lambda_\varphi \left(\int_{\mathcal{S}} \varphi^{2/3}(\mathbf{x}) d\mathbf{x} \right)^3 \quad \text{subject to} \quad \varphi \in \mathcal{F},$$

admits a unique optimal solution $\varphi^* : \mathcal{S} \to \mathbb{R}_+$ given by,

$$\varphi^*(\mathbf{x}) = \frac{1}{(K_1 + K_2\|\mathbf{x} - \mathbf{s}\|)^3}, \tag{5.1}$$

where K_1 and K_2 are positive constants.

Figure 5-6 illustrates the optimal system time for unbiased and biased policies, when the speed v of the vehicle is unitary; as expected, the former is larger. The intuition behind this is simple. Under the optimal unbiased policy π^*, the few targets that are placed very far away from the depot $\mathbf{s}$ are visited with the same frequency as the larger fraction located closer to $\mathbf{s}$. These tours, although unusual, are very time consuming and cause large delays in the service of targets near the depot. The ultimate result is an increase in the average waiting time of a typical demand, the system time. On the other hand, in the optimal biased policy σ^* targets located very distant from $\mathbf{s}$ are left unserviced until their number cease to be small. This way larger trips are used more efficiently, allowing more time to visit target locations closer to $\mathbf{s}$. The overall result is an improvement on the system time, which becomes arbitrarily large as $\rho \to \infty$.

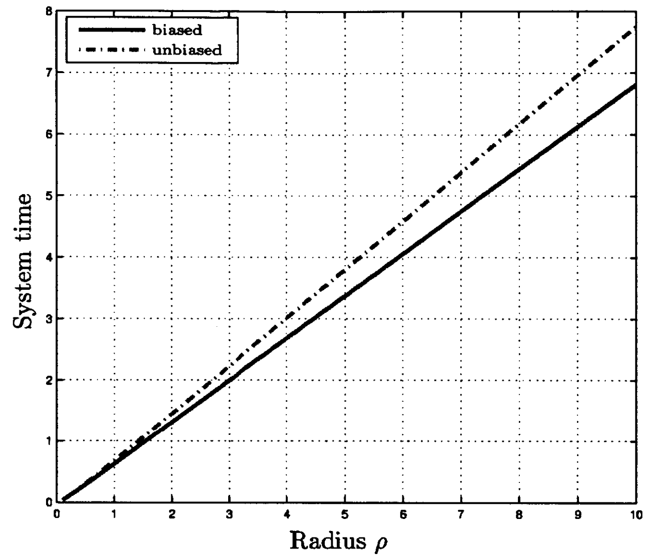

Figure 5-6: Optimal system times for spatially unbiased/biased routing policies.

In fact, the above is intimately related to the reason why the power-law in (5.1) decays faster than the one defined in (3.28): given that target locations far away from the depot are visited not very often, the way to make the system time as large as possible is to increase the density of targets in the vicinity around $\mathbf{s}$.

Chapter 6

Concluding Remarks

In this thesis we have studied the problem of strategic dynamic vehicle routing where demands were carried and place in a region S by an adversarial agent operating from a central depot with unitary capacity. We have analyzed the problem from a zero-sum game theoretic perspective, with the average waiting time of a typical demand as utility function, and characterized equilibria in the heavy load regime.

We discriminated between two cases: bounded and unbounded domains. In both cases, a routing policy based on performing successive TSP tours through outstanding demands was found to be optimal. In order to find the optimal point process according to which the agent should place targets, which would fully characterize the equilibrium, we faced the problem of maximizing a non-convex functional over the infinite-dimensional space of probability densities with support over S. Through a logarithmic change of variables we decoupled the spatio-temporal dependence induced by the physical constraints imposed on the agent's behavior, and split the problem into two components: one to determine an optimal parametric family of spatial densities, and another one to determine the optimal temporal parameter that would identify the optimal density, solution to the problem, within the previously found family.

Deriving the optimal parametric family entailed the maximization of a nowhere differentiable concave integral functional subject to linear equality constraints. To solve this problem, duality seemed to be ideal since it would transformed an infinite-dimensional maximization into an $n-$dimensional minimization of a convex function; however, the commonly used approach of differentiating the associated Lagrangian could not be rigorously formalized because derivatives would not exist. For bounded domains, we bypassed this issue through the use of Fenchel conjugate duality and results from partially finite programming. For unbounded domains, integral conjugates were not well-defined and instead, we exploited the structure of the objective function and the linear constraints to find the solution. With the optimal parametric family, Weierstrass theorem would guarantee the existence of an optimal parameter; other geometric arguments were needed to establish the uniqueness. Remarkably, all the results obtained hold for any region $\mathcal{S}$ with a sufficiently smooth boundary. This is an important fact, since it allows to introduce support constraints into the problem. Also, the extension to the multi-vehicle case is straightforward.

Regarding avenues for future research, we believe it would be interesting to apply the optimization tools used in this thesis to the field of geographic profiling and optimal foraging. Geographic profiling was conceived as a tool to study spatial patterns of serial crime. Given the locations of a series of crimes attributed to a single criminal, it determines the most probable area for the criminals home base or "anchor point". More recently, it has been used to study predator-prey interactions (see [28] and references therein). Such foraging problems seem amenable to be analyzed under the same framework with more meaningful utility functions that, for example, model the balance between prey density and competition.

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