Families of p-adic Galois Representations

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by

Fucheng Tan

Submitted to the Department of Mathematics
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PHD

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Abstract

In this thesis, I first generalize Kisin's theory of finite slope subspaces to arbitrary p-adic fields, and then apply it to the generic fibers of Galois deformation spaces. I study the finite slope deformation rings in details by computing the dimensions of their Zariski cotangent spaces via Galois cohomologies. It turns out that the Galois cohomologies tell us not only the formal smoothness of finite slope deformation rings, but also the behavior of the Sen operator near a generic de Rham representation. Applying these results to the finite slope subspace of two dimensional Galois representations of the absolute Galois group of a p-adic field, we are able to show that a *generic* (indecomposible) de Rham representation lies in the finite slope subspace.

It follows from the construction of the finite slope subspace that the complete local ring of a point in the finite slope subspace is closely related to the finite slope deformation ring at the same point. As a consequence, we manage to show the flatness of the weight map near generic de Rham points, and accumulation and smoothness of generic de Rham points. In particular, we have a precise dimension formula for the finite slope subspace. Taking into account twists by characters, we define the nearly finite slope subspace, which is believed to serve as the local eigenvariety, as is suggested by Colmez's theory of trianguline representation. Following Gouvêa-Mazur and Kisin, we construct an infinite fern in the local Galois deformation space. Moreover, we define the global eigenvariety for GL_2 over any number field, and give a lower bound of its dimension.

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Contents

Chapter 1

Introduction

Throughout the paper, p is a fixed prime number. We fix the p-adic valuation $v_p(\cdot)$ with $v_p(p) = 1$ on $\mathbb{Q}_p$, which extends uniquely to any finite extension $K/\mathbb{Q}_p$ (which we call a *p-adic field*), where it is written as $v_K(\cdot)$. The valuation then extends to the algebraic closure $\bar{\mathbb{Q}}_p$, whose p-adic completion is denoted as $\mathbb{C}_p$. We write $G_K = \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$ for the absolute Galois group of K for K a p-adic field, and write G_F for the absolute Galois group of a number field F similarly.

For K a p-adic field, let ϖ_K be a fixed uniformizer of K. We let K_0 denote the maximal unramified extension of $\mathbb{Q}_p$ inside K, and let $f = [K_0 : \mathbb{Q}_p]$ be the residue degree. Denote by $\chi_K \colon G_K \to \mathcal{O}_K^\times$ the cyclotomic character. We choose an element $\mathrm{Frob}_p \in G_{\mathbb{Q}_p}$, a lifting of the map on $\bar{\mathbb{F}}_p$ raising an element to its p-th power, such that $\mathrm{Frob}_p^f := \mathrm{Frob}_K \in G_K$ is mapped to ϖ_K via the cyclotomic character. We call it (and its powers) the *arithmetic* Frobenius, whose inverse is called the *geometric* Frobenius.

Our convention is that χ_K has Hodge-Tate weight -1, as a 1-dimensional continuous representation of G_K.

Let X be a rigid analytic space (in the sense of Tate) over a p-adic field E. We say that a subset $S \subset X$ is a *closed subset* of X if there exists an admissible covering $\{U_i\}_{i\in I}$ of X such that for each $i \in I$, $S \cap U_i$ is the zero-locus of some ideal $I_i \in \mathcal{O}(U_i)$. We say that a subset S is *Zariski-dense* if the only closed subset of X containing S is X itself. For a point $x \in X(E')$ with E' a finite extension of $\mathbb{Q}_p$, let $\bar{x}$ be the image of x in X, and $\hat{\mathcal{O}}_{X,\bar{x}}$ the complete local ring at this closed point. We denote by $\hat{\mathcal{O}}_{X,x}$ the extension of scalars to E' of $\hat{\mathcal{O}}_{X,\bar{x}}$ from the residue field of $\bar{x}$. We identify closed points in X with residue field E' and E'-valued points of X.

1.1 Hecke operator on modular forms and φ-operator on Fontaine's modules

Let f be a (classical) cuspidal modular eigenform on $\Gamma_1(N)$ with $N \in \mathbb{Z}_{\geq 1}$, and let $k \geq 2$ and ϵ be its weight and nebentypus character. Denote by $a_n(f)$ the n-th coefficient of the q-expansion of f. By the well-known theorem of Deligne, there is a (unique) Galois representation

$$G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\bar{\mathbb{Q}})$$

which, after extension by scalars via an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ that we will fix, gives rise to the p-adic representation

$$\rho_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$$

for (any) prime p. The representation ρ_f is characterized by the condition that for any prime $\ell \nmid Np$ the Hecke polynomial of $\rho_f(\mathrm{Frob}_\ell)$ is

$$X^2 - a_\ell(f)X + \epsilon(\mathrm{Frob}_\ell)\ell^{k-1}.$$

At the prime p, we have the following results of Falthings, Tsuji and Saito about $V_f := \rho_f^*|_{G_{\mathbb{Q}_p}}$. First of all, V_f is de Rham with Hodge-Tate weights $0, k-1$. Moreover,

- If $p \nmid N$, then V_f is crystalline.
- If $p \mid N$ and $a_p(f) \neq 0$, then V_f is semi-stable but not crystalline.
- If $p \mid N$ and $a_p(f) = 0$, then V_f is potentially crystalline.

More precisely, we have the following result.

Theorem 1.1.1 (T. Saito, [31]). *Let U_p denote the Atkin-Lehner operator on f. Let $\lambda = a_p(f)$ be the (only) U_p-eigenvalue if $p \mid N$ and $a_p(f) \neq 0$, and be one of the two roots of $X^2 - a_p(f)X + \epsilon(\mathrm{Frob}_p)p^{k-1}$ if $p \nmid N$.*

If $\lambda \neq 0$, then it is an eigenvalue of the φ operator on Fontaine's module $D_{\mathrm{cris}}(\rho_f^*|_{G_{\mathbb{Q}_p}})$.

In particular, for any prime $p \nmid N$, if the two roots of $X^2 - a_p(f)X + \epsilon(\mathrm{Frob}_p)p^{k-1}$ are non-zero and distinct, then they are the φ-eigenvalues on $D_{\mathrm{cris}}(\rho_f^*|_{G_{\mathbb{Q}_p}})$. We refer the reader to Section 2.1 for more details about Fontaine's mudules.

In the following, we will see that an analogue of Theorem 1.1.1 holds for families of (overconvergent) modular forms.

1.2 The Coleman-Mazur eigencurve

Let p be an odd prime . For f a (classical elliptic) cuspidal eigenform of weight $k \geq 2$ and level $\Gamma_1(p^n N)$, where $p \nmid N$ and $n \geq 1$, the *slope* of f is by definition the p-adic valuation of the U_p-eigenvalue on f. Let

$$\Lambda_N = \varprojlim \mathbb{Z}_p[(\mathbb{Z}/p^n N\mathbb{Z})^\times] \simeq \mathbb{Z}_p[(\mathbb{Z}/pN\mathbb{Z})^\times] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\![1 + p\mathbb{Z}_p]\!].$$

Denote by $\mathcal{W}$ the $\mathbb{Q}_p$-analytic space associated to Λ_N.

Assume the tame level $N = 1$ in the rest of this subSection for simplicity. The construction of eigencurve has been generalized to arbitrary tame level by Kevin Buzzard [6].

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \bar{\mathbb{F}}_p$ be a two dimensional *odd* representation, where S is a finite set of primes containing the infinite place and all the primes dividing pN. We assume that $\bar{\rho}$ has only scalar endomorphisms. Denote by ρ^{univ} the universal representation, $R_{\bar{\rho}}^{\mathrm{univ}}$ the associated universal deformation ring, and Z the rigid analytic space associated to $R_{\bar{\rho}}^{\mathrm{univ}}$. As is explained in Section 4 of [12] (in the case $N = 1$) the ring $R_{\bar{\rho}}^{\mathrm{univ}}$ has a natural Λ_1-algebra structure, which gives rise to a weight map

$$w : Z \to \mathcal{W}.$$

Set $\mathcal{H}$ to be the commutative polynomial (topological) Λ_1-algebra generated by the formal variables X_q for all prime numbers $q \in \mathbb{Z}$ and let $\mathcal{H}'$ be the sub-ring generated by the formal variables X_q for $q \notin S$. We have a well-defined action of $\mathcal{H}$ on the space $M_{Np^\infty}(r)$ of r-overconvergent modular forms of tame level N by letting X_q act as the Hecke operator T_q for each $q \notin S$ and as U_q for $q \in S$. We refer the reader to [12] for the definition of overconvergent modular forms and the Hecke algebras.

Let

$$\iota : \mathcal{H}' \to R_{\bar{\rho}}^{\mathrm{univ}}$$

be the map sending X_q to $\mathrm{trace}(\rho^{\mathrm{univ}}(\mathrm{Frob}_q))$ for q outside S, which is well-defined by a theorem of Gouvêa-Hida (cf. Section 5.2 [12]).

For r sufficiently close to 1, by Theorem 4.3.1 of [12], the Atkin-Lehner operator U_p acts on $M_{p^\infty N}(r)$ *completely continuously.* By Sec.4 of [12], we can, for each $\alpha \in \mathcal{H}'$, form a Fredholm series with coefficients in Λ_1

$$P_{\alpha U_p}(Y) = \det(1 - \alpha U_p Y | M_{Np^\infty}(r))$$

which is independent of the choice of r (for r sufficiently close to 1). For each $\alpha \in \mathcal{H}'$ such that $\iota(\alpha)$ is a unit, define a natural map

$$r_\alpha : Z \times \mathbb{G}_m \to \mathcal{W} \times \mathbb{G}_m, (z, y) \mapsto (w(z), \frac{y}{\iota(\alpha)(z)}),$$

where $\iota(\alpha)(z)$ means the residue of $\iota(\alpha)$ in $R_{\bar{\rho}}^{\mathrm{univ}}/z$, $z \in Z$ being identified with the corresponding maximal ideal in $R_{\bar{\rho}}^{\mathrm{univ}}$.

Define the *Coleman-Mazur eigencurve* (of tame level $N = 1$) to be

$$\mathcal{C} = \bigcap_r r_\alpha^{-1}(Z_{\alpha U_p}), \quad \forall \alpha \in \mathcal{H}' \quad \text{such that} \quad \iota(\alpha) \in (R_{\bar{\rho}}^{\text{univ}})^\times,$$

where $Z_{\alpha U_p}$ is the spectral curve defined as the zero locus of the Fredholm series $P_{\alpha U_p}(Y)$ which we regard as function on $\mathcal{M} \times \mathbb{G}_m$. We refer the reader to Chapter 4 of [12] for the details about spectral curves.

We may and do assume $\mathcal{C}$ to be reduced, by taking its nilreduction. It is proved by Coleman and Mazur in their seminal paper [12] (for $N = 1$) that

Theorem 1.2.1. *(1)The points in $\mathcal{C}(\mathbb{C}_p)$ are in one-to-one correspondence with normalized overconvergent eigenforms of finite slope of tame level* 1*, such that the image under the weight map of a point in $\mathcal{C}(\mathbb{C}_p)$ coincides with the weight of the corresponding overconvergent modular form.*

(2) The (reduced) curve $\mathcal{C}$ is the rigid Zariski-closure of the classical points, i.e. the points corresponding to the classical eigenforms.

(3) The image of the restriction to any irreducible component of the curve $\mathcal{C}$ of the weight map consists of (at least) all but finitely many weights.

(4) There is an admissible affinoid covering $\{U\}$of $\mathcal{C}$ such that the image under the weight map of (each) U is an affinoid subdomain in the weight space $\mathcal{W}$ and the weight map is finite flat over U.

1.3 Kisin's Galois theoretic eigencurve

It is asked by Coleman and Mazur in [12] how to construct a version of their eigencurve in terms of Fontaine's modules, which are purely local. They also ask if the eigencurve is smooth, among other questions. The first question is answered by Kisin in his paper [26] and [23], and the second is answered (for most classical points) in [26], combined with his main result in [25].

In the following, we will use the notation introduced in Chapter 2.

1.3.1 Theory of finite slope subspaces: continuity of φ-operator

In the paper [26], Kisin shows the following:

For a $\mathbb{Q}_p$-rigid analytic space X, a finite free $\mathcal{O}_X$-module M with a continuous $\mathcal{O}_X$-linear $G_{\mathbb{Q}_p}$-action and an invertible function Y on X, the existence of a non-zero $G_{\mathbb{Q}_p}$-equivariant $\mathcal{R}$-linear map

$$\theta^* M(\operatorname{Sp}\mathcal{R}) \to (B^+_{\mathrm{cris}}\hat{\otimes}_{\mathbb{Q}_p}\mathcal{R})^{\varphi=Y}$$

for *certain* maps of rigid spaces $\theta : \operatorname{Sp}\mathcal{R} \to X$ characterizes a Zariski closed subspace $X_{fs} \subset X$.

It is rather important to specify the word "certain", i.e. to what kind of maps Kisin's theorem applies. We explain the idea below and refer the reader to Theorem 4.3.3 for a detailed construction, as well as the generalization to the case where $G_{\mathbb{Q}_p}$ is replaced by G_K for any finite extension K of $\mathbb{Q}_p$, among other technical improvements.

To cut a rigid subspace X_{fs} out of the given rigid space X, one would like to show that existence of crystalline periods on *any* affinoid admissible open subset in X is a closed condition. This is quite hard and is answered by Kisin for certain affinoids, namely the ones that satisfy the so called *Y-small* condition, on which one of the Hodge-Tate weights is fixed and all the others are not constant. Here the Hodge-Tate weights are given by Sen theory on affinoids, as is recalled in Section 4.1. We explain the three conditions.

(1) On a Y-small affinoid $\mathcal{R}$, there is a non zero constant in $\lambda \in \bar{\mathbb{Q}}_p$ such that (on each connected component) $Y\lambda - 1$ is topologically nilpotent.

(2) It is a technical point in this theory that one needs to fix one Hodge-Tate weight on $M_{\mathcal{R}} =: \theta^* M(\operatorname{Sp}\mathcal{R})$.

(3) Then one assumes further that all the other Hodge-Tate weights on $M_{\mathcal{R}}$ are not constant.

The condition (3) guarantees that the Hodge-Tate period obtained in (2) lifts to a (non-zero) de Rham period on $M_{\mathcal{R}}$, i. e., a Galois-equivariant $\mathcal{R}$-linear map $h_{\mathcal{R}}$ from $M_{\mathcal{R}}$ into the p-adic Banach space $B^+_{\mathrm{dR}}/t^j B^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}\mathcal{R}$ for any j large enough.

Then condition (1) of Y-smallness will give that the Y-eigen subspace of crystalline periods $(B^+_{\text{cris}}\hat{\otimes}_{\mathbb{Q}_p}\mathcal{R})^{\varphi=Y}$ is closed in $B^+_{\text{dR}}/t^j B^+_{\text{dR}}\hat{\otimes}_{\mathbb{Q}_p}\mathcal{R}$. We denote the quotient by $U_{\mathcal{R}}$.

Finally, one could ask that this de Rham period factor through $(B^+_{\text{cris}}\hat{\otimes}_{\mathbb{Q}_p}\mathcal{R})^{\varphi=Y}$, which turns out to be a closed condition. More precisely, we can write, for any element $m \in M_{\mathcal{R}}$, the image of $h_{\mathcal{R}}(m)$ in $U_{\mathcal{R}}$ in terms of an orthonormal basis in $U_{\mathcal{R}}$, whose coefficients form an ideal $I_{X_{fs}}$ that essentially gives rise to X_{fs}.

1.3.2 Construction of the Kisin eigencurve

It is a general expectation that the Weil-Deligne representations via the local Langlands correspondence are compatible with those via p-adic Hodge theory, namely that the Hecke eigenvaues of a classical automorphic form coincide with the eigenvalues of the Frobenius operator φ on Fontaine's module of the Galois representation associated to the automorphic form (which is expected to exist). For instance, such compatibility is known in the $\text{GL}_{2/\mathbb{Q}}$ case, thanks to Theorem 1.1.1.

Combining this with the fact that the Atkin-Lehner operator U_p is (locally) completely continuous on the Coleman-Mazur eigencurve, one would expect the analogue holds for families of (p-adic) automorphic forms (resp. Galois representations). For this, one needs an analogue of the Coleman-Mazur eigencurve whose points correspond to p-adic Galois representations on the Fontaine modules on which the φ operator acts (locally) continuously.

Instead of φ-eigenvalues, it would be convenient to consider the φ-eigenvectors in Fontaine's modules. Each of these is a nonzero Galois equivariant map from the representation to the period ring B_{cris} of Fontaine, i.e. a *crystalline period*, as in the theory of finite slope subspaces in the previous subsection .

Kisin then applies the general theory of finite slope subspaces to the rigid analytification Z_p of the generic fibre of a (versal) local Galois deformation space associated to a (fixed) 2-dimensional mod p representation of $G_{\mathbb{Q}_p}$. Namely, he takes $X = Z_p \times \mathbb{G}_m$, M the pull-back to X of the (fixed) versal representation, and Y the canonical coordinate on $\mathbb{G}_m$. The finite slope subspace associated to (X, M, Y) is denoted by $X_{fs,p}(\text{GL}_{2/\mathbb{Q}_p})$. This should be regarded as a p-adic eigenvariety for $\text{GL}_{2/\mathbb{Q}_p}$, which turns out to be a 3-dimensional rigid analytic space. When it is clear that we are in a local situation, we may simply write $X_{fs,p}$ as X_{fs}.

Now go back to the global picture. Suppose that we are given the rigid analytification Z of the generic fibre of a Galois deformation space associated to a 2-dimensional mod p representation $\bar{\rho}$ of $G_{\mathbb{Q}}$. We write Z_p for the rigid space associated to $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ as before. Then we have a map $Z \to Z_p$ induced by the restriction of a $G_{\mathbb{Q}}$-representation to the decomposition group $G_{\mathbb{Q}_p}$ at p, via the (fixed) embedding of $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Kisin takes the fibre product

$$X_{fs}(\mathrm{GL}_{2/\mathbb{Q}}) = (Z \times \mathbb{G}_m) \times_{(Z_p \times \mathbb{G}_m)} X_{fs,p},$$

which is a Zariski closed subspace of $Z \times \mathbb{G}_m$. We call $X_{fs}(\mathrm{GL}_{2/\mathbb{Q}})$ the *Kisin eigencurve.*

Using Saito's result and Theorem 1.2.1, Kisin manages to show that there exists a crystalline period

$$h : \rho_f|_{G_{\mathbb{Q}_p}} \to (B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)^{\varphi = a_p(f)}$$

for any cuspidal overconvergent finite slope eigenform f of a fixed tame level. Here $E/\mathbb{Q}_p$ is the p-adic field where f is defined.

It is expected that the local finite slope subspace $X_{fs,p}(\mathrm{GL}_{2/\mathbb{Q}_p})$ (hence $X_{fs}(\mathrm{GL}_{2/\mathbb{Q}})$) is exactly characterized by the existence of a crystalline period on any 2-dimensional p-adic $G_{\mathbb{Q}_p}$-representation (with the same mod p representation). By the general theory of finite slope subspace, the above condition is necessary. However, it is difficult to prove that the condition is also sufficient. Kisin is able to prove the sufficiency for 2-dimensional p-adic $G_{\mathbb{Q}_p}$-representations V with Hodge-Tate weights $(0, k)$ in the following two cases:

(1) k is a negative integer or is non-integral.

(2) $k \geq 0$, V is de Rham with only scalar endomorphisms, and if V is crystalline then the two φ-eigenvalues on $D_{\mathrm{cris}}(V^*)$ are distinct and they are ordered by increasing p-adic valuations in the case $D_{\mathrm{cris}}(V \otimes V^*(1))^{\varphi=1} \neq 0$.

Using the result for case (1) above and the Zariski density of non-classical points in the Coleman-Mazur eigencurve $\mathcal{C}$, one sees that $\mathcal{C}$ is inside $X_{fs}(\mathrm{GL}_{2/\mathbb{Q}})$ (cf. Theorem 11.6(2) [26]). Years later, Kisin proves that the two subspaces $\mathcal{C}$ and $X_{fs}(\mathrm{GL}_{2/\mathbb{Q}})$ of $Z \times \mathbb{G}_m$ coincide (at the same residual part), as a consequence of his modularity theorem (for trianguline representations) in his remarkable paper [22]. This gives the complete answer to the first question raised by Coleman and Mazur mentioned at the beginning of this Section .

It is thus reasonable to expect that the eigencurve (both in the Coleman-Mazur language and Kisin language) is characterized by the condition that each point in the eigencurve (regarded as a representation) is, up to twist by a character, equipped with one crystalline period. Such an expectation is supported by Colmez's study of trianguline representations [8], where he (cf. Proposition 9.2.5) proves that a 2-dimensional $G_{\mathbb{Q}_p}$-representation is trianguline if and only if some twist of it carries a crystalline period. It is hoped that trianguline is the right property characterizing eigenvarieties p-adic locally.

1.3.3 Local rings via Galois deformations

To show that a de Rham point $x = (z, \lambda)$ in case (2) of the previous subSection lies in the local eigenvariety X_{fs}, one has to study its local ring $\hat{R}_x$ in X as well as the condition that this point carries a crystalline period. Let's assume the residual representation for V_z has only scalar endomorphisms in this Section , for simplicity. Then the local ring $\hat{R}_x$ is given by the Galois deformation ring $R_{V_z}^{\mathrm{univ}}$ (which we assume to exist, again for simplicity). Thus one may expect that the existence of a crystalline period will cut out a quotient ring of $R_{V_z}^{\mathrm{univ}}$, which is supposed to be closely related to the (potential) local ring of x in X_{fs}.

In fact, Kisin develops what we call *finite slope deformation* theory (cf. Section 5). He proves that lifting a crystalline period on a fixed characteristic 0 representation gives rise to a deformation functor, which is representable in the situations we consider. Furthermore, for the de Rham point x above the finite slope deformation ring $R_{V_z}^{h,\varphi}$ is formally smooth of dimension 3, which one can see by computing the Galois cohomologies related to the cotangent space of $R_{V_z}^{h,\varphi}$. Let κ_z be the ideal of

$R_{V_z}^{\mathrm{univ}}$ corresponding to the quotient ring $R_{V_z}^{h,\varphi}$. One may then expect that the quotient $\hat{R}_x/\kappa_z$ provides the local ring of x in X_{fs}.

It is not hard to see that the natural map $\hat{R}_x \to \hat{R}_x/\kappa_z$ factors through $\hat{R}_x^0$, which is by definition the quotient of $\hat{R}_x$ on which the Sen polynomial has a constant weight 0. This corresponds to the condition (2) in Section 1.3.1. One can find the details in Proposition 6.4.5.

It turns out that the Galois cohomology computations also imply that the weight map at such a point is not constant! (see Proposition 6.4.5 for a proof.) This corresponds to the condition (3) in Section 1.3.1.

Notice that the universal property of $R_{V_z}^{h,\varphi}$ provides a compatible system of crystalline periods

$$h_r : M \to (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} \hat{R}_x^0/(\kappa_z + \mathfrak{m}_x^r))^{\lambda_r}, \quad r \geq 1$$

where $\mathfrak{m}_x \subset \hat{R}_x^0$ is the maximal ideal and λ_r's are determined by λ. Moreover, $\hat{R}_x^0/(\kappa_z + \mathfrak{m}_x^r)^{\lambda_r}$ are Y-small affinoids in the sense of theory of finite slope subspace. This, together with the previous two results, enables one to show that the point $x \in X$ lies in X_{fs}. We refer the reader to the proof of Theorem 7.1.1 for details (in a more general situation).

Meanwhile, the similar proof (cf. Proposition 7.1.3) gives a Galois theoretical description of the local ring $\hat{\mathcal{O}}_{X_{fs},x}$. Concretely, at a point $x = (z, \lambda) \in X_{fs}(\mathrm{GL}_{2/\mathbb{Q}_p})$ as in case (2), the complete local ring $\hat{\mathcal{O}}_{X_{fs}(\mathrm{GL}_{2/\mathbb{Q}_p}),(z,\lambda)}$ is canonically isomorphic to the finite slope deformation ring $R_{V_z}^{h,\varphi}$, which is formally smooth of dimension 3. As a consequence, the weight map on the local Galois eigenvariety $X_{fs,p}(\mathrm{GL}_{2/\mathbb{Q}_p})$ is *flat* at the de Rham points in case (2). This is analogous to the global result Theorem 1.2.1(4) about flatness of the weight map on the Coleman-Mazur eigencurve.

Taking into account the main result in [25] about smoothness of modular points in (the generic fibre of) the universal deformation space for $\mathrm{GL}_{2/\mathbb{Q}}$, Kisin shows that the geometric points with local properties as in case (2) on $\mathcal{C} = X_{fs}(\mathrm{GL}_{2/\mathbb{Q}})$ are smooth. Here by *geometric* we mean being unramified at most places and de Rham at p, which is in most cases equivalent to modular, by Kisin's proof [23] of the Fontaine-Mazur conjecture in this case.

1.4 Mazur's idea of refinements

In Kisin's theory, as well as its generalization (cf. Theorem 7.1.1), one reduces to the computations of Galois cohomologies for de Rham E-representations (E is the coefficient field). More precisely, we want to show that the map induced by the crystalline period $h : V \to (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)^{\varphi^f=\lambda}$

$$H^1(G_{\mathbb{Q}_p}, h \otimes 1) : H^1(G_{\mathbb{Q}_p}, V \otimes_E V^*) \to H^1(G_{\mathbb{Q}_p}, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V^*)^{\varphi^f=\lambda}$$

is surjective. This being the case, one obtains the formal smoothness of the finite slope deformation ring $R_{V_z}^{h,\varphi}$ (cf. Section 5 for the reason). In fact, one is able to prove (2) and (3) in Section 1.3.1 at the same time: see Theorem 6.4.4 and Proposition 6.4.5. As we have mentioned in the previous subsection, they play essential roles in the proof of the general theorem, Theorem 7.1.1.

The surjectivity holds for the points in case (2) of Section 1.3.2, because the assumption that $H^0(G_{\mathbb{Q}_p}, V \otimes_E V^*) = E$ (essentially made for the use of deformation theory) turns out to imply that $h : V \to (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)^{\varphi^f=\lambda}$, considered as a map $h_{cris} : V \to B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E$, does not factor through $\mathrm{Fil}^1 B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E$. The choice of a crystalline period h and the key assumption that h_{cris} does not factor through $\mathrm{Fil}^1 B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E$ in fact coincide with Mazur's consideration of refinement and its property which is called *non-critical*, respectively, which are originally made for modular forms but apply to Galois representations in a natural way as we will explain in Section 3 in details.

As is emphasized in Mazur's paper [27], although one can find a family of p-adic modular forms such that the Hecke eigenvalues at any good prime p vary p-adically, one has to specify which one of the two Hecke eigenvalues is varying. This leads to the concept of refinement. For an elliptic newform f of level N, the choice of one of the Hecke eigenvalues at a good prime is called a *refinement* for f at this prime, by Mazur originally.

At such a good prime p we may also apply p-adic Hodge theory to the p-adic representation V_f of $G_{\mathbb{Q}}$ associated to f to make sense of the meaning of refinement. In this case, V is a two dimensional crystalline $G_{\mathbb{Q}_p}$-representation over a finite extension $E/\mathbb{Q}_p$, we call a choice of one of the two φ-eigenvectors in $D_{\text{cris}}(V^*)$ a *refinement* of V. This coincides with Mazur's definition, thanks to Theorem 1.1.1 and the general expectation that the two Hecke eigenvalues are always distinct.

Furthermore, one has to consider non-crystalline representations (for example, those coming from classical eigenforms at bad primes and from non-classical points on the eigencurve) as well. The definition of a refinement of any 2-dimensional $G_{\mathbb{Q}_p}$-representation is given by Emerton [13], inspired by Colmez's study of trianguline representations (cf. Proposition 9.2.5).

The picture is clearer if one thinks of refinements in term of families of (p-adic) modular forms (resp. Galois representations). Here we keep in mind that for a point on the eigencurve corresponding to an overconvergent finite slope cusp form f, the U_p-eigenvalue $a_p(f)$ is equal to the φ-eigenvalue on $D_{\text{cris}}(\rho_f^*|_{G_{\mathbb{Q}_p}})$, or rather, the analytic function a_p on the Coleman-Mazur eigencurve interpolating U_p-eigenvalues coincides with the analytic function φ on the Kisin eigencurve interpolating the φ-eigenvalues on Fontaine's modules $D_{\text{cris}}(\cdot)$.

In the language of Coleman-Mazur eigencurve, for a classical modular eigenform f, if $p \nmid N$, then there are two oldforms f_1, f_2 on $\Gamma_1(pN)$ corresponding to f such that (f_1, λ_1) and (f_2, λ_2), with λ_1, λ_2 roots of the Hecke polynomial at p, both lie on the eigencurve. By Theorem 1.2.1, in a sufficiently small affinoid neighborhood in $\mathcal{C}$ of such a point, say (f_1, λ_1), the weights vary in an infinite set, while the slope $v_p(a_p)$ is constant. It is thus impossible that the two Hecke eigenvalues vary p-adically continuously simultaneously, because the two slopes add up to be the weight at each point in the neighborhood. We remark that, as p is a bad prime for f_1 and f_2, there is only one point on the eigencurve corresponding to each old form, since the Hecke polynomial is of degree 1.

In the language of Kisin eigencurve, for a classical modular eigenform f both of the φ-eigenvalues on $D_{\text{cris}}(\rho_f^*|_{G_{\mathbb{Q}_p}})$ can vary p-adically continuously, but separately, at

$p \nmid N$, while for a classical form at $p|N$ or an overconvergent finite slope cusp form, the vector space (over the p-adic field E where f is defined) $D_{\mathrm{cris}}(\rho_f^*|_{G_{\mathbb{Q}_p}})$ has dimension 1, since it is nonzero by the result of Saito and Kisin and since the representation $\rho_f|_{G_{\mathbb{Q}_p}}$ is not crystalline.

The most important property of a refinement is criticality, which is also posed by Mazur in [27]. The motivation is the Coleman inequality:

Let N be a positive integer such that $p \nmid N$ and let r be a positive integer. An overconvergent elliptic eigenform f of level $p^r N$, and weight $k \in \mathbb{Z}_{\geq 2}$ is classical if

$$v_p(\lambda) < k - 1.$$

This implies, by Lemma 2.2.2, that the crystalline period determined by λ on $V_f|_{G_{\mathbb{Q}_p}}$ does not factor through $\mathrm{Fil}^1 B_{\mathrm{cris}} \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$, as observed by Kisin. We may regard Lemma 2.2.2 as a p-adic analogue of the Coleman inequality.

1.5 Main results

1.5.1 Modification of theory of finite slope subspace

Kisin's theorem on finite slope subspace does not (obviously) hold for G_K-representations, if $K/\mathbb{Q}_p$ has residue degree f strictly bigger than one. The reason is that in the case $f > 1$, the φ action is *not* linear on the K_0-vector space $D_{\mathrm{cris}}(\cdot)$.

The substitute of φ is just φ^f which is a K_0-linear operator, as is suggested by the compatibility of Weil-Deligne representations via classical Langlands correspondence and those via p-adic Hodge theory, which is explained in details in Section 2.1.

Then one realizes that Kisin's original proof of Proposition 5.4 [26] is so general that one needs only to transfer the B_{cris}-picture to the $B_{\mathrm{cris},K} := B_{\mathrm{cris}} \otimes_{K_0} K$-picture. The basic properties of $B_{\mathrm{cris},K}$ we need were already obtained by Colmez [9]. Such kind of generalization is obtained also in [30]. See Theorem 4.3.3 for the general statement and proof.

We remark that one can generalize the theory in other ways, depending on the purpose. For example, we can require the existence of multiple crystalline periods on X_{fs}, in which case the invertible function Y on X will be replaced by an ordered set of invertible functions $\{Y_i\})_{i\in I}$. This is why we need to define refinement and criticality in general.

1.5.2 Refinements in general

It is natural to ask for the generalization of the concept of refinement to higher dimensions, which is at least necessary for the study of finite slope deformations in general, i.e., in the case that one lifts more than one crystalline period. Inspired by the works of Kisin, Colmez, Emerton and Bellaïche-Chenevier, we give the general definition of refinement.

Let V be an N-dimensional G_K-representation over E with K, E finite extensions of $\mathbb{Q}_p$. A *refinement* of V is nothing but an ordered set $\mathfrak{h} = \{h_i\}_{0\le i\le j}$ of nonzero E-linear G_K-equivariant $K\otimes_{\mathbb{Q}_p} E$-linearly independent maps

$$h_i : V\otimes_{\mathbb{Q}_p} \eta_i \to (B_{\mathrm{cris},K}\otimes_{\mathbb{Q}_p} E)^{\varphi^f=\lambda_i}, 0\le i\le j,$$

where $\lambda_i \in E$ and

$$\eta_J = \{\eta_i : G_K \to E^\times\}_{0\le i\le j}$$

is an ordered set of continuous characters. Here the appearance of the ordered set of characters is due to the consideration of trianguline representations. See Proposition 9.2.5 of Colmez for the $\mathrm{GL}_{2/\mathbb{Q}_p}$-case.

A refinement for a global Galois representation will be the collection of refinements at all finite places.

Suppose that the set η_J consists of trivial characters. A refinement is *non-critical* if the ordered crystalline periods are in the general position compared to the Hodge-Tate filtration of Fontaine's module. The precise definition is in Section 3.2.

Refinements and criticality are closely related to the geometry of finite slope subspace, as will be seen below.

1.5.3 Eigenvariety of GL_2 over p-adic fields

Let $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be a (fixed) residual representation with $K/\mathbb{Q}_p$ and $\mathbb{F}/\mathbb{F}_p$ finite extensions. Fix a versal deformation ring $R_{\bar{\rho}}^{\mathrm{ver}}$ associated to $\bar{\rho}$.

Denote by $Z = \mathrm{Sp}(R_{\bar{\rho}}^{\mathrm{ver}}[1/p])$ the rigid analytic space over $\mathbb{Q}_p$ associated to $R_{\bar{\rho}}^{\mathrm{ver}}$, and M_Z the universal G_K-representation on Z. We have the characteristic polynomial $P_\phi(T) \in \mathcal{O}_Z(Z)[T]$ of the Sen operator ϕ on M_Z. Let $Z^0 \subset Z$ be the closed subspace of Z cut out by $P_\phi(0)$. Set

$$X^0 = Z^0 \times \mathbb{G}_m.$$

Let M^0 denote the pullback to X^0 of $M_Z|_{Z^0}$. Write Y for the canonical coordinate on $\mathbb{G}_m$.

Applying the theory of finite slope subspace (Theorem 4.3.3) to X^0, we obtain a closed subspace

$$X_{fs} := X_{fs,0}(X^0, M^0, Y) \subset Z^0 \times \mathbb{G}_m.$$

Theorem 1.5.1. *Let E be a finite extension of $\mathbb{Q}_p$, and $\lambda_0 \in E^\times$.*

(i) If $(z, \lambda_0) \in X_{fs}(E)$ then there is a crystalline period

$$h_0 : V_z \to (B^+_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda_0},$$

and 0 is a Hodge-Tate weight of V_z.

(ii) Suppose that an element $(z, \lambda_0) \in Z \times \mathbb{G}_m(E)$ is equipped with a crystalline period h_0 as above. Then

(1) If the Hodge-Tate weights of V_z are $0, k_1$ with $k_1 \in \bar{\mathbb{Q}}_p - \mathbb{Z}_{\geq 0}$, then (z, λ_0) lies in $X_{fs}(E)$.

(2) Suppose that V_z is de Rham of Hodge-Tate weights $0 = k_0 \leq k_1$ such that

$$H^0(G_K, V_z \otimes_E V_z^*) = E,$$

and the φ^f-eigenvalue λ_0 is of multiplicity one. Then (z, λ_0) lies in $X_{fs}(E)$.

The proof follows the same line as Kisin's (when $K = \mathbb{Q}_p$). The point is to analyze the finite slope deformation rings by computing certain Galois cohomologies. The results are summarized as Theorem 6.4.4 and Proposition 6.4.5, which correspond to the conditions (2) and (3) of Section 1.3.1. In fact, we do the computations for Galois representations of higher dimensions as well, which we hope to be useful for the study of families of Galois representations in higher dimensions.

Write $K_\infty = \bigcup_{n\geq 0} K(\varepsilon^{(n)})$ with $\varepsilon^{(n)}$ a primitive p^n-th root of unity, and let $\Gamma_{K,1}$ be the maximal pro-p quotient of $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. Denote S_0 as the rigid space parametrizing the deformations of the trivial representation $1_{\mathbb{F}} : \Gamma_{K,1} \to \mathbb{F}^\times$.

Set

$$X_{nfs} = X_{fs} \times S_0.$$

This definition is related to our dimension counting and Colmez's theory of trianguline representations (cf. Proposition 9.2.5). Both X_{fs} and X_{nfs} are the local Galois eigenvariety in our mind, since they contain the same information. Sometimes we call X_{fs} the finite slope subspace and X_{nfs} the *nearly* finite slope subspace in order to avoid ambiguity.

Denote by $\hat{\mathcal{O}}_{X_{nfs},(z,\lambda_0,\eta_0)}$ the complete local ring at a point $(z,\lambda_0,\eta_0) \in X_{nfs}(E)$ for $\eta_0 \in S_0$.

Combining Theorem 1.5.1 with Theorem 6.4.4 and Proposition 6.4.5 (again), we have the following results.

Corollary 1.5.2. *For $(z,\lambda_0,\eta_0) \in X_{nfs}(E)$ with (z,λ_0) as in Theorem 1.5.1(ii)(2) and having distinct Hodge-Tate weights, the complete local ring $\hat{\mathcal{O}}_{X_{nfs},(z,\lambda_0,\eta_0)}$ is formally smooth of dimension $1+3n$.*

Corollary 1.5.3. *Let $(z,\lambda_0,\eta_0) \in X_{nfs}(E)$ be as in Corollary 1.5.2. Then the weight map on X_{nfs} given by Sen theory is flat at this point.*

Denote by F^0 the subset of $Z^0 \times \mathbb{G}_m$ consisting of crystalline points which satisfies the conditions in Theorem 1.5.1(ii)(2). Thus F^0 is a subset of X_{fs}.

Denote by X^o_{fs} (resp. X^o_{nfs}) the union of irreducible components of X_{fs} (resp. X_{nfs}) each of which contains a point in F^0 (resp. a point (z,λ_0,η_0) with $(z,\lambda_0) \in F^0$).

As a p-adic analogue of the accumulation of classical points in the Coleman-Mazur eigencurve (Theorem 1.2.1), we have

Proposition 1.5.4. *(1) The subset $\tilde{F}^0$ consisting of the points (z, λ_0, η_0), with $(z, \lambda_0) \in F^0$ and $\eta_0 = \chi_K^k$ $(k \in \mathbb{Z})$, is accumulation in X^o_{nfs}.*

(2) Suppose $\bar{\rho}$ admits a universal deformation ring. Any irreducible component of X^o_{nfs} has dimension $1 + 3n$.

Combining these results above, we obtain

Theorem 1.5.5. *Suppose that $\bar{\rho}$ admits a universal deformation ring. Let $(z, \lambda_0, \eta_0) \in X^o_{nfs}(E)$ with E a finite extension of $\mathbb{Q}_p$.*

If (z, λ_0) is as in Theorem 1.5.1(ii)(2), then (z, λ_0) is smooth in X_{fs}. If in addition the Hodge-Tate weights of V_z are distinct, then (z, λ_0, η_0) is smooth in X_{nfs}.

At last, we remark that the GL_1 case fits into our picture, as the G_K-characters with a crystalline period are nothing but Tate twists of unramified characters, and the weight space on X_{fs} is a point since we normalize the weight of a character to be zero.

1.5.4 Eigenvariety of GL_2 over number fields

Let F be a number field of degree n, and S a finite set of places containing p and all the archimedean places. Let $\bar{\rho} : G_{F,S} \to \mathrm{GL}_2(\mathbb{F})$ be a residual representation with $\mathbb{F}$ a finite field of characteristic p. Assume that $\bar{\rho}$ admits a universal deformation ring $R^{\mathrm{univ}}_{\bar{\rho}}$. Set $Z = \mathrm{Sp}\, R^{\mathrm{univ}}_{\bar{\rho}}[1/p]$.

We define

$$X = Z \times \prod_{v|p} \mathbb{G}_m, \quad X_v = Z_v \times \mathbb{G}_m, \quad X_p := \prod_{v|p} X_v.$$

We have the nearly finite slope subspaces

$$X_{v,nfs} = X_{nfs}(X_v, M_v, Y_v)$$

for each place $v|p$ in F.

By the versal property of the local deformation rings of $\bar{\rho}|_{G_{F_v}}$, we get a map between analytic spaces $\tau_p : X \to X_p$. On the other hand, we have the natural inclusion $\prod_{v|p} X_{v,nfs} \to X_p$.

We define $\mathfrak{X} = \mathfrak{X}(\mathrm{GL}_{2/F})$ to be the fibre product of X and $\prod_{v|p} X_{v,nfs}$ over X_p via the natural maps above, which we call the *Galois eigenvariety* of $\mathrm{GL}_{2/F}$.

Define $n_v = [F_v : \mathbb{Q}_p]$ and $f_v := [F_{v,0} : \mathbb{Q}_p]$ for $v \mid p$ a place in F, where $F_{v,0}$ is the maximal unramified extension of $\mathbb{Q}_p$ inside F_v. By Theorem 1.5.1 and the definition of global Galois eigenvariety, we have

Theorem 1.5.6. *Let $z \in Z(E)$ with E a finite extension of $\mathbb{Q}_p$. Let $\lambda = (\lambda_v)_{v|p}$ with $\lambda_v \in E^\times$, and $\eta = (\eta_v)_{v|p}$ with $\eta_v \in S_{0,v}$.*

(i) If $(z, \lambda, \eta) \in \mathfrak{X}(E)$, then for any place v above p, there exists a crystalline period

$$h_v : V_z|_{G_{F_v}} \otimes_{\mathbb{Q}_p} \eta_v \to (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)^{\varphi^{f_v} = \lambda_v}.$$

(ii) Suppose that for any $v|p$, the element $(z_v, \lambda_v) \in Z_v \times \mathbb{G}_m(E)$ is equipped with a crystalline period as above. If for any embedding $F_v \hookrightarrow \bar{\mathbb{Q}}_p$ the data above satisfy either Theorem 1.5.1(i) or (ii), then (z, λ, η) lies in $\mathfrak{X}$.

We note that, in Theorem 1.5.6(ii) whether $(z, \lambda, \eta) \in X$ lies in $\mathfrak{X}$ is independent of the choice of map $R^{ver}_{\bar{\rho}|_{G_{F_v}}} \to R^{\mathrm{univ}}_{\bar{\rho}}$, as the existence of the crystalline periods is defined globally.

Let $\mathfrak{X}^o \subset \mathfrak{X}$ be the union of irreducible components of $\mathfrak{X}$ containing a closed point as in Theorem 1.5.6(ii).

Theorem 1.5.7. *If $\bar{\rho}$ is odd, then an irreducible component of $\mathfrak{X}^o$ has dimension at least $1 + n$.*

By a consideration along the line of the Bloch-Kato conjecture, we expect the lower bound above to be the true dimension of the global Galois eigenvariety.

1.5.5 Infinite fern in local Galois deformation space of GL_2: an application

For a modular newform which is unramified at p, one has to specify one of the two Hecke eigenvalues in order to relate f to a point on the eigencurve, which motivates the concept of refinement. On the other hand, given such a modular form, we have two different points on the eigencurve related to it, corresponding to different refinements. In another word, the image of the natural projection from the eigencurve $\mathcal{C}$ to (the analytification of) the generic fibre of universal deformation space contains infinitely many double points, which is the "infinite fern" in the language of Mazur.

Originally, Gouvêa and Mazur [18] prove that cusp forms on $\Gamma_1(N)$ with N running over integers not divisible by p give a Zariski dense set of points in (the generic fibre) of a global universal deformation space. The local analogue of infinite fern is given by Kisin [21], aimed at the study of p-adic Langlands correspondence. Kisin's proof is based on his construction of the finite slope subspace $X_{fs}(\mathrm{GL}_{2/\mathbb{Q}_p})$. We will apply Kisin's method to obtain an infinite fern for $\mathrm{GL}_{2/K}$, with K a finite extension of $\mathbb{Q}_p$. A similar result is also obtained in [30] via the deformations of B-pairs in the sense of Berger.

Theorem 1.5.8. *For a representation $\bar{\rho}: G_K \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ such that the framed universal deformation space is irreducible and smooth, the crystalline points are Zariski dense in the generic fibre Z of the Galois deformation space.*

We pick the set $F \subset Z \times \mathbb{G}_m$ of all 2-dimensional crystalline G_K-representations with distinct Hodge-Tate weights satisfying the conditions in Theorem 1.5.1(ii)(2) and the condition

$$v_p(\lambda_0) < k,$$

as well as their Tate twists. The last condition implies, by Lemma 2.2.2 and Theorem 7.1.1 , that $F \subset X_{nfs}$. The point is, the same representation with the other φ^f-eigenvalue, is also an element in F.

We can show that F is non-empty, thanks to the crystalline deformation theory of Kisin [22] and the Fontaine-Laffaille theory.

Then the flatness of weight map we proved before implies that the points in F are Zariski dense in any irreducible component of X_{nfs} whose interSection with F is non-empty.

Finally, we use the natural projection from X_{nfs} to Z to get an infinite fern. Assuming the irreducibility and smoothness of Z, we reduce to counting the dimensions of complete local rings of points in the Zariski closure $Y \subset Z$ of the crystalline points z coming from F, via the projections. It turns out that $\dim \mathrm{Spf}\hat{\mathcal{O}}_{Y,z} = \dim \mathrm{Spf}\hat{\mathcal{O}}_{Z,z}$. Then we are forced to have $\dim Y = \dim Z$ since the regular locus is open in an affinoid.

1.6 Outline of the paper

In §2, we first describe the construction of Weil-Deligne representations via Fontaine's module, which leads to the definition of refinements of Galois representations. It suggests how to generalize Kisin's theory of finite slope subspace, as is carried out in §3. We recall the necessary Galois deformation theory from [26] in §4, which, combined with the computations of Galois cohomology in §5, gives a precise description of the Galois deformation ring of a representation equipped with a refinement. The main results about local and global Galois eigenvarieties are in §6. Using the construction of local eigenvariety, we get the local infinite fern for GL_2 in §7. Finally, in the last Section we recall the results on trianguline representations of Colmez and Bellaïche-Chenevier, which provide a parallel way for the study of families of Galois representations.

Chapter 2

Preliminaries in p-adic Hodge theory

We recall some basic facts about p-adic Hodge theory, and refer the reader to [17], [16] and [9] for more details.

Denote by $\mathcal{O}_{\mathbb{C}_p}$ the ring of integers of $\mathbb{C}_p$. Let $R = \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p$ be Fontaine's ring, which is equipped with a (continuous) $G_{\mathbb{Q}_p}$-action. Set $x^{(n)} = \lim_{k\to\infty} \tilde{x}_{n+k}^{p^k}$ for $x = (x_i)_{i\geq 0} \in R$, where $\tilde{x}_k \in \mathcal{O}_{\mathbb{C}_p}$ is a lifting of x_k. Then

$$R = \{x = (x^{(i)})_{i\geq 0} | x^{(i)} \in \mathcal{O}_{\mathbb{C}_p}, (x^{(i+1)})^p = x^{(i)}\}$$

with the addition and multiplication

$$(x+y)^{(i)} = \lim_{k\to+\infty} (x^{(i+k)} + y^{(i+k)})^{p^k}, \quad (xy)^{(i)} = (x)^{(i)}(y)^{(i)}.$$

The ring R is perfect of characteristic p and is equipped with a valuation

$$v_R(x) := v_p(x^{(0)}).$$

The projection

$$R \to \mathcal{O}_{\mathbb{C}_p}/p, \quad (x_i)_{i\geq 0} \mapsto x_0$$

lifts uniquely to a projection $\theta : W(R) \to \mathcal{O}_{\mathbb{C}_p}$ by the universal property of the Witt ring $W(R)$. Define

$$B_{\mathrm{dR}}^+ = \varprojlim_n W(R)[1/p]/(\mathrm{Ker}(\theta))^n$$

as the $\mathrm{Ker}(\theta)$-adic completion of $W(R)[1/p]$. The topology in B_{dR}^+ which is induced from the inverse limit and the canonical topology in $W(R)$ is called the *weak topology.* On the other hand, as an abstract ring B_{dR}^+ is a discrete valuation ring with uniformizer

$$t = \log[\varepsilon] = \sum_{n=1}^{\infty}(-1)^{n-1}([\varepsilon]-1)^n/n,$$

whose residue field is $\mathbb{C}_p$, where $\varepsilon = (1, \varepsilon^{(1)}, \cdots) \in R$ with $\varepsilon^{(n)}$ a primitive p^n-th root of unity. Set $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[1/t]$, the ring of de Rham periods. It comes with a decreasing filtration $\{\mathrm{Fil}^i B_{\mathrm{dR}}\}_{i\in\mathbb{Z}}$ with

$$\mathrm{Fil}^i B_{\mathrm{dR}} = t^i B_{\mathrm{dR}}^+.$$

Let A_{cris} be the p-adic completion of the divided power envelope of $W(R)$ with respect to the ideal $\mathrm{Ker}(\theta)$. We have $t \in A_{\mathrm{cris}}$. The Frobenius map on $W(R)$ extends to a map φ on A_{cris}, which acts on t by

$$\varphi(t) = pt,$$

because the action of $G_{\mathbb{Q}_p}$ on ε is via the cyclotomic character. Set

$$B_{\mathrm{cris}}^+ = A_{\mathrm{cris}}[1/p], \quad B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/t].$$

The φ operator on A_{cris} extends to B_{cris}^+ and B_{cris}. The natural map $W(R) \to B_{\mathrm{dR}}^+$ extends to an embedding $A_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}^+$, whence embeddings $B_{\mathrm{cris}}^+, B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}^+$.

Let $\tilde{p} \in R$ be with $\tilde{p}^{(0)} = p$. Define an element in B_{dR}^+

$$u = \sum_{n\geq 1}(-1)^{n-1}([\tilde{p}]/p - 1)^n/n,$$

which is transcendental over the fraction field of B_{cris} and gives an embedding of

$$B_{\text{st}} := B_{\text{cris}}[u]$$

into B_{dR}. Then B_{st} is equipped with the monodromy operator $N : \sum_{i\geq 0} b_i u^i \mapsto -\sum_{i\geq 0} i b_i u^{i-1}$, the unique B_{cris}-derivation such that $N(u) = -1$.

The filtration on B_{dR} induces filtrations on B_{cris} and B_{st}. Namely

$$\mathrm{Fil}^i B_{\text{cris}} := \mathrm{Fil}^i B_{\text{dR}} \cap B_{\text{cris}}, \quad \mathrm{Fil}^i B_{\text{st}} := \mathrm{Fil}^i B_{\text{dR}} \cap B_{\text{st}}.$$

We define

$$\mathrm{Fil}^i B^+_{\text{cris}} := \mathrm{Fil}^i B_{\text{dR}} \cap B^+_{\text{cris}}, \quad \forall i \in \mathbb{Z}_{\geq 0}.$$

Write $B_{\text{HT}} = \oplus_{i\in\mathbb{Z}} \mathbb{C}_p(i)$. For $K/\mathbb{Q}_p$ a finite extension, we have that

$$B_{\text{HT}}^{G_K} = (B^+_{\text{dR}})^{G_K} = B_{\text{dR}}^{G_K} = K, \quad (B^+_{\text{cris}})^{G_K} = B_{\text{cris}}^{G_K} = B_{\text{st}}^{G_K} = K_0.$$

For $* = \text{st}, \text{cris}$, we define

$$B^+_{*,K} = B^+_* \otimes_{K_0} K, \quad B_{*,K} = B_* \otimes_{K_0} K$$

which are equipped with the induced filtration. One may choose (cf. Prop.8.10 [9]) an element $t_K \in B_{\text{cris},K}$ such that

$$\varphi^f(t_K) = \varpi_K t_K$$

and

$$g \cdot t_K = \chi_K(g) t_K$$

for $g \in G_K$.

Remark 2.0.1. *Note that although $B^+_{\text{cris}} \subsetneq \mathrm{Fil}^0 B_{\text{cris}}$, we have for any φ-module D*

(i.e. a finite dimensional K_0-vector space equipped with a K_0-semi-linear φ-action)

$$\mathrm{Hom}_{K_0[\varphi]}(D, \mathrm{Fil}^i B_{\mathrm{cris}}) = \mathrm{Hom}_{K_0[\varphi]}(D, \mathrm{Fil}^i B^+_{\mathrm{cris}}), \forall i \geq 0$$

by (5.3.7) [16], whence

$$\mathrm{Hom}_{K_0[\varphi^f]}(D_K, \mathrm{Fil}^i B_{\mathrm{cris},K}) = \mathrm{Hom}_{K_0[\varphi^f]}(D_K, \mathrm{Fil}^i B^+_{\mathrm{cris},K}), \forall i \geq 0$$

for $D_K = D \otimes_{K_0} K$, keeping in mind that multiplication by t^i_K induces an isomorphism $\mathrm{Fil}^0 B_{\mathrm{cris},K} \simeq \mathrm{Fil}^i B_{\mathrm{cris},K}$.

Let E be a p-adic field. Let V be a (continuous) E-representation of G_K. For $* = \mathrm{HT}, \mathrm{dR}, \mathrm{st}, \mathrm{cris}$ denote

$$D_*(V) = (B_* \otimes_{\mathbb{Q}_p} V)^{G_K},$$

and for $* = \mathrm{st}, \mathrm{cris}$, we denote

$$D_{*,K}(V) = (B_{*,K} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Definition 2.0.2. Let $* = \mathrm{HT}, \mathrm{dR}$ or $* = \mathrm{st}, \mathrm{cris}$. We say V is *Hodge-Tate, de Rham* (resp. *semi-stable, crystalline*) if $\dim_E D_*(V) = \dim_E V$ (resp. $\dim_E D_{*,K}(V) = \dim_E V$).

We say V is *potentially semi-stable* (resp. *potentially crystalline*) if $V|_{G_{K'}}$ is semi-stable (resp. crystalline) for some finite extension K'/K.

It follows from the construction of the period rings that potential semi-stability implies de Rhamness. Conversely, de Rhamness implies potential semi-stability by the main result of [2].

Let $\mathcal{R}$ be an affinoid algebra over E. Let V be a locally free $\mathcal{R}$-module of finite rank with a continuous G_K-action. For $* = \mathrm{HT}, \mathrm{dR}, \mathrm{st}, \mathrm{cris}$ denote

$$D_*(V) = (B_* \otimes_{\mathbb{Q}_p} V)^{G_K},$$

and for $* = \mathrm{st}, \mathrm{cris}$, we denote

$$D_{*,K}(V) = (B_{*,K} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

By the results of [3], they are locally free $K \otimes_{\mathbb{Q}_p} \mathcal{R}$-module of finite rank. Similarly, for $* = \mathrm{HT}, \mathrm{dR}, \mathrm{st}, \mathrm{cris}$. We say V is *Hodge-Tate*, *de Rham* (resp. *semi-stable*, *crystalline*) if $\mathrm{rank}_{K \otimes_{\mathbb{Q}_p} \mathcal{R}} D_{*,K}(V) = \mathrm{rank}_{\mathcal{R}} V$.

2.1 Fontaine's construction of Weil-Deligne representations

Let V be an E-representation of G_K with E a finite extension of $\mathbb{Q}_p$. Suppose V is de Rham. Then its E-dual V^* is also de Rham. Define

$$D_{\mathrm{pst}}(V^*) = \varinjlim_L (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^*)^{G_L}$$

with L runs over finite extensions of K. Note $D_{\mathrm{pst}}(V^*)$ is a finite free $\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} E$-module with a $\mathbb{Q}_p^{\mathrm{ur}}$-semi-linear action of G_K, which is linear for the action of the inertia subgroup I_K. It is also equipped with the monodromy operator N induced from that on B_{st}, and the Frobenius-semi-linear operator φ, i.e. φ acts on $\mathbb{Q}_p^{\mathrm{ur}} = W(\bar{\mathbb{F}}_p)$ by Frob_p, the arithmetic Frobenious.

Let $\deg : W_K \to \mathbb{Z}$ be the canonical map such that $w \in W_K$ acts on $\mathbb{Q}_p^{\mathrm{ur}}$ as $\mathrm{Frob}_p^{-\deg(w)}$. Write $\sigma \in \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/K_0)$ for the *inverse* of the relative Frobenius of $\mathbb{Q}_p^{\mathrm{ur}}$ over K_0. Then $\deg(\sigma) = f$.

Note that $J = D_{\mathrm{pst}}(V^*)^{N=0, I_K}$ is a discrete $\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} E$-module with semi-linear G_K/I_K-action, which is generated by its $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/K_0)$-invariants $J' = J^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/K_0)}$. On the other hand, we observe that

$$J' = D_{\mathrm{pst}}(V^*)^{N=0, I_K, \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/K_0)}$$

$$= (B_{\mathrm{st}} \otimes_E V^*)^{N=0, G_K} = D_{\mathrm{st}}(V^*)^{N=0} = D_{\mathrm{cris}}(V^*).$$

Then

$$J \simeq J' \otimes_{K_0} \mathbb{Q}_p^{\mathrm{ur}} = D_{\mathrm{cris}}(V^*) \otimes_{K_0} \mathbb{Q}_p^{\mathrm{ur}}.$$

Note that $w \in W_K$ acts trivially on $D_{\mathrm{cris}}(V^*)$ and acts as $\mathrm{Frob}_p^{-\deg(w)}$ on $\mathbb{Q}_p^{\mathrm{ur}}$, and φ acts on $\mathbb{Q}_p^{\mathrm{ur}}$ as Frob_p. Therefore we can define a new $\mathbb{Q}_p^{\mathrm{ur}}$-*linear* action of W_K on $D_{\mathrm{pst}}(V^*)$ by twisting $\varphi^{\deg(w)}$ to the semi-linear action of $w \in W_K$.

The new (linear) action of W_K, together with the operator N, defines a representation of the Weil-Deligne group WD_K on $D_{\mathrm{pst}}(V^*)$. In particular, the σ-action on $J' = D_{\mathrm{cris}}(V^*)$ is $\mathrm{Id} \cdot \varphi^{\deg(\sigma)} = \varphi^f$.

Noting $D_{\mathrm{cris}}(V^*) = \mathrm{Hom}_{E[G_K]}(V, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)$, we get

Proposition 2.1.1. *Keep the notation above. Assume $\lambda \in E$ is an eigenvalue of σ on $D_{\mathrm{pst}}(V^*)$. There exists a non-zero G_K-equivariant E-linear map*

$$V \longrightarrow (B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda}.$$

In particular, if the Hodge-Tate weights of V are all non-negative, the map above factors as

$$V \longrightarrow (\mathrm{Fil}^0 B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda},$$

hence as

$$V \longrightarrow (B^+_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda}.$$

Proof. This follows from the above discussion, with the last assertion following from Remark 2.0.1.

□

Example 2.1.2. We know the compatibility between Weil-Deligne representations via Fontaine's construction and those via the classical local Langlands correspondence in $\mathrm{GL}_{2/\mathbb{Q}}$ case (see the main result of Saito [31]).

Let N be a positive integer prime to p. Let f be a classical elliptic modular eigenform of level $p^r N, r \geq 0$, and of weight $k \geq 2$. Denote by λ the U_p-eigenvalue of f if $r \neq 0$, and one of the roots of the Hecke polynomial of f at p if $r = 0$. Let V_f be the E-representation of $G_{\mathbb{Q}}$ associated to f, for some finite extension $E/\mathbb{Q}_p$. By the compatibility between the Weil-Deligne representation $D_{\mathrm{pst}}(V_f^*|_{G_{\mathbb{Q}_p}})$ and that via the classical local Langlands correspondence, the local L-factor

$$\det(1-\sigma p^{-s}|(D_{\mathrm{pst}}(V_f|^*_{G_{\mathbb{Q}_p}})\otimes_{E\otimes_{\mathbb{Q}_p}\mathbb{Q}_p^{\mathrm{ur}}}\bar{K})^{N=0,I_K})^{-1}$$

coincides with the local L-factor attached to the modular form f. This implies λ is an eigenvalue of σ on $D_{\mathrm{pst}}(V|^*_{G_{\mathbb{Q}_p}})\otimes_{E\otimes_{\mathbb{Q}_p}\mathbb{Q}_p^{\mathrm{ur}}}\bar{K}$, hence is an eigenvalue of φ on $D_{\mathrm{cris}}(V_f|^*_{G_{\mathbb{Q}_p}})$ by Prop. 2.1.1.

We thus have, keeping in mind that the Hodge-Tate weights of $V_f|_{G_{\mathbb{Q}_p}}$ are non-negative, a non-zero $G_{\mathbb{Q}_p}$-equivariant E-linear map

$$V_f|_{G_{\mathbb{Q}_p}}\longrightarrow(B^+_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}E)^{\varphi=\lambda}.$$

2.2 Weak admissibility of a filtered (φ, N)-module

Recall that E and K are finite extensions of $\mathbb{Q}_p$, and $f=[K_0:\mathbb{Q}_p]$ with K_0 the maximal unramified extension of $\mathbb{Q}_p$ inside K.

Lemma 2.2.1. *Let $\mathcal{R}$ be a Banach algebra over $\mathbb{Q}_p$. We extend the φ^f-action on $(\hat{\mathbb{Q}}_p^{\mathrm{ur}})^\times$ to $(\hat{\mathbb{Q}}_p^{\mathrm{ur}}\hat{\otimes}_{\mathbb{Q}_p}\mathcal{R})^\times$ by $\mathcal{R}$-linearity. For any unit $\alpha\in\mathcal{R}^\times$ such that $\alpha-1$ is topologically nilpotent, there exists a unit $u\in(\hat{\mathbb{Q}}_p^{\mathrm{ur}}\hat{\otimes}_{\mathbb{Q}_p}\mathcal{R})^\times$ such that $\varphi^f(u)=\alpha u$.*

Proof. We just need to apply the proof of Lemma 3.6 [26] to φ^f. □

Corollary 2.2.2. *For any $\lambda\in\mathbb{C}_p$ and $k\in\mathbb{Z}_{\geq 1}$ such that $v_K(\lambda)<k$, we have*

$$\mathrm{Fil}^k(B^+_{\mathrm{cris},K}\otimes_{\mathbb{Q}_p}E)^{\varphi^f=\lambda}=0.$$

Proof. Let $x\in\mathrm{Fil}^k(B^+_{\mathrm{cris},K}\otimes_{\mathbb{Q}_p}E)^{\varphi^f=\lambda}$. Pick an integer q so that $qv_K(\lambda)$ is an integer, which we denote by n. Thus $\varphi^f(x^q)=\lambda^q x^q=\varpi_K^n\alpha x^q$ for some $\alpha\in\mathcal{O}_E^\times$. By Lemma 2.2.1 we may choose an element $u\in(\hat{\mathbb{Q}}_p^{\mathrm{ur}}\otimes_{\mathbb{Q}_p}E)^\times$ so that $\varphi^f(u)=\alpha u$. Replacing x^q by $x^q u$ we have $\varphi^f(x^q)=\varpi_K^n x^q$. Note that $x^q\in\mathrm{Fil}^{qk}(B^+_{\mathrm{cris},K}\otimes_{\mathbb{Q}_p}E)^{\varphi^f=\varpi_K^n}$.

We claim that the assumption $n = qv_K(\lambda) < qk$ implies that $\mathrm{Fil}^{qk} B_{\mathrm{cris}}^{+,\varphi=p^n} = 0$. For this we just need to show that $B_{\mathrm{cris}}^{+,\varphi=p^{-s}} = 0$ for any $s \in \mathbb{Z}_{\geq 1}$. Let $y \in B_{\mathrm{cris}}^{+,\varphi=p^{-s}}$ then we see $y = p^{sk}\varphi^k(y)$ for any $k \in \mathbb{Z}_{\geq 1}$. Thus $y \in \cap_{k \in \mathbb{Z}_{\geq 1}} p^{sk} B_{\mathrm{cris}}^+ = 0$ with the last equality being seen by definition of B_{cris}^+. This proves the claim.

Hence

$$\mathrm{Fil}^{qk}(B_{\mathrm{cris},K}^+ \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \varpi_K^n} = \mathrm{Fil}^{qk} B_{\mathrm{cris}}^{+,\varphi=p^n} \otimes_{\mathbb{Q}_p} K \otimes_{\mathbb{Q}_p} E = 0.$$

□

Lemma 2.2.3. *Let V be an E-representation of G_K which is equipped with nonzero G_K-equivariant E-linear maps*

$$h' : V \to \mathrm{Fil}^k B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E, \quad k \in \mathbb{Z}_{\geq 1}$$

and

$$h : V \to (B_{\mathrm{cris},K}^+ \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda}, \quad \lambda \in E.$$

If $v_K(\lambda) < k$, then the $E \otimes_{\mathbb{Q}_p} K$-module $D_{\mathrm{dR}}(V^*)$ is of rank at least 2.

Proof. We denote by h_{dR} the composition of h with the natural inclusion $(B_{\mathrm{cris},K}^+ \otimes_{\mathbb{Q}_p} E)^{\varphi^f=\lambda} \to B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$. By Lemma 2.2.2, $(\mathrm{Fil}^k B_{\mathrm{cris},K}^+ \otimes_{\mathbb{Q}_p} E)^{\varphi^f=\lambda} = 0$. This implies that h_{dR} does not factor through $\mathrm{Fil}^k B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$, hence is $E \otimes_{\mathbb{Q}_p} K$-linearly independent of h'. This concludes the proof.

□

Proposition 2.2.4. *Let V be an N-dimensional E-representation of G_K. Suppose there is a non-zero E-linear G_K-equivariant map*

$$h : V \to (B_{\mathrm{cris},K}^+ \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda}$$

with $\lambda \in E$.

If h induces, via the inclusion $(B_{\mathrm{cris},K}^+ \otimes_{\mathbb{Q}_p} E)^{\varphi^f=\lambda} \to B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$, a non-zero E-linear G_K-equivariant map $h_{HT} : V \to \mathbb{C}_p(k) \otimes_{\mathbb{Q}_p} E$, then $v_K(\lambda) \geq k$. Moreover, *(1) If $v_K(\lambda) = k$, then there is an element $\alpha \in (\hat{\mathbb{Q}}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} E)^\times$ such that V has a one dimensional quotient representation $\mathrm{ur}_\alpha \chi_K^k$, where ur_α is the unramified character sending the relative geometric Frobenius Frob_K^{-1} to α.*

(2) If V is de Rham with Hodge-Tate weights $0 \leq k_0 \leq \cdots \leq k_{N-1}$ and $k = k_{N-1}$, then V has a one dimensional quotient as in (1).

Proof. Twisting h by χ_K^{-k}, we get a non-zero E-linear G_K-equivariant map $V(-k) \to (B^+_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda\varpi_K^{-k}}$. Since the target lies in $\mathrm{Fil}^0 B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$, we must have $v_K(\lambda\varpi_K^{-k}) \geq 0$. The first assertion follows.

For (1), since $\lambda\varpi_K^{-k}$ is by assumption a unit, we must have

$$(B^+_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda\varpi_K^{-k}} = E \cdot \alpha$$

for some $\alpha \in (\hat{\mathbb{Q}}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} E)^\times$ such that $\varphi^f(\alpha) = \lambda\varpi_K^{-k}\alpha$, by Lemma 2.2.1. This gives rise to a non-zero E-linear G_K-equivariant surjective map

$$V(-k) \to E \cdot \alpha,$$

hence the desired quotient after twisting by χ_K^k. This proves (1).

For (2), we just need to show $v_K(\lambda) \leq k_{N-1}$, by (1). As V is de Rham, that $v_K(\lambda) \leq k_{N-1}$ follows from the weak admissibility of $D_{\mathrm{st}}(V|_{G_{K'}})$, for some K' which is a finite extension of K so that $V|_{G_{K'}}$ is semi-stable. $\square$

Lemma 2.2.5. *Let $i < j \in \mathbb{Z} \cup \{\pm\infty\}$. Let $B_* = B_{\mathrm{dR}}$ or $B_{\mathrm{cris},K}$. For any finite extension $K/\mathbb{Q}_p$, we have for $r = 0, 1$ that*

$$H^r(G_K, \mathrm{Fil}^i B_* / \mathrm{Fil}^j B_*) \simeq \begin{cases} 0, & i \geq 1 \text{ or } j \leq 0; \\ K, & i \leq 0 \text{ and } j \geq 1. \end{cases}$$

and

$$H^r(G_K, \mathrm{Fil}^i B_* / \mathrm{Fil}^j B_*) = 0, \quad \forall r \geq 2.$$

Proof. We just need to show the Lemma for $* = dR$, since $\mathrm{gr}(B_{\mathrm{dR}}) = \mathrm{gr}(B_{\mathrm{cris},K})$.

Recall by Theorem 1, 2 of Tate ([34]) we have for $r = 0, 1$

$$H^r(G_K, \mathbb{C}_p) = K, \qquad H^r(G_K, \mathbb{C}_p(k)) = 0, \forall k \in \mathbb{Z}\backslash\{0\}.$$

and $H^r(G_K, \mathbb{C}_p(k)) = 0$ for any integers k and $r \geq 2$.

Consider the short exact sequence

$$0 \longrightarrow \mathbb{C}_p(j) \longrightarrow \mathrm{Fil}^i B_{\mathrm{dR}}/\mathrm{Fil}^{j+1} B_{\mathrm{dR}} \longrightarrow \mathrm{Fil}^i B_{\mathrm{dR}}/\mathrm{Fil}^j B_{\mathrm{dR}} \longrightarrow 0$$

and the long exact sequence it induces. Then an induction on $j - i$ gives the lemma for i, j finite. The rest are obtained by taking limit.

□

Proposition 2.2.6. *Let V be a de Rham (resp. crystalline) E-representation of G_K with Hodge-Tate weights $k_0 \leq \cdots \leq k_{N-1}$. Then a nonzero E-linear G_K-equivariant map*

$$h : V \to \mathbb{C}_p(k_{N-1}) \otimes_{\mathbb{Q}_p} E$$

induces a nonzero E-linear G_K-equivariant map

$$\tilde{h} : V \to \mathrm{Fil}^{k_{N-1}} B_* \otimes_{\mathbb{Q}_p} E,$$

where $* = \mathrm{dR}$ *or* cris, K.

Proof. We show the case $* = dR$, as the other case goes over verbatim.

For any $0 \leq j \leq N - 1$, we have a short exact sequence

$$0 \longrightarrow \mathrm{Fil}^{k_j+1} B_{\mathrm{dR}} \longrightarrow \mathrm{Fil}^{k_j} B_{\mathrm{dR}} \longrightarrow \mathrm{Fil}^{k_j} B_{\mathrm{dR}}/\mathrm{Fil}^{k_j+1} B_{\mathrm{dR}} \longrightarrow 0.$$

It induces the exact sequence

$$H^0(G_K, \mathrm{Fil}^{k_j} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*) \to H^0(G_K, \mathrm{Fil}^{k_j} B_{\mathrm{dR}}/\mathrm{Fil}^{k_j+1} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*)$$

$$\xrightarrow{b_V} H^1(G_K, \mathrm{Fil}^{k_j+1} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*).$$

The obstruction class for lifting $h : V \to \mathbb{C}_p(k_j) \otimes_{\mathbb{Q}_p} E = \mathrm{Fil}^{k_j} B_{\mathrm{dR}}/\mathrm{Fil}^{k_j+1} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$ to $\mathrm{Fil}^{k_j} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$ is the image of h under the natural map b_V.

On the other hand, we have

$$H^1(G_K, \mathrm{Fil}^{k_j+1} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*) \simeq H^1(G_K, \mathrm{Fil}^{k_j+1}(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(V^*)))$$

as V^* is de Rham. As the latter is a successive extension of $H^1(G_K, \mathrm{Fil}^{k_j+1-k_i} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E)$ for $i = 0, \cdots, N-1$, each of which is trivial when $j = N-1$ by Prop. 2.2.5, the result follows.

$\square$

Corollary 2.2.7. *Let V be a 2-dimensional E-representation of G_K with Hodge-Tate weights $0, k, k \geq 1$. Suppose there is a non-zero E-linear G_K-equivariant map*

$$h : V \to (B^+_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda}$$

for some $\lambda \in E$.

(1) If $v_p(\lambda) = 0$, then V is de Rham and we have an extension of representations

$$0 \to \varepsilon\chi_K^k \to V \to \mathrm{ur}_\alpha \to 0$$

with ur_α the unramified character sending the relative geometric Frobenius Frob_K^{-1} to some $\alpha \in (\hat{\mathbb{Q}}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} E)^\times$ and ε a character such that $\varepsilon(I_K)$ is finite. The extension above may not split.

(2) If $v_p(\lambda) = k$, then we have an extension of representations

$$0 \to \varepsilon \to V \to \mathrm{ur}_\alpha \chi_K^k \to 0$$

with ur_α and ε as in (1). If V is in addition de Rham, then the above extension splits.

Proof. In (1), we get, by Prop. 2.2.4, a non-zero E-linear G_K-equivariant surjective map

$$V \longrightarrow E \cdot \alpha$$

for some $\alpha \in (\hat{\mathbb{Q}}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} E)^\times$. The kernel of the above map is a character of Hodge-Tate weight k, hence of the form $\varepsilon \chi_K^k$ with ε a character under which the inertia has finite image.

Then V is de Rham by Prop. 2.2.6 and Cor. 2.2.3. The obstruction for the extension to split lies in $H_g^1(G_{K'}, \mathbb{Q}_p(k-1))$ by Hochschild-Serre, with K'/K a finite extension such that $V|_{G_{K'}}$ is semi-stable. On the other hand,

$$\dim H_g^1(G_{K'}, \mathbb{Q}_p(k)) = \begin{cases} n', & k = 1; \\ n' + 1, & k > 1. \end{cases}$$

with $n' = [K' : \mathbb{Q}_p]$, by (3.9) of [4] (see also Prop. 6.2.1). Therefore the extension is not necessarily split.

In (2), we get the extension similarly, again by the use of Prop. 2.2.4.

When V is de Rham, the obstruction for the extension to split lies in $H_g^1(G_{K'}, \mathbb{Q}_p(-k))$ (for some finite extension K'/K so that $V|_{G_{K'}}$ is semi-stable) by Hochschild-Serre again. In this case, however,

$$H_g^1(G_{K'}, \mathbb{Q}_p(-k)) = 0, \quad \forall k \geq 1$$

by (3.9) of [4].

$\square$

Chapter 3

Refinements of a Galois representation

3.1 Definition

Let V be a (continuous) N-dimensional E-representation of G_K. For a *crystalline period* (resp. *de Rham period*, *Hodge-Tate period*) of V we mean a nonzero E-linear G_K-equivariant map $V \to B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E$ (resp. $V \to B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$, $V \to B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} E$).

Note that φ^f is a K_0-linear operator on the $K_0[\varphi]$-module $D_{\mathrm{cris},K}(V^*)$. Given $\lambda \in E$, a nonzero E-linear G_K-equivariant map

$$h : V \to (B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f=\lambda}$$

induces a crystalline period via the natural Galois-equivariant embedding

$$(B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f=\lambda} \hookrightarrow B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E,$$

which we denote by

$$h_{cris} : V \to B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E.$$

When we say some crystalline periods factoring through different φ^f-eigenspaces are E-linearly independent (or simply *distinct*), we always mean they are E-linearly independent as elements in $D_{\mathrm{cris},K}(V^*)$. We use similar conventions for de Rham periods and Hodge-Tate periods.

Definition 3.1.1. Let V be an N-dimensional E-representation of G_K. For $0 \leq j \leq N-2$ a positive integer, write $J = \{0, \cdots, j\}$. A *J-refinement* of V is a triple $\mathcal{R} = (\eta_J, \mathfrak{h}_J, \lambda_J)$ with

$$\eta_J = \{\eta_i : G_K \to E^\times\}_{0 \leq i \leq j}$$

an ordered set of continuous characters,

$$\lambda_J = (\lambda_i)_{0 \leq i \leq j} \in E^{j+1}$$

and

$$\mathfrak{h}_J = \{h_i : V \otimes_{\mathbb{Q}_p} \eta_i \to (B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda_i}\}_{0 \leq i \leq j}$$

an ordered set of $E \otimes_{\mathbb{Q}_p} K_0$-linearly independent crystalline periods.

We will simply call a J-refinement of V a refinement when it is obvious what J is.

Let $\mathcal{R} = (\eta_J, \mathfrak{h}_J, \lambda_J)$ and $\mathcal{R}' = (\eta'_J, \mathfrak{h}'_J, \lambda'_J)$ be two J-refinements of V. We say they are *equivalent* if for any $h_i \in \mathfrak{h}$, $h'_i \in \mathfrak{h}'$, there is a unit $\alpha_i \in \mathcal{O}_E^\times$ and a non-zero element $\mu_i \in (\mathcal{O}_{\mathbb{Q}_p^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} E)^{\varphi^f = \alpha_i}$, such that $h'_i = h_i \cdot \mu_i$. Note that, this being the case, we have that η'_i equals to the product of η_i and the unramified character sending the relative geometric Frobenius to α_i, and that $v_p(\lambda_i) = v_p(\lambda'_i)$.

A J-refinement is called *ordinary* if there exist integers $\{k_i\}_{i \in J}$ such that each $h_i \in \mathfrak{h}$ factors through $\mathrm{Fil}^{k_i}(B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda_i}$, and the natural composition

$$V \otimes_{\mathbb{Q}_p} \eta_i \xrightarrow{h_i} \mathrm{Fil}^{k_i}(B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda_i} \hookrightarrow \mathrm{Fil}^{k_i} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E \twoheadrightarrow \mathbb{C}_p(k_i) \otimes_{\mathbb{Q}_p} E$$

is non-zero. This being the case, we have, for any $i \in J$,

$$\mathrm{Fil}^{k_i}(B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda_i} \simeq E \cdot u$$

for some $u_i \in K_0 \otimes_{\mathbb{Q}_p} \mathcal{O}_E^\times$ such that $\varphi^f(u_i) = \lambda_i u_i$, by Lemma 2.2.1. As the above composition is G_K-equivariant, the twist by η_i^{-1} of the unramified character sending the relative geometric Frobenius to u_i is a quotient representation of V. In particular, a G_K-representation (of dimension at least two) which can be equipped with an ordinary refinement has to be reducible.

3.2 Criticality of refinements

Assume further that V is with integral Hodge-Tate weights $k_0 \leq \cdots \leq k_{N-1}$. If the characters η_i in $\mathcal{R}$ are all trivial, let $D_{i,cris}(V^*)$ denote the $K_0 \otimes_{\mathbb{Q}_p} E$-submodule of $D_{\mathrm{cris},K}(V^*)$ generated by the crystalline periods $\{h_r\}_{0 \leq r \leq i}$. Then a J-refinement $\mathcal{R}$ being *non-critical* means

$$D_{\mathrm{cris},K}(V^*) = D_{i,cris}(V^*) \oplus \mathrm{Fil}^{k_i+1} D_{\mathrm{cris},K}(V^*), \quad i = 0, \cdots, j$$

for all i such that $k_i \neq k_{i+1}$. This is used in [1] for $G_{\mathbb{Q}_p}$-representations with distinct Hodge-Tate weights.

In particular, in the case that V is of dimension 2 with $0 = k_0 \leq k_1$, the refinement $\mathcal{R} = (1_E, \{h_0\}, \{\lambda_0\})$ is non-critical if h_0 does not factor through $\mathrm{Fil}^1 B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E$. Notice that the refinement is always non-critical when V has equal Hodge-Tate weights.

Example 3.2.1. For V a 2-dimensional crystalline $G_{\mathbb{Q}_p}$-representation with Hodge-Tate weights $0, k > 0$, such that the p-adic valuation of any φ-eigenvalue on $D_{\mathrm{cris}}(V^*)$ is strictly smaller that k , there are always two non-equivalent non-critical refinements of V.

(2) The restriction to $G_{\mathbb{Q}_p}$ of the p-adic Galois representation associated to an elliptic modular form of weight 1 has Hodge-Tate weights $(0, 0)$, hence admits only ordinary refinements, which have to be non-critical.

In $\mathrm{GL}_{2/\mathbb{Q}}$ case, the following results of Coleman and Breuil-Emerton explain the meaning of criticality on the automorphic side.

Theorem 3.2.2 (Coleman inequality). *Let N be a positive integer such that $p \nmid N$. An overconvergent elliptic eigenform f of level $p^r N, r \geq 0$, and integral weight $k \geq 2$ is classical if*

$$v_p(\lambda) < k-1, \text{ or } f \text{ is not in the image of the map } \Theta^{k-1}.$$

Here λ is the U_p-eigenvalue of f if $r \neq 0$, and either of the roots of the Hecke polynomial of f at p if $r = 0$, and Θ is the operator acting as $q\frac{d}{dq}$ on q-expansions.

(We remark that in Thm. 3.2.2, the condition has to be checked for both of the Hecke eigenvalues in the case $r = 0$.)

In the first case of Thm. 3.2.2, the refinement of $V_f|_{G_{\mathbb{Q}_p}}$ given by the crystalline period corresponding to λ (cf. Example 2.1.2) is non-critical, by Lemma 2.2.2. In the second case, $v_p(\lambda) = k-1$ and it is not easy to see if the refinement is non-critical.

Theorem 3.2.3 (Thm.1.1.3, [5]). *Let f be as in Thm. 3.2.2. The refinement of $V_f|_{G_{\mathbb{Q}_p}}$ given by the crystalline period corresponding to λ is critical if and only if f is in the image of the operator Θ^{k-1}.*

Chapter 4

Finite slope subspace: the general theory

4.1 Sen theory on p-adic Hodge structures in families

We recall Sen theory from the summary (2.2)-(2.6) of [26].

For L a finite extension of $\mathbb{Q}_p$, we denote $\tilde{L}_\infty := \bigcup_{n\in\mathbb{N}} L(\varepsilon^{(n)})$ with $\varepsilon^{(n)}$ a primitive p^n-th root of unity, and write $\Gamma_L = \mathrm{Gal}(\tilde{L}_\infty/L)$. Let $\Gamma_{L,1}$ be the free (quotient) part of Γ_L and let L_∞ be the subfield of $\tilde{L}_\infty$ corresponding to $\Gamma_{L,1}$. Denote by $H_L = \mathrm{Gal}(\bar{\mathbb{Q}}_p/L_\infty)$ and $\hat{L}_\infty$ the completion of L_∞. Recall that

$$\mathbb{C}_p^{H_L} = \hat{L}_\infty$$

by Ax-Sen-Tate.

Let E and K be two finite extensions of $\mathbb{Q}_p$ such that $E \subset K$. Let X be an E-analytic space and M a locally free $\mathcal{O}_X$-module of rank n, with a continuous G_K-action.

Let $\mathrm{Sp}\,\mathcal{R} \subset X$ be an affinoid admissible open subset. Again denote by M the restriction to $\mathrm{Sp}\,\mathcal{R}$ and write $\hat{\mathcal{R}}_\infty := \hat{K}_\infty \hat{\otimes}_E \mathcal{R}$. We may choose (by Prop.6 of

[33]) K' a Galois extension of K which is big enough so that the $\hat{K}'_\infty \hat{\otimes}_E \mathcal{R}$-module $(\mathbb{C}_p \hat{\otimes}_E M_{K'})^{H_{K'}}$ is free of rank n, and we have a natural isomorphism

$$(\mathbb{C}_p \hat{\otimes}_E M_{K'})^{H_{K'}} \hat{\otimes}_{\hat{K}'_\infty} \mathbb{C}_p \xrightarrow{\sim} M_{K'} \otimes_E \mathbb{C}_p.$$

The étale descent shows $(\mathbb{C}_p \hat{\otimes}_E M)^{H_K}$ is a finite flat $\hat{\mathcal{R}}_\infty$-module, and we have a natural isomorphism

$$(\mathbb{C}_p \hat{\otimes}_E M)^{H_K} \hat{\otimes}_{\hat{K}_\infty} \mathbb{C}_p \xrightarrow{\sim} M \otimes_E \mathbb{C}_p.$$

Then Prop.6 of [33] allows us to choose a finite extension K'/K so that $(\mathbb{C}_p \hat{\otimes}_E M_{K'})^{H_{K'}}$ is free over $\hat{K}'_\infty \otimes_E \hat{\mathcal{R}}$, and to find a basis of $(\mathbb{C}_p \hat{\otimes}_E M_{K'})^{H_{K'}}$ such that the $K' \otimes_E \mathcal{R}$-module W_* generated by the basis is $\Gamma_{K',1}$-stable. Let γ be a topological generator of $\Gamma_{K',1} \simeq \mathbb{Z}_p$. Let $r \in \mathbb{N}$ be big enough and define an element $\phi \in \mathrm{End}_{K' \otimes_E \mathcal{R}} W_*$ by

$$\phi := \frac{\log(\gamma^{p^r}|_{W_*})}{\log \chi_{K'}(\gamma^{p^r})}.$$

One checks that ϕ is well-defined and is independent of the choice of r for r big enough. By extension of scalars to $\hat{K}'_\infty \otimes_{\hat{K}_\infty} \hat{\mathcal{R}}_\infty$ we get an operator on $(\mathbb{C}_p \hat{\otimes}_E M_{K'})^{H_{K'}}$, which is again denoted by ϕ. By P. 659 of [33], the characteristic polynomial $P_\phi(T)$ of ϕ lies in $(K \otimes_E \mathcal{R})[T]$.

We will globalize Sen's operator ϕ to get a polynomial $P_\phi(T) \in \mathcal{O}(X)[T]$ attached to the $\mathcal{O}_X$-module M, once we check the compatibility with change of coefficients of the above construction. That is, we need that for any map of E-Banach algebras $\mathcal{R} \to \mathcal{R}'$, the natural map

$$(\mathbb{C}_p \hat{\otimes}_E M_{K'})^{H_{K'}} \hat{\otimes}_{\mathcal{R}} \mathcal{R}' \to (M_{K'} \hat{\otimes}_{\mathcal{R}} \mathcal{R}')^{H_{K'}}$$

is an isomorphism. This is seen easily by using the previous isomorphism

$$(\mathbb{C}_p \hat{\otimes}_E M_{K'})^{H_{K'}} \hat{\otimes}_{\hat{K}'_\infty} \mathbb{C}_p \xrightarrow{\sim} M_{K'} \otimes_E \mathbb{C}_p.$$

Remark 4.1.1. *To an E-representation V of G_K, we can attach Sen's polynomial*

$P_\phi(T) \in (E \otimes_{\mathbb{Q}_p} K)[T]$ by the discussion above. For a given embedding $\iota : E \hookrightarrow \bar{K}$, the zeros of the specialization $P_{\phi,\iota}(T) \in \bar{K}[T]$ via $\iota \otimes_{\mathbb{Q}_p} 1 : E \otimes_{\mathbb{Q}_p} K \hookrightarrow \bar{K}$ are called the Hodge-Tate weights of V with respect to ι. *As ι runs over all the embeddings $E \hookrightarrow \bar{K}$, we get a collection of numbers in $\bar{K}$, which are the Hodge-Tate weights of V regarded as a $\mathbb{Q}_p$-representation of G_K. When $K = \mathbb{Q}_p$ and $\mu \in \mathbb{Q}_p$, the multiplicity with which μ appears as a Hodge-Tate weight is independent of the embedding ι.*

4.2 Periods on families of Galois representations

We summarize and modify some results in Sec. 2, Sec. 3 and Sec. 4 of [26].

Let $k < i$ be non-negative integers. Keep the notation in the previous subsection.

Proposition 4.2.1. *Suppose $P_\phi(T) = (T-k)^r Q_k(T)$ for some $r \in \mathbb{Z}_{\geq 1}$ and some polynomial $Q_k(T) \in \mathcal{O}_X(X)[T]$.*

(1) The $\mathcal{R} \otimes_E K$-module $H^0(G_K, \mathbb{C}_p(k)\hat{\otimes}_E M^*)$ is finitely generated and is killed by $\det(\phi + k)$, which is finite flat of rank r if $Q_k(k)$ is a unit.

For any map $\theta : \mathcal{R} \to \mathcal{R}'$ between noetherian Banach E-algebras such that $\theta(Q_k(k))$ is a unit, we have a natural isomorphism

$$H^0(G_K, \mathbb{C}_p(k)\hat{\otimes}_E M^*) \otimes_{\mathcal{R}} \mathcal{R}' \simeq H^0(G_K, \mathbb{C}_p(k)\hat{\otimes}_E (M^* \otimes_{\mathcal{R}} \mathcal{R}')).$$

(2) Let $i \geq k+1$ be an integer. $H^0(G_K, \mathrm{Fil}^k B_{\mathrm{dR}}/t^i B_{\mathrm{dR}} \hat{\otimes}_E M^*)$ is a finitely generated $\mathcal{R} \otimes_E K$-module. If $Q_k(\nu)$ is a unit for any $k \leq \nu \leq i-1$, then

$$H^0(G_K, \mathrm{Fil}^k B_{\mathrm{dR}}/t^i B_{\mathrm{dR}} \hat{\otimes}_E M^*) \simeq H^0(G_K, \mathbb{C}_p(k)\hat{\otimes}_E M^*).$$

If $Q_k(\nu)$ is a unit for all $\nu \geq k$, then

$$H^0(G_K, \mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_E M^*) \simeq H^0(G_K, \mathrm{Fil}^k B_{\mathrm{dR}}/t^i B_{\mathrm{dR}} \hat{\otimes}_E M^*) \simeq H^0(G_K, \mathbb{C}_p(k)\hat{\otimes}_E M^*).$$

Proof. These are easily seen by the proofs of Prop. 2.4 and Prop. 2.5 [26]. ∎

Let $f = [K_0 : \mathbb{Q}_p]$ and $k \geq 0$ be integers.

Proposition 4.2.2. *For any $\lambda \in \bar{\mathbb{Q}}_p^\times$, there exists a finite extension E/K and a positive integer i_0 depending only on $v_p(\lambda)$ and k, such that for any $i \geq i_0$, the natural map*

$$(\mathrm{Fil}^k B^+_{\mathrm{cris},K} \otimes_K E)^{\varphi^f = \lambda} \hookrightarrow \mathrm{Fil}^k B_{\mathrm{dR}}/\mathrm{Fil}^i B_{\mathrm{dR}} \otimes_K E$$

is an injection with closed image.

Proof. One deduces this by the use of the argument (3.3)-(3.5) [26], replacing $B_{\mathrm{cris}}, \varphi$ *loc.cit* with $B_{\mathrm{cris},K}, \varphi^f$. □

Corollary 4.2.3. *Let $\mathcal{R}$ be a K-Banach algebra and $Y \in \mathcal{R}^\times$. If there exists an element $\lambda \in E^\times$ with E a finite extension of K contained in $\mathcal{R}$ such that $Y\lambda^{-1} - 1$ is topologically nilpotent, then for $i \gg 0$ we have an injection*

$$(\mathrm{Fil}^k B^+_{\mathrm{cris},K} \hat{\otimes}_K \mathcal{R})^{\varphi^f = Y} \hookrightarrow \mathrm{Fil}^k B_{\mathrm{dR}}/\mathrm{Fil}^i B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}$$

with closed image.

Proof. Using Lemma 2.2.1, we have $\exists u \in (\widehat{\mathbb{Q}}_p^{\mathrm{ur}} \hat{\otimes}_{\mathbb{Q}_p} \mathcal{R})^\times$ such that $\varphi^f(u) = (Y^{-1}\lambda)u$. Then by multiplying u on both sides of the above map we reduce to the case $Y = \lambda$. Applying $\hat{\otimes}_E \mathcal{R}$ to the map

$$(\mathrm{Fil}^k B^+_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda} \hookrightarrow \mathrm{Fil}^k B_{\mathrm{dR}}/\mathrm{Fil}^i B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$$

obtained in Prop. 4.2.2, we are done. □

Let $x = (x_i)_{i \geq 0} \in R$, Fontaine's ring. Suppose $v_R(x) > 0$ and define the power series

$$P(x, T)_K = \sum_{i \in \mathbb{Z}} [\varphi^{fi}(x)] T^i,$$

where $[y] \in W(R)$ denotes the Teichmüller lifting of an element $y \in R$. Note that one has the identity

$$\varphi^f(P(x, T))_K = T^{-1} P(x, T)_K.$$

Proposition 4.2.4. *Let $\mathcal{R}$ be a Banach algebra over K and $\alpha \in \mathcal{R}$ such that $\alpha - 1$ is topologically nilpotent. Then the $\mathcal{R}$-submodule of* $(\mathrm{Fil}^k B^+_{\mathrm{cris},K} \hat{\otimes}_K \mathcal{R})^{\varphi^f = \varpi_K^{k+1}\alpha}$ *spanned by*

$$\{t_K^k P(x, \varpi_K^{-1}\alpha^{-1})_K \| |x^{(0)}| < 1|\}$$

is dense.

Proof. One notices that in the case $k = 0$ the Proposition follows from the same proof as that of Cor. 4.6 [26]. For any k, the result follows since we have an isomorphism of $\mathbb{Q}_p$-topological spaces

$$B^{+,\varphi^f = \varpi_K}_{\mathrm{cris},K} \simeq (\mathrm{Fil}^k B^+_{\mathrm{cris},K})^{\varphi^f = \varpi_K^{k+1}},$$

given by multiplication by t_K^k. □

4.3 Construction of finite slope subspace

Let E be a finite extension of $\mathbb{Q}_p$. Write $X_{E'} = X \otimes_E E'$ for X a rigid analytic space over E and E'/E a finite extension. For a map $X' \to X$ of rigid spaces and Y an analytic function on X, we denote by Y' the pullback to X' of Y. For Q an analytic function on a rigid space X, we denote by X_Q the complementary of the vanishing locus of Q.

Definition 4.3.1. Let X be a K-analytic space and M a locally free $\mathcal{O}_X$-module of finite rank, equipped with a continuous G_K-action. For a given invertible function $Y \in \mathcal{O}_X^\times(X)$, we say a map $\theta : X' \to X$ between two analytic spaces over K is Y-*small* if there is a finite extension K' of K, and an element $\lambda \in \mathcal{O}_{X'_{K'}}(X'_{K'})^\times$ such that $K'(\lambda)$ is a product of finite extensions of K and $Y'\lambda^{-1} - 1$ is topologically nilpotent on $X'_{K'}$.

We say a rigid space X is Y-small if the identity map on X is Y-small.

For example, in the proof of Thm. 4.3.3, we consider the case that $X' = \mathrm{Sp}\,\mathcal{R}'$ is a connected affinoid over K and $\lambda \in K' \subset \mathcal{R}'$, so that we will be able to apply Cor.

4.2.3.

Remark 4.3.2. *One sees that in the following two cases we have Y-small maps, which will be the situations we work in.*

- *For a point $x \in X$ and a positive integer k the inclusion*

$$\mathrm{Sp}(\mathcal{O}_{X,x}/\mathfrak{m}_x^k) \to X$$

  is Y-small for any invertible function Y on X, where one can take $K' = K$ and $\lambda = Y(x)$. Here $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$.

- *If $x \in X$ is a closed point and $\mathrm{Sp}\,\mathcal{R}$ is a sufficiently small affinoid neighborhood of x, then the inclusion $\mathrm{Sp}\,\mathcal{R} \to X$ is Y-small for any invertible function Y on X, where one can take $\lambda = Y(x)$ and take K' to be the residue field of x.*

Let X be a separated K-analytic space and M a finite free $\mathcal{O}_X$-module with a continuous $\mathcal{O}_X$-linear G_K-action. We have Sen operator ϕ and Sen polynomial $P_\phi(T) \in \mathcal{O}_X(X)[T]$ attached to M.

Theorem 4.3.3. *Let $k \geq 0$ be an integer, and let $X^k \subset X$ be the closed subspace on which $P_\phi(T) = (T-k)Q_k(T)$ for some polynomial $Q_k(T) \in \mathcal{O}_X(X)[T]$. Let $Y \in \mathcal{O}_X(X)^\times$ be an invertible function on X.*

There exists a rigid Zariski closed subspace $X_{fs,k} = X_{fs,k}(X, M, Y)$ of X^k which is characterized by the following two conditions:

(i) For any $\nu \in \mathbb{Z}_{\geq k}$, $Q_k(\nu)$ is not a zero-divisor on $X_{fs,k}$.

(ii) For $\theta : \mathrm{Sp}\mathcal{R}' \to X$ any Y-small map of K-analytic spaces factoring through $X^k_{Q_k(\nu)}$ for every integer $\nu \geq k$, it factors through $X_{fs,k}$ if and only if any $\mathcal{R}'$-linear G_K-equivariant map

$$h_{\mathcal{R}'} : \theta^* M(\mathrm{Sp}\mathcal{R}') \to \mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}'$$

factors through $(\mathrm{Fil}^k B^+_{\mathrm{cris},K} \hat{\otimes}_K \mathcal{R}')^{\varphi^f = Y'}$.

Moreover, the formation of $X_{fs,k}$ commutes with base change by flat maps.

Remark 4.3.4. *In the setting of Thm. 4.3.3, we fix the p-adic field K and take X to be an rigid analytic space over K, which does not lose any generality. Precisely, suppose that X is defined over a p-adic field E and M a finite free $\mathcal{O}_X$-module with a continuous $\mathcal{O}_X$-linear G_K-action. In the case $E \subset K$, we may apply Thm. 4.3.3 to the base change X_K to K of X_E and regard the resulting subspace as an E-analytic space via the natural morphism $\operatorname{Sp} K \to \operatorname{Sp} E$. In the case $K \subset E$, we may apply Thm. 4.3.3 to $X_{\tau(K)}$ (and the Sen polynomial on it) for each embedding $\tau : K \hookrightarrow E$ and obtain the finite slope subspace $X_{fs,k} \subset X_{\tau(K)}$ which depends on the embedding τ. Alternatively, we could obtain an analogous subspace independent of the choice of embedding τ, by simply requiring the conditions (1) and (2) of Thm. 4.3.3 hold for any $\tau(K)$. We choose to state the theorem as in Thm. 4.3.3 in order to simplify the notations hereafter.*

Proof of Theorem 4.3.3. The uniqueness and that the formation of $X_{fs,k}$ commutes with base change by flat maps are formality, as are shown in (5.7), (5.8) and (5.9) of [26].

For the existence of $X_{fs,k}$, we may assume $X = \operatorname{Sp} \mathcal{R}$ is an affinoid. This can be seen by replacing B_{cris}, φ in (5.9) [26] by $B_{\text{cris},K}, \varphi^f$.

Furthermore one can assume $\operatorname{Sp}\mathcal{R}$ to be small enough, so that $|Y||Y^{-1}| < |\varpi_K^{-1}|$. Here $|\cdot|$ denotes a residue norm on $\operatorname{Sp}\mathcal{R}$, which is a semi-multiplicative K-algebra norm. Fix an element $\lambda \in \bar{\mathbb{Q}}_p$ such that $|Y^{-1}|^{-1} \leq |\lambda| \leq |Y|$ and fix a finite Galois extension $E_{\mathcal{R}}/K$ which contains λ. Then we check easily that $|\varpi_K \lambda Y^{-1}| < 1$, which, together with Cor. 4.4 of [26], implies $t_K^k P(x, \varpi_K^{-1} Y \lambda^{-1})$ is a well-defined element in $(\operatorname{Fil}^k B^+_{\text{cris},K} \hat{\otimes}_K E_{\mathcal{R}} \otimes_K \mathcal{R})^{\varphi^f = \varpi_K^{k+1} Y^{-1} \lambda}$, where $x^{(0)} \in \mathcal{O}_{\mathbb{C}_p}$ with norm strictly smaller than 1.

First note that a map $h_{\mathcal{R}'}$ in question factors through $(\operatorname{Fil}^k B^+_{\text{cris},K} \hat{\otimes}_K \mathcal{R}')^{\varphi^f = Y'}$ if and only if this holds for its composition with the projection from $\operatorname{Fil}^k B_{\text{dR}}$ to $\operatorname{Fil}^k B_{\text{dR}}/\operatorname{Fil}^i B_{\text{dR}}$ for i big enough, by the assumption that θ factors through $X_{Q_k(\nu)}$ for every integer $\nu \geq k$.

Now we consider a non-zero $\mathcal{R}$-linear G_K-equivalent map

$$h : M \to \mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}. \tag{4.1}$$

We further choose by Cor. 4.2.3 a sufficiently large integer i (depending on $v_p(\lambda)$ and k) so that the natural map

$$(\mathrm{Fil}^k B^+_{\mathrm{cris},K} \otimes_K E_{\mathcal{R}})^{\varphi^f = \varpi_K^{k+1}\lambda} \hookrightarrow \mathrm{Fil}^k B_{\mathrm{dR}}/\mathrm{Fil}^i B_{\mathrm{dR}} \otimes_K E_{\mathcal{R}}$$

is injective with closed image. We denote its cokernel by U_λ, which admits an orthogonal basis, which in turn gives rise to an orthogonal basis of the $\mathcal{R} \otimes_K E_{\mathcal{R}}$-module $U_\lambda \hat{\otimes}_K \mathcal{R}$ by a result of Coleman (A1.3, [10]).

Now consider the composition

$$h' : M \otimes_K E_{\mathcal{R}} \xrightarrow{h \otimes_K 1} \mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R} \otimes_K E_{\mathcal{R}} \twoheadrightarrow \mathrm{Fil}^k B_{\mathrm{dR}}/\mathrm{Fil}^i B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R} \otimes_K E_{\mathcal{R}}$$

$$\xrightarrow{t_K^k P(x, \varpi_K^{-1} Y \lambda^{-1})} \mathrm{Fil}^k B_{\mathrm{dR}}/\mathrm{Fil}^i B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R} \otimes_K E_{\mathcal{R}} \twoheadrightarrow U_\lambda \hat{\otimes}_K \mathcal{R}.$$

We define the ideal $I_{\mathcal{R} \otimes_K E_{\mathcal{R}}}(\lambda, x, m, h) \subset \mathcal{R} \otimes_K E_{\mathcal{R}}$ to be generated by the coefficients of $h'(m \otimes 1)$ ($m \in M$) written in the orthogonal basis of the $\mathcal{R} \otimes_K E_{\mathcal{R}}$-module $U_\lambda \hat{\otimes}_K \mathcal{R}$. Here we note that $\sum_{g \in \mathrm{Gal}(E_{\mathcal{R}}/K)} I_{\mathcal{R} \otimes_K E_{\mathcal{R}}}(g \cdot \lambda, x, m, h)$ is invariant under the action of $\mathrm{Gal}(E_{\mathcal{R}}/K)$, hence descends to an ideal $I_{\mathcal{R}}(\lambda, x, m, h) \subset \mathcal{R}$.

We define $X'_{fs,k}$ to be the K-analytic subspace of $\mathrm{Sp}\mathcal{R}$ with respect to the ideal $\mathfrak{a}'_k = \sum_{\lambda, x, m, h} I_{\mathcal{R}}(\lambda, x, m, h)$ with λ, m, x, h varying in obvious ranges.

In the following, we check $X'_{fs,k}$ satisfies condition (ii) of the Theorem. Now suppose we are in (ii). We first remark that for the corresponding map $\theta^\sharp : \mathcal{R} \to \mathcal{R}'$ of affinoid algebras, the image in $\mathcal{R}'$ of $\sum_{x,m} I_{\mathcal{R}}(\lambda, x, m, h)$ is zero if and only if (for any x, m as above) the extension by $\hat{\otimes}_{\mathcal{R}} \mathcal{R}'$ of the map h' factors through $(\mathrm{Fil}^k B^+_{\mathrm{cris},K} \hat{\otimes}_K \mathcal{R}' \otimes_K E_{\mathcal{R}})^{\varphi^f = \varpi_K^{k+1}\lambda}$.

Now suppose the map $h_{\mathcal{R}'}$ in question factors through $(\mathrm{Fil}^k B^+_{\mathrm{cris},K} \hat{\otimes}_K \mathcal{R}')^{\varphi^f = Y'}$. We may assume (by Prop. 4.2.1) its specialization to $\mathrm{Fil}^k B_{\mathrm{dR}}/\mathrm{Fil}^i B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}'$ is induced by the map h above, via the map $\theta^\sharp : \mathcal{R} \to \mathcal{R}'$. Then (for all x, λ as above) the composition

$$M \otimes_{\mathcal{R}} \mathcal{R}' \xrightarrow{h_{\mathcal{R}'}} \mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}' \twoheadrightarrow \mathrm{Fil}^k B_{\mathrm{dR}} / \mathrm{Fil}^i B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}'$$

$$\hookrightarrow \mathrm{Fil}^k B_{\mathrm{dR}} / \mathrm{Fil}^i B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}' \otimes_K E_{\mathcal{R}} \xrightarrow{t_K^k P(x, \varpi_K^{-1} Y' \lambda^{-1})} \mathrm{Fil}^k B_{\mathrm{dR}} / \mathrm{Fil}^i B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}' \otimes_K E_{\mathcal{R}}$$

factors through $(\mathrm{Fil}^k B^+_{\mathrm{cris}, K} \hat{\otimes}_K \mathcal{R}' \otimes_K E_{\mathcal{R}})^{\varphi^f = \varpi_K^{k+1} \lambda}$. Hence $\theta \in X'_{fs,k}(\mathcal{R}')$. Here we have used the fact that

$$\varphi^f P(x, \varpi_K^{-1} Y' \lambda^{-1}) = \varpi_K Y'^{-1} \lambda P(x, \varpi_K^{-1} Y' \lambda^{-1})).$$

Conversely, suppose the given Y-small map $\theta : \mathrm{Sp}\,\mathcal{R}' \to \mathrm{Sp}\,\mathcal{R}$ lies in $X'_{fs,k}(\mathcal{R}')$. We may assume that $\mathrm{Sp}\,\mathcal{R}'$ is connected and the field K' in the definition of Y-smallness is contained in $\mathcal{R}'$ and is Galois over K (see the end of (5.9), [26]). Then we have, by Definition 4.3.1, $Y' \lambda_0^{-1} - 1 \in \mathcal{R}'$ is topologically nilpotent, for some $\lambda_0 \in K'$.

For any map $h_{\mathcal{R}'} : M \otimes_{\mathcal{R}} \mathcal{R}' \to \mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}'$ in question, the assumption that $Q_k(\nu)$ $(\forall \nu \geq k)$ is a unit implies that the $\mathcal{R}'$-module $H^0(G_K, \mathrm{Fil}^k B_{\mathrm{dR}} / \mathrm{Fil}^i B_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} M \otimes_{\mathcal{R}} \mathcal{R}')$ is finite flat of rank 1. Hence we my assume that the specialization to $\mathrm{Fil}^k B_{\mathrm{dR}} / \mathrm{Fil}^i B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}'$ of $h_{\mathcal{R}'}$ is the scalar extension via the map $\theta^\sharp : \mathcal{R} \to \mathcal{R}'$ of the map h in the construction of $X'_{fs,k}$. Thus the composition

$$M \otimes_{\mathcal{R}} \mathcal{R}' \xrightarrow{h_{\mathcal{R}'}} \mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}' \twoheadrightarrow \mathrm{Fil}^k B_{\mathrm{dR}} / \mathrm{Fil}^i B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}'$$

$$\hookrightarrow \mathrm{Fil}^k B_{\mathrm{dR}} / \mathrm{Fil}^i B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}' \otimes_K K' \xrightarrow{t_K^k P(x, \varpi_K^{-1} Y' \lambda_0^{-1})} \mathrm{Fil}^k B_{\mathrm{dR}} / \mathrm{Fil}^i B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}' \otimes_K K'$$

factors through $(\mathrm{Fil}^k B^+_{\mathrm{cris}, K} \hat{\otimes}_K \mathcal{R}' \otimes_K K')^{\varphi^f = \varpi_K^{k+1} \lambda_0}$, by the definition of $X'_{fs,k}$.

On the other hand, by Lemma 2.2.1 there is an element $u \in (\hat{\mathbb{Q}}_p^{\mathrm{ur}} \otimes_K \mathcal{R}')^\times$ such that $\varphi^f(u) = Y'^{-1} \lambda_0 u$, as $Y' \lambda_0^{-1} - 1$ is topologically nilpotent. These together show that

$$u t_K^{k+1} \in (\mathrm{Fil}^k B^+_{\mathrm{cris}, K} \hat{\otimes}_K \mathcal{R}' \otimes_K K')^{\varphi^f = \varpi_K^{k+1} Y'^{-1} \lambda_0}.$$

Now use the multiplication map $\mathcal{R}' \otimes_K K' \to \mathcal{R}'$. For any $m' = m \otimes 1 \in M \otimes_{\mathcal{R}} \mathcal{R}'$

with $m \in M$,

$$ut_K^{k+1}h_{\mathcal{R}'}(m') \in (\mathrm{Fil}^k B^+_{\mathrm{cris},K}\hat{\otimes}_K\mathcal{R}')^{\varphi^f=\varpi_K^{k+1}\lambda_0},$$

as $t_K^k P(x, \varpi_K^{-1}Y'\lambda_0^{-1})h_{\mathcal{R}'}(m')$ lies in the right by the construction of $X'_{fs,k}$, and the elements $t_K^k P(x, \varpi_K^{-1}Y'\lambda_0^{-1})$ are dense in $(\mathrm{Fil}^k B^+_{\mathrm{cris},K}\hat{\otimes}_K\mathcal{R}')^{\varphi^f=\varpi_K^{k+1}Y'^{-1}\lambda_0}$ by Prop. 4.2.4. Therefore

$$h_{\mathcal{R}'}(m') \in (\mathrm{Fil}^k B^+_{\mathrm{cris},K}\hat{\otimes}_K\mathcal{R}')^{\varphi^f=Y'}, \forall m' \in M \otimes_{\mathcal{R}} \mathcal{R}'.$$

This concludes the proof that $X'_{fs,k}$ is characterized by condition (2) in the Theorem.

Now write $X^i_{fs,k}$ for the Zariski closure of $X'_{fs,k,\prod_{\nu=k}^{i-1}Q_k(\nu)}$ in $X'_{fs,k}$. Then $X^i_{fs,k}$ satisfies condition (i) of the Theorem. For integers $i \geq 1$ we have a decreasing sequence of closed subspaces of $X'_{fs,k}$, which is stationary since $\mathcal{R}$ is noetherian. We define $X_{fs,k} = X^i_{fs,k}$ for i sufficiently large, which is characterized by condition (i) and (ii) in the Theorem, as desired. □

Proposition 4.3.5. *Keep the notation in Thm. 4.3.3. We have a natural isomorphism*

$$X_{fs,k}(X, M, Y) \simeq X_{fs,0}(X, M(k), Y),$$

where $M(i)$ the twist by χ_K^i of M, for any $i \in \mathbb{Z}$.

Proof. Adopt the notation in the proof of Thm. 4.3.3. First note that the twist by χ_K^{-k} of the map h' (the composition of $h \otimes_{\mathbb{Q}_p} 1$ and multiplication by $t_K^k P(x, \varpi_K^{-1}Y\lambda^{-1})_K$) is

$$h'(-k) : M(-k) \otimes_K E_{\mathcal{R}} \overset{h(-k)\otimes_K 1}{\longrightarrow} \mathrm{Fil}^0 B_{\mathrm{dR}}\hat{\otimes}_K\mathcal{R}\otimes_K E_{\mathcal{R}} \twoheadrightarrow \mathrm{Fil}^0 B_{\mathrm{dR}}/\mathrm{Fil}^i B_{\mathrm{dR}}\hat{\otimes}_K\mathcal{R}\otimes_K E_{\mathcal{R}}$$

$$\overset{P(x,\varpi_K^{-1}Y\lambda^{-1})_K}{\longrightarrow} \mathrm{Fil}^0 B_{\mathrm{dR}}/\mathrm{Fil}^i B_{\mathrm{dR}}\hat{\otimes}_K\mathcal{R} \otimes_K E_{\mathcal{R}}.$$

Applying the construction of $X'_{fs,0}$ using $h'(-k)$, we see $I_{\mathcal{R}}(\lambda, x, m, h) \simeq I_{\mathcal{R}}(\lambda, x, m, h(-k))$, whence $\mathfrak{a}'_k = \sum_{\lambda,x,m,h} I_{\mathcal{R}}(\lambda, x, m, h) \simeq \mathfrak{a}'_0 = \sum_{\lambda,x,m,h} I_{\mathcal{R}}(\lambda, x, m, h(-k))$. Hence we have

$$X'_{fs,k}(X, M, Y) \simeq X'_{fs,0}(X, M(k), Y).$$

Note that the Sen polynomial for $(M(-k))^*$ is equal to $P_\phi(T + k)$, with $Q_0(T) :=$

$P_\phi(T+k)/T^r = Q_k(T+k)$. Thus Thm. 4.3.3(i) is equivalent to saying that, for any $\nu \in \mathbb{Z}_{\geq 0}$, $Q_0(\nu)$ is not a zero-divisor on $X_{fs,k}$. Now we have that $X^i_{fs,0}$, the Zariski closure of $X'_{fs,0,\prod_{\nu=0}^{i-1} Q_0(\nu)}$ in $X'_{fs,0}(X, M(k), Y) \simeq X'_{fs,k}(X, M, Y)$, is isomorphic to $X^i_{fs,k}$. Therefore we have the desired isomorphism.

$\square$

The following result is completely analogous to (5.16) [26]. We include it here for later use.

Proposition 4.3.6. *Let X, M, Y, k and $X_{fs,k}$ be as in Theorem 4.3.3. Let* $\mathrm{Sp}\mathcal{R}$ *be a Y-small affinoid subdomain of $X_{fs,k}$.*

(i) For i larger enough, any $\mathcal{R}$-linear G_K-equivariant map

$$M(\mathrm{Sp}\mathcal{R}) \to \mathrm{Fil}^k B_{\mathrm{dR}}/t^i B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}$$

factors through $(\mathrm{Fil}^k B^+_{\mathrm{cris},K} \hat{\otimes}_K \mathcal{R})^{\varphi^f = Y}$.

(ii) Let $H \subset \mathcal{R}$ be the smallest ideal such that any map as in (i) factors through $\mathrm{Fil}^k B_{\mathrm{dR}}/t^i B_{\mathrm{dR}} \hat{\otimes}_K H$. *Then the complementary of the vanishing locus of H is Zariski dense in* $\mathrm{Sp}\mathcal{R}$.

(iii) For $E \subset \mathcal{R}$ a closed subfield with a continuous map $\mathcal{R} \to E$, there exists a non-zero E-linear G_K-equivariant map

$$M \otimes_{\mathcal{R}} E \to (\mathrm{Fil}^k B^+_{\mathrm{cris},K} \hat{\otimes}_K E)^{\varphi^f = Y}.$$

Proof. These follow from the arguments of (5.14)-(5.16) [26].

Let $x \in \mathrm{Sp}\,\mathcal{R}$ be a closed point such that $Q_k(\nu)(x) \neq 0, \forall \nu \geq k$. Noting that for r a positive integer the map $\hat{\mathcal{R}}_x/\mathfrak{m}_x^r \to \mathrm{Sp}\,\mathcal{R}$ is Y-small (cf. Remark 4.3.2), where $\hat{\mathcal{R}}_x$ denotes the $\mathfrak{m}_x$-adic completion of $\mathcal{R}$. Denote I to be the index set $\{x, r\}$ with x running over $\mathrm{Sp}\,\mathcal{R}$ and $r \in \mathbb{Z}_{\geq 1}$. Write $\mathcal{R}_\alpha = \hat{\mathcal{R}}_x/\mathfrak{m}_x^r$ for $\alpha = (x, r) \in I$. Set $I_\nu = I$ for any $\nu \in \mathbb{Z}_{\geq k}$.

One checks (cf. the proof of (5.16) [26]) that

(1) For any $\alpha \in I_\nu$, any $\mathcal{R}_\alpha$-linear G_K-equivariant map

$$h : M \otimes_{\mathcal{R}} \mathcal{R}_\alpha \longrightarrow \mathrm{Fil}^k B_{\mathrm{dR}}/t^i B_{\mathrm{dR}} \hat{\otimes}_K \mathcal{R}_\alpha$$

factors through $(\mathrm{Fil}^k B^+_{\mathrm{cris},K} \hat{\otimes}_K \mathcal{R}_\alpha)^{\varphi^f = Y}$.

(2) For any $\alpha \in I_\nu$, $\prod_{\nu=k}^{i-1} Q_k(\nu) \in \mathcal{R}_\alpha$ is a unit.

(3) The map $\mathcal{R} \to \prod_{\alpha \in I_\nu} \mathcal{R}_\alpha$ is an injection.

Now using (1)-(3) we can apply the proof of (5.14) [26] to deduce (i) and (ii). More precisely, the results (3.7) and (2.6) *loc. cit* will be replaced by Cor. 4.2.3 and Prop. 4.2.1. The assertion (iii) is a consequence of (ii) (cf. the proof of (5.15) [26]).

$\square$

Chapter 5

Finite slope deformations

Let's recall some basic concepts and results from the fundamental papers [29], [28] and [26].

Let G be a profinite group satisfying *p-finiteness* condition, that is any finite dimensional (continuous) $\mathbb{F}_p$-representation of G has finite dimensional cohomologies. This condition in turn implies that any finite dimensional (continuous) $\mathbb{Q}_p$-representation of G has finite dimensional cohomologies.

Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$ and $\bar{V}$ an N-dimensional (continuous) $\mathbb{F}$-representation of G. Fix a basis of $\bar{V}$. We then get a representation $\bar{\rho} : G \to \mathrm{GL}_N(\mathbb{F})$.

We define $D_{\bar{\rho}}$ to be the functor which assigns to a local Artinian $W(\mathbb{F})$-algebra A with residue field $\mathbb{F}$ the set of (continuous) liftings of $\bar{\rho}$ to A, up to *strict equivalence*, where we say two lifts are strictly equivalent if they are conjugate by an element in the kernel of reduction map $\mathrm{GL}_N(A) \to \mathrm{GL}_N(\mathbb{F})$. As usual, we call a strict equivalence class of lifts of $\bar{\rho}$ a *deformation* of $\bar{V}$.

Similarly, we define $D^{\square}_{\bar{V}}$ to be the functor which assigns to a local Artinian $W(\mathbb{F})$-algebra A with residue field $\mathbb{F}$ the set of deformations of $\bar{\rho}$ to A which are equipped with a A-basis lifting the chosen basis on $\bar{V}$.

By (1.2) of [29], $\bar{\rho}$ admits a versal deformation to a complete local noetherian $W(\mathbb{F})$-algebra with residue field $\mathbb{F}$. The versal deformation ring $R^{\mathrm{ver}}_{\bar{\rho}}$ is universal if $(\bar{V} \otimes_{\mathbb{F}} \bar{V}^*)^G = \mathbb{F}$. Note that the framed deformation functor $D^{\square}_{\bar{\rho}}$ is always (pro-)represented by a complete local noetherian $W(\mathbb{F})$-algebra, which we denote by $R^{\square}_{\bar{V}}$.

We have analogues for the generic fibre of a (uni-)versal deformation ring. Let A be a (fixed) local Artinian $\mathbb{Q}_p$-algebra whose residue field E is a finite extension of $\mathbb{Q}_p$. Let V be a (fixed) finite free A-module with an A-linear continuous G-action, which are equipped with a fixed A-basis. We define the deformation functor D_V (resp. framed deformation functor $D_V^\square$) on the category of local Artinian $\mathbb{Q}_p$-algebras A' equipped with a local map $A' \to A$ which reduces to an isomorphism on E modulo the maximal ideals, which assigns to such an A' the set of deformations $V_{A'}$ of V to A' (resp. deformations $V_{A'}$ of V which is equipped with an A'-basis lifting the chosen basis on V.)

Let $E/\mathbb{Q}_p$ be a finite extension. The same arguments of (1.2) of [29] show that, an E-representation V of G admits a versal deformation to a complete local noetherian $\mathbb{Q}_p$-algebra with residual field E. The versal deformation ring is universal if $(V \otimes_E V^*)^G = E$. The framed deformation functor $D_V^\square$ is (pro-)represented by a complete local noetherian $\mathbb{Q}_p$-algebra, which we denote by $R_V^\square$.

5.1 Galois deformations in characteristic p and in characteristic 0

Let $\bar{\rho} : G \to \mathrm{GL}_N(\mathbb{F})$ be an N-dimensional $\mathbb{F}$-representation of G whose underlying $\mathbb{F}$-vector space is denoted by $\bar{V}$.

Let E be a finite extension of $\mathbb{Q}_p$. Suppose we are given a map $z : R_{\bar{\rho}}^{\mathrm{ver}}[1/p] \to E$ (resp. $z : R_{\bar{V}}^\square[1/p] \to E$) whose kernel is $\mathfrak{m}_z$ and whose image is E_z. This by continuity induces a map $R_{\bar{\rho}}^{\mathrm{ver}} \to \mathcal{O}_E$ (resp. $R_{\bar{V}}^\square \to \mathcal{O}_E$). Denote $V_{z,\mathcal{O}_E} = V_{R_{\bar{\rho}}^{\mathrm{ver}}} \otimes_{R_{\bar{\rho}}^{\mathrm{ver}}} \mathcal{O}_E$ (resp. $V_{z,\mathcal{O}_E} = V_{R_{\bar{V}}^\square} \otimes_{R_{\bar{V}}^\square} \mathcal{O}_E$) and $V_{z,E} = V_{z,\mathcal{O}_E} \otimes_{\mathcal{O}_E} E$, where $V_{R_{\bar{\rho}}^{\mathrm{ver}}}$ (resp. $V_{R_{\bar{V}}^\square}$) is the versal (resp. universal) $R_{\bar{\rho}}^{\mathrm{ver}}[G]$-module (resp. $R_{\bar{V}}^\square[G]$-module).

Denote by $R_{V_{z,E}}^\square$ the complete noetherian local ring representing the deformation functor $D_{V_{z,E}}^\square$. If $H^0(G, V_{z,E} \otimes_E V_{z,E}^*) = E$, we have the universal deformation ring $R_{V_{z,E}}^{\mathrm{univ}}$ representing the deformation functor $D_{V_{z,E}}$.

The $\mathfrak{m}_z$-adic completion $\hat{R}_{z,E_z}$ (resp. $\hat{R}_{z,E_z}^\square$) of $R_{\bar{\rho}}^{\mathrm{ver}}[1/p]$ (resp. $R_{\bar{V}}^\square[1/p]$) is a complete noetherian local ring with residual field E_z. Write $\hat{R}_z = \hat{R}_{z,E_z} \otimes_{E_z} E$ and $\hat{R}_z^{\square} = \hat{R}_{z,E_z}^{\square} \otimes_{E_z} E$. Then $V_{R_V^{\square}} \otimes_{R_V^{\square}} \hat{R}_z^{\square}$ is a deformation of $V_{z,E}$, hence gives us a natural map $R_{V_{z,E}}^{\square} \to \hat{R}_z^{\square}$, which respects the projections to E. Similarly, when $H^0(G, V_{z,E} \otimes_E V_{z,E}^*) = E$, we have a natural map $R_{V_{z,E}}^{\mathrm{univ}} \to \hat{R}_z$ respecting the projections to E.

Proposition 5.1.1 (Kisin, (9.5) [26], (2.3.5) [24]). *The natural map $R_{V_{z,E}}^{\square} \to \hat{R}_z^{\square}$ is an isomorphism.*

If $H^0(G, V_{z,E} \otimes_E V_{z,E}^*) = E$, then the natural map $R_{V_{z,E}}^{\mathrm{univ}} \to \hat{R}_z$ is formally smooth, which is an isomorphism if $R_{\bar{\rho}}^{\mathrm{ver}}$ is universal.

5.2 Smoothness and representability of finite slope deformation functors

Let G be a profinite group satisfying p-finiteness condition. Let B_i $(0 \le i \le j)$ be topological $\mathbb{Q}_p$-algebras equipped with continuous G-actions such that each B_i is a topological sub-$\mathbb{Q}_p[G]$-algebra of $B =: B_0$. Suppose for any $\mathbb{Q}_p$-representation V of G, $\dim_{\mathbb{Q}_p}(B \otimes_{\mathbb{Q}_p} V)^G$ is finite. We may regard elements in $(B_i \otimes_{\mathbb{Q}_p} V)^G$ for different i's as elements in $(B \otimes_{\mathbb{Q}_p} V)^G$ via the inclusions $(B_i \otimes_{\mathbb{Q}_p} V)^G \subset (B \otimes_{\mathbb{Q}_p} V)^G$, so that it makes sense to talk about linear independence.

5.2.1 Smoothness of p-adic period deformation functors

Let V be a fixed finite free A-module, with A a fixed local Artinian $\mathbb{Q}_p$-algebra whose residue field E is a finite extension of $\mathbb{Q}_p$.

Given an ordered set of (non-zero) A-linearly independent G-equivariant A-linear maps

$$\mathfrak{h} = \{h_i : V \to B_i \otimes_{\mathbb{Q}_p} A\}_{0 \le i \le j},$$

we define a functor $D_V^{\mathfrak{h}}$ on the category of local Artinian $\mathbb{Q}_p$-algebras A' equipped with a local map $A' \to A$ which reduces to an isomorphism on E modulo the maximal ideals, which assigns to such an A' the set of deformations $V_{A'}$ of V to A' such that for each $h_i \in \mathfrak{h}$ there exists an A'-linear G-equivariant lifting $\tilde{h}_i : V_{A'} \to B_i \otimes_{\mathbb{Q}_p} A'$ of h_i. The functor $D_V^{\mathfrak{h}}$ extends to the category of complete noetherian local $\mathbb{Q}_p$-algebras.

Let $\mathfrak{h}'$ be an ordered subset of $\mathfrak{h}$. One checks that $D_V^{\mathfrak{h}}$ is a sub-functor of $D_V^{\mathfrak{h}'}$. In particular, they are sub-functors of D_V. By definition, we have $D_V^{\mathfrak{h}} = \prod_{i=0}^{j} D_V^{h_i}$ with the product taken over D_V.

For each i, $h_i : V \to B_i \otimes_{\mathbb{Q}_p} A$ induces the obvious map $h_i \otimes_A 1 : V \otimes_A V^* \to B_i \otimes_{\mathbb{Q}_p} V^*$, hence the maps on cohomologies. Denote the map $\{H^1(G, h_i \otimes_A 1)\}_{h_i \in \mathfrak{h}}$ by $H^1(G, \mathfrak{h} \otimes_A 1)$. Define

$$H^1_{\mathfrak{h}}(G, V \otimes_A V^*) = \mathrm{Ker}(H^1(G, V \otimes_A V^*) \overset{H^1(G,\mathfrak{h}\otimes_A 1)}{\longrightarrow} \oplus_i H^1(G, B_i \otimes_{\mathbb{Q}_p} V^*)).$$

It is a standard result (cf., e.g., Page 288, [28]) that there are canonical isomorphisms

$$D_V(A[\epsilon]) \simeq H^1(G, V \otimes_A V^*) \simeq \mathrm{Ext}^1_{A[G]}(V, V),$$

where $\epsilon \neq 0, \epsilon^2 = 0$.

Lemma 5.2.1. *We have canonical isomorphisms*

$$D_V^{\mathfrak{h}}(A[\epsilon]) \simeq H^1_{\mathfrak{h}}(G, V \otimes_A V^*) \simeq \mathrm{Ext}^1_{A[G],\mathfrak{h}}(V, V)$$

with the last term the subspace of $\mathrm{Ext}^1_{A[G]}(V, V)$ *consisting of extensions* $0 \to V \to W \to V \to 0$ *such that there are* $A \otimes_{\mathbb{Q}_p} B^G$*-linearly independent* A*-linear* G*-equivariant maps* $g_i : W \to B_i \otimes_{\mathbb{Q}_p} A$ *with* $g_i|V = h_i, \forall h_i \in \mathfrak{h}$.

Proof. We just need to apply the proof of Prop. 8.2 [26]. First, given an extension $0 \to V \to W \to V \to 0$ in $D_V(A[\epsilon]) \simeq \mathrm{Ext}^1_{A[G]}(V, V)$, for each i we already have a map $W \to V \to B_i \otimes_{\mathbb{Q}_p} A$, the composition of the projection $W \to V$ with h_i. Now it is easy to see lifting h_i to $\tilde{h}_i : W \to B_i \otimes_{\mathbb{Q}_p} A[\epsilon] = A \oplus A \cdot \epsilon$ is equivalent to giving an A-linear G-equivariant map $g_i : W \to B_i \otimes_{\mathbb{Q}_p} A \cdot \epsilon$ whose restriction to $V \subset W$ is ϵh_i. This tells us $D_V^{\mathfrak{h}}(A[\epsilon]) \simeq \mathrm{Ext}^1_{A[G],\mathfrak{h}}(V, V)$.

We are left to show $\mathrm{Ext}^1_{A[G],\mathfrak{h}}(V, V) \simeq H^1_{\mathfrak{h}}(G, V \otimes_A V^*)$. With the notation and discussion *loc. cit*, the extension $0 \to V \to W \to V \to 0$ in $H^1(G, V \otimes_A$

$V^*) \simeq \mathrm{Ext}^1_{A[G]}(V,V)$ lies in $\mathrm{Ext}^1_{A[G],\{h_i\}}(V,V)$ if and only if $\delta_W(h_i) = 0$, and the map $W \to \delta_W(h_i)$ is just the map $H^1(G, h_i \otimes_A 1)$. Then the extension class $[W]$ lies in $\mathrm{Ext}^1_{A[G],\mathfrak{h}}(V,V)$ if and only if $\delta_W(h_i) = H^1(G, h_i \otimes_A 1)([W]) = 0$ for all $0 \le i \le j$, i. e. if and only if it lies in $H^1_{\mathfrak{h}}(G, V \otimes_A V^*)$. $\square$

For each i, denote by $\bar{h}_i$ the reduction of h_i modulo the maximal ideal $\mathfrak{m}_A$ of A, and write $\bar{V} = V \otimes_A E$. Then we have the maps

$$\bar{h}_i \otimes_E 1 : \bar{V} \otimes_E \bar{V}^* \to B_i \otimes_{Q_p} \bar{V}^*,$$

hence the induced map

$$H^1(G, \bar{\mathfrak{h}} \otimes_E 1) : H^1(G, \bar{V} \otimes_E \bar{V}^*) \to \oplus_i H^1(G, B_i \otimes_{\mathbb{Q}_p} \bar{V}^*).$$

Proposition 5.2.2. *Suppose the map $H^1(G, \bar{\mathfrak{h}} \otimes_E 1)$ is surjective. Then the functor $D_V^{\mathfrak{h}}$ is smooth if D_V is.*

Proof. One gets this by checking the proofs of (8.3) and (8.4) [26]. Precisely, we replace the map η in (8.3.2) *loc. cit* by

$$\eta = (\eta_i)_i : D_V(A') \to \oplus_i H^1(G, B_i \otimes_{\mathbb{Q}_p} \bar{V}^*) \otimes_E I, \quad [V_{A'}] \mapsto (\delta_{V_{A'}}(h_i))_i,$$

and then notice that $[V_{A'}] \in D_V^{\mathfrak{h}}(A')$ if and only if $\delta_{V_{A'}}(h_i) = 0$ for all i. $\square$

One may apply these results to the p-adic period rings and the Galois equivariant maps to them.

5.2.2 Finite slope deformations

Now suppose B is equipped with a continuous B^G-linear endomorphism ξ. Suppose we are given elements $\lambda_i \in A^\times$ and an ordered set of A-linearly independent G-equivariant A-linear maps

$$\mathfrak{h} = \{h_i : V \to (B_i \otimes_{\mathbb{Q}_p} A)^{\xi=\lambda_i}\}_{0 \le i \le j}.$$

We define a functor $D_V^{\mathfrak{h},\xi}$ on the category of local Artinian $\mathbb{Q}_p$-algebras A' equipped with a local map $A' \to A$ which reduces to an isomorphism on E modulo the maximal ideals, which assigns to such an A' the set of deformations $V_{A'}$ of V to A' such that for each i there is a lifting $\tilde{\lambda}_i \in A'$ of λ_i and an A'-linear G-equivariant map $\tilde{h}_i : V_{A'} \to (B_i \otimes_{\mathbb{Q}_p} A')^{\xi=\tilde{\lambda}_i}$ lifting h_i. Again, the functor $D_V^{\mathfrak{h},\xi}$ extends to the category of complete noetherian local $\mathbb{Q}_p$-algebras.

Let $\mathfrak{h}'$ be an ordered subset of $\mathfrak{h}$. One checks that $D_V^{\mathfrak{h},\xi}$ is a sub-functor of $D_V^{\mathfrak{h}',\xi}$. They are sub-functors of $D_V^{\mathfrak{h}'}$, with each h_i regarded as a map to $B_i \otimes_{\mathbb{Q}_p} A$ via the inclusion $(B_i \otimes_{\mathbb{Q}_p} A)^{\xi=\lambda_i} \subset B_i \otimes_{\mathbb{Q}_p} A$. By definition, we have $D_V^{\mathfrak{h},\xi} = \prod_{i=0}^{j} D_V^{h_i,\xi}$ with the product taken over D_V.

Let $\bar{\lambda}_i$ be the image of λ_i in $A/\mathfrak{m}_A$ for each i. We have the map

$$\bar{h}_i \otimes_E 1 : \bar{V} \otimes_E \bar{V}^* \to (B_i \otimes_{\mathbb{Q}_p} \bar{V}^*)^{\xi=\bar{\lambda}_i},$$

hence the induced map

$$H^1(G, \bar{\mathfrak{h}} \otimes_E 1) : H^1(G, \bar{V} \otimes_E \bar{V}^*) \to \oplus_i H^1(G, B_i \otimes_{\mathbb{Q}_p} \bar{V}^*)^{\xi=\bar{\lambda}_i}.$$

Proposition 5.2.3. *Let A' be a local Artinian $\mathbb{Q}_p$-algebra equipped with a surjective local map $A' \to A$ with kernel I such that $I \cdot \mathfrak{m}_A = 0$, which reduces to an isomorphism on E modulo the maximal ideals. If each ξ-eigenvalue $\bar{\lambda}_i$ on $(B \otimes_{\mathbb{Q}_p} \bar{V}^*)^G$ is of multiplicity one, then the natural map $D_V^{\mathfrak{h},\xi}(A') \to D_V^{\mathfrak{h}}(A')$ is an isomorphism.*

Proof. By (8.9) [26] this is true for each $D_V^{h_i,\xi}$, so is for $D_V^{\mathfrak{h},\xi}$. □

Proposition 5.2.4. *Suppose each ξ-eigenvalue $\bar{\lambda}_i$ on $(B \otimes_{\mathbb{Q}_p} \bar{V}^*)^G$ is of multiplicity one, and suppose the map $H^1(G, \bar{\mathfrak{h}} \otimes_E 1)$ is surjective. Then $D_V^{\mathfrak{h},\xi}$ is smooth if D_V is.*

Proof. With the notation of Prop. 5.2.2, we have (cf. Prop. 8.10 [26])

$$\xi(\eta_i([V_{A'}])) = \lambda_i \eta_i([V_{A'}]).$$

Thus the surjectivity of $H^1(G, \bar{\mathfrak{h}} \otimes_E 1)$ shows that there is a class $[c] \in H^1(G, \bar{V} \otimes_E \bar{V}^*)$

mapping to $\eta([V_{A'}])$, which implies that $[V_{A'}] - [c] \in D_V^{\mathfrak{h}}(A')$. On the other hand, we have by Prop. 5.2.3 that $D_V^{\mathfrak{h},\xi}(A') \simeq D_V^{\mathfrak{h}}(A')$. □

Proposition 5.2.5. *Let $E/\mathbb{Q}_p$ be a finite extension. Fix an ordered set of E-linearly independent G-equivariant E-linear maps*

$$\mathfrak{h} = \{h_i : V \to (B_i \otimes_{\mathbb{Q}_p} E)^{\xi=\lambda_i}\}_{0 \le i \le j}$$

with $\lambda_i \in E^\times$.

Suppose $\dim_E(B_i \otimes_{\mathbb{Q}_p} V^*)^{G,\xi=\lambda_i} = 1$ for any $0 \le i \le j$. Then for each i, there exists a unique lifting $\tilde{\lambda}_i \in A^\times$ of λ_i such that the A-module $(B_i \otimes_{\mathbb{Q}_p} V_A^*)^{G,\xi=\tilde{\lambda}_i}$ contains an lifting of h, and the A-module above is free of rank one.

Proof. By the E-linear independence of elements in $\mathfrak{h}$, we are reduced to show the Proposition for a single h_i, which is the case of (8.12), [26]. □

Proposition 5.2.6. *Keep the notation and assumptions as in Prop. 5.2.5. Then the functor $D_V^{\mathfrak{h},\xi}$ is representable if D_V is.*

Proof. It follows from (8.13) [26] that if D_V is representable then each $D_V^{h_i,\xi}$ is representable, so is $D_V^{\mathfrak{h},\xi}$. □

For example, we can apply these results to the case $G = G_K, B = B_{\text{cris}}^+, \xi = \varphi^f$, where f is the residue degree of K.

Chapter 6

Deformations of a Galois representation with refinement

In what follows, $N \geq 2$ is an integer, $K/\mathbb{Q}_p$ is a finite extension of degree n with residue degree f, and E is a finite extension of $\mathbb{Q}_p$.

Recall that for an E-representation V of G_K with $E \subset \bar{E} = \bar{\mathbb{Q}}_p$, we can attach Sen polynomial $P_\phi(T) \in K \otimes_{\mathbb{Q}_p} E[T]$. For a given embedding $\iota : K \hookrightarrow \bar{\mathbb{Q}}_p$, the zeros of the specialization $P_{\phi,\iota}(T) \in \bar{\mathbb{Q}}_p[T]$ via $\iota \otimes_{\mathbb{Q}_p} 1 : K \otimes_{\mathbb{Q}_p} E \hookrightarrow \bar{\mathbb{Q}}_p$ are called the (generalized) *Hodge-Tate weights of V with respect to ι.* When we use the term *Hodge-Tate weight*, we mean Hodge-Tate weight with respect to a fixed embedding $K \hookrightarrow \bar{\mathbb{Q}}_p$.

6.1 Obstruction for deforming a de Rham representation

Denote $\dim_E H^0(G_K, V \otimes_E V^*) = h^0$.

Lemma 6.1.1. *Let V be an N-dimensional E-representation of G_K. Then*

$$\dim_E H^1(G_K, V \otimes_E V^*) = h^0 + nN^2 + \dim_E H^2(G_K, V \otimes_E V^*).$$

Proof. One gets the equality by the local Tate duality, noting $H^i(G_K, V \otimes_E V^*) =$

$0, \forall i > 2.$ $\square$

Lemma 6.1.2. *Let V be a 2-dimensional de Rham representation of G_K. If $H^0(G_K, V \otimes_E V^*) = E$, then $H^2(G_K, V \otimes_E V^*) = 0$.*

Proof. By the local Tate duality,

$$H^2(G_K, V \otimes_E V^*) \simeq (V \otimes_E V^*(1))^{G_K} \simeq (V(1) \otimes_E V^*)^{G_K},$$

where $V^*(1) = \mathrm{Hom}(V, \mathbb{Q}_p(1))$ is the Tate twist of the representation contragredient to V.

If $H^2(G_K, V \otimes_E V^*) \neq 0$, then there is a non-trivial map of $E[G_K]$-modules

$$r : V \longrightarrow V(1)$$

which forces V to be reducible because otherwise $V = V(1)$ as $E[G_K]$-modules. Then V contains a non-trivial sub-representation $\mathrm{Ker}(r) := U \subsetneq V$. Then V/U is a sub-representation of $V(1)$, which we call $W(1)$. This way W is a sub-representation of V.

In particular, if V is two dimensional indecomposable over E (satisfying $H^2(G_K, V \otimes_E V^*) \neq 0$), we must have $U = W$. Hence V is an extension of $U(1)$ by U. If V is in addition de Rham, then the extension class lies in $H^1_g(G_K, E(-1))$, which is trivial by Prop. 6.2.1, since $E(-1)$ has Hodge-Tate weight 1. Therefore $H^2(G_K, V \otimes_E V^*) = 0$ if V is a 2-dimensional indecomposable de Rham representation. $\square$

6.2 Selmer groups

For an E-representation W of G_K, we have the natural maps

$$W \to W \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}, \quad W \to W \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$$

which induce maps on cohomologies.

Following Bloch-Kato ([4]), we define

$$H^1_f(G_K, W) := \mathrm{Ker}(H^1(G_K, W) \to H^1(G_K, W \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})),$$

$$H^1_g(G_K, W) := \mathrm{Ker}(H^1(G_K, W) \to H^1(G_K, W \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}))$$

via the induced maps on cohomologies.

Proposition 6.2.1 ([4]). *If W is a de Rham E-representation of G_K, we have*

$$\dim_E H^1_f(G_K, W) = \dim_E H^0(G_K, W) + n \dim_K D_{\mathrm{dR}}(W)/\mathrm{Fil}^0 D_{\mathrm{dR}}(W),$$

$$\dim_E H^1_g(G_K, W) = \dim_E H^1_f(G_K, W) + \dim_E D_{\mathrm{cris}}(W^*(1))^{\varphi=1}.$$

Proof. We summarize the results in different places of [4] to give a proof of the Proposition.

By Prop. 1.17 [4], there is a short exact sequence

$$0 \to \mathbb{Q}_p \xrightarrow{\alpha} B_{\mathrm{cris}} \oplus \mathrm{Fil}^0 B_{\mathrm{dR}} \xrightarrow{\beta} B_{\mathrm{cris}} \oplus B_{\mathrm{dR}} \to 0$$

where $\alpha(x) = (x, x)$ and $\beta(x, y) = (x - \varphi(x), x - y)$.

Tensoring the above exact sequence with W, and taking G_K cohomology, we get an exact sequence

$$0 \to H^0(G_K, W) \to D_{\mathrm{cris}}(W) \oplus \mathrm{Fil}^0 D_{\mathrm{dR}}(W) \to D_{\mathrm{cris}}(W) \oplus D_{\mathrm{dR}}(W) \to H^1_f(G_K, W) \to 0,$$

where we have used the fact (Lemma 3.8.1, [4]) that

$$H^1(G_K, \mathrm{Fil}^0 B_{\mathrm{dR}} \otimes_E V) \subset H^1(G_K, B_{\mathrm{dR}} \otimes_E V).$$

Keeping in mind $\dim_K D_{\mathrm{dR}}(W) = \dim_E W$, we get the first equality.

Similarly, Prop. 1.17 [4] gives the short exact sequence

$$0 \to \mathbb{Q}_p \xrightarrow{\alpha} B_{\mathrm{cris}}^{\varphi=1} \oplus B_{\mathrm{dR}}^{+} \xrightarrow{\gamma} B_{\mathrm{dR}} \to 0$$

where $\alpha(x) = (x, x)$ and $\gamma(x, y) = x - y$. We then get an exact sequence

$$0 \to H^0(G_K, W^*(1)) \to D_{\mathrm{cris}}(W^*(1))^{\varphi=1} \oplus \mathrm{Fil}^0 D_{\mathrm{dR}}(W^*(1)) \to D_{\mathrm{dR}}(W^*(1)) \to H^1_e(G_K, W^*(1)) \to 0.$$

Since

$$H^1_e(G_K, W^*(1)) = \mathrm{Ker}(H^1(G_K, W^*(1)) \to H^1(G_K, W^*(1) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}^{\varphi=1})),$$

which is dual to $H^1_g(G_K, W)$ (Prop. 3.8 [4])under the perfect duality

$$H^1(G_K, W) \times H^1(G_K, W^*(1)) \xrightarrow{\cup} H^2(G_K, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p$$

as W is de Rham. One sees easily, by the exact sequence and the first equality, that

$$\dim_E H^1_e(G_K, W^*(1)) = \dim H^1_f(G_K, W^*(1)) - \dim_E D_{\mathrm{cris}}(W^*(1))^{\varphi=1}.$$

Now the second equality follows as $H^1_f(G_K, W^*(1))$ is the perfect dual of $H^1_f(G_K, W)$, under the above perfect duality .

□

For V an E-representation of G_K with integral Hodge-Tate weights $k_0 \leq \cdots \leq k_{N-1}$, let s_i be the multiplicity of k_i in the multiset of Hodge-Tate weights. Denote

$$s_V = \sum_{0 \leq i \leq N-1, k_i \neq k_{i-1}} s_i(s_i - 1)/2$$

where we set $k_{-1} = -\infty$ and $k_N = +\infty$. Write

$$\dim_E D_{\mathrm{cris}}(V \otimes_E V^*(1))^{\varphi=1} = c.$$

(whence $\dim_E D_{\mathrm{cris},K}(V \otimes_E V^*(1))^{\varphi^f=1} \otimes_{K_0} K = nc$.)

Corollary 6.2.2. *Let V be an N-dimensional de Rham E-representation of G_K with*

Hodge-Tate weights $0 = k_0 \leq \cdots \leq k_{N-1}$. *Then*

$$\dim_E H^1_f(G_K, V \otimes_E V^*) = h^0 + nN(N-1)/2 - ns_V,$$

$$\dim_E H^1_g(G_K, V \otimes_E V^*) = h^0 + nN(N-1)/2 - ns_V + nc.$$

Proof. Easy computations show

$$\dim_E D_{\mathrm{dR}}(V \otimes_E V^*)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V \otimes_E V^*) = nN(N-1)/2 - ns_V.$$

Then the result follows from Prop. 6.2.1.

□

6.3 Deforming a Galois representation equipped with p-adic periods

For V an E-representation of G_K with integral Hodge-Tate weights $k_0 \leq k_1 \leq \cdots \leq k_{N-1}$, equipped with distinct Hodge-Tate periods

$$h_{i,HT} : V \to \mathbb{C}_p(k_i) \otimes_{\mathbb{Q}_p} E, i = 0, \cdots, N-1,$$

we use the notation

$$\mathfrak{h}_{I,HT} = \{h_{i,HT}\}_{i \in I}$$

for an ordered subset $I \subset \{0, \cdots, N-1\}$.

Lemma 6.3.1. *Let V be an E-representation of G_K with integral Hodge-Tate weights $k_0 \leq \cdots \leq k_{N-1}$.*

(1) If k_i has multiplicity s_i in the multiset of Hodge-Tate weights, then

$$\dim_E H^1(G_K, \mathbb{C}_p(k_i) \otimes_E V^*) \leq ns_i$$

with equality holds if and only if V is Hodge-Tate.

(2) Assume further that V is de Rham (resp. crystalline). For $* = \mathrm{dR}$ (resp. $* = \mathrm{cris}, K$), we have

$$\dim_E H^1(G_K, \mathrm{Fil}^{k_i} B_* \otimes_E V^*) = n(N - i + m_i), \quad 0 \leq i \leq N-2,$$

with m_i the number of Hodge-Tate weights k_r such that $r < i$ and $k_r = k_i$.

Proof. For (1), note that $\mathbb{C}_p(k_i) \otimes_{\mathbb{Q}_p} V^*$ is a successive extension of $\mathbb{C}_p(k_i - k_r) \otimes_{\mathbb{Q}_p} E$, $r = 0, \cdots, N-1$. Thus $\dim_E H^1(G_K, \mathbb{C}_p(k_i) \otimes_E V^*) \leq ns_i$ by Lemma 2.2.5. We have

$$\mathbb{C}_p(k_i) \otimes_{\mathbb{Q}_p} V^* \simeq \mathbb{C}_p(k_i - k_0) \otimes_{\mathbb{Q}_p} E \oplus \cdots \oplus \mathbb{C}_p(k_i - k_N) \otimes_{\mathbb{Q}_p} E$$

if and only if the Sen operator is semi-simple, i. e. if and only if V is Hodge-Tate.

For (2), we treat the case $* = \mathrm{dR}$, as the other case is completely analogous. The dimension of $H^1(G_K, \mathrm{Fil}^{k_i} B_{\mathrm{dR}} \otimes_E V^*)$ can be calculated as follows:

The short exact sequence

$$0 \longrightarrow \mathrm{Fil}^{k_i+1} B_{\mathrm{dR}} \longrightarrow \mathrm{Fil}^{k_i} B_{\mathrm{dR}} \longrightarrow \mathbb{C}_p(k_i) \longrightarrow 0$$

induces the long exact sequence

$$H^0(G_K, \mathrm{Fil}^{k_i} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*) \xrightarrow{\beta} H^0(G_K, \mathbb{C}_p(k_i) \otimes_E V^*) \to H^1(G_K, \mathrm{Fil}^{k_i+1} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*)$$

$$\to H^1(G_K, \mathrm{Fil}^{k_i} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*) \to H^1(G_K, \mathbb{C}_p(k_i) \otimes_{\mathbb{Q}_p} V^*) \to 0,$$

where the surjectivity follows from Prop. 2.2.5. The map β is surjective because $h_{i,HT}$ is the image of $h_{i,dR}$ via the map $\mathrm{Fil}^{k_i} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^* \to \mathbb{C}_p(k_i) \otimes_{\mathbb{Q}_p} V^*$. Hence we have a short exact sequence

$$0 \to H^1(G_K, \mathrm{Fil}^{k_i+1} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*) \to H^1(G_K, \mathrm{Fil}^{k_i} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*) \to H^1(G_K, \mathbb{C}_p(k_i) \otimes_{\mathbb{Q}_p} V^*) \to 0.$$

As V^* is de Rham, hence Hodge-Tate, we have

$$H^1(G_K, \mathbb{C}_p(k_i) \otimes_{\mathbb{Q}_p} V^*) \simeq H^1(G_K, \oplus_{r=0}^{N-1} \mathbb{C}_p(k_i - k_r) \otimes_{\mathbb{Q}_p} E)$$

Then we get, for i such that $k_{i-1} \neq k_i$, a short exact sequence

$$0 \to H^1(G_K, \mathrm{Fil}^{k_i+1} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*)$$

$$\to H^1(G_K, \mathrm{Fil}^{k_i} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*) \to H^1(G_K, \oplus_{r=i}^{N-1} \mathbb{C}_p(k_i - k_r) \otimes_{\mathbb{Q}_p} E) \to 0,$$

since $H^1(G_K, \oplus_{r=0}^{i-1} \mathbb{C}_p(k_i - k_r) \otimes_{\mathbb{Q}_p} E) = 0$ by Prop. 2.2.5.

Note that

$$H^1(G_K, \mathrm{Fil}^{k_0} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*) \simeq H^1(G_K, B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*) \simeq H^1(G_K, (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E)^N),$$

whence has E-dimension nN, by the assumption that V is de Rham and Lemma 2.2.5. Then one checks by the use of the last short exact sequence and by induction on i that

$$\dim_E H^1(G_K, \mathrm{Fil}^{k_i} B_{\mathrm{dR}} \otimes_E V^*) = n(N - i + m_i), \quad 0 \leq i \leq N-1.$$

□

Recall the definition of $H^1_{\mathfrak{h}}$ for $\mathfrak{h}$ an ordered set of p-adic periods from Sec. 5.2 .

Lemma 6.3.2. *Let V be an E-representation of G_K with integral Hodge-Tate weights $k_0 \leq \cdots \leq k_{N-1}$. Given an E-linear G_K-equivariant map*

$$h_{i,HT} : V \to \mathbb{C}_p(k_i) \otimes_{\mathbb{Q}_p} E,$$

we have

(1) If V is Hodge-Tate and k_i is of multiplicity one in the multiset of Hodge-Tate

weights, then the natural map

$$H^1(G_K, h_{i,HT} \otimes_E 1) : H^1(G_K, V \otimes_E V^*) \to H^1(G_K, \mathbb{C}_p(k_i) \otimes_E V^*)$$

is surjective with n-dimensional image over E.

(2) If k_i has multiplicity $s_i > 1$ in the multiset of Hodge-Tate weights, then $H^1_{h_{i,HT}}(G_K, V \otimes_E V^*) \subset H^1(G_K, V \otimes_E V^*)$ has codimension $\leq ns_i$. If V is Hodge-Tate, then the image of the natural map $H^1(G_K, h_{i,HT} \otimes_E 1)$ has dimension at least n over E.

Proof. First note that the first assertion in (2) is clear by Lemma 6.3.1. For the rest of the lemma, we just need to compare the dimension of the image of $H^1(G_K, h_{i,HT} \otimes_E 1)$ and that of $H^1(G_K, \mathbb{C}_p(k_i) \otimes_E V^*)$.

Now assume V is Hodge-Tate. In either case, for the dimension of the image of $H^1(G_K, h_{i,HT} \otimes_E 1)$, note that the image of $H^1(G_K, E)$ under the map on H^1 induced by the inclusions

$$\iota : E \subset \mathbb{C}_p \otimes_{\mathbb{Q}_p} E \hookrightarrow \mathbb{C}_p(k_i) \otimes_{\mathbb{Q}_p} V^*$$

generates a subspace of $H^1(G_K, \mathbb{C}_p(k_i) \otimes_{\mathbb{Q}_p} V^*)$ whose dimension is at least n over E. On the other hand, since the direct summand $E \subset V \otimes_E V^*$ (via the trace map) is mapped, under the map $h_{i,HT} \otimes_E 1 : V \otimes_E V^* \to \mathbb{C}_p(k_i) \otimes_E V^*$, to a G_K-invariant subspace of $\mathbb{C}_p(k_i) \otimes_E V^*$, it has to map isomorphically to $\iota(E) \subset \mathbb{C}_p(k_i) \otimes_{\mathbb{Q}_p} V^*$. That is, the image of $H^1(G_K, h_{i,HT} \otimes_E 1)$ contains $H^1(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} E)$, which has E-dimension at least n by Prop. 2.2.5. □

Let $0 \leq j \leq N-1$ be an integer. Denote $* = \mathrm{dR}$ or cris, K. If V is in addition equipped with an ordered set of $E \otimes_{\mathbb{Q}_p} K$-linearly independent G_K-equivariant E-linear maps

$$\{h_{i,*} : V \to B_* \otimes_{\mathbb{Q}_p} E\}_{0 \leq i \leq j},$$

we write $D_{i,*}(V^*)$ $(0 \leq i \leq j)$ for the $E \otimes_{\mathbb{Q}_p} K$-submodule of $D_*(V^*)$ generated by

$h_{r,*}$ for $r = 0, \cdots, i$. We use the notation

$$\mathfrak{h}_{I,*} = \{h_{i,*}\}_{i \in I}$$

for an ordered subset $I \subset \{0, \cdots, j\}$.

Proposition 6.3.3. *Let V be a de Rham (resp. crystalline) E-representation of G_K with Hodge-Tate weights $0 = k_0 \leq \cdots \leq k_{N-1}$, which is equipped with an ordered set of $E \otimes_{\mathbb{Q}_p} K$-linearly independent de Rham (resp. crystalline) periods*

$$\{h_{i,*} : V \to \mathrm{Fil}^{k_i} B_* \otimes_{\mathbb{Q}_p} E\}_{0 \leq i \leq N-2}$$

such that

$$D_*(V^*) = D_{i,*}(V^*) \oplus \mathrm{Fil}^{k_i+1} D_*(V^*),$$

for all $0 \leq i \leq N-2$ such that $k_i \neq k_{i+1}$.

Suppose

$$H^2(G_K, V \otimes_E V^*) = 0,$$

and suppose further that $D_{\mathrm{cris}}(V \otimes_E V^*(1))^{\varphi=1} = 0$ when V is de Rham but not crystalline. Then

(1) The map ($\forall 0 \leq i \leq N-2$)

$$H^1(G_K, h_{i,*} \otimes_E 1) : H^1(G_K, V \otimes_E V^*) \to H^1(G_K, \mathrm{Fil}^{k_i} B_* \otimes_E V^*)$$

and the map ($\forall 0 \leq i \leq N-1$)

$$H^1(G_K, h_{i,HT} \otimes_E 1) : H^1(G_K, V \otimes_E V^*) \to H^1(G_K, \mathbb{C}_p(k_i) \otimes_E V^*)$$

are surjective.

(2) If the Hodge-Tate weights of V are distinct, then

$$\dim_E H^1_{\mathfrak{h}_{I,*}}(G_K, V \otimes_E V^*) = h^0 + nN^2 - n \sum_{i \in I} (N - i)$$

for $I \subset \{0, \cdots, N-2\}$ an ordered subset.

Proof. We write $\mathfrak{h}_{I,*} = \mathfrak{h}_{\leq N-2,*}$ for $I = \{0, \cdots, N-2\}$.

We first consider the case $* = \mathrm{dR}$.

By Lemma 5.2.1, an extension class $0 \to V \to W \to V \to 0$ in $H^1(G_K, V \otimes_E V^*)$ lies in $H^1_{\mathfrak{h}_{\leq N-2,dR}}(G_K, V \otimes_E V^*)$ if and only if there is a non-zero E-linear G_K-equivariant map $g_{i,dR} : W \to \mathrm{Fil}^{k_i} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$ such that $g_{i,dR}|_V = h_{i,dR}$, for any $0 \leq i \leq N-2$. Similarly, such an extension class lies in $H^1_{h_{N-1,HT}}(G_K, V \otimes_E V^*)$ if and only if there is a non-zero E-linear G_K-equivariant map $g_{N-1,HT} : W \to \mathbb{C}_p(k_{N-1}) \otimes_{\mathbb{Q}_p} E$ such that $g_{N-1,HT}|_V = h_{N-1,HT}$.

By Prop. 2.2.6, the Hodge-Tate period $h_{N-1,HT} : V \to \mathbb{C}_p(k_{N-1}) \otimes_{\mathbb{Q}_p} E$ (resp. $g_{N-1,HT}$) lifts to be a de Rham period $h_{N-1,dR} : V \to \mathrm{Fil}^{k_{N-1}} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$ (resp. $g_{N-1,dR} : V \to \mathrm{Fil}^{k_{N-1}} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$).

Thus we get N (distinct) de Rham periods on W by using the elements in $H^1_{\mathfrak{h}_{\leq N-2,dR}}(G_K, V \otimes_E V^*)$ and $H^1_{h_{N-1,HT}}(G_K, V \otimes_E V^*)$, which, together with the compositions

$$W \to V \overset{h_{i,dR}}{\to} \mathrm{Fil}^{k_i} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E, \quad 0 \leq i \leq N-1,$$

give rise to $2N$ de Rham periods on W, that is an element in $H^1_g(G_K, V \otimes_E V^*)$. Conversely, one sees easily that an extension class $[W]$ in $H^1_g(G_K, V \otimes_E V^*)$ can produce such $2N$ de Rham periods on W. Therefore we have

$$H^1_{\mathfrak{h}_{\leq N-2,dR}}(G_K, V \otimes_E V^*) \cap H^1_{h_{N-1,HT}}(G_K, V \otimes_E V^*) = H^1_g(G_K, V \otimes_E V^*).$$

This equality implies that the codimension of $H^1_{\mathfrak{h}_{\leq N-2,dR}}(G_K, V \otimes_E V^*) \subset H^1(G_K, V \otimes_E V^*)$ is at least

$$nN(N+1)/2 + ns_V - ns_{N-1},$$

because the codimension of $H^1_g(G_K, V \otimes_E V^*) \subset H^1(G_K, V \otimes_E V^*)$ is $nN(N+1)2 + ns_V$ by Cor. 6.2.2, and that of $H^1_{h_{N-1,HT}}(G_K, V \otimes_E V^*) \subset H^1(G_K, V \otimes_E V^*)$ is at most ns_{N-1} by Lemma 6.3.1.

Thus to see the surjectivity of the maps $H^1(G_K, h_{i,dR} \otimes_E 1)$ $(0 \leq i \leq N -$

2) and $H^1(G_K, h_{N-1,HT} \otimes_E 1)$, we just need to show that the target of the map $H^1(G_K, \mathfrak{h}_{\leq N-2,dR} \otimes_E 1)$ has dimension $nN(N+1)/2 + ns_V - ns_{N-1}$, that is

$$\dim_E \oplus_{i=0}^{N-2} H^1(G_K, \mathrm{Fil}^{k_i} B_{\mathrm{dR}} \otimes_E V^*) = nN(N+1)/2 + ns_V - ns_{N-1}$$

$$= \sum_{i=0}^{N-2} n(N-i) + \sum_{0 \leq i \leq N-2, k_i \neq k_{i-1}} ns_i(s_i - 1)/2 + n(s_{N-1} - 1)(s_{N-1} - 2)/2$$

where the last equality is written for later use.

On the other hand, by Lemma 6.3.1(2) we have

$$\dim_E H^1(G_K, \mathrm{Fil}^{k_i} B_{\mathrm{dR}} \otimes_E V^*) = n(N - i + m_i), \quad 0 \leq i \leq N-1,$$

with m_i the number of Hodge-Tate weights k_r such that $r < i$ and $k_r = k_i$. Then we check easily that

$$\sum_{i=0}^{N-2} \dim_E H^1(G_K, \mathrm{Fil}^{k_i} B_{\mathrm{dR}} \otimes_E V^*)$$

$$= \sum_{i=0}^{N-2} n(N-i) + \sum_{0 \leq i \leq N-2, k_i \neq k_{i-1}} ns_i(s_i - 1)/2 + n(s_{N-1} - 1)(s_{N-1} - 2)/2,$$

as desired.

The surjectivity of the map $H^1(G_K, h_{i,HT} \otimes_E 1)$ $(0 \leq i \leq N-2)$ then follows from that of $H^1(G_K, h_{i,dR} \otimes_E 1)$ and the surjectivity of $H^1(G_K, \mathrm{Fil}^{k_i} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*) \to H^1(G_K, \mathbb{C}_p(k_i) \otimes_{\mathbb{Q}_p} V^*)$, where is the latter is seen (cf. proof of Lemma 6.3.1(2)) by the short exact sequence

$$0 \to H^1(G_K, \mathrm{Fil}^{k_i+1} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*) \to H^1(G_K, \mathrm{Fil}^{k_i} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*) \to H^1(G_K, \mathbb{C}_p(k_i) \otimes_{\mathbb{Q}_p} V^*) \to 0.$$

This concludes the proof of (1).

Similarly, for $* = \mathrm{cris}, K$, we have

$$H^1_{\mathfrak{h}_{\leq N-2,cris}}(G_K, V \otimes_E V^*) \cap H^1_{h_{N-1,HT}}(G_K, V \otimes_E V^*) = H^1_f(G_K, V \otimes_E V^*).$$

Then dimension counting remains the same because

$$\dim_E H^1_f(G_K, V \otimes_E V^*) = \dim_E H^1_g(G_K, V \otimes_E V^*)$$

by Cor. 6.2.2.

The assertion (2) follows from the surjectivity of $H^1(G_K, h_{i,*} \otimes_E 1)$ and Lemma 6.3.1(2), which says that $\dim_E H^1(G_K, \mathrm{Fil}^{k_i} B_* \otimes_E V^*) = n(N-i)$ when the Hodge-Tate weights are distinct.

□

Lemma 6.3.4. *Let V be a finite dimensional E-representation of G_K and $\lambda \in E^\times$. The $(K_0 \otimes_{\mathbb{Q}_p} E)[G_K]$-module $(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\varphi^f = \lambda}$ has finite dimensional cohomologies which vanish in degree > 1, with the Euler characteristic*

$$\chi_E((B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\varphi^f=\lambda}) = -[K_0 : \mathbb{Q}_p] \dim_E V.$$

Proof. This is (7.9) of [26] when $K = \mathbb{Q}_p$. One sees immediately that the analogue holds for any finite extension K of $\mathbb{Q}_p$, as indicated in the Lemma. Precisely, we replace M *loc. cit* by $\oplus_{i=1}^r (K_0 \otimes_{\mathbb{Q}_p} E)(\lambda_i^{-1})$ with $\lambda_i \in E^\times$, by K_0-linearity. Then the same construction *loc. cit* produces an irreducible weakly admissible $K_0 \otimes_{\mathbb{Q}_p} E[\varphi]$-module. Now replace W *loc. cit* by $\mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} M)^{\varphi^f=1}$, which is an $K_0 \otimes_{\mathbb{Q}_p} E$-module of rank r. Then the fundamental short exact sequence

$$0 \to \mathbb{Q}_p \to B_{\mathrm{cris}}^{\varphi=1} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \to 0$$

gives the short exact sequence of $K_0 \otimes_{\mathbb{Q}_p} E[G_K]$-modules

$$0 \to W \to B_{\mathrm{cris}}^{\varphi^f=1} \otimes_{K_0} W \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \otimes_{K_0} W \to 0,$$

which, tensored by V over E, gives the exact sequence of $K_0 \otimes_{\mathbb{Q}_p} E[G_K]$-modules

$$0 \to W \otimes_E V \to \oplus_{i=1}^r (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\varphi^f=\lambda_i} \to (B_{\mathrm{dR}} \otimes_{K_0} M)/(B_{\mathrm{dR}}^+ \otimes_{K_0} M) \otimes_E V \to 0.$$

We see that the third non-zero term has cohomologies vanishing in degree ≥ 2 and has zero Euler characteristic, by Lemma 2.2.5. Note that

$$H^2(G_K, W \otimes_E V) \simeq \mathrm{Hom}_{E[G_K]}(W, V(1)) = 0$$

as we chose $\dim_E W > \dim_E V$ and W is irreducible. Therefore we have

$$\sum_{i=1}^{r} \chi_E(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\varphi^f=\lambda_i} = \chi_E(W \otimes_E V) = -fr \dim_E V$$

with the last equality given by the local Tate duality. Thus

$$\chi_E(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\varphi^f=\lambda_i} = -f \dim_E V, \quad \forall i$$

because the λ_i can vary independently.

□

For V an N-dimensional E-representation of G_K equipped with a non-zero crystalline period h_0 with φ^f-eigenvalue $\lambda \in E$:

$$h_0 : V \to (B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f=\lambda_0},$$

let $h_{0,cris}$ (resp. $h_{0,dR}$) denote the crystalline period (resp. de Rham period) induced by h_0 via the natural inclusions $\tau : (B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f=\lambda_0} \hookrightarrow B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E$ and $\iota : B_{\mathrm{cris},K} \hookrightarrow B_{\mathrm{dR}}$.

Lemma 6.3.5. *Let V be an E-representation of G_K equipped with a crystalline period*

$$h_0 : V \to (B^+_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f=\lambda_0}$$

with $\lambda_0 \in E^\times$. We have

$$\dim_E H^1(G_K, B^+_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} V^*)^{\varphi^f=\lambda_0} = nN.$$

Proof. First note we have the short exact sequence

$$0 \to (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)^{\varphi^f=\lambda_0} \to B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E \overset{\varphi^f-\lambda_0}{\to} B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E \to 0$$

with G_K-equivariant maps, where one sees the surjectivity of $\varphi^f - \lambda_0$ by the use of the admissible $K_0 \otimes_{\mathbb{Q}_p} E[\varphi]$-module M and the surjectivity of

$$\varphi^f - 1 : B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} W \to B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} W.$$

from Lemma 6.3.4. This in turn gives a short exact sequence of $K_0 \otimes_{\mathbb{Q}_p} E$-modules

$$0 \to D_{\mathrm{cris}}(V^*)/(\varphi^f-\lambda_0) \to H^1(G_K, (B_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}V^*)^{\varphi^f=\lambda_0}) \to H^1(G_K, B_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}V^*)^{\varphi^f=\lambda_0} \to 0.$$

Then

$$\begin{aligned}&\dim_E H^1(G_K, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V^*)^{\varphi^f=\lambda_0}\\ &= \dim_E H^1(G_K, (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V^*)^{\varphi^f=\lambda_0}) - \dim_E H^0(G_K, (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V^*)^{\varphi^f=\lambda_0}),\end{aligned}$$

where we have used the fact that $D_{\mathrm{cris}}(V^*)/(\varphi^f - \lambda_0) = H^0(G_K, (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V^*)^{\varphi^f=\lambda_0})$. Thus we have

$$\dim_E H^1(G_K, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V^*)^{\varphi^f=\lambda_0} = -\chi_E((B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V^*)^{\varphi^f=\lambda_0}) = fN$$

by Lemma 6.3.4. The result follows by taking extension by scalars via $\otimes_{K_0} K$. $\square$

Remark 6.3.6. *For any $r \leq 0$,*

$$t^{r-1} B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V^*/t^r B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V^* \simeq \mathbb{C}_p(r-1) \otimes_{\mathbb{Q}_p} V^*,$$

with the right hand side having trivial cohomologies by the assumption that all the $k_i \geq 0$ and by Prop. 2.2.5. Therefore Lemma 6.3.1, lemma 6.3.2, Prop. 6.3.3, Lemma 6.3.4 and Lemma 6.3.5 hold with B^+_{cris} in place of B_{cris}.

6.4 Finite slope deformations of 2-dimensional Galois representations

From now on, we consider the case $N = 2$.

For the convenience of the reader, we specify the previous results on Galois cohomolgy in the case $N = 2$.

Proposition 6.4.1 (Proposition 6.2.1, Lemma 6.3.1, Proposition 6.3.3). *Denote* $* = \mathrm{dR}$ *or* cris, K. *Let V be a de Rham (resp. crystalline) E-representation of G_K with Hodge-Tate weights $0 = k_0 \leq k_1$, which is equipped with an E-linear de Rham (resp. crystalline) period*

$$h_{0,*} : V \to \mathrm{Fil}^0 B_* \otimes_{\mathbb{Q}_p} E$$

such that

$$D_*(V^*) = D_{0,*}(V^*) \oplus \mathrm{Fil}^1 D_*(V^*).$$

Suppose

$$H^2(G_K, V \otimes_E V^*) = 0.$$

Then

(1)

$$\dim_E H^1_f(G_K, V \otimes_E V^*) = \begin{cases} h^0 + n, & 0 < k_1; \\ h^0, & 0 = k_1. \end{cases}$$

$$\dim_E H^1_g(G_K, V \otimes_E V^*) = \dim_E H^1_f(G_K, V \otimes_E V^*) + D_{\mathrm{cris}}(V \otimes_E V^*(1))^{\varphi=1}.$$

(2)

$$\dim_E H^1(G_K, \mathbb{C}_p \otimes_E V^*) = \begin{cases} n, & 0 < k_1; \\ 2n, & 0 = k_1. \end{cases}$$

$$\dim_E H^1(G_K, \mathrm{Fil}^0 B_* \otimes_E V^*) = 2n.$$

(3) Suppose $D_{\mathrm{cris}}(V \otimes_E V^*(1))^{\varphi=1} = 0$ when V is de Rham but not crystalline.

The map

$$H^1(G_K, h_{0,*} \otimes_E 1) : H^1(G_K, V \otimes_E V^*) \to H^1(G_K, \mathrm{Fil}^0 B_* \otimes_E V^*)$$

and the maps

$$H^1(G_K, h_{i,HT} \otimes_E 1) : H^1(G_K, V \otimes_E V^*) \to H^1(G_K, \mathbb{C}_p(k_i) \otimes_E V^*)$$

are surjective, where $i = 0, 1$.

Let V be a 2-dimensional E-representation of G_K equipped with a non-zero crystalline period h_0 with φ^f-eigenvalue $\lambda \in E$:

$$h_0 : V \to (B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda_0}.$$

Recall that $h_{0,cris}$ (resp. $h_{0,dR}$) denotes the crystalline period (resp. de Rham period) induced by h_0 via the natural inclusions $\tau : (B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda_0} \hookrightarrow B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E$ and $\iota : B_{\mathrm{cris},K} \hookrightarrow B_{\mathrm{dR}}$.

We remark that if V is de Rham and if $D_{\mathrm{cris}}(V \otimes_E V^*(1))^{\varphi=1} \neq 0$ then V is crystalline. This is because, as V is de Rham (hence potentially semi-stable), there is a finite totally ramified extension L/K such that

$$D_{\mathrm{cris}}(V \otimes_E V^*(1))^{\varphi=1} = \mathrm{Hom}_{(E \otimes_{\mathbb{Q}_p} L_0)[\varphi, N]}((B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^*)^{G_L}, (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^*)^{G_L}(-1))^{\mathrm{Gal}\, L/K}.$$

Then a nonzero element α in $D_{\mathrm{cris}}(V \otimes_E V^*(1))^{\varphi=1}$ forces N to be zero (hence $V|_{G_L}$ is crystalline) and the image of $h_0|_{G_L}$ under α is another nonzero element $h_1 \in (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^*)^{G_L}$ which is E-linearly independent of $h_0|_{G_L}$. As h_0 is $\mathrm{Gal}(L/K)$-invariant, so is h_1. In this case, we denote the other φ^f-eigenvalue on $D_{\mathrm{cris}}(V^*)$ by λ_1. Of course, by the observation above, we must have $\lambda_0 = p^{\pm 1}\lambda_1$.

Note that the induced map

$$H^1(G_K, h_0 \otimes_E 1) : H^1(G_K, V \otimes_E V^*) \to H^1(G_K, (B_{\mathrm{cris},K} \otimes_E V^*)^{\varphi^f = \lambda_0})$$

factors through $H^1(G_K, B_{\mathrm{cris},K} \otimes_E V^*)^{\varphi^f=\lambda_0}$ by assumption.

Proposition 6.4.2. *Let V be a de Rham E-representation of G_K with Hodge-Tate weights $0 = k_0 \leq k_1$, equipped with a non-zero crystalline period*

$$h_0 : V \to (B^+_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f=\lambda_0}$$

with $\lambda_0 \in E^\times$, such that

$$D_{\mathrm{cris},K}(V^*) = D_{0,cris,K}(V^*) \oplus \mathrm{Fil}^1 D_{\mathrm{cris},K}(V^*)$$

when $k_1 \geq 1$.

Assume $H^2(G_K, V \otimes_E V^*) = 0$.

Then the map

$$H^1(G_K, h_0 \otimes_E 1) : H^1(G_K, V \otimes_E V^*) \to H^1(G_K, B^+_{\mathrm{cris},K} \otimes_E V^*)^{\varphi^f=\lambda_0}$$

and the map $H^1(G_K, h_{i,HT} \otimes_E 1)$ ($\forall 0 \leq i \leq 1$) are surjective. As a consequence,

$$\dim_E H^1_{h_{0,cris}}(G_K, V \otimes_E V^*) = h^0 + 2n.$$

Proof. First we deal with the case $D_{\mathrm{cris}}(V \otimes_E V^*(1))^{\varphi=1} = 0$.

Note that there is a natural map $\iota \otimes_E 1 : B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} V^* \to B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*$, hence a map

$$H^1(G_K, \iota \otimes_E 1) : H^1(G_K, B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} V^*) \to H^1(G_K, B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*),$$

which is compatible with filtration. By Prop. 6.3.3 the map $H^1(G_K, h_{0,dR} \otimes_E 1)$ is surjective. We just need to show that the map $H^1(G_K, h_0 \otimes_E 1)$ coincides with the map

$$H^1(G_K, h_{0,dR} \otimes_E 1)$$

via the induced map $H^1(G_K, \iota \otimes_E 1)$. By definition $H^1(G_K, h_{0,dR} \otimes_E 1)$ is the composition

$$H^1(G_K, V\otimes_E V^*) \overset{H^1(G_K,h_0\otimes_E 1)}{\longrightarrow} H^1(G_K, B^+_{\mathrm{cris},K}\otimes_E V^*)^{\varphi^f=\lambda_0}$$

$$\overset{H^1(G_K,\iota\otimes_E 1)}{\longrightarrow} H^1(G_K, B^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} V^*).$$

Then the map $H^1(G_K,\iota\otimes_E 1)$ is surjective. Hence it is an isomorphism, as the source and the target have the same dimension by Lemma 6.3.5.

Next we look at the case $D_{\mathrm{cris}}(V\otimes_E V^*(1))^{\varphi=1}\neq 0$. We have seen that V is crystalline. Then replacing $h_{0,dR}$ by $h_{0,cris}$ in the argument above, we get the desired result, Precisely, again by Prop. 6.3.3 the map $H^1(G_K, h_{0,cris}\otimes_E 1)$ is surjective. We just need to show that the map $H^1(G_K, h_0\otimes_E 1)$ coincides with the map

$$H^1(G_K, h_{0,cris}\otimes_E 1)$$

via the natural map induced by the inclusion $\tau : (B_{\mathrm{cris},K}\otimes_{\mathbb{Q}_p} E)^{\varphi^f=\lambda_0}\hookrightarrow B_{\mathrm{cris},K}\otimes_{\mathbb{Q}_p} E$. By definition $H^1(G_K, h_{0,cris}\otimes_E 1)$ is the composition

$$H^1(G_K, V\otimes_E V^*) \overset{H^1(G_K,h_0\otimes_E 1)}{\longrightarrow} H^1(G_K, B^+_{\mathrm{cris},K}\otimes_E V^*)^{\varphi^f=\lambda_0}$$

$$\overset{H^1(G_K,\tau\otimes_E 1)}{\longrightarrow} H^1(G_K, B^+_{\mathrm{cris}}\otimes_{\mathbb{Q}_p} V^*).$$

Then the map $H^1(G_K,\tau\otimes_E 1)$ is surjective. Hence it is an isomorphism, as the dimensions of the source and the target are the same.

$\square$

Remark 6.4.3. *In Prop. 6.4.2, in the case $D_{\mathrm{cris}}(V\otimes_E V^*(1))^{\varphi=1}\neq 0$ we could assume that $\lambda_1 = p^f\lambda_0$ as is in (7.13) [26], instead of*

$$D_{\mathrm{cris},K}(V^*) = D_{0,cris,K}(V^*)\oplus \mathrm{Fil}^1 D_{\mathrm{cris},K}(V^*).$$

Note that the two slopes are $\frac{k-1}{2}$ and $\frac{k+1}{2}$. Then both of them are strictly smaller than k if $k>1$ and the smaller one is always smaller than k. In either case, Lemma 2.2.2 *gives*

$$D_{\mathrm{cris},K}(V^*) = D_{0,cris,K}(V^*) \oplus \mathrm{Fil}^1 D_{\mathrm{cris},K}(V^*).$$

In the following, we recall the nice argument in [26] to obtain Prop. 6.4.2 in the case that $D_{\mathrm{cris}}(V \otimes_E V^*(1))^{\varphi=1} \neq 0$ *under the assumption that* $\lambda_1 = p^f \lambda_0$. *This argument also shows that* $H^1(G_K, h_{0,dR} \otimes_E 1)$ *is not surjective.*

Now we assume $D_{\mathrm{cris}}(V \otimes_E V^*(1))^{\varphi=1} \neq 0$, *i.e.* $\dim_E D_{\mathrm{cris}}(V \otimes_E V^*(1))^{\varphi=1} := c = 1$.

We denote the last crystalline period by

$$h_1 : V \to (B^+_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f=\lambda_1}.$$

First, we have by the proof of Prop. 6.3.3 that

$$H^1_{h_{0,dR}}(G_K, V \otimes_E V^*) \cap H^1_{h_{1,HT}}(G_K, V \otimes_E V^*) = H^1_g(G_K, V \otimes_E V^*),$$

which (combined with Cor. 6.2.2) gives

$$\dim_E H^1_{h_{0,dR}}(G_K, V \otimes_E V^*)$$

$$\leq h^0 + nN(N-1)/2 - ns_V + ns_{N-1} + c = h^0 + 2n + c := h^1_{h_{0,dR}} + nc,$$

where we have used that the target of the map $H^1(G_K, h_{1,HT} \otimes_E 1)$ *is of dimension* ns_{N-1} *over* E, *by Lemma 6.3.1. If the equality holds, one sees the surjectivity of* $H^1(G_K, h_{1,HT} \otimes_E 1)$, *because then its image has dimension equal to that of the target.*

Meanwhile, we have

$$\dim_E H^1_{h_{0,cris}}(G_K, V \otimes_E V^*)$$

$$\geq \dim_E H^1(G_K, V \otimes_E V^*) - \dim_E H^1(G_K, B^+_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} V^*)^{\varphi^f=\lambda_0} = h^1_{h_{0,dR}},$$

where we obtian the last equality by recalling Lemma 6.3.1(3).

Therefore to show the surjectivity of $H^1(G_K, h_0 \otimes_E 1)$, *it suffices to show that*

$$\dim_E H^1_{h_{0,dR}}(G_K, V \otimes_E V^*) - \dim_E H^1_{h_{0,cris}}(G_K, V \otimes_E V^*) \geq nc,$$

which will make the inequalities above equalities.

We claim that

$$H^1_g(G_K, V \otimes_E V^*) \subset H^1_{h_{1,cris}}(G_K, V \otimes_E V^*).$$

To see this, let $0 \to V \to W \to V \to 0$ *be an extension of de Rham representations, i.e. an element in* $H^1_g(G_K, V \otimes_E V^*)$. *Replacing* K *by a finite extension if needed, we may assume this is a semi-stable extension, which in turn induces an extension*

$$0 \to D_{\text{cris}}(V^*) \to D_{\text{st}}(W^*) \overset{res}{\to} D_{\text{cris}}(V^*) \to 0$$

of $K_0[\varphi]$*-modules, where the map res sends a map on* W *to its restriction on* V. *The monodromy operator* N_W *on* $D_{\text{st}}(W^*)$ *induces a map of* $K_0[\varphi]$*-modules* $\bar{N}_W$: $D_{\text{cris}}(V^*) \to D_{\text{cris}}(V^*)(-1)$. *The claim means, by Lemma 5.2.1, that any* G_K*-equivariant* E*-linear map* $w_1 : W \to B_{\text{st}} \otimes_{\mathbb{Q}_p} E$ *whose restriction to* V *is* $h_{1,cris}$, *factors through* $B_{\text{cris}} \otimes_{\mathbb{Q}_p} E$. *Hence we just need to show* $N_W(w_1) = \bar{N}_W(h_1) = 0$ *in* $D_{\text{cris}}(V^*)(-1)$. *Suppose* $\bar{N}_W(h_1) \neq 0$. *On one hand, the* φ^f*-eigenvalues on* $D_{\text{cris}}(V^*)(-1)$ *are*

$$\{p^{-f}\lambda_0, p^{-f}\lambda_1\}.$$

On the other hand, the assumption that $\varphi^f(h_1) = \lambda_1 h_1$ *and that* $\bar{N}_W$ *is* φ*-equivariant imply that* $\bar{N}_W(h_1) \in D_{\text{cris}}(V^*)(-1)$ *has* φ^f*-eigenvalues* λ_1, *which is a contradiction to the assumption that* $\lambda_1 \neq p^{-f}\lambda_0$. *This proves the claim.*

It is easy to see (cf. the proof of Lemma 5.2.1) that

$$H^1_{h_{0,cris}}(G_K, V \otimes_E V^*) \cap H^1_{h_{1,cris}}(G_K, V \otimes_E V^*) = H^1_f(G_K, V \otimes_E V^*).$$

Intersecting it with $H^1_g(G_K, V \otimes_E V^*)$ and using the claim shown above, one gets

$$H^1_{h_{0,cris}}(G_K, V \otimes_E V^*) \cap H^1_g(G_K, V \otimes_E V^*) = H^1_f(G_K, V \otimes_E V^*).$$

Since the three terms are subspaces of $H^1_{h_{1,dR}}(G_K, V \otimes_E V^*)$, the codimension of $H^1_{h_{1,cris}}(G_K, V \otimes_E V^*) \subset H^1_{h_{1,dR}}(G_K, V \otimes_E V^*)$ is at least

$$\dim_E H^1_g(G_K, V \otimes_E V^*) - \dim_E H^1_f(G_K, V \otimes_E V^*) = nc,$$

where the last equality holds by Prop. 6.2.1. This finishes the proof of surjectivity of $H^1(G_K, h_{0,cris} \otimes_E 1)$ (and that of $H^1(G_K, h_{1,HT} \otimes_E 1)$ at the same time).

Now we relate the results on Galois cohomologies above to finite slope deformation functors in Sec. 5.

Theorem 6.4.4. *Let V be as in Prop. 6.4.2. Suppose the φ^f-eigenvalues λ_0 are of multiplicity one. Then*

(1) The functor $D_V^{h_0,\varphi^f}$ is represented by a $\mathbb{Q}_p$-algebra $R_V^{h_0,\varphi^f}$, if D_V is representable.

(2) If $H^0(G_K, V \otimes_E V^*) = E$, then the finite slope deformation ring $R_V^{h_0,\varphi^f}$ is a formally smooth quotient of R_V^{univ}, with (relative) dimension equal to $1 + 2n$.

Proof. That the representability of the deformation functor D_V implies that of its sub-functor $D_V^{h_0,\varphi^f}$ is by our assumptions and Prop. 5.2.6. This concludes the proof of (1).

For (2), first note that the deformation functor D_V is representable as $H^0(G_K, V \otimes_E V^*) = E$ and is smooth since $H^2(G_K, V \otimes_E V^*) = 0$. The smoothness of $D_V^{h_0,\varphi^f}$ follows from that of D_V and Prop. 5.2.4, as the map $H^1(G_K, h_0 \otimes_E 1)$ is surjective by Prop. 6.4.2. Hence the finite slope deformation ring $R_V^{h_0,\varphi^f}$ obtained in (1) is formally smooth. One sees the dimension by Prop. 6.4.2, recalling from Lemma 5.2.1 that

$$D_V^{h_0,\varphi^f}(E[\epsilon]) \simeq D_V^{h_0}(E[\epsilon]) \simeq H^1_{h_{0,cris}}(G_K, V \otimes_E V^*),$$

where the first isomorphism is by Prop. 5.2.3.

□

Proposition 6.4.5. *Let V be as in Prop. 6.4.4(2). Denote by $P_\phi(T)$ the Sen polynomial on the universal representation on $R_V^{h_0,\varphi^f}$. Then $P_\phi(T)$ takes the form $P_\phi(T) = TQ_0(T)$ for some $Q_0(T) \in R_V^{h_0,\varphi^f}[T]$, and $Q_0(\nu)$ is non-zero on $R_V^{h_0,\varphi^f}$ for any $\nu \geq 0$.*

Proof. First, Prop. 5.2.2 implies that $D_V^{h_1,HT}$ is a smooth functor, by the surjectivity of the map $H^1(G_K, h_{1,HT} \otimes_E 1)$ obtained in Prop. 6.4.2.

For $r \in Z_{\geq 1}$, let V_r be the universal deformation to $R_V^{h_0,\varphi^f}/\mathfrak{m}^r$, where $\mathfrak{m} \subset R_V^{h_0,\varphi^f}$ is the maximal ideal. By assumption, h_0 lies in $B^+_{\text{cris},K} \otimes_{\mathbb{Q}_p} E$ but does not factor through $\mathrm{Fil}^1 B^+_{\text{cris},K} \otimes_{\mathbb{Q}_p} E$, i.e. it has non-zero image under the map

$$B^+_{\text{cris},K} \otimes_{\mathbb{Q}_p} V^* \to \mathbb{C}_p \otimes_{\mathbb{Q}_p} V^*. \tag{6.1}$$

By the definition of $R_V^{h_0,\varphi^f}$, h_0 lift to $R_V^{h_0,\varphi^f}/\mathfrak{m}^r$. It is easy to check that h_0 lifts to an element $\tilde{h}_0 \in H^0(G_K, \mathrm{Fil}^0 B^+_{\text{cris},K} \otimes_{\mathbb{Q}_p} V_r^*)$. The image $\tilde{h}_{0,HT} \in H^0(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} V_r^*)$ of $\tilde{h}_0$ has to be non-zero, as its image in $H^0(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} V^*)$ is the specialization of h_i via (6.1) which is non-zero. Note that $\tilde{h}_{0,HT} \cdot R_V^{h_0,\varphi^f}/\mathfrak{m}^r$ is a free $R_V^{h_0,\varphi^f}/\mathfrak{m}^r$-module. On the other hand, by Prop. 4.2.1 the module $H^0(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} V_r^*)$ is killed by $\det(\phi)$, where by abuse of notation we write ϕ as the Sen operator on the universal deformation V_r on $R_V^{h_0,\varphi^f}/\mathfrak{m}^r$. In particular, $\det(\phi) \cdot \tilde{h}_{0,HT} = 0$, hence $\det(\phi) = 0$, which means 0 is a root of $P_\phi[T] \in R_V^{h_0,\varphi^f}/\mathfrak{m}^r[T]$. Letting $r \to \infty$ we get that 0 is a root of Sen polynomial on $R_V^{h_0,\varphi^f}$.

We are left to show that none of the $\nu \in \mathbb{Z}_{\geq 0}$ are roots of $Q_0(T)$ on $R_V^{h_0,\varphi^f}$, for which it is enough to show that $D_V^{h_0,\varphi^f} \times_{D_V} D_V^{h_1,HT} \neq D_V^{h_0,\varphi^f}$, as we already have $Q_0(T)(V) = T - k_1$. Suppose the converse is true. We must have that $H^1_{h_{0,cris}}(G_K, V \otimes_E V^*)$ is contained in $H^1_{h_{1,HT}}(G_K, V \otimes_E V^*)$, then by Lemma 5.2.1 and Prop. 5.2.3 we should have

$$H^1_{h_{0,cris}}(G_K, V \otimes_E V^*) \cap H^1_{h_{1,HT}}(G_K, V \otimes_E V^*) = H^1_{h_{0,cris}}(G_K, V \otimes_E V^*).$$

However, the left hand side is just $H^1_f(G_K, V \otimes_E V^*)$, which is at least n dimensional smaller than the right hand side, a contradiction.

$\square$

Chapter 7

Local and global eigenvarieties of GL_2

7.1 Local Galois eigenvarieties of GL_2

Let $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be a (fixed) residual representation with $\mathbb{F}/\mathbb{F}_p$ a finite extension. Fix a versal deformation ring $R_{\bar{\rho}}^{\mathrm{ver}}$ associated to $\bar{\rho}$.

Denote by $Z = \mathrm{Sp}(R_{\bar{\rho}}^{\mathrm{ver}}[1/p])$ the rigid analytic space over $\mathbb{Q}_p$ associated to $R_{\bar{\rho}}^{\mathrm{ver}}$, and M_Z the universal G_K-representation on Z. We have the characteristic polynomial $P_\phi(T) \in \mathcal{O}_Z(Z)[T]$ of Sen operator ϕ on M_Z.

We let $Z^0 \subset Z$ be the closed subspace of Z cut out by $P_\phi(0)$. Set

$$X^0 = Z^0 \times \mathbb{G}_m.$$

Let M^0 denote the pullback to X^0 of $M_Z|_{Z^0}$. Write Y for the canonical co-oordinate on $\mathbb{G}_m$.

Applying Thm. 4.3.3 to X^0, we obtain a closed subspace

$$X_{fs} := X_{fs,0}(X^0, M^0, Y) \subset Z^0 \times \mathbb{G}_m.$$

For $z \in Z(E)$ (resp. $Z^0(E)$) an E-valued point with $E/\mathbb{Q}_p$ a finite extension, write

V_z for the corresponding E-representation of G_K and $\hat{R}_z$ (resp. $\hat{R}_z^0$) for the (scalar extension from its coefficient field to E of) complete local ring at $z \in Z$ (resp. Z^0). For $\lambda_0 \in E^\times$, if $(z, \lambda_0) \in X(E)$ (resp. $X^0(E)$) we define $\hat{R}_{z,\lambda_0}$ (resp. $\hat{R}^0_{z,\lambda_0}$) similarly.

Recall from Sec. 5 that we have the universal deformation ring $R^{\mathrm{univ}}_{V_z}$ when $H^0(G_K, V_z \otimes_E V_z^*) = E$. In this case, we have, by Prop. 5.1.1, a canonical formally smooth map $R^{\mathrm{univ}}_{V_z} \to \hat{R}_z$, which in turn gives a map $R^{\mathrm{univ}}_{V_z} \to \hat{R}_{z,\lambda_0}$ by the composition with $\hat{R}_z \to \hat{R}_{z,\lambda_0}$.

Theorem 7.1.1. *Let $0 = k_0 \leq k_1$ be non-negative integers. Let E be a finite extension of $\mathbb{Q}_p$, and $\lambda_0 \in E^\times$.*

(i) If $(z, \lambda_0) \in X_{fs}(E)$ then there is a non-zero G_K-equivariant E-linear map

$$h_0 : V_z \to (B^+_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda_0},$$

and 0 is a Hodge-Tate weight of V_z.

(ii) Suppose an element $(z, \lambda_0) \in Z \times \mathbb{G}_m(E)$ is equipped with a crystalline period h_0 as above. Then

(1) If the Hodge-Tate weights of V_z are $0, k_1$ with $k_1 \in \bar{\mathbb{Q}}_p - \mathbb{Z}_{\geq 0}$, then (z, λ_0) lies in $X_{fs}(E)$.

(2) Suppose V_z is de Rham of Hodge-Tate weights $0 = k_0 \leq k_1$ such that

$$H^0(G_K, V_z \otimes_E V_z^*) = E,$$

and the φ^f-eigenvalue λ_0 is of multiplicity one. Then (z, λ_0) lies in $X_{fs}(E)$.

Remark 7.1.2. *In the case (ii)(2), we actually have that the assumption*

$$H^0(G_K, V_z \otimes_E V_z^*) = E$$

implies

$$D_{\mathrm{cris},K}(V_z^*) = D_{0,cris,K}(V_z^*) \oplus \mathrm{Fil}^1 D_{\mathrm{cris},K}(V_z^*),$$

by Corollary 2.2.7.

Proof of Thm. 7.1.1. If $(z,\lambda_0)\in X_{fs}(E)$ then the existence of h_0 follows from Prop. 4.3.6. This shows (i).

In what follows, we adopt the notations in the proof of Thm. 4.3.3.

That the existence of h_0 implies $(z,\lambda_0)\in X_{fs}(E)$ in case (1) of (ii) is seen by Thm. 4.3.3, as $Q_0(\nu)$ is not a zero-divisor in X_{fs} for any $\nu\in\mathbb{Z}_{\geq 0}$ by assumption.

To prove $(z,\lambda_0)\in X_{fs}(E)$ in case (2), we may assume $E=k(z)$ for simplicity. We will use the same notation for M^0 and its pullbacks when no confusion arises.

Let κ_z be the kernel of the natural projection $R^{\mathrm{univ}}_{V_z}\twoheadrightarrow R^{h_0,\varphi^f}_{V_z}$, which exists by Thm. 6.4.4.

First we recall that the natural projection $\hat{R}_{z,\lambda_0}\to\hat{R}_{z,\lambda_0}/\kappa_z$ (resp. $\hat{R}_z\to\hat{R}_z/\kappa_z$) factors through $\hat{R}^0_{z,\lambda_0}/\kappa_z$ (resp. $\hat{R}^0_z/\kappa_z$). This is what we have seen in the proof of Prop. 6.4.5.

Recall there are canonical (formally smooth) maps $R^{\mathrm{univ}}_{V_z}\to\hat{R}_z$ and $R^{\mathrm{univ}}_{V_z}\to\hat{R}_{z,\lambda_0}$. We have for any positive integer r, the composition of the obvious maps

$$R^{h_0,\varphi^f}_{V_z}=R^{\mathrm{univ}}_{V_z}/\kappa_z\to\hat{R}^0_z/\kappa_z\to\hat{R}^0_z/(\kappa_z+\mathfrak{m}^r_{z,E})\to\hat{R}^0_{z,\lambda_0}/(\kappa_z+\mathfrak{m}^r_{z,\lambda_0})$$

with $\mathfrak{m}_{z,E}$ (resp. $\mathfrak{m}_{z,\lambda_0}$) the maximal ideal of $\hat{R}^0_z$ (resp. $\hat{R}^0_{z,\lambda_0}$). Then we have the maps

$$h_{0,r}:M^0\to(B^+_{\mathrm{cris},K}\otimes_{\mathbb{Q}_p}\hat{R}^0_z/(\kappa_z+\mathfrak{m}^r_{z,E})^{\varphi^f=\lambda_{0,r}}\to$$

$$(B^+_{\mathrm{cris},K}\otimes_{\mathbb{Q}_p}\hat{R}^0_{z,\lambda_0}/(\kappa_z+\mathfrak{m}^r_{z,\lambda_0}))^{\varphi^f=\lambda_{0,r}}$$

by the universal property of $R^{h_0,\varphi^f}_{V_z}$, where $\lambda_{0,r}$ denotes the image in $\hat{R}^0_z/(\kappa_z+\mathfrak{m}^r_{z,E})$ of the universal element $\lambda^{\mathrm{univ}}_0\in R^{h_0,\varphi^f}_{V_z}$ lifting λ_0. Note by Prop. 5.2.5 that $\lambda^{\mathrm{univ}}_0$ is uniquely determined, which enables us to choose $\{h_{0,r}\}_{r\geq1}$ to be a compatible system in the sense that $\lambda_{0,r+1}=\lambda_{0,r}$ modulo $\mathfrak{m}^r_{z,E}$ (this is condition (2.7)(3) of [26]). Let

$$\hat{h}_0:=\varprojlim h_{0,r}:M^0\to\mathrm{Fil}^0B^+_{\mathrm{cris},K}\hat{\otimes}_{\mathbb{Q}_p}\hat{R}^0_{z,\lambda_0}/\kappa_z.$$

As in the proof of Thm. 4.3.3, let $\mathrm{Sp}\mathcal{R}$ be an affinoid neighborhood of $(z,\lambda_0)\in Z\times\mathbb{G}_m$ so that $|Y||Y^{-1}|<|\varpi_K^{-1}|$ on $\mathrm{Sp}\mathcal{R}$. Take a non-zero $\mathcal{R}$-linear G_K-equivariant map

$$h_0' : M^0 \to B_{\mathrm{dR}}^+/\mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} \mathcal{R}$$

so that

$$(B_{\mathrm{cris},K}^+ \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \varpi_K \lambda_0} \hookrightarrow \mathrm{Fil}^0 B_{\mathrm{dR}}/\mathrm{Fil}^k B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$$

is an injection with closed image for $k \gg 0$, where we take $E_{\mathcal{R}} = E$ as $\lambda_0 \in E$. The map h_0' gives rise to the composition

$$\hat{h_0'} : M^0 \to \mathrm{Fil}^0 B_{\mathrm{dR}}/\mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} \mathcal{R} \to \mathrm{Fil}^0 B_{\mathrm{dR}}/\mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} \hat{R}^0_{z,\lambda_0}/\kappa_z.$$

We claim that the $K \otimes_{\mathbb{Q}_p} \hat{R}^0_{z,\lambda_0}/\kappa_z$-module

$$H^0(G_K, B_{\mathrm{dR}}^+/\mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} (M^0)^*)$$

is free of rank 1.

This follows from Prop. 2.8 [26]. To do this, we need to check the Conditions (1)-(5) (2.7) [26], with $\hat{\mathcal{R}}_{\mathfrak{m}}, J, \hat{h}$ *loc. cit* being $\hat{R}^0_{z,\lambda_0}, \kappa_z, \hat{h}_0$ in our case.

First, we have $R_{V_z}^{\mathrm{univ}}$ is formally smooth as $H^2(G_K, V_z \otimes_E V_z^*) = 0$. The local ring $\hat{R}_{z,\lambda_0}$ is formally smooth over $R_{V_z}^{\mathrm{univ}}$, because $\hat{R}_z$ is, by Prop. 5.1.1. Hence $\hat{R}_z/\kappa_z = \hat{R}_z^0/\kappa_z$ and $\hat{R}_{z,\lambda_0}/\kappa_z = \hat{R}^0_{z,\lambda_0}/\kappa_z$ are formally smooth over $R_{V_z}^{\mathrm{univ}}/\kappa_z := R_{V_z}^{h_0,\varphi^f}$. On the other hand, $R_V^{h_0,\varphi^f}$ is formally smooth by Prop. 6.4.4. Therefore $\hat{R}_z^0/\kappa_z$ and $\hat{R}^0_{z,\lambda_0}/\kappa_z$ are formally smooth rings. Then $K \otimes_{\mathbb{Q}_p} \hat{R}^0_{z,\lambda_0}/\kappa_z$ is a domain. This is Condition (1).

Condition (3) is already seen by our construction, and Condition (4) is the claim we proved at the beginning.

Note that Condition (2) (resp. Condition (5)) is equivalent to saying that $\nu \in \mathbb{Z}_{\geq 1}$ (resp. $\nu = 0$) is not a root of the Sen polynomial in $K \otimes_{\mathbb{Q}_p} \hat{R}^0_{z,\lambda_0}/\kappa_z$. They follow from Prop. 6.4.5. This concludes the proof of the claim.

By the claim, $\hat{h_0'}$ and $\hat{h}_0$ only differ by multiplication by some element $a_0 \in$

$\hat{R}^0_{z,\lambda_0}/\kappa_z$. Thus the specialization of h'_0,

$$h'_{0,r} : M^0 \xrightarrow{h'_0} B^+_{\mathrm{dR}}/\mathrm{Fil}^k B_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}\mathcal{R} \to B^+_{\mathrm{dR}}/\mathrm{Fil}^k B_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}\hat{R}^0_{z,\lambda_0}/(\kappa_z + \mathfrak{m}^r_{z,\lambda_0}),$$

factors through $(B^+_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}\hat{R}^0_{z,\lambda_0}/(\kappa_z+\mathfrak{m}^r_{z,\lambda_0}))^{\varphi^f=\lambda'_{0,r}}$, with $\lambda'_{0,r} = \lambda_{0,r}$ by the uniqueness of $\lambda_0^{\mathrm{univ}}$.

Recall the fact (cf. Remark 4.3.2) that the map

$$\mathrm{Sp}\hat{R}^0_{z,\lambda_0}/(\kappa_z + \mathfrak{m}^r_{z,\lambda_0}) \to \mathrm{Sp}\mathcal{R}$$

is Y-small. Now Prop. 6.4.5 enable us to use Thm. 4.3.3, which implies the map above factors through $X'_{fs} := X'_{fs,0}(\mathrm{Sp}\mathcal{R}))$, i.e. the ideal $\mathfrak{a}'_0 \subset \mathcal{R}$ cutting X'_{fs} out of $\mathrm{Sp}\mathcal{R}$ satisfies

$$\mathfrak{a}'_0\hat{R}^0_{z,\lambda_0} \subset (\tilde{\kappa}_z + \mathfrak{m}^r_{z,\lambda_0})\hat{R}^0_{z,\lambda_0}$$

for *each* $r \geq 1$, where

$$\tilde{\kappa}_z := \kappa_z\hat{R}^0_{z,\lambda_0} + (Y - \tilde{\lambda}_0) \subset \hat{R}^0_{z,\lambda_0}$$

with $\tilde{\lambda}_0 \in \hat{R}^0_z$ a lifting of the image in $\hat{R}^0_z/\kappa_z$ of $\lambda_0^{\mathrm{univ}} \in R^{h_0,\varphi^f}_{V_z}$. Hence $\mathfrak{a}'_0\hat{R}^0_{z,\lambda_0} \subset \tilde{\kappa}_z\hat{R}^0_{z,\lambda_0}$, which means $(z,\lambda_0) \in X'_{fs}$ and we have a map

$$\mathcal{R}/\mathfrak{a}'_0 \to \hat{R}^0_{z,\lambda_0}/\tilde{\kappa}_z.$$

This map must factor through $\mathcal{R}/\mathfrak{a}_0$ with $\mathfrak{a}_0$ the ideal cutting out $X_{fs} := X_{fs,0}(\mathrm{Sp}\mathcal{R}))$, because of the existence of the formally smooth map $R^{h_0,\varphi^f}_{V_z} \to \hat{R}^0_z/\kappa_z \simeq \hat{R}^0_{z,\lambda_0}/\tilde{\kappa}_z$, and Prop. 6.4.5, which imply that there is at least one component of X'_{fs} containing (z,λ_0) where $\det(\phi - \nu)$ is non-zero for any $\nu \in \mathbb{Z}_{\geq 0}$. That is, $(z,\lambda_0) \in X_{fs}$.

$\square$

Keep the notation in Thm. 7.1.1. Denote by $\hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)}$ the complete local ring at $(z,\lambda_0) \in X_{fs}(E)$.

Proposition 7.1.3. *(1) If (z, λ_0) is as in Thm. 7.1.1(ii)(1) such that the φ^f-eigenvalue λ_0 is of multiplicity one, then there is a canonical isomorphism*

$$\hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)} \xrightarrow{\sim} \hat{R}_z \otimes_{R^{\mathrm{univ}}_{V_z}} R^{h_0,\varphi^f}_{V_z}.$$

Moreover, the complete local ring $\hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)}$ is formally smooth over $R^{h_0,\varphi^f}_{V_z}$ if $H^0(G_K, V_z \otimes_E V_z^*) = E$.

(2) If (z, λ_0) is as in Thm. 7.1.1(ii)(2), then there is a canonical isomorphism

$$\hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)} \xrightarrow{\sim} \hat{R}_z \otimes_{R^{\mathrm{univ}}_{V_z}} R^{h_0,\varphi^f}_{V_z},$$

and the complete local ring $\hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)}$ is formally smooth quotient of $\hat{R}_z$ with codimension $2n$.

Proof. This follows from the argument of (10.6) [26]. Keep the notations in the proof of Thm. 7.1.1.

The existence of canonical isomorphism $\hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)} \to \hat{R}_z \otimes_{R^{\mathrm{univ}}_{V_z}} R^{h_0,\varphi^f}_{V_z}$ implies, combined with Prop. 5.1.1 and Thm. 6.4.4, the assertions about smoothness and dimensions.

First suppose we are in case (1) of Thm. 7.1.1(ii), where $Q_0(\nu)$ is not a zero divisor in $\hat{R}^0_{z,\lambda_0}/(\kappa_z + \mathfrak{m}^r_{z,\lambda_0})$, for any $\nu \in \mathbb{Z}_{\geq 0}$. Consider the specialization

$$h'_{0,r} : M^0 \to \mathrm{Fil}^0 B_{\mathrm{dR}}/\mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} \hat{R}^0_{z,\lambda_0}/(\kappa_z + \mathfrak{m}^r_{z,\lambda_0})$$

of the fixed map h'_0 by the map $\mathcal{R} \to \hat{R}^0_{z,\lambda_0}/\kappa_z \to \hat{R}^0_{z,\lambda_0}/(\kappa_z + \mathfrak{m}^r_{z,\lambda_0})$. Then by Prop. 4.2.1 the above map coincides (up to a scalar) with the map $h_{0,r}$ obtained by the universal property of $R^{h_0,\varphi^f}_{V_z}$ (recall the proof of Thm. 7.1.1). Hence it factors through $(B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} \hat{R}^0_{z,\lambda_0}/(\kappa_z + \mathfrak{m}^r_{z,\lambda_0}))^{\varphi^f = \lambda_{0,r}}$. Then the map

$$M^0 \xrightarrow{h'_{0,r}} \mathrm{Fil}^0 B_{\mathrm{dR}}/\mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} \hat{R}^0_{z,\lambda_0}/(\kappa_z + \mathfrak{m}^r_{z,\lambda_0}) \twoheadrightarrow \mathrm{Fil}^0 B_{\mathrm{dR}}/\mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} \hat{R}^0_{z,\lambda_0}/(\tilde{\kappa}_z + \mathfrak{m}^r_{z,\lambda_0})$$

factors through $\hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)}$ by Thm. 4.3.3(ii), noting $\hat{R}^0_{z,\lambda_0}/(\tilde{\kappa}_z + \mathfrak{m}^r_{z,\lambda_0})$ is Y-small.

This gives, passing to the limit over r, a natural map

$$\hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)} \to \hat{R}^0_{z,\lambda_0}/\tilde{\kappa}_z \simeq \hat{R}_z/\kappa_z \simeq \hat{R}_z \otimes_{R^{\mathrm{univ}}_{V_z}} R^{h_0,\varphi^f}_{V_z}.$$

We now need to find the inverse map. Write the maximal ideal of $\hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)}$ by $\mathfrak{n}_{z,\lambda_0}$. Note that $Q_0(\nu)$ (for any $\nu \in \mathbb{Z}_{\geq 0}$) is not a zero-divisor on $\hat{\mathcal{O}}_{X_{fs}(z,\lambda_0)}/\mathfrak{n}^r_{z,\lambda_0}$ for $r \geq 1$, since k_1 is assumed to be away from $\mathbb{Z}_{\geq 0}$. Then by Thm. 4.3.3 (ii), there exists (for any $r \geq 1$) some $\lambda'_{0,r} \in \hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)}/\mathfrak{n}^r_{z,\lambda_0}$ and an $\hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)}$-linear, G_K-equivariant map

$$h'_{0,r} : M^0 \otimes_{\mathcal{R}} \hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)}/\mathfrak{n}^r_{z,\lambda_0} \to (B^+_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} \hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)}/\mathfrak{n}^r_{z,\lambda_0})^{\varphi^f = \lambda'_{0,r}}$$

lifting the given map $h_0 : V_z \to (B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)^{\varphi^f = \lambda_0}$. Then the composition

$$R^{\mathrm{univ}}_{V_z} \to \hat{R}_z \to \hat{R}_{z,\lambda_0} \to \hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)} \to \hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)}/\mathfrak{n}^r_{z,\lambda_0}$$

factors through $R^{h_0,\varphi^f}_{V_z}$ by the universal property of the latter. One takes the limit over r and gets a map $\hat{R}_z \otimes_{R^{\mathrm{univ}}_{V_z}} R^{h_0,\varphi^f}_{V_z} \to \hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)}$, which easily seen to be the inverse of the map constructed at the beginning. This concludes the proof in case (1).

Now suppose we are in case Thm. 7.1.1(ii)(2). Recall from the proof of Thm. 7.1.1 the map $\mathcal{R}/\mathfrak{a}_0 \to \hat{R}^0_{z,\lambda_0}/\tilde{\kappa}_z$. This provides us a natural map

$$\hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)} \to \hat{R}^0_{z,\lambda_0}/\tilde{\kappa}_z \simeq \hat{R}_z/\kappa_z \simeq \hat{R}_z \otimes_{R^{\mathrm{univ}}_{V_z}} R^{h_0,\varphi^f}_{V_z}.$$

To show this is an isomorphism, as in the proof for case (1), we just need to show the natural map

$$R^{\mathrm{univ}}_{V_z} \to \hat{R}_z \to \hat{R}_{z,\lambda_0} \to \hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)}$$

factors through $R^{h_0,\varphi^f}_{V_z}$. That is, we need to show $\kappa_z \hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)} = 0$.

Let $\operatorname{Sp}\mathcal{R} \subset Z^0 \times \mathbb{G}_m$ be an affinoid subdomain which is Y-small. Then Prop.

4.3.6 implies (for $k \gg 0$) any $\mathcal{R}/\mathfrak{a}_0$-linear G_K-equivariant map

$$(h_0)_{\mathcal{R}/\mathfrak{a}_0} : M^0 \otimes_{\mathcal{R}} \mathcal{R}/\mathfrak{a}_0 \to B^+_{\mathrm{dR}}/\mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} \mathcal{R}/\mathfrak{a}_0$$

factors through $(B^+_{\mathrm{cris},K} \hat{\otimes}_{\mathbb{Q}_p} \mathcal{R}/\mathfrak{a}_0)^{\varphi^f = Y}$. Let H be the smallest ideal in $\mathcal{R}/\mathfrak{a}_0$ such that $(h_0)_{\mathcal{R}/\mathfrak{a}_0}$ factors through $B^+_{\mathrm{dR}}/\mathrm{Fil}^k B_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} H$. Applying Prop. 4.3.6, we have that $\mathrm{Sp}(\mathcal{R}/\mathfrak{a}_0) \backslash V(H)$ is Zariski dense in $\mathrm{Sp}(\mathcal{R}/\mathfrak{a}_0)$. Let T_0 be the blow-up of H on $\mathrm{Sp}(\mathcal{R}/\mathfrak{a}_0)$ and $\tilde{x} \in T_0$ a point over $x = (z, \lambda_0) \in \mathrm{Sp}(\mathcal{R}/\mathfrak{a}_0)$. Then the argument on P. 62 [26] shows that

$$\kappa_z \mathcal{O}_{T_0, \tilde{x}} = 0$$

and

$$\hat{\mathcal{O}}_{X_{fs,(z,\lambda_0)}} \hookrightarrow \prod_{\tilde{x}} \mathcal{O}_{T_0, \tilde{x}}$$

with $\tilde{x}$ running over the points above x. We then have $F_0 \hat{\mathcal{O}}_{X_{fs,(z,\lambda_0)}} = 0$, as wanted.

□

Write $K_\infty = \cup_{n \geq 0} K(\varepsilon^{(n)})$ with $\varepsilon^{(n)}$ a primitive p^n-th root of unity, and $\Gamma_{K,1}$ the maximal pro-p quotient of $\Gamma_K = \mathrm{Gal}(K_\infty/K)$, for which the cyclotomic character χ_K gives a canonical injection $\Gamma_K \hookrightarrow \mathbb{Z}_p^\times$.

Denote by $D_{1_{\mathbb{F}}}$ and T_0 the deformation functor and universal deformation ring of the trivial representation of $\Gamma_{K,1}$ over $\mathbb{F}$, respectively. Then T_0 is formally smooth over $W(\mathbb{F})$ of relative dimension $n = [K : \mathbb{Q}_p]$. Write $S_0 = (\mathrm{Spec}\, T_0[1/p])^{\mathrm{an}}$ for the associated rigid space and M_{S_0} for the universal representation on S_0. Denote by $s_0 \in S_0$ the point corresponding to the trivial representation $W(\mathbb{F})[1/p]$ of $\Gamma_{K,1}$.

Set

$$X_{nfs} = X_{fs} \times S_0.$$

Following (1.3.1) [21], we have

Proposition 7.1.4. *Let $(z, \lambda_0) \in X_{fs}(E)$ be as in Prop. 7.1.3 such that the Hodge-*

Tate weights k_i of V_z are distinct. Then the natural map

$$\nu_{h_0} : R_{V_z}^{\mathrm{univ}} \longrightarrow R_{V_z}^{h_0,\varphi^f} \hat{\otimes}_{W(\mathbb{F})[1/p]} \hat{\mathcal{O}}_{S_0,s_0}$$

(inducing the map $(V_A, \chi) \mapsto V_A \otimes_{\mathbb{Q}_p} \chi$ for a local Artinian E-algebra A with residue field E) is surjective.

In particular, in the case of Prop. 7.1.3 (2) the image $\tilde{R}_{V_z}^{h_0,\varphi^f}$ of ν_{h_0} is formally smooth over E of dimension

$$\dim_E R_{V_z}^{h_0,\varphi^f} + n.$$

Proof. It suffices to show the map induces an injection on A-points with A any Artinian local E-algebra whose residue field is E. For this, recall we have Sen polynomial $P_\phi(T)$ on any finite free $A[G_K]$-module. Let (V_A, χ) and (V'_A, χ') be two points having the same image under $\mathrm{Spec}\, \nu_{h_0}$. We may assume $\chi = \eta_0$. The Sen polynomials of V_A (resp. V'_A) takes the form $T(T - k_1 + a_1)$ (resp. $T(T - k_1 + a'_1)$) with $a_1, a'_1 \in \mathfrak{m}_A$, the maximal ideal of A. Pick a topological generator γ of $\Gamma_{K,1}$ (or that of the pro-cyclic open subgroup of $\Gamma_{K,1}$ and set $a = \log\chi'(\gamma)/\log\chi_K(\gamma)$. Then the Sen polynomial on $V'_A \otimes_{\mathbb{Q}_p} \chi'$ is $(T + a)(T - k_1 + a'_1 + a)$. Since k_i's are distinct and $a, a_1, a'_1 \in \mathfrak{m}_A$, a has to be zero. Hence $\chi' = \eta_0$ and $V_A = V'_A$. □

Denote by $\hat{\mathcal{O}}_{X_{nfs},(z,\lambda_0,\eta_0)}$ the complete local ring at a point $(z, \lambda_0, \eta_0) \in X_{nfs}(E)$ for $\eta_0 \in S_0$.

Corollary 7.1.5. *Assume (z, λ_0) is as in Prop. 7.1.4. Then we have a canonical isomorphism*

$$\hat{\mathcal{O}}_{X_{nfs},(z,\lambda_0,\eta_0)} \xrightarrow{\sim} \hat{R}_z \otimes_{R_{V_z}^{\mathrm{univ}}} \tilde{R}_{V_z}^{h_0,\varphi^f}.$$

If in addition (z, λ_0) is as in Prop. 7.1.3(2), then the complete local ring $\hat{\mathcal{O}}_{X_{nfs},(z,\lambda_0,\eta_0)}$ is a formally smooth quotient of $\hat{R}_z$ of codimension n.

Proof. The isomorphism follows from Prop. 7.1.3 and Prop. 7.1.4, which implies, in particular, that

$$\dim \hat{R}_z - \dim \hat{\mathcal{O}}_{X_{nfs},(z,\lambda_0,\eta_0)} = \dim R_{V_z}^{\mathrm{univ}} - \dim R_{V_z}^{h_0,\varphi^f} - n.$$

Then the quantity about codimension can be seen by Thm. 6.4.4. □

7.1.1 Flatness of weight map on the local eigenvariety

We have the Sen polynomial $P_\phi(T) \in \mathcal{O}_Z(Z)[T]$, the zeros of whose evaluation at a point $z \in Z$ are the (generalized) Hodge-Tate weights. The coefficients of these polynomials give rise to a map

$$\underline{w} := \{w_i\}_{0\leq i\leq 1} : Z \longrightarrow \mathbb{A}^2 := \mathcal{W}^2.$$

We use the same notation $\underline{w} : X_{fs} \to \mathcal{W}^2$ for the composite of $\underline{w}$ with the projection $X_{fs} \to Z$, and call it the *weight map* on X_{fs}. Of course, the factor w_0 is constantly zero on X_{fs} by our construction.

Proposition 7.1.6. *Let $(z, \lambda_0) \in X_{fs}(E)$ be as in Thm. 7.1.1(ii)(2). Then the weight map w_1 is flat at this point.*

Proof. By Prop. 7.1.3(2), the complete local ring $\hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)}$ is formally smooth over the finite slope deformation ring $R_{V_z}^{h_0,\varphi^f}$. Then the flatness of w_1 follows from Prop. 6.4.5.

□

7.1.2 Dimension of the local eigenvariety

Denote by F^0 the subset of $Z^0 \times \mathbb{G}_m$ consisting of points (z, λ), where V_z is a 2-dimensional crystalline representation of G_K over a finite extension $E/\mathbb{Q}_p$ with distinct Hodge-Tate weights $0 = k_0 < k_1$ and distinct φ^f-eigenvalues λ_0, λ_1, such that

$$H^0(G_K, V_z \otimes_E V_z^*) = E,$$

$$v_K(\lambda_0) < k_1.$$

Note that we have

$$v_K(\lambda_1) < k_1$$

as well, because $v_K(\lambda_0) + v_K(\lambda_0) = k_1$ by admissibility.

The condition between slopes and Hodge-Tate weights implies that the refinement determined by λ_0 and λ_1 are both non-critical, by Lemma 2.2.2. Thus Thm. 7.1.1(ii)(2) implies that $(z, \lambda_0), (z, \lambda_1) \in X_{fs}$.

Lemma 7.1.7. *The subset $F^0 \subset Z^0 \times \mathbb{G}_m$ is non-empty.*

Proof. As is explained in [21], one can find, by replacing $\bar{V}$ by a twist of the mod p cyclotomic character (if necessary), and by using the Fontaine-Laffaille theory and (3.3.8) of [22], a two dimensional crystalline representation satisfying the conditions we need. For example, there is such a 2-dimensional crystalline representation of Hodge-Tate weights $(0, 1)$ with $\lambda_0 \neq \lambda_1$ and $v_K(\lambda_0) = v_K(\lambda_1) = 1/2$.

□

Lemma 7.1.8. *For any point $x \in F^0 \subset X_{fs}$, there is a quasi-compact admissible open subset U containing x, such that if a closed point $(z, \lambda_0) \in U$ has image under w_1 being a sufficiently large integer, then (z, λ_0) is in F^0.*

Proof. Let $(z, \lambda_0) \in X_{fs}(E)$ with $E/\mathbb{Q}_p$ a finite extension, such that V_z has Hodge-Tate weights $0, k_1$, and

(1) $v_K(\lambda_0) < k_1$,

Note that the φ^f-eigenvalue λ_0 determines a crystalline period h_0, which, by condition (1) and Lemma 2.2.2, does not factor through $\mathrm{Fil}^{k_1} B_{\mathrm{dR}} \otimes E$. On the other hand, the Hodge-Tate period $V_z \to \mathbb{C}_p(k_1)$ lifts to a de Rham period $h' : V_z \to \mathrm{Fil}^{k_1} B_{\mathrm{dR}} \hat{\otimes} E$ by Prop. 2.2.6, which is then $K \otimes_{\mathbb{Q}_p} E$-linearly independent of the other de Rham (crystalline) period obtained above. Hence the V_z's with (z, λ_0) satisfying (1) are de Rham.

Now look at the Weil-Deligne representation $D_{\mathrm{pst}}(V_z^*)$. By the existence of the crystalline period h_0, $D_{\mathrm{pst}}(V_z^*)$ contains an unramified characters χ_0, as a direct summand. The remaining character χ_1 corresponding to the de Rham period h' is also unramified because $\chi_0\chi_1$ is unramified, which follows from the fact that $\det V_z|_{I_K} = \chi_K^{k_1}$ is crystalline. Therefore V_z is semi-stable.

Suppose further that V_z satisfies

(2) $k_1 > 2v_K(\lambda_0) + 1$.

Then V_z is in fact crystalline, because otherwise the monodromy operator N on $D_{\mathrm{st}}(V_z^*)$ is non-zero, hence $\lambda_1 = p^{\pm f}\lambda_0$, which contradicts (2).

We note here that (2) also implies that the φ^f-eigenvalues on $D_{\mathrm{cris}}(V_z^*)$ are distinct, because otherwise $k_1 = 2v_K(\lambda_0)$.

Let $U = \mathrm{Sp}\,\mathcal{R}$ be an affinoid neighborhood of x in X_{fs}, which consists of points $(z, \lambda_0) \in X_{fs}$ such that if $w_1((z, \lambda_0))$ is an integer, then (z, λ_0) verifies $H^0(G_K, V_z \otimes_{\mathbb{Q}_p} V_z^*) = E$ and conditions (1), (2). Then one sees that U is the desired admissible open, by Thm. 7.1.1(ii)(2).

□

Denote by X_{fs}^o (resp. X_{nfs}^o) the union of irreducible components of X_{fs} (resp. X_{nfs}) each of which contains a point in F^0 (resp. a point (z, λ_0, η_0) with $(z, \lambda_0) \in F^0$).

For X a rigid analytic space, we say a subset $S \subset X$ is an *accumulation subset* if for any $x \in S$ and any affinoid neighborhood $U \subset X$ of x, there is an affinoid neighborhood $V \subset U$ of x such that $S \cap V$ is Zariski-dense in V.

Proposition 7.1.9. *(1) The subset F^0 is an accumulation subset of X_{fs}^o. In particular, X_{fs}^o is the rigid Zariski-closure of F^0.*

(2) The subset $\tilde{F}^0$ consisting of the points (z, λ_0, η_0), with $(z, \lambda_0) \in F^0$ and $\eta_0 = \chi_K^k$ $(k \in \mathbb{Z})$, is accumulation in X_{nfs}^o. In particular, X_{nfs}^o is the rigid Zariski-closure of $\tilde{F}^0$.

Proof. By the construction of X_{nfs} via X_{fs}, we see easily that (1) implies (2). We now prove (1).

For a point $x = (z, \lambda_0) \in F^0 \subset X_{fs}^o$, there is only one irreducible component in X_{fs}^o passing through this point and it is smooth, by Prop. 7.1.3(2). For (1), we just need to show this irreducible component is contained in $\bar{F}^0$, the rigid Zariski-closure of F^0 in X_{fs}^o.

Consider the quasi-compact admissible open subspace $U \subset X_{fs}$ containing the point x, obtained from Lemma 7.1.8. We may assume further that that U is connected and smooth, as the (unique) irreducible component passing through x is smooth.

Set $T = \bar{D}^0 \cap U$. It is enough to show $T = U$, because if so then $\bar{F}^0$ contains U, hence contains the irreducible component of X_{fs} passing through x.

By Prop. 7.1.6 the weight map $w_1 : U \to \mathcal{W}$ on X_{fs} is flat at any point in F^0. As $T = \bar{F}^0 \cap U$ is quasi-compact and the image of T under w_1 contains infinitely many points in $\mathcal{W}$, there are infinitely many points in $\mathcal{W}$ over which w_1 is flat. Let $y \in \mathcal{W}$ be a point at which $w_1|_T$ is flat. We then get

$$\dim T \geq \dim(w_1)^{-1}(y) + \dim \mathcal{W} = \dim U,$$

which shows $T = U$, as $T \subset U$ is a closed subspace and U is irreducible.

$\square$

Corollary 7.1.10. *Suppose $\bar{\rho}$ admits a universal deformation ring. Then any irreducible component of X^o_{fs} has dimension*

$$1 + 2n$$

and any irreducible component of X^o_{nfs} has dimension

$$1 + 3n.$$

Proof. First for X^o_{fs}. Consider an irreducible component in question. Since a point in F^0 satisfies the conditions in Prop. 7.1.3(2), such a component has dimension

$$\dim_E \hat{R}_z - \dim_E R^{\mathrm{univ}}_{V_z} + \dim_E R^{h_0,\varphi^f}_{V_z} = \dim_E \hat{R}_z - 2n$$

at a point $(z, \lambda_0) \in F^0$. Moverover, by Prop. 5.1.1 $\hat{R}_z \simeq R^{\mathrm{univ}}_{V_z}$, and the latter has dimension $\dim_E H^0(G_K, V_z \otimes_E V_z^*) + nN^2 = 1 + 4n$. On the other hand, by Prop. 7.1.9, such an irreducible component is the rigid Zariski-closure of the points in F^0, hence the result follows.

The dimension of X^o_{nfs} is then seen by the definition of X_{nfs}, which is $\dim X^o_{fs} + n$.

$\square$

7.1.3 Smoothness of de Rham points in the local eigenvariety

Theorem 7.1.11. *Suppose $\bar{\rho}$ admits a universal deformation ring. Let $(z, \lambda_0, \eta_0) \in X^o_{nfs}(E)$ with E a finite extension of $\mathbb{Q}_p$.*

If (z, λ_0) is as in Thm. 7.1.1(ii)(2), then (z, λ_0) is smooth in X_{fs}. If in addition the Hodge-Tate weights of V_z are distinct, then (z, λ_0, η_0) is smooth in X_{nfs}.

Proof. By Prop. 7.1.3 (2), Cor. 7.1.5 and Cor. 7.1.10, under the assumptions in the Theorem, the dimension of the complete local ring at a point in question is equal to the dimension of the corresponding rigid space.

□

7.2 Global Galois Eigenvariety of GL_2

As before, $N = 2$.

Let F be a number field of degree n, and S a finite set of places containing p and all the archimedean places. Let $\bar{\rho} : G_{F,S} \to \mathrm{GL}_2(\mathbb{F})$ be a residual representation with $\mathbb{F}$ a finite field of characteristic p. Assume that $\bar{\rho}$ admits a universal deformation ring $R^{\mathrm{univ}}_{\bar{\rho}}$.

Denote by Z the rigid analytic space associated to $R^{\mathrm{univ}}_{\bar{\rho}}[1/p]$ and by M_Z the universal Galois module over Z. For v a place of F above p, F_v denotes the completion of F at v. Then G_{F_v} is a subgroup of $G_{F,S}$ by choosing an embedding $\bar{F} \hookrightarrow \bar{F}_v$. Denote by Z_v the analytic space associated to $R^{\mathrm{ver}}_{\bar{\rho}|_{G_{F_v}}}[1/p]$ for $R^{\mathrm{ver}}_{\bar{\rho}|_{G_{F_v}}}$ the versal deformation ring associated to the local residual representation $\bar{\rho}|_{G_{F_v}}$. Note that we have Sen operator ϕ_v on Z_v, for any place v in F.

We denote

$$X = Z \times \prod_{v|p} \mathbb{G}_m, \quad X_v = Z_v \times \mathbb{G}_m, \quad X_p := \prod_{v|p} X_v.$$

Let M_v denote the pullback to X_v of $M_Z|_{Z_v}$. For each $v|p$, write Y_v for the canonical co-oordinate on $\mathbb{G}_m$. As in the local case, let $S_{0,v}$ be the rigid space parameterizing the characters on $\Gamma_{F_v,1}$ whose reduction is the trivial character $1_{\mathbb{F}}$.

Now we have the nearly finite slope subspaces

$$X_{v,nfs} = X_{nfs}(X_v, M_v, Y_v)$$

for each place $v|p$ in F.

By the versal property of $R^{\mathrm{ver}}_{\bar{\rho}|_{G_{F_v}}}$, we have a map of local $W(\mathbb{F})$-algebras $R^{\mathrm{ver}}_{\bar{\rho}|_{G_v}} \to R^{\mathrm{univ}}_{\bar{\rho}}$ (may not be unique), which induces a map between analytic spaces $Z \to Z_v$, for any place $v|p$. Then we have the induced map $\tau_p : X \to X_p$.

We define $\mathfrak{X} = \mathfrak{X}(\mathrm{GL}_{2/F})$ to be the fibre product

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \prod_{v|p} X_{v,nfs} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\tau_p} & X_p \end{array}$$

where the right vertical map is the natural inclusion.

We call $\mathfrak{X}$ the Galois *eigentower* of $\mathrm{GL}_{2/F}$.

As usual, if $z \in Z(E)$ is an E-valued point, V_z denotes the corresponding N-dimensional E-representation of $G_{F,S}$. We denote by z_v the image of z in the local deformation space Z_v.

Denote $n_v = [F_v : \mathbb{Q}_p]$ and $f_v := [F_{v,0} : \mathbb{Q}_p]$ for $v \mid p$ a place in F, where $F_{v,0}$ is the maximal unramified extension of $\mathbb{Q}_p$ inside F_v.

Theorem 7.2.1. *Let $z \in Z(E)$ with E a finite extension of $\mathbb{Q}_p$. Let $\lambda = (\lambda_v)_{v|p}$ with $\lambda_v \in E^\times$, and $\eta = (\eta_v)_{v|p}$ with $\eta_v \in S_{0,v}$.*

(i) If $(z, \lambda, \eta) \in \mathfrak{X}(E)$, then for any place v above p, there exists a non-zero G_{F_v}-equivariant E-linear map

$$h_v : V_z|_{G_{F_v}} \to (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)^{\varphi^{f_v} = \lambda_v}\}.$$

(ii) Suppose for any $v|p$, the element $(z_v, \lambda_v) \in Z_v \times \mathbb{G}_m(E)$ is equipped with a non-zero G_{F_v}-equivariant E-linear map

$$h_v : V_z|_{G_{F_v}} \to (B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)^{\varphi^{f_v} = \lambda_v}.$$

Then (z,λ,η) lies in $\mathfrak{X}$, if for any embedding $F_v \hookrightarrow \bar{\mathbb{Q}}_p$ the data above satisfy either Thm. 7.1.1(ii)(1) or Thm. 7.1.1(ii)(2).

Proof. This follows from Thm. 7.1.1 and the definition of global Galois eigentower immediately. $\square$

We note that, in Thm. 7.2.1(ii) whether $(z,\lambda,\eta)\in\mathfrak{X}$ lies in $\mathfrak{X}$ is independent of the choices of maps $R^{ver}_{\bar{\rho}|_{G_{F_v}}} \to R^{\mathrm{univ}}_{\bar{\rho}}$ we make by Thm. 7.2.1, as the existence of $\mathfrak{h}$ is defined globally.

Keep the notation in Thm. 7.2.1. Set the functors

$$D_{V_z,p} := \prod_{v|p} D_{V_z|_{G_{F_v}}} \supset D^{\mathfrak{h}}_{V_z,p} := \prod_{v|p}(D^{\mathfrak{h}}_{V_z|_{G_{F_v}}} \times D_{1_{\mathbb{F}}}) \supset D^{\mathfrak{h},\varphi}_{V_z,p} := \prod_{v|p}(D^{\mathfrak{h},\varphi^{f_v}}_{V_z|_{G_{F_v}}} \times D_{1_{\mathbb{F}}}).$$

$$D^{\mathfrak{h}}_{V_z} := D_{V_z} \times_{D_{V_z,p}} D^{\mathfrak{h}}_{V_z,p} \supset D^{\mathfrak{h},\varphi}_{V_z} := D_{V_z} \times_{D_{V_z,p}} D^{\mathfrak{h},\varphi}_{V_z,p}.$$

Proposition 7.2.2. *If (z,λ,η) is as in cases (1) or (2) of Theorem 7.2.1(ii) such that the φ^{f_v}-eigenvalues λ_v are of multiplicity one, then the functor $D^{\mathfrak{h},\varphi}_{V_z}$ is represented by the complete local ring $\hat{\mathcal{O}}_{\mathfrak{X},(z,\lambda,\eta)}$.*

In particular, the cotangent space of $\mathfrak{X}$ at (z,λ,η) is canonically isomorphic to

$$H^1_{\mathfrak{h}}(G_{F,S}, V_z \otimes_{\mathbb{Q}_p} V_z^*) :=$$

$$\mathrm{Ker}(H^1(G_{F,S}, V_z\otimes_E V_z^*) \to \prod_{v|p}\mathrm{Im}(H^1(G_{F_v}, h_v\otimes_E 1))) \oplus_E \prod_{v|p} H^1(G_{F_v}, V_{s_{0,v}}\otimes_E V^*_{s_{0,v}}),$$

where $s_{0,v}\in S_{0,v}$ denotes the point corresponding to the trivial representation $W(\mathbb{F})[1/p]$ of $\Gamma_{F_v,1}$.

Proof. The first assertion is formality, for which one can refer to (11.3) [26] for details. The second assertion then follows from Lemma 5.2.1 and Cor. 7.1.5.

$\square$

Assume $\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}[G_{F,S}]}\bar{\rho} = 1$.

Proposition 7.2.3. *If the number field F is of degree $n = r_1 + 2r_2$ with r_1 (resp. r_2) the number of real (resp. imaginary) embeddings, we have*

(1) Assume $N = 2m$. If the image of any complex conjugation is equivalent to $\begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix}$, *then the Krull dimension*

$$\dim R_{\bar{\rho}}^{\mathrm{univ}}/pR_{\bar{\rho}}^{\mathrm{univ}} \geq 1 + nmN.$$

(2) Assume $N = 2m+1$. If the image of any complex conjugation is equivalent to $\begin{pmatrix} I_m & 0 \\ 0 & -I_{m+1} \end{pmatrix}$, *then the Krull dimension*

$$\dim R_{\bar{\rho}}^{\mathrm{univ}}/pR_{\bar{\rho}}^{\mathrm{univ}} \geq 1 + nmN + r_2N.$$

(3) For any positive integer N, if the image of the complex conjugation is equivalent to I_N, then the Krull dimension

$$\dim R_{\bar{\rho}}^{\mathrm{univ}}/pR_{\bar{\rho}}^{\mathrm{univ}} \geq 1 + r_2N^2.$$

Proof. It follows from the proof of Prop. 5 of [29], which uses the global Tate duality. □

A representation $\bar{\rho} : G_{F,S} \to \mathrm{GL}_N(\mathbb{F})$ is said to be *odd* (resp. *even*) if it satisfies (1) or (2) of Prop. 7.2.3 (resp. Prop. 7.2.3(3)).

Let $\mathfrak{X}^o \subset \mathfrak{X}$ be the union of irreducible components of $\mathfrak{X}$ containing a closed point as in Thm. 7.2.1(2).

Theorem 7.2.4. *(1) If $\bar{\rho}$ is odd, then an irreducible component of $\mathfrak{X}^o$ has dimension at least*

$$1 + n.$$

(2) If $\bar{\rho}$ is even, then an irreducible component of $\mathfrak{X}^o$ has dimension at least

$$1 + 4r_2 - n.$$

Proof. We show (1), since (2) is seen similarly. One sees by Prop. 7.2.3 that

$$\dim Z \geq 1 + 2n, N = 2m = 2$$

at any point in Z.

Let v be a place of F above p. By Cor. 7.1.5, the surjective map $\hat{\mathcal{O}}_{Z_v,z} \to \hat{\mathcal{O}}_{X_{v,nfs},(z_v,\lambda_v,\eta_v)}$ has kernel generated by n_v elements, whence the kernel of $\hat{\mathcal{O}}_{Z,z} \to \hat{\mathcal{O}}_{\mathfrak{X},(z,\lambda,\eta)}$ is generated by at most n elements, keeping in mind $\sum_{v|p} n_v = n$. Thus the dimension of an irreducible component in question is at least $\dim Z - n$, hence the desired quantities by Prop. 7.2.3.

□

Chapter 8

Infinite fern in a local Galois deformation space

Let K be a finite extension of $\mathbb{Q}_p$ of degree n and $\mathbb{F}$ a finite extension of $\mathbb{F}_p$. For a 2-dimensional $\mathbb{F}$-representation $\bar{V}$ of G_K with a fixed basis, denote by $R^{\square}_{\bar{V}}$ the framed universal deformation ring.

We will construct, for certain $\bar{V}$'s, an infinite fern (in the sense of Mazur) in the generic fibre $\operatorname{Spec} R^{\square}_{\bar{V}}[1/p]$ of the universal deformation space $\operatorname{Spec} R^{\square}_{\bar{V}}$, using Kisin's method in [21] and the construction of the local Galois eigenvariety for $\mathrm{GL}_{2/K}$.

Lemma 8.0.5. *The scheme* $\operatorname{Spec} R^{\square}_{\bar{V}}[1/p]$ *is of dimension* $d = 4(1+n)$ *over* $W(\mathbb{F})[1/p]$ *at every irreducible component.*

Proof. To see this, first note that $R^{\square}_{\bar{V}}$ is a power series ring whose number of generators minus the number of relations is $d = 1 + nN^2 + N^2 - 1$ by the argument of [29]. Hence each irreducible components has dimension at least d. On the other hand, it is easy to see the irreducible locus of $\operatorname{Spec} R^{\square}_{\bar{V}}[1/p]$ is a dense subspace. For a closed point z in $\operatorname{Spec} R^{\square}_{\bar{V}}[1/p]$ whose corresponding representation is irreducible, we have $H^2(G_K, V_z \otimes_{\mathbb{Q}_p} V_z^*) \simeq H^0(G_K, V_z \otimes_{\mathbb{Q}_p} V_z^*(1)) = 0$, which implies that $R^{\square}_{V_z}[1/p]$ is formally smooth of dimension d by the local Tate duality. By Prop. 5.1.1, the complete local ring at the point z is isomorphic $R^{\square}_{V_z}[1/p]$, hence is of dimension d. Then any irreducible components of $\operatorname{Spec} R^{\square}_{\bar{V}}[1/p]$ is of dimension d.

$\square$

Theorem 8.0.6. *If the generic fibre of the universal Galois deformation space associated to $\bar{V}$ is irreducible and smooth, then the crystalline points in it are Zariski dense.*

Proof. Denote by $Z = \mathrm{Sp}(R^{\square}_{\bar{V}}[1/p])$ the $\mathbb{Q}_p$-analytic space associated to the complete noetherian local ring $R^{\square}_{\bar{V}}[1/p]$.

We apply the construction of X_{fs} and X_{nfs} to Z. The proofs in Sec. 7.1 go over verbatim, by Prop. 5.1.1. In particular, with $\hat{R}_z$ (resp. $R^{\mathrm{univ}}_{V_z}$) replaced by $\hat{R}^{\square}_z$ (resp. $R^{\square}_{V_z}$), Prop. 7.1.3 and Cor. 7.1.5 hold for framed deformation rings. Thus we have (with the notation there) natural isomorphisms

$$R^{h_0,\varphi^f}_{V_z} \simeq \hat{\mathcal{O}}_{X_{fs},(z,\lambda_0)}, \quad \tilde{R}^{h_0,\varphi^f}_{V_z} \simeq \hat{\mathcal{O}}_{X_{nfs},(z,\lambda_0,\eta_0)}.$$

Hence the isomorphisms above hold for points $(z,\lambda_i,\eta_i) \in \tilde{F}^0$ (η_i is an integral power of χ_K). In particular,

$$\dim \hat{\mathcal{O}}_{X_{nfs},(z,\lambda_i,\eta_i)} = d - n, i = 0, 1.$$

Consider the projection

$$\pi_1 : X_{nfs} \to Z : (z,\lambda,\eta) \mapsto V_z \otimes \eta.$$

Let $\bar{F}$ be the Zariski closure in Z of F, the subset of crystalline points of Z coming from the elements in F^0 (via π_1) and their Tate twists. Note that the sets $F \subset Z$ and the set $\tilde{F}^0 \subset X_{nfs}$ are in one to one correspondence. By Lemma 7.1.7, $\bar{F}$ contains a crystalline point coming from F^0, hence is non-empty.

By Prop. 7.1.9, the preimage of $\bar{F}$ under the projection π_1 is X^o_{nfs}, the union of all the irreducible components of X_{nfs} passing through a point (z,λ,η) with $(z,\lambda) \in F^0$. Then, at a point $z \in \bar{F}$ coming from F^0, $\mathrm{Spf}\hat{\mathcal{O}}_{\bar{F},z}$ contains two formally smooth subspace $\mathrm{Spf}R_i \simeq \mathrm{Spf}\hat{\mathcal{O}}_{X_{nfs},(z,\lambda_i,\eta_i)}$, whose dimensions are equal to

$$d - n.$$

Moreover, we see easily $\cap_{i=0}^{1}\mathrm{Spf}R_i$ is the formal scheme associated to a twist of the crystalline deformation ring $R_{V_z}^{\mathrm{cr}}$ associated to V_z ([22]), noting that

$$(\otimes_{i=0}^{1}\tilde{R}_{V_z}^{h_i,\varphi^f}) \simeq (\otimes_{i=0}^{1}R_{V_z}^{h_i,\varphi^f}) \otimes_{W(\mathbb{F})[1/p]} \hat{\mathcal{O}}_{S_0,s_0}.$$

The crystalline deformation ring is formally smooth of dimension $d-3n$ by (3.3.8) *loc. cit*, as V_z has distinct Hodge-Tate weights. Thus $\cap_{i=0}^{1}\mathrm{Spf}R_i$ is formally smooth of dimension

$$d - 2n.$$

Consider the tangent space (at any z coming from F^0) $\mathfrak{m}_{\bar{F},z}/\mathfrak{m}_{\bar{F},z}^2$, where $\mathfrak{m}_{\bar{F},z}$ is the maximal ideal of $\hat{\mathcal{O}}_{\bar{F},z}$. One must have that

$$\sum_{i=0}^{1}(\dim \mathfrak{m}_{\bar{F},z}/\mathfrak{m}_{\bar{F},z}^2 - \dim \mathrm{Spf}R_i) \geq \dim \mathfrak{m}_{\bar{F},z}/\mathfrak{m}_{\bar{F},z}^2 - \dim \cap_{i=0}^{1}\mathrm{Spf}R_i,$$

which gives, by the quantities obtained before, that

$$\dim \mathfrak{m}_{\bar{F},z}/\mathfrak{m}_{\bar{F},z}^2 \geq d.$$

Note that twisting by χ_K^k ($k \in \mathbb{Z}$) is an automorphism of $\bar{F}$, provided $\bar{V}\otimes_{\mathbb{F}}\omega_K^k \sim \bar{V}$ where ω_K is the mod p cyclotomic character. We then have that $\dim \mathfrak{m}_{\bar{F},z}/\mathfrak{m}_{\bar{F},z}^2 \geq d$ for any $z \in \bar{F}$.

Pick a point $z \in F$. Let $\mathrm{Sp}\mathcal{R}$ be an affoid neighborhood of z in $\bar{F}$. Since $\mathcal{R}$ is excellent, the regular locus U in $\mathrm{Sp}\mathcal{R}$ is open. There must exist a point $z' \in F \cap U$, as F is Zariski dense in $\mathrm{Sp}\mathcal{R}$. Now $\bar{F}$ is *smooth* at z', as the local ring of $\bar{F}$ at z' is the local ring of $\mathrm{Sp}\mathcal{R}$ at the same point. Thus $\dim \hat{\mathcal{O}}_{\bar{F},z'} = \dim \mathfrak{m}_{\bar{F},z'}/\mathfrak{m}_{\bar{F},z'}^2 \geq d$. Then the irreducible component $\bar{F}_{z'}$ of $\bar{F}$ passing through z' is of dimension at least

d, hence is of dimension d. Therefore $\bar{F}_{z'} = \bar{F} = Z$ as Z is irreducible.

$\square$

Chapter 9

Trianguline representations: another point of view

Let $K/\mathbb{Q}_p$ be a finite extension. We define $H_K = \mathrm{Gal}(\bar{\mathbb{Q}}_p/K_\infty)$ where $K_\infty = \bigcup_{n\in\mathbb{N}} K(\varepsilon^{(n)})$. Write $\Gamma_K = G_K/H_K$. The cyclotomic character then gives a canonical map $\chi_K : \Gamma_K \hookrightarrow \mathbb{Z}_p^\times$.

9.1 (φ, Γ)-modules over Fontaine's rings and overconvergence

Write $\pi = [\varepsilon] - 1$. Let $A_{\mathbb{Q}_p} \in W(\mathrm{Fr}R)$ be the completion of $\mathbb{Z}_p[\![\pi]\!][1/\pi]$ under the p-adic topology on $W(\mathrm{Fr}R)$. Then $B_{\mathbb{Q}_p} = A_{\mathbb{Q}_p}[1/p]$ is a field complete for the valuation v_p with ring of integers $A_{\mathbb{Q}_p}$ and residue field $\mathbb{F}_p((\bar{\pi}))$.

Let E^s be the separable closure of $\mathbb{F}_p((\bar{\pi}))$ in $\mathrm{Fr}R$. The theorem of Fontaine and Wintenberger ([14], [35]) says that E^s is dense in $\mathrm{Fr}R$ and stable under $G_{\mathbb{Q}_p}$ and

$$\mathrm{Gal}(\bar{\mathbb{Q}}_p/K_{0,\infty}) \simeq \mathrm{Gal}(E^s/k((\bar{\pi})))$$

where $K_{0,\infty} = \bigcup_{n\in\mathbb{N}} K_0(\varepsilon^{(n)})$, and k is the residue field of K.

Then $W(\mathrm{Fr}R)[1/p]$ contains a unique extension B_K of $B_{\mathbb{Q}_p}$ whose residue field is $E_K = (E^s)^{H_K}$ and whose ring of integers is $A_K = B_K \cap W(\mathrm{Fr}R)$. By uniqueness, B_K

is stable under the commutative actions of φ and Γ_K. The field $B^{\mathrm{ur}} = \bigcup_{[K:\mathbb{Q}_p]<\infty} B_K$ is the maximal unramified extension of $B_{\mathbb{Q}_p}$. We denote by B the p-adic completion of B^{ur}. Then its ring of integers is $A = B \cap W(\mathrm{Fr}R)$ and its residue field is $A/p = E^s$.

Definition 9.1.1. A (φ, Γ_K)*-module over* A_K (resp. B_K) is a finitely generated A_K- (resp. B_K-) module D with semi-linear continuous (for the canonical topology) and commutative actions of φ and Γ_K. It is said to be *étale* if $\varphi(D)$ generates D as an A_K-module. A (φ, Γ_K)-module D over B_K is said to be étale if it has an A_K-lattice which is étale as a (φ, Γ_K)-module over A_K.

A fundamental theorem of Fontaine ([15]) says that the correspondence

$$V \mapsto D(V) = (A \otimes_{\mathbb{Z}_p} V)^{H_K}$$

is an equivalence of tensor categories from the category of $\mathbb{Z}_p$-representations of G_K to that of étale (φ, Γ_K)-modules over A_K, and the inverse functor is

$$D \mapsto (A \otimes_{A_K} D)^{\varphi=1}.$$

For a real number $r > 0$, we define a subring $\tilde{A}^{(0,r]} \subset W(\mathrm{Fr}R)$ as

$$\tilde{A}^{(0,r]} = \{x = \sum_{i \geq 0} p^i [x_i] \in W(\mathrm{Fr}R) |, \lim_{k \to +\infty} v_R(x_k) + k/r = +\infty\}.$$

Set $A^{(0,r]} = \tilde{A}^{(0,r]} \cap B$. For a $\mathbb{Z}_p$-representation V of G_K, define

$$D^{(0,r]}(V) = (A^{(0,r]} \otimes_{\mathbb{Z}_p} V)^{H_K} \subset D(V).$$

Theorem 9.1.2 (Chebonier-Colmez [7]). *For any $\mathbb{Z}_p$-representation V of G_K, there exists an $r_V > 0$ such that*

$$A_K \otimes_{A^{(0,r_V]}} D^{(0,r_V]}(V) \xrightarrow{\sim} D(V).$$

9.2 (φ, Γ)-modules over the Robba ring and slope filtration

In the following, let A denote a finite dimensional local $\mathbb{Q}_p$-algebra.

The *Robba ring* $\mathcal{R}_A$ with coefficients in A is the topological ring of power series

$$f(x) = \sum_{n \in \mathbb{Z}} a_n (x-1)^n, a_n \in A$$

which are convergent on some annulus $r(f) \leq |z-1| < 1$ of $\mathbb{C}_p$, with the natural A-algebra topology. A (φ, Γ_K)*-module over the Robba ring* $\mathcal{R}_A$ is a finite free $\mathcal{R}_A$-module D with $\mathcal{R}_A$-semi-linear continuous commutative actions of φ and Γ_K, and $\varphi(D)$ generates D as an $\mathcal{R}_A$-module. A (φ, Γ_K)-module over $\mathcal{R}_A$ is called *étale* if its underlying φ-module is purely of slope 0 in the sense of slope filtration theory of Kedlaya [20] [19].

Write $\tilde{A}^\dagger = \bigcup_{r>0} \tilde{A}^{(0,r]}, \tilde{B}^\dagger = \bigcup_{r>0} \tilde{B}^{(0,r]}$ with $\tilde{B}^{(0,r]} = \tilde{A}^{(0,r]}[1/p]$, and $A^\dagger = \tilde{A}^\dagger \cap B, B^\dagger = \tilde{B}^\dagger \cap B$. The ring $B_K^\dagger = (B^\dagger)^{G_K}$ is isomorphic to the ring of bounded analytic functions $f(\pi_K)$ in the variable π_K on the annulus $0 < v_p(T) \leq r(f)$, which is in fact a field. The (φ, Γ_K)-action on $B_K^\dagger$ extends by continuity to $\mathcal{R}_K$. For V a $\mathbb{Q}_p$-representation of G_K, define

$$D^\dagger(V) = (B^\dagger \otimes_{\mathbb{Q}_p} V)^{H_K}, \quad D_{\text{rig}}(V) = \mathcal{R}_K \otimes_{B_K^\dagger} D^\dagger(V).$$

Theorem 9.2.1 (L. Berger, [2]). *Let V be a $\mathbb{Q}_p$-representation of G_K. We have*

$$D_{\text{cris}}(V) = (D_{\text{rig}}(V)[1/t])^{\Gamma_K}.$$

If V is crystalline, then we have isomorphism of (φ, Γ_K)-modules

$$\mathcal{R}_K[1/t] \otimes_{K_0} D_{\text{cris}}(V) \simeq \mathcal{R}_K[1/t] \otimes_{B_K^\dagger} D^\dagger(V).$$

Theorem 9.2.2 ([8]). *The functor*

$$D_{\mathrm{rig}} : V \mapsto (B \otimes_{\mathbb{Q}_p} V)^{H_K}$$

induces a tensor equivalence of categories from the category of A-representations of G_K to that of étale (φ, Γ_K)-modules over $\mathcal{R}_A$.

9.2.1 Refinements and triangulations of $G_{\mathbb{Q}_p}$-representations

Definition 9.2.3. A rank N (φ, Γ_K)-module D over $\mathcal{R}_A$ is *triangular* if it is equipped with a strictly increasing filtration

$$0 \subsetneq \mathrm{Fil}^0 D \subsetneq \mathrm{Fil}^1 D \subsetneq \cdots \subsetneq \mathrm{Fil}^{N-1} D = D$$

by (φ, Γ_K)-submodules $\mathrm{Fil}^i D$ which are free and direct summands as $\mathcal{R}_A$-modules. The filtration is called the *triangulation* of D. We say an A-representation V of G_K is *trianguline* if $D_{\mathrm{rig}}(V)$ can be equipped with a triangulation.

In the rest we only consider the case $K = \mathbb{Q}_p$.

For a continuous character $\delta : \mathbb{Q}_p^\times \to A^\times$, we define, following [8], the $(\varphi, \Gamma_{\mathbb{Q}_p})$-module $\mathcal{R}_A(\delta)$ over $\mathcal{R}_A$ whose underlying $\mathcal{R}_A$-module is $\mathcal{R}_A$ and the $(\varphi, \Gamma_{\mathbb{Q}_p})$-action on which is defined as

$$\varphi(1) = \delta(p), \quad \gamma(1) = \delta(\gamma), \quad \forall \gamma \in \Gamma_{\mathbb{Q}_p} = \mathbb{Z}_p^\times.$$

We then regard δ as a continuous homomorphism $W_{\mathbb{Q}_p} \to E^\times$ via the Artin map, by setting

$$\delta(g) = \delta(p)^{-\deg(g)} \delta(\chi_{\mathbb{Q}_p}(g)).$$

The character δ extends continuously to $G_{\mathbb{Q}_p}$ if and only if $v_p(\delta(p)) = 0$, in which case

$$\mathcal{R}_A(\delta) = D_{\mathrm{rig}}(\delta).$$

Theorem 9.2.4 (Thm.0.2, [8]). *Any rank* 1 $(\varphi, \Gamma_{\mathbb{Q}_p})$*-module over* $\mathcal{R}_A$ *is isomorphic to* $\mathcal{R}_A(\delta)$ *for a unique character* $\delta : \mathbb{Q}_p^\times \to A^\times$. *Such a module is isocline of slope* $v_p(\delta(p))$.

We note that an extension of two rank 1 $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules can be étale even if they are not étale.

Suppose D is a triangular $(\varphi, \Gamma_{\mathbb{Q}_p})$-module of rank N. Then each

$$\mathrm{gr}^i D = \mathrm{Fil}^i D / \mathrm{Fil}^{i-1} D, 0 \le i \le N-1$$

is isomorphic to $\mathcal{R}_A(\delta_i)$ for some character $\delta_i : \mathbb{Q}_p^\times \to A^\times$. We define the *parameter* of the triangulation $\mathcal{T} = (\mathrm{Fil}^i D)_{0 \le i \le N-1}$ to be the continuous homomorphism

$$\delta = (\delta_i)_{0 \le i \le N-1} : \mathbb{Q}_p^\times \to (A^\times)^N.$$

The following result is Prop.5.10 of [8].

Proposition 9.2.5. *Let* V *be a* 2*-dimensional continuous* E*-representation of* $G_{\mathbb{Q}_p}$. *Suppose there is a* 1*-refinement*

$$h : V \otimes \eta \to (B_{\mathrm{cris}}^+ \otimes_{\mathbb{Q}_p} E)^{\varphi = \lambda}$$

with $\lambda \in E^\times$.

Then there is a sequence $0 \subsetneq D_0 \subsetneq D = D_{\mathrm{rig}}(V^*)$ *of* $(\varphi, \Gamma_{\mathbb{Q}_p})$*-submodules of* $D_{\mathrm{rig}}(V^*)$*, where* D_0 *is a free and direct summand of* $D_{\mathrm{rig}}(V^*)$ *as* $\mathcal{R}_E$*-modules.*

Proof. Choose sufficiently large integers c, and replacing η (resp. λ) by $\eta\chi_p^{-c}$ (resp. λp^c), so that, keeping in mind Thm. 9.2.1, we may assume

$$D_{\mathrm{cris}}(V(\eta)^*)^{\varphi=\lambda} = D_{\mathrm{rig}}(V(\eta)^*)^{\Gamma=1, \varphi=\lambda}.$$

The E-vector space $D_{\mathrm{rig}}(V(\eta)^*)^{\Gamma=1,\varphi=\lambda}$ is generated by a non-zero element x, which is stable by φ and fixed by Γ. The $\mathcal{R}_E$-submodule of $D_{\mathrm{rig}}(V(\eta)^*)$ generated by x is stable by φ and Γ. We then have (φ, Γ)-submodules D_0 of $D_{\mathrm{rig}}(V^*)$. The $(\varphi, \Gamma_{\mathbb{Q}_p})$-submodule D_0 is *saturated* by Lemma 4.4 of [8], which is equivalent to saying that D/D_0 is free and direct summands of D as $\mathcal{R}_E$-modules. We are done.

$\square$

We recall the following result from [1].

Remark 9.2.6. *There is a one to one correspondence between refinements of an N-dimensional $G_{\mathbb{Q}_p}$-representation V and triangulations on $D_{\mathrm{rig}}(V)$ when V is crystalline, essentially by Thm. 9.2.1.*

First $D_{\mathrm{cris}}(V)$ admits a φ-stable filtration $\mathcal{F} = \{D_i(V)\}_{0 \leq i \leq N-1}$:

$$0 = D_0(V) \subsetneq D_1(V) \subsetneq \cdots \subsetneq D_{N-1}(V) = D_{\mathrm{cris}}(V).$$

Define a triangulation $\mathcal{T} = \{\mathrm{Fil}^i D\}$ on the $(\varphi, \Gamma_{\mathbb{Q}_p})$-module $D = D_{\mathrm{rig}}(V)$ over the Robba ring $\mathcal{R}_L$ by setting

$$\mathrm{Fil}^i D_{\mathrm{rig}}(V) := (D_i(V) \otimes \mathcal{R}_L[1/t]).$$

On the other hand, given a $(\varphi, \Gamma_{\mathbb{Q}_p})$-module $D = D_{\mathrm{rig}}(V)$ over the Robba ring $\mathcal{R}_E$ with triangulation $\mathcal{T} = \{\mathrm{Fil}^i D\}$, one defines a φ-stable filtration on $D_{\mathrm{cris}}(V)$ by setting

$$D_i(V) := (\mathrm{Fil}^i(D_{\mathrm{rig}}(V)) \otimes \mathcal{R}_E[1/t])^{\Gamma_{\mathbb{Q}_p}}.$$

Noting $D_{\mathrm{rig}}(V)[1/t]^{\Gamma_{\mathbb{Q}_p}} = D_{\mathrm{cris}}(V)$, and the isomorphism in Thm. 9.2.1

$$D_{\mathrm{cris}}(V) \otimes_E \mathcal{R}_E[1/t] \simeq D_{\mathrm{rig}}(V)[1/t]$$

one sees the above two maps define a bijection between the set of refinements of V and that of triangulations of $D_{\mathrm{rig}}(V)$.

9.2.2 Trianguline deformations of $G_{\mathbb{Q}_p}$-representations of Bellaïche-Chenevier

Denote by $\mathcal{AR}_E$ the category of local Artinian $\mathbb{Q}_p$-algebras A with an isomorphism $A/\mathfrak{m}_A \xrightarrow{\sim} E$ whose restriction to E is the identity.

Let D be a $(\varphi, \Gamma_{\mathbb{Q}_p})$-module over $\mathcal{R}_E$ of rank N which is equipped with a triangulation $\mathcal{T} = \{\mathrm{Fil}^i D_A)\}_{0\leq i\leq N-1}$ with parameter $\{\delta_i\}_{0\leq i\leq N-1}$.

Consider the following functors:

- $$\mathcal{M}_D : \mathcal{AR}_E \to \mathrm{Sets} : A \mapsto \text{isomorphism classes of pairs } (D_A, \pi_A)$$

  where D_A is a $(\varphi, \Gamma_{\mathbb{Q}_p})$-module over $\mathcal{R}_A$ and $\pi_A : D_A \otimes_A E \xrightarrow{\sim} D$ as $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules.

- $$\mathcal{M}_{D,\mathcal{T}} : \mathcal{AR}_E \to \mathrm{Sets} : A \mapsto \text{isomorphism classes of triples } (D_A, \pi_A, \{\mathrm{Fil}^i D_A\})$$

  where $(D_A, \{\mathrm{Fil}^i D_A)\})$ is a $(\varphi, \Gamma_{\mathbb{Q}_p})$-module over $\mathcal{R}_A$ and $\pi_A : D_A \otimes_A E \xrightarrow{\sim} D$ is a $\mathcal{R}_A$-linear map of $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules such that $\pi_A(\mathrm{Fil}^i D_A) = \mathrm{Fil}^i D, 0 \leq i \leq N-1$.

- $$\mathcal{F}_V : \mathcal{AR}_E \to \mathrm{Sets} : A \mapsto \text{isomorphism classes of pairs } (V_A, \pi_A)$$

  where V_A is an A-representation of $G_{\mathbb{Q}_p}$ and $\pi_A : V_A \otimes_A E \xrightarrow{\sim} V$ as $A[G_{\mathbb{Q}_p}]$-modules.

- $$\mathcal{F}_{V,\mathcal{T}} : \mathcal{AR}_E \to \mathrm{Sets} : A \mapsto \text{isomorphism classes of triples } (V_A, \pi_A, \{\mathrm{Fil}^i D_{\mathrm{rig}}(V_A)\})$$

  where V_A is an A-representation of $G_{\mathbb{Q}_p}$ with $\pi_A : V_A \otimes_A E \xrightarrow{\sim} V$ as $A[G_{\mathbb{Q}_p}]$-modules, and $(D_{\mathrm{rig}}(V_A), D_{\mathrm{rig}}(\pi_A), \{\mathrm{Fil}^i D_{\mathrm{rig}}(V_A)\})$ is an element of $\mathcal{M}_{D,\mathcal{T}}(A)$.

We have the following results from Sec.2.3 and Sec.2.5 of [1].

Proposition 9.2.7. *(1) If $\delta_i\delta_j \notin x^{\mathbb{N}}, \forall i < j$, then $\mathcal{M}_{D,\mathcal{T}}$ is a subfunctor of $\mathcal{M}_D$ and $\mathcal{M}_{D,\mathcal{T}} \to \mathcal{M}_D$ is relatively representable.*

(2) The functor D_{rig} induces natural isomorphisms of functors

$$\mathcal{F}_V \simeq \mathcal{M}_D, \quad \mathcal{F}_{V,\mathcal{T}} \simeq \mathcal{M}_{D,\mathcal{T}}.$$

Let V be an N-dimensional crystalline E-representation of $G_{\mathbb{Q}_p}$ with a refinement $(\underline{1}_E, \mathfrak{h}, \{\lambda_i\})$ and denote the corresponding triangulation on $D_{\mathrm{rig}}(V)$ by $\mathcal{T}$.

Proposition 9.2.8 (Thm.2.5.9 [1]). *Suppose the refinement $(\underline{1}_E, \mathfrak{h}, \{\lambda_i\})$ is non-critical such that $\lambda_i\lambda_j \notin \{1, p^{-1}\}$ for $i < j$, and suppose $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(V, V(-1)) = 0$. Then the functor $\mathcal{F}_{V,\mathcal{T}}$ is formally smooth of dimension*

$$\dim_E H^0(G_{\mathbb{Q}_p}, V \otimes_E V^*) + N(N+1)/2.$$

Note that, when $N = 2$, the formally smoothness and the dimension above coincide with those of the complete local ring of the nearly finite slope subspace $X_{nfs}(\mathrm{GL}_{2/\mathbb{Q}_p})$ at this crystalline point.

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