Locating point diffractors in layered media by spatial dynamics

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Abstract

We present a new approach to the problem of detecting point diffractors from active source surface seismic data. We formulate an optimization problem in the configuration space of possible collections of scatterers and construct a birth-and-death spatial dynamic, which converges to the optimal solution. By design, this dynamic does not have resolution limits typical of migration based techniques, which allows for subwavelength sensing.

1 Introduction

In this paper, we consider a problem of detecting point diffractor in a layered elastic medium. Detecting small artifacts buried inside a known background has been and remains an important class of problem particularly in seismic imaging but also in other areas, such as in medical imaging.

Many imaging techniques proposed to date, such as reverse time migration and interferometry, rely on the principle of time reversal. In those methods, a recorded wavefield or functionals thereof are propagated in the reverse direction of time yielding an image. The image so produced often suffers from artifacts whose nature is manifold and due to several different reasons. Among the problems that lead to a poor quality image are incomplete source coverage, spatially aliased acquisition, instrumental noise, lack of precise knowledge of the background medium, and the non-zero wavelength of the probing pulse.

Resolution of the image is the smallest scale at which coherent and meaningful information can be extracted, and it is directly related to the source wavelength in virtually all conventional setups. As a result imaging techniques mentioned above have natural resolution limits, which do not allow to resolve details at a subwavelength scale (*Schuster*, 1996).

If subwavelength resolution is of the essence, one has little choice but to resort to modeling. By modeling, we here understand a process of selecting a configuration and simulating a relevant physical process with the explicit purpose of comparing the result of the simulations with observed data. A precise description of a measure of fitness enables one in principle to construct an iterative scheme by which the best possible configuration out of available alternatives is found. The end result is an optimization problem, which is to be solved in order to find the optimal configuration.

In this paper, we assume that we have a source and receivers located at the surface. The background elastic medium extends down, and it is assumed to be layered and known. The object of interest is a finite collection of point scatterers (or diffractors) located inside the layered medium. The total number of scatterers as well as their coordinates are unknown. Of particular interest is a situation where diffractors are positioned sufficiently close to each other, namely closer than the spatial wavelength of the probing pulse. This is a situation where conventional imaging methods fail, and the proposed technique will prove to be a possible help. We, however, stress that from a conceptual viewpoint, a tight arrangement of scatterers is an interesting case but not a necessary assumption.

The goal is to find the total number of all diffractors as well as their individual locations as closely as possible, and within a reasonable computational time. At the heart of the method that we propose in this paper lie the notions of a spatial dynamic and simulated annealing. A spatial dynamic is a birth-and-death stochastic process, which randomly generates new diffractors and removes old ones. The rates of the birth and death processes are carefully tailored to the measure of goodness of fit. Configurations of scatterers that are produced as a result tend to generate data resembling the recorded observations. Simulated annealing (*Geman and Geman*, 1984) is a generic process of controlled "cooling" of a probabilistic system under investigation so as to allow it to settle in an optimal state. As elaborated below, this process will manifest itself in a parameter inside the algorithm, which goes to zero to ensure the eventual convergence of a dynamic to its globally optimal state.

The rate of cooling or the "cooling schedule" is a subject of a certain controversy. Many simulated annealing-based optimization techniques (e.g. diffusions) achieve in theory a global optimum. Yet the cooling schedule required for that is so slow that it makes the convergence time be extremely large and thus renders the algorithm practically useless (*Poliannikov et al.*, 2007). Spatial dynamics are manifestly different. They have been tried as a tool to tackle problems in image processing (*Descombes et al.*, 2007; *Descamps et al.*, 2008; *Descombes and Zhizhina*, 2008) where it has been shown that an exponentially decaying "temperature" may guarantee the convergence to the global minimum. While a complete theoretical development of the mathematical theory of the proposed dynamic remains to be completed, numerical simulations presented in this report provide sufficient ground for optimism that a similar situation takes place here.

Our paper is organized as follows. In Section 2, we introduce relevant notation and give a precise formulation of the problem. We also outline problems with conceivable alternative approaches. In Section 3, we introduce a discrete spatial dynamic along with the birth and death maps, as well as the cooling schedule. In Section 4, we show the viability of the proposed ideas with numerical simulations. Section 5 concludes the paper and offers a discussion of the results and possible future directions for further research.

2 Problem setup

For simplicity of notation, we will consider a two-dimensional version of the problem (see Fig. 1). Consider a medium

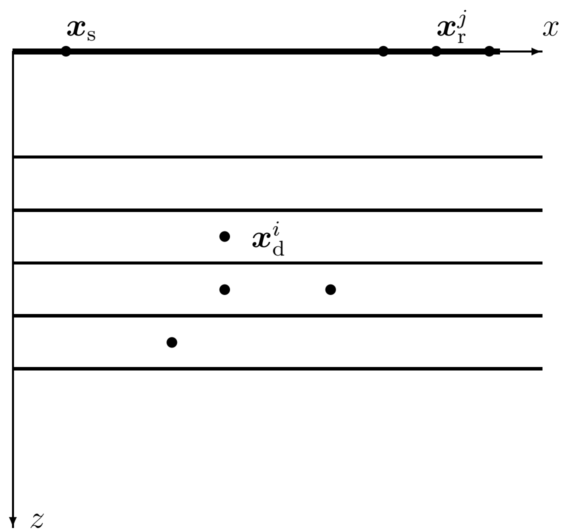

Figure 1: Problem setup

$$D \equiv \{\boldsymbol{x} = (x, z) \mid -X < x < X,\ 0 < z < Z\} \subset \mathbb{R}^2. \tag{1}$$

We will assume that it is layered, i.e. p,s-velocities as well as other physical characteristics of an elastic medium vary only in the z-direction:

$$\alpha(x, z) = \alpha(z),\ \beta(x, z) = \beta(z), \ldots. \tag{2}$$

The source $\boldsymbol{x}_\mathrm{s} = (x_\mathrm{s}, 0)$ and receivers $\{\boldsymbol{x}_\mathrm{r}^j\}_{j=1}^{N_\mathrm{r}}$, where

$$\boldsymbol{x}_\mathrm{r}^j = (x_\mathrm{r}^j, 0),\ j = 1, \ldots, N_\mathrm{r}, \tag{3}$$

are located at the surface. A finite collection of point diffractors inside the medium

$$\boldsymbol{x}_\mathrm{d}^i = (x_\mathrm{d}^i, z_\mathrm{d}^i) \in D,\ i = 1, \ldots, N_\mathrm{d} \tag{4}$$

will be called *a configuration* $\mathscr{C}_\mathrm{d}$. The size of a configuration is not prescribed *a priori* and may vary from 0 to infinity.

A source at time $t = 0$ emits a wave, which propagates into the medium and reflects off diffractors as well as layer boundaries and is recorded at the receiver location to form a shot-gather

$$W(\mathscr{C}_\mathrm{d}; \boldsymbol{x}_\mathrm{r}^j, t),\ j = 1, \ldots, N_\mathrm{r},\ t \in [0, T]. \tag{5}$$

Here we take W to be the vertical component of the particle velocity. The shot-gather is a function of a receiver and a time, as well as implicitly of a configuration of scatterers in the medium. We assume that a true and unknown configuration $\mathscr{C}_\mathrm{d}^*$ is physically present in the medium and it generates the observed shot-gather

$$d(\boldsymbol{x}_\mathrm{r}^j, t) = W(\mathscr{C}_\mathrm{d}^*; \boldsymbol{x}_\mathrm{r}^j, t). \tag{6}$$

The problem is to find $\mathscr{C}_\mathrm{d}^*$. Toward that end, we define a measure of goodness of fit of an arbitrary configuration of scatterers (4) to the observed data (6). In reality, several factors including the data quality may play a role in selecting the goodness of fit measure. In the interest of staying focused, we will forego any discussion of this important issue and simply assume that the measure is given by the Euclidean norm

$$\varrho(\mathscr{C}_\mathrm{d}, \mathscr{C}_\mathrm{d}^*) = \sum_{j=1}^{N_\mathrm{r}} \int_0^T \left| W(\mathscr{C}_\mathrm{d}; x_\mathrm{r}^j, t) - d(x_\mathrm{r}^j, t) \right|^2 dt. \tag{7}$$

The distance ϱ so defined measures how close the wavefield W generated by a test configuration $\mathscr{C}_\mathrm{d}$ is to the observed data d. A perfect find $\mathscr{C}_\mathrm{d} = \mathscr{C}_\mathrm{d}^*$ will clearly yield $\varrho = 0$, and any other choice of $\mathscr{C}_\mathrm{d}$ will give a positive value for the distance. The problem of detecting point scatters inside the medium based on the observed shot-gather can therefore be written as an optimization problem

$$\begin{cases} \min \varrho(\mathscr{C}_{\mathrm{d}}, \mathscr{C}_{\mathrm{d}}^{*}) \\ \text{s.t. } \mathscr{C}_{\mathrm{d}} = \left\{ \boldsymbol{x}_{\mathrm{d}}^{1}, \ldots, \boldsymbol{x}_{\mathrm{d}}^{N_{\mathrm{d}}} \right\} \subset D. \end{cases} \tag{8}$$

Now that the problem is well-defined, we discuss possible approaches to solving it. A standard first approach to an optimization problem is to try a local optimizer, such as the Steepest Descent method or Newton's method. While these methods are powerful and relatively fast, they certainly are capable of producing of only a local minimizer, and thus require a quality initial guess. In the context the problem at hand, for a local method to have a chance of finding the global optimum, this initial guess would have to include a correct number of point diffractors, which is very unrealistic.

Diffusion type stochastic optimization methods address the problem of getting stuck in local minima by adding to a local minimizer a stochastic term. This term introduces random fluctuations of a test configuration and thus avoid the problem of local minima. Unfortunately the problem of having to guess correctly the total number of scatterers is not resolved with these methods. Additionally, if theoretically assured convergence to the global minimum is required, the speed of convergence of such methods is very slow, which makes them impractical for applications.

3 Spatial dynamics

3.1 Basic definitions

Consider a space of all possible finite configurations $\mathscr{C}_{\mathrm{d}}$ of point scatterers inside the domain D. A birth is a transition from one configuration to another of the form

$$\mathscr{C}_{\mathrm{d}} = \left\{\boldsymbol{x}_{\mathrm{d}}^{1}, \ldots, \boldsymbol{x}_{\mathrm{d}}^{N_{\mathrm{d}}}\right\} \to \mathscr{C}_{\mathrm{d}}' = \left\{\boldsymbol{x}_{\mathrm{d}}^{1}, \ldots, \boldsymbol{x}_{\mathrm{d}}^{N_{\mathrm{d}}}, \underbrace{\boldsymbol{x}_{\mathrm{d}}^{N_{\mathrm{d}}+1}}_{\text{new diffractor}}\right\}. \tag{9}$$

A death is a transition from one configuration to another of the form

$$\mathscr{C}_{\mathrm{d}} = \left\{\boldsymbol{x}_{\mathrm{d}}^{1}, \ldots, \boldsymbol{x}_{\mathrm{d}}^{N_{\mathrm{d}}}\right\} \to \mathscr{C}_{\mathrm{d}}' = \left\{\boldsymbol{x}_{\mathrm{d}}^{1}, \ldots, \underbrace{\widehat{\boldsymbol{x}_{\mathrm{d}}^{j},}}_{\text{removed diffractor}} \ldots, \boldsymbol{x}_{\mathrm{d}}^{N_{\mathrm{d}}}\right\}, \tag{10}$$

where the hat denotes that the j-th diffractor has been removed from the configuration.

Let $\mathscr{C}_{\mathrm{d}}^{0} = \{\}$ be an initial empty configuration containing no diffractors. A birth-and-death spatial dynamic is an iterative process by which at each iteration new diffractors are added and/or existing ones are removed with some probabilities that may depend on an iteration number as well as the location of a diffractor. For this process to converge to the optimal configuration, the birth and death probabilities have to be chosen extremely carefully. We now begin to describe how it is done.

3.2 Main algorithm

To once again simplify the presentation we will consider a discrete version of the dynamic. It is also directly suitable for implementation, although perhaps not so convenient for theoretical analysis. We will define a mesh on the domain D of an arbitrary grid size $(\Delta x, \Delta z)$. Although the total number of grid points $N_{\mathrm{g}} = \frac{2XZ}{\Delta x \Delta z}$ is finite, the number of possible configurations $2^{N_{\mathrm{g}}}$ grows exponentially, so checking all of them individually is still impossible.

Our algorithm proceed as follows:

- Pick an initial value δ_1. Let the inverse temperature $\beta_1 = \frac{C_\beta}{\delta_i}$, where C_β is another fixed constant.
- For each iteration number i we define a random sequence of visiting each node.
- While at each node, if there is no diffractor there, one is born with a *constant* probability p_{b}.
- If there is a diffractor, one is removed with a probability p_{d} computed below:
  - Let $\mathscr{C}_{\mathrm{d}}$ be a configuration with this diffractor left intact, and $\mathscr{C}'_{\mathrm{d}}$ be a configuration with it removed.
  - The difference $\varrho(\mathscr{C}_{\mathrm{d}}, \mathscr{C}^*_{\mathrm{d}}) - \varrho(\mathscr{C}'_{\mathrm{d}}, \mathscr{C}^*_{\mathrm{d}})$ is the relative cost or gain that we pay or obtain by keeping this diffractor.
  - The probability of death for this diffractor is computed using this relative cost as follows:

$$p_{\mathrm{d}} = \frac{\delta_i e^{\beta_i \left(\varrho(\mathscr{C}_{\mathrm{d}}, \mathscr{C}^*_{\mathrm{d}}) - \varrho(\mathscr{C}'_{\mathrm{d}}, \mathscr{C}^*_{\mathrm{d}})\right)}}{1 + \delta_i e^{\beta_i \left(\varrho(\mathscr{C}_{\mathrm{d}}, \mathscr{C}^*_{\mathrm{d}}) - \varrho(\mathscr{C}'_{\mathrm{d}}, \mathscr{C}^*_{\mathrm{d}})\right)}}. \tag{11}$$

- The temperature is lowered

$$\delta_{i+1} = K_\delta \cdot \delta_i,\ \beta_{i+1} = \frac{\beta_i}{K_\delta}. \tag{12}$$

  where $0 < K_\delta < 1$.

- We return to the next step of the iteration.

- The dynamic has converged if all newly born diffractors die out immediately.

A formal and careful justification of the above algorithm lies outside of the scope of this paper. Here we limit ourselves to a few comments intended to shed some light at how the dynamic defined above operates.

1. Random visitation of nodes at each iteration prevents the dynamic from introducing a bias towards one direction at the expense of all others.

2. It is easy to see that the space available for births is much larger than the number of diffractors existing at any step of the iterative process. It is therefore much more efficient to keep the birth map defined by the probability p_{b} as simple as possible and only update the death map given by probabilities p_{d}.

3. If the potential gain from removing a diffractor is large, i.e.

$$\varrho(\mathscr{C}_{\mathrm{d}}, \mathscr{C}_{\mathrm{d}}^*) - \varrho(\mathscr{C}'_{\mathrm{d}}, \mathscr{C}_{\mathrm{d}}^*) \gg 0$$

   then $p_{\mathrm{d}} \approx 1$. Altenatively, if the potential cost from removing the diffractor is large, i.e.

$$\varrho(\mathscr{C}_{\mathrm{d}}, \mathscr{C}_{\mathrm{d}}^*) - \varrho(\mathscr{C}'_{\mathrm{d}}, \mathscr{C}_{\mathrm{d}}^*) \ll 0$$

then $p_\mathrm{d} \approx 0$. In other cases, the dynamic behaves randomly removing the diffractor only with some non-trivial probability.

4. By introducing a decreasing "temperature" δ_i, we embed our dynamic into a simulated annealing process. For a fixed temperature, the dynamic would adopt a behavior governed by so-called Gibbs measures. The latter may be thought of here as being equivalent to neighborhoods of the optimal configuration. The decreasing temperature ensures that the Gibbs measures converge to a limiting measure whose support rests solely on the set of global minimizers of ϱ. This means the convergence of an evolving configuration to the true solution.

4 Implementation and results

4.1 Computing shot-gathers in a layered medium

As is clearly seen from the definition of the death map (11), success or failure of the proposed dynamic or virtually any other method of stochastic optimization largely depends on one's ability to compute the distance function ϱ quickly and accurately. That means that an efficient numerical scheme is required for evaluation of shot-gathers (5) corresponding to test configurations $\mathscr{C}_\mathrm{d}$. Producing such a scheme for an arbitrary elastic medium remains a challenge. In the case of a layered medium, however, a single propagation can be performed and stored. All shot gathers then can be calculated easily using this information.

Indeed, suppose the Green's function of the elastic layered medium

$$G_0(\boldsymbol{x}_\mathrm{s}, \boldsymbol{x}, t),\ x \in D \tag{13}$$

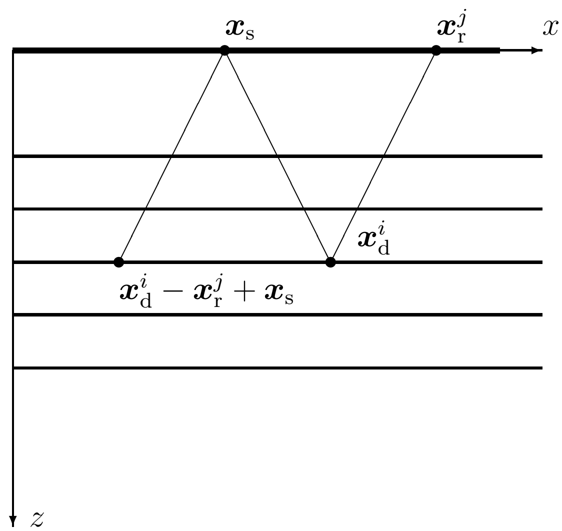

Figure 2: Computing shot-gathers from a single propagation

from the source at the surface to any point within the medium has been computed numerically or is otherwise given. Under a single scattering assumption, using geometrical consideration (see Fig. 2) and the reciprocity of the Green's function, a shot-gather that corresponds to a test configuration $\mathscr{C}_\mathrm{d}$ is written as a convolution of the source $f(t)$ with a Green's function given by

$$G(\boldsymbol{x}_\mathrm{s}, \boldsymbol{x}_\mathrm{r}^j, t) = \sum_{i=1}^{N_\mathrm{d}} \Big(G_0(\boldsymbol{x}_\mathrm{s}, \boldsymbol{x}_\mathrm{d}^i, \cdot) \star G_0(\boldsymbol{x}_\mathrm{s}, \boldsymbol{x}_\mathrm{d}^i - \boldsymbol{x}_\mathrm{r}^j + \boldsymbol{x}_\mathrm{s}, \cdot) \Big)(t). \tag{14}$$

4.2 Numerical simulations

The following numerical examples (see Fig. 4, 5, 6) illustrate the performance of the proposed algorithm for a range of true configurations.

The exact setup is as follows. The domain is a rectangle $[0, 11] \times [0, 5] km$. The source is located at $(0, 0)$ and its wavelength is $0.4s$. We have 10 equidistant receivers from $(1, 0)km$ to $(10, 0)km$. The medium contains three layers of thickness: $1, 2, 2km$, the p-velocities are: $3.5, 4.3, 5.5km/s$, the s-velocities are: $2.0, 2.5, 3.2km/s$, and the densities are $2.7, 2.9, 3.1g/cc$ (see Fig 3).

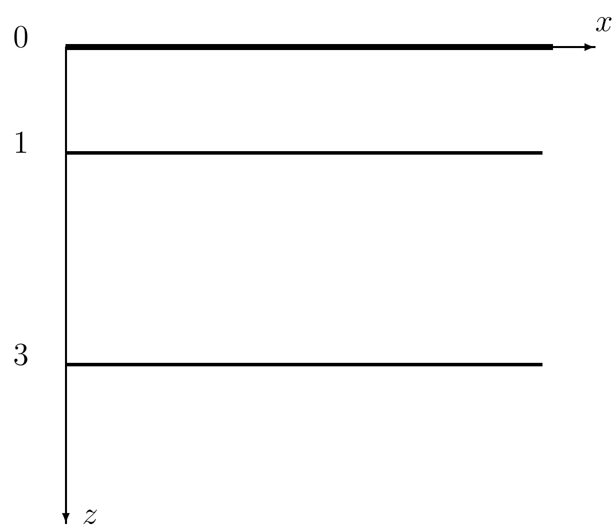

Figure 3: Numerical setup

The dynamic is run on the grid with the cell size $0.25km$, which is much smaller than the spatial wavelength of the pulse. We use Michel Bouchon's elastic wave propagation code based on (*Bouchon and Aki*, 1977; *Müller*, 1985) and record the vertical component of the velocity. The initial temperature is taken to be 0.01 for all examples below and it is decreased by a factor 1.001 at each iteration. The initial configuration is empty.

Cyan squares denote true location of diffractors and blue circles are the result provided by the algorithm. We see the the dynamic automatically finds all point scatterers for various cases of their number and inter-arrangement. The total number of configurations tested by the dynamic before it converged to the solution was of the order of $300-500$. To put this number in the proper context, we note that the brute force approach to our problem on a 3×3 grid would require testing of 512 different configurations. This number grows exponentially as the grid becomes finer.

5 Conclusions

In this report we have proposed a novel approach to identifying point scatterers inside a layered medium based on a single observed shot-gather. This is done by formulating and solving an optimization problem on the space of all possible configurations of diffractors within a domain under study. The birth-and-death dynamic presented in this report converges to the true configuration by generating diffractors that produce a response similar to the observed data and by removing ones that do not.

A dynamic so defined inherently lacks any limitation on the distances between scatterers and is thus capable of resolving information at subwavelength scales. This differentiates it from more conventional counterparts prone to resolution limits determined largely by the source wavelength.

A vast amount of literature (*Fouque et al.*, 2007, and references therein) is devoted to imaging in random media where the background velocity is ran-

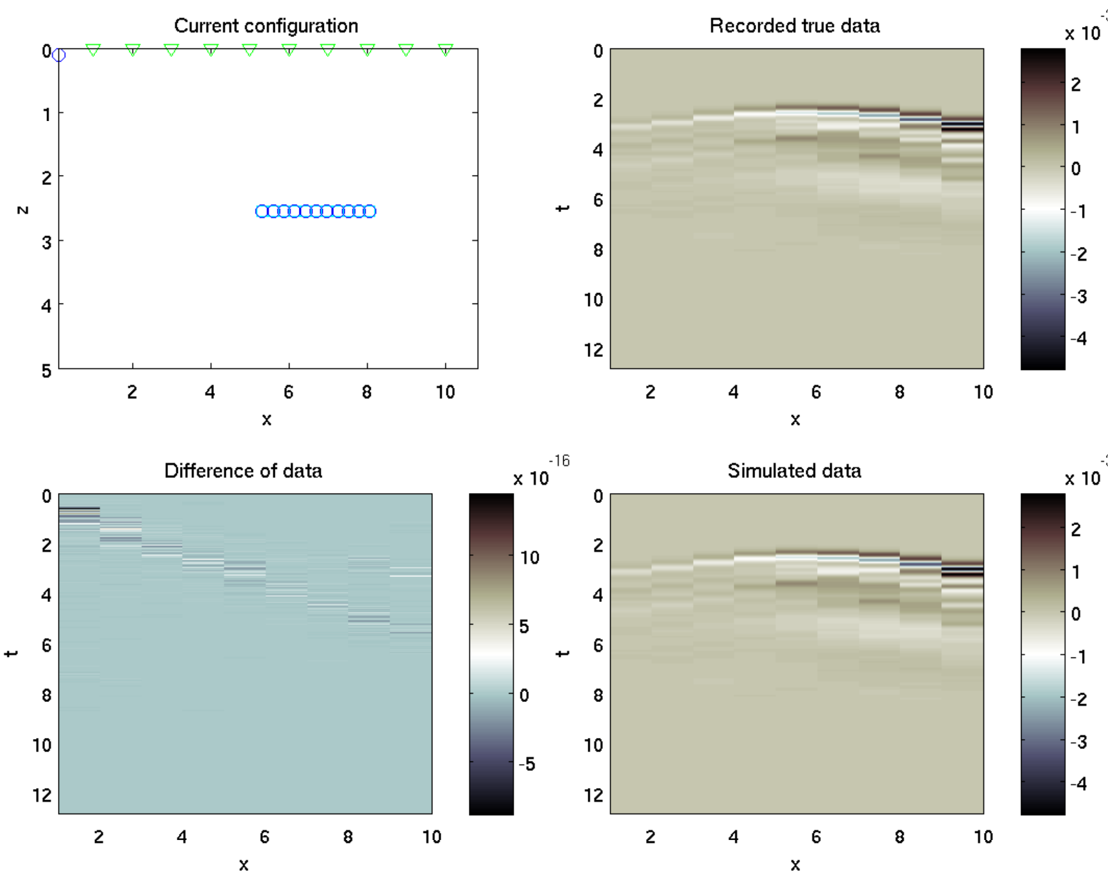

Figure 4: Small reflector

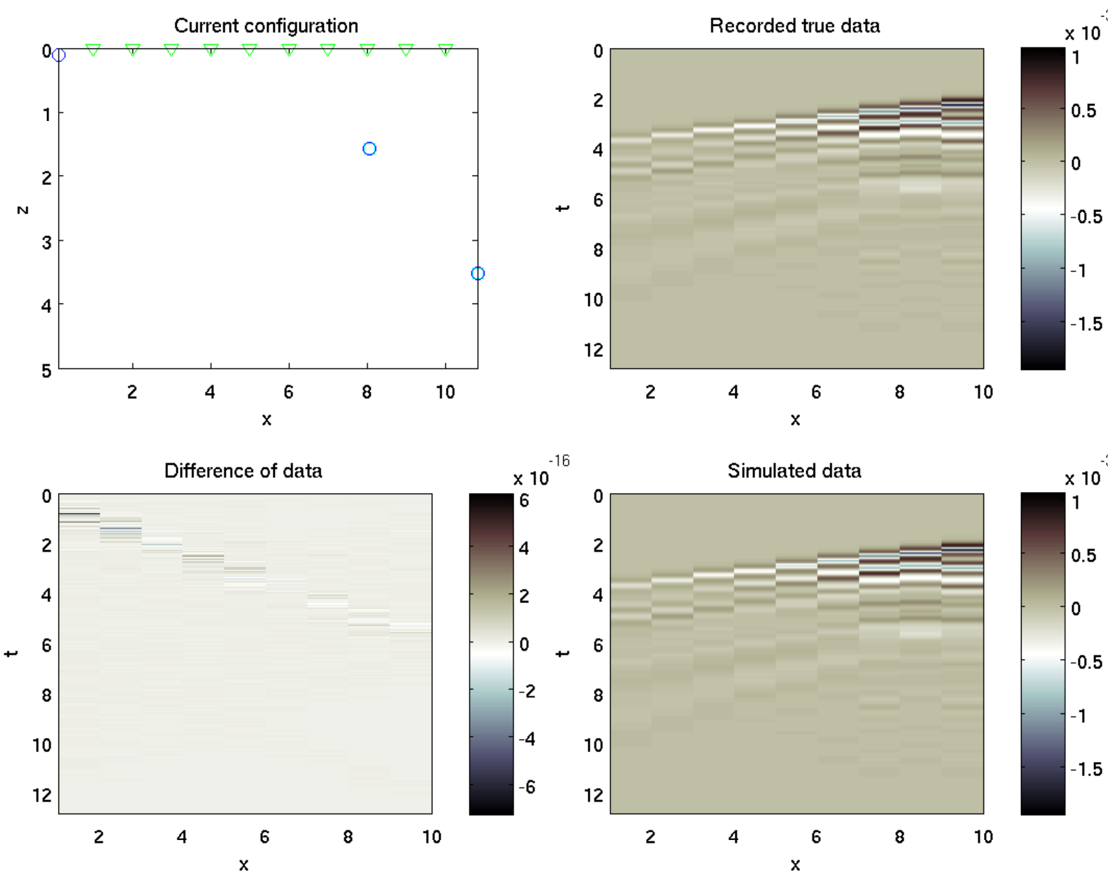

Figure 5: Two distant scatterers

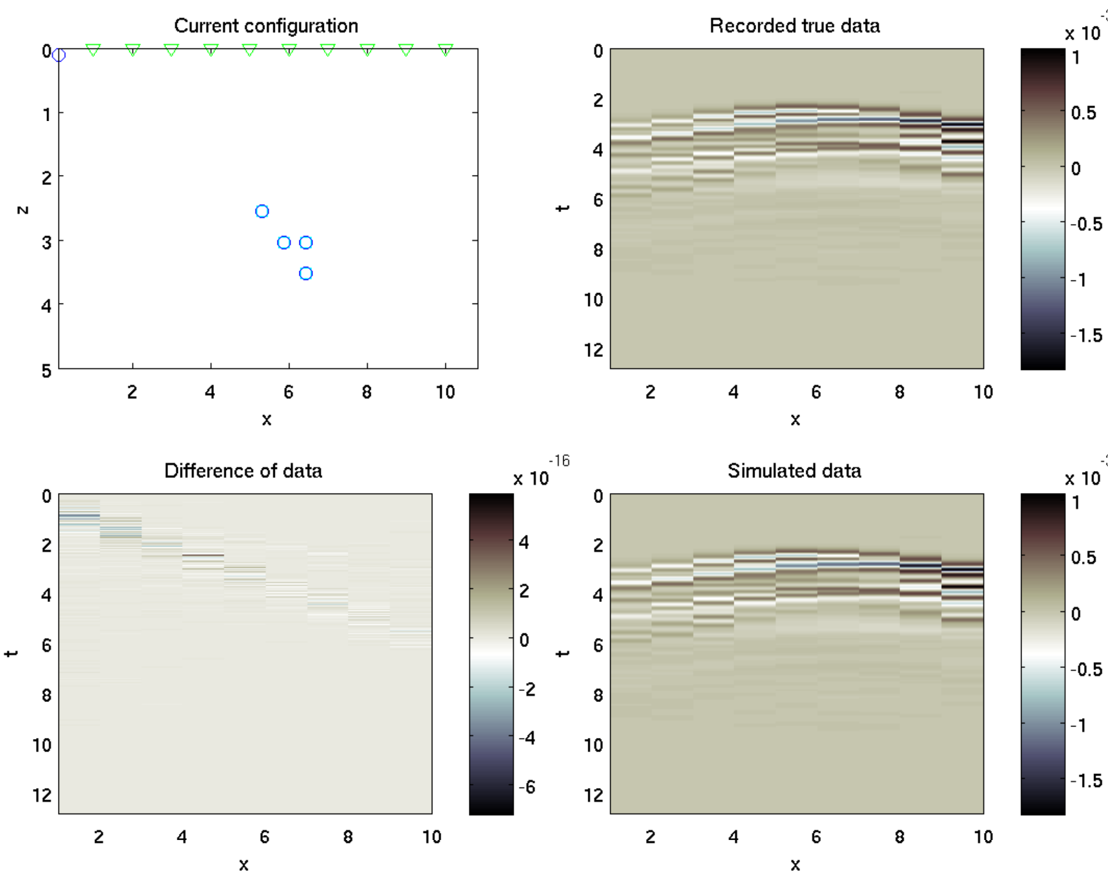

Figure 6: Four scatterers nearby

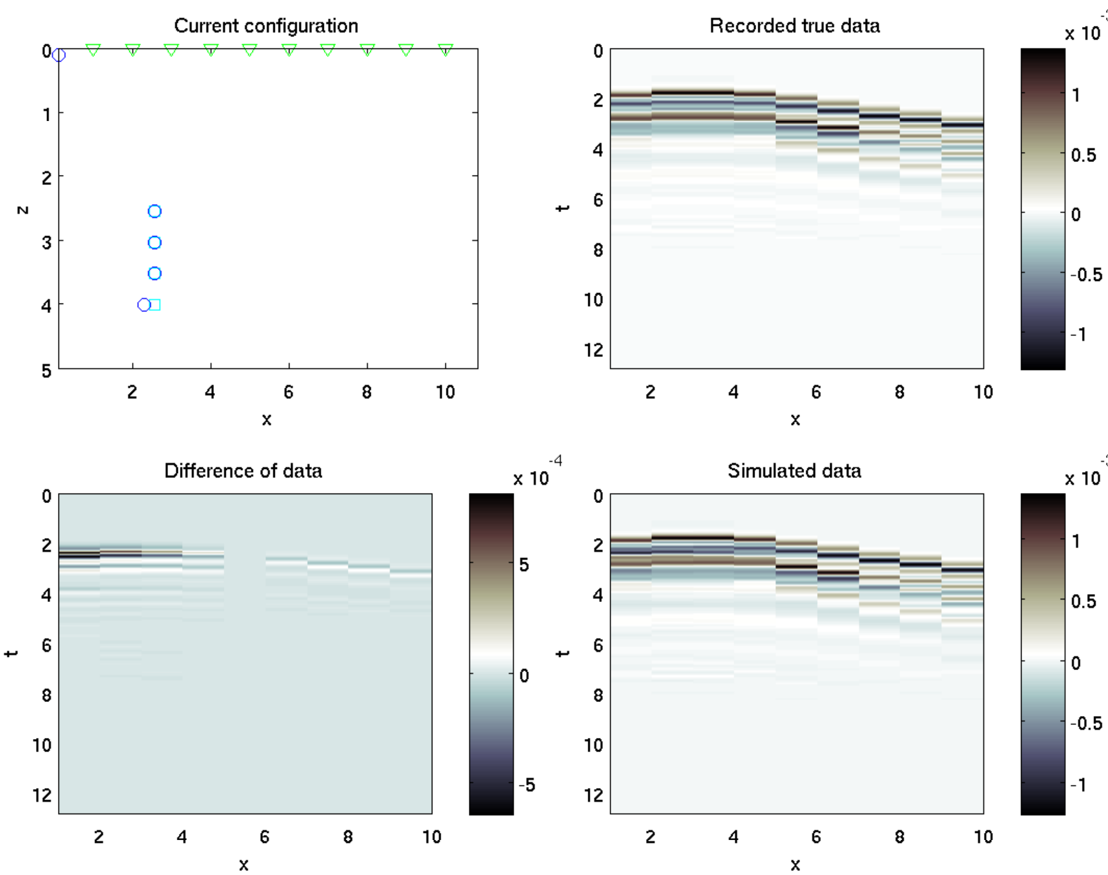

Figure 7: Small vertical reflector

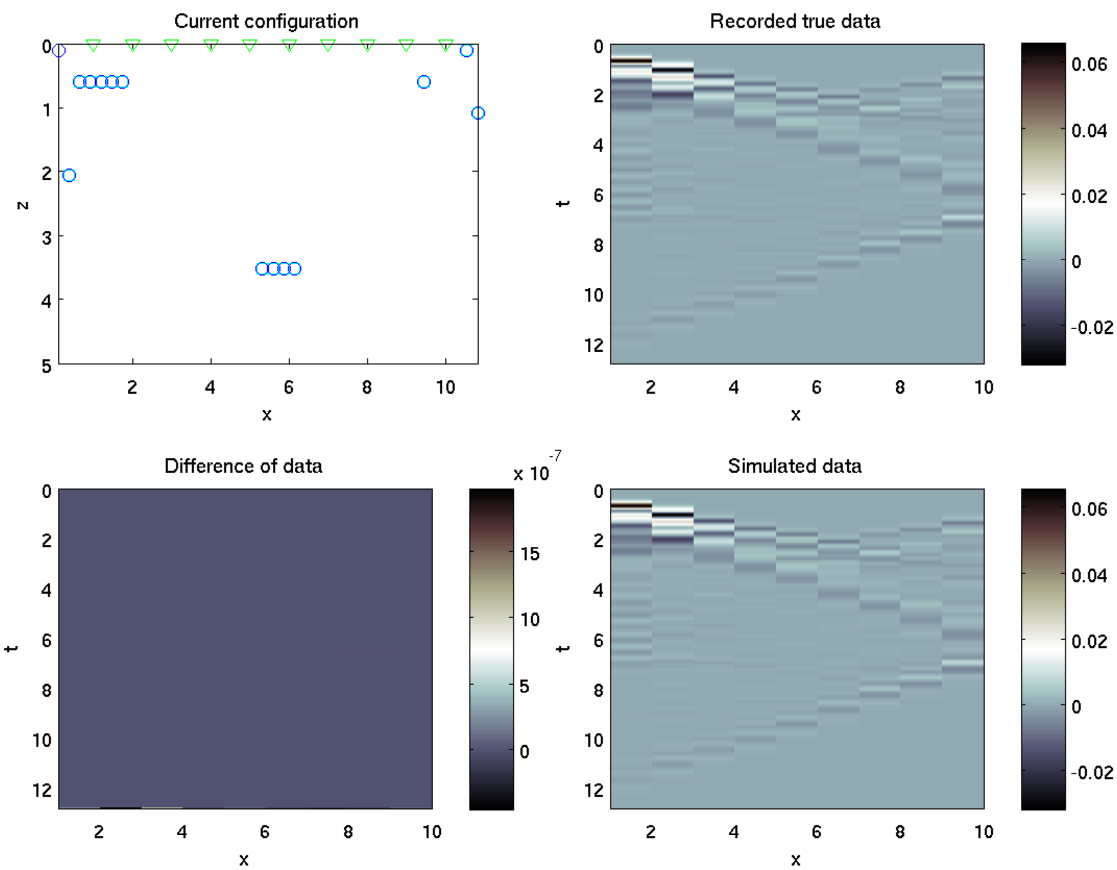

Figure 8: Scattered diffractors

domly perturbed by a locally correlated noise representing our uncertainty in the background model. It has been shown that cross-correlations of traces from neighboring receivers are stable and coherent even if shot-gathers themselves are not. Consequently, it appears that a natural modification of the goodness-of-fit measure to match correlograms instead of receiver wavefields would produce a dynamic that is virtually insensitive to this kind of noise. The effect of other kinds of uncertainties including large scale perturbations and possible ways to mitigate their influence remains a challenging problem for future research.

The approach presented here can be applied to other geophysical problems so long as they involve a natural configuration space and the measure of goodness of fit can be computed for an individual configuration quickly and reliably.

6 Acknowledgement

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