

Multifractional splines: from seismic singularities to geological transitions

Felix Herrmann
Earth Resources Laboratory
Dept. of Earth, Atmospheric, and Planetary Sciences
Massachusetts Institute of Technology
Cambridge, MA
and
Maarten de Hoop
Colorado School of Mines
Golden, CO

Abstract

A matching pursuit technique in conjunction with an imaging method is used to obtain quantitative information on geological records from seismic data. The technique is based on a greedy, non-linear search algorithm decomposing data into atoms. These atoms are drawn from a redundant dictionary of seismic waveforms. Fractional splines are used to define this dictionary, whose elements are not only designed to match the observed waveforms but also to span the appropriate family of geological patterns. Consequently, the atom's parameterization provides localized scale, order and direction information that reveals the stratigraphy and the type of geological transitions. Besides a localized scaling characterization, the atomic decomposition allows for an accurate denoised reconstruction of data with only a small number of atoms. Application of this approach to angles gathers allows us to track geological singularities from seismic data. Our characterization bridges the gap between the analysis of the main features within geologic processes, i.e. the geologic patterns, and the interpretation of their associated seismic response. A case study of Valhall data is presented.

1 Introduction

The history of geological processes is stored in geological records. Transitions in the geological record reflect changes in sedimentary processes induced by major depositional and erosional events. These transitions are abrupt and manifest themselves as singularities in the elastic moduli, mostly along the depth or vertical direction. Seismic measurements provide access to these features because seismic waves predominantly interact at these singularities in the elastic moduli and/or density.

In this paper we present a method of recovering the geologic record from seismic data. First, by means of the Generalized Radon Transform (GRT), we transform seismic data into a collection of subsurface images parametrized by angles. Under certain conditions, these angles correspond to the angles under which the subsurface singularities are being insonified by the seismic waves, i.e. which are the scattering angles ([de Hoop and Bleistein, 1997](#); [de Hoop et al., 1999](#); [de Hoop and Brandsberg-Dahl, 2000](#)). A common image gather (CIG) is a collection of traces that pertain to the same horizontal (surface) position, sorted for increasing scattering angles. Second, we subject these images to an adaptive atomic decomposition in one spatial direction, generally the vertical, and then repeat this procedure for each of the remaining spatial directions, generally the horizontal. On the one hand, inherent redundancy of the angles domain data is utilized by subjecting the angles CIGs to a non-adaptive wavelet thresholding ([Donoho and Johnstone, 1998](#)) along the angles coordinates. This thresholding denoises the images and enforces smoothness in the angle directions. At the same time this thresholding suppresses the artifacts generated by the GRT in the presence of caustics.

On the other hand, we preserve the singularities in the depth direction by a decomposition into transition sharpness-adaptive atoms. Noise in the data is removed by selecting those atoms which correlate best with coherent events in the image gathers. By merit of the generalized convolutional model these atoms are a proxy for the important patterns within the geologic record.

As shown by Muller et al. (1992) and later by Saucier et al. (1997); Herrmann (1997, 1998); Marsan and Bean (1999), sedimentary records show evidence of multifractal behavior and these records reveal a wide variety of transitions with differing scaling/singular behavior (Herrmann, 2001b; Herrmann et al., 2001). The Hölder scale exponents characterize the transition sharpness and vary discontinuously with position. Finding these exponents from bandwidth-limited data is difficult due to the lack of scales and the presence of noise. Therefore, we propose a method capable of estimating singularity orders from noisy data with limited scale content. The method is based on the identification of geological patterns by searching a redundant dictionary for waveforms that optimally correlate with seismic reflection events. The success of such an atomic decomposition (Mallat, 1997) depends on how well the constructed elements, within the dictionary, match with generic geological transitions and their induced reflectivity in the CIGs.

Multifractal scaling of sedimentary records motivates a data-adaptive atomic representation by fractional spline wavelets, indexed by location, dyadic scale, direction, and a continuously varying scaling/regularity parameter. (Fractional splines approximate via an embedding argument elements in Zygmund classes of positive order to which multifractals belong.) The scaling parameter measures the abruptness of geological transitions, their scaling and seismic response. The best correlating atoms are selected from the dictionary via a greedy search algorithm called Matching Pursuit (Mallat, 1997). The dictionary itself consists of a multitude of *translation-invariant* discrete wavelet transforms (Coifman and Donoho, 1995) computed with respect to order and direction varying fractional spline wavelets. Compared to earlier work (Herrmann, 2001b; Herrmann et al., 2001; Herrmann, 2001a), this approach restores the translation-invariance and allows, via composition, for a singularity/edge preserving reconstruction. Moreover, the atom’s parameterization provides information on the location (stratigraphy) and transition abruptness (lithology), via the location and singularity order of the knots.

First, we introduce the details of the parametric representation. Then, we summarize our method of seismic imaging, followed by the singularity characterization based on the greedy search. We conclude with a multi-component ocean bottom data example (Valhall) and discuss the implications of our findings.

2 Parametric representation of geologic transitions

We consider media that consist of accumulations of varying order transitions of the type (Herrmann, 1997, 2001b; Herrmann et al., 2001):

$$\chi_{\pm}^{\alpha}(z) = \begin{cases} 0 & z \lesssim 0 \\ \frac{z^{\alpha}}{\Gamma(\alpha+1)} & z \gtrsim 0 \end{cases} \quad (1)$$

with Γ the Gamma function. These depth- z and index- α onset functions define causal $\chi_{+}^{\alpha}(z)$, anti-causal $\chi_{-}^{\alpha}(z)$ and symmetric $\chi_{*}^{\alpha}(z) = \chi_{+}^{\alpha}(z) + \chi_{-}^{\alpha}(z)$ singularities and are characterized by the parameter α which determines local scale-invariance of the type, $\sigma^{-\alpha}\chi_{\pm}^{\alpha}(\sigma t) = \chi_{\pm}^{\alpha}(t)$, $\sigma > 0$. For $\alpha = 0$, the onset becomes a jump discontinuity (piece-wise constant record) and for $\alpha = 1$ a ramp function (piece-wise linear record). See Fig. 1 for examples. Mathematically, the exponent α corresponds to the Hölder exponent describing the local irregularity of a function at the transition (Mallat, 1997; Herrmann et al., 2001). The smaller the exponent, the sharper, more abrupt the transition.

3 Mapping of geologic singularities by seismic imaging

Recently, a theory of seismic imaging has been developed that in the single scattering approximation transforms data,

$$\mathbf{d}(s, r, t) = \mathbf{K}\mathbf{m} + \mathbf{n} \quad (2)$$

as a function of source (s), receiver (r) positions and time (t) to an image at the scattering position (z), angle and azimuth ($\mathbf{e} = (\Psi, \Phi)$)

$$\mathbf{u}_{\mathbf{x}}(z, \mathbf{e}) = \mathbf{K}^*(\mathbf{e})\mathbf{d}(\mathbf{x}, z), \quad (3)$$

with $\mathbf{u}_{\mathbf{x}}$ common angle pre-stack migrated data (the formulation separates the different wave modes). This transformation follows from restricting the adjoint \mathbf{K}^* to a fixed angle and azimuth, \mathbf{e} . From now on we suppress references to the horizontal position (\mathbf{x}) in our notation. The noise (\mathbf{n}) polluted data in Eq. 2 is assumed to be given by the action of the linearized forward scattering operator (\mathbf{K} acting on the medium elastic perturbations and hence on the changes the geologic record (\mathbf{m})). Pre-stack migration, as defined in Eq. 3, corresponds to a partial solution of the least-squares inverse problem

$$\langle \mathbf{m} \rangle = (\mathbf{K}^* \mathbf{K})^{-1} \mathbf{K}^* \mathbf{d} \quad (4)$$

with $\langle \mathbf{m} \rangle$ the estimated spatial distribution of the medium perturbation. Formulating the inverse scattering problem in the angles domain naturally reveals the singularity structure even for complicated subsurface models where the waves form caustics. In the ray-Born approximation for scattering from non-intersecting smooth interfaces, seismic traces can be represented by a convolutional model (de Hoop and Bleistein, 1997; Herrmann, 2001b). In a convolutional model setting, the \mathbf{u} is of the form

$$\mathbf{u}(z, \mathbf{e}) \sim \partial_z \mathbf{M}(\cdot, \mathbf{e}) \overset{(z)}{*} \varphi(t(\cdot, \mathbf{e})), \quad (5)$$

where φ represents the bandwidth-limited seismic signature, t is the time-to-depth conversion, and \mathbf{M} is derived from \mathbf{m} . The parameterization of Eq. 1 pertains to \mathbf{m} . The CIG traces, \mathbf{u} , should be the same for each \mathbf{e} . Because of this redundancy, the CIGs provide an appropriate domain for denoising and singularity analysis.

4 Singularity detection and characterization

We build a redundant dictionary with atoms that are designed to fit events in a seismic trace in the angles domain of the form Eq. 5.

4.1 The fractional spline dictionary

Conventional degree- m splines are piecewise m^{th} -order polynomials smoothly (for $m > 0$) joined together at the knots. At these knots the m^{th} -order derivative contains a jump discontinuity. For uniform unit spacing of the knots, splines can uniquely be characterized by a summation of B-splines, which are given by m^{th} -order auto-convolutions of the boxcar function, i.e.

$$\beta^m(z) = \underbrace{(\beta^0 * \beta^0 * \dots * \beta^0)}_{(m+1) \text{ times}}(z) \quad (6)$$

with $\beta^0(z)$ being the indicator function on $z \in (-\frac{1}{2}, +\frac{1}{2})$. In the Fourier domain this definition corresponds to raising the sinc-function to the $(m+1)^{\text{th}}$ -power. Following Unser and Blu (2000); Blu and Unser (2000), we can generalize integer order splines to fractional α -order by raising the sinc-function to fractional powers. In the space domain this generalization can be written as the $(\alpha+1)$ -order fractional difference of the onset function defined in Eq. 1,

$$\beta_{\pm}^{\alpha}(z) = (\Delta_{\pm}^{\alpha+1} \chi_{\pm}^{\alpha})(z), \quad \beta_{*}^{\alpha}(z) = (\beta_{+}^{\frac{\alpha-1}{2}} * \beta_{-}^{\frac{\alpha-1}{2}})(z), \quad (7)$$

where

$$(\Delta_{\pm}^{\alpha} f)(z) = \sum_{k \geq 0} (-1)^k \binom{\alpha}{k} f(z \mp k). \quad (8)$$

As with ordinary splines (Mallat, 1997), fractional splines can be used to construct a wavelet basis via an orthogonalization process (Unser and Blu, 2000), yielding an expression for the refinement filter. Given this filter we can define the fractional spline wavelets as (see Blu and Unser (2000) for detail):

$$\psi_{\pm,*}^{\alpha}(\frac{z}{2}) = \sum_{k \in \mathbb{Z}} g_{\pm,*}^{\alpha}[k] \beta_{\pm,*}^{\alpha}(z - k). \quad (9)$$

We use $\hat{g}(\omega) = -e^{-i\omega} \hat{h}^*(\omega + \pi)$ to derive the high-pass filter from the low-pass filter.

Compared to ordinary m^{th} -order spline wavelets, α -order spline wavelets have a prescribed fractional degree of regularity (α) and act as smoothed fractional $(\alpha + 1)^{\text{th}}$ -derivative operators, for the low frequencies (ζ): $\hat{\psi}_{\pm}^{\alpha}(\zeta) \sim (\pm j \zeta)^{\alpha+1}$ and $\hat{\psi}_{*}^{\alpha}(\zeta) \sim \zeta^{\alpha+1}$ as $\zeta \rightarrow 0$. See Kane *et al.* in these proceedings for the application of fractional spline wavelets in the non-linear solution of linear inverse problems using thresholding.

To maintain shift-invariance, we use a wavelet transform with the fractional spline wavelets given by Eq. 9 and without down-sampling. For each α and direction ($\{\pm, *\}$), we wavelet transform our data, i.e. $f \mapsto \langle f, \psi_{\pm,*}^{\alpha} \rangle$ with $\langle f, \psi_{\pm,*}^{\alpha} \rangle$ the empirical wavelet coefficients of f with respect to the different wavelets.

Compared to the ordinary discrete wavelet transform, the multiple non-downsampled wavelet transforms offer flexibility to find the optimal basis function representation. Optimality in this case refers to both the regularity-adaptiveness and approximation error as a function of the number of atoms. To reduce the computational cost of finding the best decomposition, we store for each different order and direction the wavelet coefficients into binary trees similar to wavelet packet trees (Coifman and Donoho, 1995), yielding a dictionary $\mathcal{D} = \{g_{\gamma}\}_{\gamma \in \Gamma}$ with g_{γ} the different α -order (anti)-causal/symmetric fractional spline wavelets as defined in Eq. 9 (here g_{γ} represent the actual wavelets and not the refinement filters).

4.2 Atomic decomposition by matching pursuit

Following Mallat (1997) we use a greedy search algorithm for the decomposition of f from the redundant dictionary, \mathcal{D} . This data-adaptive algorithm is based on minimizations of the L^2 -norm difference between reductions of f and the atoms. Reductions are projections of f

$$f = \langle f, g_{\gamma_0} \rangle g_{\gamma_0} + Rf, \quad (10)$$

where Rf is the residue and g_{γ_0} the particular γ_0 -indexed atom that maximizes the correlation, i.e. $|\langle Rf, g_{\gamma_0} \rangle| \geq \sup_{\gamma \in \Gamma} |\langle Rf, g_{\gamma} \rangle|$ (Mallat, 1997). The index $\gamma \in \Gamma$ refers to the location in the tree (dyadic scale (2^j) and sample number (i)), the direction ($\{\pm, *\}$) and the order (α). By repeating the above reductions, we obtain the atomic decomposition. Compared to the ordinary wavelet transform, the atomic decomposition specifically depends on the data f , hence the term data-adaptive.

4.3 Denoising and reconstruction

In our imaging procedure, the singularities are confined to one spatial direction. Since there are no singularities in the angular directions \mathbf{e} , we may consider the images to be smooth in those directions. In that situation the non-adaptive wavelet transform can be used, and good denoising results are obtained upon non-linearly thresholding the wavelet coefficients computed for a smooth as possible wavelet. The non-linear denoised estimates (denoted by the symbol $\hat{\cdot}$) are given by

$$\hat{\mathbf{u}}(\cdot, \mathbf{e}) = \sum_{\lambda} \theta_T \{ \langle \mathbf{u}(\cdot, \cdot), \psi_{\lambda} \rangle \} \psi_{\lambda}(\mathbf{e}) \quad (11)$$

with \mathbf{e} the angles, $\theta_T \{ \cdot \}$ the thresholding operator. We use the threshold, $T = \sigma \sqrt{2} \log N$, with N the number of samples and σ the noise level, as given by (Donoho and Johnstone, 1998; Mallat, 1997). ψ_{λ} is an orthogonal λ -indexed (location (i) and scale (j)) wavelet. After thresholding, the trace by trace continuity is enhanced, a property that is consistent with the data's redundancy. The non-linear thresholding suppresses the noise and therefore improves the signal to noise ratio in the angles image as well as in the angular stack and inversion.

Since the singularities are located in the spatial direction we apply our matching pursuit algorithm to each angular trace. This algorithm gives us, for each angle, a decomposition into a noise level dependent number (M) of atoms. Reconstruction of $\mathbf{u}(z)$ by these first M atoms,

$$\hat{\mathbf{u}}(z, \cdot) = \sum_{m=1}^M \langle \mathbf{u}(\cdot, \cdot), g_{\gamma_m} \rangle g_{\gamma_m}(z), \quad (12)$$

gives us the denoised estimate as long as we choose M such that $\frac{|\langle R^M \mathbf{u}, g_{\gamma_M} \rangle|}{\|R^M \mathbf{u}\|} > \text{noise level}$. By construction, the selected atoms can be interpreted as data-adaptive “principle geologic components”. For seismic reflectivity, the wavelet coefficients of the noise and reflectivity are very close and hence **not** separable, rendering non-data adaptive wavelet thresholding techniques ineffective. However, redundancy in our dictionary allows us to find the coherent structures. In Fig. 3, we present the results of denoising followed by stacking. We can see that the denoising significantly improves the resolution especially at the reservoir (just below sample # 100).

4.4 Singularity characterization

The atom’s parameterization provides estimates for the location, order, direction and scale of the reflectors. These parameters are given by the parameterization of the selected atoms. Fig.’s 4-5 summarize the estimates which correspond to the first 100 selected atoms. Notice the reasonable lateral consistency.

5 Implications for modeling and interpretation

We conclude from our example that discontinuities other than zero-order steps and first-order ramp functions occur in sedimentary basins. This observation has important consequences for the interpretation of geological boundaries, which can no longer be considered as strictly local, as is the case for jump discontinuities. By construction, our decomposition implies a *multifractional* spline representation for the geological record, where the location and singularity orders of the knots correspond to the location (stratigraphy) and transition abruptness (lithology). We conjecture that fractional splines play a more fundamental role in connection between the geological process and transitions in the elastic parameters. We paired wavelets and seismology to fractional splines and sedimentary processes. Since the atoms represent the coherent structures in the data, we can suppress GRT artifacts and hence improve image quality, while preserving the singularities.

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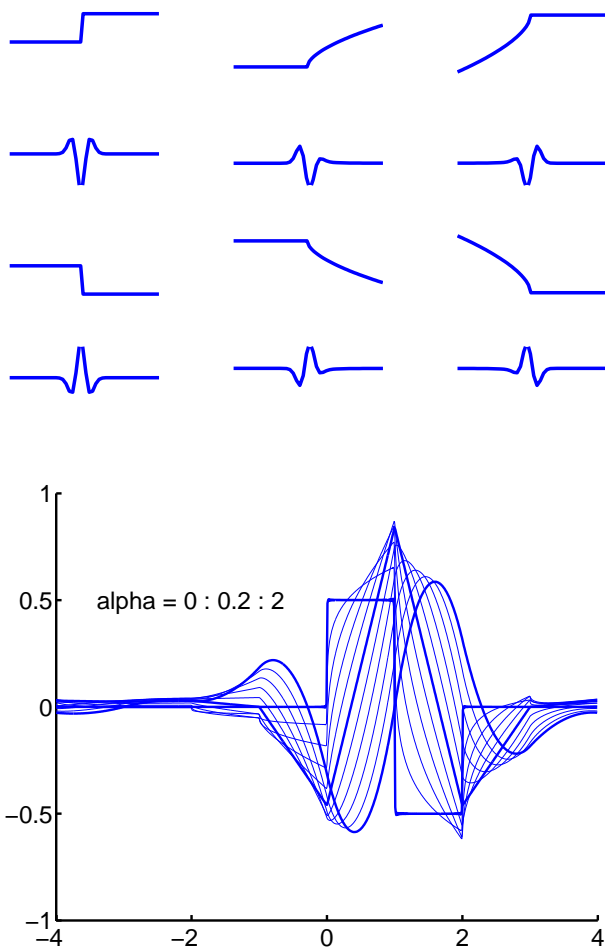


Figure 1: Example of cusplike singularities and their seismic response (left) and part of the dictionary (right).

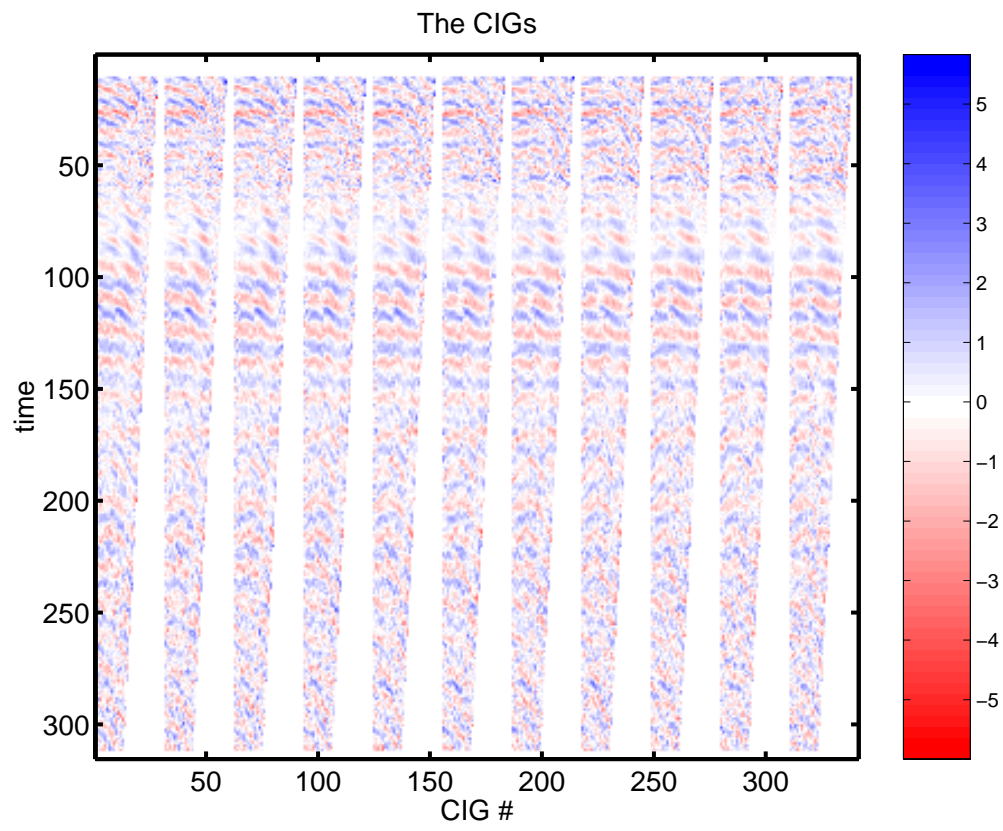


Figure 2: PP Scattering Angles Common Image Gathers.

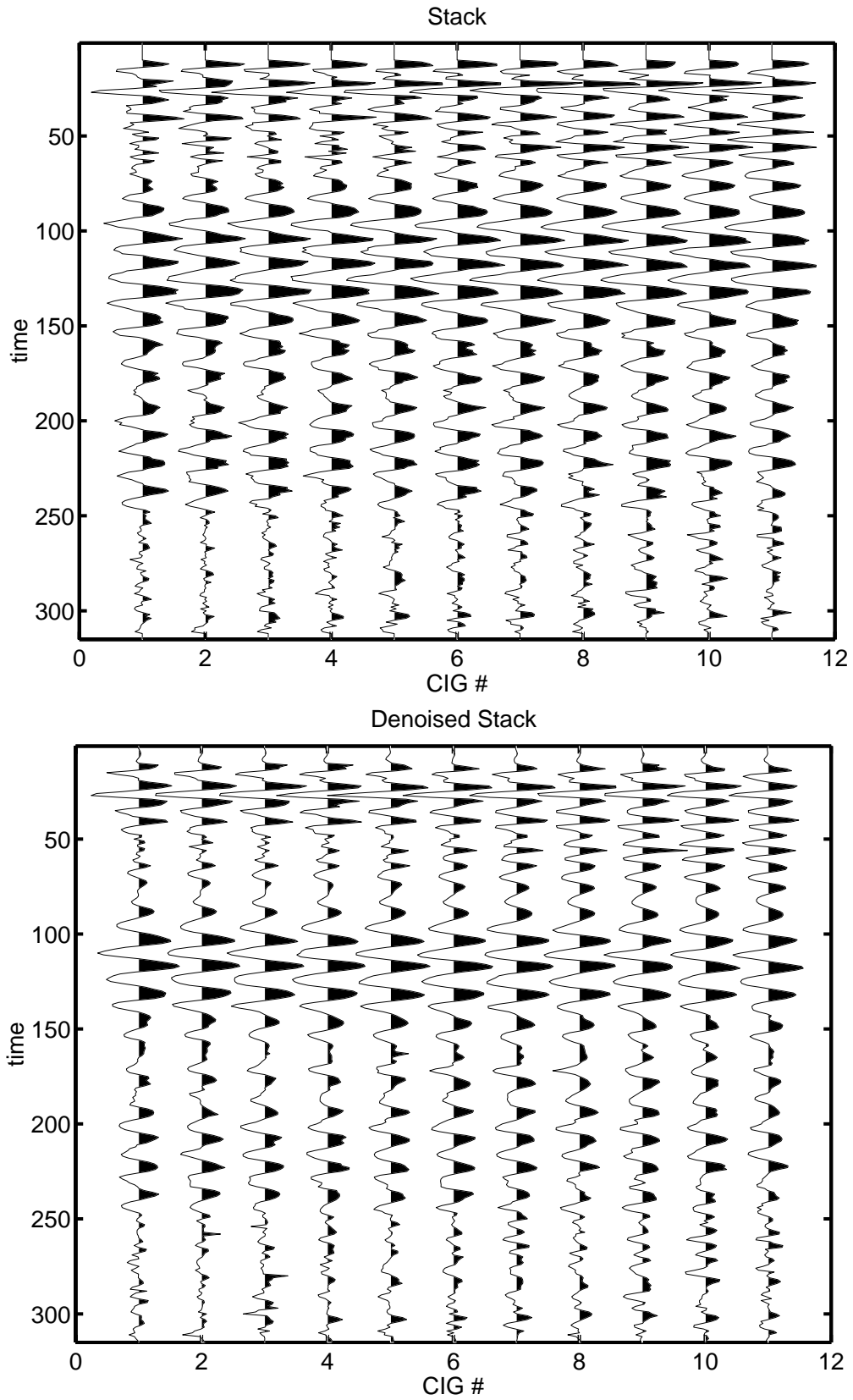


Figure 3: Comparison ordinary stack and stack after non-linear denoising.

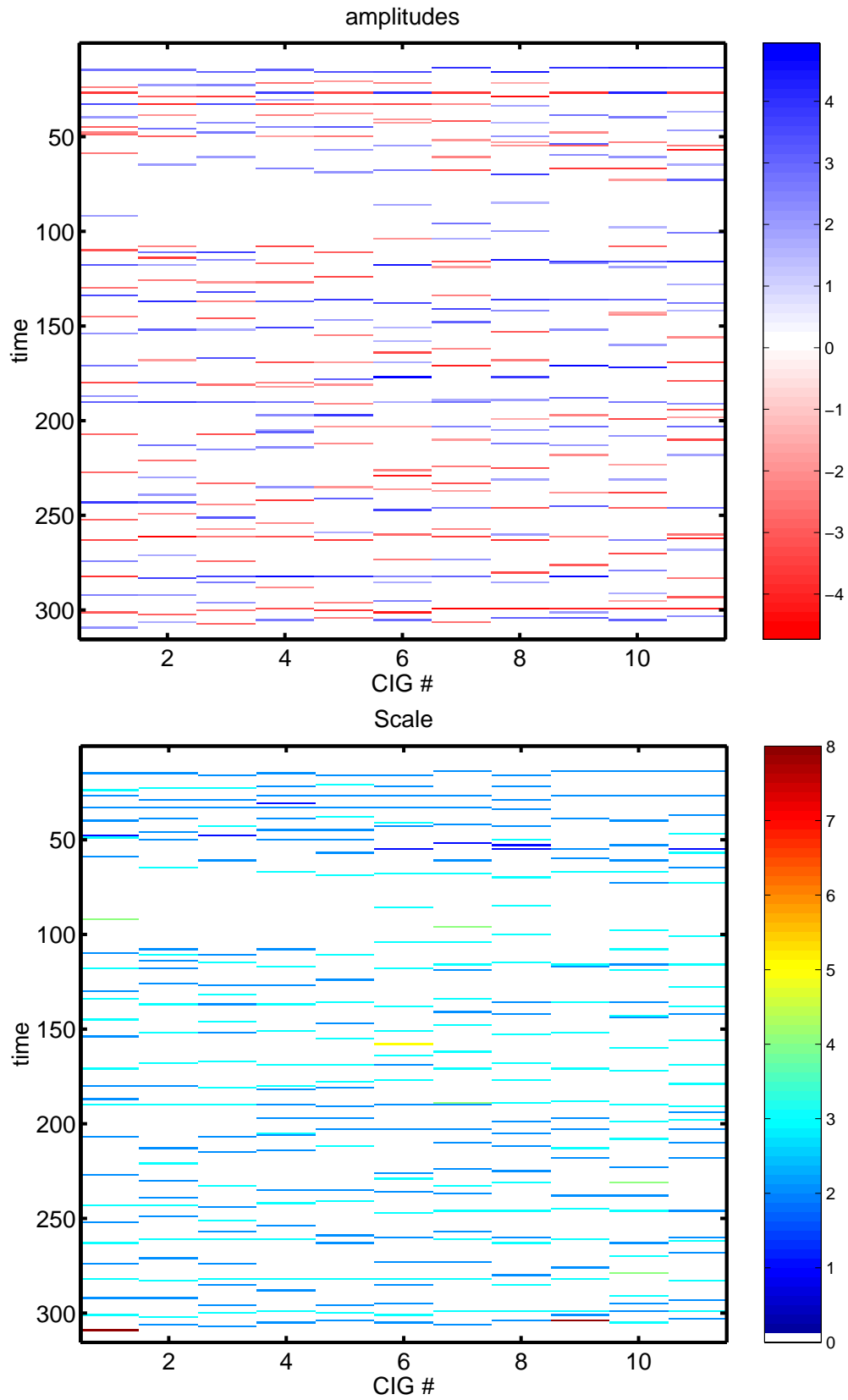


Figure 4: Estimated amplitudes and scale.

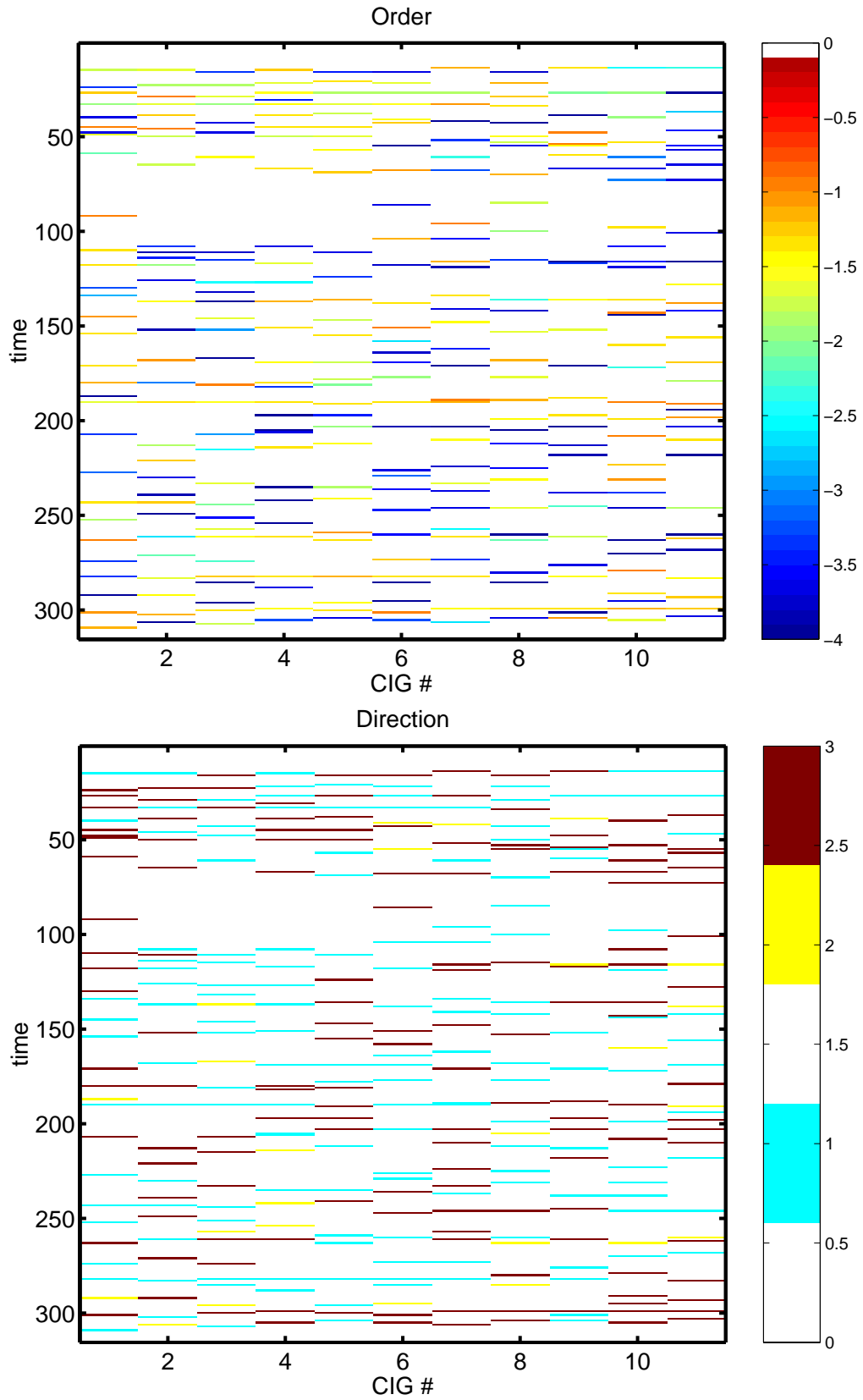


Figure 5: Estimated order and direction (1 anti-causal, 2 symmetric and 3 causal) respectively.