Combinatorics in Schubert varieties and Specht modules

by

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Abstract

This thesis consists of two parts. Both parts are devoted to finding links between geometric/algebraic objects and combinatorial objects.

In the first part of the thesis, we link Schubert varieties in the full flag variety with hyperplane arrangements. Schubert varieties are parameterized by elements of the Weyl group. For each element of the Weyl group, we construct certain hyperplane arrangement. We show that the generating function for regions of this arrangement coincides with the Poincaré polynomial if and only if the Schubert variety is rationally smooth. For classical types the arrangements are (signed) graphical arrangements coming from (signed) graphs. Using this description, we also find an explicit combinatorial formula for the Poincaré polynomial in type A.

The second part is about Specht modules of general diagram. For each diagram, we define a new class of polytopes and conjecture that the normalized volume of the polytope coincides with the dimension of the corresponding Specht module in many cases. We give evidences to this conjecture including the proofs for skew partition shapes and forests, as well as the normalized volume of the polytope for the toric staircase diagrams. We also define new class of toric tableaux of certain shapes, and conjecture the generating function of the tableaux is the Frobenius character of the corresponding Specht module. For a toric ribbon diagram, this is consistent with the previous conjecture. We also show that our conjecture is intimately related to Postnikov's conjecture on toric Specht modules and McNamara's conjecture of cylindric Schur positivity.

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Contents

Chapter 1

Hyperplane arrangements and Schubert varieties

This part of my thesis is based on the joint work with Suho Oh and Alexander Postnikov [19]. It also contains the results of [20].

1.1 Introduction

The purpose of this chapter is to link Schubert varieties with hyperplane arrangements. Schubert varieties are widely studied in algebraic geometry, representation theory and combinatorics. In the full flag manifold, they are orbit closures of Borel group action, and parameterized by elements of the corresponding Weyl group W:

$$X_w = \overline{BwB/B}, \ w \in W \tag{1.1}$$

Many geometric properties of Schubert varieties can be studied via combinatorial properties of the Weyl group elements. In particular, it is well known that the Poincaré polynomial of X_w coincides with the rank generating function of the lower interval $[id, w]$ in the strong Bruhat order.

Coxeter arrangement of a Weyl group is the arrangement of root hyperplanes. For instance in type A_{n-1}, it is the arrangement of hyperplanes

$$x_i - x_j = 0, \ 1 \leq i < j \leq n. \tag{1.2}$$

Each of $n!$ regions of this arrangement corresponds to an element of symmetric group S_n, and each hyperplane corresponds to a reflection. Given $w \in S_n$, the *inversion hyperplane arrangement* $\mathcal{A}_w$ is the subarrangement of the Coxeter arrangement that consists of the hyperplanes corresponding to inversions of w:

$$x_i - x_j = 0, \ 1 \leq i < j \leq n \text{ and } w(i) > w(j) \tag{1.3}$$

Numerical relationship between the set of regions of $\mathcal{A}_w$ and the lower Bruhat interval $[id, w]$ was discovered by Postnikov in [23]. He proved that, for Grassmannian permutations, both sets are in bijection with totally nonnegative cells in the corresponding Schubert vatirey in the Grassmannian. In particular, they have the same cardinality. The exact conditions for permutations to have this property was conjectured by Postnikov using pattern avoidance, and proved by Hultman, Linusson, Shareshian and Sjöstrand in [2]. Our main result on the symmetric group is a q-analogue of this numerical coincidence for permutations whose Schubert varieties are smooth. Namely,

Theorem 1.1.1. *In type A, the generating function of regions of an inversion hyperplane arrangement coincides with the Poincaré polynomial of the corresponding Schubert variety if and only if the Schubert variety is smooth.*

For any graph G on the vertex set $\{1, \ldots, n\}$, we can assign a graphical hyperplane arrangement $\mathcal{A}_G$ with hyperplanes

$$x_i - x_j = 0, \text{ for all edges}(i, j) \text{ of } G \tag{1.4}$$

Given a permutation w, the inversion graph G_w is the graph whose edges are (i, j) where $w(i) > w(j)$. It is clear that the inversion hyperplane arrangement is the graphical arrangement coming from the inversion graph, $\mathcal{A}_w = \mathcal{A}_{G_w}$.

Our main technical tools to prove the theorem for type A are chordal graphs and perfect elimination orderings. A graph is called *chordal* if each of its cycles with four or more vertices has a *chord* which is an edge joining two vertices that are not adjacent in the cycle. We show that the inversion graph of a permutation w whose corresponding Schubert variety is smooth is chordal with some additional properties, and this leads the generating function of regions of $\mathcal{A}_{G_w}$ to be factorized into product of q-numbers in the same way that the Poincaré polynomial is.

As a byproduct of the proof, we get an explicit combinatorial formula of the factorization of the Poincaré polynomial of the smooth Schubert variety into q-numbers:

$$P_w(q) = [e_1 + 1]_q \cdots [e_n + 1]_q \tag{1.5}$$

where $e_i' s$ are the roots of characteristic polynomial of the inversion hyperplane arrangement $\mathcal{A}_w$. We get a formula that directly reads $e_i' s$ from the permutation w.

Schubert varieties are defined for all crystallographic Coxeter groups, and it is also natural to define the inversion hyperplane arrangement for other types. Indeed, the numerical relation between the set of regions of the inversion hyperplane arrangement and the lower Bruhat interval was recently provided by Hultman in [12] for all finite Coxeter groups.

Although, smoothness is too strong condition for general types of Weyl groups. For non-simply laced root systems, smoothness is different from rational smoothness, and by the theorem of Carrell and Peterson, we know that the Poincaré polynomial is palindromic if and only if the Schubert variety is rationally smooth. As shown by Deodhar [10] and Peterson, smoothness and rational smoothness are equivalent for simply-laced root systems. Hence rational smoothness is the correct condition to generalize our theorem.

The classical Weyl groups of type B(=C) and D can be represented as signed permutations. Using this representation, the inversion graph of each type can be defined analogously to symmetric groups. On the other hand, Billey gives a rule to factorize Poincaré polynomials of rationally smooth Schubert varieties in [5]. We show that the generating function of regions of the inversion hyperplane arrangement also follows this rule, hence coincides with the Poincaré polynomial when the Schubert variety is rationally smooth.

For more general types of Weyl groups, there is a result of Billey and Postnikov [6] on the parabolic decomposition of Poincaré polynomial:

Theorem 1.1.2. *[6] Let J be any subset of simple roots. Assume $w \in W$ has parabolic decomposition $w = uv$ with $u \in W_J$ and $v \in W^J$ and furthermore, u is the maximal element in the strong Bruhat order below w. Then*

$$P_w(t) = P_u(t)P_v^{W^J}(t)$$

where $P_v^{W^J} = \sum_{z \in W^J, z \leq v} t^{\ell(z)}$ is the Poincaré polynomial for v in the quotient. Moreover, if w is rationally smooth then w or w^{-1} admits such a parabolic decomposition.

By showing that the inversion hyperplane arrangement allows the same factorization, we extend our result to other types:

Theorem 1.1.3. *For any finite Weyl group, the generating function of regions of an inversion hyperplane arrangement coincides with the Poincaré polynomial of the corresponding Schubert variety if and only if the Schubert variety is rationally smooth.*

It is to be noted that the Poincaré polynomial of a Schubert variety is the rank generating function of the strong Bruhat order while the regions in the inversion hyperplane arrangement form a certain truncation of the weak Bruhat order. Hence our result finds an interesting connection between the strong and weak Bruhat orders.

The rest of introduction is devoted to outline how this chapter is organized. The first half of this chapter will be about classical Weyl groups, type A,B(=C) and D. Our approach in these cases depends on the combinatorics of (signed) permutations and inversion graphs. The remaining half will be about a general approach to other types. Although the second half implies the main result about symmetric groups, we include them both separately because there are more combinatorics involved in symmetric groups, and the former approach is more suited to generalize to affine type A as well as other classical types.

In Section 1.2, we will review the basic definitions and theorems on the Bruhat order and Schubert varieties. In Section 1.3 we will define inversion graphs of classical types, and their relation to hyperplane arrangements. Section 1.3.1 will be about graphical arrangements and type A, and Section 1.3.2 and Section 1.3.3 will be about inversion graphs of type B and D,respectively, defined in terms of signed permutations. In Section 1.3.4 we review the properties of chordal graphs and perfect elimination ordering, and define nice perfect elimination ordering. It is used to factorize the generating function of regions in Section 1.3.5. In Section 1.3.6, we define simple perfect elimination ordering and use it to get a direct factorization formula of Poincaré polynomials. Section 1.4 is going to be about general definition of inversion hyperplane arrangement for all Weyl groups. In Section 1.5, we review Billey and Postnikov's result on factorization of Poincaré polynomials using parabolic decomposition. Their result will play a key role in the proof of our main theorem for Weyl groups in Section 1.6. In Section 1.7, we state further questions and conjectures.

1.2 Bruhat order and Poincaré polynomials

We begin with overview of basic definitions and theorems from Coxeter group theory. In this section, we define the Bruhat order on Weyl groups and its relation to Schubert varieties. For more details, see [7],[4].

Let G be a semisimple simply-connected complex Lie group, B a Borel subgroup and $\mathfrak{h}$ the corresponding Cartan subalgebra. Let W be the corresponding Weyl group, $\Delta \subset \mathfrak{h}^*$ be the set of roots and $\Pi \subset \Delta$ be the set of simple roots. The choice of simple roots determines the set of positive roots. We will write $\alpha > 0$ for $\alpha \in \Delta$ being a positive root. Let S be the set of simple reflections and $T := \{wsw^{-1} : s \in S, w \in W\}$ be the set of reflections.

The *strong Bruhat order* "$\leq$" on W is the partial order generated by the relations $w < w \cdot t$ if $\ell(w) < \ell(w \cdot t)$. Here $t \in T$ is a reflection; and $\ell(w)$ denotes the *length* of $w \in W$, i.e. number of inversions of w. Equivalently, for $u, v \in W$, $u < v$ if any reduced word of w contains a reduced word of u as a substring.

In this thesis, unless stated otherwise, we refer to the strong Bruhat order when we use the term *Bruhat order*. Intervals in the Bruhat order play an important role in Schubert calculus and Kazhdan-Lusztig theory. We concentrate on the intervals of the form $[id, w] := \{u \in W \mid u \leq w\}$ where id is the identity element in W, that is, on the lower order ideals of the Bruhat order. They are related to geometry of *Schubert varieties* $X_w = \overline{BwB/B}$ in the flag manifold G/B. Namely, the *Poincaré polynomial* of the cohomology ring of X_w is known to coincide with the rank generating function of the interval $[id, w]$, e.g., see [4]:

$$P_w(q) = \sum_{u \leq w} q^{\ell(u)}.$$

More precisely, the cohomology group of odd rank vanishes and the dimension of $H^{2i}(X_w)$ is the coefficient of q^i in $P_w(q)$. Although $P_w(q^2)$ is the Poincaré polynomial of X_w, we will refer $P_w(q)$ as the Poincaré polynomial as well. For convenience, we will say that $P_w(q)$ is the Poincaré polynomial of w.

An algebraic variety is *rationally smooth* if its rational locus is smooth. Any smooth variety is necessarily rationally smooth but the converse is not true in general, although they are equivalent for Schubert varieties of simply-laced root systems. For Schubert varieties, rational smoothness can be checked by studying its Poincaré polynomial. In fact, we can regard the next theorem of Carrell and Peterson as a definition of rational smoothness for Schubert varieties.

Theorem 1.2.1. (Carrell-Peterson [8], see also [4, Sect. 6.2]) *For any Weyl group W and $w \in W$, the Schubert variety X_w is rationally smooth if and only if the Poincaré polynomial $P_w(q)$ is palindromic.*

Here, a polynomial $P(q)$ of degree n is *palindromic* if the coefficient of q^i is the same as the coefficient of q^{n-i} for all i, i.e. $P(q) = q^n P(q^{-1})$. We will say that w is *rationally smooth* if $P_w(q)$ is palindromic, or equivalently, X_w is rationally smooth.

1.3 Inversion graphs of classical Weyl groups

In this section we recall the combinatorics of Weyl groups of type A,B(=C),D in terms of (signed) permutations. We define inversion graphs and associate hyperplane arrangements to them. We will state and prove our main theorems for classical types by showing some recurrences for the generating function of acyclic orientations on the inversion graphs.

1.3.1 Weyl group of type A and graphical arrangements

The Weyl group of type A_{n-1} is the symmetric group S_n whose elements are permutations written in one-line notation as $w(1)w(2)\dots w(n)$. It is generated by simple reflections σ_i for $1 \le i \le n-1$ where $w\sigma_i$ is obtained by interchanging $w(i)$ and $w(i+1)$. For example, if $w = 413526$ then $w\sigma_2 = 431526$. Each of simple reflections corresponds to a simple root.

There is a well-known smoothness criterion for Schubert varieties of type A, due to Lakshmibai and Sandhya, based on pattern avoidance. A permutation $w \in S_n$ contains a *pattern* $\sigma \in S_k$ if there is a subword with k letters in w with the same relative order of the letters as in the permutation σ. A permutation w *avoids the pattern* σ if w does not contain this pattern.

Theorem 1.3.1. (Lakshmibai-Sandhya [32]) *For a permutation $w \in S_n$, the Schubert variety X_w is smooth if and only if w avoids the two patterns* 3412 *and* 4231.

Now let us attach an hyperplane arrangement to each element of S_n using inversion graph of type A. The set of inversions of $w \in S_n$ is

$$\{(i,j) \in [n] \times [n] \mid i < j, w(i) > w(j)\}$$

We connect every i and j with an edge whenever (i,j) is an inversion. More precisely,

Definition 1.3.2. The *inversion graph* of $w \in S_n$ is the graph G_w on the vertex set $\{1,\dots,n\}$ with the set of edges $\{(i,j) \mid i < j,\ w(i) > w(j)\}$. The *inversion hyperplane arrangement* of $w \in S_n$ is the hyperplane arrangement $\mathcal{A}_w$ in $\mathbb{R}^n$ with hyperplanes $x_i - x_j = 0$ for all edges (i,j) of G_w.

In fact, inversion hyperplane arrangement of type A is a special case of *graphical arrangement.* For any graph G on the vertex set $\{1,\dots,n\}$, the graphical arrangement $\mathcal{A}_G$ is the hyperplane arrangement in $\mathbb{R}^n$ with hyperplanes $x_i - x_j = 0$ for all edges (i,j) in G. Clearly $\mathcal{A}_w$ is the graphical arrangement of G_w. The *characteristic polynomial* $\chi_G(t)$ of the graphical arrangement $\mathcal{A}_G$ is also the *chromatic polynomial* of the graph G. The value of $\chi_G(t)$ at a positive integer t equals the number of ways to color the vertices of the graph G in t colors so that all neighboring pairs of vertices have different colors. The value $(-1)^n\chi_G(-1)$ is the number of regions of $\mathcal{A}_G$. The regions of $\mathcal{A}_G$ are in bijection with acyclic orientations of the graph G. Recall that an *acyclic orientation* is a way to direct edges of G so that no directed cycles are formed. The region of $\mathcal{A}_G$ associated with an acyclic orientation $\mathcal{O}$ is described by the inequalities $x_i < x_j$ for all directed edges $i \to j$ in $\mathcal{O}$.

Now let R_w be the number of regions in the inversion arrangement $\mathcal{A}_w$, and let $B_w := \#[id, w] = P_w(1)$ be the number of elements in the Bruhat interval $[id, w]$. Interestingly, the numbers R_w and B_w are related to each other.

Theorem 1.3.3. (Hultman-Linusson-Shareshian-Sjöstrand [2])

(1) *For any permutation $w \in S_n$, we have $R_w \leq B_w$.*

(2) *The equality $R_w = B_w$ holds if and only if w avoids the following four patterns* 4231, 35142, 42513, 351624.

This result was conjectured in [23] and announced as an open problem in a workshop in Oberwolfach in January 2007. A. Hultman, S. Linusson, J. Shareshian, and J. Sjöstrand proved the conjecture after the workshop.

Remark 1.3.4. It was shown in [23] that $R_w = B_w$ for all Grassmannian permutations w, which agrees with the above result. In this case, B_w counts the number of totally nonnegative cells in the corresponding Schubert variety in the Grassmannian.

Remark 1.3.5. The four patterns from Theorem 1.3.3 came up earlier in the literature in at least two places. Firstly, Gasharov and Reiner [31] showed that the Schubert variety X_w can be described by simple inclusion conditions exactly when w avoids these four patterns. Secondly, Sjöstrand [27] showed that the Bruhat interval $[id, w]$ can be described as the set of permutations associated with rook placements that fit inside a skew Ferrers board if and only if w avoids the same four patterns.

Remark 1.3.6. Note that each of the four patterns from Theorem 1.3.3 contains one of the two patterns from Lakshmibai-Sandhya's smoothness criterion. Thus the theorem implies the equality $R_w = B_w$ for all smooth permutations w.

Let us define the q-analogue of the number of regions of the graphical arrangement $\mathcal{A}_G$, where G is a graph on the vertex set $\{1, \ldots, n\}$. For two regions r and r' of the arrangement $\mathcal{A}_G$, let $d(r, r')$ be the number of hyperplanes in $\mathcal{A}_G$ that separate r and r'. In other words, $d(r, r')$ is the minimal number of hyperplanes we need to cross to go from r to r'. Let r_0 be the region of $\mathcal{A}_G$ that contains the point $(1, \ldots, n)$. Define

$$R_G(q) := \sum_r q^{d(r, r_0)},$$

where the sum is over all regions r of the arrangement $\mathcal{A}_G$. Equivalently, the polynomial $R_G(q)$ can be described in terms of acyclic orientations of the graph G. For an acyclic orientation $\mathcal{O}$, let $\mathrm{des}(\mathcal{O})$ be the number of edges of G oriented as $i \to j$ in $\mathcal{O}$ where $i > j$ (descent edges). Then

$$R_G(q) = \sum_{\mathcal{O}} q^{\mathrm{des}(\mathcal{O})},$$

where the sum is over all acyclic orientations $\mathcal{O}$ of G. Indeed, for the acyclic orientation $\mathcal{O}$ associated with a region r we have $\mathrm{des}(\mathcal{O}) = d(r, r_0)$.

For $w \in S_n$, let $R_w(q) := R_{G_w}(q)$ be the polynomial that counts the regions of the inversion arrangement $\mathcal{A}_w = A_{G_w}$.

We are now ready to formulate the first main result of this chapter for type A. Recall that $P_w(q) := \sum_{u \le w} q^{\ell(u)}$ is the Poincaré polynomial of the Schubert variety.

Theorem 1.3.7. *For a permutation $w \in S_n$, we have $P_w(q) = R_w(q)$ if and only if w is a smooth permutation, i.e., if and only if w avoids the patterns* 3412 *and* 4231.

The "only if" part of Theorem 1.3.7 is straightforward. Indeed, if w is not smooth then by Carrell-Peterson's smoothness criterion (Theorem 1.2.1) the Poincaré polynomial $P_w(q)$ is not palindromic. On the other hand, the polynomial $R_w(q)$ is always palindromic, which follows from the involution on the regions induced by the map $x \mapsto -x$. Thus $P_w(q) \neq R_w(q)$ in this case. We will prove the "if" part of Theorem 1.3.7 in Section 1.3.5.

Our second result on type A is an explicit non-recursive formula for the polynomials $P_w(q) = R_w(q)$, when w is smooth.

Let us say that an index $r \in \{1, \dots, n\}$ is a *record position* of a permutation $w \in S_n$ if $w(r) > \max(w(1), \dots, w(r-1))$. The values $w(r)$ are called the *records* or *left-to-right maxima* of w. For $i = 1, \dots, n$, let r and r' be the record positions of w such that $r \le i < r'$ and there are no other record positions between r and r'. (Set $r' = +\infty$ if there are no record positions greater than i.) Let

$$e_i := \#\{j \mid r \le j < i,\ w(j) > w(i)\} + \#\{k \mid r' \le k \le n,\ w(k) < w(i)\}.$$

Theorem 1.3.8. *Let w be a smooth permutation in S_n, and let $e_1, \dots, e_n$ be the numbers constructed from w as above. Then*

$$P_w(q) = R_w(q) = [e_1 + 1]_q\, [e_2 + 1]_q \cdots [e_n + 1]_q\, .$$

Here $[a]_q := (1-q^a)/(1-q) = 1 + q + q^2 + \cdots + q^{a-1}$. We will prove Theorem 1.3.8 in Section 1.3.6.

Example 1.3.9. *Let $w = 5\,1\,6\,4\,7\,3\,2$. The record positions of w are $1, 3, 5$. We have*

$$(e_1, \dots, e_7) = (0+3,\ 1+0,\ 0+2,\ 1+2,\ 0+0,\ 1+0,\ 2+0).$$

Theorem 1.3.8 says that $P_w(q) = R_w(q) = [4]_q\, [2]_q\, [3]_q\, [4]_q\, [1]_q\, [2]_q\, [3]_q$.

Remark 1.3.10. It was known before that the Poincaré polynomial $P_w(q)$ for smooth w factors as a product of q-numbers $[a]_q$. Gasharov [11] (see Proposition 1.3.28 below) gave a recursive construction for such factorization. On the other hand, Carrell gave a closed non-recursive expression for $P_w(q)$ as a ratio of two polynomials, see [8] and [4, Thm. 11.1.1]. However, it is not immediately clear from that expression that its denominator divides the numerator. One benefit of the formula in Theorem 1.3.8 is that it is non-recursive and it involves no division. Another combinatorial formula for $P_w(q)$ that has these features was given by Billey, see [5] and [4, Thm. 11.1.8].

1.3.2 Weyl groups of type B

Let S_n^B be the Weyl group of type B. It can be realized as the group of all bijections w of the set $[\pm n] = \{-n, \dots, -1, 1, \dots, n\}$ in itself such that

$$w(-a) = -w(a)$$

for all $a \in [\pm n]$, with composition as group operation. The *window notation* of w is $w(1)w(2)\dots w(n)$. S_n^B is called the group of all signed permutations of $[n]$. For simplicity, we will write $\bar{i}$ instead of $-i$ in the window notation. We take the set of generators of S_n^B as $\{\sigma_0^B, \sigma_1^B, \dots, \sigma_{n-1}^B\}$, where σ_i^B is the simple transposition that interchanges $w(i)$ and $w(i+1)$ for $i = 1, \dots, n-1$, and σ_0^B changes the sign of $w(1)$ in the window notation. For example, if $w = \bar{3}2\bar{6}14\bar{5}$ then $w\sigma_2^B = \bar{3}\bar{6}214\bar{5}$ and $w\sigma_0^B = 32\bar{6}14\bar{5}$. Naturally each of the generators corresponds to a simple root of the root system of type B.

There is a criterion for rationally smoothness of Schubert varieties in type B, analogous to the theorem of Lakshmibai and Sandhya. Billey [5] showed that, for $w \in S_n^B$, the Schubert variety X_w is rationally smooth if and only if w avoids the following patterns in its window notation:

$$\begin{array}{ccccccc}
\bar{1}2\bar{3} & 1\bar{2}\bar{3} & 12\bar{3} & 1\bar{3}\bar{2} & \bar{2}\bar{1}\bar{3} & \bar{2}1\bar{3} & 2\bar{1}\bar{3} \\
2\bar{3}\bar{1} & \bar{3}1\bar{2} & \bar{3}\bar{2}\bar{1} & \bar{3}\bar{2}1 & \bar{3}2\bar{1} & 3\bar{2}\bar{1} & 3\bar{2}1 \\
\bar{2}\bar{4}31 & 2\bar{4}31 & \bar{3}\bar{4}\bar{1}\bar{2} & \bar{3}4\bar{1}2 & \bar{3}412 & 34\bar{1}2 & 3412 \\
4\bar{1}3\bar{2} & 413\bar{2} & \bar{4}231 & 423\bar{1} & 4231 & &
\end{array} \tag{1.6}$$

The set of inversions of $w \in S_n^B$ is

$$\{(i,j) \in [n] \times [n] \mid i < j, w(i) > w(j)\} \cup \{(i,j) \in [n] \times [n] \mid i \leq j, w(-i) > w(j)\}$$

Again as in type A, we construct the inversion graph of type B by essentially connecting every inversions with edges. Let us make it more precise.

Definition 1.3.11. The *inversion graph* of $w \in S_n^B$ is the graph G_w^B on the vertex set $[\pm n]$. We attach edges to $\{(i,j) \in [\pm n] \times [\pm n] \mid i < j, i \neq -j \text{ and } w(i) > w(j)\}$, and double edges to $\{(-i, i) \mid i = 1, \dots, n \text{ and } w(-i) > w(i)\}$.

Notice that every edge (i,j) is paired with another edge $(-j,-i)$ in G_w^B. We will always consider this pair of edges together. For instance, the length of $w \in S_n^B$, or equivalently the number of inversions of w is the number of pairs of edges in G_w^B. Also the inversion hyperplanes are attached to each of the pairs:

Definition 1.3.12. For $w \in S_n^B$, *inversion hyperplane arrangement* $\mathcal{A}_w^B$ is the hyperplane arrangement in $\mathbb{R}^n$ with the following hyperplanes:

- $x_i - x_j = 0$ for all pair of edges $\{(-j,-i),(i,j)\}$ of G_w^B where $0 < i < j$,
- $x_{-i} + x_j = 0$ for all pair of edges $\{(-j,-i),(i,j)\}$ of G_w^B where $i < 0 < j$,

Asymmetric orientation of the graph G_w^B is an orientation that the direction of (i,j) and $(-j,-i)$ are the same. The direction of the pair $\{(-j,-i),(i,j)\}$ determines which side a point is in with respect to the hyperplane attached to the pair. The regions of $\mathcal{A}_w^B$ are naturally in bijection with *acyclic asymmetric orientations* of the graph G_w^B.

Let us define the q-analogue of the number of regions of $\mathcal{A}_w^B$ as in type A. Note that $\mathcal{A}_w^B$ is a subarrangement of $\mathcal{A}_{w_0}^B$ where w_0 is the longest element of S_n^B. Clearly $\mathcal{A}_{w_0}^B$ is the Coxeter arrangement of type B, and the regions of $\mathcal{A}_{w_0}^B$ correspond to elements of S_n^B. Let r_0 be the region of $\mathcal{A}_w$ that contains the region of $\mathcal{A}_{w_0}^B$ corresponding to the identity element. Define

$$R_w(q) := \sum_r q^{d(r,r_0)},$$

where the sum is over all regions r of the arrangement $\mathcal{A}_w^B$. Equivalently, the polynomial $R_w(q)$ can be described in terms of acyclic asymmetric orientations of the graph G_w^B. For an acyclic asymmetric orientation $\mathcal{O}$, let $\mathrm{des}^B(\mathcal{O})$ be the number of pairs $\{(-j,-i),(i,j)\}$ oriented as $-j \to -i$ and $i \to j$ in $\mathcal{O}$ where $i > j$ (descent edge pairs). Then

$$R_w(q) = \sum_{\mathcal{O}} q^{\mathrm{des}^B(\mathcal{O})},$$

where the sum is over all acyclic asymmetric orientations $\mathcal{O}$ of G_w^B. Indeed, for the acyclic asymmetric orientation $\mathcal{O}$ associated with a region r we have $\mathrm{des}^B(\mathcal{O}) = d(r, r_0)$.

Our main result on type B is,

Theorem 1.3.13. *For a signed permutation $w \in S_n^B$, we have $P_w(q) = R_w(q)$ if and only if w is rationally smooth, i.e., if and only if w avoids the patterns in (1.6).*

By the same reason as in type A, "only if" part follows from that $P_w(q)$ is palindromic if and only if w is rationally smooth while $R_w(q)$ always is. We will prove "if" part in Section 1.3.5.

1.3.3 Weyl groups of type D

Let S_n^D be the Weyl group of type D. This is just a subgroup of S_n^B consisting of signed permutations having an even number of negative entries in their window notation. As a set of generators of S_n^D, we have $\{\sigma_0^D, \sigma_1^{\cdot} \ldots, \sigma_{n-1}^D\}$. Here $\sigma_i^D = \sigma_i^B$ for all $i = 1, \ldots, n-1$, and σ_0^D sends $w(1)w(2)w(3)\ldots w(n)$ to $(-w(2))(-w(1))w(3)\ldots w(n)$ in the window notation. For example, if $w = \bar{3}2\bar{6}145$ then $w\sigma_0^D = \bar{2}3\bar{6}145$. The generators are in bijection with the simple roots of the root system of type D.

In [5], Billey gives a criterion for rationally smoothness of Schubert varieties of type D as well. She showed that $w \in S_n^D$ is rationally smooth if and only if w avoids the following patterns in its window notation:

$$
\begin{array}{llllll}
12\bar{3} & \bar{1}2\bar{3} & \bar{1}\bar{3}\bar{2} & 1\bar{3}\bar{2} & \bar{2}\bar{1}\bar{3} & \bar{3}\bar{2}\bar{1} \\
\bar{1}4\bar{3}2 & \bar{2}1\bar{3}\bar{4} & 2\bar{1}\bar{3}\bar{4} & 21\bar{3}\bar{4} & \bar{2}\bar{3}1\bar{4} & 2\bar{3}1\bar{4} \\
2\bar{4}31 & \bar{2}\bar{4}3\bar{1} & \bar{2}4\bar{3}\bar{1} & 24\bar{3}\bar{1} & 2\bar{4}3\bar{1} & \bar{2}4\bar{3}1 \\
\bar{2}\bar{4}31 & 3\bar{1}\bar{2}\bar{4} & 31\bar{2}\bar{4} & 3\bar{2}1\bar{4} & 3\bar{2}\bar{4}1 & \bar{3}\bar{4}1\bar{2} \\
3\bar{4}\bar{1}\bar{2} & \bar{3}412 & 34\bar{1}2 & \bar{3}4\bar{1}2 & 3412 & \bar{3}\bar{4}\bar{1}\bar{2} \\
3\bar{4}1\bar{2} & \bar{3}\bar{4}\bar{2}1 & 34\bar{2}\bar{1} & \bar{3}4\bar{2}1 & 3\bar{4}\bar{2}1 & \bar{4}\bar{1}\bar{3}2 \\
413\bar{2} & \bar{4}\bar{1}3\bar{2} & 4\bar{1}3\bar{2} & \bar{4}13\bar{2} & 4\bar{1}\bar{3}2 & \bar{4}1\bar{3}2 \\
4\bar{2}1\bar{3} & 4\bar{2}\bar{3}\bar{1} & \bar{4}\bar{2}\bar{3}1 & \bar{4}231 & 423\bar{1} & \bar{4}23\bar{1} \\
4231 & 4\bar{2}\bar{3}1 & 4\bar{3}\bar{1}\bar{2} & 4\bar{3}\bar{1}2 & \bar{4}\bar{3}12 & 4\bar{3}1\bar{2} \\
4\bar{3}\bar{2}1 & & & & &
\end{array}
\tag{1.7}
$$

The set of inversions of $w \in S_n^D$ is very similar with that of type B, namely

$$\{(i,j) \in [n] \times [n] \mid i < j, w(i) > w(j)\} \cup \{(i,j) \in [n] \times [n] \mid i < j, w(-i) > w(j)\}$$

The inversion graph of type D is almost the same with that of type B, except that we delete all the double edges from G_w^B. More precisely,

Definition 1.3.14. The *inversion graph* of $w \in S_n^D$ is the graph G_w^D on the vertex set $[\pm n]$ with edges $\{(i,j) \in [\pm n] \times [\pm n] \mid i < j, i \neq -j \text{ and } w(i) > w(j)\}$.

Notice that, as in type B, every edge (i,j) is paired with another edge $(-j,-i)$ in G_w^B. The number of inversions of w is the number of pairs of edges in G_w^D. Also the inversion hyperplanes are attached to each of the pairs:

Definition 1.3.15. For $w \in S_n^D$, *inversion hyperplane arrangement* $\mathcal{A}_w^D$ is the hyperplane arrangement in $\mathbb{R}^n$ with the following hyperplanes:

- $x_i - x_j = 0$ for all pair of edges $\{(-j,-i),(i,j)\}$ of G_w^D where $0 < i < j$,
- $x_{-i} + x_j = 0$ for all pair of edges $\{(-j,-i),(i,j)\}$ of G_w^D where $i < 0 < j$,

Given an asymmetric orientation of G_w^D, the direction of the pair $\{(-j,-i),(i,j)\}$ determines which side a point is in with respect to the hyperplane attached to the pair. The regions of $\mathcal{A}_w^D$ are naturally in bijection with *acyclic asymmetric orientations* of the graph G_w^D.

The q-analogue of the number of regions of $\mathcal{A}_w^D$ is also defined in the same way. Let r_0 be the region of $\mathcal{A}_w$ that contains the region of Coxeter arrangement of type D corresponding to the identity element of S_n^D. Define

$$R_w(q) := \sum_r q^{d(r,r_0)},$$

where the sum is over all regions r of the arrangement $\mathcal{A}_w^D$. We can describe $R_w(q)$ in terms of acyclic asymmetric orientations of the graph G_w^D. For an acyclic asymmetric orientation $\mathcal{O}$, let $\text{des}^D(\mathcal{O})$ be the number of pairs $\{(-j,-i),(i,j)\}$ oriented as $-j \to -i$ and $i \to j$ in $\mathcal{O}$ where $i > j$ (descent edge pairs). Then

$$R_w(q) = \sum_{\mathcal{O}} q^{\text{des}^D(\mathcal{O})},$$

where the sum is over all acyclic asymmetric orientations $\mathcal{O}$ of G_w^D. For the acyclic asymmetric orientation $\mathcal{O}$ associated with a region r we have $\text{des}^D(\mathcal{O}) = d(r, r_0)$.

Now, our main result on type D is,

Theorem 1.3.16. *For $w \in S_n^D$, we have $P_w(q) = R_w(q)$ if and only if w is rationally smooth, i.e., if and only if w avoids the patterns in (1.7).*

Again, "only if" part follows directly from Theorem 1.2.1, and we will prove "if" part in Section 1.3.5.

Remark 1.3.17. Type B and D can be unified into a single scheme using *signed graph* of Zaslavsky [30],[29]. In fact, our inversion graph is the modified covering graph of the corresponding signed graph appearing in [30]. The inversion graphs of type B and D can be rewritten in signed graph language, and the inversion hyperplane arrangements are special cases of "signed graphical arrangement". Given a signed graph G, the characteristic polynomial $\chi_G(t)$ of the corresponding signed graphical arrangement is the (signed) chromatic polynomial of G. This approach will give us a parallel story to our results on type A, including direct factorization of $P_w(q)$ for rationally smooth w. In particular it will prove that $P_w(q) = \prod_{i=1}^n [e_i + 1]_q$ where e_i's are the roots of $\chi_{G_w}(t)$ for rationally smooth w. We will develop this story in a separate forthcoming paper.

1.3.4 Chordal graphs and perfect elimination orderings

In this section we review properties of choral graphs. We define "nice perfect elimination ordering" on chordal graphs, and use it to factorize $R_w(q)$ of graphical arrangements.

A graph is called *chordal* if each of its cycles with four or more vertices has a *chord,* which is an edge joining two vertices that are not adjacent in the cycle. A *perfect elimination ordering* in a graph G is an ordering of the vertices of G such that, for each vertex v of G, all the neighbors of v that precede v in the ordering form a clique (i.e., a complete subgraph).

Theorem 1.3.18. (Fulkerson-Gross [9]) *A graph is chordal if and only if it has a perfect elimination ordering.*

It is easy to calculate the chromatic polynomial $\chi_G(t)$ of a chordal graph G. Let us pick a perfect elimination ordering $v_1, \ldots, v_n$ of the vertices of G. For $i = 1, \ldots, n$, let e_i be the number of the neighbors of the vertex v_i among the preceding vertices $v_1, \ldots, v_{i-1}$. The numbers $e_1, \ldots, e_n$ are called the *exponents* of G. The following formula is well-known.

Proposition 1.3.19. *The chromatic polynomial of the chordal graph G equals $\chi_G(t) = (t-e_1)(t-e_2)\cdots(t-e_n)$. Thus the graphical arrangement $\mathcal{A}_G$ has $(-1)^n\chi_G(-1) = (e_1+1)(e_2+1)\cdots(e_n+1)$ regions.*

For completeness, we include the proof, which is also well-known.

Proof. It is enough to prove the formula for a positive integer t. Let us count the number of coloring of vertices of G in t colors. The vertex v_1 can be colored in $t = t - e_1$ colors. Then the vertex v_2 can be colored in $t - e_2$ colors, and so on. The vertex v_i can be colored in $t - e_i$ colors, because the a_i preceding neighbors of v_i already used a_i *different* colors. □

Remark 1.3.20. A chordal graph can have many different perfect elimination orderings that lead to different sequences of exponents. However, the multiset (unordered sequence) $\{e_1, \dots, e_n\}$ of the exponents does not depend on a choice of a perfect elimination order. Indeed, by Proposition 1.3.19, the exponents e_i are the roots of the chromatic polynomial $\chi_G(t)$.

Lemma 1.3.21. (cf. Björner-Edelman-Ziegler [1]) *Suppose that a graph G on the vertex set $\{1, \dots, n\}$ has a vertex v adjacent to m vertices that satisfy the two conditions:*

1. *The set of all neighbors of v is a clique in G.*
2. *(a) All neighbors of v are less than v, or*
   (b) all neighbors of v are greater than v.

Then $R_G(q) = [m+1]_q\, R_{G\setminus v}(q)$, where $G\setminus v$ is the graph G with the vertex v removed.

This claim follows from general results of [1] on supersolvable hyperplanes arrangements. For completeness, we give a simple proof.

Proof. The polynomials $R_G(q)$ and $R_{G\setminus v}(q)$ are des-generating functions for acyclic orientations of the graphs G and $G \setminus v$.

Let us fix an acyclic orientation $\mathcal{O}$ of the graph $G\setminus v$, and count all ways to extend $\mathcal{O}$ to an acyclic orientation of G. The vertex v is connected to a subset S of m vertices of the graph $G \setminus v$, which forms the clique $G|_S \simeq K_m$. Clearly, there are $m+1$ ways to extend an acyclic orientation of the complete graph K_m to an acyclic orientation of K_{m+1}. Moreover, for each $j = 0, \dots, m$, there is a unique extension of $\mathcal{O}$ to an acyclic orientation $\mathcal{O}'$ of G such that there are exactly j edges oriented towards the vertex v in $\mathcal{O}'$ (and $m-j$ edges oriented away from v).

All vertices in S are less than v or all of them are greater than v. In both cases we have $\sum_{\mathcal{O}'} q^{\mathrm{des}(\mathcal{O}')} = [m+1]_q\, q^{\mathrm{des}(\mathcal{O})}$, where the sum is over extensions $\mathcal{O}'$ of $\mathcal{O}$. Thus $R_G(q) = [m+1]_q\, R_{G\setminus v}(q)$. □

Definition 1.3.22. For a chordal graph G on the vertex set $\{1, \dots, n\}$, we say that a perfect elimination ordering $v_1, \dots, v_n$ of the vertices of G is *nice* if it satisfies the following additional property. For $i = 1, \dots, n$, all neighbors of the vertex v_i among the vertices $v_1, \dots, v_{i-1}$ are greater than v_i (in the usual order on $\mathbb{Z}$), or all neighbors of v_i among $v_1, \dots, v_{i-1}$ are less than v_i.

For a nice perfect elimination ordering $v_1, \ldots, v_n$ of G, the last vertex $v = v_n$ satisfies the conditions of Lemma 1.3.21. Moreover, $v_1, \ldots, v_{n-1}$ is a nice perfect elimination ordering of the graph $G \backslash v_n$. In this case, we can inductively use Lemma 1.3.21 to completely factor the polynomial $R_G(q)$ as $R_G(q) = [m+1]_q \, [m'+1]_q \cdots$. The numbers $m, m', \ldots$ are exactly the exponents $e_n, e_{n-1}, \ldots$ (written backwards) coming from this perfect elimination ordering.

Corollary 1.3.23. *Suppose that G has a nice perfect elimination ordering of vertices. Let $e_1, \ldots, e_n$ be the exponents of G. Then we have*

$$R_G(q) = [e_1+1]_q \, [e_2+1]_q \cdots [e_n+1]_q \, .$$

1.3.5 Recurrence for polynomials $R_w(q)$

The main purpose of this section is find a recurrence for $R_w(q)$ which is true for $P_w(q)$ as well. We will begin with analyzing type A.

It is convenient to represent a permutation $w \in S_n$ as the *rook diagram* D_w, which the placement of n non-attacking rooks into the boxes $(w(1), 1), (w(2), 2), \ldots, (w(n), n)$ of the $n \times n$ board. See an example on Figure 1-1. We assume that boxes of the board are labelled by pairs (i, j) in the same way as matrix elements. The rooks are marked by $\times$'s.

	×						
						×	
×							
		×					
							×
					×		
				×			
			×				

Figure 1-1: The rook diagram D_w of the permutation $w = 3\,1\,4\,8\,7\,6\,2\,5$.

The inversion graph G_w contains an edge (i, j), with $i < j$, whenever the rook in the i-th column of D_w is located to the South-West of the rook in the j-th column. In this case, we say that this pair of rooks *forms an inversion.*

Here are the rook diagrams of the two forbidden patterns 3412 and 4231 for smooth permutations:

		×	
			×
×			
	×		

			×
	×		
		×	
×			

A permutation w is smooth if and only if its diagram D_w does not contain four rooks located in the same relative order as in one of these diagrams D_{3412} or D_{4231}.

Let a be the rook located in the last column of D_w, and let b be the rook located in the last row of D_w. The row containing a and the column containing b subdivide the diagram D_w into the four sectors A, B, C, D, as shown on Figure 1-2. In the case when $w(n) = n$, we assume that $a = b$ and the sectors B, C, D are empty.

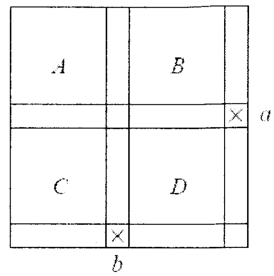

Figure 1-2:

Lemma 1.3.24. *Let w be a smooth permutations. Then its rook diagram D_w has the following two properties. (1) Each pair of rooks located in the sector D forms an inversion. (2) At least one of the sectors B or C contains no rooks.*

For example, for the rook diagram $D_{31487625}$ shown on Figure 1-1, the sector B contains one rook, the sector C contains no rooks, and the sector D contains two rooks that form an inversion.

Proof. (1) If the sector D contains a pair of rooks that do not form an inversion, then these two rooks together with the rooks a and b form a forbidden pattern as in the diagram D_{4231}. (2) If the sector B contains at least one rook and the sector C contains at least one rook, then these two rooks together with the rooks a and b form a forbidden pattern as in the diagram D_{3412}. □

Let $v_a = n$ and v_b be the vertices of the inversion graph G_w corresponding to the rooks a and b. Also let $v_1, \ldots, v_k$ be the vertices of G_w corresponding to the rooks inside the sector D.

If the sector B of the rook diagram D_w is empty, then the vertex v_b is connected only with the vertices $v_1, \ldots, v_k, v_a$, that form a clique in the graph G_w, and all these vertices are greater than v_b. On the other hand, if the sector C of the rook diagram D_w is empty, then the vertex v_a is connected only with the vertices $v_b, v_1, \ldots, v_k$, that form a clique, and all these vertices are less than v_a.

In both cases, the inversion graph G_w satisfies the conditions of Lemma 1.3.21, where $v = v_b$ if B is empty, and $v = v_a$ if C is empty. (If both B and C are empty then we can pick $v = v_a$ or $v = v_b$.)

For $w \in S_n$ and $k \in \{1, \ldots, n\}$, let $w' = \mathrm{flat}(w, k) \in S_{n-1}$ be the *flattening* of the sequence $w(1), \ldots, w(k-1), w(k+1), \ldots, w(n)$, that is, the permutation w' has the same relative order of elements as in this sequence. Equivalently, the rook diagram $D_{w'}$ is obtained from the rook diagram D_w by removing its k-th column and the $w(k)$-th row.

Lemma 1.3.21, together with the above discussion, implies the following recurrence relations for the polynomials $R_w(q)$.

Proposition 1.3.25. *Let $w \in S_n$ be a smooth permutation, and assume that $w(d) = n$ and $w(n) = e$. Then (at least) one of the following two statements is true:*

1. $w(d) > w(d+1) > \cdots > w(n)$, *or*
2. $w^{-1}(e) > w^{-1}(e+1) > \cdots > w^{-1}(n)$.

In both cases, the polynomial $R_w(q)$ factors as

$$R_w(q) = [m+1]_q \, R_{w'}(q),$$

where $w' = \mathrm{flat}(w, d)$ and $m = n - d$ in case (1), or $w' = \mathrm{flat}(w, n)$ and $m = n - e$ in case (2).

In this proposition, case (1) means that the sector B of the rook diagram D_w is empty, and case (2) mean that the sector C is empty.

Clearly, if w is smooth, then the flattening $w' = \mathrm{flat}(w, k)$ is smooth as well. The inversion graph $G_{w'}$ is isomorphic to the graph $G \setminus k$. This means that, for smooth $w \in S_n$, one can inductively use Proposition 1.3.25 to completely factor the polynomial $R_w(q)$ as in Corollary 1.3.23.

Corollary 1.3.26. *For a smooth permutation $w \in S_n$, the inversion graph G_w is chordal and, moreover, it has a nice perfect elimination ordering. We have $R_w(q) = [e_1+1]_q \, [e_2+1]_q \cdots [e_n+1]_q$, where $e_1, \ldots, e_n$ are the exponents of the inversion graph G_w. On the other hand, for any permutation $w \in S_n$, if the inversion graph G_w has a nice perfect elimination ordering, then w is smooth.*

To prove the last claim, note that if G_w has a nice perfect elimination ordering then so does any induced subgraph of G_w. Hence G_w cannot contain G_{3412} and G_{4231} as induced subgraphs, so w is smooth.

Remark 1.3.27. It is not true that G_w is chordal exactly when w is smooth. For instance G_{4231} is chordal.

Interestingly, Gasharov [11] found exactly the same recurrence relations for the Poincaré polynomials $P_w(q)$.

Proposition 1.3.28. (Gasharov [11], cf. Lascoux [16]) *The Poincaré polynomials $P_w(q)$, for smooth permutations w, satisfy exactly the same recurrence relation as in Proposition 1.3.25.*

Note that Lascoux [16] gave a factorization of the Kazhdan-Lusztig basis elements, that implies Proposition 1.3.28.

Propositions 1.3.25 and 1.3.28, together with the trivial claim $P_{\mathrm{id}}(q) = R_{\mathrm{id}}(q) = 1$, imply that $P_w(q) = R_w(q)$ for all smooth permutations w. This finishes the proof of Theorem 1.3.7.

The proofs for types B and D are similar to the proof for type A given above. Billey [5] found analogous set of conditions to Proposition 1.3.25 and Proposition 1.3.28 for types B and D.

Proposition 1.3.29. (Billey [5],Theorem 3.3) *Let $w \in S_n^B$, and assume $w(d) = \pm n$ and $w(n) = \pm e$. Then $P_w(q)$ factors in the form*

$$P_w(q) = P_{w'}(q)[\mu + 1]_q$$

under the following circumstances:

1. *If $w(d) = n$ and $w(d) > w(d+1) > \cdots > w(n)$, then $w' = w\sigma_d^B \ldots \sigma_{n-1}^B$ and $\mu = n - d$.*
2. *If w^{-1} is in the previous situation, then $w' = \sigma_{n-1}^B \ldots \sigma_{e+1}^B \sigma_e^B w$ and $\mu = n - e$.*
3. *If each $w(i)$ is negative and $w(1) > w(2) > \ldots \hat{w(d)} \cdots > w(n)$ (decreasing after removing $w(d)$), then $w' = w\sigma_{d-1}^B \ldots \sigma_1^B \sigma_0^B \sigma_1^B \ldots \sigma_{n-1}^B$ and $\mu = d + n - 1$.*
4. *If w^{-1} is in the previous situation, then $w' = \sigma_{n-1}^B \ldots \sigma_1^B \sigma_0^B \sigma_1^B \ldots \sigma_{e-1}^B w$ and $\mu = e + n - 1$.*
5. *If each $w(i)$ is positive except for $w(d) = \bar{n}$ and $w(1) > w(2) > \cdots > w(d)$, then $w' = w\sigma_{d-1}^B \ldots \sigma_1^B \sigma_0^B$ and $\mu = d$.*

Moreover, if w is rationally smooth then it falls into one of the above circumstances, so $P_w(q)$ factors into q-numbers.

Proposition 1.3.30. (Billey [5],Theorem 6.3) *Let $w \in S_n^D$, and assume $w(d) = \pm n$ and $w(n) = \pm e$. Then $P_w(q)$ factors in the form*

$$P_w(q) = P_{w'}(q)[\mu + 1]_q$$

under the following circumstances:

1. *If $w = w_0$ is the longest element in S_n^D, namely $w = \pm 1\bar{2} \ldots \bar{n}$, then*

$$P_{w_0}(q) = \prod_{k=1}^{n-1} (1 + q + \cdots + q^{k-1} + 2q^k + q^{k+1} + \cdots + q^{2k})$$

2. *If $w(d) = n$ and $w(d) > w(d+1) > \cdots > w(n)$, then $w' = w\sigma_d^D \ldots \sigma_{n-1}^D$ and $\mu = n - d$.*
3. *If w^{-1} is in the previous situation, then $w' = \sigma_{n-1}^D \ldots \sigma_{e+1}^D \sigma_e^D w$ and $\mu = n - e$.*
4. *If $w(1) < 0$ and $w(d) = \bar{n}$ are the only two negatives in the window notation and $-w(1) > w(2) > \cdots > w(d)$, then $w' = w\sigma_{d-1}^D \ldots \sigma_2^D \sigma_0^B$ and $\mu = d - 1$.*
5. *If $\bar{n}$ and $\bar{1}$ are the only two negatives in the window notation and $w(1) > \cdots > w(d)$, then $w' = w\sigma_{d-1}^D \ldots \sigma_2^D \sigma_1^B$ and $\mu = d - 1$.*

Moreover, if w is (rationally) smooth then it falls into one of the above circumstances. Since it is well known that $P_{w_0}(q)$ factors into q-numbers, $P_w(q)$ factors into q-numbers for all (rationally) smooth w.

Using these two propositions, we can show that $R_w(q)$ factors in the same way under each of the circumstances. This finishes the proof of Theorem 1.3.13 and Theorem 1.3.16. Let us begin with the lemma that follows from the proof of Lemma 1.3.21.

Lemma 1.3.31. *Let K_{m+1} be the complete graph on the vertex set $[m+1]$, and K_m be the complete graph on the vertex set $[m]$ naturally embedded in K_{m+1}. Given an acyclic orientation $\mathcal{O}'$ of K_m and $k = 0, \cdots, m$, there is a unique acyclic orientation $\mathcal{O}_k$ of K_{m+1} that extends $\mathcal{O}'$ and $\operatorname{des}(\mathcal{O}_k) - \operatorname{des}(\mathcal{O}') = k$. Moreover each $\mathcal{O}_k$ is obtained by flipping the orientation of an edge from $\mathcal{O}_{k-1}$.*

Factorization of $R_w(q)$ for type B Let w be a rationally smooth element and w' be the one as in Proposition 1.3.29. In all cases, it is clear that any descent of w' is a descent of w. Therefore $G^B_{w'}$ is a subgraph of G^B_w. Let $\mathcal{O}'$ be an acyclic asymmetric orientation of $G^B_{w'}$. To prove that $R_w(q)$ follows the same factorization rule as $P_w(q)$, it is enough to show that there is a unique acyclic asymmetric orientation $\mathcal{O}_k$ of G^B_w that extends $\mathcal{O}'$ and $\operatorname{des}(\mathcal{O}_k) - \operatorname{des}(\mathcal{O}') = k$ for any $k = 0, \cdots, \mu$ in such a way that $\mathcal{O}_k$ is obtained by flipping the orientation of a pair $\{(-j,-i),(i,j)\}$. This follows from a careful analysis of all cases of Proposition 1.3.29:

1. In the first case, the additional edges of G^B_w compared to $G^B_{w'}$ are those that are connected to n and $-n$, i.e. $\{(n, w(d+1)), \cdots, (n, w(n)), (-n, -w(d+1)), \cdots, (-n, -w(n))\}$. Observe that the neighbor of n (resp., $-n$) forms a clique of size $n-d$ in G^B_w. Hence the claim follows from Lemma 1.3.31.

2. The second case follows from the first case using w^{-1} instead of w.

3. In the third case, the additional edges of G^B_w compared to $G^B_{w'}$ are those that are connected to n and $-n$. Moreover, $G^B_{w'}$ is the inversion graph of the longest element in S^B_{n-1}. Hence, choosing an acyclic asymmetric orientation $\mathcal{O}'$ of $G^B_{w'}$ corresponds to choosing an element of S^B_{n-1}, namely $v = v(1)\cdots v(n-1)$ in the window notation. If we rearrange the vertices of $G^B_{w'}$ in the order

$$-n, -v(n-1), \cdots, -v(1), v(1), \cdots, v(n-1), n$$

then all the edges connecting $\pm v(i)$ and $\pm v(j)$ are oriented from left to right. Now the neighbor of n (resp., $-n$) except $-n$ (resp., n) forms a clique of size $n-d-2$ directed from left to right. Therefore given an acyclic asymmetric orientation $\mathcal{O}'$ of $G^B_{w'}$, the unique $\mathcal{O}_k$ of G^B_w is constructed in the following way:

   (a) when $k = 0$, direct all pairs of edges $\{(-n,-i),(i,n)\}$ so that $(-n) \to i$ and $i \to n$.

   (b) if the k outermost pairs of edges don't share any vertex except n and $-n$, then direct $(-i) \to (-n), n \to i$ for those k outermost pairs, $(-n) \to (-i), i \to n$ for all the others.

   (c) otherwise, direct outermost $(k-1)$th pairs of edges so that $(-i) \to (-n)$ and $n \to i$, direct $n \to (-n)$, and direct $(-n) \to (-i), i \to n$ for all the others.

By Lemma 1.3.31 it is easy to see that $\mathcal{O}_k$ is unique. We are done.

4. The fourth case follows from the third case using w^{-1} instead of w.

5. In the fifth case, the additional edges of G_w^B compared to $G_{w'}^B$ are those that connect n with $-w(1), \cdots, -w(d-1)$, that connect $-n$ with $w(1), \cdots, w(d-1)$, and that connects $-n$ with n. Observe that the new edges connected to n (resp., $-n$) are connected to a clique of size d. Therefore the claim directly follows from Lemma 1.3.31.

Factorization of $R_w(q)$ for type D The setting is the same as in type B. Again, let w be a rationally smooth element and w' be the one as in Proposition 1.3.30. In all cases, it is clear that any descent of w' is a descent of w. Therefore $G_{w'}^D$ is a subgraph of G_w^D. Let $\mathcal{O}'$ be an acyclic asymmetric orientation of $G_{w'}^D$. To prove that $R_w(q)$ follows the same factorization rule as $P_w(q)$, it is enough to show that there is a unique acyclic asymmetric orientation $\mathcal{O}_k$ of G_w^D that extends $\mathcal{O}'$ and $\mathrm{des}(\mathcal{O}_k) - \mathrm{des}(\mathcal{O}') = k$ for any $k = 0, \cdots, \mu$ in such a way that $\mathcal{O}_k$ is obtained by flipping the orientation of a pair $\{(-j, -i), (i, j)\}$. This follows from a careful analysis of all cases of Proposition 1.3.30:

1. For the longest element, $R_{w_0}(q) = P_{w_0}(q)$ since the weak and strong Bruhat posets have the same Poincaré polynomials.

2. The second case is the same as in the first case of type B.

3. The third case follows from the second case using w^{-1} instead of w.

4. As in the fifth case of type B, additional edges of G_w^D compared with $G_{w'}^D$ are connected to n and $-n$, and those connected to n (resp., $-n$) form a clique of size $d-1$. Hence the claim follows from Lemma 1.3.31.

5. Similarly, the additional edges are connected to n and $-n$, and those connected to n (resp., $-n$) form a clique of size $d-1$. The claim follows from Lemma 1.3.31.

1.3.6 Simple perfect elimination ordering

The purpose of this section is to find a non-recursive factorization of $P_w(q)$ for smooth $w \in S_n$.

In Section 1.3.5 we gave a recursive construction for a nice perfect elimination ordering of the graph G_w, for smooth w. In this section we give a simple non-recursive construction for another perfect elimination ordering of G_w. This simple ordering may not be nice (see Definition 1.3.22). However, one still can use it for calculating the exponents of the graph G_w and factorizing the polynomials $P_w(q) = R_w(q)$ as in Corollary 1.3.26. Indeed, the multiset of the exponents does not depend on a choice of a perfect elimination ordering (see Remark 1.3.20).

Recall that a *record position* of a permutation $w \in S_n$ is an index $r \in \{1, \ldots, n\}$ such that $w(r) > \max(w(1), \ldots, w(r-1))$. Let $[a, b]$ denote the interval $\{a, a+1, \ldots, b\}$ with the usual $\mathbb{Z}$-order of entries.

Lemma 1.3.32. *For a smooth permutation $w \in S_n$ with record positions $r_1 = 1 < r_2 < \cdots < r_s$, the ordering*

$$[r_s, n],\ [r_{s-1}, r_s - 1], \ldots, [r_2, r_3 - 1],\ [r_1, r_2 - 1]$$

of the set $\{1, \ldots, n\}$ is a perfect elimination ordering of the inversion graph G_w.

Example 1.3.33. *(cf. Example 1.3.9) The permutation $w = 5\,1\,6\,4\,7\,3\,2$ has records* $5, 6, 7$ *and record positions* $1, 3, 5$. *Lemma 1.3.32 says that the ordering* $\underline{5,\ 6,\ 7},\ \underline{3,\ 4},\ \underline{1,\ 2}$ *is a perfect elimination ordering of the inversion graph G_w. Figure 1-3 displays this inversion graph G_w. For each vertex $i = 1, \ldots, 7$ of G_w, we wrote i inside a circle and $w(i)$ below it. The exponents of this graph (i.e., the numbers of edges going to the left from the vertices) are* $0, 1, 2, 2, 3, 3, 1$.

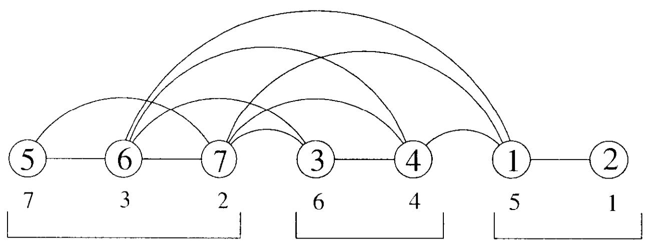

Figure 1-3:

Proof of Lemma 1.3.32. Suppose that this ordering of vertices of G_w is not a perfect elimination ordering. This means that there is a vertex i connected in G_w with vertices j and k, preceding i in the order, such that the vertices j and k are not connected by an edge in G_w. Let us consider three cases.

I. The vertices i, j, k belong to the same interval $I_p := [r_p, r_{p+1} - 1]$, for some $p \in \{1, \ldots, s\}$. (Here we assume that $r_{s+1} = n + 1$.) We have $k < j < i$ and $w(k) > w(i)$, $w(j) > w(i)$, but $w(k) < w(j)$, because (k, i) and (j, k) are edges of G_w but (k, j) is not an edge. The value $w(r_p)$ is the maximal value of w on the interval I_p. Since $w(k) < w(j)$ is not the maximal value of w on I_p, we have $r_p \neq k$ and so $r_p < k$. Thus $r_p < k < j < i$ and the values $w(r_p), w(k), w(j), w(i)$ form a forbidden 4231 pattern in w. So w is not smooth. Contradiction.

II. The vertices i, j are in the same interval I_p and the vertex k belongs to a different interval I_q. Then $q > p$, because the vertex k precedes i in the order. In this case we have $j < i < k$, $w(j) > w(i)$, $w(i) > w(k)$. This implies that $w(j) > w(k)$ that is (j, k) is an edge in the inversion graph G_w. Contradiction.

III. The vertex i belongs to the interval I_p and the vertices j, k do not belong to I_p. Assume that $j < k$ and that j belongs to I_q. Then $q > p$. In this case, $i < j < k$, $w(i) > w(j)$, $w(i) > w(k)$, and $w(j) < w(k)$. The record value $w(r_q)$ is greater than $w(i)$. This implies that $w(r_q) > w(i) > w(j)$. In particular, $w(r_q) \neq w(j)$ and, thus, $r_q \neq j$. We have $i < r_q < j < k$ and the values $w(i), w(r_q), w(j), w(k)$ form a forbidden 3412 pattern. Contradiction. $\square$

Proof of Theorem 1.3.8. Let us calculate the exponents of the inversion graph G_w for a smooth permutation $w \in S_n$ using the perfect elimination ordering from Lemma 1.3.32. Suppose that $i \in I_p$. Then the exponent e_i of the vertex i equals the number of neighbors of the vertex i in the graph G_w among the preceding vertices, that is among the vertices in the sets $\{r_p, \ldots, i-1\}$ and $I_{p+1} \cup I_{p+2} \cup \ldots$. In other words, the exponent e_i equals

$$\#\{j \mid r_p \leq j < i,\ w(j) > w(i)\} + \#\{k \mid k \geq r_{p+1},\ w(k) < w(i)\}.$$

This is exactly the expression for e_i from Theorem 1.3.8. The result follows from Corollary 1.3.26. □

1.4 Inversion Hyperplane arrangements and chamber graphs

In this section we define inversion hyperplane arrangements $\mathcal{A}_w$ for every Weyl group. There is no such interpretation as signed permutations for non-classical Weyl groups, hence we need to develop another approach to generalize our main result to other types. We will define chamber graph which will be our main tool to study $R_w(q)$. We follow the notations in Section 1.2

To assign the inversion hyperplanes to each $w \in W$, we first need the definition of the inversion set of w. The inversion set Δ_w of w is defined as the following:

$$\Delta_w := \{\alpha | \alpha \in \Delta, \alpha > 0, w(\alpha) < 0\}.$$

For classical types, this gives the usual definition of an inversion set. The *inversion hyperplane arrangement* $\mathcal{A}_w$ is the collection of hyperplanes $\alpha(x) = 0$ for all roots $\alpha \in \Delta_w$. Let r_0 be the *fundamental chamber* of $\mathcal{A}_w$, the chamber that contains the points satisfying $\alpha(x) > 0$ for all $\alpha \in \Delta_w$. Then we can define the following polynomial:

$$R_w(q) := \sum_r q^{d(r_0,r)},$$

where the sum is over all chambers of the arrangement $\mathcal{A}_w$ and $d(r_0, r)$ is the number of hyperplanes separating r_0 and r.

Given a subarrangement $\mathcal{A}$ of the Coxeter arrangement, and its subarrangement $\mathcal{A}'$, let c be a chamber of $\mathcal{A}'$. Then a *chamber graph* of c with respect to $\mathcal{A}$ is defined as a directed graph $G_{c,\mathcal{A}} = (V, E)$ where

- The vertex set V consists of vertices representing each chambers of $\mathcal{A}$ contained in c,
- we have an edge directed from vertex representing chamber c_1 to a vertex representing chamber c_2 if c_1 and c_2 are adjacent and $d(r_0, c_1) + 1 = d(r_0, c_2)$.

The chamber graph can be seen as a Hasse diagram of a poset of chambers in c, ranked by the distance from r_0, and directed from lower to higher rank elements. Let us write the Poincaré polynomial of the chamber graph $G_{c,\mathcal{A}}$ as $R^c_{\mathcal{A}}(q)$. We will say that $\mathcal{A}$ *uniformly divides* $\mathcal{A}'$ if every chamber graphs with respect to $\mathcal{A}_w$ are isomorphic in all chambers of $\mathcal{A}'$. The following lemma shows how the chamber graph can be used to analyze $R_w(q)$.

Lemma 1.4.1. *If $\mathcal{A}_u$ is a subarrangement of $\mathcal{A}_w$ and $\mathcal{A}_w$ uniformly divides $\mathcal{A}_u$, then $R_w(q)$ is divided by $R_u(q)$, and $R_w(q) = R_u(q)R^u_w(q)$. Here $R^u_w(q) = R^c_{\mathcal{A}_w}(q)$ for some chamber c of $\mathcal{A}_u$.*

Proof. Straightforward and omitted. □

1.5 Parabolic decomposition

In this section, we introduce a theorem of Billey and Postnikov [6] regarding parabolic decomposition that will serve as a key tool to study $P_w(q)$. We follow the notations in the previous section.

Let's first recall the definition of the parabolic decomposition. Given a Weyl group W and $J \subset S$, denote W_J to be the subgroup of the Weyl group generated by J. Subgroups of this form are called *parabolic*. Let W^J be the set of minimal length coset representatives of $W_J \backslash W$. Then it is a well-known fact that every $w \in W$ has a unique parabolic decomposition $w = uv$ where $u \in W_J, v \in W^J$ and $\ell(w) = \ell(u) + \ell(v)$. Also,

Lemma 1.5.1. *[33] For any $w \in W$ and subset J of simple roots, W_J has a unique maximal element below w.*

We will denote the maximal element of W_J below w as $m(w, J)$. The following theorem of Billey and Postnikov allows us to factor the Poincaré polynomials.

Theorem 1.5.2. *[6] Let J be any subset of simple roots. Assume $w \in W$ has parabolic decomposition $w = uv$ with $u \in W_J$ and $v \in W^J$ and furthermore, $u = m(w, J)$. Then*

$$P_w(t) = P_u(t)P^{W^J}_v(t)$$

where $P^{W^J}_v = \sum_{z \in W^J, z \leq v} t^{\ell(z)}$ is the Poincaré polynomial for v in the quotient.

We will say that $J = \Pi \setminus \{\alpha\}$ is *leaf-removed* if α corresponds to a leaf in the Dynkin diagram of Π. It is enough to look at leaf-removed parabolic subgroups for our purpose.

Theorem 1.5.3. *[6] Let $w \in W$ be a rationally smooth element. Then there exists a maximal proper subset $J = \Pi \setminus \{\alpha\}$ of simple roots, such that*

1. *we have a decomposition of w or w^{-1} as in Theorem 1.5.2,*

2. *α corresponds to a leaf in the Dynkin diagram of Π.*

We will call the parabolic decompositions that satisfies the conditions of the above theorem as *BP-decompositions*. For Weyl groups of type A,B and D, there is a stronger result of Billey:

Lemma 1.5.4 ([5]). *Let W be a Weyl group of type A,B or D. Let $w \in W$ be a rationally smooth element. When w is not the longest element of W, there exists a BP-decomposition of w or w^{-1} with respect to J such that $P^J(v)$ is of the form $q^l + q^{l-1} + \cdots + q + 1$, where l is the length of v.*

If v satisfies the conditions of the above lemma, we will say that v is a *chain element* of W^J. Let I be the set of simple reflections appearing in v. When the root system is simply-laced and v is a chain element of W^J, I corresponds to a type A Dynkin sub-diagram. And in this case, it is clear that v is the longest element of $W_I^{I \cap J}$. Hence we get the following corollary:

Corollary 1.5.5. *Let W be a Weyl group of type A or D. If $w \in W$ is rationally smooth then either w is the longest element or there exists a BP-decomposition of w or w^{-1} with respect to $J = \Pi \setminus \{\alpha\}$ such that v is the longest element of $W_I^{I \cap J}$ where I is the set of simple reflections appearing in v.*

We were also able to check using computer the following property of palindromic intervals in quotient Bruhat poset of type E.

Proposition 1.5.6. *Let W be a Weyl group of type E and let J be a leaf-removed subset of S and let $w = uv$ be a parabolic decomposition. Then v has a palindromic lower interval in W^J if and only if v is the longest element in $W_I^{I \cap J}$ where I is the set of simple reflections appearing in v.*

In fact, we believe that the above proposition is true for all simply-laced types. Let's look at an example. Choose D_6 to be our choice of Weyl group and label the simple roots $\Pi = \{\alpha_1, \cdots, \alpha_6\}$ so that the labels match the corresponding nodes of the Dynkin diagram in Figure 1-4. If we set $J = \Pi \setminus \{\alpha_1\}$, then the list of $v \in W^J$ such that the lower interval in W^J being palindromic is:

$$id, s_1, s_1s_2, s_1s_2s_3, s_1s_2s_3s_4, s_1s_2s_3s_4s_5, s_1s_2s_3s_4s_6, s_1s_2s_3s_4s_5s_6s_4s_3s_2s_1.$$

Each of them are the longest elements of $W_I^{I \cap J}$, where I is the set of simple reflections appearing in v. One can see that the set of nodes I is connected inside the Dynkin diagram of D_6.

Now we will study how $R_w(q)$ behaves with respect to the BP-decomposition. Following the above notations, our first step is to prove that every reflection formed by simple reflections in $I \cap J$ is in $T_R(u)$ when $w = uv$ is a BP-decomposition.

Lemma 1.5.7. *Let $w \in W$ be a rationally smooth element and $w = uv$ be a BP-decomposition. Then every simple reflection in J appearing in the reduced word of v is a right descent of u.*

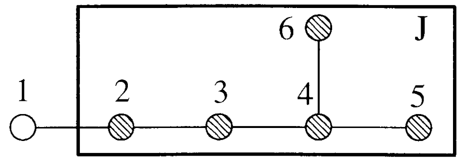

Figure 1-4: Dynkin diagram of D_6 and $J = \Pi \setminus \{\alpha_1\}$

Proof. Multiplying $t \in T_L(w)$ to w corresponds to deleting one simple reflection in a certain reduced word of w. If we delete every simple reflection appearing in v but one in J, then the resulting element is in W_J and is below w. Hence by maximality of u, it is below u. □

Actually we can state much more about u in terms of simple reflections of J appearing in v.

Lemma 1.5.8. *Let $w = uv$ be a BP-decomposition with respect to J. Let I be the subset of S that appears in the reduced word of v. Then every reflection formed by simple reflections in $I \cap J$ is a right inversion reflection of u. In fact, there is a minimal length decomposition $u = u'u_{I\cap J}$ where $u_{I\cap J}$ is the longest element of $W_{I\cap J}$.*

Proof. Take the parabolic decomposition of u under the right quotient by $W_{I\cap J}$. Say, $u = u'u_{I\cap J}$. Then u' is the minimal length representative of u in $W/W_{I\cap J}$. For any simple reflection $s \in I \cap J$, the minimal length representative of us in $W/W_{I\cap J}$ is still u', hence the parabolic decomposition of us is $us = u'(u_{I\cap J}s)$. Since s is a right descent of u by Lemma 1.5.7, s is a right descent of $u_{I\cap J}$. Therefore $u_{I\cap J}$ is the longest element in $W_{I\cap J}$. The rest follows from this. □

The above lemma tells us that for each rationally smooth $w \in W$, we can decompose w or w^{-1} to $u'u_{I\cap J}v$ where uv is the BP-decomposition with respect to J, $u = u'u_{I\cap J}$ and $u_{I\cap J}$ is the longest element of $W_{I\cap J}$. We have a decomposition

$$\Delta_w = \Delta_{u'} \sqcup u' \cdot \Delta_{u_{I\cap J}} \sqcup u \cdot \Delta_v.$$

Recall that we denote by Δ_w the inversion set of $w \in W$. For $I \subset S$, we will denote Δ_I the set of roots of W_I. One can see that $\Delta_{u_{I\cap J}} = \Delta_{I\cap J}$ and $\Delta_v \subseteq \Delta_I \setminus \Delta_{I\cap J}$. And this tells us that $u' \cdot \Delta_{u_{I\cap J}} = u \cdot \Delta_{I\cap J}$. Let us assume we have decomposed some rationally smooth w as above. Let $\mathcal{A}_1, \mathcal{A}_0, \mathcal{A}_2$ denote the hyperplane arrangements coming from $u^{-1} \cdot \Delta_{u'}, \Delta_{I\cap J}, \Delta_v$. We can study $\mathcal{A} := \mathcal{A}_1 \sqcup \mathcal{A}_0 \sqcup \mathcal{A}_2$ instead of looking at $\mathcal{A}_w$.

Lemma 1.5.9. *Let c be some chamber inside $\mathcal{A}_1 \sqcup \mathcal{A}_0$. Let c' be the chamber of $\mathcal{A}_0$ that contains c. Then the chamber graph $G_{c\mathcal{A}}$ is isomorphic to the chamber graph of $G_{c',\mathcal{A}_0 \sqcup \mathcal{A}_2}$.*

Proof. Let c_1 and c_2 be two different chambers of $\mathcal{A}$ contained in c. They are separated by a hyperplane in $\mathcal{A}_2$. Let c_1' (resp., c_2') be the chamber of $\mathcal{A}_0 \sqcup \mathcal{A}_2$ that contains c_1

(resp., c_2). c_1' and c_2' are different chambers since they are separated by the hyperplane that separates c_1 and c_2. If c_1 and c_2 are adjacent, then c_1' and c_2' are adjacent. If c_1' and c_2' are adjacent but c_1 and c_2 are not, that means there is a hyperplane of $\mathcal{A}_1$ that separates c_1 and c_2. But that contradicts the fact that c_1 and c_2 are both contained in the same chamber of $\mathcal{A}_1 \sqcup \mathcal{A}_0$. So c_1 and c_2 are adjacent if and only if c_1' and c_2' are. From the fact that the distance from the fundamental chamber is equal to the number of hyperplanes that separate the chamber from the fundamental chamber, we see that the direction of the corresponding edges in the chamber graphs are the same.

Hence it is enough to show that the number of chambers of $\mathcal{A}$ in c equals number of chambers of $\mathcal{A}_0 \sqcup \mathcal{A}_2$ in c'. And this follows from that any chamber of $\mathcal{A}_0 \sqcup \mathcal{A}_2$ shares a common interior point with a chamber of $\mathcal{A}_0 \sqcup \mathcal{A}_1$ as long as they are contained in the same chamber of $\mathcal{A}_0$. To show this, we can safely include additional hyperplanes to $\mathcal{A}_1$ and $\mathcal{A}_2$. So let $\mathcal{A}_2$ be the hyperplane arrangement of $\Delta_I \setminus \Delta_{I\cap J}$ and $\mathcal{A}_1$ hyperplane arrangement of $\Delta \setminus \Delta_I$. Now $\mathcal{A}$ is just the Coxeter arrangement of W, and each chamber of $\mathcal{A}$ is indexed by $w \in W$.

We have a parabolic decomposition of W by $W_{I\cap J} W_I^{I\cap J} W^I$. Fixing a chamber c of $\mathcal{A}_0$ corresponds to fixing an element of $W_{I\cap J}$. In c, fixing a chamber x (resp., y) of $\mathcal{A}_0 \sqcup \mathcal{A}_2$ (resp., $\mathcal{A}_0 \sqcup \mathcal{A}_1$) corresponds to fixing an element of $W_I^{I\cap J}$ (W^I). So given any such chamber x and y, we can find a chamber of $\mathcal{A}$ contained in them. This concludes the argument.

□

Corollary 1.5.10. *Let $w = uv = u'u_{I\cap J}v$ be the decomposition as in the above. If v is the longest element in $W_I^{I\cap J}$ where I is the set of simple reflections appearing in v, then $P_u(q) = R_u(q)$ implies $P_w(q) = R_w(q)$.*

Proof. Since v is the longest element of $W_I^{I\cap J}$, $w' := u_{I\cap J}v$ is the longest element of W_I. Then it is obvious that $\mathcal{A}_{w'}$ uniformly divides $\mathcal{A}_{u_{I\cap J}}$. Now it follows from Lemma 1.5.9 that $\frac{R_{w'}(q)}{R_{u_{I\cap J}}(q)} = \frac{R_w(q)}{R_u(q)}$. By the induction on the rank of W, we also know that the left hand side equals $P_v^{W^J}(q)$, hence $R_u(q) = P_u(q)$ implies $R_w(q) = P_u(q)P_v^{W^J}(q) = P_w(q)$. □

In the next section, we will use the above lemma and corollary to prove the main theorem for type A,B,D and E.

1.6 Proof of the main theorem for Weyl groups

In this section, we prove the main theorem. Type F and G cases are very small and checked by a computer.

Proposition 1.6.1. *Let W be a Weyl group of type A, D or E. Let w be a rationally smooth element. Then $R_w(q) = P_w(q)$.*

Proof. Decompose w or w^{-1} as in the Corollary 1.5.5 or Proposition 1.5.6. We see that v is the longest element of $W_I^{I\cap J}$. Now we can apply Corollary 1.5.10. So we can replace w with some rationally smooth u that is contained in some Weyl group of type A or D with strictly smaller rank. Now the result follows from an obvious induction argument. $\square$

For type B, we will use Lemma 1.5.4 and Lemma 1.5.9. Let us denote $\Pi = \{\alpha_0 = x_1, \alpha_1 = x_1 - x_2, \cdots, \alpha_n = x_{n-1} - x_n\}$. We will study $W^{\Pi\setminus\{\alpha_0\}}$ and $W^{\Pi\setminus\{\alpha_n\}}$. In both of them, if there is an adjacent commuting letters in the reduced word of v, then v is not a chain element. So when $J = \Pi \setminus \{\alpha_0\}$, the chain elements are

$$id, s_0, s_0s_1, s_0s_1s_2, \cdots, s_0s_1\cdots s_n, s_0s_1s_0.$$

And when $J = \Pi \setminus \{\alpha_n\}$, the chain elements are

$$id, s_n, s_ns_{n-1}, \cdots, s_ns_{n-1}\cdots s_1s_0, s_ns_{n-1}\cdots s_1s_0s_1, \cdots, s_ns_{n-1}\cdots s_1s_0s_1\cdots s_{n-1}s_n.$$

Proposition 1.6.2. *Let W be a Weyl group of type B. Let w be a rationally smooth element. Then $R_w(q) = P_w(q)$.*

Proof. By Lemma 1.5.4, we may assume w or w^{-1} decomposes to uv where $u \in W_J$, $v \in W^J$, J is leaf-removed and v is a chain-element. Using $P_w(q) = P_{w^{-1}}(q)$ and $R_w(q) = R_{w^{-1}}(q)$, we can assume that $w = uv$. Let us first show that when u is the longest element of W_J, $\mathcal{A}_w$ uniformly divides $\mathcal{A}_u$, and each chamber graph $G_{c,\mathcal{A}_w}$ for any chamber of $\mathcal{A}_u$ are chain graph with length $\ell(v)$, hence $R_w(q) = P_w(q)$. Instead of looking at hyperplane arrangement coming from $\Delta_w = \Delta_u \sqcup u \cdot \Delta_v$, we can look at the hyperplane arrangement coming from $u^{-1} \cdot \Delta_w = \Delta_u \sqcup \Delta_v$. So $\mathcal{A}_u$ consists of hyperplanes coming from Δ_u and $\mathcal{A}_v$ consists of hyperplanes coming from Δ_v.

When $J = \Pi \setminus \{\alpha_0\}$, we have $\Delta_v \subset \{x_1, \cdots, x_n\}$ and $|\Delta_v| = \ell(v)$. Choosing a chamber in $\mathcal{A}_u$ is equivalent to giving a total ordering on $\{x_1, \cdots, x_n\}$. Choosing a chamber in $\mathcal{A}_v$ is equivalent to assigning signs to roots of Δ_v. Given any total ordering on $\{x_1, \cdots, x_n\}$, there is a unique way to assign t number of "+"s and $|v| - t$ number of "−"s to Δ_v so that it is compatible with the total order on Δ_v. This tells us that $\mathcal{A}_w$ uniformly divides $\mathcal{A}_u$ and $R_w(q) = R_u(q)(1+q+\cdots+q^{|v|}) = R_u(q)P_v^{W^J}(q)$. When $J = \Pi \setminus \{\alpha_n\}$, the proof is pretty much similar and is omitted.

Now let's return to the general case. Using Lemma 1.5.9 and the above argument, we can replace w with some rationally smooth u that is contained in some Weyl group of type A or B with strictly smaller rank. Then the result follows from an obvious induction argument.

$\square$

1.7 Final remarks

Combining the results in the previous sections, we get

Theorem 1.7.1. *For any finite Weyl group W and $w \in W$ such that X_w is rationally smooth, $P_w(q) = R_w(q)$.*

Our proof of the theorem is based on a recurrence relation. It would be interesting to give a proof based on a bijection between elements of the Bruhat interval $[id, w]$ and regions of the arrangement $\mathcal{A}_w$.

It would be interesting to better understand the relationship between Bruhat intervals $[id, w]$ and the hyperplane arrangement $\mathcal{A}_w$. One can construct a directed graph Γ_w on the regions of $\mathcal{A}_w$. Two regions r and r' are connected by a directed edge (r, r') if these two regions are adjacent (i.e., separated by a single hyperplane) and r is more close to r_0 than r'. For example, for the longest element w_0, the graph Γ_{w_0} is the Hasse diagram of the weak Bruhat order. Hence our result is saying that there is a rank preserving bijection between the lower interval $[id, w]$ of the strong Bruhat order and the elements of certain contraction of the weak Bruhat poset.

The statement of our main theorem can be extended to other Coxeter groups. Although we don't have Schubert varieties for general Coxeter groups, the Poincaré polynomial $P_w(q)$ can still be defined as the rank generating function of the interval $[id, w]$. The first class of such Coxeter groups is the class of finite Coxeter groups. They are finite reflection groups, hence naturally we can associate them with hyperplane arrangements. Inversion hyperplanes are also defined analogously. The second class is the Coxeter groups of crystallographic root systems. There we have the notion of roots, hence the inversion hyperplane arrangements are well defined. Furthermore there are Schubert varieties associated to each element of crystallographic Coxeter groups, and Carrell-Peterson theorem applies in this case.

Conjecture 1.7.2. *Let W be a finite or crystallographic Coxeter group. Then $[id, w]$ is palindromic if and only if $P_w(q) = R_w(q)$.*

It would be also interesting to generalize the notion of inversion hyperplanes to other Coxeter groups than the ones in conjecture.

By Theorem 1.3.26 and Proposition 1.3.19, we see that, when w is a smooth permutation, $R_w(q)$ factors into a product of q-numbers $[e_i+1]_q$'s where e_i's are roots of the characteristic polynomial of $\mathcal{A}_w$. We conjecture that this is true for other types as well. In fact, if w_0 is the longest element of a Weyl group, the conjecture follows from a well-known factorization of the Poincaré polynomial.

Conjecture 1.7.3. *Let W be a Weyl group of rank n, and $w \in W$. If w is rationally smooth, then the characteristic polynomial $\chi_w(t)$ of $\mathcal{A}_w$ factors over nonnegative integers, and $R_w(q) = P_w(q) = \prod_{i=1}^{n}[e_i+1]_q$ where e_i's are the roots of $\chi_w(t)$.*

This conjecture has an interesting interpretation. Given a hyperplane arrangement $\mathcal{A}$ of rank n, let $C_{\mathcal{A}}$ be the complement of the complexified arrangement $\mathcal{A} \otimes \mathbb{C}$, i.e. $C_{\mathcal{A}} = \mathbb{C}^n \setminus (\cup_{H\in\mathcal{A}} H \otimes \mathbb{C})$. Let $Poin_{\mathcal{A}}(t)$ be the Poincaré polynomial of $C_{\mathcal{A}}$. Then by the fundamental result of Orlik and Solomon [21], we know the relation between the characteristic polynomial $\chi_{\mathcal{A}}(t)$ and $Poin_{\mathcal{A}}(t)$, namely $\chi_{\mathcal{A}}(t) = t^n Poin_{\mathcal{A}}(-t^{-1})$. Therefore, combined with Conjecture 1.7.3 and our main result, we get:

Conjecture 1.7.4 (Another interpretation of Conjecture 1.7.3). *If W is a Weyl group of rank n and $w \in W$ is rationally smooth, then*

$$Poin_{\mathcal{A}_w}(t) = \prod_{i=1}^{n}(1 + e_i t)$$

$$P_w(q) = \prod_{i=1}^{n}(1 + [e_i]_q q)$$

where e_i's are the roots of the characteristic polynomial $\chi_w(t)$.

Note that there is a similarity between $Poin_{\mathcal{A}_w}(t)$ and $P_w(q)$. It would be interesting to find a deformation of Orlik-Solomon algebra of $\mathcal{A}_w$ in which the polynomial $P_w(q;t) := \prod_{i=1}^{n}(1 + [e_i]_q t)$ can be realized.

Our second proof for the Weyl group case relied heavily on Theorem 1.5.3 and Proposition 1.5.6. Described a bit roughly, the former helps us to find the recurrence for $P_w(q)$ and the latter helps us to find the recurrence for $R_w(q)$. So the key would be to extending these two statements. For Proposition 1.5.6, it is easy to see that one direction holds for all Weyl groups. We give a slightly weakened statement that seems to hold for all Weyl groups.

Conjecture 1.7.5. *Let W be a Weyl group and let J be a maximal proper subset of the simple roots. Then, v has palindromic lower interval in W^J if and only if the interval is isomorphic to a maximal parabolic quotient of some Weyl group.*

Chapter 2

Specht modules and polytopes

2.1 Introduction

The main purpose of this chapter is to connect Specht modules of general diagrams to other combinatorial objects such as polytopes. In particular, we study two families of diagrams generalizing toric staircase shapes or equivalently a bipartite cycle graphs.

Specht modules are a special family of representations of symmetric group S_n parameterized by partitions $\lambda \vdash n$. They lie in the heart of algebraic combinatorics, providing fundamental insights through the connection to the algebra of symmetric functions and the geometry of Grassmannians.

The combinatorics of partitions and Young tableaux play a central role in the theory of representations of S_n. The set of Specht modules S^λ, where λ is a partition of n, forms a complete set of irreducible representations of S_n. They are the ideals of the symmetric group algebra $\mathbb{C}[S_n]$ defined as follows:

$$S^\lambda := \mathbb{C}[S_n]C(\lambda)R(\lambda)$$

where $C(\lambda)$ and $R(\lambda)$ are column and row symmetrizers, respectively. The dimension of S^λ equals to the number of standard Young tableaux of shape λ, and the Frobenius character of S^λ is the Schur function $s_\lambda(x)$ in the ring of symmetric functions. The set of Schur functions $s_\lambda(x)$ forms a basis of the ring of symmetric functions.

Specht modules can be defined for general diagrams not only for partitions. The most well-studied diagram other than partition shape is skew partition shape λ/μ. The dimension of $S^{\lambda/\mu}$ is the number of standard Young tableaux of shape λ/μ and the Frobenius character of $S^{\lambda/\mu}$ is expanded by Schur functions so that the coefficients $c^\lambda_{\mu,\nu}$ are expressed by the famous Littlewood-Richardson rule.

$$s_{\lambda/\mu}(x) = \sum_\nu c^\lambda_{\mu\nu} s_\nu(x) \tag{2.1}$$

Although Specht modules of general diagrams have been studied in a number of contexts ([17],[14],[22],[25]), they are not completely understood except some special cases. The most general result is due to Reiner and Shimozono on percentage-avoiding diagrams. They gave a combinatorial rule to decompose Specht modules of percentage-avoiding shapes. Another family for which some results are known is three-rowed shapes. Magyar and van der Kallen [14] gave character formula and dimension formula of Schur modules of all diagrams with at most three rows. Here, Schur module is a GL_n module corresponding to the Specht module by Schur-Weyl duality.

Every diagram contained in a $k \times (n-k)$ rectangular box can be interpreted as a (2-colored) bipartite graph with the vertex set $[k] \sqcup [n-k]$. Each box (i,j) in the diagram corresponds to the edge connecting vertex i and j. This correspondence is clearly bijective, hence we can identify a general diagram with the corresponding bipartite graph.

Recently Liu studied Specht modules of forests under this identification. He proved that the dimension of the Specht module is the same as the normalized volume of the matching polytope of the forest. Our first main conjecture generalizes this relationship between Specht modules and polytopes. We define a polytope, called *matching ensemble polytope* $\mathcal{M}(G,\mathcal{E})$, for each pair of a diagram (or equivalently bipartite graph G) and a certain collection of matchings $\mathcal{E}$ of G. We conjecture that the normalized volume of the polytope equals the dimension of the corresponding Specht module in many cases. We will give evidences that support this conjecture. In particular we prove,

Theorem 2.1.1. *Let $\mathcal{C}^{2k}$ be the cycle graph with $2k$ vertices. The normalized volume of the polytope $\mathcal{M}(\mathcal{C}^{2k},\mathcal{E})$ is $k(E_{2k-1} - C_{k-1})$ independent of the choice of $\mathcal{E}$. Here E_{2k-1} is the Euler number that is the number of alternating permutations of length $2k-1$ and C_{k-1} is the $(k-1)$-th Catalan number.*

The second part of this chapter will be about Specht modules of toric diagrams. It is closely related to quantum geometry of the Grassmannian.

The cohomology ring of the Grassmannian is isomorphic to a certain quotient of the ring of symmetric functions. The fundamental cohomology classes of Schubert varieties form a linear basis of the ring, and they correspond to the Schur polynomials under this isomorphism. Their structure constants are the Littlewood-Richardson coefficients in (2.1).

There is a quantum version of this geometric picture, namely quantum cohomology ring of the Grassmannian and Gromov-Witten invariants. The quantum cohomology ring of the Grassmannian is a q-deformation of the usual cohomology ring. The Schur polynomials form a linear basis in the quantum ring as well, and the structure coefficients are the 3-point Gromov-Witten invariants which count the numbers of certain rational curves of fixed degree.

Postnikov [24] showed that we can construct a quantum analogue of the skew Schur polynomials by generalizing the skew partition shapes to toric skew shapes. Indeed, the coefficients of the toric Schur polynomials in the basis of ordinary Schur polynomials are the Gromov-Witten invariants:

$$s_{\lambda/d/\mu}(x) = \sum_{\nu} C_{\mu\nu}^{\lambda,d} s_\nu(x) \tag{2.2}$$

He conjectured that the Specht module of toric skew shape decomposes into irreducibles in the same way that the corresponding toric Schur polynomial expands with respect to the ordinary Schur polynomials. We provide evidences to this conjecture combined with our first conjecture.

Actually, our second conjecture is an expression of the Frobenius character χ^D of a certain Specht module S^D of toric shape D as a generating function of what we call proper semistandard toric tableaux (PSSTT).

$$\chi^D = \sum_{T:PSSTT} x^T$$

Hence the number of proper standard toric tableaux in this case will be the dimension of the Specht module. Toric staircase diagram of length $2k$ corresponds to $\mathcal{C}^{2k}$, and the number of proper standard toric tableaux in this case is $k(E_{2k-1} - C_{k-1})$, hence agrees with Theorem 2.1.1. We also show how our second conjecture is related to McNamara's cylindric Schur positivity conjecture [18].

The rest of the introduction is devoted to outline how this chapter is organized. In Section 2.2.1, we review basic theory on symmetric functions and Young tableaux. We also review Postnikov's theory on cylindric Schur functions and toric Schur polynomials in Section 2.2.2. In Section 2.2.3, we define Specht modules for general diagrams, and represent any diagram in terms of a 2-colored bipartite graph. Section 2.3.1 and Section 2.3.2 will be devoted to definitions and properties of the chain polytope and the alcoved polytope. They will be used to study matching ensemble polytope, which will be defined as the convex hull of matching ensembles in Section 2.4. In Section 2.4.1, we state our main conjecture on the relation between matching ensemble polytopes and Specht modules. We also give some evidences to the conjecture including the proof for two families of diagrams. In Section 2.4.2, we study matching ensemble polytopes for toric ribbon diagrams. In particular, we calculate the normalized volume of the polytope for toric staircase diagrams. We review Postnikov's conjecture on toric Specht module in Section 2.5, and provide some evidences to it. In Section 2.5.1, we define a combinatorial game called cylindric jeu de taquin, and use it to define proper semistandard toric tableau for toric shapes $\lambda/1/\mu$. We conjecture that the generating function of those tableaux is the character of the corresponding toric Specht module. We also show, in Section 2.5.2, that it is equivalent to Postnikov's conjecture combined with McNamara's conjecture on cylindric Schur positivity.

2.2 Symmetric functions and Specht modules

2.2.1 Symmetric functions

In this section we overview basic theory of symmetric functions. We will recall properties of (skew) Schur functions, jeu de taquin, and Littlewood-Richardson rule. For more details, see [26].

Let D be a finite collection of unit lattice boxes in the south-east quadrant of the plane. We will denote the position of a box placed at the i-th row and j-th column as (i, j). Such a collection is called *diagram*. For instance, a *Young diagram* (or *partition diagram*) of the partition $\lambda = (\lambda_1, \lambda_2, \cdots) \vdash n$ is the collection of all boxes (i, j) such that $j \leq \lambda_i$. When two partitions λ, μ satisfy $\mu_i \leq \lambda_i$ for all i, *skew Young diagram* (or *skew partition diagram*) of shape λ/μ is the collection of all boxes (i, j) such that $\mu_i < j \leq \lambda_i$.

For any diagram D, *tableau* of shape D is a filling of boxes of D with natural numbers. In case of (skew) partition shape, a tableau is called *semistandard* if the numbers in each row weakly increase and the numbers in each column strictly increase. The *weight* of a skew Young tableau is $\alpha = (\alpha_1, \alpha_2, \cdots)$ where α_i is the number of i appearing in the tableau. A semistandard skew Young tableau is called *standard* if the weight is $(1, 1, \cdots, 1)$.

The generating function of weights of semistandard Young tableaux of shape λ is given as follows:

$$s_\lambda(x) := \sum_{T:SSYT(\lambda)} x^T = \sum_{T:SSYT(\lambda)} x_1^{\alpha_1} x_2^{\alpha_2} \cdots \tag{2.3}$$

where the sum ranges over all semistandard Young tableaux T of shape λ and α is the weight of T. This formal sum of monomials is called *Schur function.* It is not immediately clear that $s_\lambda(x)$ is a symmetric function in this definition, but in fact we can say the following:

Proposition 2.2.1. *For any partition λ, $s_\lambda(x)$ is a symmetric function of homogeneous degree $|\lambda|$. Furthermore, they form a linear basis of the ring Λ of symmetric functions.*

The structure coefficients of Λ with respect to the basis $\{s_\lambda(x)\}$ are called *Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$.*

$$s_\mu(x) s_\nu(x) = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda(x)$$

The coefficients are nonnegative integers, and there is a combinatorial way, called Littlewood-Richardson rule, to calculate each $c^\lambda_{\mu\nu}$ by enumerating certain tableaux. Below, we briefly describe the rule here.

Given a semistandard skew Young tableaux of shape λ/μ, the following combinatorial game is called *jeu de taquin slide.*

1. Pick a box c to be an inner corner of μ.
2. While c is not an inner corner of λ do
   (a) if $c = (i, j)$ then let c' be either the box $(i+1, j)$ or $(i, j+1)$ so that we can slide c' into the position of c retaining semistandard-ness.
   (b) slide c' into the position of c and let $c := c'$.

Definition 2.2.2. A sequence of boxes $(c_1, \cdots, c_\ell)$ is a *slide sequence* for a tableau T if we can legally form $T = T_0, T_1, \cdots, T_\ell$ where $T_i$is obtained from T_{i-1} by performing jeu de taquin slide from the box c_i. Given a semistandard skew Young tableau T, we play *jeu de taquin* by choosing an arbitrary slide sequence that brings T to a partition shape and then applying the slides. The resulting tableau is denoted by $j(T)$.

It is not obvious that $j(T)$ is independent of the slide sequence. However it turns out that we always get the same standard Young tableau, namely the Robinson-Schensted P-tableau for the row word of T. Now we can describe the Littlewood-Richardson rule as follows.

Theorem 2.2.3 (Littlewood-Richardson rule). *The value of $c^\lambda_{\mu\nu}$ is equal to the number of semistandard Young tableaux T such that*

1. *T has shape λ/μ and its weight is ν.*
2. *$j(T)$ is the semistandard Young tableau of shape ν with weight ν.*

In fact, we have

$$\sum_T x^T = c^\lambda_{\mu\nu} s_\nu(x)$$

where the sum ranges over semistandard Young tableaux of shape λ/μ such that $j(T)$ has the shape ν.

The Littlewood-Richardson coefficients appear in a different context as well. Let $s_{\lambda/\mu}(x) = \sum x^T$ be the generating function of weights of semistandard Young tableaux of shape λ/μ. It is called *skew Schur function* of shape λ/μ. $s_{\lambda/\mu}(x)$ is also a symmetric function by the same argument proving that $s_\lambda(x)$ is a symmetric function. Hence we can expand it with respect to the basis $s_\lambda(x)$. Surprisingly the coefficients are the Littlewood-Richardson coefficients.

$$s_{\lambda/\mu}(x) = \sum_\nu c^\lambda_{\mu\nu} s_\nu(x)$$

This is still true when we specialize to the variables $x_1, \cdots, x_k$ and consider (skew) Schur polynomials instead of (skew) Schur functions.

2.2.2 Toric Schur polynomials

In this section we review Postnikov's result on toric Schur polynomials. We will recall the definitions of cylindric and toric diagrams. For more details, see [24],[18].

Let us begin with the overview on the connection of geometry of the Grassmannian with the ring of symmetric functions. It is well known that the cohomology ring of the Grassmannian Gr_{kn} is isomorphic to a certain quotient of the ring Λ of symmetric functions. Let P_{kn} be the set of partitions that fit inside the $k \times (n-k)$ rectangle. In other words,

$$P_{kn} = \{\lambda = (\lambda_1, \cdots, \lambda_k) \mid n-k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0\}$$

The Schubert varieties in Gr_{kn} are parameterized by partitions $\lambda \in P_{kn}$, and their fundamental cohomology classes σ_λ form a $\mathbb{Z}$-linear basis of the cohomology ring $H^*(Gr_{kn})$. Under the isomorphism to the quotient ring of symmetric functions, σ_λ corresponds to the Schur polynomial $s_\lambda(x_1, \cdots, x_{|\lambda|})$.

The (small) quantum cohomology ring $QH^*(Gr_{kn})$ of the Grassmannian is an algebra over $\mathbb{Z}[q]$ where q is a variable of degree n. $QH^*(Gr_{kn}) = H^*(Gr_{kn}) \otimes \mathbb{Z}[q]$ as a linear space, hence the Schubert classes $\sigma_\lambda, \lambda \in P_{kn}$ form a $\mathbb{Z}[q]$-linear basis. The product in $QH^*(Gr_{kn})$ is a q-deformation of the product in $H^*(Gr_{kn})$. It is defined as

$$\sigma_\mu * \sigma_\nu = \sum_{d,\lambda} q^d C_{\mu\nu}^{\lambda,d} \sigma_\lambda$$

where the sum ranges over nonnegative integers d and partitions $\lambda \in P_{kn}$ such that $|\lambda| = |\mu| + |\nu| - dn$. The structure constants are called *Gromov-Witten invariants.* Geometrically, they count the number of rational curves of degree d in Gr_{kn} that meet certain conditions. On the side of Λ, we can similarly define quantum product of Schur polynomials so that the quantum quotient ring of symmetric functions is isomorphic to $QH^*(Gr_{kn})$. Recall that we have an interesting symmetric polynomial $s_{\lambda/\mu}(x_1, \cdots, x_k)$ whose expansion coefficients with respect to the basis $\{s_\lambda(x_1, \cdots, x_k)\}$ is the structure constants of the ring. Postnikov [24] defined the analogous symmetric function in the quantum ring. It is called *toric Schur polynomial.* To state his result, let us review the definitions of cylindric and toric tableaux. We follow the notations in [24].

For two fixed positive integers n, k such that $n > k > 1$, the *cylinder* $\mathcal{C}_{kn}$ is the quotient

$$\mathcal{C}_{kn} = \mathbb{Z}^2/(-k, n-k)\mathbb{Z}.$$

Let $\langle i, j \rangle = (i, j) + (-k, n-k)\mathbb{Z}$ be an element of $\mathcal{C}_{kn}$. The partial order "$\preceq$" on $\mathcal{C}_{kn}$ is generated by the following covering relations: $\langle i, j \rangle \prec \langle i, j+1 \rangle$ and $\langle i, j \rangle \prec \langle i+1, j \rangle$. For two points $a, b \in \mathcal{C}_{kn}$, the interval $[a, b]$ is the set $\{c \in \mathcal{C}_{kn} \mid a \preceq c \preceq b\}$.

Definition 2.2.4. A *cylindric diagram D* is a finite subset of $\mathcal{C}_{kn}$ closed with respect to taking intervals.

We say that an integer sequence $(\cdots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \cdots)$, infinite in both directions, is (k,n)-periodic if $\alpha_i = \alpha_{i+k} + (n-k)$ for all $i \in \mathbb{Z}$. For a partition $\lambda \in P_{kn}$, let $\lambda[r]$ be the (k,n)-periodic sequence defined by $\lambda[r]_{i+r} = \lambda_i + r$ for all $i = 1, \cdots, k$. The order ideals of $\mathcal{C}_{kn}$ are of the form $D_{\lambda[r]} = \{\langle i,j \rangle \in \mathcal{C}_{kn} \mid (i,j) \in \mathbb{Z}^2, j \leq \lambda[r]_i\}$, and every cylindric diagram is a set theoretic difference of two order ideals

$$D_{\lambda[r]/\mu[s]} := D_{\lambda[r]} \setminus D_{\mu[s]}$$

We say that $D_{\lambda[r]/\mu[s]}$ is a cylindric diagram of type (k,n) and shape $\lambda[r]/\mu[s]$. For two partitions $\lambda, \mu \in P_{kn}$ and a nonnegative integer d, we will use the notation $\lambda/d/\mu$ instead of $\lambda[d]/\mu[0]$. In particular, $\lambda/0/\mu$ is the ordinary skew partition shape λ/μ inside a $k \times (n-k)$ rectangle.

For a cylindric diagram of shape $\lambda/d/\mu$, the cylindric tableau of shape $\lambda/d/\mu$ is the assignment of natural numbers to each of elements (or boxes) of the diagram. The tableau is called semistandard if it weakly increases on each of the rows and strictly increases on each of the columns of the diagram.

Let $\mathcal{T}_{kn} = \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/(n-k)\mathbb{Z}$ be the integer $k \times (n-k)$ torus. It is a quotient of the cylinder $\mathcal{T}_{kn} = \mathcal{C}_{kn}/(k,0)\mathbb{Z} = \mathcal{C}_{kn}/(0,n-k)\mathbb{Z}$. A cylindric diagram D is called *toric shape* if restriction of the natural projection $\mathcal{C}_{kn} \to \mathcal{T}_{kn}$ to D is an injective embedding $D \hookrightarrow \mathcal{T}_{kn}$. A *toric tableau* is a cylindric tableau of toric shape. Note that a toric shape can be embedded into the $k \times (n-k)$ rectangle so that rows and columns are cyclicly connected if we identify both of vertical and horizontal boundaries respectively. A semistandard toric tableau is the assignment of natural numbers on the boxes in the toric shape that strictly increases on each of rows and columns cyclicly.

For $\lambda, \mu \in P_{kn}$ and a nonnegative integer d, let us define the *cylindric Schur function* $s_{\lambda/d/\mu}(x)$ to be the generating function of semistandard cylindric tableaux of shape $\lambda/d/\mu$

$$s_{\lambda/d/\mu}(x) = \sum_{T:SSCT(\lambda/d/\mu)} x^T$$

The toric Schur polynomial is defined to be the specialization of $s_{\lambda/d/\mu(x)}$ to the first k variables:

$$s_{\lambda/d/\mu}(x_1, \cdots, x_k) = s_{\lambda/d/\mu}(x_1, \cdots, x_k, 0, 0, \cdots)$$

It is a symmetric function and nonzero only if $\lambda/d/\mu$ is a toric shape. Now we can formulate Postnikov's theorem on toric Schur polynomials.

Theorem 2.2.5 (Postnikov). *For $\lambda, \mu \in P_{kn}$ and a nonnegative integer d,*

$$s_{\lambda/d/\mu}(x_1, \cdots, x_k) = \sum_{\nu \in P_{kn}} C_{\mu\nu}^{\lambda,d} s_\nu(x_1, \cdots, c_k)$$

where $C_{\mu\nu}^{\lambda,d}$ is the Gromov-Witten invariants.

Note that this theorem generalizes the expansion of $s_{\lambda/\mu}(x_1, \cdots, x_k)$ with respect to the Schur polynomials since $C_{\mu\nu}^{\lambda,0} = c_{\mu\nu}^{\lambda}$.

2.2.3 Specht modules of general diagrams

In this section we define Specht modules of general diagrams. We review the conjecture of Postnikov on toric Specht modules. Also we identify general diagrams with bipartite graphs in order to generalize toric ribbon shapes to bipartite graphs with unique cycle.

Recall that a diagram D is a finite collection of unit lattice boxes in the southeast quadrant of the plane. Let $|D| = N$, and order the boxes in D arbitrarily. The symmetric group S_N acts on the ordering of boxes by permuting them naturally. Let C_D and R_D be the subgroups of S_N consisting of $\sigma \in S_N$ that stabilizes each columns and rows of D, respectively. Then column and row symmetrizers of D are defined by

$$C(D) = \sum_{\sigma \in C_D} sgn(\sigma)\sigma, \; R(D) = \sum_{\sigma \in R_D} \sigma$$

and the Specht module S^D is the left ideal of the group algebra $\mathbb{C}[S_N]$

$$S^D := \mathbb{C}[S_N]C(D)R(D)$$

It becomes an S_N module by multiplying elements of S_N from the left. Let us denote the Frobenius character of S^D by χ^D.

When D is a skew partition diagram of shape λ/μ, then the Specht module $S^{\lambda/\mu}$ has the character $\chi^{\lambda/\mu} = s_{\lambda/\mu}(x)$. In particular the dimension of $S^{\lambda/\mu}$ is the number of standard skew Young tableaux.

Now let D be the diagram that comes from embedding a toric shape $\lambda/d/\mu$ in a $k \times (n-k)$ rectangle. We will denote the Specht module S^D by $S^{\lambda/d/\mu}$, and call it the *toric Specht module*. The conjecture of Postnikov on toric Specht module is as follows.

Conjecture 2.2.6 (Postnikov). *The coefficients of irreducible components in the toric Specht module are the Gromov-Witten invariants:*

$$S^{\lambda/d/\mu} = \bigoplus_{\nu \in P_{kn}} C_{\mu\nu}^{\lambda,d} S^{\nu}$$

So it implies that

$$\chi^{\lambda/d/\mu} = \sum_{\nu \in P_{kn}} C_{\mu\nu}^{\lambda,d} s_{\nu}(x).$$

If we specialize RHS of this sum to the first k variables $x_1, \cdots, x_k$ then we get toric Schur polynomial. Although, the cylindric Schur function of the toric shape $\lambda/d/\mu$ is not Schur positive, i.e. not a positive sum of Schur functions. Hence $s_{\lambda/d/\mu}(x)$ is not $\chi^{\lambda/d/\mu}$. We will formulate the conjectural formula of $\chi^{\lambda/1/\mu}$ as the generating function of weights of certain semistandard toric tableaux of shape $\lambda/1/\mu$ in Section 2.5.1.

Let us construct a 2-colored bipartite graph G_D from a diagram D in the following way. Each row of the diagram corresponds to a black vertex, each column of the diagram corresponds to a white vertex of G_D, and each box in the diagram corresponds to the edge connecting black and white vertices corresponding to its row and column. This correspondence between D and G_D is bijective once we fix the orders of rows and columns of D, or equivalently the black and white vertices of G_D. Recall that the Specht module S^D is invariant under permuting the rows and columns, hence S^D only depends on G_D.

Other than the forest, the simplest bipartite graph is the cycle graph $\mathcal{C}^{2k}$ with $2k$ vertices. The diagram corresponding to $\mathcal{C}^{2k}$ is the *toric staircase diagram* that fits inside a $k \times k$ square. In the notation of previous section, it is the toric diagram of type $(k, 2k)$ and shape $\lambda/1/\lambda$ where $\lambda = (k-1, k-2, \cdots, 1, 0)$.

A *toric ribbon shape* is a shape of a diagram that comes from embedding a toric diagram which is connected and does not contain any 2×2 square. Here, a toric diagram is *connected* if one can move from one box to any other box by walking through adjacent boxes, or equivalently the corresponding bipartite graph is connected. In the notation of previous section, it is the toric diagram of shape $\lambda/1/\lambda$ for some partition λ. The bipartite graph corresponding to a toric ribbon diagram is a slight generalization of the cycle graph. In fact, it is the graph we get by attaching multiple number of single edges at each vertex of the cycle graph.

One can further generalize this graph to those bipartite graphs that contain unique cycle. One attach trees to each vertex of the cycle graph to obtain such a graph. The diagrams corresponding to graphs with unique cycle are not in general toric shape anymore.

2.3 Polytope Miscellanea

In this section we review definitions and properties of some polytopes that will be used later in our proof. For further references, see [28],[15].

2.3.1 Chain polytopes

Let P be a finite poset with N elements. A *linear extension* of P is an order-preserving bijection from P to $[N]$. On the plane, we can naturally define a poset structure generated by the following relations:

$$(i, j) > (i, j+1) \text{ and } (i, j) > (i+1, j)$$

For any skew partition diagram on the plane, we can think of it as a poset by inheriting the poset structure on the plane. Each of linear extension of the skew partition diagram corresponds to a standard skew Young tableau by definition.

The *chain polytope* $\mathcal{C}(P)$ of P is a convex polytope defined as follows:

$$\mathcal{C}(P) := \{f : P \to [0,1] \mid f(p_1)+f(p_2)+\cdots+f(p_r) \leq 1 \text{ for all chains } p_1 < p_2 < \cdots < p_r\}$$

Let $\chi_I : P \to [0,1]$ denote the characteristic function of I, i.e. $\chi_I(x) = 1$ if $x \in I$ and $\chi_I(x) = 0$ otherwise. Stanley [28] showed that the vertices of $\mathcal{C}(P)$ are exactly the characteristic functions of antichains in P. It is clearly a 01-polytope, and the volume of a 01-polytope in N dimensional space is a integer multiple of $1/N!$. The value $N!$ times the volume of a such polytope is called *normalized volume.* Let us denote the normalized volume of a polytope Q by $Vol(Q)$.

Stanley also showed that the normalized volume of $\mathcal{C}(P)$ is the number of linear extensions of P. In particular, when P is the skew partition diagram of shape λ/μ then the normalized volume of $\mathcal{C}(P)$ is the number of standard skew Young tableaux, which is the same as the dimension of the Specht module $S^{\lambda/\mu}$. As a special case, when P is the zig-zag diagram (or staircase diagram) of length N, the normalized volume of $\mathcal{C}(P)$ is the number of alternating permutations of length N, hence the famous Euler number E_N.

Lemma 2.3.1. *Let P be the poset of toric staircase diagram of length $2k$. The poset structure is defined in Section 2.2.2. Then*

$$Vol(\mathcal{C}(P)) = kE_{2k-1}$$

Proof. When P is the poset of a cylindric diagram, then the normalized volume of $\mathcal{C}(P)$ is the number of corresponding standard cylindric tableaux. In particular when P is the toric staircase diagram of length $2k$, the normalized volume is the number of cyclicly alternating permutations of length $2k$. Note that there are k positions where $2k$ can be put in. Once we fix the place of $2k$, the number of permutations of remaining numbers is the Euler number E_{2k-1}. Hence the result follows. □

Remark 2.3.2. The number kE_{2k-1} appeared in other context as well. It is called *Eulerian-Catalan number* in [3].

2.3.2 Alcoved polytopes

The regions of affine Coxeter arrangements are called *alcoves.* An *alcoved polytope* is a convex polytope that is the union of several alcoves. Postnikov and Lam studied alcoved polytopes in [15]. They found a special triangulation of any alcoved polytope, and combinatorial method to calculate the normalized volume of an alcoved polytope by enumerating certain permutations.

Every alcoved polytope can be defined to lie within the hypersimplex $\Delta_{k,n}$. In this case, alcoved polytope $Q \subset \mathbb{R}^n$ is defined by the hyperplane $x_1 + x_2 + \cdots + x_n = k$, the inequalities $0 \leq x_i \leq 1$ together with inequalities of the form

$$b_{ij} \leq x_{i+1} + \cdots + x_j \leq c_{ij}$$

for integers b_{ij} and c_{ij} for each pair (i, j) satisfying $0 \leq i < j \leq n-1$. Let $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection that sends x_n to 0.

Theorem 2.3.3 (Postnikov-Lam). *The normalized volume of the projection $p(Q)$ is the number of permutations $w = w(1)w(2)\cdots w(n-1) \in S_{n-1}$ satisfying the following conditions:*

1. w has k-1 descents,

2. The sequence $w(i)\cdots w(j)$ has at least b_{ij} descents. Furthermore, if $w(i)\cdots w(j)$ has exactly b_{ij} descents, then $w(i) < w(j)$,

3. The sequence $w(i)\cdots w(j)$ has at most c_{ij} descents. Furthermore, if $w(i)\cdots w(j)$ has exactly c_{ij} descents, then $w(i) > w(j)$.

In the above conditions, we assume $w(0) = 0$.

Since Q lies on the hyperplane $x_1 + \cdots + x_n = k$, we see that the volume of Q is $\frac{\sqrt{n}}{(n-1)!}$ times the number of permutations satisfying the above conditions.

2.4 Matching Ensemble Polytopes

In this section we define matching ensembles and matching ensemble polytopes of a general diagram. We state our main conjecture and look at evidences that supports the conjecture. In particular, we calculate the volume of the polytope coming from toric staircase diagram.

Given a finite graph G, a *matching* M of G is a collection of edges of G such that no two edges in M share a vertex. Any subset of edges in M is called a *submatching*. A matching M is called *perfect matching* of G if every vertex of G is covered by an edge in M. An induced subgraph H of G is called a *minor* if it admits a perfect matching. For a set of edges I of G, let χ_I be the characteristic function of I, i.e. $\chi_I(x) = 1$ if x is an edge of G, $\chi_I(x) = 0$ otherwise. Now let us define the central object of this section.

Definition 2.4.1. Let G be a finite graph. The *matching ensemble* $\mathcal{E}$ is a collection of matchings of G that satisfies the following conditions:

1. For every minor H of G, exactly one perfect matching of H belongs to $\mathcal{E}$,
2. If a matching M is in $\mathcal{E}$, then any submatching of M is also in $\mathcal{E}$.

Given G and a matching ensemble $\mathcal{E}$, the convex hull of characteristic functions of matchings in $\mathcal{E}$ is called *matching ensemble polytope* $\mathcal{M}(G,\mathcal{E})$.

It is clear that $\mathcal{M}(G,\mathcal{E})$ is a 01-polytope and every matchings in $\mathcal{E}$ correspond to vertices of the polytope. We will denote the normalized volume of the polytope by $Vol(\mathcal{M}(G,\mathcal{E}))$.

Proposition 2.4.2. *Let G be a disjoint union of graphs G_1 and G_2, where G_1 has n_1 edges and G_2 has n_2 edges. Any matching ensemble $\mathcal{E}$ of G induces matching ensembles on G_1 and G_2 by restriction, namely $\mathcal{E}_1$ and $\mathcal{E}_2$. We have*

$$Vol(\mathcal{M}(G,\mathcal{E})) = \binom{n_1+n_2}{n_1} Vol(\mathcal{M}(G_1,\mathcal{E}_1)) \cdot Vol(\mathcal{M}(G_2,\mathcal{E}_2))$$

Proof. Any matching of G is uniquely represented as a disjoint union of matchings of G_1 and G_2. It follows that $\mathcal{M}(G,\mathcal{E}) = \mathcal{M}(G_1,\mathcal{E}_1) \times \mathcal{M}(G_2,\mathcal{E}_2)$. Therefore,

$$\frac{Vol(\mathcal{M}(G,\mathcal{E}))}{(n_1+n_2)!} = \frac{Vol(\mathcal{M}(G_1,\mathcal{E}_1))}{n_1!} \cdot \frac{Vol(\mathcal{M}(G_2,\mathcal{E}_2))}{n_2!}$$

The claim follows from this. $\square$

2.4.1 Main conjecture

Let us state our conjecture on the matching ensemble polytopes.

Conjecture 2.4.3. *Let D be a diagram, and G_D the bipartite graph corresponding to D in the sense of Section 2.2.3. There exists a matching ensemble $\mathcal{E}$ of G_D such that $Vol(\mathcal{M}(G_D,\mathcal{E})) = dim S^D$, where S^D is the Specht module of the diagram D.*

Using Proposition 2.4.2, it is easy to see that the conjecture is consistent with taking direct sum of two Specht modules. Hence we can restrict ourselves to the cases where G_D is connected.

Note that not all matching ensemble gives the correct normalized volume. Although, we conjecture that good matching ensembles are "generic" in the sense that the normalized volume of $\mathcal{M}(G,\mathcal{E})$ is maximal when the equality holds.

There are two special families of diagrams that support the conjecture. Namely the skew partition diagrams and forests. Below we give the proofs of the conjecture for these two families.

Proof for skew partition shapes. Let D be a skew partition diagram. The rows and columns are naturally ordered, and this order induces a lexicographic order on the set of matchings of G_D. More precisely, let $M = \{(i_1,j_1),\cdots,(i_r,j_r)\}$ and $N = \{(i'_1,j'_1),\cdots,(i'_r,j'_r)\}$ be two perfect matchings of the same minor. Here (i,j) is the edge connecting i-th row vertex with j-th column vertex, and $i_1 < \cdots < i_r$, $i'_1 < \cdots < i'_r$. The lexicographic order between M and N is defined so that $M < N$ if there exists $1 \le s \le r$ such that $j_t = j'_t$ for all $t < s$ and $j_s < j'_s$. This is a total order on the set of perfect matchings of the given minor. Now we choose the lexicographically maximal perfect matching from each minor of G. Then this collection of matchings clearly form a matching ensemble $\mathcal{E}_L$. On the other hand, since D is a skew partition shape, if we have two edges (i,j) and (i',j') such that $i < i'$ and $j < j'$ then we also have (i,j') and (i',j) in G_D. Hence the lexicographically maximal matchings are exactly the antichains of the poset of D. Therefore $\mathcal{M}(G_D,\mathcal{E}_L)$ is the chain polytope of the poset of D, and its normalized volume is the number of standard Young tableaux which coincides with the dimension of S^D. $\square$

Proof for forests. Let D be a diagram such that G_D is a forest. It is clear that for any minor of G_D there is a unique perfect matching which is the set of all edges of the minor. Hence there is a unique matching ensemble for G_D. In this case, the matching ensemble polytope is just the matching polytope of G_D whose vertices are the characteristic functions of every matchings of G_D. Liu [13] showed that the normalized volume of the matching polytope of G_D is the same as the dimension of S^D. Hence we are done. □

Remark 2.4.4. The lexicographic ordering can be defined on any diagram, and gives the lexicographically maximal matching ensemble. Although, the choice of ensemble heavily depends on the ordering of rows and columns, and permuting them gives us a different matching ensemble. The matching ensemble polytope also changes if we choose a different ordering. This is unlike that the Specht module is invariant under the permuting rows and columns.

Remark 2.4.5. There is other interesting families of diagrams, namely 3-row diagrams and graphs with a unique cycle. 3-row diagrams are the ones that have at most 3 rows. The character and dimension formula of Schur modules of 3-row diagrams are completely spelled out by Magyar and van der Kallen in [14]. They actually constructed the Schur modules geometrically using Borel-Weil type method. Interestingly, we were able to check using computer for large number of diagrams in this family that not only our conjecture is true but also the volume of $\mathcal{M}(G_D, \mathcal{E})$ does not depend on the choice of $\mathcal{E}$. The tricky part of this observation is that the combinatorial type of the polytope $\mathcal{M}(G_D, \mathcal{E})$ actually changes very much when we choose different ensemble $\mathcal{E}$, nevertheless the volume does not.

Any graph with a unique cycle has only two matching ensembles. This family of diagrams contain toric ribbon diagrams, and in particular, toric staircase diagrams which will be studied in the next section. We conjecture that the volume of a matching ensemble polytope for any graph with a unique cycle does not depend on the choice of the ensemble, and it is always the dimension of the Specht module.

Remark 2.4.6. The matching ensemble appears in other contexts as well. Let us assign weights generically on each edge of the graph G, and pick the perfect matching from each minor that maximizes the sum of weights. Then the collection of matchings form a matching ensemble. Such matching ensembles are called *regular*. Postnikov showed that the regular matching ensembles of G are in bijection with regular central triangulations of the root polytope R_G, and regular noncrossing matching ensembles of G are in bijection with regular central noncrossing triangulations of R_G. They are also related to polypositroids.

2.4.2 Toric ribbon shapes

The next simplest graph other than forest is the cycle graph. Let $\mathcal{C}^{2k}$ be the cycle graph with $2k$ vertices. The corresponding diagram is the toric staircase diagram of length $2k$. It fits inside a $k \times k$ square. In the notation of toric diagrams, it is the toric shape $\lambda/1/\lambda$ where $\lambda = (k-1, k-2, \cdots, 1, 0)$. See Figure 2-1 for an example.

Let us start with the observation that the matching ensemble polytope is independent of the choice of the ensemble in this case.

Proposition 2.4.7. *There are only two matching ensembles for $\mathcal{C}^{2k}$, and the corresponding matching ensemble polytopes are isomorphic.*

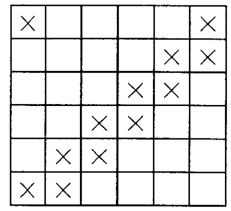

Figure 2-1: The toric staircase diagram of length 12

Proof. For any strict minor, there is a unique perfect matching. The only minor that admits more than one perfect matching is $\mathcal{C}^{2k}$ itself. Since there are two perfect matchings of $\mathcal{C}^{2k}$, there are two matching ensembles as well. Let us denote them by $\mathcal{E}_1$ and $\mathcal{E}_2$. The obvious graph automorphism that flips the graph along its diameter interchanges $\mathcal{E}_1$ and $\mathcal{E}_2$. Therefore it induces a coordinate permutation which interchanges $\mathcal{M}(\mathcal{C}^{2k}, \mathcal{E}_1)$ and $\mathcal{M}(\mathcal{C}^{2k}, \mathcal{E}_2)$. It follows that the two matching polytopes are isomorphic. $\square$

Using this proposition, let us denote the matching ensemble polytope of $\mathcal{C}^{2k}$ by $\mathcal{M}(\mathcal{C}^{2k})$. The following theorem gives us the volume of this polytope.

Theorem 2.4.8. *The normalized volume of $\mathcal{M}(\mathcal{C}^{2k})$ is $k(E_{2k-1} - C_{k-1})$, where E_{2k-1} is the number of alternating permutations of length $2k-1$ and C_{k-1} is the $(k-1)$-th Catalan number.*

Observe that the matchings of $\mathcal{C}^{2k}$ are exactly the antichains of the poset of the toric staircase diagram. Therefore the matching polytope of $\mathcal{C}^{2k}$ is the chain polytope of the poset of the corresponding toric staircase diagram. The chain polytope is a convex polytope in $\mathbb{R}^{2k}$ defined by the following set of inequalities:

$$0 \leq x_i, \forall i$$

$$\begin{gathered} x_1 + x_2 \leq 1 \\ x_2 + x_3 \leq 1 \\ \vdots \\ x_{2k-1} + x_{2k} \leq 1 \\ x_{2k} + x_1 \leq 1 \end{gathered}$$

All of these inequalities are tight, i.e. each of them corresponds to a facet of the chain polytope. The polytope $\mathcal{M}(\mathcal{C}^{2k})$ is different from the matching polytope of $\mathcal{C}^{2k}$ just by missing one vertex, namely $(1, 0, 1, 0, \cdots, 1, 0)$. Now let us describe $\mathcal{M}(\mathcal{C}^{2k})$ by similar inequalities.

Lemma 2.4.9. *The polytope $\mathcal{M}(\mathcal{C}^{2k})$ is a convex polytope in $\mathbb{R}^{2k}$ defined by the in-*

equalities listed above together with the following additional set of inequalities:

$$\sum_{i=1}^{2k} x_i - x_{2s-1} \leq k-1$$

for all $s = 1, \cdots, k$.

Proof. Let Q be the polytope defined by the inequalities of the lemma. It is easy to see that Q is unimodular, hence a lattice polytope. Furthermore, by the obvious restriction $0 \leq x_i \leq 1$ for all i, it is a 01-polytope. In other words, every lattice point is a vertex. Since it is an intersection of the matching polytope with certain half spaces, it follows that the set of vertices of Q is a subset of vertices of the matching polytope. On the other hand, every matching but one corresponding to the vertex $(1, 0, 1, 0, \cdots, 0, 1, 0)$ still satisfy the additional inequalities. Therefore the vertices of Q are exactly the same as the vertices of $\mathcal{M}(\mathcal{C}^{1k})$. So the claim follows. □

Note that the additional inequalities make some of previous inequalities redundant. However the new inequalities are tight since we can easily find enough number of vertices satisfying each of the new inequalities. As there are k new inequalities, they produce k new facets. Hence the difference between $\mathcal{M}(\mathcal{C}^{2k})$ and the matching polytope of $\mathcal{C}^{2k}$ is decomposed into k number of cones whose bottem facets are the newly produced facets and the upper vertex is the missing vertex $(1, 0, \cdots, 1, 0)$.

Lemma 2.4.10. *The newly produced facets are cones whose bottom facets are alcoved polytopes and the upper vertex is* $(0, 1, 0, 1, \cdots, 0, 1)$. *The bottom facets are isomorphic to the alcoved polytope defined by the following inequalities:*

$$0 \leq x_i, \forall i$$

$$x_1 + x_2 \leq 1$$

$$\vdots$$

$$x_{2k-2} + x_{2k-1} \leq 1$$

$$x_1 + \cdots + x_{2k-1} = k-1$$

Proof. First, we slice the polytope $\mathcal{M}(\mathcal{C}^{2k})$ by the hyperplanes $\sum_{i=1}^{2k} x_i = s$. When $s \leq k-1$, the additional inequalities in Lemma 2.4.9 does not contribute anything. Hence the part of $\mathcal{M}(\mathcal{C}^{2k})$ below the hyperplane $\sum_{i=1}^{2k} x_i = k-1$ is the same as the matching polytope. In other words, the each of newly produced facets is a cone whose the bottom facet lies on the hyperplane $\sum_{i=1}^{2k} x_i = k-1$, because there is only one vertex $(0, 1, \cdots, 0, 1)$ above it. Let us look at one particular newly produced facet (cone), namely the one on the hyperplane $x_2 + \cdots + x_{2k} = k-1$. On the bottom facet of this cone, $x_1 = 0$. Therefore after shifting the indices by -1, we get the inequalities given in the lemma. The set of inequalities define an alcoved polytope, see Section 2.3.2. In fact, it is easy to see that the polytope is defined by the inequalities

$$\lfloor\frac{j-1}{2}\rfloor - \lfloor\frac{i+1}{2}\rfloor \le x_{i+1} + \cdots + x_j \le \lfloor\frac{j-i+1}{2}\rfloor$$

for all $0 \le i < j \le 2k-2$. It is also clear that any other newly produced facet has the same set of inequalities after permuting the indices appropriately. Hence we are done. $\square$

Lemma 2.4.11. *The volume of the alcoved polytope in Lemma 2.4.10 is* $\frac{\sqrt{2k-1}}{(2k-2)!} \times C_{k-1}$.

Proof. By Theorem 2.3.3, it is enough to show that the number of permutations in S_{2k-2} satisfying the following conditions is C_{k-1}:

1. w has $k-2$ descents,
2. The sequence $w(i)\cdots w(j)$ has at least $\lfloor\frac{j-1}{2}\rfloor - \lfloor\frac{i+1}{2}\rfloor$ descents. Furthermore, if $w(i)\cdots w(j)$ has exactly $\lfloor\frac{j-1}{2}\rfloor - \lfloor\frac{i+1}{2}\rfloor$ descents, then $w(i) < w(j)$,
3. The sequence $w(i)\cdots w(j)$ has at most $\lfloor\frac{j-i+1}{2}\rfloor$ descents. Furthermore, if $w(i)\cdots w(j)$ has exactly $\lfloor\frac{j-i+1}{2}\rfloor$ descents, then $w(i) > w(j)$.

It is easy to check that the descent set is $\{2, 4, \cdots, 2k-4\}$ if a permutation satisfies the above conditions. Moreover, the permutation has to satisfy $w(2) > w(4) > \cdots > w(2k-2)$ and $w(1) > w(3) > \cdots > w(2k-3)$ because of the condition 2. and 3. Now, the number of such permutations is the number of standard Young tableaux of $2\times(k-1)$ rectangular shape. By the hook length formula, the number is the Catalan number C_{k-1}. $\square$

Now we are ready to give a proof of Theorem 2.4.8.

Proof of the Theorem 2.4.8. As we have seen before, the difference between $\mathcal{M}(\mathcal{C}^{2k})$ and the matching polytope is decomposed into k cones whose common upper vertex is $(1, 0, \cdots, 1, 0)$. By Lemma 2.4.10 and Lemma 2.4.11, each of cones' bottom facet is also a cone over an alcoved polytope that has the volume $\frac{\sqrt{2k-1}}{(2k-2)!} \times C_{k-1}$ with the upper vertex $(0, 1, \cdots, 0, 1)$. The distance between the alcoved polytope and the vertex $(0, 1, \cdots, 0, 1)$ is $\frac{1}{\sqrt{2k-1}}$ and the distance between the vertex $(1, 0, \cdots, 1, 0)$ and the bottom facet is 1. Therefore, the volume of each cone is

$$\frac{\sqrt{2k-1}}{(2k-2)!} \times C_{k-1} \times \frac{1}{\sqrt{2k-1}} \times \frac{1}{2k-1} \times \frac{1}{2k} = \frac{C_{k-1}}{(2k)!}$$

and the normalized volume is C_{k-1}.

Recall that, by Lemma 2.3.1, the normalized volume of the matching polytope is the number of standard toric tableaux, which is kE_{2k-1}. Since there are k isomorphic cones, the normalized volume of $\mathcal{M}(\mathcal{C}^{2k})$ is $kE_{2k-1} - kC_{k-1} = k(E_{2k-1} - C_{k-1})$. $\square$

Remark 2.4.12. The toric staircase diagram is a special case of toric ribbon diagram. Let λ be any partition in P_{kn}. Then the toric diagram $D^{\lambda/1/\lambda}$ of shape $\lambda/1/\lambda$ is the toric ribbon diagram contained in a $k \times (n-k)$ rectangle. The corresponding graph is the one we get by attaching multiple number of single edges to each vertex of a cycle graph. There are also only two matching ensembles by the same reason as for toric staircase diagram. It is not entirely clear that the corresponding matching ensemble polytopes have the same volume, but we believe that a very similar technique using cone decomposition of the difference from the matching polytope can calculate the volume of the matching ensemble polytopes:

$$Vol(\mathcal{M}(D^{\lambda/1/\lambda}, \mathcal{E})) = \#STT - \binom{n-2}{k-1} \text{ for any } \mathcal{E}$$

where $\#STT$ is the number of standard toric tableaux of shape $\lambda/1/\lambda$, i.e. the volume of the matching polytope of $G_{D^{\lambda/1/\lambda}}$.

2.5 Toric Specht modules

In this section we study the toric Specht modules of shape $\lambda/1/\mu$. We will define a combinatorial game called cylindric jeu de taquin and proper semistandard toric tableau for these diagrams. We will use them to express the character of the toric Specht module as a generating function of proper semistandard toric tableaux.

2.5.1 Cylindric jeu de taquin and proper semistandard toric tableaux

Recall the definition of jeu de taquin in Section 2.2.1. Given a semistandard cylindric tableau of shape $\lambda/d/\mu$, the *cylindric jeu de taquin* is a combinatorial game similar to the jeu de taquin except that we do it on the cylindric setting. More precisely, *cylindric jeu de taquin slide* is the following process:

1. Pick a box c to be an inner corner of μ.
2. While c is not an inner corner of λ do
   (a) if $c = \langle i, j\rangle$ then let c' be either $\langle i+1, j\rangle$ or $\langle i, j+1\rangle$ so that we can slide c' into the position of c retaining semistandard-ness.
   (b) slide c' into the position of c and let $c := c'$.

Remember that a box $c = \langle i, j\rangle$ is a coset $(i, j) + (-k, n-k)\mathbb{Z}$. So each step of a cylindric jeu de taquin slide applies simultaneously to all boxes contained in the coset $\langle i, j\rangle$ periodically.

Definition 2.5.1. The *cylindric jeu de taquin* is a sequence of cylindric jeu de taquin slides that ends when the box $\langle 1, 1\rangle$ is filled after the last slide.

Unfortunately, this definition of cylindric jeu de taquin slide is not as good as the ordinary jeu de taquin. In particular, the resulting tableau is not independent of the choice of slides, hence there is no analogue of Robinson-Schensted P-tableau in this definition. (See Figure 2-2 for an example of proper tableau of type $(3,6)$ that has two different resulting tableaux. 5 is initially outside of the 3×3 rectangle.) Nevertheless, we can use it to define certain "bad" toric tableaux in some cases. Below we describe how to do it.

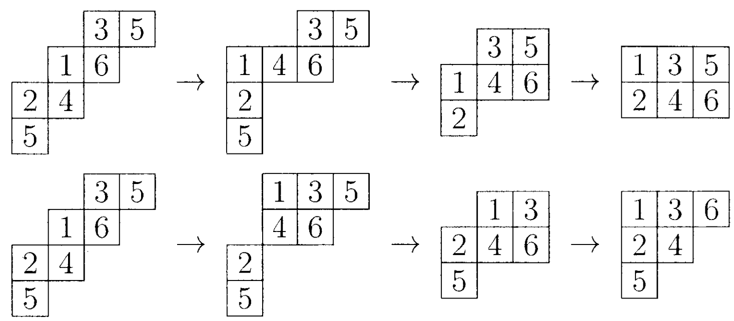

Figure 2-2: Proper tableau of type $(3,6)$ with two different resulting tableaux

Given $0 < k < n$, let us consider a toric diagram of type (k,n) with the shape $\lambda/1/\mu$ such that $\lambda_1, \mu_1 < n-k$ and $\lambda_k = \mu_k = 0$. When embedded in the $k \times (n-k)$ rectangle R_{kn}, the diagram looks like a skew partition shape with one extra unit box placed at the left upper corner. See Figure 2-3 for an example.

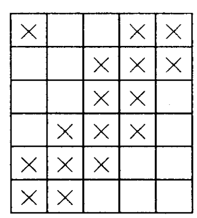

Figure 2-3: The shape $(4,3,3,2,1,0)/1/(3,2,2,1,0,0)$

Let T be a semistandard toric tableau of shape $\lambda/1/\mu$. After doing a cylindric jeu de taquin on T, the resulting tableau is either embedded inside R_{kn} or not.

Definition 2.5.2. A semistandard toric tableau of shape $\lambda/1/\mu$ is called *proper* if the resulting tableau after a cylindric jeu de taquin fits inside R_{kn}.

Since the resulting tableau depends on the sequence of cylindric jeu de taquin slides, it is not clear that the proper semistandard toric tableau is well-define. The following lemma takes care of the problem.

Lemma 2.5.3. *Let T be a semistandard toric tableau of shape $\lambda/1/\mu$. If there is a slide sequence such that the resulting tableau is embedded in R_{kn}, then for any sequence of cylindric jeu de taquin slides, the resulting tableau fits inside R_{kn}.*

Proof. There is only one box that is initially outside of R_{kn}, the one placed at the left upper corner. If, at some step of a cylindric jeu de taquin, the box slides into R_{kn}, then the cylindric jeu de taquin from that point is just the same as ordinary jeu de taquin. Hence the resulting tableu will be a partition shape inside R_{kn}. Therefore if T does not end up contained in R_{kn} after some cylindric jeu de taquin, the left upper corner box never comes inside R_{kn} during the process. The only situation that can happen is when the number written inside the left upper corner box is the largest number appearing in the tableau, and it is the only largest number. In this case, cylindric jeu de taquin on T is essentially the same as ordinary jeu de taquin applied on the skew Young tableau we get by removing the left upper corner box from T. Therefore, the resulting tableau is independent of the choice of slide sequence. We are done. $\square$

Remark 2.5.4. In the above proof, notice we proved that the resulting (cylindric) tableau of a nonproper semistandard toric tableau of shape $\lambda/1/\mu$ is independent of the slide sequence. See Figure 2-4 for an example of cylindric jeu de taquin of a nonproper tableau. N is the largest number placed outside of 3×3 rectangle.

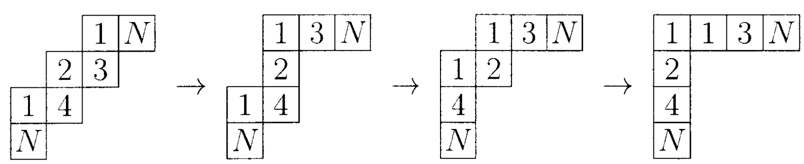

Figure 2-4: Nonproper semistandard toric tableau of type $(3, 6)$

Let us now formulate our second conjecture on the Frobenius character of the Specht module $S^{\lambda/1/\mu}$.

Conjecture 2.5.5. *For a toric shape $\lambda/1/\mu$ as above, the Frobenius character of $S^{\lambda/1/\mu}$ is the generating function of proper semistandard toric tableaux (PSSTT) of shape $\lambda/1/\mu$:*

$$\chi^{\lambda/1/\mu} = \sum_{T:PSSTT_{\lambda/1/\mu}} x^T$$

In particular, the dimension of the Specht module $S^{\lambda/1/\mu}$ is the number of proper standard toric tableaux (PSTT) of shape $\lambda/1/\mu$:

$$dim S^{\lambda/1/\mu} = \#PSTT_{\lambda/1/\mu}$$

It is not obvious that RHS of the conjecture is a symmetric function. Recall that the cylindric Schur function $s_{\lambda/1/\mu}(x)$ is the generating function of all cylindric toric tableaux of shape $\lambda/1/\mu$. Hence,

$$s_{\lambda/1/\mu}(x) = \sum_{T:PSSTT_{\lambda/1/\mu}} x^T + \sum_{T:NPSSTT_{\lambda/1/\mu}} x^T$$

We will prove in Section 2.5.2 that the latter sum over nonproper semistandard toric tableaux is a symmetric function, so the former sum is as well.

Combining this conjecture with Conjecture 2.2.6, we get

$$\sum_{T:PSSTT_{\lambda/1/\mu}} x^T = \sum_{\nu \in P_{kn}} C_{\mu\nu}^{\lambda,1} s_\nu(x)$$

In particular, if we look at the coefficient of $x_1 x_2 \cdots x_N$, where $N = |\lambda| - |\mu| + n$ is the number of boxes in the diagram, we get another combinatorial equation

$$\# PSTT_{\lambda/1/\mu} = \sum_{\nu \in P_{kn}} C_{\mu\nu}^{\lambda,1} f^\nu$$

where $\# PSTT_{\lambda/1/\mu}$ is the number of proper standard toric tableaux of shape $\lambda/1/\mu$, and f^ν is the number of standard Young tableaux of shape ν.

Recall that we get a toric ribbon diagram by taking $\lambda = \mu$.

Proposition 2.5.6. *The number of proper standard toric tableaux of shape $\lambda/1/\lambda$ is $\# STT - \binom{n-2}{k-1}$. In particular when $n = 2k$ and $\lambda = (k-1, k-2, \cdots, 1, 0)$, it is $k(E_{2k-1} - C_{k-1})$.*

Proof. Every nonproper standard toric ribbon tableaux go to the cylindric hook diagram after cylindric jeu de taquin. Furthermore, this correspondence is bijective since there is only one tableau of a given ribbon shape whose Robinson-Schensted P-tableau has the hook shape. The number of standard cylindric hook tableaux is just $\binom{n-2}{k-1}$ because after removing the box containing N we get an ordinary hook shape $(n-k, 1^{k-1})$. Hence the result follows. □

Remark 2.5.7. Together with Theorem 2.4.8 and Remark 2.4.12, this gives another evidence to Conjecture 2.4.3.

2.5.2 Cylindric Schur positivity

In this section we show how Conjecture 2.5.5 is related to McNamara's conjecture of cylindric Schur positivity. In particular we prove that $\sum_{T:PSSTT} x^T$ in the previous section is a symmetric function by showing that $\sum_{T:NPSSTT} x^T$ is cylindric Schur positive.

A cylindric Schur function $s_{\lambda/d/\mu}(x)$ is not Schur positive in general. For instance, McNamara [18] showed for toric ribbon diagrams that

$$s_{\lambda/1/\lambda}(x) = \sum_{\nu \in P_{kn}} C_{\lambda\nu}^{\lambda,1} s_\nu(x) + s_{H_{nk}}(x)$$

where $s_{H_{nk}}(x)$ is the cylindric Schur function of the cylindric hook shape, and

$$s_{H_{nk}}(x) = s_{(n-k,1^k)}(x) - s_{(n-k-1,1^{k+1})}(x) + \cdots + (-1)^{n-k-2} s_{(2,1^{n-2})}(x) + (-1)^{n-k-1} s_{(1^n)}(x)$$

Although not Schur positive, it is a positive sum of Schur functions and a cylindric hook Schur function. McNamara conjectures that every cylindric Schur function is cylindric Schur positive, i.e. a positive sum of cylindric Schur functions of the form $s_{\lambda/d/\phi}(x)$ for some partition λ and a nonnegative integer d. This is the cylindric Schur positivity conjecture.

Recall that nonproper semistandard toric ribbon tableaux are in bijection with the cylindric hook diagram by the cylindric jeu de taquin. It follows that

$$\sum_{T:NPSSTT_{\lambda/1/\lambda}} x^T = s_{H_{kn}}(x).$$

Therefore we have,

Proposition 2.5.8.
$$\sum_{T:PSSTT_{\lambda/1/\lambda}} x^T = \sum_{\nu\in P_{kn}} C_{\lambda\nu}^{\lambda,1} s_\nu(x).$$

In fact, the same technique can be used to all shapes $\lambda/1/\mu$. For a partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ such that $\lambda_1 < n-k$ and $\lambda_k = 0$, let $\lambda+1$ be the partition $(n-k, \lambda_1+1, \cdots, \lambda_{k-1}+1)$. Also for a partition $\nu = (\nu_1, \cdots, \nu_k)$ such that $\nu_1 = n-k$ and $\nu_k \geq 1$, let $\nu - 1$ be the partition $(\nu_2 - 1, \cdots, \nu_k - 1, 0)$. The following proposition shows that the generating function of proper semistandard toric tableaux is indeed a symmetric function.

Proposition 2.5.9. *We have*

$$s_{\lambda/1/\mu}(x) = \sum_{T:PSSTT_{\lambda/1/\mu}} x^T + \sum_\nu c_{\mu\nu}^{\lambda+1} s_{\nu-1/1/\phi}(x).$$

where the second sum ranges over all partition $\nu \in P_{kn}$ such that $\nu_1 = n-k$ and $\nu_k \geq 1$. Assuming Conjecture 2.5.5, this implies cylindric Schur positivity of $s_{\lambda/1/\mu}(x)$.

Proof. By Theorem 2.2.3 and Remark 2.5.4, we have $\sum_T x^T = c_{\mu\nu}^{\lambda+1} s_{\nu-1/1/\phi}(x)$ where the sum ranges over all NPSST of shape $\lambda/1/\mu$ that goes to the shape $\nu - 1/1/\phi$ after cylindric jeu de taquin. Therefore

$$\sum_{T:NPSSTT_{\lambda/1/\mu}} x^T = \sum_\nu c_{\mu\nu}^{\lambda+1} s_{\nu-1/1/\phi}(x).$$

The result follows. □

McNamara also conjectured the form of cylindric Schur expansion, and in his conjecture the case of $\lambda/1/\mu$ is as follows:

$$s_{\lambda/1/\mu}(x) = \sum_{\nu\in P_{kn}} C_{\mu\nu}^{\lambda,1} s_\nu(x) + \sum_\nu c_{\mu\nu}^{\lambda} s_{\nu/1/\phi}(x).$$

Comparing this with Proposition 2.5.9, we expect the following equality.

Proposition 2.5.10. $c^{\lambda}_{\mu,\nu} = c^{\lambda+1}_{\mu,\nu+1}$.

Proof. It is enough to find a bijection between the set of corresponding Littlewood-Richardson tableaux that count each side of the equation. Let $T^{\lambda}_{\mu,\nu}$ be a Littlewood-Richardson tableau of shape λ/μ whose Robinson-Schensten tableau has the shape ν. We construct $T^{\lambda+1}_{\mu,\nu+1}$ by the following algorithm:

1. Write 1's in the boxes at the top of each column of the shape $(\lambda+1)/\mu$.

2. Write i in the box at the rightmost place of i-th row of the shape $(\lambda+1)/\mu$.

3. The remaining boxes form a shape λ/μ, hence write in the boxes the corresponding numbers of $T^{\lambda}_{\mu,\nu}$ added by 1.

We illustrate the correspondence below in Figure 2-5 using an example of $T^{(3,2,2)}_{(1,1)(2,2,1)}$ and $n=10$, $k=5$. In this example, $(3,2,2)+1=(5,4,3,3,1)$ and $(2,2,1)+1=(5,3,3,2,1)$. It is straightforward that the correspondence is bijective for any λ, μ, ν, and Robinson-Schensted tableaux have the correct shapes. $\square$

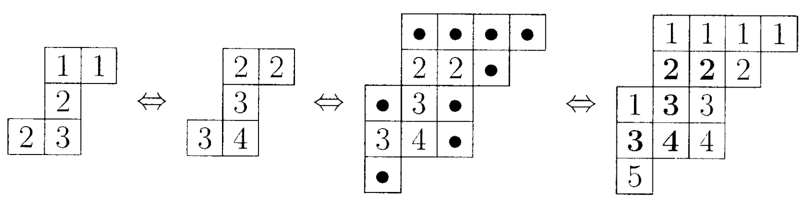

Figure 2-5: Correspondence between $T^{(3,2,2)}_{(1,1)(2,2,1)}$ and $T^{(5,4,3,3,1)}_{(1,1)(5,3,3,2,1)}$

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