Quantized Multiplicative Quiver Varieties and Actions of Higher Genus Braid Groups

by

David Andrew Jordan

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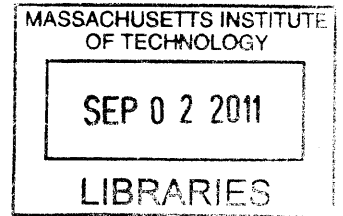in partial fulfillment of the requirements for the degree of

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Author . . . .
Department of Mathematics
April 11, 2011

Certified by. . . . .
Pavel Etingof
Professor of Mathematics
Thesis Supervisor

Accepted by . . . .
Bjorn Poonen
Chairman, Department Committee on Graduate Students

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Abstract

In this thesis, a new class of algebras called quantized multiplicative quiver varieties $A_{\mathbf{d}}^{\xi}(Q)$, is constructed, depending upon a quiver Q, its dimension vector $\mathbf{d}$, and a certain "moment map" parameter ξ. The algebras $A_{\mathbf{d}}^{\xi}(Q)$ are obtained via quantum Hamiltonian reduction of another algebra $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ relative to a quantum moment map $\mu_q^{\#}$, both of which are also constructed herein. The algebras $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ and $A_{\mathbf{d}}^{\xi}(Q)$ bear relations to many constructions in representation theory, some of which are spelled out herein, and many more whose precise formulation remains conjectural.

When Q consists of a single vertex of dimension N with a single loop, the algebra $\mathcal{D}_q(\mathrm{Mat}_N(Q))$ is isomorphic to the algebra of quantum differential operators on $G = GL_N$. In this case, for any $n \in \mathbb{Z}_{\geq 0}$, we construct a functor from the category of $\mathcal{D}_q$-modules to representations of the type A double affine Hecke algebra of rank n. This functor is an instance of a more general construction which may be applied to any quasi-triangular Hopf algebra H, and yields representations of the elliptic braid group of rank n.

Acknowledgments

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Chapter 1

Overview and Outlook

1.1   Quantized multiplicative quiver varieties

Let us consider a reductive algebraic group G, with Lie algebra $\mathfrak{g}$. Central to the representation theory of G is the following organizing diamond:

$$
\begin{array}{ccc}
 & U_q(\mathfrak{g}) & \\
\text{quasi-classical} & & \text{classical} \\
\text{limit } q \to 1 & & \text{limit } q \to 1 \\
U(\mathfrak{g}) & & O(G) \\
\text{associated} & & \text{rational} \\
\text{graded} & & \text{degeneration} \\
 & S(\mathfrak{g}) &
\end{array}
\tag{1.1}
$$

The quantum group $U_q(\mathfrak{g})$ (more precisely, its locally finite subalgebra $U'_q(\mathfrak{g})$) depends on a quantum parameter q, as well as a Rees parameter t; degenerating q and t appropriately, we can recover either the universal enveloping algebra $U(\mathfrak{g})$, or the coordinate algebra $O(G)$ of the group G. The main idea of this thesis is to apply the same organizing diamond to the algebro-geometric constructions surrounding representations of quivers, with the end-goal of producing a new quantization of the coordinate algebra of a multiplicative quiver variety.

Let $Q = (V, E)$ denote a connected quiver, with vertex set V, and directed edge set E. For $e \in E$, let $\alpha = \alpha(e)$ and $\beta = \beta(e)$ denote the tail and head of e, respectively. Fix a function $\mathbf{d} : V \to \mathbb{Z}_{\geq 0}, v \mapsto d_v$ (called a dimension vector), and consider the

following affine variety and affine algebraic group, respectively:

$$\mathrm{Mat}_{\mathbf{d}}(Q) := \prod_{e \in E} \mathrm{Mat}(\mathbb{C}^{d_\alpha}, \mathbb{C}^{d_\beta}), \quad \mathbb{G}^{\mathbf{d}} := \prod_{v \in V} GL(\mathbb{C}^{d_v}).$$

We let $\mathbb{G}^{\mathbf{d}}$ act on $X \in \mathrm{Mat}_{\mathbf{d}}(Q)$ by change of basis at each vertex,

$$(g.X)_e := g_{\beta(e)} X_e g_{\alpha(e)}^{-1}.$$

The doubled quiver $\overline{Q} = (V, \overline{E} = E \cup E^\vee)$ is built from Q by adding an adjoint arrow $\beta(e) \xrightarrow{e^\vee} \alpha(e) \in E^\vee$, for each $e \in E$. We have canonical isomorphisms,

$$T^* \mathrm{Mat}(\mathbb{C}^{d_\alpha}, \mathbb{C}^{d_\beta}) \cong \mathrm{Mat}(\mathbb{C}^{d_\alpha}, \mathbb{C}^{d_\beta}) \times \mathrm{Mat}(\mathbb{C}^{d_\beta}, \mathbb{C}^{d_\alpha}),$$

with the standard symplectic pairing given by:

$$(X, Y) = \mathrm{tr}\,(X_e Y_{e^\vee} - Y_e X_{e^\vee}).$$

Taken together, these give an identification $T^* \mathrm{Mat}_{\mathbf{d}}(Q) \cong \mathrm{Mat}_{\mathbf{d}}(\overline{Q})$.

The construction of multiplicative quiver varieties involves a certain open subset $T^* \mathrm{Mat}_{\mathbf{d}}(Q)^\circ$ defined by a non-degeneracy condition, while the construction of quantized quiver varieties involves the algebra $\mathcal{D}(\mathrm{Mat}_{\mathbf{d}}(Q))$ of polynomial differential operators on $\mathrm{Mat}_{\mathbf{d}}(Q)$, a quantization of the the symplectic structure on $T^* \mathrm{Mat}_{\mathbf{d}}(Q)$. We begin Chapter 2 by seeking an algebra $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$, situated as follows:

$$\begin{array}{ccc} & \mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q)) & \hspace{3cm} (1.2) \\[1em] \text{quasi-classical} \nearrow & & \searrow \text{classical} \\ \text{limit } q \to 1 & & \text{limit } q \to 1 \\[1em] \mathcal{D}(\mathrm{Mat}_{\mathbf{d}}(Q)) & & \mathcal{O}(T^* \mathrm{Mat}_{\mathbf{d}}(Q)^\circ) \\[1em] \searrow \text{associated} & & \nearrow \text{rational} \\ \text{graded} & & \text{degeneration} \\[1em] & \mathcal{O}(T^* \mathrm{Mat}_{\mathbf{d}}(Q)) & \end{array}$$

That is, the algebra $\mathcal{D}_q = \mathcal{D}_q(\mathrm{Mat_d}(Q))$ that we construct depends on a parameter q, as well as a Rees parameter t; by degenerating q and t appropriately, we can recover either the algebra $D(\mathrm{Mat_d}(Q))$ or $\mathcal{O}(T^* \mathrm{Mat_d}(Q)^\circ)$. We give a straightforward presentation of $\mathcal{D}_q$ by generators and relations, and show that it admits a PBW basis of ordered monomials analogous to the usual basis of $\mathcal{D}(\mathrm{Mat_d}(Q))$.

There is more than simply an analogy between the diagrams (1.1) and (1.2): indeed each vertex of (1.2) receives a "moment map" from the corresponding vertex of (1.1). We briefly recall each of these maps here.

Recall that the action of $\mathbb{G}^\mathbf{d}$ on $T^* \mathrm{Mat_d}(Q)$ is Hamiltonian; on the level of coordinate algebras, this means we have a $\mathbb{G}^\mathbf{d}$-equivariant homomorphism,

$$\mu^\# : S(\mathfrak{g}) \to \mathcal{O}(T^* \mathrm{Mat_d}(Q)),$$

of Poisson algebras, inducing the $\mathbb{G}^\mathbf{d}$-action, via the Poisson bracket. Likewise, there is a quantized moment map, a $\mathbb{G}^\mathbf{d}$-equivariant homomorphism,

$$\hat{\mu}^\# : U(\mathfrak{g}) \to \mathcal{D}(\mathrm{Mat_d}(Q)),$$

of algebras, inducing the $\mathbb{G}^\mathbf{d}$-action by via the Lie bracket. Finally, the "group-valued" moment map for the action of $\mathbb{G}^\mathbf{d}$ on $T^* \mathrm{Mat_d}(Q)^\circ$ is, on the level of coordinate algebras, a $\mathbb{G}^\mathbf{d}$-equivariant homomorphism,

$$\widetilde{\mu}^\# : \mathcal{O}(\mathbb{G}^\mathbf{d}) \to \mathcal{O}(T^* \mathrm{Mat_d}(Q)^\circ)$$

of quasi-Poisson algebras, which again induces the action of $\mathbb{G}^\mathbf{d}$ on $T^* \mathrm{Mat_d}(Q)^\circ$.

In Section 2.6, we construct an equivariant homomorphism,

$$\mu_q^\# : U_q'(\mathfrak{g}^\mathbf{d}) \to \mathcal{D}_q(\mathrm{Mat_d}(Q)),$$

which is a quantum moment map in the sense of [L], [VV]: the action of $U_q(\mathfrak{g}^\mathbf{d})$ on $\mathcal{D}_q(\mathrm{Mat_d}(Q))$ is given by a variant of the adjoint action. Each of these moment

maps is compatible with the simultaneous degenerations of source and target; we can express these relationships in the following diagram (we abbreviate $X = \mathrm{Mat}_{\mathbf{d}}(Q)$):

$$
\begin{array}{ccc}
U_q(\mathfrak{g}^{\mathbf{d}}) & \xrightarrow{\ \mu_q^{\#}\ } & \mathcal{D}_q(X) \\
U(\mathfrak{g}^{\mathbf{d}}) \xrightarrow{\ \hat{\mu}^{\#}\ } \mathcal{D}(X) & & \\
O(\mathbb{G}^{\mathbf{d}}) \xrightarrow{\ \tilde{\mu}^{\#}\ } O(T^*X^{\circ}) & & \\
S(\mathfrak{g}^{\mathbf{d}}) \xrightarrow{\ \mu^{\#}\ } O(T^*X) & &
\end{array}
\tag{1.3}
$$

Each of these moment maps allows us to construct the so-called "Hamiltonian reduction" of the target algebra with respect to a character of the source algebra. The reduction of $O(T^*X)$ by $\mu^{\#}$ at a central coadjoint orbit λ yields the algebra $\Gamma(\mathcal{M}_{\mathbf{d}}^{\lambda}(Q), \mathcal{O}_{\mathcal{M}_{\mathbf{d}}^{\lambda}(Q)})$ of global sections of structure sheaf of the quiver variety $\mathcal{M}_{\mathbf{d}}^{\lambda}(Q)$ constructed by Lusztig. Likewise, the reduction of $O(T^*X^{\circ})$ by $\tilde{\mu}^{\#}$ at a central adjoint orbit ξ of $\mathbb{G}^{\mathbf{d}}$ yields the algebra $\Gamma(\widetilde{\mathcal{M}}_{\mathbf{d}}^{\xi}(Q), \mathcal{O}_{\widetilde{\mathcal{M}}_{\mathbf{d}}^{\xi}(Q)})$ of global sections of the structure sheaf on the multiplicative quiver variety $\widetilde{\mathcal{M}}_{\mathbf{d}}^{\xi}(Q)$ constructed by Crawley-Boevey and Shaw [C-BS]. The quantized Hamiltonian reduction of $\mathcal{D}(X)$ by $\hat{\mu}^{\#}$ at a character of $U(\mathfrak{g}^{\mathbf{d}})$ yields the a quantization $\widehat{\Gamma}(\mathcal{M}_{\mathbf{d}}^{\lambda}(Q), \mathcal{O}_{\mathcal{M}_{\mathbf{d}}^{\lambda}(Q)})$ of the symplectic structure on $\mathcal{M}_{\mathbf{d}}^{\lambda}$. Finally, the quantum Hamiltonian reduction of $\mathcal{D}_q(X)$ by $\mu_q^{\#}$ at a character ξ of $U_q'(\mathfrak{g}^{\mathbf{d}})$ is a new algebra, which we denote $A_{\mathbf{d}}^{\xi}(Q)$. We obtain the following organizing diamond, echoing (1.1), (1.2):

$$
\begin{array}{ccc}
& A_{\mathbf{d}}^{\xi}(Q) & \tag{1.4} \\
\overset{\text{quasi-classical}}{\underset{\text{limit } q \to 1}{\swarrow}} & & \overset{\text{classical}}{\underset{\text{limit } q \to 1}{\searrow}} \\
\widehat{\Gamma}(\mathcal{M}_{\mathbf{d}}^{\lambda}(Q), \mathcal{O}_{\mathcal{M}_{\mathbf{d}}^{\lambda}(Q)}) & & \Gamma(\widetilde{\mathcal{M}}_{\mathbf{d}}^{\xi}(Q), \mathcal{O}_{\widetilde{\mathcal{M}}_{\mathbf{d}}^{\xi}(Q)}) \\
\overset{\text{associated}}{\underset{\text{graded}}{\searrow}} & & \overset{\text{rational}}{\underset{\text{degeneration}}{\swarrow}} \\
& \Gamma(\mathcal{M}_{\mathbf{d}}^{\lambda}(Q), \mathcal{O}_{\mathcal{M}_{\mathbf{d}}^{\lambda}(Q)}) &
\end{array}
$$

The construction of the algebra $A_{\mathbf{d}}^{\xi}(Q)$, and enumeration of its basic properties, is the primary focus of the first chapter. At the end, we discuss degenerations, and give an application to the representation theory of the spherical double affine Hecke algebra of type A_n, and those associated to star-shaped quivers.

1.2 Representations of the elliptic braid group coming from quantum D-modules on G

In Chapter 3, we study the algebra $\mathcal{D}_q(G)$ of quantum differential operators on the group $G = SL_N$. This algebra is a close relative of $\mathcal{D}_q(\mathrm{Mat}(\overset{N}{\bullet}\,\circlearrowright))$, to which our results apply equally well. This chapter is based on the publication [J]. We recall the introduction here:

Let $G = SL_N(\mathbb{C})$, and V be the vector representation of G. In [AS], Arakawa and Suzuki construct a functor F_n from the category of $U(\mathfrak{g})$-bimodules to the category of representations of the degenerate affine Hecke algebra (AHA) $\mathcal{H}_n^{deg}$. Namely, as a vector space, $F_n(M) = (V^{\otimes n} \otimes M)^{\mathfrak{g}}$ (where $\mathfrak{g} = \mathrm{Lie}(G)$ acts on M by the adjoint action), and the generators of $\mathcal{H}_n^{deg}$ act on $F_n(M)$ by certain explicit formulas.

In the paper [CEE], Calaque, Enriquez, and Etingof extended this construction to the double affine case. Namely, they upgraded the Arakawa-Suzuki functor to a functor F_n from the category of $D(G)$-modules to the category of representations of the degenerate (i.e., trigonometric) double affine Hecke algebra (DAHA) $\mathcal{H}_n^{deg}(k)$, $k = N/n$. They also considered the rational degeneration of this construction, in which one uses $D(\mathfrak{g})$-modules instead of $D(G)$-modules, and the rational DAHA $\mathcal{H}_n^{rat}(k)$ instead of the trigonometric one.

The goal of this chapter is to generalize both of these constructions to the case of quantum groups and nondegenerate AHA and DAHA. Namely, let U be a ribbon Hopf algebra with enough finite dimensional representations (i.e. the common annihilator of all finite dimensional representations is zero), and let V be an irreducible finite dimensional representation of U. Then we define a functor $F_{n,V}$ from the category of

U-bimodules to the category of representations of the affine braid group B_n^{tr}, given by the formula $F_{n,V}(M) = (V^{\otimes n} \otimes M)^{\text{inv}}$, where the invariants are taken with respect to the adjoint action of U on M, and the action of the generators of the affine braid group is defined using R-matrices. These representations are similar to those considered in [EtGe]. If V satisfies the Hecke condition (i.e. the braiding on $V \otimes V$ satisfies the equation $(\beta - q^{-1}t)(\beta + q^{-1}t^{-1}) = 0$), then $F_{n,V}(M)$ descends to a representation of the AHA $\mathcal{H}_n(t)$. More interestingly, we upgrade the functor $F_{n,V}$ to a functor from the category of D_U-modules (D-modules on the quantum group corresponding to U) to the category of representations of the elliptic braid group B_n^{ell}, which in the Hecke case lands in the category of representations of the DAHA $\mathcal{HH}_n(q,t)$. If $U = U_t(sl_N)$, V the vector representation, and $t = q^{nk}$, then in the quasiclassical limit $q \to 1$ we recover the functors from [AS] and [CEE], respectively.

Our construction is also a generalization of the work of Lyubashenko and Majid [Ly], [LyMa], where an action of the elliptic braid group is obtained on $(V^{\otimes n} \otimes A)^{\text{inv}}$, where A is the dual Hopf algebra of U. Indeed, A is the most basic example of a D_U-module (the module of functions on the quantum group).

1.3 Future directions

One motivation of Chapter 2 is to extend the unifying structure of quivers to a myriad of constructions in "quantum" algebraic geometry, such as quantum planes, and their q-Weyl algebras, FRT algebras, reflection equation algebras, differential operator algebras on quantum groups, and perhaps most importantly, double affine Hecke algebras.

In particular, for certain explicit quivers and parameters, we are able to identify our algebras $A_{\mathbf{d}}^{\xi}(Q)$ with spherical double affine Hecke algebras of type A, and also with generalized double affine Hecke algebras associated to star-shaped quivers (see Example 2.3.12). We anticipate relations with Gan-Ginzburg algebras [GG1], as q-deformations of Montarani's constructions [Mo], which relate symplectic reflection algebras to D-modules on quivers. Given the ubiquity of quiver formalism in Lie

theory, we expect to be able to apply our deformation procedure in a wide class of examples to obtain deformations of Lie theoretic objects.

When the quantum parameter q is a non-trivial root of unity, the algebras $\mathcal{D}_q$ and their reductions $A_{\mathbf{d}}^{\xi}(Q)$ obtain large "q-centers;" in simple examples, $\mathcal{D}_q$ is Azumaya over its q-center, which turns out to be identified with $O(T^*X^\circ)$. In such examples, the algebra $A_{\mathbf{d}}^{\xi}(Q)$ is Azumaya over its q-center, which is identified with the corresponding multiplicative quiver variety. We intend to explore this property in general, and thus obtain a second "quasi-classical" degeneration of our algebra, analogous to the well-known "p-center" phenomenon for varieties in characteristic p.

The results of Chapter 3 also point towards many directions of future research. In subsequent papers, we plan to study the representation-theoretic properties of the functor $F_{n,V}$, i.e. what it does to particular U-bimodules and D_U-modules; in particular, it would be interesting to consider the case of roots of unity. Also, our construction paves way for a number of further generalizations, some of which we have already explored [JM], and some of which plan to explore in the future. One direction where some progress has been made is the quantum generalization of the paper [EFM], which generalizes the construction of [CEE] from the type A case to the type BC case: it defines a functor from twisted D-modules on the symmetric space $GL_N/GL_p \times GL_q$ $(p+q=N)$ to representations of degenerate DAHA of type BC_n. This generalization involves quantum symmetric spaces and the Sahi-Stokman BC_n DAHA; [JM] is related to the paper [OS] in a way similar to the relation between the present paper and the construction of [VV], explained above. Another interesting direction is the generalization to the case of an arbitrary ribbon category $\mathcal{C}$, which is not necessarily the category of representations of a ribbon Hopf algebra. For example, if this category is semisimple, the role of the dual A of U will be played by $\oplus_X X \otimes X^*$, where X runs over all simple objects of $\mathcal{C}$. In this case, if $M = A$, our construction would recover the natural elliptic braid group action on the genus 1 modular functor, described in the book [BK]. We note that this elliptic braid group action comes with a compatible action of the modular group $SL_2(\mathbb{Z})$, and we expect that under some conditions on the D-module M, such an action will exist on $F_{n,V}(M)$; in the Hecke

case this will recover the difference Fourier transform of Cherednik [Ch1].

More generally, we hope to combine the techniques of Chapter 3 with the new examples of Chapter 2 to construct representations of braid groups of punctured, Riemann surfaces of higher genus.

Chapter 2

Quantized multiplicative quiver varieties

2.1 Introduction

The aim of this chapter is to construct "q-quantizations" of Lusztig's quiver varieties, which are simultaneously q-deformations of quantized quiver varieties constructed in [GG2], and quantizations of the moduli of representations of multiplicative pre-projective algebras of Crawley-Boevey and Shaw [C-BS]. Here, "quantization" is with respect to some (quasi-) Poisson structure, e.g. the replacement of a cotangent space T^*X with the algebra $\mathcal{D}(X)$ of differential operators on X, while "q-deformation" involves the introduction of quantum groups and, with them, the parameter q. The combination of these two techniques we call "q-quantization": the output is an algebra of multiplicatively deformed differential operators on the space X.

To begin, we build an algebra $\mathcal{D}_q = \mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ of quantum differential operators on the space of matrices associated to a quiver Q and its dimension vector $\mathbf{d}$. This algebra admits an action by a quantum group $U_q(\gg^{\mathbf{d}})$, acting by change of basis at each vertex. There is a "quantum moment map" $\mu_q^{\#}$ for this action, and we construct a "quantum Hamiltonian reduction" $A_{\mathbf{d}}^{\xi}(Q)$, relative to $\mu_q^{\#}$ and character ξ. A basic example of a representation of $\mathcal{D}_q$ is the algebra $\mathcal{O}_q$ of quantum coordinates on the matrix space of Q. Its ξ-invariant subalgebra $\mathcal{O}_q^{\mathbb{G}_q, \xi}$ is a basic example of a

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representation of $A_{\mathbf{d}}^{\xi}(Q)$.

Applying work of Crawley-Boevey on the flatness of moment maps for classical quiver varieties, we are able to show in a large class of examples that the algebras $A_{\mathbf{d}}^{\xi}(Q)$ are flat non-commutative deformations of their classical counterparts, and that they quantize well-known classical Poisson structures.

In this extended introduction, we begin by reviewing the geometric constructions which underpin our work; we then summarize our results in the q-deformed setting, and outline the future directions we intend to pursue.

2.1.1 Moduli spaces of quiver representations

Let $Q = (V, E)$ denote a connected quiver, with vertex set V, and directed edge set E. For $e \in E$, let $\alpha = \alpha(e)$ and $\beta = \beta(e)$ denote the tail and head of e, respectively. The subject of much study is the category $\operatorname{Rep} Q$, of representations of Q. An object X of $\operatorname{Rep} Q$ is an assignment of a finite dimensional vector space X_v over $\mathbb{C}$ to each $v \in V$, and a linear operator $X_e : X_\alpha \to X_\beta$ to each $e \in E$. A morphism ϕ between X and Y is a collection of linear maps $X_v \to Y_v$, which satisfy $Y_e \circ \phi_\alpha = \phi_\beta \circ X_e$, for all $e \in E$.

Fix a function $\mathbf{d} : V \to \mathbb{Z}_{\geq 0}, v \mapsto d_v$ (called a dimension vector), and consider the following affine variety and affine algebraic group, respectively:

$$\operatorname{Mat}_{\mathbf{d}}(Q) := \prod_{e \in E} \operatorname{Mat}(\mathbb{C}^{d_\alpha}, \mathbb{C}^{d_\beta}), \quad \mathbb{G}^{\mathbf{d}} := \prod_{v \in V} GL(\mathbb{C}^{d_v}).$$

We let $\mathbb{G}^{\mathbf{d}}$ act on $M \in \operatorname{Mat}_{\mathbf{d}}(Q)$ by change of basis at each vertex,

$$(g.M)_e := g_{\beta(e)} M_e g_{\alpha(e)}^{-1}.$$

Many important applications of the representation theory of quivers involve the doubled quiver $\overline{Q} = (V, \overline{E} = E \cup E^{\vee})$, built from Q by adding an adjoint arrow

$\beta(e) \xrightarrow{e^\vee} \alpha(e) \in E^\vee$, for each $e \in E$. We have canonical isomorphisms,

$$T^* \mathrm{Mat}(\mathbb{C}^{d_\alpha}, \mathbb{C}^{d_\beta}) \cong \mathrm{Mat}(\mathbb{C}^{d_\alpha}, \mathbb{C}^{d_\beta}) \times \mathrm{Mat}(\mathbb{C}^{d_\beta}, \mathbb{C}^{d_\alpha}),$$

with the standard symplectic pairing given by:

$$(M, N) = \mathrm{tr}\left(M_e N_{e^\vee} - N_e M_{e^\vee}\right).$$

Taken together, these give an identification $T^* \mathrm{Mat}_{\mathbf{d}} Q \cong \mathrm{Mat}_{\mathbf{d}} \overline{Q}$. Clearly, $\mathbb{G}^{\mathbf{d}}$ acts by symplectomorphisms; moreover, the action admits a moment map:

$$\mu : \mathrm{Mat}_{\mathbf{d}}(\overline{Q}) \to \mathfrak{g}^{\mathbf{d}},$$

$$M \mapsto \sum_{e \in E} [M_e, M_{e^\vee}],$$

where we set $\mathfrak{g}^{\mathbf{d}} := \mathrm{Lie}(\mathbb{G}^{\mathbf{d}})$. Thus we may construct the Hamiltonian reduction along $\mu^{-1}(0)$:

$$\mathcal{M}_{\mathbf{d}}(Q) := \mathrm{Mat}_{\mathbf{d}}(\overline{Q}) /\!\!/_{\mu,0} \mathbb{G}^{\mathbf{d}},$$

a Poisson affine algebraic variety. That is, we first impose the condition on $M \in \mathrm{Mat}_{\mathbf{d}}(\overline{Q})$ that:

$$\sum_{e \in E} [M_e, M_{e^\vee}] = 0, \tag{2.1}$$

and we then take the categorical quotient of the subvariety of such M by the action of $\mathbb{G}^{\mathbf{d}}$. On the level of coordinate functions, we have:

$$\mathcal{O}(\mathcal{M}_{\mathbf{d}}(Q)) := \left(\mathcal{A} \Big/ \mathcal{A}\mu^{\#}(\mathfrak{g}^{\mathbf{d}})\right)^{\mathbb{G}^{\mathbf{d}}},$$

where $\mathcal{A} = \mathcal{O}(T^* \mathrm{Mat}_{\mathbf{d}}(Q))$.

2.1.2   Deformed pre-projective algebras

The *preprojective algebra*, $\Pi_0(Q)$ of Q [GP], is the quotient of the path algebra $\mathbb{C}\overline{Q}$ by relation,

$$\sum_{e \in E} [e, e^{\vee}] = 0,$$

corresponding to equation (2.1). The variety $\mathcal{M}_{\mathbf{d}}^{\mathbf{0}}(Q)$ may thus be interpreted as a moduli space of $\mathbf{d}$-dimensional semi-simple representations of $\Pi_0(Q)$. More generally, given a vector $\lambda : V \to \mathbb{C}$, we may construct the Hamiltonian reduction $\mathcal{M}_{\mathbf{d}}^{\lambda}$ along $\mu^{-1}(\sum \lambda_v \operatorname{id}_v)$. That is, we first impose the condition on $M \in \operatorname{Mat}_{\mathbf{d}}(Q)$ that:

$$\sum_{e \in E} [M_e, M_{e^{\vee}}] = \sum_v \lambda_v \operatorname{id}_v, \tag{2.2}$$

and then take the categorical quotient of the subvariety of such M by the action of $\mathbb{G}^{\mathbf{d}}$. We assume that $\lambda \cdot d = 0$, as otherwise equation (2.2) implies that $\mathcal{M}_{\mathbf{d}}^{\lambda}$ is empty. The *deformed pre-projective algebras*, $\Pi_{\lambda}(Q)$, were constructed by Crawley-Boevey and Holland in [C-BH], and have since received wide attention. These algebras are quotients of the path algebra $\mathbb{C}\overline{Q}$ by the relation

$$\sum_{e \in E} [e, e^{\vee}] = \sum_{v \in V} \lambda_v \iota_v,$$

corresponding to equation (2.2). The variety $\mathcal{M}_{\mathbf{d}}^{\lambda}$ may be interpreted as a moduli space of semi-simple representations of $\Pi_{\lambda}(Q)$.

In the present work, we will be concerned with certain flat non-commutative deformations of the variety $\mathcal{M}_{\mathbf{d}}^{\lambda}(Q)$. The flatness of our deformations depends, in turn, on the flatness of the classical moment map μ. Fortunately, there is a completely explicit criterion for the flatness of μ, due to Crawley-Boevey. Let A denote the Cartan matrix associated to Q, and let $p : \mathbb{Z}^V \to \mathbb{C}$ denote the function:

$$p(d) := 1 - \frac{1}{2}(d, Ad) = 1 + \sum_{e \in E} d_{\alpha(e)} d_{\beta(e)} - \sum_{v \in V} d_v^2.$$

We have:

Theorem 2.1.1. *[C-B1] The following are equivalent:*

1. *μ is a flat morphism of algebraic varieties.*

2. *$\mu^{-1}(0)$ has dimension $(d,d) - 1 + 2p(d)$.*

3. *$p(d) \geq \sum_i p(r_i)$, for any decomposition $d = \sum_i r_i$ into positive roots r_i.*

Moreover, if it happens that d satisfies the strict inequality in (3) for all possible non-trivial decompositions $d = \sum_i r_i$ into positive roots r_i, then it is shown in [C-BEG], Theorem 11.3.1, that the fibers, $\mu^{-1}(\sum_v \lambda_v \operatorname{id}_v)$, are all reduced and irreducible complete intersections. In this case, $\mathcal{M}_\mathbf{d}^\lambda$ coincides with its smooth resolution for generic λ, and in particular, both are actually affine.

For Dynkin quivers Q, Theorem 2.1.1 asserts that μ is flat if and only if $\mathbf{d}$ is a positive root for Q; in this case the classical Hamiltonian reduction is zero-dimensional, so these are not interesting examples from the point of view of deformation theory.

For affine Dynkin quivers Q, let δ denote the positive generator of the imaginary root lattice. In this case, Theorem 2.1.1 asserts that μ is flat in one of two cases: when $\mathbf{d} = r_i + \delta$, for a root r_i of the ordinary Dynkin quiver associated to Q, or when $\mathbf{d} = \delta$. In the former case, the classical Hamiltonian reduction is zero-dimensional, while in the latter case it is two-dimensional, and gives the Kleinian singularity associated with Q.

The most interesting examples come from quivers Q, which are neither of Dynkin nor affine-Dynkin type. For such quivers, it is shown in [C-BEG], Lemma 11.3.3, that the strict version of condition (3) above is satisfied by a Zariski-dense set Σ_0 of dimension vectors $\mathbf{d} \in \mathbb{Z}_{\geq 0}^V$. Thus, such Q produce a rich family of examples of flat Hamiltonian reductions of positive dimension. Of particular interest are the so-called "Calogero-Moser" quivers obtained by adding a "base" vertex $\tilde{v}$ to an affine Dynkin quiver, whose unique edge connects it to the extending vertex. In this case, the dimension vector $n\delta + \tilde{v}$ satisfies the strict version of condition (3) in Theorem 2.1.1 for any $n \geq 0$.

2.1.3 Multiplicative deformed pre-projective algebras

The deformed pre-projective algebra admits a multiplicative deformation, which may be described as follows. Extend $e \mapsto e^{\vee}$ to an involution on $\overline{E}$, by setting $e^{\vee\vee} := e$, and define $\epsilon(e) = 1$, if $e \in E$, -1 else. We choose an ordering on the edges $e \in \overline{E}$, and a function $\xi : V \to \mathbb{C}^{\times}$. First, we restrict our attention to the set $T^* \operatorname{Mat}_{\mathbf{d}}(Q)^{\circ}$ of $X \in T^* \operatorname{Mat}_{\mathbf{d}}(Q) \cong \operatorname{Mat}_{\mathbf{d}}(\overline{Q})$ such that, for each $e \in \overline{E}$, the matrices $(\operatorname{id}_{\alpha} + X_{e^{\vee}} X_e)$ are invertible. Further, we impose the following restriction, which is a multiplicative version of equation $(2.2)^1$:

$$\overrightarrow{\prod_{e \in \overline{E}}} (\operatorname{id}_{\alpha} + M_{e^{\vee}} M_e)^{\epsilon(e)} = \sum \xi_v \operatorname{id}_v . \qquad (2.3)$$

Taking the categorical quotient by the action of $\mathbb{G}^{\mathbf{d}}$, we obtain the space $\widetilde{\mathcal{M}}_{\mathbf{d}}^{\xi}$, which again has an interpretation as moduli of $\mathbf{d}$-dimensional semi-simple representations for a certain localization of $\mathbb{C}\overline{Q}$, known as the multiplicative deformed pre-projective algebra. As has been noted in [C-BS] and [VdB1, VdB2], the variety $\widehat{\mathcal{M}}_{\mathbf{d}}^{\xi}$ is in fact an instance of quasi-Hamiltonian reduction along a "group-valued" moment map,

$$\widetilde{\mu} : T^* \operatorname{Mat}_{\mathbf{d}}(Q)^{\circ} \to \mathbb{G}^{\mathbf{d}}$$

$$X \mapsto \overrightarrow{\prod_{e \in E}} (\operatorname{id}_{\alpha} + X_{e^{\vee}} X_e)^{\epsilon(e)},$$

with respect to the central element $(\xi_v \operatorname{id}_v)_{v \in V} \in \mathbb{G}^{\mathbf{d}}$. Quasi-Hamiltonian reduction is a multiplicative analog of Hamiltonian reduction, as defined by Alekseev and Kosmann-Schwarzbach in [AK-S]. See also [AMM], [AK-SM], for foundational development of quasi-Poisson geometry, and group-valued moment maps.

2.1.4 Quantized quiver varieties

Finally, there is a quantization of the variety $\mathcal{M}_{\mathbf{d}}^{\lambda}$ [GG2], which involves replacing the cotangent bundle to $\operatorname{Mat}_{\mathbf{d}}(Q)$ with its quantization, the algebra $\mathcal{D}(\operatorname{Mat}_{\mathbf{d}}(Q))$

---

[1]here, $\overrightarrow{\prod}$ denotes ordered product; see Section 2.2.3.

of differential operators. The algebra $\mathcal{D}(\mathrm{Mat}_{\mathbf{d}}(Q))$ quantizes the symplectic form on $T^* \mathrm{Mat}_{\mathbf{d}}(Q)$, and one constructs its quantum Hamiltonian reduction $\widehat{\Gamma}(\mathcal{M}_{\mathbf{d}}^{\lambda}(Q), \mathcal{O}_{\mathcal{M}_{\mathbf{d}}^{\lambda}(Q)})$ along the homomorphism,

$$\widehat{\mu}^{\#} : U(\mathfrak{g}^{\mathbf{d}}) \to \mathcal{D}(\mathrm{Mat}_{\mathbf{d}}(Q)),$$

$$X \in \mathfrak{g} \mapsto L_X,$$

where L_X is the vector field generated by the action of X. The reduction is taken with repsect to a character $\lambda : U(\mathfrak{g}^{\mathbf{d}}) \to \mathbb{C}$. For an exposition of quantum Hamiltonian reduction, see [L]. For some applications, see [E1], [Mo].

2.1.5 The multiplicative Deligne-Simpson problem

The applications of quiver varieties and (multiplicative, deformed) pre-projective algebras to diverse areas of mathematics are too many to list here; as such we mention only one important application, due to Crawley-Boevey and Shaw [C-BS]. Given conjugacy classes $C_1, \ldots, C_n \subset GL(V)$, the Deligne-Simpson problem asks when there exists an irreducible local system on $\mathbb{P}^1 \backslash \{p_1, \ldots p_n\}$ with monodromy around each p_i given by a matrix $A_i \in C_i$. Thus, the Deligne-Simpson problem concerns the classification of n-tuples of matrices $A_i \in C_i$ without common fixed subspace, satisfying:

$$A_1 \cdots A_n = \mathrm{id},$$

up to simultaneous conjugation of the A_i.

Crawley-Boevey and Shaw were able to answer this question rather concretely in terms of the root data of a certain star-shaped quiver Q, which encodes the conjugacy classes C_i. They determine for which Q, with the relevant dimension vector $\mathbf{d}$, the variety $\widehat{\mathcal{M}_{\mathbf{d}}^q}$ is non-empty and, in this case, what is its dimension. Still, the finer geometry of these varieties is not completely well-understood. The connection between multiplicative quiver varieties and fundamental groups of Riemann surfaces is a major motivation for the present work.

In particular, there is a well-known symplectic structure on the space of bundles with flat connections on a compact, closed oriented two-manifold with boundary of genus g. A quantization of this symplectic structure has been considered in [FR], and constructed in [RS]; our results provide another construction, and a generalization to arbitrary quivers.

2.1.6 Outline of results

In Section 3.4, beginning with the data of a quiver Q and its dimension vector $\mathbf{d}$, we construct an algebra $\mathcal{D}_q = \mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$. The algebra $\mathcal{D}_q$ is a braided tensor product of the algebras $\mathcal{D}_q(e)$ associated to each edge e of Q, while each $\mathcal{D}_q(e)$ is a straightforward q-deformation of the Weyl algebra associated to the standard affine space $\mathrm{Mat}(d_\alpha, d_\beta)$. The relations of $\mathcal{D}_q(e)$ are given in a formal way designed to make their equivariance properties evident; the reader interested in a hands-on, RTT-type presentation can skip ahead to Sections 2.3.4 and 2.3.5.

Our first theorem is Theorem 2.4.3, which states that the algebra $\mathcal{D}_q$ is a flat q-deformation of the algebra $\mathcal{D}(\mathrm{Mat}_{\mathbf{d}}(Q))$ of differential operators on the matrix space of the quiver Q. Our proof is modeled on Theorem 1.5 of [GZ], and consists of constructing an explicit PBW basis of ordered monomials, which clearly deforms the usual basis of $\mathcal{D}(\mathrm{Mat}_{\mathbf{d}}(Q))$.

The defining relations for $\mathcal{D}_q$ in examples related to quantum groups are similar to the *FRT*-construction of quantum coordinate algebras, and are also closely related to the algebras $\mathcal{D}_q(GL_N)$ of quantum differential operators on GL_N, which have been studied by many authors. In Sections 2.3.4 and 2.3.5, we list out the relations in detail for these examples of interest, and explain their relation to known constructions.

The algebra $\mathcal{D}_q$ possesses certain elements $\det_q(e)$, for each edge $e \in E$, which conjugate standard monomials in $\mathcal{D}_q$ by powers of q (the proof of this assertion is delayed until Section 4, Corollaries 2.6.13 and 2.6.15). We therefore localize $\mathcal{D}_q$ at the multiplicative Ore set generated by these q-determinants, to obtain an algebra $\mathcal{D}_q^\circ$, in which certain quantum matrices become invertible.

In Section 2.5, we construct a q-analog, $\mathcal{F}$, of the classical Fourier transform

map on the algebra $\mathcal{D}(\mathrm{Mat}_\mathbf{d}(Q))$, which allows us to prove the independence of $\mathcal{D}_q^\circ(\mathrm{Mat}_\mathbf{d}(Q))$ on the orientation of Q. Our main results in this section are Definition-Propositions 2.5.7 and 2.5.11, where $\mathcal{F}$ is defined explicitly on generators; necessary relations are checked directly. As a warmup, we work out one-dimensional examples in Definition-Propositions 2.5.4 and 2.5.9, whose proofs foreshadow the general one.

In Section 2.6, we define a q-deformed, braided analog $\mu_q^\#$ of the multiplicative moment map underlying relation (2.3). We subsequently define an analog of Hamiltonian reduction in this context, which is closely related to Lu's notion [L] for Hopf algebras, and is also inspired by the quantum moment maps appearing in [VV]. The output of this Hamiltonian reduction is an algebra $A_\mathbf{d}^\lambda(Q)$, which q-quantizes the space $\mathcal{M}_\mathbf{d}^\lambda$. The main results of Section 6 are Definition-Propositions 2.6.14 and 2.6.11, and Propositions 2.6.12 and 2.6.21, in which the moment map is defined, and the moment map condition is verified.

In Section 2.8, we consider relations between the algebra $A_\mathbf{d}^\lambda(Q)$ and well-known constructions in representation theory - specifically quiver varieties and spherical double affine Hecke algebras. To begin, we study flatness properties of $A_\mathbf{d}^\lambda(Q)$ as the parameter q varies. While the flatness of the algebra $\mathcal{D}_q$ is proven directly, the flatness of the algebra $A_\mathbf{d}^\lambda(Q)$ is considerably more subtle. This is because the argument we give for $\mathcal{D}_q$ relies upon the existence of a $\mathbb{Z}$-grading with finite dimensional graded components; this grading does not descend to $A_\mathbf{d}^\lambda(Q)$.

For this reason, we restrict ourselves to situations where the classical moment map μ is flat (as in Theorem 2.1.1), and we consider the question of formal flatness of $A_\mathbf{d}^\lambda(Q)$. That is, we set $q = e^\hbar$, and consider the algebras $\mathcal{D}_q$ and $U_q(\mathfrak{g}^\mathbf{d})$, moment map μ_q, and Hamiltonian reduction $A_\mathbf{d}^\lambda(Q)$ all in the category of $\mathbb{C}[[\hbar]]$-modules. We prove that $A_\mathbf{d}^\lambda(Q)$ is a topologically free $\mathbb{C}[[\hbar]]$-module, so that the deformation is flat in the formal neighborhood of $q = 1$.

Next, we address the question: what algebra does $A_\mathbf{d}^\lambda(Q)$ deform? In answering this question, we must explain that there is a unifier in the construction of $\mu_q^\#$, and in its simultaneous relation to (classical, multiplicative, and quantized) moment maps μ. Recall the two variations of Hamiltonian reduction in classical geometry: "quan-

tized Hamiltonian reduction" and "quasi-Hamiltonian reduction". In the former, the moment map is a homomorphism of algebras

$$\widehat{\mu}^{\#} : U(\mathfrak{g}^{\mathbf{d}}) \to \mathcal{D}(\mathrm{Mat}_{\mathbf{d}}(Q)),$$

while in the latter, we have a morphism of varieties $\widetilde{\mu} : T^{*}\mathrm{Mat}_{\mathbf{d}}(Q)^{\circ} \to \mathbb{G}^{\mathbf{d}}$, or equivalently a map of algebras

$$\widetilde{\mu}^{\#} : \mathcal{O}(\mathbb{G}^{\mathbf{d}}) \to \mathcal{O}(T^{*}\mathrm{Mat}_{\mathbf{d}}(Q)^{\circ}).$$

In classical geometry, there are analogies between these moment maps, but not a precise connection. We will see that the map $\mu_{q}^{\#}$ bears a precise relationship to both maps $\widetilde{\mu}^{\#}$ and $\widehat{\mu}^{\#}$, under degeneration.

Recall that the Hopf algebra $U = U_{q}(\mathfrak{g}^{\mathbf{d}})$ has a large co-ideal subalgebra U', consisting of the elements which are locally finite under the adjoint action of U on itself (see [JL] for details, and for the sense in which U' is "large"). The homomorphism $\mu_{q}^{\#}$ maps out of U', and is a q-deformed quantum moment map, as considered by Lu [L] and [VV]. On the other hand, we have Majid's covariantized coordinate algebra $A_{q}(\mathbb{G}^{\mathbf{d}})$, a flat deformation of $O(\mathbb{G}^{\mathbf{d}})$, and we have the Rosso isomorphism $\kappa : A_{q}(G) \xrightarrow{\sim} U'$ (see [Ma]). Thus, we may also view $\mu_{q}^{\#}$ as a quantization of the group-valued moment map underlying equation (2.3). We summarize these relationships in the following diagram:

$$
\begin{array}{ccccccc}
\mathcal{O}(\mathbb{G}^{\mathbf{d}}) & \xleftarrow[q \to 1]{\rightsquigarrow} & A_{q}(\mathbb{G}^{\mathbf{d}}) & \xleftarrow[\kappa]{\sim} & U'_{q}(\mathfrak{g}^{\mathbf{d}}) & \xrightarrow[q \to 1]{\rightsquigarrow} & U(\mathfrak{g}) \\
{\scriptstyle \mu^{\#}}\big\downarrow & & {\scriptstyle \mu_{q}^{\#}}\big\downarrow & & {\scriptstyle \mu_{q}^{\#}}\big\downarrow & & {\scriptstyle \widehat{\mu}}\big\downarrow \\
\mathcal{O}(\mathrm{Mat}_{\mathbf{d}}(Q)^{\circ}) & \xleftarrow[q \to 1]{\rightsquigarrow} & \mathcal{D}_{q} & = \!\!=\!\!=\!\!=\!\!= & \mathcal{D}_{q} & \xrightarrow[q \to 1]{\rightsquigarrow} & \mathcal{D}(\mathrm{Mat}_{\mathbf{d}}(\overline{Q}))
\end{array}
$$

Thus, taking quasi-Hamiltonian reduction along $\mu^{\#}$, q-deformed quantum Hamiltonian reduction along $\mu_{q}^{\#}$, and quantum Hamiltonian reduction along $\widehat{\mu}$, we have the "commutative diagram" (1.4) of deformations and degenerations of the corresponding Hamiltonian reductions.

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As an application, we show in Theorem 2.9.4 that the algebra $A_{\mathbf{d}}^\lambda(Q)$ is isomorphic to the spherical DAHA of type A_n, when Q and d are the Calogero-Moser quiver and dimension vector:

$$(Q, d) = \overset{1}{\bullet} \to \overset{n}{\bullet} \circlearrowright,$$

which allows us to give a new description of the representation category of the spherical DAHA as a quotient of the category of equivariant $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$-modules by a certain Serre subcategory of aspherical modules. This assertion follows from generalities about flat deformations, together with the fact that the spherical DAHA is the universal deformation of the corresponding rational Cherednik algebra, which itself may be built by quantum Hamiltonian reduction from $\widehat{\Gamma}(\mathcal{M}_{\mathbf{d}}^\lambda(Q), \mathcal{O}_{\mathcal{M}_{\mathbf{d}}^\lambda(Q)})$. In fact, because we have restricted to formal parameters $q = e^\hbar$, this result is not very valuable, as the spherical DAHA is actually a trivial deformation over $\mathbb{C}[[\hbar]]$ of the spherical rational Cherednik algebra. However, we expect this isomorphism to hold also numerically, for generic q and λ.

Likewise, if Q is a so-called star-shaped quiver (meaning all vertices are uni- or bi-valent, except for a single vertex, called the node) we have an isomorphism between $A_{\mathbf{d}}^\lambda(Q)$ and the generalized spherical DAHA, defined in [EOR]. Star-shaped quivers play a central role in the approach to the Deligne-Simpson problem in [C-B2], [C-BS]. These applications suggest that the algebras $A_{\mathbf{d}}^\lambda(Q)$ may be viewed as further generalizations of the (spherical) DAHA, to an arbitrary quiver Q.

2.2 Construction of $\mathcal{O}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ and $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$

2.2.1 Discussion

The constructions in this section are phrased in the language of braided tensor categories, while all that is essential for our primary example is a vector space V, the tensor flip $\tau : v \otimes w \mapsto w \otimes v$, and a Hecke R-matrix, $R : V \otimes V \to V \otimes V$, satisfying

the "quantum Yang Baxter" equation (QYBE) ,

$$\tau_{12}R_{12}\tau_{23}R_{23}\tau_{12}R_{12} = \tau_{23}R_{23}\tau_{12}R_{12}\tau_{23}R_{23} : V \otimes V \otimes V \to V \otimes V \otimes V,$$

and the quadratic "Hecke" relations:

$$\tau \circ R - R^{-1} \circ \tau = (q - q^{-1})\,\mathrm{id} \otimes \mathrm{id}.$$

There are nevertheless several practical reasons for adopting the tensor categorical formalism over the more concrete data of Hecke R-matrices.

First, when deforming algebras with geometrical significance, it is often not clear at the outset precisely how to proceed: the set of "bad" definitions is open dense in the space of all possible definitions. That is, given only the goal of producing some new algebra with similar generators and relations, which "degenerates" to the classical algebra when $q \to 1$, there is far too much flexibility, and many pathologies can arise (as regards flatness, zero-divisors, localizations, etc.). However, in the present work, we require that our algebras $\mathcal{D}_q(\mathrm{Mat_d}(Q))$ enjoy the following properties:

1. $\mathcal{D}_q(\mathrm{Mat_d}(Q))$ is a algebraically flat deformation of $\mathcal{D}(\mathrm{Mat_d}(Q))$. This means we exhibit an explicit PBW-basis for $\mathcal{D}_q(\mathrm{Mat_d}(Q))$ specializing to the standard monomials when $q = 1$. This condition is much stronger than being formally flat.

2. $\mathcal{D}_q(\mathrm{Mat_d}(Q))$ carries an action of the quantum group $U_q(\mathfrak{g^d})$, which quantizes $\mathbb{G^d}$:
$$U_q(\mathfrak{g^d}) := \bigotimes_{v \in V} U_q(\mathfrak{gl}_{d_v}).$$

3. There exists a "quantum moment map" $\mu_q^{\#}$, simultaneously quantizing and q-deforming the classical moment map μ.

Requirements (1) and (2) suggest that the algebra $\mathcal{D}_q(\mathrm{Mat_d}(Q))$ necessarily is an algebra in the braided tensor category $\mathcal{C} = U_q(\mathfrak{g^d})$-lfmod of locally finite modules for $U_q(\mathfrak{g^d})$. This drastically restricts which sorts of algebras we may consider, namely

to those whose generators and relations express as the image of morphisms in the braided tensor category $\mathcal{C}$.

Secondly, in condition (3), we require the moment map itself to be equivariant for the quantum group, which means that it is a homomorphism of algebras in $\mathcal{C}$. Since we wish the construction to be uniform for different dimension vectors $\mathbf{d}$, it is natural to allow ourselves only the axioms of a braided tensor category, together with the Hecke relation on the braiding. This turns out to be a useful restriction, as it narrows our focus sufficiently such that the "right" definitions are essentially the only ones we are able to write down.

A third practical benefit from working with braided tensor categories has already surfaced in [J], [JM], where we studied interplay between certain algebraic constructions in Lie theory and geometry of spaces of configurations of points on Riemann surfaces. These constructions are greatly clarified by the use of braided tensor categorical language and quantum groups, in the same way that the language of braided tensor categories clarifies the connections between quantum groups and knot invariants.

In addition to the practical motivations above, there are two more substantive motivations for working with braided tensor categories. The first is that there are more braided tensor categories besides $U_q(\mathfrak{g}^{\mathbf{d}})$-lfmod that we can associate to Q. Two particularly tantalizing examples are:

1. Fusion categories associated to quantum groups at roots of unity

2. Deligne's categories $U_q(\mathfrak{g}_\nu)$-mod, where $\nu : V \to \mathbb{C}$ has as values arbitrary complex numbers, rather than positive integers.

We hope that the methods of this paper will go through in these settings more or less intact, which would open the door for connections to modular categories and invariants of links and knots on higher genus surfaces. The second motivation is related to the notion of a quasi-symmetric tensor category, which is a braided tensor

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category over $\mathbb{C}[[\hbar]]$ such that the braiding satisfies:

$$\sigma_{W,V}\sigma_{V,W} = \mathrm{id}_{V\otimes W} \quad \mathrm{mod}\ \hbar.$$

It is well-known how to degenerate such categories into symmetric tensor categories. In case $\mathcal{C}$ is the representation category of a quantum group $U_q(\mathfrak{g})$, the first-order term in $\hbar$ often carries some interesting data for Lie theory: for instance, the first non-trivial term of $\sigma_{W,V}\sigma_{V,W}$ is essentially the Casimir operator $\Omega \in \mathrm{Sym}^2(\mathfrak{g})^{\mathfrak{g}}$, while the first non-trivial term of the associator is the unique invariant alternating 3-form, $\phi \in \Lambda^3(\mathfrak{g})^{\mathfrak{g}}$.

As an application of these ideas one can recover the axioms of quasi-Poisson geometry as first-order degenerations of the axioms for algebras in braided tensor categories. It is our hope that the axioms of "group-valued moment maps" can also be obtained as degenerations of the notion of quantum Hamiltonian reduction.

2.2.2   Reminders on braided tensor categories

In this section, we recall some basic constructions involving braided tensor categories, in order to fix notations. As such, we do not discuss all details, but only those we will use explicitly. For clarity's sake, we supress instances of the associativity and unit isomorphisms in definitions and commutative diagrams, as they can be inserted uniquely, if necessary.

Recall that a tensor category is a $\mathbb{C}$-linear abelian category $\mathcal{D}$, together with a biadditive functor,

$$\otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D},$$

linear on Hom's, together with a unit $\mathbf{1} \in \mathcal{C}$, associativity isomorphism α, and unit isomorphisms. These are required to satisfy a well-known list of axioms, which we do not recall here. A tensor functor $F = (F, J)$ between tensor categories $\mathcal{D}_1$ and $\mathcal{D}_2$ is an exact functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ of underlying abelian categories, together with a

functorial isomorphism,

$$J : F(-) \otimes F(-) \to F(- \otimes -),$$

respecting units and associators in the appropriate sense. The opposite tensor category $\mathcal{D}^\vee$ is the same underlying abelian category, with tensor product $V \otimes^{op} W := W \otimes V$, and associator α^{-1}. A braided tensor category is a tensor category $\mathcal{D}$, together with a natural isomorphism $\sigma : \otimes \to \otimes^{op}$, satisfying the so-called hexagon relations.

Deligne's external product of abelian categories

Recall that a $\mathbb{C}$-linear abelian category $\mathcal{D}$ is called *locally finite*, if all Hom spaces are finite dimensional, and every object $V \in \mathcal{D}$ has finite length. We use the symbol $\boxtimes$ to denote Deligne's tensor product of locally finite categories (see, e.g. [EGNO]). In this article, we will consider semisimple abelian categories; in this case, the external tensor product $\mathcal{D}_1 \boxtimes \mathcal{D}_2$ of $\mathcal{D}_1$ and $\mathcal{D}_2$ is just a semisimple abelian category with simple objects $X \boxtimes Y$, where X and Y are simples in $\mathcal{D}_1$, $\mathcal{D}_2$. External tensor products may be defined for non-semisimple categories - this will be needed when considering q to be a root of unity - but we will not need them here.

Example 2.2.1. Let A be a (possibly infinite-dimensional) $\mathbb{C}$-algebra. Then the category A-fmod of finite dimensional A-modules is a locally finite $\mathbb{C}$-linear abelian category. For two such algebras A and B, we have a natural equivalence,

$$A\text{-fmod} \boxtimes B\text{-fmod} \sim (A \otimes B)\text{-fmod}.$$

In all our examples the external tensor products of categories we consider are of this sort.

The Deligne tensor product $\mathcal{D}_1 \boxtimes \cdots \boxtimes \mathcal{D}_n$ of (braided) locally finite tensor categories is again a (braided) locally finite tensor category, with structure functors defined diagonally: we set $\otimes := \otimes_1 \boxtimes \cdots \boxtimes \otimes_n$ (and $\sigma := \sigma_1 \boxtimes \cdots \boxtimes \sigma_n$).

2.2.3 Primary objects

In this section we construct the algebras $\mathcal{O}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ and $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ as algebras in a braided tensor category $\mathcal{C}$ associated to Q.

Quiver notation

We resume the notation for quivers from the Introduction. We choose, once and for all, an ordering on $\overline{E} = E \cup E^\vee$: we will emphasize dependence on this ordering in later definitions with an over-arrow decoration, e.g $\overrightarrow{\otimes}, \overrightarrow{\coprod}$. For $v \in V$, we define E_v^α and E_v^β as the subsets of non-loop edges $e \in E$ such that $\alpha(e) = v$ or $\beta(e) = v$, respectively; we define E_v° as the subset of self-loops based at v. Each obtains an induced ordering from E.

For each $v \in V$, we fix a locally finite braided tensor category $\mathcal{C}_v$, and a distinguished object $W_v \in \mathcal{C}_v$. This data we encode in the function $\mathbf{d}$, by defining $\mathbf{d}(v) := (W_v, \mathcal{C}_v)$. Thus, $\mathbf{d}$ is a generalized dimension vector, specifying which object is associated to each vertex.

Definition 2.2.2. We let $\mathcal{C} := \underset{v \in V}{\boxtimes} \mathcal{C}_v$, with tensor product and braiding defined diagonally. We regard any object $X_v \in \mathcal{C}_v$ as an object in $\mathcal{C}$ by putting the tensor unit $\mathbf{1}_w := \mathbf{1}_{\mathcal{C}_w}$ in the omitted tensor components. Strictly speaking, $\mathcal{C}$ depends on an implicit choice of ordering on V; however the categories associated to different orderings are canonically equivalent by the obvious functors of transposition of factors; it is the ordering on edges which is more significant in these constructions.

Defining relations

The defining relations for the algebras $\mathcal{O}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ and $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ are most naturally expressed as the image of certain canonical morphisms built from the braiding. We define those morphisms here for use later. For $e \in \overline{E}$, we let $\mathrm{Mat}(e) := W_\alpha^* \otimes W_\beta \in \mathcal{C}$. Choose a parameter $t \in \mathbb{C}$.

Definition 2.2.3. For $e \in E$ with $\alpha(e) \neq \beta(e)$, we define:

$$R(e) : W_\alpha^* \otimes W_\alpha^* \boxtimes W_\beta \otimes W_\beta \to \mathrm{Mat}(e) \otimes \mathrm{Mat}(e),$$

$$R(e) := \sigma_{W_\alpha^*, W_\alpha^*} - \sigma_{W_\beta, W_\beta}.$$

$$S(e, e^\vee) : W_\alpha^* \otimes W_\alpha \boxtimes W_\beta^* \otimes W_\beta \to \mathrm{Mat}(e^\vee) \otimes \mathrm{Mat}(e) \oplus \mathrm{Mat}(e) \otimes \mathrm{Mat}(e^\vee) \oplus \mathbb{C},$$

$$S(e, e^\vee) := \sigma_{W_\alpha^*, W_\alpha} - \sigma_{W_\beta, W_\beta^*}^{-1} - t \cdot (\mathrm{ev}_{W_\alpha} \boxtimes \mathrm{ev}_{W_\beta}),$$

where $\alpha = \alpha(e) = \beta(e^\vee)$, $\beta = \beta(e) = \alpha(e^\vee)$.

Definition 2.2.4. For $e \in E$ with $\alpha(e) = \beta(e)$, we define:

$$R(e) : W_\alpha^* \otimes W_\alpha^* \otimes W_\alpha \otimes W_\alpha \to \mathrm{Mat}(e) \otimes \mathrm{Mat}(e),$$

$$R(e) := \sigma_{W_\alpha, W_\alpha^*}^{-1} \circ (\sigma_{W_\alpha^*, W_\alpha^*} - \sigma_{W_\alpha, W_\alpha}).$$

$$S(e, e^\vee) : W_\alpha^* \otimes W_\alpha^* \otimes W_\alpha \otimes W_\alpha \to \mathrm{Mat}(e) \otimes \mathrm{Mat}(e^\vee) \oplus \mathrm{Mat}(e^\vee) \otimes \mathrm{Mat}(e),$$

$$S(e, e^\vee) := \sigma_{W_\alpha, W_\alpha^*}^{-1} (\sigma_{W_\alpha, W_\alpha} - \sigma_{W_\alpha^*, W_\alpha^*}^{-1}).$$

Recall that, for any object X in a tensor category $\mathcal{D}$, the tensor algebra,

$$T(X) := \bigoplus_{k=0}^{\infty} X^{\otimes k}$$

is an naturally an algebra in $\mathcal{D}$. Given $Y \subset T(X)$, we let $\langle Y \rangle$ denote the two-sided ideal in $T(X)$ generated by Y.

Definition 2.2.5. For $e \in E$, we define the two-sided ideals:

$$I(e) := \langle \mathrm{Im}\, R(e) \rangle \subset T(\mathrm{Mat}(e)), \text{ and}$$

$$I(e, e^\vee) := \langle \mathrm{Im}\, S(e, e^\vee) \rangle \subset T(\mathrm{Mat}(e) \oplus \mathrm{Mat}(e^\vee)).$$

The braided coordinate and differential operator algebras of Q

Definition 2.2.6. The edge coordinate algebra $\mathcal{O}_q(e)$ is the quotient of the tensor algebra $T(\mathrm{Mat}(e))$ by its two-sided quadratic ideal $I(e)$.

Definition 2.2.7. The braided quiver coordinate algebra $\mathcal{O}_q = \mathcal{O}_q(\mathrm{Mat}_\mathbf{d}(Q))$ is the braided tensor product of algebras,

$$\mathcal{O}_q(\mathrm{Mat}_\mathbf{d}(Q)) := \overrightarrow{\bigotimes_{e \in E}} \mathcal{O}_q(e).$$

Definition 2.2.8. The edge differential operator algebra $\mathcal{D}_q(e)$ is the quotient of the tensor algebra $T(\mathrm{Mat}(e) \oplus \mathrm{Mat}(e^\vee))$ by its two sided quadratic ideal

$$\mathcal{I} := I(e) + I(e^\vee) + I(e, e^\vee).$$

Definition 2.2.9. The braided differential operator algebra $\mathcal{D}_q(\mathrm{Mat}_\mathbf{d}(Q))$ is the braided tensor product of algebras,

$$\mathcal{D}_q(\mathrm{Mat}_\mathbf{d}(Q)) := \overrightarrow{\bigotimes_{e \in E}} \mathcal{D}_q(e).$$

Remark 2.2.10. Recall that for the tensor product of algebras in a braided tensor category the component subalgebras do not commute trivially; rather, they commute by the braiding:

$$\mu_{A \otimes B} := (\mu_A \otimes \mu_B) \circ \sigma_{B,A} : A \otimes B \otimes A \otimes B \to A \otimes B.$$

Note, however, that edge algebras do commute trivially if they share no common vertex, since in this case they occupy distinct $\boxtimes$-components of $\mathcal{C}$. However, we have an isomorphism $A \otimes B \to B \otimes A$ of $\mathcal{C}$-algebras given by $\sigma_{B,A}^{-1}$; thus $\mathcal{O}_q$ and $\mathcal{D}_q$ are defined independently of the ordering of $v \in E$, up to isomorphism.

Remark 2.2.11. The dependence on the parameter t appearing in the definition of $\mathcal{D}_q(\mathrm{Mat}_d(Q))$ is inessential in the following sense: for $t_1, t_2 \neq 0$, the two algebras

obtained by using t_1 or t_2 are isomorphic, by a simple rescaling of the generators (this phenomenon is common in the undeformed setting as well). Thus, to ease notation, we set $t = 1$, for the remainder of the paper. The exception, however comes when we compute degenerations in Section 2.9, when we will work with formal power series in $\hbar$, and will set t to $\hbar$, to compute the degeneration.

2.3 Quantum groups and an RTT-type presentation for $\mathcal{O}_q$ and $\mathcal{D}_q$

In this section, we unfold the definitions of $\mathcal{O}_q$ and $\mathcal{D}_q$ in our primary examples of interest, namely those coming from $\mathcal{C}_v = U_q(\mathfrak{gl}_{d_v})$-lfmod, so that $\mathcal{C} = U_q(\mathfrak{g}^\mathbf{d})$-lfmod. We will see that for certain simple quivers Q, $\mathcal{O}_q$ and $\mathcal{D}_q$ are related to well-known constructions in the theory of quantum groups. To begin, let us recall the quasi-triangular Hopf algebra $U_q(\mathfrak{gl}_N)$. The discussion here has been adapted from [JM], [KS], where the relation to the Serre presentation is explained.

2.3.1 The R-matrix on $\mathbb{C}^N$

We fix, for the remainder of this article, the following endomorphism R, of $\mathbb{C}^N \otimes \mathbb{C}^N$:

$$R := q \sum_i E_i^i \otimes E_i^i + \sum_{i \neq j} E_i^i \otimes E_j^j + (q - q^{-1}) \sum_{i > j} E_i^j \otimes E_j^i. \qquad (2.4)$$

We note that R satisfies the QYBE, and the Hecke condition from Section 2.2.1. We define $R_{ij}^{kl}, (R^{-1})_{ij}^{kl} \in \mathbb{C}$, for $i, j, k, l = 1, \ldots, N$ by:

$$R(e_i \otimes e_j) = \sum_{k,l} R_{ij}^{kl}(e_k \otimes e_l), \quad R^{-1}(e_i \otimes e_j) = \sum_{k,l} (R^{-1})_{ij}^{kl}(e_k \otimes e_l).$$

We have:

$$R_{kl}^{ij} = q^{\delta_j^i} \delta_k^i \delta_l^j + (q - q^{-1})\theta(i - j)\delta_l^i \delta_k^j,$$

$$(R^{-1})_{kl}^{ij} = q^{-\delta_j^i} \delta_k^i \delta_l^j - (q - q^{-1})\theta(i - j)\delta_l^i \delta_k^j,$$

where $\delta_j^i = 1$ if $i = j$, 0 else, and $\theta(k) = 1$ if $k > 0$, 0 else.

2.3.2 The Drinfeld-Jimbo quantum group $U_q(\mathfrak{gl}_N)$

Let $\widetilde{U}$ denote the free algebra with generators l_j^{+i}, and l_l^{-k}, where $i, j, k, l = 1 \ldots N$. We organize the generators into matrices $L^+, L^- \in \mathrm{Mat}_N(\widetilde{U}) \cong \widetilde{U} \otimes \mathrm{Mat}_N(\mathbb{C})$:

$$L^+ = \sum_{i,j} l_j^{+i} \otimes E_i^j, \quad L^- = \sum_{k,l} l_l^{-k} \otimes E_k^l.$$

For $M \in \mathrm{Mat}_N(\tilde{U})$, we define:

$$M_1 := M \otimes \mathrm{id}, \quad M_2 = \mathrm{id} \otimes M \in \mathrm{Mat}_N(\tilde{U}) \otimes \mathrm{Mat}_N(\tilde{U}).$$

Definition 2.3.1. The Drinfeld-Jimbo quantum group $U_q(\mathfrak{gl}_N)$ is the quotient of $\widetilde{U}$ by the relations:

$$L_1^\pm L_2^\pm R = R L_2^\pm L_1^\pm, \quad L_1^- L_2^+ R = R L_2^+ L_1^-, \tag{2.5}$$

$$l_i^{+i} l_i^{-i} = l_i^{-i} l_i^{+i} = 1, \quad i = 1, \ldots, N, \tag{2.6}$$

$$l_j^{+i} = l_i^{-j} = 0, \quad i > j. \tag{2.7}$$

U is a Hopf algebra with the antipode S, coproduct Δ and counit ϵ given by:

$$S(L^\pm) = (L^\pm)^{-1}, \quad \Delta(l_j^{\pm i}) = \sum_k l_k^{\pm i} \otimes l_j^{\pm k}, \quad \text{and} \quad \epsilon(l_j^{\pm i}) = \delta_j^i.$$

Remark 2.3.2. Each of the relations in line (2.5) above is actually an $N^2 \times N^2$ matrix of relations. For instance equation (2.5) asserts, for all $i, j, m, n \in 1 \cdots N$, the

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relations:

$$\sum_{k,l} l_k^{+i} l_l^{+j} R_{mn}^{kl} = \sum_{o,p} R_{op}^{ij} l_m^{+p} l_n^{+o}.$$

We shall use such notation frequently in this and future sections without further comment.

Definition 2.3.3. The vector representation $\rho : U \to \mathrm{End}(\mathbb{C}^N)$ is defined on generators by:

$$\rho_V(l_j^{+i}) = \sum_{\alpha,\beta} R_{\beta j}^{\alpha i} E_\alpha^\beta, \quad \rho_V(l_j^{-i}) = \sum_{\alpha,\beta} (R^{-1})_{j\beta}^{i\alpha} E_\alpha^\beta.$$

2.3.3 The locally finite part U' of U

The Hopf algebra U acts on itself via the adjoint action:

$$x \rhd y := x_{(1)} y S(x_{(2)}).$$

Definition 2.3.4. *The locally finite subalgebra U' is the subalgebra of U of vectors which generate a finite dimensional vector space under the adjoint action.*

We will make use of the following explicit description of U'. We define $\widetilde{l}_j^i \in U$ by $\widetilde{l}_j^i := \sum_k l_k^{+i} S(l_j^{-k})$. We define $\widetilde{L} := L^+ S(L^-)$, so that $\widetilde{L} = \sum_{i,j} \widetilde{l}_j^i E_i^j$.

Theorem 2.3.5. *(see [KS])*

1. *U' is generated by the $\widetilde{l}_j^i$, and the inverse $\det_q^{-1} = l_1^{-1} \ldots l_N^{-N}$ of the q-determinant.*

2. *U' is a left co-ideal: we have $\Delta(U') \subset U \otimes U'$. The coproduct on U' is given by:*

$$\Delta(\widetilde{l}_j^i) = \sum_{s,t} l_s^{+i} S(l_k^{-t}) \otimes \widetilde{l}_t^s, \quad \Delta(\det_q^{-1}) = \det_q^{-1} \otimes \det_q^{-1}$$

Let U^+ denote the subalgebra generated by the $\widetilde{l}_j^i$. Item (2) above implies that U^+ is a co-ideal subalgebra in U'.

2.3.4 Braided quiver coordinate algebra

Fix a quiver $Q = (V, E)$, and a dimension vector $\mathbf{d} : V \to \mathbb{Z}_{\geq 0}$. We specialize $\mathcal{C}_v = U_q(\mathfrak{gl}_{d_v})$-mod, with $W_v = \mathbb{C}^{d_v}$, its defining representation. In this case, the matrix representation $(R^v)^{ij}_{kl}$ of the universal R-matrix is defined relative to the standard basis of $\mathbb{C}^{d_v}$, so that $\sigma_{W_v, W_v}(w_i \otimes w_j) = (R^v)^{kl}_{ij} w_l \otimes w_k$.

Recall that the identities $(S \otimes \mathrm{id})(R) = R^{-1}$ and $(S \otimes S)(R) = R$ imply the formulas:

$$\sigma_{W_v^*, W_v}(v^i \otimes v_j) = \sum_{\alpha, \beta} (R^{-1})^{i\alpha}_{\beta j} v_\alpha \otimes v^\beta,$$

$$\sigma^{-1}_{W_v, W_v^*}(v^i \otimes v_j) = \sum_{\alpha, \beta} R^{i\alpha}_{\beta j} v_\alpha \otimes v^\beta,$$

$$\sigma_{W_v^*, W_v^*}(v^i \otimes v^j) = \sum_{\alpha, \beta} R^{ij}_{\alpha \beta} v^\beta \otimes v^\alpha.$$

In the definitions to follow, we introduce the following four matrices (where the $a(e)^i_j$ are formal symbols):

$$R := \sum_{i,j} R^{ij}_{kl}(E^k_i \otimes E^l_j), \quad R_{21} := \sum_{i,j} R^{ji}_{lk}(E^k_i \otimes E^l_j), \tag{2.8}$$

$$A^e_1 := \sum_{ij} a(e)^i_j(E^j_i \otimes \mathrm{id}), \quad A^e_2 := \sum_{ij} a(e)^i_j(\mathrm{id} \otimes E^j_i).$$

The following is a more concrete reformulation of Definition 2.2.7:

Definition 2.3.6. The braided quiver coordinate algebra, $\mathcal{O}_q(\mathrm{Mat}_\mathbf{d}(Q))$, is the algebra generated by elements $a(e)^i_j$, for $e \in E$, $i = 1, \ldots d_{\alpha(e)}$, and $j = 1, \ldots, d_{\beta(e)}$, subject to:

1. Relations between generators on the same edge:

$\overset{v}{\bullet} \overset{e}{\to} \overset{w}{\bullet}:$
$$R^v A_2^e A_1^e = A_1^e A_2^e R_{21}^w,$$

$\overset{v}{\bullet} \overset{e}{\circlearrowright}:$
$$R_{21}^v A_1^e R^v A_2^e = A_2^e R_{21}^v A_1^e R^v,$$

2. Relations between generators on distinct edges (assume $e < f$):

$\bullet \overset{f}{\to} \bullet \quad \bullet \overset{e}{\to} \bullet:$
$$A_1^f A_2^e = A_2^e A_1^f$$

$\overset{v}{\bullet} \underset{f}{\overset{e}{\rightrightarrows}} \overset{w}{\bullet}:$
$$A_1^f A_2^e = R^v A_2^e A_1^f R^w$$

$\overset{v}{\bullet} \underset{f}{\overset{e}{\rightleftarrows}} \overset{w}{\bullet}:$
$$A_1^f R^v A_2^e = A_2^e (R^w)^{-1} A_1^f$$

$\bullet \overset{e}{\to} \overset{v}{\bullet} \overset{f}{\to} \bullet:$
$$A_1^f A_2^e = A_2^e (R^v)^{-1} A_1^f,$$

$\bullet \overset{e}{\leftarrow} \overset{v}{\bullet} \overset{f}{\leftarrow} \bullet:$
$$A_1^f R^v A_2^e = A_2^e A_1^f$$

$\bullet \overset{e}{\to} \overset{v}{\bullet} \overset{f}{\leftarrow} \bullet:$
$$A_1^f A_2^e = A_2^e A_1^f R^v$$

$\bullet \overset{e}{\leftarrow} \overset{v}{\bullet} \overset{f}{\to} \bullet:$
$$A_1^f A_2^e = R^v A_2^e A_1^f,$$

$\bullet \overset{e}{\to} \overset{v}{\bullet} \overset{f}{\circlearrowright}:$
$$A_1^f A_2^e = A_2^e (R^v)^{-1} A_1^f R^v$$

$\bullet \overset{e}{\leftarrow} \overset{v}{\bullet} \overset{f}{\circlearrowright}:$
$$A_1^f R^v A_2^e = R^v A_2^e A_1^f$$

$\bullet \overset{f}{\to} \overset{v}{\bullet} \overset{e}{\circlearrowright}:$
$$A_1^f R^v A_2^e = A_2^e A_1^f R^v$$

$\bullet \overset{f}{\leftarrow} \overset{v}{\bullet} \overset{e}{\circlearrowright}:$
$$A_1^f A_2^e = R^v A_2^e (R^v)^{-1} A_1^f$$

$\overset{e}{\circlearrowleft} \overset{v}{\bullet} \overset{f}{\circlearrowright}:$
$$A_1^f R^v A_2^e (R^v)^{-1} = R^v A_2^e (R^v)^{-1} A_1^f$$

2.3.5 Braided quiver differential operator algebra

To the notation of equation (2.8), we add:

$$D_1^e := \sum_{k,l} \partial(e)_l^k (E_k^l \otimes \mathrm{id}), \quad D_2^e := \sum_{k,l} \partial(e)_l^k (\mathrm{id} \otimes E_k^l), \quad \Omega := \sum_{i,j} E_j^i \otimes E_i^j.$$

Definition 2.3.7. The braided quiver differential operator algebra, $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$, is the algebra generated by elements $a(e)^i_j$ and $\partial(e)^k_l$, for $e \in E$, with $i, l = 1, \ldots, d_{\alpha(e)}$, and $j, k = 1, \ldots, d_\beta(e)$, subject to:

1. The generators $a(e)^i_j$ satisfy the same relations amongst themselves as $a(e)^i_j$ in (1) and (2) of Definition 2.3.6.

2. The generators $\partial(e)^k_l$ satisfy the same relations amongst themselves as $a(e^\vee)^k_l$ in (1) and (2) of Definition 2.3.6.

3. For $e \neq f$, the generators $a(e)^i_j$ and $\partial(e)^k_l$ satisfy the same cross relations as $a(e)^i_j$ and $a(e^\vee)^k_l$, respectively in (2) of Definition 2.3.6.

4. For generators $a(e)^i_j$ and $\partial(e)^k_l$ on the same edge, we have the cross relations:

$$\overset{v \;\; e \;\; w}{\bullet \to \bullet} : \qquad\qquad D^e_2(R^v)^{-1}A^e_1 = A^e_1 R^w D^e_2 + \Omega,$$

$$\overset{v \;\; e}{\bullet \circlearrowright} : \qquad\qquad R^v_{21}D^e_1 R^v A^e_2 = A^e_2 R^v_{21} D^e_1 (R^v_{21})^{-1},$$

2.3.6 Familiar examples

Definition 2.2.7 encompasses many standard examples in the theory of quantum groups when applied to small quivers; these are illustrated below. To simplify notation, we do not specify ranges of free indices in equations, when the range is clear from context.

Example 2.3.8. The Kronecker quiver. Let $Q = \overset{\alpha \;\; e \;\; \beta}{\bullet \to \bullet}$. Choose dimensions d_α, d_β and let $\mathcal{C}_\alpha = U_q(\mathfrak{gl}_{d_\alpha})$, and $\mathcal{C}_\beta = U_q(\mathfrak{gl}_{d_\beta})$. Let $V_\alpha = \mathbb{C}^{d_\alpha}$, $V_\beta = \mathbb{C}^{d_\beta}$. Then $\mathcal{O}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ is the quotient of the free algebra with generators a^i_j, with $i = 1, \ldots, d_\alpha$, and $j = 1, \ldots, d_\beta$, by the relations:

$$\sum_{k,l} R^{ji}_{kl} a^l_m a^k_n = \sum_{k,l} a^j_l a^i_k R^{kl}_{mn},$$

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or equivalently:

$$a_m^i a_n^j = q^{\delta_{mn}} a_n^j a_m^i + \theta(m-n)(q-q^{-1}) a_m^j a_n^i, \qquad (i>j), \qquad (2.9)$$

$$a_m^i a_n^i = q^{-1} a_n^i a_m^i, \qquad (m>n).$$

Similarly, $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ is the quotient of the free algebra with generators a_j^i, ∂_l^k with $i, l = 1, \ldots, d_\alpha$, and $j, k = 1, \ldots, d_\beta$, by the relations:

$$\sum_{k,l} R_{kl}^{ij} a_m^l a_n^k = \sum_{k,l} a_l^i a_k^j R_{mn}^{kl}, \quad \sum_{k,l} R_{kl}^{ij} \partial_m^l \partial_n^k = \sum_{k,l} \partial_l^i \partial_k^j R_{mn}^{kl}$$

$$\sum_{k,l} \partial_k^i (R^{-1})_{lm}^{jk} a_n^l = \sum_{k,l} a_k^j R_{nl}^{ki} \partial_m^l + q \delta_n^i \delta_m^j,$$

Equivalently, the generators a_j^i and ∂_l^k satisfy relations (2.9) amongst themselves, and the cross relations:

$$\partial_m^i a_n^j = q^{\delta_{in}+\delta_{jm}} a_n^j \partial_m^i + \delta_{in} q^{\delta_{jm}} (q-q^{-1}) \sum_{p>i} a_p^j \partial_m^p + \delta_{jm}(q^2-1) \sum_{p>j} \partial_p^i a_n^p + \delta_{in}\delta_{jm}$$

We observe that $\mathcal{O}_q$ is the *equivariant FRT algebra* (see Proposition 2.4.1), while $\mathcal{D}_q$ is, apparently, new.

Example 2.3.9. The quantum plane. Let Q be the Kronecker quiver with $d_\alpha = 1$, and $d_\beta = N \in \mathbb{N}$. The defining representation for $U_q(\mathfrak{gl}_1)$ has $R_{V,V} = q \in \mathbb{C}^\times$, so that setting $x_j := a_j^1$, we have that $\mathcal{O}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ is a quotient of the free algebra generated by $x_1, \ldots x_N$ by the relations:

$$x_i x_j = q^{-1} x_j x_i, \qquad (i>j). \qquad (2.10)$$

Likewise, $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ is the quotient of the free algebra generated by x_j, ∂^k, with

$j, k = 1, \ldots, N$, by the relations (2.10), and:

$$\partial^i \partial^j = q \partial^j \partial^i, \qquad\qquad (i > j)$$

$$\partial^i x_j = q x_j \partial^i \qquad\qquad (i \neq j).$$

$$\partial^i x_i = q^2 x_i \partial^i + (q^2 - 1) \sum_{k > i} x_k \partial^k + q.$$

In this case, the relations essentially reduce to the relations for the "quantum Weyl algebra" from [GZ].

Example 2.3.10. The Jordan normal form quiver. Let Q have a single vertex v, and loop $e : v \to v$. Let $\mathcal{C} = U_q(\mathfrak{gl}_N)$-mod, and $V = \mathbb{C}^N$. Then $\mathcal{O}_q(\mathrm{Mat_d}(Q))$ is the quotient of the free algebra with generators a_j^i, for $i, j = 1, \ldots N$, with relations:

$$\sum_{k,l,o,p} R_{kl}^{ji} a_p^l R_{mo}^{pk} a_n^o = \sum_{p,q,s,t} a_p^j R_{sq}^{pi} a_t^q R_{mn}^{ts},$$

or equivalently:

$$(2.11)$$

$$a_m^i a_n^j = q^{\delta_{in} + \delta_{mn} - \delta_{mj}} a_n^j a_m^i + q^{\delta_{im} - \delta_{jm}} (q - q^{-1}) \theta(n - m) a_m^j a_n^i \qquad (i > j)$$

$$+ q^{\delta_{mn} - \delta_{jm}} \delta_{in} (q - q^{-1}) \sum_{p > i} a_p^j a_m^p - \delta_{mj} (1 - q^{-2}) \sum_{p > j} a_p^i a_n^p$$

$$+ q^{-\delta_{jm}} \delta_{im} (q - q^{-1})^2 \theta(n - m) \sum_{p > i} a_p^j a_n^p.$$

$$a_m^i a_n^i = q^{\delta_{in} - \delta_{im} - 1} a_n^j a_m^i + q^{-1 - \delta_{jm}} \delta_{in} (q - q^{-1}) \sum_{p > i} a_p^i a_m^p \qquad (m > n)$$

Likewise, $\mathcal{D}_q(\mathrm{Mat_d}(Q))$ is the quotient of the free algebra with generators a_j^i, ∂_l^k, for $i, j, k, l = 1, \ldots, N$, and relations:

$$\sum_{k,l,o,p} R_{kl}^{ji} a_p^l R_{mo}^{pk} a_n^o = \sum_{p,q,s,t} a_p^j R_{sq}^{pi} a_t^q R_{mn}^{ts},$$

$$\sum_{k,l,o,p} R_{kl}^{ji} \partial_p^l R_{mo}^{pk} \partial_n^o = \sum_{p,q,s,t} \partial_p^j R_{sq}^{pi} \partial_t^q R_{mn}^{ts},$$

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$$\sum_{k,l,o,p} R^{ji}_{kl}\partial^l_p R^{pk}_{mo}a^o_n = \sum_{k,l,o,p} a^j_l R^{li}_{pk}\partial^k_o (R^{-1})^{po}_{nm}.$$

Equivalently, the a^i_j and ∂^i_j satisfy the relations (2.11) amongst themselves, and the cross relations (for i,j,m,n distinct):

$$(2.12)$$

$$\partial^i_m a^j_n = q^{\delta_{i,n}-\delta_{m,n}-\delta_{m,j}-\delta_{i,j}} a^j_n \partial^i_m - \theta(j-i)q^{\delta_{i,m}-\delta_{m,j}-\delta_{i,j}}(q-q^{-1})\partial^j_m a^i_n$$

$$- \theta(m-n)(q-q^{-1})q^{\delta_{i,m}-\delta_{m,j}-\delta_{i,j}} a^j_m \partial^i_n - (q-q^{-1})q^{\delta_{m,j}}\delta_{m,j}\sum_{p>k}\partial^i_p a^p_n$$

$$- (q-q^{-1})^2 q^{-\delta_{m,j}-\delta_{i,j}}\delta_{i,m}\theta(j-i)\sum_{p>k}\partial^j_p a^p_n$$

$$- (q-q^{-1})q^{-\delta_{m,n}-\delta_{m,j}-\delta_{i,j}}\delta_{i,n}\sum_{p>i} a^j_p \partial^p_m$$

$$+ (q-q^{-1})^2 q^{-\delta_{m,j}-\delta_{i,j}}\theta(m-n)\delta_{i,m}\sum_{p>i} a^j_p \partial^p_n$$

In this case, $\mathcal{O}_q$ is the well-known reflection equation algebra [Do], [Ma], [Mu], while $\mathcal{D}_q$ is the algebra $\mathbb{D}^+$ of polynomial quantum differential operators on quantum GL_n, as studied in [VV].

2.3.7 New examples

New examples of interest are detailed below. For two $\mathbb{C}$-algebras A, B, we let $A * B$ denote their free product, and we use the notation $\overset{*}{\prod_{i\in I}} A_i$ for iterated free products.

Example 2.3.11. The Calogero-Moser quiver. Let Q and d be the Calogero-Moser quiver and dimension vector, $(Q,d) = \overset{1}{\bullet} \to \overset{n}{\bullet} \circlearrowleft$. Then $\mathcal{O}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ is the quotient of the free product, $\mathcal{O}_q(\overset{1}{\bullet} \to \overset{n}{\bullet}) * \mathcal{O}_q(\overset{n}{\bullet}\circlearrowleft)$, by the relations:

$$\sum_{k,l} x_k R^{ki}_{jl} a^l_m = \sum_{k,l} a^i_k x_l R^{lk}_{jm}.$$

Likewise, $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ is the quotient of the free product, $\mathcal{D}_q(\overset{1}{\bullet} \to \overset{n}{\bullet}) * \mathcal{D}_q(\overset{n}{\bullet}\circlearrowleft)$, by

the relations:

$$\sum_{k,l} x_k R^{ki}_{jl} a^l_m = \sum_{k,l} a^i_k x_l R^{lk}_{jm}, \quad \sum_{k,l} x_k R^{ki}_{jl} \partial^l_m = \sum_{k,l} \partial^i_k x_l R^{lk}_{jm}$$

$$\partial^i a^j_m = \sum_{k,l,o,p} R^{ij}_{kl} a^l_o (R^{-1})^{ko}_{pm} \partial^p, \quad \partial^i \partial^j_m = \sum_{k,l,o,p} R^{ij}_{kl} \partial^l_o (R^{-1})^{ko}_{pm} \partial^p$$

Example 2.3.12. Star shaped quiver. Let Q be the star-shaped quiver, with legs of length $l_1, \ldots, l_m$, and with nodal vertex v_0. We adopt the following labelling convention on Q. The vertex set of Q is:

$$V := \{ v_{\alpha\beta} \mid \alpha = 1, \ldots, m, \ \beta = 0, \ldots l_i \},$$

where each $v_{\alpha\beta}$ is on the αth leg, at a distance of β edges from the node, and $v_0 = v_{\alpha,0}$, for all $\alpha = 1 \ldots, m$. The edge set of Q is:

$$E := \{ e_{\alpha,\beta} : v_{\alpha,\beta+1} \to v_{\alpha,\beta} \mid \alpha = 1, \ldots m, \beta = 0, \ldots, l_\alpha - 1 \}.$$

The labelling is depicted below:
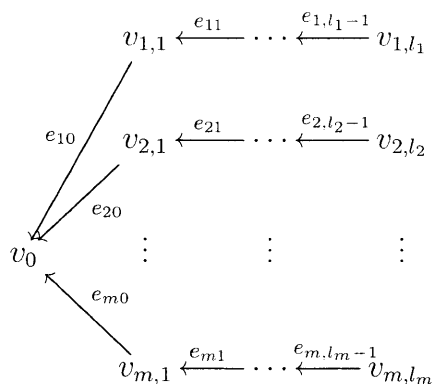
We choose for the ordering on the edges the natural lexicographic ordering on the indices. We set $d_v = 1$ for all $v \neq v_0$, and $d_{v_0} = n$; we call such $\mathbf{d}$ the Calogero-moser dimension vector for Q. By Example 2.3.8, for $\alpha = 1, \ldots m, \beta = 1 \ldots l_i - 1$, each edge algebra $\mathcal{D}_q(e_{ij})$ has two generators, which we denote x_α and ∂_β. Likewise, each edge algebra $\mathcal{D}_q(e_{i,0})$ has $2n$ generators, which we denote $y_{\alpha 1}, \ldots, y_{\alpha n}, \xi^1_\alpha, \ldots, \xi^n_\alpha$. Then,

$\mathcal{D}_q(\mathrm{Mat}_\mathbf{d}(Q))$ is the quotient of the free product, $\overset{*}{\underset{e_{\alpha\beta}\in E}{\prod}} \mathcal{D}_q(\bullet \xrightarrow{e_{\alpha\beta}} \bullet)$, by the relations that all generators without a common vertex commute, and cross-relations on the remaining edges:

$$x_{\alpha,\beta-1}x_{\alpha\beta} = qx_{\alpha\beta}x_{\alpha,\beta-1}, \quad \partial_{\alpha,\beta-1}\partial_{\alpha\beta} = q^{-1}\partial_{\alpha\beta}\partial_{\alpha,\beta-1}$$

$$\partial_{\alpha\beta}x_{\alpha,\beta-1} = qx_{\alpha,\beta-1}\partial_{\alpha\beta}, \quad x_{\alpha\beta}\partial_{\alpha,\beta-1} = q^{-1}\partial_{\alpha,\beta-1}x_{\alpha\beta},$$

$$y_{\beta i}y_{\alpha j} = \sum_{k,l} y_{\alpha k}y_{\beta l}R_{ij}^{lk}, \quad \xi_\beta^i\xi_\alpha^j = \sum_{k,l} R_{kl}^{ij}\xi_\beta^l\xi_\beta^k, \quad \text{(for } \alpha < \beta\text{)},$$

$$\xi_\alpha^i y_{\beta j} = \sum_{k,l} y_{\beta k}R_{jl}^{ki}\xi_\alpha^l, \quad \xi_\beta^i y_{\alpha j} = \sum_{k,l} y_{\alpha k}(R^{-1})_{lj}^{ik}\xi_\beta^l.$$

Remark 2.3.13. It has been suggested to us by B. Webster that the case when Q is arbitrary non-Dynkin, but $d_v = 1$ for all v should yield quantizations of hypertoric varieties associated to Q. We hope to study such examples in the future.

2.3.8   Monomial notation

In order to denote monomials in the generators of $\mathcal{O}_q$ and $\mathcal{D}_q$, we introduce the following shorthand. Let I be an ordered list of triples $I = (e_i \in E, m_i \in \{1, \ldots d_{\alpha(e)}\}, n_i \in \{1, \ldots, d_{\beta(e)}\})_i$, and J an ordered list of triples $J = (f_i \in E^\vee, o_i \in \{1, \ldots d_{\beta(e)}\}, p_i \in \{1, \ldots, d_{\alpha(e)}\})_i$, we denote the products

$$a_I := a(e_1)_{n_1}^{m_1} \cdots a(e_k)_{n_k}^{m_k}$$

$$\partial_J := \partial(f_1)_{p_1}^{o_1} \cdots \partial(f_l)_{p_l}^{o_l}.$$

When there is no risk of confusion, we omit the specification of the edge in the notation (e.g, we write a_j^i instead of $a(e)_j^i$). The list I will be said to be ordered, if for all $i < j$, either $e_i < e_j$. or $e_i = e_j$ and $m_i < m_j$, or $e_i = e_j, m_i = m_j$ and $n_i \le n_j$. Likewise the list J will be said to be ordered, if for all $i < j$, either $f_i < f_j$. or $f_i = f_j$ and $o_i < o_j$, or $f_i = f_j, o_i = o_j$ and $p_i \le p_j$. Monomials $a_I\partial_J$, for I, J ordered, will be called *standard monomials*.

2.4 Flatness of the algebras $\mathcal{O}_q$ and $\mathcal{D}_q$

In the present section, we prove that the algebras $\mathcal{O}_q$ and $\mathcal{D}_q$ constructed in previous sections are flat noncommutative deformations of their classical counterparts, the algebras $\mathcal{O}(\mathrm{Mat}_\mathbf{d}(Q))$ and $\mathcal{D}(\mathrm{Mat}_\mathbf{d}(Q))$. More precisely, we show that the set of standard monomials form a basis of $\mathcal{O}_q$ and $\mathcal{D}_q$.

Proposition 2.4.1. *We have the following descriptions for $\mathcal{O}_q(e)$:*

1. *If $\alpha(e) \neq \beta(e)$, then $O_q(e)$ is twist equivalent to the FRT algebra via the tensor equivalence $\sigma \boxtimes \mathrm{id} : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}^\vee \boxtimes \mathcal{C}$.*

2. *If $\alpha(e) = \beta(e)$, then $O_q(e)$ is isomorphic to the reflection equation algebra.*

Proof. The $\mathcal{C}^\vee \boxtimes \mathcal{C}$-algebra $\mathcal{O}'_q(e)$, twist-equivalent to $\mathcal{O}_q(e)$, has the same underlying vector space as $\mathcal{O}_q(e)$, with multiplication given by $m' := m \circ (R^{-1} \boxtimes \mathrm{id})$, where m denotes the product in $\mathcal{O}_q(e)$. In particular, $\mathcal{O}'_q(e)$ is generated by elements $\tilde{a}^i_j$, $i = 1, \ldots, n, j = 1, \ldots, m$, with relations:

$$R^{ij}_{op}\tilde{a}^o_m\tilde{a}^p_n = a^i_m a^j_n = (R^{ji}_{op})^{-1} a^o_k a^p_l R^{lk}_{mn} = \tilde{a}^j_k\tilde{a}^i_l R^{lk}_{mn},$$

which are the relations of the FRT algebra. On the other hand if $\alpha = \beta$, we have seen in Example 2.3.10 that we recover the relations of the reflection equation algebra. $\square$

$\mathcal{O}_q$ is defined as a braided tensor product of the edge algebras $\mathcal{O}_q(e)$, which are flat by Proposition 2.4.1, together with the well-known flatness of the FRT and RE algebras (see, e.g. [KS]). More precisely, we have:

Corollary 2.4.2. *The algebra $\mathcal{O}_q$ is a flat deformation of the algebra $\mathcal{O}(\mathrm{Mat}_\mathbf{d}(Q))$. A basis of $\mathcal{O}_q$ is given by the set of standard monomials a_I.*

In fact, the analogous statement holds for $\mathcal{D}_q$, as well. The proof is modeled on the proof of Theorem 1.5 of [GZ], which is a special case. We have:

Theorem 2.4.3. *The algebra $\mathcal{D}_q$ is a flat deformation of the algebra $\mathcal{D}(\mathrm{Mat}_\mathbf{d}(Q))$. A basis of $\mathcal{D}_q$ is given by the set of standard monomials a_I, ∂_J.*

Proof. Since we have defined $\mathcal{D}_q$ as a braided tensor product of its edge algebras, we need only to prove flatness for each edge algebra $\mathcal{D}_q(e)$. By Theorem 2.4.1, it suffices to prove that $\mathcal{D}_q(e) \cong \mathcal{O}_q(e) \otimes \mathcal{O}_q(e^\vee)$, as a vector space. We make use of the following lemma, which generalizes [GZ], Lemma 1.6.

Lemma 2.4.4. *In the tensor algebra* $T(\mathrm{Mat}(e) \oplus \mathrm{Mat}(e^\vee))$, *we have the following containments:*

1. $T(\mathrm{Mat}(e^\vee))I(e) \subset I(e)T(\mathrm{Mat}(e^\vee)) + I(e, e^\vee)$.

2. $I(e^\vee)T(\mathrm{Mat}(e)) \subset T(\mathrm{Mat}(e))I(e^\vee) + I(e, e^\vee)$.

Proof. We prove (1) by direct computation; (2) then follows by a similar proof, due to the symmetry in the definition of $I(e, e^\vee)$. For the first claim, it suffices to show that, for all o, p, i, j, m, n, we have

$$\partial_p^o(R_{kl}^{ij}a_m^l a_n^k - a_l^i a_k^j R_{mn}^{kl}) \in I(e)T(e^\vee) + I(e, e^\vee).$$

This is equivalent to showing that $A_{nmv}^{suo} \in I(e)T(e^\vee) + I(e, e^\vee)$, for all u, s, o, v, m, n, where:

$$A_{nmv}^{suo} := (R^{-1})_{jv}^{ut}(R^{-1})_{it}^{sp}\partial_p^o(R_{kl}^{ij}a_m^l a_n^k - a_l^i a_k^j R_{mn}^{kl}),$$

as these differ by an invertible linear transformation, and so generate the same subspace. We let

$$A := \sum_{u,s,o,v,m,n} A_{nmv}^{suo}(E_s^n \otimes E_u^m \otimes E_o^v)$$

$$= D_3 R_{13}^{-1} R_{23}^{-1} R_{12} A_2 A_1 - D_3 R_{13}^{-1} R_{23}^{-1} A_1 A_2 R_{21},$$

in the notation of Section 2.3.2, so that the matrix coefficients of A are precisely the

A_{nmv}^{suo}. We compute:

$$A = D_3 \underbrace{R_{13}^{-1} R_{23}^{-1} R_{12}}_{QYBE} A_2 A_1 - \underbrace{D_3 R_{13}^{-1} A_1}_{I(e,e^\vee)} R_{23}^{-1} A_2 R_{21}$$

$$= R_{12} \underbrace{D_3 R_{23}^{-1} A_2}_{I(e,e^\vee)} R_{13}^{-1} A_1 - A_1 R_{13} \underbrace{D_3 R_{23}^{-1} A_2}_{I(e,e^\vee)} R_{21} - \Omega_{13} R_{23}^{-1} A_2 R_{21}$$

$$= R_{12} A_2 R_{23} \underbrace{D_3 R_{13}^{-1} A_1}_{I(e,e^\vee)} + R_{12}\Omega_{23} R_{13}^{-1} A_1 - A_1 A_2 \underbrace{R_{13} R_{23} R_{21}}_{QYBE} D_3$$

$$\quad - A_1 R_{13}\Omega_{23} R_{21} - \Omega_{13} R_{23}^{-1} A_2 R_{21}$$

$$= R_{12} A_2 A_1 R_{23} R_{13} D_3 + R_{12} A_2 R_{23} \Omega_{13} + \underbrace{R_{12}\Omega_{23} R_{13}^{-1}}_{\text{cancel inv.}} A_1 - A_1 A_2 R_{21} R_{23} R_{13} D_3$$

$$\quad - A_1 R_{13}\Omega_{23} R_{21} - \Omega_{13} R_{23}^{-1} A_2 R_{21}$$

$$= (R_{12} A_2 A_1 - A_1 A_2 R_{21}) R_{23} R_{13} D_3 + \underbrace{(R_{12} - R_{21}^{-1})}_{\text{Hecke reln.}} A_2 R_{23}\Omega_{13} + \Omega_{23} A_1$$

$$\quad - A_1 R_{13}\Omega_{23} R_{21}$$

$$= (R_{12} A_2 A_1 - A_1 A_2 R_{21}) R_{23} R_{13} D_3 + (q - q^{-1})\Omega_{12} A_2 R_{23}\Omega_{13} + \Omega_{23} A_1$$

$$\quad - A_1 R_{13}\Omega_{23} R_{21}$$

$$= (R_{12} A_2 A_1 - A_1 A_2 R_{21}) R_{23} R_{13} D_3$$

$$\quad A_1 \Omega_{23} \underbrace{((q - q^{-1}) R_{12}\Omega_{12} + 1 - R_{12} R_{21})}_{\text{Hecke reln.}}$$

$$= (R_{12} A_2 A_1 - A_1 A_2 R_{21}) R_{23} R_{13} D_3.$$

Comparing matrix coefficients, the above reads:

$$A_{nmv}^{suo} = R_{nj}^{kl} R_{ml}^{io} (R_{pa}^{su} a_i^a a_k^p - a_p^s a_t^u R_{ik}^{tp}) \partial_v^j \subset I(e) O_{e^\vee},$$

.

as claimed.

$\square$

To finish the proof of the theorem, we first observe that $\mathcal{D}_q(e) \cong S/(I(e) + I(e^\vee))$,

where $S = T(\mathrm{Mat}(e) \oplus \mathrm{Mat}(e^\vee))/I(e, e^\vee)$. Every element of S can be uniquely reduced to a sum $\sum C_{IJ} a_I \partial_J$, where $C_{IJ} \in \mathbb{C}$, by relations $I(e, e^\vee)$, by straightforward application of the diamond lemma. Thus the multiplication $m : T(\mathrm{Mat}(e)) \otimes T(\mathrm{Mat}(e^\vee)) \to S$ is a linear isomorphism. By the lemma, the two-sided ideal $\langle I(e) \otimes 1 + 1 \otimes I(e^\vee) \rangle \subset S$ lies in the image under m of the linear subspace $I(e) \otimes T(\mathrm{Mat}(e^\vee)) + T(\mathrm{Mat}(e)) \otimes I(e^\vee)$, which implies that

$$m : T(\mathrm{Mat}(e))/I(e) \otimes T(\mathrm{Mat}(e^\vee))/I(e^\vee) \to \mathcal{D}_q(e)$$

is a linear isomorphism, as desired. $\qquad\square$

Corollary 2.4.5. *The identification* $\mathcal{O}_q \cong \mathcal{D}_q/(\sum_{e \in E} \mathrm{Mat}(e^\vee))\mathcal{D}_q(e)$ *as objects of* $\mathcal{C}$ *makes* $\mathcal{O}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ *a* $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$-*module in* $\mathcal{C}$, *q-deforming the usual* $\mathbb{G}^{\mathbf{d}}$-*equivariant action of* $\mathcal{D}(\mathrm{Mat}_{\mathbf{d}}(Q))$ *on* $\mathcal{O}(\mathrm{Mat}_{\mathbf{d}}(Q))$.

2.5 Independence of $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ on the orientation of Q

The algebra of differential operators on a finite dimensional vector space, $V = \langle e_1, \ldots, e_n \rangle$ with dual basis $V^* = \langle f_1, \ldots, f_n \rangle$, has a Fourier transform automorphism $\mathcal{F}$, induced by the symplectomorphism on the symplectic vector space $V \oplus V^*$, $e_i \mapsto f_i$, $f_i \mapsto -e_i$. In this section we show that a certain localization $\mathcal{D}_q(e)^\circ$ of the edge differential operator algebras $\mathcal{D}_q(e)$ admit analogous isomorphisms, which may be extended to (a localization $\mathcal{D}_q(\mathrm{Mat}_d(Q))^\circ$ of) $\mathcal{D}_q(\mathrm{Mat}_d(Q))$, by the identity on the other $\mathcal{D}_q(f)$ subalgebras. In particular, this implies that the algebra $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ does not depend on the orientation of Q, up to isomorphism. The results of this section should also be compared to Section 2 of [C-BS], of which they are a quantization.

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2.5.1 Braided Fourier transform on $\mathcal{D}_q(e)$ when e is not a loop

Easy case:

$e = \overset{1}{\bullet} \to \overset{1}{\bullet}$. We work this example out for the sake of clarity, before considering the general situation. In this case, we have:

$$\mathcal{D}_q(e) = \mathbb{C}\langle \partial, a \rangle \Big/ \langle \partial q^{-1} a = aq\partial + 1 \rangle.$$

We introduce the elements:

$$g^\alpha := (1 + (q - q^{-1})\partial a), \quad g^\beta := (1 + (q - q^{-1})a\partial).$$

Proposition 2.5.1. *We have the relations:*

1. *$g^\alpha \partial = \partial g^\beta$,*

2. *$g^\beta a = ag^\alpha$,*

3. *$g^\alpha a = q^2 ag^\alpha$,*

4. *$g^\alpha \partial = q^{-2}\partial g^\alpha$.*

Proof. Items (1) and (2) are self-evident. For (3), we compute:

$$
\begin{aligned}
g^\alpha a &= (1 + (q - q^{-1})\partial a)a \\
&= (1 + (q - q^{-1})(q^2 a\partial + q))a \\
&= q^2(1 + (q - q^{-1})a\partial)a \\
&= q^2 a(1 + (q - q^{-1})\partial a) \\
&= q^2 ag^\alpha,
\end{aligned}
$$

as desired. The computation for (4) is similar to (3). $\qquad\square$

Remark 2.5.2. We note in passing that (3) and (4) are special cases of Corollary 2.6.13, which is proven independently.

Definition 2.5.3. We let $\mathcal{D}_q(e)^\circ$ denote the non-commutative localization of $\mathcal{D}_q(e)$ at the multiplicative Ore set $S := \{g_\alpha^k g_\beta^l \mid k, l \in \mathbb{Z}_{\geq 0}\}$.

Definition-Proposition 2.5.4. There exists a unique isomorphism:

$$\mathcal{F} : \mathcal{D}_q(e)^\circ \to \mathcal{D}_q(e^\vee)^\circ,$$

$$a \mapsto \partial, \quad \partial \mapsto -a g_\alpha^{-1}.$$

Proof. Clearly we have a homomorphism $\mathcal{F} : T(\mathrm{Mat}(e) \oplus \mathrm{Mat}(e^\vee)) \to \mathcal{D}_q(e)^\circ$ given on generators as above. We have only to check that the relations defining $\mathcal{D}_q(e)$ are mapped to zero by $\mathcal{F}$. We compute:

$$\mathcal{F}(\partial q^{-1} a - a q \partial - 1) = -a g_\alpha^{-1} q^{-1} \partial + \partial q a g_\alpha^{-1} - 1$$

$$= -q a \partial g_\alpha^{-1} + q \partial a g_\alpha^{-1} - 1$$

$$= q(\partial a - a \partial) g_\alpha^{-1} - 1$$

$$= (1 + (q - q^{-1})\partial a) g_\alpha^{-1} - 1 = 0,$$

as desired. $\square$

General case:

$e = \overset{n}{\bullet} \to \overset{m}{\bullet}$. Following the notation of Section 2.3 equation (2.8), we introduce the matrices:

$$g^\alpha := (I + (q - q^{-1})DA), \quad g^\beta := (I + (q - q^{-1})AD).$$

Proposition 2.5.5. *We have the relations:*

1. $g^\alpha D = D g^\beta,$

2. $g^\beta A = A g^\alpha,$

3. $g_1^\alpha R^\beta D_2 = (R^\beta)_{21}^{-1} D_2 g_1^\alpha,$

4. $g_1^\beta (R^\alpha)_{21}^{-1} A_2 = R^\alpha A_2 g_1^\beta,$

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5. $g_1^\beta D_2 R_{21}^\alpha = D_2 (R^\alpha)^{-1} g_1^\beta$,

6. $g_1^\alpha A_2 (R^\beta)^{-1} = A_2 R_{21}^\beta g_1^\alpha$,

7. $g_1^\beta g_2^\alpha = g_2^\alpha g_1^\beta$.

Proof. Items (1) and (2) are self-evident. For (3), we compute:

$$
\begin{aligned}
g_1^\alpha R^\beta D_2 &= (I + (q - q^{-1}) D_1 A_1) R^\beta D_2 \\
&= R^\beta D_2 + (q - q^{-1}) D_1 D_2 (R^\alpha)^{-1} A_1 - (q - q^{-1}) D_1 \Omega \\
&= R^\beta D_2 + (q - q^{-1})(R^\beta)_{21}^{-1} D_2 D_1 A_1 - (q - q^{-1}) D_1 \Omega \\
&= (R^\beta - (q - q^{-1}) \Omega^\beta) D_2 + (q - q^{-1})(R^\beta)_{21}^{-1} D_2 D_1 A_1 \\
&= (R^\beta)_{21}^{-1} D_2 (I + (q - q^{-1}) D_1 A_1) \\
&= (R^\beta)_{21}^{-1} D_2 g_1^\alpha.
\end{aligned}
$$

Similar computations prove (4)-(6). For (7), we compute:

$$
\begin{aligned}
\frac{[g_1^\beta, g_2^\alpha]}{(q - q^{-1})^2} &= A_2 D_2 D_1 A_1 - D_1 A_1 A_2 D_2 \\
&= A_2 R_{21}^\beta \underbrace{(R^\beta)_{21}^{-1} D_2 D_1}_{I(e^\vee)} A_1 - D_1 A_1 A_2 D_2 \\
&= \underbrace{A_2 R_{21}^\beta D_1}_{I(e,e^\vee)} \underbrace{D_2 (R^\alpha)^{-1} A_1}_{I(e,e^\vee)} - D_1 A_1 A_2 D_2 \\
&= (D_1 (R^\alpha)_{21}^{-1} A_2 - \Omega)(A_1 R^\beta D_2 + \Omega) - D_1 A_1 A_2 D_2 \\
&= \underbrace{D_1 (R^\alpha)_{21}^{-1} A_2 A_1 R^\beta D_2}_{\text{cancel this}} - \Omega A_1 R^\beta D_2 + D_1 (R^\alpha)_{21}^{-1} A_2 \Omega - 1 - \underbrace{D_1 A_1 A_2 D_2}_{\text{with this}} \\
&= \Omega (D_2 (R^\alpha)^{-1} A_1 - A_1 R^\beta D_2 - \Omega) \\
&= 0.
\end{aligned}
$$

$\square$

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Definition 2.5.6. We let $\mathcal{D}_q(e)^\circ$ denote the non-commutative localization of $\mathcal{D}_q(e)$ at the quantum determinant $\det_q$ of the matrices g_α and g_β.

In Section 2.6, Corollary 2.6.13 (which is independent of the present section), we prove that the powers of $\det_q$ form a multiplicative Ore set in $\mathcal{D}_q(e)$, so that the localization is straightforward to construct (in particular, $\mathcal{D}_q^\circ(e)$ gives rise to a flat deformation of the localized cotangent bundle to $\mathrm{Mat}_\mathbf{d}(e)$, which appears in [C-BS].

Definition-Proposition 2.5.7. There exists a unique isomorphism:

$$\mathcal{F} : \mathcal{D}_q(e)^\circ \to \mathcal{D}_q(e^\vee)^\circ,$$
$$A \mapsto D, D \mapsto -A(g^{\alpha(e^\vee)})^{-1}.$$

Proof. Clearly we have a homomorphism $\mathcal{F} : T(\mathrm{Mat}(e) \oplus \mathrm{Mat}(e^\vee)) \to \mathcal{D}(e)^\circ$ given on generators by the above formula. We have to check that the relations defining $\mathcal{D}_q(e)$ are mapped to zero by $\mathcal{F}$. In the formulas below, for each edge $e \in E$, and its adjoint edge $e^\vee \in E^\vee$, we abbreviate $\alpha = \alpha(e) = \beta(e^\vee) = \beta^\vee$, $\beta = \beta(e) = \alpha(e^\vee) = \alpha^\vee$. We first compute the image of the relations between the $a(e)_j^i$:

$$\mathcal{F}(R^\alpha A_2 A_1 - A_1 A_2 R_{21}^\beta) = R^{\beta^\vee} D_2 D_1 - D_1 D_2 R_{21}^{\alpha^\vee} = 0.$$

Next, we compute the image of the relations between the $\partial(e)_j^i$:

$$S := \mathcal{F}(R^\beta D_2 D_1 - D_1 D_2 R^\alpha_{21})$$

$$= R^{\alpha^\vee} \underbrace{A_2(g_2^{\alpha^\vee})^{-1}}_{2.5.5\,(2)} A_1(g_1^{\alpha^\vee})^{-1} - A_1(g_1^{\alpha^\vee})^{-1} \underbrace{A_2(g_2^{\alpha^\vee})^{-1}}_{2.5.5\,(2)} R^{\beta^\vee}_{21}$$

$$= R^{\alpha^\vee}(g_2^{\beta^\vee})^{-1} \underbrace{A_2 A_1}_{I(e)}(g_1^{\alpha^\vee})^{-1} - A_1(g_1^{\alpha^\vee})^{-1}(g_2^{\beta^\vee})^{-1} A_2 R^{\beta^\vee}_{21}$$

$$= \underbrace{R^{\alpha^\vee}(g_2^{\beta^\vee})^{-1} R^{\alpha^\vee}_{21} A_1}_{2.5.5\,(5)} \underbrace{A_2(R^{\beta^\vee})^{-1}(g_1^{\alpha^\vee})^{-1}}_{2.5.5\,(6)} - A_1(g_1^{\beta^\vee})^{-1}(g_2^{\alpha^\vee})^{-1} A_2 R^{\beta^\vee}_{21}$$

$$= A_1 g_2^{-1} g_1^{-1} A_2 R^\alpha_{21} - A_1 g_1^{-1} g_2^{-1} A_2 R^\alpha_{21}$$

$$= 0,$$

by part (7) of Proposition 2.5.5.

Finally, to compute the image, $\mathcal{F}(D_2 R^{-1} A_1 - A_1 R D_2 - \Omega)$, of the cross relations, we flip tensor factors, and compute:

$$\mathcal{F}(D_1(R_{21})^{-1} A_2 - A_2 R_{21} D_1 - \Omega) = -A_1 \underbrace{g_1^{-1} R_{21}^{-1} D_2}_{2.5.5\,(3)} + D_2 R_{21} A_1 g_1^{-1} - \Omega$$

$$= (D_2 R_{21} A_1 - \underbrace{A_1 R D_2}_{I(e,\,e^\vee)}) g_1^{-1} - \Omega$$

$$= (\Omega + D_2(\underbrace{R_{21} - R^{-1}}_{\text{Hecke reln.}}) A_1) g_1^{-1} - \Omega$$

$$= (\Omega + D_2(q - q^{-1})\Omega A_1) g_1^{-1} - \Omega$$

$$= \Omega \left((I + (q - q^{-1}) D_1 A_1) g_1^{-1} - I \right)$$

$$= 0,$$

by definition of g_1. $\qquad\square$

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2.5.2 Braided Fourier transform on $\mathcal{D}_q(e)$ when e is a loop

Easy case:

$e = \overset{1}{\bullet}\, \circlearrowleft$. We again consider the $d_v = 1$ case first for the sake of clarity, before moving on to the general situation. In this case, we have:

$$D_q(e) = \mathbb{C}\langle \partial, a\rangle \big/ \langle a\partial = q^2\partial a\rangle.$$

Definition 2.5.8. We let $D_q(e)^\circ$ denote the noncommutative localization at the multiplicative Ore set $S := \{a^k \partial^l \mid k, l \in \mathbb{Z}_{\geq 0}\}$.

Definition-Proposition 2.5.9. There exists a unique homomorphism:

$$\mathcal{F} : D_q(e)^\circ \to D_q(e^\vee)^\circ,$$

$$a \mapsto \partial, \partial \mapsto \partial a^{-1}\partial^{-1}.$$

Moreover, $\mathcal{F}$ is an isomorphism.

Proof. Clearly we have a homomorphism $\mathcal{F} : T(\mathrm{Mat}(e) \oplus \mathrm{Mat}(e^\vee)) \to \mathcal{D}(e)^\circ$ given on generators by the formulas above. We have to check that the relations defining $\mathcal{D}_q(e)$ are mapped to zero by $\mathcal{F}$. We compute:

$$\mathcal{F}(a\partial - q^2\partial a) = \partial(\partial a^{-1}\partial^{-1}) - q^2(\partial a^{-1}\partial^{-1})\partial = q^2\partial a^{-1} - q^2\partial a^{-1} = 0.$$

$\square$

General case:

$e = \overset{n}{\bullet}\, \circlearrowleft$.

Definition 2.5.10. We let $D_q(e)^\circ$ denote the non-commutative localization at the quantum determinant $\det_q$ of the matrices D and A.

It is well-known that the powers of $\det_q$ form a multiplicative Ore set in $\mathcal{D}_q(e)$, so that the localization is straightforward.

Definition-Proposition 2.5.11. There exists a unique isomorphism:

$$\mathcal{F} : \mathcal{D}_q(e)^\circ \to \mathcal{D}_q(e)^\circ,$$

$$A \mapsto D, D \mapsto DA^{-1}D^{-1}.$$

Proof. Clearly we have a homomorphism $\mathcal{F} : T(\mathrm{Mat}(e) \oplus \mathrm{Mat}(e^\vee)) \to \mathcal{D}(e)^\circ$ given on generators by the formulas above. We have to check that the relations defining $\mathcal{D}_q(e)$ are mapped to zero by $\mathcal{F}$. Clearly the relations between the $a(e)^i_j$ are sent to zero, as $\mathcal{F}(A) = D$ still satisfies the reflection equations. We compute the image of the relations between the $\partial(e)^i_j$.

$$\mathcal{F}(D_2 R_{21} D_1 R) = D_2 A_2^{-1} D_2^{-1} R_{21} D_1 A_1^{-1} D_1^{-1} R$$

$$= D_2 A_2^{-1} \underbrace{D_2^{-1} R_{21} D_1 R}_{I(e^\vee)} R^{-1} A_1^{-1} D_1^{-1} R$$

$$= D_2 \underbrace{A_2^{-1} R_{21} D_1 R}_{I(e,e^\vee)} \underbrace{D_2^{-1} R^{-1} A_1^{-1}}_{I(e,e^\vee)} D_1^{-1} R$$

$$= D_2 R_{21} D_1 \underbrace{R_{21}^{-1} A_2^{-1} R^{-1} A_1^{-1}}_{I(e)} \underbrace{R_{21}^{-1} D_2^{-1} R^{-1} D_1^{-1} R}_{I(e^\vee)}$$

$$= \underbrace{D_2 R_{21} D_1 R}_{I(e^\vee)} R^{-1} A_1^{-1} \underbrace{R_{21}^{-1} A_2^{-1} R^{-1} D_1^{-1} R_{21}^{-1}}_{I(e,e^\vee)} D_2^{-1}$$

$$= R_{21} D_1 \underbrace{R D_2 R^{-1} A_1^{-1}}_{I(e,e\vee)} D_1^{-1} R_{21}^{-1} A_2^{-1} D_2^{-1}$$

$$= R_{21} D_1 A_1^{-1} \underbrace{R_{12} D_2 R_{21} D_1^{-1} R_{21}^{-1}}_{I(e^\vee)} A_2^{-1} D_2^{-1}$$

$$= R_{21} D_1 A_1^{-1} D_1^{-1} R_{12} D_2 A_2^{-1} D_2^{-1}$$

$$= \mathcal{F}(R_{21} D_1 R D_2).$$

Finally, we compute the image of the cross relations. We find:

$$
\begin{aligned}
\mathcal{F}(A_1 R D_2) &= \underbrace{D_1 R D_2}_{I(e^\vee)} A_2^{-1} D_2^{-1} \\
&= R D_2 \underbrace{R D_1 R^{-1} A_2^{-1}}_{I(e,e^\vee)} D_2^{-1} \\
&= R D_2 A_2^{-1} \underbrace{R_{21} D_1 R D_2^{-1}}_{I(e,e^\vee)} \\
&= R D_2 A_2^{-1} D_2^{-1} R_{21} D_1 R \\
&= \mathcal{F}(R D_2 R_{21} A_1 R).
\end{aligned}
$$

$\square$

2.5.3 Independence of $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))^\circ$ on the orientation of Q.

For a quiver Q, and $e \in E$, let $\tau_e(Q)$ denote the quiver obtained from Q by reversing the orientation of e.

Let Q_1 and Q_2 be quivers whose underlying undirected graphs are isomorphic. Choose an isomorphism, by which we can identify the sets V_1, V_2 of vertices, and $\widetilde{E}_1$, $\widetilde{E}_2$ of undirected edges. We have the following:

Theorem 2.5.12. *Let $e_1, \ldots, e_n$ be a sequence of edges of Q_1, such that $\tau_{e_n} \cdots \tau_{e_1}(Q_1) \cong Q_2$ as oriented graphs. Then there is an induced isomorphism,*

$$
\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q_1))^\circ \cong \mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q_2))^\circ.
$$

Proof. Clearly, it suffices to assume that the orientations on Q_1 and Q_2 differ at exactly one edge. In this case, the isomorphism $\mathcal{D}_q(e) \to \mathcal{D}_q(e^\vee)$ constructed in the previous section can be extended to an isomorphism $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q_1)) \to \mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q_2))$, as the relations between $\mathcal{D}_q(e)$ (resp, $\mathcal{D}_q(e^\vee)$) and the rest of $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q_1))$ (resp, $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q_2))$) are just the tensor product relations, which are preserved by $\mathcal{F}$, which is a morphism in $\mathcal{C}$. $\square$

2.6 Construction of the q-deformed quantum moment map

In this section we construct the q-analog of the moment map in the classical geometric construction of the quiver variety.

2.6.1 Bialgebras and Hopf algebras in braided tensor categories

We recall some basic constructions involving Hopf algebras in braided tensor categories, which we will use later.

Definition 2.6.1. A bialgebra in $\mathcal{C}$ is a 5-tuple,

$$(A \in \mathcal{C},\ \mu : A \otimes A \to A,\ \eta : \mathbf{1} \to A,\ \Delta : A \to A \otimes A,\ \epsilon : A \to \mathbf{1}),$$

such that (A, μ, η) is a unital algebra in $\mathcal{C}$, (A, Δ, ϵ) is a co-unital coalgebra in $\mathcal{C}$, Δ is a homomorphism to the tensor product algebra $A \otimes A$. Homomorphisms are defined in the obvious way, and we denote by $\mathcal{C}$-biAlg the category of bialgebras in $\mathcal{C}$.

Definition 2.6.2. A Hopf algebra in $\mathcal{C}$ is a bialgebra in $\mathcal{C}$, with a (necessarily unique) convolution inverse S to the identity, called the antipode: either composition,

$$S * \mathrm{id} : A \xrightarrow{\Delta} A \otimes A \xrightarrow{S \otimes \mathrm{id}} A \otimes A \xrightarrow{\mu} A,$$

$$\mathrm{id} * S : A \xrightarrow{\Delta} A \otimes A \xrightarrow{\mathrm{id} \otimes S} A \otimes A \xrightarrow{\mu} A,$$

coincides with the convolution unit $\eta \circ \epsilon : A \to A$. We define the category $\mathcal{C} - \mathrm{Hopf\text{-}Alg}$ as the full subcategory of $\mathcal{C}$-biAlg consisting of bialgebras with antipode.

Let H be a Hopf algebra (in Vect), A be an algebra, and $\phi : H \to A$ be a homomorphism of algebras. To simplify notation, we omit the explicit application of ϕ here and in the definitions to follow. H acts on A via the induced adjoint action, $h \triangleright a = h_{(1)} a S(h_2) \in A$. For $\mathcal{C}$-Hopf-Alg, there is an analogous construction:

Definition 2.6.3. Let $H \in \mathcal{C}$-Hopf-Alg, and let $A \in \mathcal{C}$-Alg. Let $\phi : H \to A$ be a homomorphism of $\mathcal{C}$-algebras. The regular action of $H \otimes H$ on A is defined by:

$$\mathrm{act}_2 : H \otimes H \otimes A \xrightarrow{\mathrm{id} \otimes \sigma_{H,A}} H \otimes A \otimes H \xrightarrow{\mathrm{id}_H \otimes \mathrm{id}_A \otimes S} H \otimes A \otimes H \xrightarrow{\mu \circ (\mathrm{id} \otimes \mu)} A.$$

The adjoint action of H on A is given by

$$\mathrm{ad} : H \otimes A \xrightarrow{\Delta \otimes \mathrm{id}} H \otimes H \otimes A \xrightarrow{\mathrm{act}_2} A.$$

It is a standard exercise to check that these are indeed actions, i.e. that

$$\mathrm{ad} \circ (\mu_H \otimes \mathrm{id}_A) = \mathrm{ad} \circ (\mathrm{id}_H \otimes \mathrm{ad}) : H \otimes H \otimes A \to A.$$

2.6.2 Hopf algebra of matrix coefficients

For a locally finite braided tensor category $\mathcal{D}$, we have its algebra $A(\mathcal{D})$ of matrix coefficients, whose general construction dates back to work of Lyubashenko and Majid [LyMa], [Ma]. We recall the construction here.

We have the functor of tensor product,

$$\mathcal{T} : \mathcal{D} \boxtimes \mathcal{D} \to \mathcal{D},$$

$$V \boxtimes W \mapsto V \otimes W.$$

The braiding endows $\mathcal{T}$ with the structure of a tensor functor:

$$J : \mathcal{T}(X \boxtimes U) \otimes \mathcal{T}(V \boxtimes W) = X \otimes U \otimes V \otimes W \xrightarrow{\sigma_{U,V}} X \otimes V \otimes U \otimes W = \mathcal{T}((X \boxtimes U) \otimes (V \boxtimes W)).$$

$\mathcal{T}$ has a right adjoint $\mathcal{T}^\vee$ taking values in the Ind-category of $\mathcal{D} \boxtimes \mathcal{D}$. We define $A(\mathcal{D}) := \mathcal{T}^\vee(\mathbf{1}_\mathcal{D})$, and call it the algebra of matrix coefficients (for reasons which will become clear below). $A(\mathcal{D})$ is thus defined uniquely, up to canonical isomorophism,

as the representing object for the functor of co-invariants,

$$\operatorname{Hom}_{\mathcal{D}\boxtimes\mathcal{D}}(-\boxtimes-, A(\mathcal{D})) \cong \operatorname{Hom}_{\mathcal{D}}(-\otimes-, \mathbf{1}).$$

This description allows us to construct $A(\mathcal{D})$ explicitly as an Ind-algebra in $\mathcal{D}\boxtimes\mathcal{D}$. We let $\widetilde{A}(\mathcal{D})$ be the sum over all objects of $\mathcal{D}$,

$$\widetilde{A}(\mathcal{D}) := \bigoplus_{V\in\mathcal{D}} V^* \boxtimes V,$$

and let $A(\mathcal{D})$ be the quotient $\widetilde{A}(\mathcal{D})/Q$, where Q denotes the sum over all morphisms,

$$Q := \sum_{\phi:V\to W} \operatorname{im}(\Delta_\phi) \subset \widetilde{A}(\mathcal{D}), \text{ where}$$

$$\Delta_\phi := (\operatorname{id}\boxtimes\phi - \phi^*\boxtimes\operatorname{id}) : W^*\boxtimes V \to W^*\boxtimes W \oplus V^*\boxtimes V.$$

To see that $A(\mathcal{D})$ does indeed satify the desired universal property, we observe that we have natural isomorphisms:

$$\operatorname{Hom}_{\mathcal{D}}(X\boxtimes Y, A(\mathcal{D})) \cong \operatorname{Hom}(X, Y^*) \cong \operatorname{Hom}(X\otimes Y, \mathbf{1}),$$

because we can write any morphism $\phi \in \operatorname{Hom}(Y, V)$ as $\phi \circ \operatorname{id}_Y$, and can then apply the relations of Q to reduce the sum over all V to the single summand $V = Y$.

We have natural morphisms $i_V : V^*\boxtimes V \to A$, and also $\mathcal{T}(i_v) : V^*\otimes V \to \mathcal{T}(A)$, for all $V \in \mathcal{C}$. We will abuse notation and call $\mathcal{T}(i_v)$ simply by i_V when context is clear.

The algebra structure on A is given on generating objects $V^*\boxtimes V$, $W^*\boxtimes W$ by

$$(V^*\boxtimes V) \otimes_2 (W^*\boxtimes W) = V^*\otimes W^*\boxtimes V\otimes W \xrightarrow{\sigma_{V^*,W^*}} W^*\otimes V^*\boxtimes V\otimes W \xrightarrow{i_{V\otimes W}} A.$$

The algebra structure on $\mathcal{T}(A)$ is given on generating objects $V^*\otimes V$, $W^*\otimes W$ by

$$(V^*\otimes V) \otimes (W^*\otimes W) \xrightarrow{\sigma_{(V^*\otimes V),W^*}} (W^*\otimes V^*) \otimes (V\otimes W) \xrightarrow{i_{V\otimes W}} \mathcal{T}(A).$$

The unit of A, (resp. $\mathcal{T}(A)$) is the subspace $\mathbf{1}^* \boxtimes \mathbf{1}$ (resp. $\mathbf{1} \cong \mathbf{1}^* \otimes \mathbf{1}$).

Remark 2.6.4. The adjoint pair of functors $(\mathcal{T}, \mathcal{T}^\vee)$ are braided tensor categorical analogs of the restriction and induction functors, $(\mathrm{Res}_G^{G \times G}, \mathrm{Ind}_G^{G \times G})$, of finite groups, and the construction given above is analogous to constructing the $G - G$-bimodule $\mathbb{C}[G]$ as $\mathrm{Ind}_G^{G \times G} \mathbb{C}$.

Remark 2.6.5. In case $\mathcal{D} = U$-mod, for some quasi-triangular Hopf algebra H, the algebra $A(\mathcal{D})$ identifies as a vector space with the subspace of H^* spanned by functionals $c_{f,v}$, for $v \in V, f \in V^*$ defined by $c_{f,v}(h) := f(hv)$. Choosing a basis $v_1, \dots v_n$ and its dual basis $f_1, \dots, f_n$, one has the functionals $c_{f_i, v_j}(h)$, which are the i, jth matrix entry of the map $H \to \mathrm{Mat}_n(\mathbb{C})$ of the representation V.

Definition 2.6.6. $\mathcal{T}(A(\mathcal{D}))$ becomes a Hopf algebra in $\mathcal{D}$ with coproduct, counit, and antipode defined on each subspace $V^* \otimes V$ by:

$$\Delta|_{V^* \otimes V} : V^* \otimes V \xrightarrow{\mathrm{id} \otimes \mathrm{coev} \otimes \mathrm{id}} V^* \otimes V \otimes V^* \otimes V \xrightarrow{i_V \otimes i_V} \mathcal{T}(A(\mathcal{D})) \otimes \mathcal{T}(A(\mathcal{D})),$$

$$\epsilon|_{V^* \otimes V} : V^* \otimes V \xrightarrow{\mathrm{ev}} \mathbf{1},$$

$$S|_{V^* \otimes V} : V^* \otimes V \xrightarrow{\sigma_{V^* \otimes V}} V \otimes V^* \xrightarrow{\theta_V \otimes \mathrm{id}} V^{**} \otimes V^* \xrightarrow{i_{v*}} \mathcal{T}(A(\mathcal{D})).$$

For the category $\mathcal{C} := \boxtimes_{v \in V} \mathcal{C}_v$ of Section 2.2.3, we have $A(\mathcal{C}) := \boxtimes_{v \in V} A(\mathcal{C}_v)$, and $\mathcal{T}(A(\mathcal{C})) := \boxtimes_{v \in V} \mathcal{T}(A(\mathcal{C}_v))$, which becomes a $\mathcal{C}$-Hopf algebra with structure morphisms defined diagonally.

The quantum determinant $\det_q$

When $\mathcal{D}$ is the braided tensor category of type I $U_q(\mathfrak{gl}_N)$-modules, the algebra $\mathcal{T}(A)$ of the previous section contains a central element called the quantum determinant. It is defined as follows:

Definition 2.6.7. The quantum determinant $\det_q$ is the unique generator of the one-dimensional subspace $\mathcal{T}((\Lambda_q^N(\mathbb{C}^N))^* \boxtimes \Lambda_q^N(\mathbb{C}^N))$, with normalization $ev_{\Lambda_q^N}(\det_q) = 1$.

Proposition 2.6.8. *The element* $\det_q$ *is central and group-like in* $\mathcal{T}(A(\mathcal{D}))$.

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Proof. That $\det_q$ is group-like is clear from the fact that $\Lambda_q^N(\mathbb{C}^N))$ is one-dimensional. It's centrality follows from the fact that $\sigma_{\Lambda_q^N(\mathbb{C}^N)),V}$ is a scalar matrix for any V. □

Remark 2.6.9. The construction of $\det_q$ above does not yield a clear formula for $\det_q$ in terms of the standard generators a_j^i. We have been unable to find such a formula in the literature, for the reflection equation algebra, although a formula for the corresponding element in the FRT algebra is well-known.

Explicit presentation of $\mathcal{T}(A(\mathcal{C}_v))$.

We have the following well-known presentation for $\mathcal{T}(A(\mathcal{C}_v))$.

Theorem 2.6.10. *We have an isomorphism:*

$$\mathcal{T}(A(\mathcal{C}_v)) \cong \mathcal{T}(A(\mathcal{C}_v))^+[(\det_q)^{-1}], \text{ where}$$

$$\mathcal{T}(A(\mathcal{C}_v))^+ := \langle l_j^i, \ i,j = 1,\ldots,d_v \mid R_{kl}^{ij}l_m^l R_{no}^{mk}l_p^o = l_l^i R_{km}^{lj}l_o^m R_{np}^{ok}\rangle. \tag{2.13}$$

In particular, there is a well-known isomorphism of algebras,

$$\kappa : \mathcal{T}(A(\mathcal{C}_v)) \to U_q'(\mathfrak{gl}_{d_v})$$

$$l_j^i \mapsto \widetilde{l}_{ij}.$$

We note in passing that $\kappa(\mathcal{T}(A(\mathcal{C}))^+) = U^+$. Henceforth, we identify $\mathcal{T}(A(\mathcal{C}_v))$ with $U_q'(\mathfrak{gl}_{d_v})$ and $\mathcal{T}(A(\mathcal{C}_v))^+$ with U^+ via the isomorphism κ.

2.6.3 Quantum moment map for $\mathcal{D}_q(e)$ when e is not a loop

In the next two sections, we construct quantum moment maps, $\mu_v^e : U_v \to \mathcal{D}_e$ for each edge $e \in E$, and $v = \alpha(e), \beta(e)$. As might be expected, the construction is quite different depending on whether or not e is a loop. As such, we treat the two cases in different sections.

First, we recall from [VV] the notion of a quantum moment map for a coideal subalgebra, a mild generalization of that in [L]. Let H be a Hopf algebra, with H'

a left coideal subalgebra; that is, H' is a subalgebra, and $\Delta(H') \subset H \otimes H'$. Then a moment map for an H-algebra A is a homomorphism $\mu : H' \to A$, such that

$$\mu(h)a = (h_{(1)} \rhd a)\mu(h_{(2)}),$$

where we denote the action of $h \in H$ on $a \in A$ by $h \rhd a$ to distinguish it from the multiplication in A.

Definition-Proposition 2.6.11. Let $e \in E$, and $v = \beta(e) \neq \alpha(e)$. The *edge map* $\mu_v^e : U_v^+ \to \mathcal{D}_e$ given on generators by:[2]

$$\mu_v^e(l_j^i) = (\delta_j^i + (q - q^{-1}) \sum_k \partial_k^i a_j^k),$$

defines a homomorphism of algebras in $\mathcal{C}$.

Proof. Following the notation of Section 2.3.2, we let M denote the matrix:

$$M := \sum_{i,j} \mu_v^e(l_j^i) E_i^j.$$

We have $M = I + (q - q^{-1})DA$. We need to show that the elements $\mu_v^e(l_j^i) \in \mathcal{D}_e$

---

[2]We have set $t = 1$ in the definition of $\mathcal{D}_q(\mathrm{Mat}_d(Q))$, for ease of notation (see Remark 2.2.11). It is easily checked that defining $\mu_v^e(l_j^i) := \delta_j^i + t(q - q^{-1})\partial_k^i a_j^k$ yields a moment map for other choices of t. This will be needed in Section 2.9.

satisfy the reflection equation relations (2.13). We compute, in matrix notation:

$$M_2 R_{21} M_1 R_{12} = (I + (q - q^{-1}) D_2 A_2) R_{21} (I + (q - q^{-1}) D_1 A_1) R_{12}$$

$$= R_{21} R_{12} + (q - q^{-1})(D_2 A_2 \underbrace{R_{21} R_{12}}_{\text{Hecke reln.}} + R_{21} D_1 A_1 R_{12})$$

$$+ (q - q^{-1})^2 D_2 \underbrace{A_2 R_{21} D_1}_{I(e, e^\vee)} A_1 R_{12}$$

$$= R_{21} R_{12} + (q - q^{-1})(D_2 A_2 + (q - q^{-1}) \underbrace{D_2 A_2 \Omega_{12} R_{12}}_{\text{cancel this}} + R_{21} D_1 A_1 R_{12})$$

$$+ (q - q^{-1})^2 (D_2 D_1 \underbrace{R_{21}^{-1} A_2 A_1 R_{12}}_{I(e)} - \underbrace{D_2 \Omega_{12} A_1 R_{12}}_{\text{with this}})$$

$$= R_{21} R_{12} + (q - q^{-1})(D_2 A_2 + R_{21} D_1 A_1 R_{12}) + (q - q^{-1})^2 D_2 D_1 A_1 A_2$$

On the other hand, we compute:

$$R_{21} M_1 R_{12} M_2 = R_{21} (I + (q - q^{-1}) D_1 A_1) R_{12} (I + (q - q^{-1}) D_2 A_2)$$

$$= R_{21} R_{12} + (q - q^{-1})(R_{21} D_1 A_1 R_{12} + \underbrace{R_{21} R_{12}}_{\text{Hecke reln.}} D_2 A_2)$$

$$+ (q - q^{-1})^2 R_{21} D_1 \underbrace{A_1 R_{12} D_2}_{I(e, e^\vee)} A_2$$

$$= R_{21} R_{12} + (q - q^{-1})(D_2 A_2 + (q - q^{-1}) \underbrace{\Omega_{12} R_{12} D_2 A_2}_{\text{cancel this}} + R_{21} D_1 A_1 R_{12})$$

$$+ (q - q^{-1})^2 (\underbrace{R_{21} D_1 D_2 R_{12}^{-1}}_{I(e^\vee)} A_1 A_2 - \underbrace{R_{21} D_1 \Omega_{12} A_2}_{\text{with this}})$$

$$= R_{21} R_{12} + (q - q^{-1})(D_2 A_2 + R_{21} D_1 A_1 R_{12}) + (q - q^{-1})^2 D_2 D_1 A_1 A_2$$

$$= M_2 R_{21} M_1 R_{12},$$

as desired. Thus the homomorphism μ_v^e is well defined. $\square$

Proposition 2.6.12. *Let* $v = \beta(e) \neq \alpha(e)$. *Regard* μ_v^e *above as a map from* U^+ *via*

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the isomorphism κ. Then μ_v^e is a quantum moment map:

$$\mu_v^e(x)y = (x_{(1)} \triangleright y)\mu_v^e(x_{(2)}),$$

for all $x \in U^+, y \in \mathcal{D}_e^\circ$.

Proof. It suffices to check this on the generators $\widetilde{l}_j^i$ of U^+, and the generators a_n^m, ∂_p^o of $\mathcal{D}_e$. By definition of the U^+ action on V, we have:

$$((\widetilde{l}_j^i)_{(1)} \triangleright a_n^m)\mu_v^e((\widetilde{l}_j^i)_{(2)}) = ((l_k^{+i}S(l_j^{-l})) \triangleright a_n^m)(\delta_l^k + (q - q^{-1})\partial_o^k a_l^o)$$
$$= R_{pk}^{qi}R_{jn}^{lp}a_q^m(\delta_l^k + (q - q^{-1})\partial_o^k a_l^o).$$

In the matrix notation of Section 2.3.2, we set

$$N := \sum_{i,j,n,m} ((\widetilde{l}_j^i)_{(1)} \triangleright a_n^m)\mu_v^e((\widetilde{l}_j^i)_{(2)})E_i^j \otimes E_m^n$$
$$= \sum_{i,j,n,m} (R_{pk}^{qi}R_{jn}^{lp}a_q^m(\delta_l^k + (q - q^{-1})\partial_o^k a_l^o))E_i^j \otimes E_m^n.$$

Then, we have:

$$N = A_2 R_{21}(I + (q - q^{-1})D_1 A_1)R_{12}$$
$$= A_2 R_{21} R_{12} + (q - q^{-1})\underbrace{A_2 R_{21} D_1}_{I(e, e^\vee)} A_1 R_{12}$$
$$= A_2 R_{21} R_{12} + (q - q^{-1})(D_1 \underbrace{R_{21}^{-1} A_2 A_1 R_{12}}_{I(e)} - \Omega_{12} A_1 R_{12})$$
$$= A_2(\underbrace{R_{21}R_{12} - \Omega_{12}R_{12}}_{\text{Hecke reln.}}) + (q - q^{-1})D_1 A_1 A_2$$
$$= A_2 + (q - q^{-1})D_1 A_1 A_2$$
$$= M_1 A_2$$

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Comparing matrix coefficients, we find:

$$((\widetilde{l_j^i})_{(1)} \rhd a_n^m) \mu_v^e((\widetilde{l_j^i})_{(2)}) = \mu_v^e(\widetilde{l_j^i}) a_n^m,$$

as desired. The computation for ∂_p^o is similar. $\qquad\square$

Corollary 2.6.13. *The image $\mu_v^e(\det_q)$ of the quantum determinant in U^+ satisfies the equation:*

$$\mu_v^e(\det_q) a^I \partial_J = q^{2(|J|-|I|)} a^I \partial_J \mu_v^e(\det_q),$$

Proof. Recall that $\det_q$ is grouplike in U^+. Thus the moment map condition reads:

$$\mu_v^e(\det_q) a^I \partial_J = (\det_q \rhd a^I \partial_J) \mu_v^e(\det_q).$$

The element $\det_q$ acts on $V \in \mathcal{C}_v$ by the scalar q^2, and $V^* \in \mathcal{C}_v$ by the scalar q^{-2}, so the claim follows. $\qquad\square$

Proposition 2.6.14. *Let $e \in E$, and $v = \alpha(e) \neq \beta(e)$. The elements $\overline{\mu}_v^e(l_j^i)$,*

$$\overline{\mu}_v^e(l_j^i) = (\delta_j^i + (q - q^{-1}) \sum_k a_k^i \partial_j^k),$$

satisfy the relation:

$$\overline{M}_2 R_{12}^{-1} \overline{M}_1 R_{21}^{-1} = R_{12}^{-1} \overline{M}_1 R_{21}^{-1} \overline{M}_2,$$

where $\overline{M}$ denotes the matrix:

$$\overline{M} := \sum_{i,j} \overline{\mu}_v^e(l_j^i) E_i^j.$$

Proof. We observe that the defining relations of $\mathcal{D}_e$ and $\mathcal{D}_{e^\vee}$ are related by interchanging each a_j^i with ∂_j^i, and replacing R_α, R_β with $(R_\alpha)_{21}^{-1}$, $(R_\beta)_{21}^{-1}$, so that this relation follows from Definition-Proposition 2.6.11 . $\qquad\square$

Corollary 2.6.15. *Let $v = \beta \neq \alpha$ (resp, $v = \alpha \neq \beta$). The powers of the q-determinant in the variables $\mu_e^v(l_j^i)$ (resp, $\overline{\mu}_v^e(l_j^i)$) form a multiplicative Ore set.*

Proof. This follows as in Corollary 2.6.13. □

Definition 2.6.16. The localized edge differential operator algebra $\mathcal{D}_q(e)^\circ$ is the localization of $\mathcal{D}_q(e)$ at the multiplicative Ore sets generated by the q-determinants in the elements $\mu_\beta^e(l_j^i)$ and $\overline{\mu}_\alpha^e(l_l^k)$.

Definition-Proposition 2.6.17. Let $e \in E$, and $v = \alpha(e) \neq \beta(e)$. The *edge map* $\mu_v^e : U_v^+ \to \mathcal{D}_e^\circ$ given on generators by:[3]

$$\mu_v^e(l_j^i) = (\delta_j^i + (q - q^{-1}) \sum_k a_k^i \partial_j^k)^{-1},$$

defines a homomorphism of algebras in $\mathcal{C}$.

Proof. The entries of the inverse matrix in the definition lie in the localized algebra $\mathcal{D}_q(e)^\circ$, where we have inverted the q-determinant. That μ_v^e defines a homomorphism follows from Proposition 2.6.14, by taking the inverses of both sides. □

Definition 2.6.18. The edge maps μ_α^e and μ_β^e extend uniquely to homomorphisms $\mu_\alpha^e : U_\alpha \to \mathcal{D}_e^\circ$ and $\mu_\beta^e : U_\beta \to \mathcal{D}_e^\circ$.

We will henceforth refer only to this extended homomorphism, and not its restriction to U^+.

2.6.4 Quantum moment map for $\mathcal{D}_q(e)$ when e is a loop

Definition 2.6.19. Let $v = \alpha(e) = \beta(e)$. The localized edge algebra $\mathcal{D}_q^\circ(e)$ is the localization of $\mathcal{D}_q(e)$ at the q-determinants in the variables a_j^i and ∂_j^i of $\mathcal{D}_q(e)$.

Definition-Proposition 2.6.20. There is a unique homomorphism,

$$\mu_v^e : \mathcal{T}(A(C_v)) \to \mathcal{D}_q(e),$$

$$l_j^i \mapsto (DA^{-1}D^{-1}A)_j^i$$

---

[3]For $t \neq 1$, we set $\mu_v^e(l_j^i) = (\delta_j^i + t(q - q^{-1})a_k^i \partial_j^k)^{-1}$ instead (see Definition 2.6.11, Remark 2.2.11).

Proof. First, we claim that there is a unique homomorphism of algebras in $\mathcal{C}$,

$$\phi : \mathcal{T}(A(\mathcal{C}_v)) \otimes \mathcal{T}(A(\mathcal{C}_v)) \to \mathcal{D}_q(e)$$

$$(l^i_j \otimes l^k_l) \mapsto (DA^{-1}D^{-1})^i_j a^k_l.$$

Once we have constructed ϕ, we can simply define $\mu^e_v := \phi \circ \Delta$.

An algebra homomorphism $\phi = f \otimes g$ out of $\mathcal{T}(A(\mathcal{C}_v)) \otimes \mathcal{T}(A(\mathcal{C}_v))$ is the same as a pair f, g of algebra homomorphism out of $\mathcal{T}(A(\mathcal{C}_v))$, such that the images of f and g braided-commute. That is, we require the following relations on $\mathcal{T}(A(\mathcal{C}_v)) \otimes \mathcal{T}(A(\mathcal{C}_v))$:

$$(1 \otimes x)(y \otimes 1) = r^- y \otimes r^+ x,$$

where we use the shorthand $R = r^+ \otimes r^-$ (sum is implicit). On generators $x = l^i_j$, $y = l^k_l$, this condition reads:

$$(1 \otimes l^i_j)(l^k_l \otimes 1) = \widetilde{R}^{mk}_{jn} R^{op}_{ml} R^{in}_{qr} (R^{-1})^{qs}_{tp} (l^r_s \otimes l^t_o),$$

or equivalently,

$$(1 \otimes l^i_j) R^{jn}_{mk} (l^k_l \otimes 1) = R^{in}_{qr} (l^r_s \otimes 1)(R^{-1})^{qs}_{tp} (1 \otimes l^t_o) R^{op}_{ml}.$$

Thus the condition we require on f and g is:

$$g(l^i_j) R^{jn}_{mk} f(l^k_l) = R^{in}_{qr} f(l^r_s)(R^{-1})^{qs}_{tp} g(l^t_o) R^{op}_{ml}$$

or, in the matrix notation of Section 2.3.2:

$$G_1 R F_2 = R F_2 R^{-1} G_1 R, \tag{2.14}$$

where F and G denote the matrices:

$$F := \sum_{i,j} f(l_j^i) E_i^j, \quad G := \sum_{i,j} g(l_j^i) E_i^j.$$

The maps $f, g : \mathcal{T}(A(\mathcal{C})) \to \mathcal{D}_e^\circ$, $f(l_j^i) = (DA^{-1}D^{-1})_j^i$ and $g(l_j^i) = A_j^i$ are each homomorphisms (they are the natural inclusion of $\mathcal{T}(A(\mathcal{C}))$, and its composition with Fourier transform, respectively). It remains to check relation (2.14). We have:

$$RF_2 R^{-1} G_1 R = RDA^{-1} \underbrace{D^{-1} R^{-1} AR}_{I(e,e^\vee)}$$

$$= RD \underbrace{A^{-1} RAR}_{I(e)} D^{-1}$$

$$= \underbrace{RDRAR}_{I(e,e^\vee)} A^{-1} D^{-1}$$

$$= ARDA^{-1} D^{-1}$$

$$= G_1 RF_2,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Proposition 2.6.21. *Let $v = \beta(e) = \alpha(e)$. Regard μ_v^e above as a map from U^+ via the isomorphism κ. Then μ_v^e is a quantum moment map for all $x \in U^+, y \in \mathcal{D}_e^\circ$.*

Proof. It suffices to check this on the generators $\widetilde{l}_j^i$ of U^+, and the generators a_n^m, ∂_p^o of $\mathcal{D}_e$. By definition of the U^+ action on V, we have:

$$((\widetilde{l}_j^i)_{(1)} \rhd a_n^m) \mu_v^e((\widetilde{l}_j^i)_{(2)}) = ((l_k^{+i} S(l_j^{-l})) \rhd a_n^m)(DA^{-1}D^{-1}A)_l^k$$

$$= (R^{-1})_{qs}^{mi} \widetilde{R}_{jx}^{tq} a_o^x R_{pk}^{os} R_{tn}^{lp} (DA^{-1}D^{-1}A)_l^k.$$

Thus, the moment map condition reads:

$$(DA^{-1}D^{-1}A) \otimes A = (R^{-1})_{qs}^{mi} \widetilde{R}_{jx}^{tq} a_o^x R_{pk}^{os} R_{tn}^{lp} (DA^{-1}D^{-1}A)_l^k E_i^j \otimes E_m^n,$$

or equivalently, moving the R^{-1} and $\widetilde{R}$ to the LHS, and re-writing the RHS in matrix notation:

$$R_{21}D_1A_1^{-1}D_1^{-1}A_1RA_2 = A_2R_{21}D_1A_1^{-1}D_1^{-1}A_1R_{12}.$$

We simplify the RHS:

$$RHS = \underbrace{A_2R_{21}D_1}_{I(e,e^\vee)} A_1^{-1}D_1^{-1}A_1R$$

$$= R_{21}D_1 \underbrace{RA_2R_{21}A_1^{-1}}_{I(e)} D_1^{-1}A_1R$$

$$= R_{21}D_1A_1^{-1} \underbrace{RA_2R_{21}D_1^{-1}}_{I(e,e^\vee)} A_1R$$

$$= R_{21}D_1A_1^{-1}D_1^{-1}R_{21}^{-1} \underbrace{A_2R_{21}A_1R}_{I(e)}$$

$$= R_{21}D_1A_1^{-1}D_1^{-1}A_1RA_2,$$

and thus the moment map condition is satisfied. $\qquad\square$

2.6.5 Quantum moment map for $\mathcal{D}_q(\mathrm{Mat_d}(Q))$

In the previous section, we defined moment maps $\mu_v^e : U_v \to \mathcal{D}_e$ for every pair (e,v), with e attached to v. In this section, we combine the edge maps into a homomorphism $\mu_q^\# : A(\mathcal{C}) \to \mathcal{D}_q(\mathrm{Mat_d}(Q))$, quantizing the moment map defined in the classical case. First, we have:

Lemma 2.6.22. *For all $v, v' \in V$ distinct, and for all $e \in E_v, e' \in E_{v'}$, we have:*

$$\mu_v^e(l_j^i)\mu_{v'}^{e'}(l_l^k) = \mu_{v'}^{e'}(l_l^k)\mu_v^e(l_j^i).$$

Proof. We claim that, for any e emanating from v and for any $w \neq v$, the image of μ_v^e is contained in a trivial isotypic component of $\mathcal{C}_w$. This is obvious if e is a loop,

and for e not a loop, it follows from the following, more canonical description of μ_v^e :

$$\mu_v^e(l_j^i) = \mathrm{ev}(v^i \otimes v_j) + (q - q^{-1})v^i \otimes v_j \boxtimes \mathrm{coev}(\mathbf{1}).$$

Since $\mathcal{D}_q$ is defined as a braided tensor product over its edge algebras $\mathcal{D}_e$, elements in the image of μ_v^e commute with those in the image of any μ_w^f, via the braiding. As the trivial representation braids trivially with any representation, the claim follows. $\square$

Remark 2.6.23. At this point, we note that the ordering on $\overline{E}$ is not used in any construction, but rather the induced ordering on each $\overline{E}_v$. This is consistent with similar observations in [C-BS], [VdB2].

Definition 2.6.24. The vertex moment map $\mu_v^{\#} : A(\mathcal{C}_v) \to \mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ is the composition:

$$\mu_v^{\#} : A(\mathcal{C}_v) \xrightarrow{\Delta^{(|E_v|)}} A(\mathcal{C})^{\otimes |E_v|} \xrightarrow{\underset{e \in E_v}{\bigotimes} \mu_v^e} \bigotimes_{e \in E_v} \mathcal{D}_e \subset \mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q)).$$

Definition 2.6.25. The moment map $\mu_q^{\#} : T(A(\mathcal{C})) \to \mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ is the external tensor product,

$$\mu_q^{\#} := \underset{v \in V}{\boxtimes} \mu_v^{\#} : T(A(\mathcal{C})) \cong \underset{v \in V}{\boxtimes} A(\mathcal{C}_v) \to \mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q)).$$

It follows by Propositions 2.6.12 and 2.6.21 that $\mu_q^{\#}$ is indeed a moment map in the sense of [L].

2.7 Construction of the quantized multiplicative quiver variety

In this section, we are finally in a position to define the quantized multiplicative quiver variety. First, we recall certain characters of $A(\mathcal{C}_v)$, where $\mathcal{C}_v = U_q(\mathfrak{gl}_{d_v})$-lfmod. For a complete classification of the characters of $A(\mathcal{C}_v)$, see [Mu].

2.7.1 Quantum trace characters

First, we observe that for all $\rho \in \mathbb{C}$, there exists a unique homomorphism of algebras:

$$\mathrm{tr}_\rho : T(A(\mathcal{C}_v)) \to \mathbb{C},$$

$$l^i_j \mapsto \rho \delta^i_j.$$

It is easily checked that the left coideal subalgebra $U' \subset U$ is stable under tr_ρ, in the following sense: for $x \in U'$, we have $x_{(1)} \mathrm{tr}_\rho(x_{(2)}) \in U'$. Thus for any $\xi : V \to \mathbb{C}^\times$, we may define the character,

$$\mathrm{tr}_\xi := \bigotimes_{v \in V} \mathrm{tr}_{\xi_v} : T(A(\mathcal{C})) \to \mathbb{C}.$$

We set $\mathcal{I}_\xi := \ker \mathrm{tr}_\xi \subset U$.

2.7.2 Multiplicative quantized quiver variety

Definition 2.7.1. Fix a quiver Q, its dimension vector $\mathbf{d}$, and character $\xi : V \to \mathbb{C}^\times$. The multiplicative, quantized quiver variety, $\mathcal{A}^\xi_\mathbf{d}(Q)$, is the quantum Hamiltonian reduction of $\mathcal{D}_q(\mathrm{Mat}_\mathbf{d}(Q))$ by the moment map $\mu^\#_q$. That is,

$$\mathcal{A}^\xi_\mathbf{d}(Q) := \mathrm{Hom}_\mathcal{C}\left(\mathbf{1}, \mathcal{D}_q(\mathrm{Mat}_\mathbf{d}(Q)) \Big/ \mathcal{D}_q(\mathrm{Mat}_\mathbf{d}(Q))\mu^\#_q(\mathcal{I}_\xi)\right).$$

Definition 2.7.2. We let $\mathcal{D}_q(\mathrm{Mat}_\mathbf{d}(Q))\text{-mod}_\mathcal{C}$ denote the category of $\mathcal{D}_q$-modules in the category $\mathcal{C}$.

The following is a localization theorem for the algebras $\mathcal{A}^\xi_\mathbf{d}(Q)$, whose proof is identical to that of [GG2], Corollary 7.2.4. We refer the reader to the excellent exposition there.

Theorem 2.7.3. *We have an essentially surjective functor,*

$$\mathbb{H} : \mathcal{D}_q(\mathrm{Mat}_\mathbf{d}(Q))\text{-}mod_\mathcal{C} \to \mathcal{A}^\xi_\mathbf{d}(Q)\text{-}mod,$$

$$M \mapsto \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, M),$$

inducing an equivalence of categories,

$$\mathbb{H} : \mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))\text{-}mod_{\mathcal{C}}/\mathrm{Ker}\,\mathbb{H} \to \mathcal{A}_{\mathbf{d}}^{\xi}(Q)\text{-}mod.$$

Here, $\mathrm{Ker}\,\mathbb{H}$ denotes the Serre subcategory of aspherical $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$-modules, i.e. those modules whose space of invariants is zero. The functor $\mathbb{H}$ is called the functor of Hamiltonian reduction.

2.8 The degeneration of $A_{\mathbf{d}}^{\xi}(Q)$

2.8.1 The Kassel-Turaev biquantization of $S(\mathfrak{g})$

In order to compute the quasi-classical limit of $\mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}(Q))$ and its moment map $\mu_q^{\#}$, we need to recall from [KT] the theory of biquantization of Lie bialgebras. For $\mathfrak{g} = \mathfrak{sl}_N$, the constructions we now recall here was also given by J. Donin [Do]. We begin with definitions.

Definition 2.8.1. A co-Poisson algebra is a cocommutative coalgebra C, together with a Lie co-bracket $\delta : C \to C \wedge C$ satisfying the compatibility condition:

$$(\mathrm{id}\otimes\Delta) \circ \delta = (\delta \otimes \mathrm{id} + (\sigma \otimes \mathrm{id}) \circ (\mathrm{id}\otimes\delta)) \circ \Delta.$$

Definition 2.8.2. A bi-Poisson bialgebra is a commutative, cocommutative bialgebra A, together with a Poisson bracket and co-bracket, satisfying the compatibility conditions:

1. $\Delta(\{a, b\}) = \{\Delta(a), \Delta(b)\}$,

2. $\delta(ab) = \delta(a)\Delta(b) + \Delta(a)\delta(b)$,

3. $\delta(\{a, b\}) = \{\delta(a), \Delta(b)\} + \{\Delta(a), \delta(b)\}$.

Recall that for any vector space V, the symmetric algebra $S(V)$ is a bialgebra with coproduct:

$$\Delta(v) = v \otimes 1 + 1 \otimes v.$$

A Lie bialgebra structure on $\mathfrak{g}$ gives rise to a bi-Poisson bialgebra structure on the symmetric algebra $S(\mathfrak{g})$ by declaring the Poisson bracket and co-bracket be the unique extensions to $S(\mathfrak{g})$ of the Lie bracket and co-bracket on $\mathfrak{g}$. Consider $\mathfrak{g} = \mathfrak{gl}_N$, and let

$$r := \sum_{i<j} E_j^i \otimes E_i^j + \frac{1}{2} \sum_i E_i^i \otimes E_i^i \in \mathfrak{g} \otimes \mathfrak{g}$$

denote the classical r-matrix for $\mathfrak{gl}_N$, associated to the trace form. Of particular interest for us is the Lie bialgebra structure on $\mathfrak{gl}_N$, with cobracket $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ given by:

$$\delta(x) := [r, x \otimes 1 + 1 \otimes x],$$

In [KT], Kassel and Turaev constructed a $\mathbb{C}[[u,v]]$ bialgebra $A(\mathfrak{g}) = A_{u,v}(\mathfrak{g})$, which is a biquantization of $S(\mathfrak{g})$. This means, firstly, that we have the following commutative diagram of bialgebras:

$$
\begin{array}{ccc}
A(\mathfrak{g}) & \longrightarrow & A(\mathfrak{g})/(v) \\
\downarrow & & \downarrow \\
A(\mathfrak{g})/(u) & \longrightarrow & A(\mathfrak{g})/(u,v)
\end{array}
$$

Secondly, we have natural isomorphisms of coalgebras, algebras, and bialgebras, respectively:

$$A(\mathfrak{g})/(v) \cong S(\mathfrak{g})[[u]], \quad A(\mathfrak{g})/(u) \cong S(\mathfrak{g})[[v]], \quad A(\mathfrak{g})/(u,v) \cong S(\mathfrak{g}).$$

In this sense, $A(\mathfrak{g})$ simultaneously quantizes the Poisson bracket and co-bracket on $S(\mathfrak{g})$: v is the deformation parameter for the coproduct, and u is the deformation parameter for the product.

Recall that the Etingof-Kazhdan quantization [EK] of the Lie bialgebra $\mathfrak{g}$ is a Hopf algebra $U_\hbar(\mathfrak{g})$, isomorphic as an algebra to $U(\mathfrak{g})$, but with coproduct which quantizes

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the co-bracket of $\mathfrak{g}$. Let $V_u(\mathfrak{g}) := A(\mathfrak{g})/(v)$, and let $A_{\hbar,\hbar}(\mathfrak{g})$ denote the quotient of $A_{u,v}(\mathfrak{g})$ by the ideal $(v - u)$ (in the quotient, we rename $\hbar := u = v$ for notational convenience). While we will not need to recall the full details of the construction of $A(\mathfrak{g})$, we will need the following descriptions of its quotients:

Proposition 2.8.3. *[KT]*

1. $A_{\hbar,\hbar}(\mathfrak{g})$ *is the Etingof-Kazhdan quantization* $U_{\hbar^2}(\mathfrak{g})$ *of* $\mathfrak{g}$.[4]

2. $V_u(\mathfrak{g}) \cong T(\mathfrak{g})\Big/\langle X \otimes Y - Y \otimes X = u[X,Y] \mid X, Y \in \mathfrak{g}\rangle$.

Claim (1) is not explicitly stated in [KT] but follows easily from the definition of $A_{u,v}(\mathfrak{g})$ given in Section 6, *loc. cit.*. Claim (2) is Theorem 2.6. Note that, by (2), we have an $\mathbb{C}$-algebra homomorphism,

$$i : V_u(\mathfrak{g}) \to U(\mathfrak{g})[[u]],$$

$$X \in \mathfrak{g} \mapsto uX.$$

It follows by the PBW theorem that i is an injection. We may therefore identify $V_u(\mathfrak{g})$ with the Rees algebra of $U(\mathfrak{g})$, where the latter is filtered by declaring the generating subspace $\mathfrak{g}$ to be degree 1.

Let $\mathcal{U}_\hbar$ denote the $\mathbb{C}[[\hbar]]$-Hopf algebra (a.k.a QUE algebra) obtained by setting $q = e^\hbar$ in Section 2.3.2. We have the following well-known proposition:

Proposition 2.8.4. *There exists an isomorphism* $\alpha : \mathcal{U}_\hbar \to U[[\hbar]]$ *of QUE algebras, such that* $\alpha = \mathrm{id} \mod \hbar$. *Moreover, we have* $\alpha(\mathcal{U}_\hbar') = V_\hbar(\mathfrak{g})$.

Proof. Recall that the generators $\tilde{l}^i_j$ of $\mathcal{U}_\hbar$ may be obtained as the matrix coefficients of the double-braiding:

$$(\mathrm{id} \otimes \rho_{\mathbb{C}^N})(R_{21}R) = \sum_{kl} \tilde{l}^k_l \otimes E^l_k.$$

The claim now follows from the fact that $\alpha \otimes \alpha(R_{21}R) \in V_\hbar(\mathfrak{g})^{\otimes 2}$. $\qquad\square$

---

[4]For $\mathfrak{g} = \mathfrak{gl}_N$, this agrees with the Drinfeld-Jimbo quantization of $\mathfrak{gl}_N$.

2.8.2 Flatness is preserved by quantum Hamiltonian reduction

Throughout this section, we assume Q and $\mathbf{d}$ satisfy the conditions of Theorem 2.1.1, so that the classical moment map $\mu : T^* \mathrm{Mat}_{\mathbf{d}}(Q) \to \mathfrak{g}^{\mathbf{d}}$ is flat. We set $q = e^{\hbar}$, and consider all algebras and categories defined in terms of q to be defined over $\mathbb{C}[[\hbar]]$, and complete in the $\hbar$-adic topology. As a consequence of the flatness of μ, we prove that the algebra $A_{\mathbf{d}}^{\xi}(Q)$ is a flat formal deformation of its classical ($\hbar = 0$) limit. We note that similar results have been proven in [Lo], Lemma 3.6.1, and [Br].

To begin, we recall the following lemma from ring theory (see, e.g. [B], Chapter 2, Proposition 3.12):

Lemma 2.8.5. *Let A_0, be a graded ring, and M_0 a flat A_0-module. Let A be a ring with an exhaustive, increasing filtration, and M an A-module with compatible filtration, such that $gr(A) \cong A_0$, and $gr(M) \cong M_0$ as A_0-modules. Then M is a flat A-module.*

Corollary 2.8.6. *Let A_0, B_0 be a graded rings, with a flat homomorphism $\phi_0 : B_0 \to A_0$ (i.e. ϕ makes A_0 into a flat left B_0-module). Let A, B be rings equipped with exhaustive, increasing filtrations, such that $gr(A) = A_0, gr(B) = B_0$. Then any filtered homomorphism $\phi : B \to A$ lifting ϕ_0 is flat.*

Lemma 2.8.7. *Let A_0 be a graded Poisson algebra with a Poisson action of a reductive group G, and $\mu_0 : S\mathfrak{g} \to A_0$ be a moment map for this action. Let A be a filtered algebra with $gr(A) = A_0$, and $\mu : U(\mathfrak{g}) \to A$ a quantum moment map that lifts μ_0 (so that the adjoint action is completely reducible). If μ_0 is flat, then so is μ (i.e. A is flat as a left $U(\mathfrak{g})$-module), and $gr(A//\mathfrak{g}) = A_0//\mathfrak{g}$.*

Proof. The flatness of A as a left $U(\mathfrak{g})$-module is an application of Lemma 2.8.6, with $B_0 = S(\mathfrak{g})$ and $B = U(\mathfrak{g})$. The Hamiltonian reduction $A//\mathfrak{g}$ proceeds in two steps: first we construct the quotient A/J of A by its left ideal $J = A\mu(U(\mathfrak{g})) \subset A$, and then we take the subspace of invariants in the quotient. We show that each step is compatible with the filtration, and commutes with the associated graded construction.

The module A/J inherits a filtration, and by flatness of μ, we have $gr(A/J) = A_0/J_0$, where $J_0 = A_0\mu_0(S(\mathfrak{g}))$. Since the adjoint action of $\mathfrak{g}$ on A is completely reducible, and J is $\mathfrak{g}$ invariant, we have that the quotient A/J embeds as a $\mathfrak{g}$-submodule of A, and likewise $J^{\mathfrak{g}}$ embeds as a submodule of $A^{\mathfrak{g}}$. Thus we have $(A/J)^{\mathfrak{g}} \cong A^{\mathfrak{g}}/J^{\mathfrak{g}}$. Finally, the action of $\mathfrak{g}$ preserves the filtration on A, so we have:

$$gr(A/J)^{\mathfrak{g}} \cong gr(A^{\mathfrak{g}}/J^{\mathfrak{g}}) \cong (A_0/J_0)^{\mathfrak{g}} = A_0//\mathfrak{g},$$

as desired. $\qquad\square$

Lemma 2.8.8. *Let $\mu_\hbar$ be a deformation of the classical moment map μ, $\mu_\hbar : U_\hbar(\mathfrak{g}) \to A_\hbar$, where $A_\hbar$ is a flat deformation of A. Assume that the adjoint action is completely reducible. Then $\mu_\hbar$ is flat, and $A_\hbar//U_\hbar(\mathfrak{g})$ is a flat formal deformation (equivalently, it is torsion-free in $\hbar$).*

Proof. First, we show that $\mu_\hbar$ is flat. For this, we recall another lemma from ring theory. While the proof is standard, we include it here for the sake of completeness.

Lemma 2.8.9. *Let S be a (not necessarily commutative) flat formal deformation of the algebra $S_0 = \mathbb{C}[x_1, \ldots, x_n]$. Let $\chi : S \to \mathbb{C}[[\hbar]]$ be a character, specializing to $\chi_0 : S_0 \to \mathbb{C}$. Finally, suppose that M is an S-module, topologically free over $\mathbb{C}[[\hbar]]$, such that $M_0 = M/\hbar M$ is flat over S_0. Then $M \otimes_S \chi$ is a flat formal deformation of $M_0 \otimes_{S_0} \chi_0$.*

Proof. We denote by $\mathbb{C}$ the one dimensional $\mathbb{C}[[\hbar]]$-module, where $\hbar$ acts by zero. We have only to check:

$$\mathrm{Tor}^i_{\mathbb{C}[[\hbar]]}(M \otimes_S \chi, \mathbb{C}) \overset{?}{=} 0.$$

Notice that we have an isomorphism, natural in M:

$$M \otimes_S \chi \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \cong M \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \otimes_{S_0} \chi_0 \cong M_0 \otimes_{S_0} \chi_0.$$

Thus, we have:

$$\mathrm{Tor}^i_{\mathbb{C}[[\hbar]]}(M \otimes_S \chi, \mathbb{C}) \cong \mathrm{Tor}^i_{S_0}(M, \chi_0) = 0,$$

by assumption of flatness on M. $\qquad\square$

We now turn to proving the flatness of $A_\hbar//U_\hbar(\mathfrak{g})$. We note that Hamiltonian reduction involves fixing a scalar action of $\mathfrak{gl}_N$, so that $A//G$ is completely reducible as a $U(\mathfrak{g})$-module. By the flatness of $\mu_\hbar$, $A_\hbar/J_\hbar$ is a flat $\mathbb{C}[[\hbar]]$-module. Finally, complete reducibility gives an isomorphism $(A_\hbar/J_\hbar)^{U_\hbar(\mathfrak{g})} \cong (A/J)^{\mathfrak{g}}[[\hbar]]$, as $\mathbb{C}[[\hbar]]$-modules, because completely reducible $\mathfrak{g}$-modules do not admit non-trivial deformations. $\qquad\square$

Proposition 2.8.10. *Let $t = \hbar$, and let $\xi_v := e^{\hbar^2 \lambda_v}$, for some $\xi : V \to \mathbb{C}$. Then quasi-classical limit of the ideal $\mathcal{I}_\xi$ is the classical moment ideal, i.e. the defining ideal of the closed set $\mu^{-1}(\sum \lambda_v \, \mathrm{id}_v)$.*

Proof. The ideal $\mathcal{I}_\xi$ is generated by elements $\mu_q^\#(u)$, for $u \in \mathcal{U}_\hbar'$.

Fix a $v \in V$, and let $r = |E_v|$. We compute the image of $l_j^i \in \mathcal{U}_\hbar(\mathfrak{gl}^{d_v})$ under the map μ_v (see Section 2.2.3 for notation concerning quivers):

$$
\mu(l_j^i) = \sum_{i_1,\ldots,i_r=1}^{d_v} \mu_v^{e_1}(l_{i_1}^i) \mu_v^{e_2}(l_{i_2}^{i_1}) \cdots \mu_v^{e_r}(l_j^{i_r})
$$

$$
= \delta_j^i + \hbar^2 \left(\sum_{e \in E_v^\beta} \sum_k \partial_k^i a_j^k - \sum_{e \in E_v^\alpha} \sum_k a_k^i \partial_j^k + \sum_{e \in E_v^\circ} \sum_k \left(\partial_k^i a_j^k - a_k^i \partial_j^k \right) \right) + O(\hbar^3).
$$

Thus the coefficient in $\hbar^2$ is precisely the LHS of equation (2.2). On the other hand, we easily compute that $\mathrm{tr}_\xi(l_j^i) = \delta_j^i + \hbar^2 \lambda_v \delta_j^i$. Thus equating $\hbar^2$ coefficients, we obtain Equation (2.2). $\qquad\square$

Corollary 2.8.11. *The algebra $A_\mathbf{d}^\xi(Q)$ is a topologically free $\mathbb{C}[[\hbar]]$-module, which is a flat formal deformation of $\mathcal{O}(\mathcal{M}_\mathbf{d}^\lambda)$.*

Proof. First, we note that in the formal setting $\mathcal{D}_q^\circ$ and $\mathcal{D}_q$ coincide, as the $\det_q(e)$ are invertible formal power series. We have shown in Theorem 2.4.3 that $\mathcal{D}_q$ is a flat formal deformation of $\mathcal{O}(T^* \mathrm{Mat}_\mathbf{d}(Q))$. By applying Proposition 2.8.10, we see that the ideal $\mathcal{I}_\xi$ deforms the classical moment ideal I_λ; the deformation is flat by our assumptions on dimension vectors, and thus $A_\mathbf{d}^\xi(Q)$ is a flat formal deformation of $\mathcal{O}(\mathrm{Mat}_d(Q))\big/\!\!\big/_\xi \mathbb{G}$ by Lemma 2.8.8. $\qquad\square$

2.9 Spherical DAHA's as quantized multiplicative quiver varieties

In this section we describe how to recover the spherical DAHA of type A_{n-1} as the algebra $A_{\mathbf{d}}^\lambda(Q)$, where Q is the Calogero-Moser quiver, $(Q,d) = \overset{1}{\bullet} \to \overset{n}{\bullet} \circlearrowleft$. We also explain that the spherical generalized DAHA of type Q is the algebra $A_{\mathbf{d}}^\lambda(Q)$, when Q is a star-shaped quiver. As we have remarked in the Introduction, the results presented in this section, with formal parameters, are not very strong; in particular it would be interesting to upgrade the claims of this section to include generic numerical values of q, and also to study the parameter correspondence between the parameter λ and the parameter $\mathbf{c}$ appearing in the definition of Cherednik algebras (see, e.g, [EG], [EOR]).

Lemma 2.9.1. *([GG2]) The classical moment map,*

$$\mu : \mathrm{Mat}_n \times \mathrm{Mat}_n \times \mathbb{C}^n \times (\mathbb{C}^n)^* \to \mathfrak{gl}_n(\mathbb{C}) \times \mathbb{C},$$

$$(A, B, i, j) \mapsto ([A, B] + i \otimes j, j(i)),$$

on the Calogero-Moser matrix space is flat.

We make use of the following lemma, which is proven in [CEE], using KZ functors, and in [Ch1], [Ch2] by direct computation.

Lemma 2.9.2. *The spherical DAHA of type A_{n-1} is isomorphic as a $\mathbb{C}[[\hbar]]$-algebra to the spherical Cherednik algebra of type A_{n-1}.*

Theorem 2.9.3. *([EG], Theorem 2.16) The spherical Cherednik algebra is the universal deformation of the algebra of invariant differential operators on $\mathbb{C}^n$ for the action of S_n.*

Theorem 2.9.4. *The algebra $A_{\mathbf{d}}^\xi(Q)$ is isomorphic to the spherical DAHA of type A_{n-1}.*

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Proof. Both algebras $A_{\mathbf{d}}^{\xi}(Q)$ and the spherical DAHA of type A_{n-1} are deformation quantizations of the Calogero-Moser variety. Moreover, the spherical DAHA is the universal such deformation. It follows that there exists a surjective homomorphism of $\mathbb{C}[[\hbar]]$-algebras from spherical DAHA to $A_{\mathbf{d}}^{\xi}(Q)$. This map is the identity modulo $\hbar$, and is thus an isomorphism. $\qquad\square$

Theorem 2.9.5. *Let Q be a star-shaped quiver, and $\mathbf{d}$ be the Calogero-Moser dimension vector of Example 2.3.12. Then the algebra $A_{\mathbf{d}}^{\xi}(Q)$ is isomorphic to the spherical GDAHA associated to Q.*

Proof. This is proven in the same way as Theorem 2.9.4. $\qquad\square$

Chapter 3

Representations of the elliptic braid group coming from quantum D-modules on G

3.1 Introduction

Recall the notations from Section 1.2

The contents of this article are laid out as follows. Section 2 consists of prelimi-
naries. In Subsection 2.1, we recall the definitions of the elliptic braid group and a
certain quotient, called the double affine Hecke Algebra. In Subsection 2.2, we recall
the notion of twisting of the comultiplication of a quasi-triangular Hopf algebra U.
This subsection is somewhat technical, and may be skipped on a first read. In Sub-
section 2.3, we recall the construction of the Reflection Equation algebra, which in a
certain sense generalizes the algebra of functions on an algebraic group to the braided
setting. To the extent possible, we give key definitions and propositions in braided-
categorical terms. In Subsection 2.4, we recall the Heisenberg double construction,
and its relation to differential operators on an algebraic group. In Subsection 2.5,
we recall how to apply this in the non-commutative (quantum, braided) context. In
particular, here is where we give the definition of quantum D-modules we will use in

this article. In Subsection 2.6 we recall the left, right, and adjoint actions of a Lie algebra $\mathfrak{g}$ on D_G-modules, and generalize this to the quantum setting.

In Section 3, we state without proof our main results; in particular, we assert the existence of a certain family of functors from the category of quantum D-modules to the category of representations of the elliptic braid groups. Section 4 comprises our primary new contribution to the subject, wherein we construct the functors $F_{n,V}$ asserted in Section 3. In Section 5, we explain that the element $\tilde{Y}$ of Section 4 and a related element $\tilde{X}$ act as scalars when V is irreducible. In Section 5, we consider the case $U = U_t$, and we show that we can recover the geometric constructions of [CEE] as a trigonometric degeneration of our constructions.

3.2 Preliminaries

3.2.1 The Elliptic Braid Group and the DAHA

In this section we define the elliptic braid group following Birman and Scott [Bir] and [Sc], and a particular quotient of its group algebra, called the double affine Hecke algebra.

Definition 3.2.1. The elliptic braid group, B_n^{Ell}, is the fundamental group of the configuration space of n points on the torus. It is generated by

- the commuting elements $X_1, \ldots, X_n$

- the commuting elements $Y_1, \ldots, Y_n$

- and the braid group of the plane,

$$
B_n = \left\langle T_1, \ldots, T_{n-1} \quad \middle| \quad \begin{array}{l} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \forall i, \\ T_i T_j = T_j T_i, |i - j| \geq 2 \end{array} \right\rangle.
$$

The cross relations are:

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- $T_i X_i T_i = X_{i+1}$,

- $T_i Y_i T_i = Y_{i+1}$,

- $X_1 Y_2 = Y_2 X_1 T_1^2$,

- $\tilde{Y} X_i = X_i \tilde{Y}$, where $\tilde{Y} = \prod_j Y_j$.

Under the usual realization of the torus T^2 as the unit square with opposite sides glued, we choose as a basepoint the configuration with all the marked points along the diagonal. The subgroup B_n is then identified with those braids which stay away from the sides, while the X_i and Y_i correspond to the horizontal and vertical global cycles, respectively. This is depicted in Figure 3-1. [1]
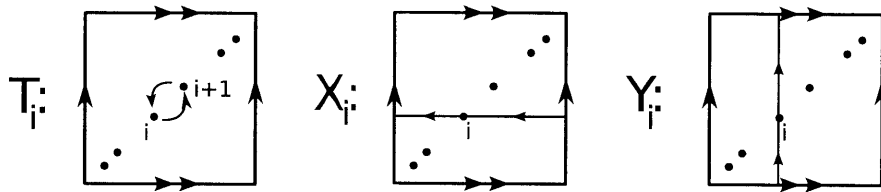

Figure 3-1: Generators for the elliptic braid group B_n^{Ell}.

Definition 3.2.2. Cherednik's double affine Hecke algebra (DAHA) $\mathcal{H}(q,t)$ is the quotient of the group algebra $\mathbb{C}B_n^{Ell}$ of the elliptic braid group by the additional relations

$$(T_i - q^{-1}t)(T_i + q^{-1}t^{-1}) = 0,$$

where q, t are complex parameters. Note that when $q, t = 1$, we have an isomorphism $\mathcal{H}(1,1) \cong \mathbb{C}[S_n \ltimes \mathbb{Z}^{2n}]$.

Remark 3.2.3. Let us discuss the precise relation between our DAHA $\mathcal{H}_n(q,t)$ and the one appearing in [?]. One can define a 3-parameter DAHA, $\mathcal{H}_n(q,t,p)$ by replacing the last relation of $\mathcal{H}_n(q,t)$ with $\tilde{Y} X_i = p X_i \tilde{Y}$. Then our algebra is $\mathcal{H}_n(q,t,1)$ and Cherednik's is $\mathcal{H}_n(1,t,p)$.

---

[1] The reader should note that these are slightly non-standard generators (e.g. [?]). In particular, the Y_i here are the inverses of the Y_i there.

3.2.2 Twistings on a Quasi-triangular Hopf algebra

Throughout this article, U denotes a quasi-triangular Hopf algebra with universal R-matrix $\mathcal{R} = \sum_k r_k^+ \otimes r_k^-$. We will denote by Δ, ϵ, S the comultiplication co-unit, and antipode in U. Let $(\mathcal{C}, \otimes, \sigma)$ denote the braided tensor category of its left modules. Here σ is given by $\tau \circ \mathcal{R}$, where τ is the flip of tensor factors.

We will need the Hopf algebras U^o, U^e, and $U^{[2]}$ as they appear in [VV]. For clarity's sake, we adopt the same notations, and recall them here.

Definition 3.2.4. Let U^o denote the algebra U, with opposite co-multiplication, $\Delta^o(x) = \tau \circ \Delta(x)$.

Definition 3.2.5. Let U^e denote the algebra $U^o \otimes U$, with coordinate-wise multiplication, co-multiplication, and antipode.

Definition 3.2.6. Let $U^{[2]}$ denote the algebra $U \otimes U$ with coordinate-wise multiplication, but with comultiplication given by $\tilde{\Delta}(x \otimes y) = \mathcal{R}_{23}^{-1} \tau_{23}(\Delta(x) \otimes \Delta(y))\mathcal{R}_{23}$, and with antipode $\tilde{S}(x \otimes y) = \mathcal{R}_{21}(S(x) \otimes S(y))\mathcal{R}_{21}^{-1}$.

In fact, U^e and $U^{[2]}$ are related by a 2-cocycle, which induces an equivalence on their tensor categories of modules. Let us recall these constructions:

Definition 3.2.7. Let H be a Hopf algebra. A normal left 2-cocycle on H is an invertible element $c \in H \otimes H$ such that

$$(\epsilon \otimes \mathrm{id})(c) = (\mathrm{id} \otimes \epsilon)(c) = 1 \otimes 1, \text{ and } (\Delta \otimes \mathrm{id})(c)(c \otimes 1) = (\mathrm{id} \otimes \Delta)(c)(1 \otimes c).$$

A 2-cocycle is sometimes called a twist. Given a 2-cocycle c on H, we can define the twisted Hopf algebra H_c to be the algebra H with twisted comultiplication $\Delta_c(h) = c^{-1}\Delta(h)c$ and antipode $S_c(h) = QS(h)Q^{-1}$, where $Q = \mu \circ (\mathrm{id} \otimes S)(c)$. We have the following standard proposition:

Proposition 3.2.8. *c induces a tensor equivalence H-mod$\to H_c$-mod.*

It is now straightforward to check that $c = \mathcal{R}_{13}\mathcal{R}_{23}$ is a 2-cocycle for U^e, and that $U^{[2]} = U_c^e$.

Definition 3.2.9. Let H be a Hopf algebra. An H-equivariant algebra A is an algebra A with an H-action on A s.t. the product $\mu : A \otimes A \to A$ is a map of H-modules.

Remark 3.2.10. In other words, A is an algebra in the category of H-modules.

Given an H-equivariant algebra A, we can define an H_c-equivariant algebra A_c as the same underlying H-module, with multiplication given by $\mu_c(a \otimes b) = \mu(c(a \otimes b))$. We call A_c the H_c-algebra *equivalent* to the H-algebra A.

For a (U, U)-bimodule, we denote the left and right actions by $\triangleright, \triangleleft$ (e.g $x \triangleright m \triangleleft y$). A (U, U) bimodule is the same thing as a U^e-module under the identification $(a \otimes b)(v) = b \triangleright v \triangleleft S(a)$. The co-multiplication on U gives the algebra maps $\Delta : U \to U^e$ (resp. $U^{[2]}$). Thus for a U^e- (resp. $U^{[2]}$-) module V, we have an action of U, denoted "ad" given by $(\mathrm{ad}\, x)(v) = \Delta(x)(v)$ (we use the symbol "ad" in both contexts).

3.2.3 The Reflection Equation Algebra

In this section, we will recall the so-called reflection equation algebra A associated to the quasi-triangular Hopf algebra U. In the case that $U = U(\mathfrak{g})$ is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ of an algebraic group G, A will be the algebra functions on G. When $U = U_t(\mathfrak{g})$ is the quantum group associated to the Lie algebra $\mathfrak{g}^2$, A will be a braided version of the algebra of functions on G, distinct from the dual quantum group O_q. Majid called A the "the braided Hopf algebra associated to O_q". The primary advantage of A from our perspective is that there is an adjoint action of U on A, for which the algebra structure on A is equivariant, and which does not exist for the usual dual quantum group O_q. This equivariance property was first observed and explained by Majid [Ma], who proposed the reflection equation algebra as a preferable replacement for O_q in the context of braided differential geometry, and showed that it was a braided-commutative braided-Hopf algebra in the category of U-modules.

---

[2]Note that U_t is not quasi-triangular, since the R-matrix lies in a completion of $U_t \otimes U_t$. However, in all our constructions, we always apply one of the components of $\mathcal{R}$ to a finite dimensional module, so its action is well-defined.

The results here are all standard, and can be found in one form or another in many sources, e.g. [Ma] or [KS]. We include them here for completeness, and to establish the diagrammatical notation which will appear in later sections. Also, the reflection equation, Proposition 3.2.19, is usually stated for the defining modules for the FRT bialgebra, but we will need it for arbitrary modules, and so we give a diagrammatical proof.

Definition 3.2.11. Let F denote the restricted dual Hopf algebra of U relative to the tensor category of its finite dimensional representations. It is the sum of all finite dimensional U^e-submodules of U^*, and is spanned by functionals $c_{f,v}$, defined by $c_{f,v}(x) = f(xv)$, where $f \in V^*, v \in V$, and V is a finite dimensional U-module.

One can easily check that $c_{f,v} c_{g,w} = c_{f \otimes g, v \otimes w}$, so that F is a sub-algebra. The U^e-action $(x \otimes y) c_{f,v} = c_{xf,yv}$ makes F into a U^e-equivariant algebra. This corresponds to the natural (U, U)-bimodule structure on U^* given by $(x \triangleright \phi \triangleleft y)(h) = \phi(S(x)hy)$.

In fact, F is a Hopf algebra with co-product $\Delta(c_{f,v}) = c_{f,e_i} \otimes c_{e_i^*,v}$, where e_i is a basis for V and e_i^* is a dual basis. The antipode on F is the adjoint to the antipode on U, defined by $(Sc_{f,v})(u) = c_{f,v}(Su)$.

Proposition 3.2.12. *Let $\phi : V \to V$ be a U-module map, and let ϕ^* denote the adjoint map. Then $c_{f,\phi v} = c_{\phi^* f, v}$.*

Proof. For $x \in U$, we compute,

$$c_{f,\phi v}(x) = f(x\phi(v)) = f(\phi(xv)) = (\phi^* f)(xv) = c_{\phi^* f, v}(x)$$

$\square$

Definition 3.2.13. A dual pairing of two Hopf algebras H and K is a map $\kappa :$

$H \otimes K \to \mathbb{C}$ s.t. for all $h, h' \in H$, $k, k' \in K$, we have

$$
\begin{aligned}
\kappa(\Delta_H(h), k \otimes k') &= \kappa(h, kk') \\
\kappa(hh', k) &= \kappa(h \otimes h', \Delta_K(k)) \\
\kappa(h, 1_K) &= \epsilon_H(h) \\
\kappa(1_H, k) &= \epsilon_K(k) \\
\kappa(S(h), k) &= \kappa(h, S(k))
\end{aligned}
$$

The pairing is called non-degenerate if its left and right kernels are zero.

Definition 3.2.14. We say that U *has enough finite-dimensional representations* if the common annihilator of all finite-dimensional representations is zero.

Remark 3.2.15. The natural pairing of vector spaces $\kappa : U \otimes U^* \to \mathbb{C}$ restricts to a dual pairing of Hopf algebras U and F. Since F was defined as a subalgebra of U^*, the right kernel of κ is automatically zero. The left kernel of κ is the set of $x \in U$ s.t. $c_{f,v}(u) = 0$ for all $f \in V^*, v \in V$, V a f.d. U-module. Thus the natural pairing is non-degenerate if, and only if, U has enough finite-dimensional representations. This is a mild condition which is certainly satisfied for any U_t, and which can be forced in general by quotienting U by the left kernel of κ, which will be a Hopf ideal. In this case, the natural pairing between F and U is nondegenerate, and we get

$$\text{Finite dimensional } U\text{-modules} \cong \text{Finite dimensional } F\text{-comodules}$$

We will assume from now on that U has enough finite dimensional representations.

Definition 3.2.16. We denote by A the $U^{[2]}$ algebra equivalent to the U^e-algebra F via the cocycle $c = R_{13}R_{23}$. It has the same co-multiplication as F, but its multiplication μ' is related to that of F by the formula:

$$
\mu'(f \otimes g) = \sum_k \mu(\mathrm{ad}\, r_k^+(f) \otimes (g \triangleleft S(r_k^-))) \tag{3.1}
$$

Remark 3.2.17. For any U-module V, we have morphisms of U-modules

$$c_{-,-} : V^* \otimes V \to A, f \otimes v \mapsto c_{f,v}.$$

Many quantities are most easily computed in the pre-image $V^* \otimes V$, applying Proposition 3.2.12 as needed. In particular, the structure maps (μ, Δ, S) for the braided-Hopf algebra A, as well as the proofs of Propositions 3.2.19 and 3.4.3 will be given in this way.

Remark 3.2.18. Let V be a U-module. By duality, V is an A-comodule, so we have a canonical element $L \in End_{\mathbb{C}}(V) \otimes A$ given by $\Delta(v) = L_1(v) \otimes L_2$. This coaction may be described invariantly under the above identification as simply $\Delta :$ $v \mapsto \sum_i e_i \otimes e_i^* v$. Here $\sum_i e_i \otimes e_i^*$ has an invariant description as the image of $1 \in \mathbb{C}$ under the coevaluation map $coev : \mathbb{C} \to V \otimes V^*$. Similarly, formula (3.1) has an invariant description in terms of the braiding. These are depicted in figure 3-2.
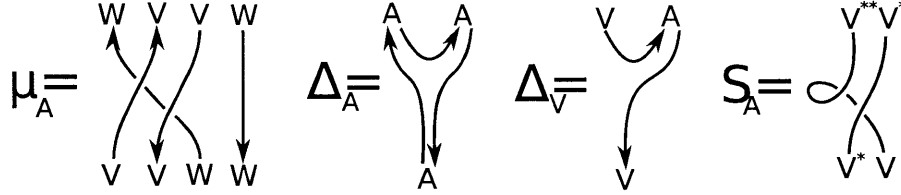

Figure 3-2: The multiplication and co-multiplication in A, the co-action on a U-module V, and the antipode in A. Diagrams are to be read from bottom up. The identity morphism $id_V : V \to V$ is denoted with a downward flowing arrow, while $id_{V^*} : V^* \to V^*$ is denoted with an upward flowing arrow. To avoid confusion, we omit arrows and explicitly label duals in the description of the antipode. An excellent reference for the diagrammatical calculus in braided tensor categories is [Ka].

Proposition 3.2.19. *[Ma] The element L defined above satisfies the reflection equation*

$$L_{01}R_{12}L_{02}R_{21} = R_{12}L_{02}R_{21}L_{01}$$

in the space of endomorphisms of the module $V \otimes V \otimes A$. (the tensor indices run from right to left, with A in index 0)

Proof. The proof is presented in figure 3-3 in the braided tensor category U-mod. The LHS diagram is $L_{01}R_{12}L_{02}R_{21}$, and the RHS is $R_{12}L_{02}R_{21}L_{01}$. Note that on the RHS we have applied Proposition 3.2.12 to the automorphism σ_{VV} of $V \otimes V$. $\qquad\square$
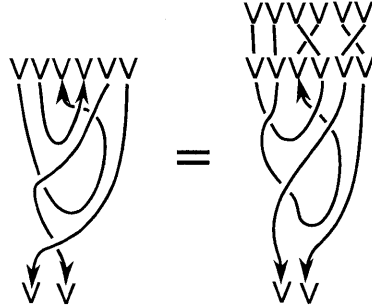

Figure 3-3: A braided-categorical proof of the "Reflection Equations".

3.2.4 The Heisenberg double and differential operators

Definition 3.2.20. Let F and U be dually paired Hopf algebras. Then F is an U^o-module under the action $x \otimes y \mapsto \kappa(S(x), y_1)y_2$, which makes F into a U^o-equivariant algebra. The Heisenberg double $H(F, U)$ is the semi-direct product of F and U^o under this action. As a vector space it is $F \otimes U$, with subalgebras $F \cong F \otimes 1$, $U \cong 1 \otimes U$, and cross relations

$$xf = (x_1 \triangleright f)x_2 = \kappa(S(x_1), f_1)f_2 x_2, \text{ for all } x \in U, f \in F$$

Example 3.2.21. D-modules on an affine algebraic group. The algebra of differential operators on an affine algebraic group has a particularly satisfactory description in terms of the Heisenberg double. Let $U = U(\mathfrak{g})$ denote the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of an algebraic group G, and let $O = O(G)$ denote the algebra of polynomial functions on G.

The Lie algebra $\mathfrak{g}$ is constructed as the sub-Lie algebra of vector fields on G which are left-translation invariant. Thus we have a pairing $\kappa : U \otimes O \to \mathbb{C}$ given by $X \otimes f \mapsto X(f)_{\mathrm{id}}$. This is a dual pairing of Hopf algebras, and so we may construct the Heisenberg double $H(O, U)$. For instance, if the group G is the affine plane $\mathbb{C}^n$,

one finds $O = \mathbb{C}[x_1, \ldots, x_n]$, $U = \mathbb{C}[\partial_1, \ldots, \partial_n]$, and $H(O, U)$ is the nth Weyl algebra W_n. More generally, we have the following standard proposition (see, e.g. [KS]):

Proposition 3.2.22. *The Heisenberg double $H(O, U)$ is isomorphic to the algebra D_G of algebraic differential operators on the group G.*

We recall the following well-known and important result:

Theorem 3.2.23. *Suppose that U has enough finite dimensional modules. Then F is a faithful $H(F, U)$-module.*

Proof. Let $\phi, \psi : F \to F$ be linear operators. Recall that the convolution $\phi * \psi : F \to F$ is defined as $(\phi * \psi)(f) = \phi(f_1)\psi(f_2)$. Then the action of $\phi := \sum f_i \otimes h_i \in H(F, U)$ on F may be written as $\rho(\phi) = \phi * 1$ (where we view ϕ as an element of $\mathrm{End}_{\mathbb{C}}(F)$ of finite rank, using that $U \subset F^*$) It is well known that the operator $\phi \mapsto \phi * 1$ on $\mathrm{End}_{\mathbb{C}}(F)$ is invertible for any Hopf algebra F, the inverse being $\phi \mapsto \phi * S$, where S is the antipode. Thus $\phi * 1 = 0$ implies $\phi = 0$, and we are done. $\square$

F is sometimes called the basic representation, since when $H(F, U)$ is differential operators on an algebraic group, F is just functions on the group.

3.2.5 Quantum D-modules on U

In this section, we want to generalize the construction of differential operators to the non-commutative setting. Our motivating example will be $U = U_t(\mathfrak{g})$, but we will make the construction for an arbitrary quasi-triangular Hopf algebra U. Key to the present construction is the isomorphism Ξ constructed in [VV], which relates two potentially different notions of differential operators, one in terms of F, the other in terms of A. Indeed, it was Propositions 1.8.1 and 1.8.2 of [VV] which first alerted the author to the relevance of the reflection equation algebra A to the present work.

Recall the reflection equation algebra A constructed previously. As A is a $U^{[2]}$-equivariant algebra, we can construct the algebra $A \rtimes U^{[2]}$.

Definition 3.2.24. [VV] The algebra D_U of differential operators on U is the subalgebra $A \otimes U \otimes 1$ of $A \rtimes U^{[2]}$. A quantum D-module for U is a module over the algebra D_U.

Remark 3.2.25. M is thus both an A-module and a U-module, such that, for $u \in U, a \in A, m \in M$, we have $x(am) = \sum_{j,k} \mu_M(((x_1 \otimes S(r_j^+)r_k^+) \rhd a) \otimes r_j^- x_2 r_k^- m)$. This commutation relation is depicted graphically in figure 3-4.
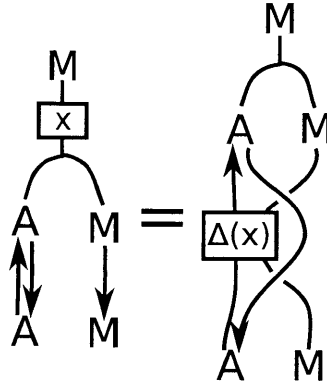

Figure 3-4: The commutation relations for differential operators.

Remark 3.2.26. Typically (e.g. [BaKr]), a quantum D-module is defined as a module over the Heisenberg double $H(F, U)$. In [VV], the authors construct an isomorphism $\Xi : D_U \cong H(F, U)$. Thus, the present notion of a quantum D-module agrees with the usual notion.[3]

3.2.6 Three actions of U on D-modules

In this section, we will recreate the left, right, and adjoint actions of vector fields on a classical D_G-module, in the non-commutative context. First, we recall the classical setup. Let G be an algebraic group, and let $U = U(\mathfrak{g})$ as before. Then the left, right and adjoint action of vector fields induce algebra maps $\partial_\rhd : U \to D_U$, $\partial_\lhd : U \to D_U$, and ad $: U \to D_U$. We have that ad $(h) = \partial_\lhd(h_1)\partial_\rhd(h_2) = \mu \circ (\partial_\lhd \otimes \partial_\rhd)\Delta(h)$. The

---

[3]In fact, it will turn out that we could assume slightly less in our constructions: namely, we could take M to be a module over the subalgebra $A \otimes U' \subset D_U$ instead of the full algebra D_U, as the constructions of Section 4 only use the action of this subalgebra. However, in this article we will ignore the distinction.

assertion that ad is a homomorphism relies on the fact that the images of $\partial_\lhd$ and $\partial_\rhd$ centralize one another in D_U, since the image of $\partial_\lhd$ is right invariant vector fields, and the image of $\partial_\rhd$ is left invariant vector fields. A more concise way of saying this is that there is a homomorphism $\phi : U^e \to D_U$, and that the left, right, and adjoint homomorphisms are given by pre-composing with the left, right, and adjoint maps $U \to U^e$. Note that because we act on functions, the left action corresponds to right translation in the group, and vice versa.

In the non-commutative situation, again following [VV], we define $\partial_\lhd$ and $\partial_\rhd$ as follows. We have the isomorphism $\Xi : D_U \cong H(F, U)$. We have the inclusion $\partial_\lhd$ as inclusion into the U^o factor.

Definition 3.2.27. The adjoint action of U on itself is given by ad $: x \otimes y \mapsto x_1 y S(x_2)$.

Definition 3.2.28. Denote by U' the sub-algebra of $x \in U$ s.t. ad $(U)x$ is finite dimensional.

Remark 3.2.29. It is straightforward to check that U' is indeed a subalgebra, and that $\Delta(U') \subset U \otimes U'$.

The adjoint action $U \otimes U' \to U'$ of U on itself yields the co-adjoint map $\partial_\rhd : U' \to F \otimes U' \subset H(F, U)$.

Definition 3.2.30. Let $U^{[2]'}$ denote the subalgebra $U \otimes U'$ of $U^{[2]}$. We define the homomorphism $\partial_2 : U^{[2]'} \to H(F, U^o)$ by $x \otimes y \mapsto \partial_\lhd(x)\partial_\rhd(y)$. Abusing notation, we denote also by $\partial_2 : U^{[2]'} \to D_U$ the map $x \otimes y \mapsto \Xi^{-1}(\partial_\lhd(x)\partial_\rhd(y))$.

Remark 3.2.31. Let $z \in U$ be a central element. z is thus ad-invariant, and so co-ad invariant. Thus we have that $\partial_\lhd(z) = 1 \otimes z = \partial_\rhd(z)$, in other words the left and right actions of z agree whenever z is central in U. We will apply this observation to the ribbon element later in the construction.

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3.3 Statement of Results

Our main result is that the data of a ribbon Hopf algebra U, a f.d. U-module V and a quantum D_U-module M together yield representations of the elliptic braid group on n strands, for any n. As D_{U_t} is a flat (in fact, trivial) deformation of D_G, this provides a rich source of such representations. Taking M to be the "basic" D_U-module A, we recover the construction of Lyubashenko and Majid [Ly], [LyMa]. Alternatively, taking a quasi-classical limit as $t \to 1$ for the quantum group $U_t(sl_N)$, we recover the geometric constructions from [CEE]. Thus our results are really an interpolation of those two papers. In a future paper, we hope to elaborate the case $U = U_t(sl_N)$ in greater detail, and extend some representation-theoretic results of [CEE].

To state the main theorem, we need to introduce some further notation. Let Z be a U-module. We denote by Z^{inv} the vector space

$$Z^{inv} = Hom_U(1, Z) = \{w \in Z \text{ s.t. } xw = \epsilon(x)w \forall x \in U\}.$$

Our main result is the following:

Theorem 3.3.1. *Let U be a ribbon Hopf algebra. Let $n \in \mathbb{N}$, V a f.d. U-module, and M a D_U-module. On the vector space*

$$W = (V^{\otimes n} \otimes M)^{inv}$$

of invariants w.r.t to the adjoint action on M, we have an action of the elliptic braid group B_n^{Ell}, which defines a functor $F_{n,V} : D_U - mod \to Rep(B_n^{Ell})$.

We will provide the construction in Section 3.4, from which will follow two easy corollaries:

Corollary 3.3.2. *Let $\nu \in U$ be the ribbon element. Suppose that V is irreducible, and that $\nu|_V = c_V \, id_V$. Then, $\tilde{Y}|_V = \tilde{X}|_V = c_V^n \, id_W$.*

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Corollary 3.3.3. *Suppose the braiding on V satisfies the Hecke relation*

$$(\sigma_{VV} - q^{-1}t)(\sigma_{VV} + q^{-1}t^{-1}).$$

Then the action of B_n^{Ell} descends to an action of the double affine Hecke algebra (DAHA) $\mathcal{HH}_n(q,t)$, and we have a functor $F_{n,V} : D_U - mod \to Rep(\mathcal{HH}_n(q,t))$.

Finally, we consider the example $U = U_t(sl_N)$, and V is the defining representation, of highest weight $(1, 0, \ldots, 0)$. It is well-known that V is Hecke with parameters $q, t = q^{nk}, k = N/n$. In [CEE], the authors considered D-modules on (the classical group) SL_N; given a D-module M, they constructed a representation of the trigonometric Cherednik algebra $\mathcal{HH}_n^{deg}(k)$ of type A_{n-1} on the space $(V^{\otimes n} \otimes M)^{inv}$, with parameter $k = N/n$. As $\mathcal{HH}_n^{deg}(k)$ is the quasi-classical limit of the DAHA as the parameter $t \to 1$, we can ask whether our construction agrees with theirs in the quasi-classical limit. Indeed, we have the following

Theorem 3.3.4. *In the quasi-classical limit as $t \to 1$ the construction in Proposition 3.3.3 recovers the $\mathcal{HH}_n^{deg}(k)$-representations constructed in [CEE].*

If we forget the A-action, and consider M only as a U^e-module, i.e. a U-bimodule, we can still define the operators Y_i, T_j. In fact, we have the following result which follows from the proof of Theorem 3.3.1

Corollary 3.3.5. *Let M be a U^e-module. The operators Y_i, T_j define an action of the affine Hecke algebra on W, whose quasi-classical limit is the Arakawa-Suzuki construction from [AS].*

3.4 The Construction

This entire section is devoted to a constructive proof of Theorem 3.3.1. We will build the representation of B_n^{Ell} by constructing first the braid group representation, then the action of the algebra $\mathbb{C}[Y]$, then the algebra $\mathbb{C}[X]$, and finally checking the commutation relations between them.

Fix U, a ribbon Hopf algebra with ribbon element ν, and let A and D_U be as in Definitions 3.2.16 and 3.2.24. Let M be a D_U-module, let V be a U-module, and consider the vector space

$$W = (V^{\otimes n} \otimes M)^{inv}.$$

This means that we take invariants with respect to the usual action on the $V^{\otimes n}$ factor, and with respect to the adjoint action on the M factor. It is important to note here that the adjoint action is only defined for elements of U', the locally finite part of U. So we will only be able to apply ad-invariance for such elements.

We index the tensor factors in W from right to left, starting with M at index 0. We will use the symbol π_V when we want to explicitly emphasize the action on V.

The action of the braid group is given by $T_i = \sigma_{i+1,i}$ acting on the $V^{\otimes n}$ factor. We define the invertible element $Y_1 = \sigma_{M,V} \circ \sigma_{V,M} = (\pi_V \otimes \partial_\lhd)(R_{01} R_{10})$ acting on the right two factors. We define operators Y_i, for $i = 1, \ldots, n-1$ by $Y_{i+1} = T_i Y_i T_i$. It follows from the QYBE that the Y_i's commute pairwise. Recall that we denote by $\tilde{Y}$ the product $\tilde{Y} = \prod_i Y_i$.

Proposition 3.4.1.

$$\tilde{Y} = (\pi_V \otimes \cdots \otimes \pi_V \otimes \partial_\lhd)\Big(\Delta^{(n)}(\nu^{-1})\underbrace{(\nu \otimes \cdots \otimes \nu)}_{n+1}\Big).$$

Proof. In any ribbon Hopf algebra, the ribbon element ν satisfies $\Delta(\nu^{-1}) = R_{21}R_{12}(\nu^{-1}\otimes \nu^{-1})$. We compute the iterated coproduct using the left expansion $\Delta^{(n)} = (\Delta \otimes id^{\otimes n-1}) \circ (\Delta \otimes id^{\otimes n-2}) \circ \cdots \circ \Delta$:

$$\Delta^{(n)}(\nu^{-1}) = (R_{n+1,\ldots,1}R_{1,\ldots,n+1})(R_{n,\ldots,1}R_{1,\ldots,n}) \cdots (R_{21}R_{12})(\nu^{-1} \otimes \cdots \otimes \nu^{-1}),$$

where we define notation $R_{k,\ldots,1} = R_{k,k-1}R_{k,k-2}\cdots R_{k1}$, and $R_{1,\ldots k} = R_{1k}\cdots R_{k-2,k}R_{k-1,k}$. Evaluating in W (and remembering our right to left indexing convention), we see the claim. $\qquad\square$

Recall that since V is an A-comodule, we have the operators $L \in A \otimes End_{\mathbb{C}}(V)$
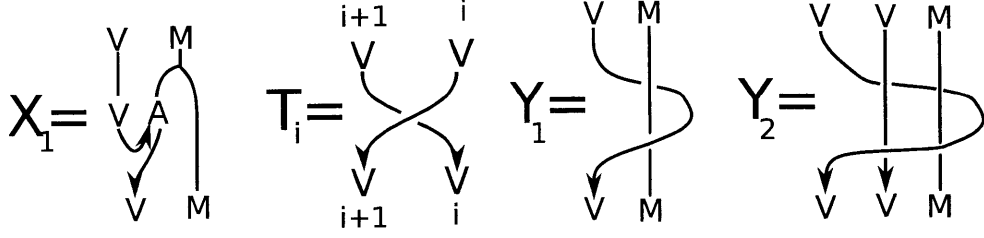
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Figure 3-5: The operators X_1, T_i, Y_1, Y_2.

(see Prop 3.2.19). We can define $X_1 = L_{01}$. Since $\Delta_0(L_{01}) = L_{02}L_{12}$, we have that $S_0(L_{01})$ gives an inverse X_1^{-1} (Here S_0 means that we apply the antipode in the A component). We set $X_{i+1} = T_i X_i T_i$, for $i = 1, \ldots, n-1$. The reflection equation implies that the X_i commute pairwise. The invariant description for X_1 is depicted in figure 3-5, and it is perhaps more enlightening. We first comultiply V as an A-module, then multiply the extra A-factor into M.

Lemma 3.4.2. *X_i, Y_i, T_i preserve the subspace of ad-invariants.*

Proof. Note that Y_i and T_i are endomorphisms of the U-module $V^{\otimes n} \otimes M$. Thus, if $v \in (V^{\otimes n} \otimes M)^{inv}$, we have that $xT_iv = T_ixv = \epsilon(x)T_iv$, and likewise for Y_i.

Only X_i requires some computation. Since $X_{i+1} = T_i X_i T_i$, we only need to check for X_1. Since D_U is a module-algebra under the adjoint action, it follows that for any $x \in U$, we have

$$
\begin{aligned}
(adx)X_1\big(\sum_i v_n^i \otimes \cdots \otimes v_1^i \otimes m^i\big) &= (adx)\mu_{21,0}\big(\sum_i v_n^i \otimes \cdots \otimes e_k^i \otimes e_k^{i*} \otimes v_1^i \otimes m^i\big) \\
&= \mu_{21,0}(adx)\big(\sum_i v_n^i \otimes \cdots \otimes e_k^i \otimes e_k^{i*} \otimes v_1^i \otimes m^i\big) \\
&= \epsilon(x)\mu_{21,0}\big(\sum_i v_n^i \otimes \cdots \otimes e_k^i \otimes e_k^{i*} \otimes v_1^i \otimes m^i\big) \\
&= \epsilon(x)X_1\big(\sum_i v_n^i \otimes \cdots \otimes v_1^i \otimes m^i\big)
\end{aligned}
$$

We used that U acts trivially on $e_k \otimes e_k^*$, as it is just the image of $1 \in \mathbb{C}$ under the co-evaluation, and that the multiplication in D_U is covariant for the adjoint action of U. $\square$

Proposition 3.4.3. $X_1 Y_2 = Y_2 X_1 T_1^2$.

Proof. We use remark 3.2.25 to express both sides as a braid diagram in figure 3-6, at which point the equality is simply an identity of tangles. □
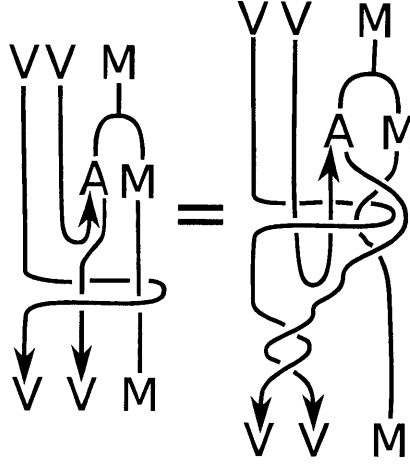

Figure 3-6: Proof of Proposition 3.4.3

This relation appears in a different form in [VV], where it is computed using the Fourier transform isomorphism: $\mathcal{F} : A \to U'$. Recall that U' is the sub-algebra of U which is locally finite under the adjoint representation. In our case, $\mathcal{F}$ applied to the zeroeth factor maps the action of X_1 onto that of Y_1, and thus lines up with the Fourier transform of the torus. Expressing X_1 in terms of the co-product of A allows us to give a diagrammatical proof instead.

Proposition 3.4.4. $\tilde{Y}X_j = X_j\tilde{Y}$

Proof. Actually, we'll show that $\tilde{Y}$ acts by $(\nu \otimes \ldots \otimes \nu)|_{V^{\otimes n}}$, so that it clearly commutes with X_j. We have by corollary 3.4.1 that

$$
\begin{aligned}
\tilde{Y}(\sum_i v_n^i \otimes \cdots \otimes v_1^i \otimes m^i) &= \Delta^{(n)}(\nu^{-1})(\sum_i \nu v_n^i \otimes \cdots \otimes \nu v_1^i \otimes \partial_\lhd(\nu)m^i) \\
&= \sum_i \nu v_n^i \otimes \cdots \otimes \nu v_1^i \otimes \partial_\lhd(\nu)\partial_\rhd(S^{-1}(\nu^{-1}))m^i, \text{ by ad invariance,} \\
&= \sum_i \nu v_n^i \otimes \cdots \otimes \nu v_1^i \otimes m^i,
\end{aligned}
$$

by remark 3.2.31, and the fact that $S^{-1}(\nu) = \nu$. Notice that this is the only place in the argument where we use ad-invariance. Because ν is central, it is in U'. Also it

is easy to see that $\Delta^{(k)}(U') \subset U^{\otimes n} \otimes U'$. Together, these justify shifting the ribbon element to the zeroeth component. $\square$

We have confirmed the necessary relations on the operators X_i, Y_i, T_j, which concludes the proof of Theorem 3.3.1.

3.5 A relation for $\tilde{X}$ and $\tilde{Y}$

As an immediate corollary of the proof of Proposition 3.4.4, we have the following

Corollary 3.5.1. *Suppose that V is irreducible, and that $\nu|_V = c_V$ id. Then $\tilde{Y} = c_V^n \operatorname{id}_W$.*

It turns out that $\tilde{X} := \prod_i X_i$ acts by the same scalar. We prove this in two steps. Recall our standing assumption that U has enough finite dimensional representations, in the sense of Definition 3.2.14.

Proposition 3.5.2. *$\tilde{X} = (\tilde{T} \otimes 1 \otimes 1) \circ \mu_{A,M} \circ \Delta_{V^{\otimes n}}$, where $\tilde{T} : V^{\otimes n} \to V^{\otimes n}$ is the double braiding, $\tilde{T} = \prod_{i=1}^{n-1} \prod_{j \leq i} T_j$, depicted in figure 3-7.*
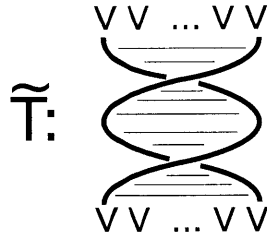

Figure 3-7: $\tilde{T}$ braids all n copies of V twice around one another.

Proof. This is a direct computation, of which the reflection equation (3.2.19) is the case $n = 2$. $\square$

Proposition 3.5.3. *Suppose that V is irreducible. Then $\tilde{X} = c_V^n$, where c_V is the same as from Proposition 3.4.4.*

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Proof. We first consider the case $n = 1$. Since we consider U-invariants, this is the same as A-coinvariants, and so we have that

$$
\begin{aligned}
\Delta_V &= \sigma_{M,A}^{-1} \circ (\mathrm{id} \otimes \mathrm{id} \otimes S^{-1}) \circ \Delta_A^{ad}, \text{ and thus,} \\
\tilde{X} &= \mu_A \circ \sigma_{M,A}^{-1} \circ (\mathrm{id} \otimes \mathrm{id} \otimes S^{-1}) \circ \Delta_A^{ad},
\end{aligned}
\tag{3.2}
$$

where Δ_A^{ad} is the co-adjoint action of A on itself. In figure 3-8, we compute that equation 3.2 is equal $\mathrm{ad}\,(\nu)$ acting on M. For concreteness, we work in $M = A$, the basic representation of D_U, which we may do because this representation is faithful, by Theorem 3.2.23, and our assumption that U has enough finite dimensional representations.

By invariance, $\mathrm{ad}\,(\nu)$ acting on M is the same as $S(\nu) = \nu$ acting on V, as desired. When $n \geq 1$, we apply this proof to $V^{\otimes n}$ instead, to conclude that

$$
\begin{aligned}
\tilde{X} &= (\tilde{T} \otimes 1 \otimes 1) \circ \mu_{A,M} \circ \Delta_{V^{\otimes n}} \\
&= \tilde{T} \circ \Delta^{(n-1)}(\nu) \otimes 1 \otimes 1, \text{ by the } n = 1 \text{ case above.} \\
&= (\nu \otimes \cdots \otimes \nu \otimes 1 \otimes 1)|_{(V^{\otimes n} \otimes A)^{inv}} \\
&= c_V^n \, \mathrm{id}
\end{aligned}
$$

$\square$

Corollary 3.5.4. *Let $B_n^{ell,0}$ be the quotient group of B_n^{ell} by the relations $\tilde{X} = \tilde{Y} = 1$, and $\mathcal{HH}_n^0(q,t)$ be the quotient of $\mathcal{HH}_n(q,t)$ by the same relations. Setting $X_i' = X_i c_V^{-1}$, $Y_i' = Y_i c_V^{-1}$, we obtain an action of the group $B_n^{ell,0}$ on W, which descends to a representation of $\mathcal{HH}_n^0(q,t)$ when V is Hecke.*

3.6 DAHA case and its degeneration

In this section, we consider in more detail the case $U = U_t(sl_N)$. We recall the degeneration process from the DAHA to trigonometric Cherednik algebra, and we
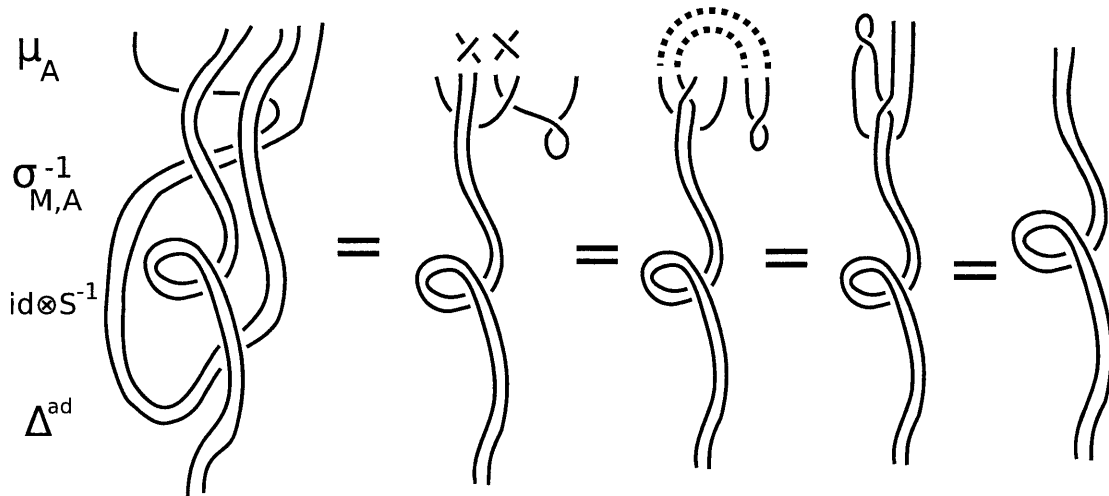
Figure 3-8: Computation of $\tilde{X}$. The second equality applies Proposition 3.2.12 to the braiding, as depicted above the diagram. The third equality applies Proposition 3.2.12 to the braiding composed with the coevaluation, as indicated by the dotted lines. The first and fourth equalities are already clear at the level of braiding diagrams.

show that our construction degenerates to the construction of [CEE] (with trivial modifications) in the quasi-classical limit.

3.6.1 A representation of $\mathcal{HH}_n(q,t)$

Proposition 3.6.1. *Let V be the defining representation (of highest weight $(1,0,\ldots,0)$) for $U_t(sl_N)$, and let M be a D_U-module. Then the operators X_i, Y_i, and T_i of the preceding section define a representation of $\mathcal{HH}_n(q,t)$ with parameters $q, t = q^{nk}, k = N/n$ on the space $W = (V^{\otimes n} \otimes M)^{inv}$, of invariants with respect to the adjoint action.*

Proof. It is well known that the operators $\sigma_{V,V}$ satisfy the Hecke relation

$$(\sigma_{VV} - t^{1-1/N})(\sigma_{VV} + t^{-1-1/N}),$$

which is the only additional relation on $\mathcal{HH}_n(q,t)$ when $t = q^{nk}, k = N/n$. $\qquad\square$

3.6.2 The trigonometric construction

In [CEE], the authors constructed a representation of the trigonometric Cherednik algebra $\mathcal{HH}_n^{deg}(k)$ on the space $(V^{\otimes n} \otimes M)^{inv}$ of invariants with respect to the adjoint

action, where V is the vector representation, and M is a D-module on G. The operators s_{ij} were defined by the usual symmetric group action on $V^{\otimes n}$, while the X_i, y_j were defined by:

$$
\begin{aligned}
X_i &= \sum_{r,s} (E^r_s \otimes A^s_r)_{i,0} \\
y_j &= k\Big(\sum_p (b_p \otimes L_{b_p})_{j,0} + \sum_{i<j} s_{ij}\Big),
\end{aligned}
$$

where A^s_r denotes multiplication by the function A^s_r in the coordinate algebra, $\{b_p\}$ are a orthonormal basis with respect to the trace form on sl_N, and L_{b_p} denotes the action by left translation along the vector field given by b_p.

3.6.3 Trigonometric degeneration of the DAHA

We recall the process of degeneration from the DAHA to the trigonometric Cherednik algebra. Let $\hbar$ denote a formal variable, let $k \in \mathbb{C}$, and let $q, t \in \mathbb{C}[[\hbar]]$ denote the power series

$$
q = e^{\hbar}, t = e^{nk\hbar}. \tag{3.3}
$$

Definition 3.6.2. $\widetilde{\mathcal{H}}_n$ is the $\mathbb{C}$-algebra freely generated by $X_i^{\pm 1}, y_i, s_j$, for $i = 1, \ldots, n$, and $j = 1, \ldots, n - 1$. Let $Y_i, T_j \in \widetilde{\mathcal{H}}_n[[\hbar]]$ denote the power series

$$
Y_i = e^{\hbar y_i}, \quad T_j = q^{-1} s_j e^{\hbar k s_j}.
$$

Let $\tilde{I}$ denote the closed ideal in $\widetilde{\mathcal{H}}_n[[\hbar]]$ generated by the DAHA relations from Definitions 3.2.1 and 3.2.2, and let I denote the saturation of $\tilde{I}$ with respect to $\hbar$,

$$
I = \{ x \in \widetilde{\mathcal{H}}_n[[\hbar]] | \hbar^m x \in \tilde{I} \text{ for some } m \geq 0 \}.
$$

Definition 3.6.3. $\widehat{\mathcal{HH}}_n(q,t)$ is the quotient:

$$\widehat{\mathcal{HH}}_n(q,t) = \widetilde{\mathcal{HH}}_n[[\hbar]]/I.$$

Definition 3.6.4. $\mathcal{HH}_n^f(q,t)$ is the complete $\mathbb{C}[[\hbar]]$-subalgebra of $\widehat{\mathcal{HH}}_n(q,t)$ generated by X_i, Y_i, $i = 1, \ldots, n$, and $T_j, j = 1 \ldots, n-1$. It is a formal version of $\mathcal{HH}_n(q,t)$ from Definition 3.2.2.

Proposition 3.6.5. *Let V be a representation of $\mathcal{HH}_n^f(q,t)$ which is flat as a $\mathbb{C}[[\hbar]]$-module, and $\rho : \mathcal{HH}_n^f(q,t) \to End(V)$ the corresponding map. Then ρ extends to a representation of $\widehat{\mathcal{HH}}_n(q,t)$ if and only if $\rho(Y_i) = 1 \mod \hbar, \forall i$. In this case the extension is unique.*

Proof. ($\Rightarrow$) is obvious. To show ($\Leftarrow$), we need to define the action of y_i and s_i on V. First we observe that since $s_i^2 = 1$, the identity $T_i = q^{-1} s_i e^{\hbar k s_i}$ can simply be solved for s_i:

$$s_i = \frac{qT_i - \sinh(\hbar k)}{\cosh(\hbar k)}.$$

We can also solve the identity $Y_i = e^{\hbar y_i}$ for y_i:

$$y_i = \frac{1}{\hbar} \log(1 - (1 - Y_i)),$$

where the RHS is a well-defined power series in $\hbar$ because we assumed $1 - Y_i$ was divisible by $\hbar$. Uniqueness follows because we have explicitly solved for the y_i's and s_i's. $\qquad\square$

3.6.4 Trigonometric degeneration of the construction of section 3.4

In this section, we again consider t, q as elements of $\mathbb{C}[[\hbar]]$, as in equation (3.3). We have the following well-known proposition

Proposition 3.6.6. *Let $U_1 = U_t/\hbar U_t$. Then $U_1 \cong U(sl_N)$, the classical enveloping algebra. Furthermore, U_t is a flat deformation of U_1.*

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Corollary 3.6.7. *Let $D_t = D_{U_t}$ and $D_1 = D_t/\hbar D_t$. Then $D_1 \cong D_{U(sl_N)}$, and D_t is a flat (in fact, trivial) deformation of D_1.*

Proof. Indeed, we have that the Hochschild cohomology of D_G is determined by the singular cohomology of G: $HH^i(D_G, D_G) = H^i(G, \mathbb{C})$. Thus, when G is semi-simple, we have in particular that $HH^2(D_G, D_G) = H^2(G, \mathbb{C}) = 0$, so that D_G admits no non-trivial formal deformations (see e.g, [E2]). $\square$

Definition 3.6.8. Let C denote the Casimir element $C = \sum_p b_p b_p \in U(sl_N)$, where $\{b_p\}$ form an orthonormal basis with respect to the trace form. Let Ω denote the canonical 2-tensor in $U(sl_N) \otimes U(sl_N)$,

$$\Omega = \frac{\Delta(C) - C \otimes 1 - 1 \otimes C}{2} = \sum_p b_p \otimes b_p.$$

Definition 3.6.9. Let r denote the classical r-matrix for $U(sl_N)$, so $\Omega = r_{10} + r_{01}$

Proposition 3.6.10. *We have the following relations between r,R,$R_{10}R_{01}$, and Ω.*

$$R = t^r = 1 + k\hbar r \mod \hbar^2,$$
$$R_{10}R_{01} = t^{r_{10}}t^{r_{01}} = 1 \otimes 1 + k\hbar\Omega \mod \hbar^2.$$

Proposition 3.6.11. *Ω acts as $\sigma_{VV} - 1/N$ on $V \otimes V$, the tensor square of the defining representation for $U(sl_N)$.*

Proof. First we compute the canonical 2-tensor $\widetilde{\Omega}$ for $U(gl_N)$. Instead of an orthonormal basis $\{b_p\}$, we can choose the basis E^i_j and dual basis E^j_i, and so

$$\widetilde{\Omega} = \sum_{i,j} E^i_j \otimes E^j_i,$$

which is exactly the flip. Since $\Omega = \widetilde{\Omega} - 1/N(id_V \otimes id_V)$, the claim follows. $\square$

Proposition 3.6.12. *The $\mathcal{H}_n(q,t)$-representation W of Proposition 3.6.1 extends to a representation of $\widehat{\mathcal{H}}_n(q,t)$.*

Proof. Y_i is expressed as a product of R-matrices, each of which is congruent to 1 mod $\hbar$. Thus the condition of Proposition 3.6.5 is satisfied. $\square$

Proposition 3.6.13. *Let $W_1 = W/\hbar W = (V_1^{\otimes n} \otimes M_1)^{inv}$. The operators s_i act on W_1 as the flip $s_{i,i+1}$ of tensor factors. The operators X_i act as*

$$X_i = \sum_{k,l} (E_l^k)_i \otimes A_k^l.$$

Proof. Straightforward computation. $\square$

Proposition 3.6.14. *The operators y_i act as*

$$y_i = k(\Omega_{i,0} + \sum_{j<i} s_{ij} - \frac{i-1}{N})$$

Proof. To see the claim for y_1, we note that

$$Y_1 = (R_{10}R_{01})_{1,0} = 1 \otimes 1 + \hbar\Omega_{1,0} \quad \mathrm{mod}\ \hbar^2.$$

We proceed by induction:

$$
\begin{aligned}
Y_i &= T_i Y_{i-1} T_i \\
&= s_i(1 + k\hbar r_{i,i-1})(1 + k\hbar(\Omega_{i-1,0} + \sum_{j<i-1} s_{i-1,j} - \frac{i-2}{N}))s_i(1 + k\hbar r_{i,i-1}) \quad \mathrm{mod}\ \hbar^2 \\
&= 1 + k\hbar\left(\sum_{j<i-1} s_{i,j} + r_{i,i-1} + r_{i-1,i} + \Omega_{i,0} - (i-2)/N \right) \quad \mathrm{mod}\ \hbar^2 \\
&= 1 + k\hbar\left(\sum_{j<i-1} s_{i,j} + s_{i,i-1} - \frac{1}{N} + \Omega_{i,0} - (i-2)/N \right) \quad \mathrm{mod}\ \hbar^2 \\
&= 1 + k\hbar\left(\sum_{j<i} s_{i,j} + \Omega_{i,0} - \frac{i-1}{N} \right) \quad \mathrm{mod}\ \hbar^2
\end{aligned}
$$

$\square$

Comparing these with the operators of [CEE], we see that the quasi-classical limit

of the present construction recovers the construction there, up to adding constants.[4]

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[4]Also, our operators L_b act by right instead of left translations, though this is just convention.

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