Fourier Transforms of Nilpotent Orbits, Limit Formulas for Reductive Lie Groups, and Wave Front Cycles of Tempered Representations

by

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Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2011

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Abstract

In this thesis, the author gives an explicit formula for the Fourier transform of the canonical measure on a nilpotent coadjoint orbit for $\mathrm{GL}(n,\mathbb{R})$. If G is a real, reductive algebraic group, and $\mathcal{O} \subset \mathfrak{g}^* = \mathrm{Lie}(G)^*$ is a nilpotent coadjoint orbit, a necessary condition is given for $\mathcal{O}$ to appear in the wave front cycle of a tempered representation. In addition, the coefficients of the wave front cycle of a tempered representation of G are expressed in terms of volumes of precompact submanifolds of certain affine spaces. In the process of proving these results, we obtain several limit formulas for reductive Lie groups.

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Acknowledgments

The author wishes to thank his advisor, David Vogan, for many conversations and corrections. The author would also like to thank Wulf Rossmann and Dan Barbasch for a few helpful tips and clarifications.

Contents

Chapter 1

Introduction

Let G be a reductive Lie group, and let τ be an irreducible, admissible representation of G with character Θ_τ. The wave front cycle of τ, denoted $\mathrm{WF}(\tau)$, is an integral linear combination of nilpotent coadjoint orbits defined in [2], [25]. Roughly speaking, the Fourier transform of the wave front cycle is a first order approximation to the character Θ_τ at the identity. Further, every orbit which occurs in the wave front cycle of τ has the same complexification [2]. In particular, if $G = \mathrm{GL}(n, \mathbb{R})$, then the leading term of an irreducible character of G is a positive integer times the Fourier transform of the canonical measure on a nilpotent coadjoint orbit.

The first result of this thesis is an explicit formula for the Fourier transform of the canonical measure on a nilpotent coadjoint orbit for $\mathrm{GL}(n, \mathbb{R})$. Given a conjugacy class of Levi subgroups, $\mathcal{L}$, for $\mathrm{GL}(n, \mathbb{R})$, fix a conjugacy class of parabolics $\mathcal{P}$ with Levi factor $\mathcal{L}$. Then we define

$$\mathcal{O}_{\mathcal{L}} \subset \mu(T^*\mathcal{P})$$

to be the unique open orbit. Here μ is the moment map (defined in section 3.1). In the next statement, we also denote the canonical measure on this orbit (defined in section 2.1) by $\mathcal{O}_{\mathcal{L}}$. Moreover, whenever G is a reductive Lie group and H is a Cartan subgroup, $W(G, H) = N_G(H)/H$ denotes the real Weyl group of G with respect to H. Here is our first result.

Theorem 1.0.1. *Fix a Cartan subgroup $H \subset G = GL(n, \mathbb{R})$, let $\mathfrak{h} = Lie(H)$, and let*

C be a connected component of the regular set $\mathfrak{h}'$ of $\mathfrak{h}$. Choose positive roots Δ^+ of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$ satisfying:

(i) If α is a positive real root and $X \in C$, then $\alpha(X) > 0$.

(ii) If α is a complex root, then α is positive iff $\overline{\alpha}$ is positive.

Then

$$\widehat{\mathcal{O}_{\mathcal{L}}}|_C = \sum_{L \in \mathcal{L},\ L \supset H} \left| \frac{W(G,H)_L}{W(L,H)} \right| \frac{\pi_L}{\pi}.$$

Here $\pi = \prod_{\alpha \in \Delta^+} \alpha$, Δ_L^+ denotes the roots of L that lie in Δ^+, $\pi_L = \prod_{\alpha \in \Delta_L^+} \alpha$, and $W(G,H)_L = \{w \in W(G,H) |\ wL = L\}$.

The above result is proved in sections 5.1 and 5.2.

Now, let G be an arbitrary real, reductive algebraic group with Lie algebra $\mathfrak{g}$. We have a couple of results about the wave front cycles of tempered representations.

Theorem 1.0.2. *Suppose $\mathcal{O}$ is an orbit contained in WF(τ) for a tempered representation τ, let $\nu \in \mathcal{O}$, and let L be a Levi factor of $Z_G(\nu)$. Then $L/Z(G)$ is compact.*

This theorem was conjectured by David Vogan. A p-adic analogue was proved by Moeglin and Waldspurger for cuspidal representations [16] and by Moeglin for tempered representations of classical p-adic groups [15].

In [21], Rossmann associated to each irreducible, tempered representation τ a finite union of regular coadjoint orbits we call $\mathcal{O}_\tau$. If τ has regular infinitesimal character, then $\mathcal{O}_\tau$ is a single coadjoint orbit. For each nilpotent coadjoint orbit, $\mathcal{O} \subset \mathfrak{g}^*$, fix $X \in \mathcal{O}$. Identify $\mathfrak{g} \cong \mathfrak{g}^*$ via a G-equivariant isomorphism, and let $\{X, H, Y\}$ be an $\mathfrak{sl}_2$-triple containing X. Put $S_X = X + Z_{\mathfrak{g}}(Y)$.

Theorem 1.0.3. *There exists a canonical measure on $\mathcal{O}_\tau \cap S_X$ such that*

$$\mathit{WF}(\tau) = \sum_{\mathcal{O}_\tau \cap S_X \ \textit{precompact}} \mathrm{vol}(\mathcal{O}_\tau \cap S_X)\mathcal{O}_X.$$

The sum is over nilpotent coadjoint orbits $\mathcal{O}_X$ such that $\mathcal{O}_\tau \cap S_X$ is precompact.

If τ has regular infinitesimal character, then 'precompact' may be replaced by 'compact' in the above theorem. In the case where G is compact, the wave front cycle of

τ is $\dim(\tau) \cdot 0$ where 0 denotes the zero orbit. In this case, our formula reduces to the well-known observation of Kirillov that the symplectic volume of the coadjoint orbit associated to τ is the dimension of τ.

These results are proven in sections 4.1 and 4.2.

In the process of proving these results, we will write down a number of limit formulas for reductive Lie groups. First, we have a limit formula for semisimple orbits.

If G is a reductive Lie group, we will write $r(G)$ (or simply r) for one half the number of roots of G with respect to any Cartan H. If we fix a Cartan H, then $q(G, H)$ will denote one half the number of non-compact, imaginary roots of G with respect to H. If H is a fundamental Cartan, then we will write $q(G)$ (or simply q) instead of $q(G, H)$.

Theorem 1.0.4 (Harish-Chandra). *Let G be a reductive Lie group, let $\xi \in \mathfrak{g}^* = Lie(G)^*$ be a semisimple element, let $L = Z_G(\xi) \subset G$ be the corresponding reductive subgroup, and let $H \subset L$ be a fundamental Cartan subgroup. Choose positive roots $\Delta_L^+ \subset \Delta_L = \Delta(\mathfrak{l}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ such that a complex root α is positive iff $\overline{\alpha}$ is positive, and put*

$$C = \{\lambda \in (\mathfrak{h}^*)' |\ \langle i\lambda, \alpha^\vee \rangle > 0 \text{ for all imaginary roots } \alpha \in \Delta_L^+\}.$$

Then

$$\lim_{\lambda \to \xi,\ \lambda \in C} \partial(\pi_L)|_\lambda \mathcal{O}_\lambda = i^{r(L)} (-1)^{q(L)} |W(L, H)| \mathcal{O}_\xi.$$

Here π_L is the product of the positive roots of L, and $\mathfrak{h}^* = Lie(H)^* \subset \mathfrak{g}^* = Lie(G)^*$ is embedded in the usual way.

We give Harish-Chandra credit for this result because he proved a group analogue with a different normalization of Haar measures on pages 33-34 of [12]. The special case where $\xi = 0$ was proved by Harish-Chandra even earlier, and the main result of [4] is a determination of the constant in the case $\xi = 0$ using modern methods. Since a proof of the above result doesn't appear in the literature, we write down a proof in sections 2.1 and 2.2, for arbitrary semisimple ξ, using well-known, classical methods.

Next, we have limit formulas for nilpotent orbits. Let $\nu \in \mathfrak{g}^*$, and let $\mathcal{O}_\nu$ denote the canonical measure on the coadjoint orbit $G \cdot \nu$. Then the limit of distributions

$$\lim_{t \to 0^+} \mathcal{O}_{t\nu} = \sum n_G(\mathcal{O}, \nu)\mathcal{O}$$

is a sum of canonical measures on nilpotent coadjoint orbits. Let $\mathcal{O}_{\mathbb{C}}$ denote the $\mathrm{Int}\mathfrak{g}_{\mathbb{C}}$-orbit $\mathrm{Int}\mathfrak{g}_{\mathbb{C}} \cdot \mathcal{O} \subset \mathfrak{g}_{\mathbb{C}}^*$.

Proposition 1.0.5. *The coefficients $n_G(\mathcal{O}, \nu)$ are positive integers. Moreover, we have the inequality $n_G(\mathcal{O}, \nu) \leq n_{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}}(\mathcal{O}_{\mathbb{C}}, \nu)$.*

In [1], Dan Barbasch gives formulas for $n_{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}}(\mathcal{O}_{\mathbb{C}}, \nu)$. Hence, the above theorem gives an upper bound for $n_G(\mathcal{O}, \nu)$. In section 3.1, we give a lower bound for $n_G(\mathcal{O}, \nu)$ in the case where $n_G(\mathcal{O}, \nu) \neq 0$. When the lower and upper bounds coincide, we get a formula for $n_G(\mathcal{O}, \nu)$. We will show that this happens for certain $\mathrm{GL}(n, \mathbb{R})$ limit formulas, and we use these formulas together with Theorem 1.0.4 to explicitly compute the formulas in Theorem 1.0.1. These bounds also coincide when $\mathcal{O}_{\mathbb{C}}$ is an even orbit. Rao proved but never published a limit formula for even nilpotent orbits. Recently in [5], Bozicevic gave a deep, modern proof of Rao's result. In section 3.2, we use the above results to give an elementary, classical proof of Rao's limit formula.

Chapter 2

Limit Formulas for Semisimple Orbits

2.1 Harish-Chandra's Limit Formula for the Zero Orbit

In this section, we prove Harish-Chandra's limit formula for the zero orbit. It was proven in [7], [8], [9], and [13], but with a different normalization of the measures on coadjoint orbits than the one we use here. In [4], using modern methods, Bozicevic gives a proof of the formula, written in terms of canonical measures on orbits. In the first section of this piece, we show how to write down such a proof using only well-known, classical methods. This is not a waste of space because many of the fundamental results we recall in this section will be needed later in this thesis for other purposes.

First, we need a couple of definitions. A Lie group G is *reductive* if there exists a real, reductive algebraic group G_1 and a Lie group homomorphism $G \to G_1$ with open image and finite kernel. Let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit for G. The Kostant-Kirillov symplectic form ω is defined on $\mathcal{O}$ by the formula

$$\omega_\lambda(\mathrm{ad}^*_X\lambda, \mathrm{ad}^*_Y\lambda) = \lambda([X, Y]).$$

The top dimensional form

$$\frac{\omega^m}{m!(2\pi)^m}$$

on $\mathcal{O}$ gives rise to the canonical measure on $\mathcal{O}$. Here $m = \frac{\dim \mathcal{O}}{2}$. We will often abuse notation and write $\mathcal{O}$ for the orbit as well as the canonical measure on the orbit. In what follows, we will denote the G-orbit through λ by $\mathcal{O}_\lambda^G$ (or sometimes just $\mathcal{O}_\lambda$).

Theorem 2.1.1 (Harish-Chandra). *Let G be a reductive Lie group, and let $H \subset G$ be a fundamental Cartan subgroup. Choose positive roots $\Delta^+ \subset \Delta = \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ such that a complex root α is positive iff $\overline{\alpha}$ is positive, and put*

$$C = \{\lambda \in (\mathfrak{h}^*)' |\ \langle i\lambda, \alpha^\vee \rangle > 0 \textit{ for all imaginary roots } \alpha \in \Delta^+\}.$$

Then

$$\lim_{\lambda \to 0,\ \lambda \in C} \partial(\pi)|_\lambda \mathcal{O}_\lambda = i^r (-1)^q |W(G,H)| \delta_0.$$

Here π is the product of the positive roots of G and $\mathfrak{h}^* = Lie(H)^* \subset \mathfrak{g}^* = Lie(G)^*$ is embedded in the usual way.

We will actually prove the Fourier transform of the above theorem. Recall the definition of the Fourier transform. Let V be a finite dimensional, real vector space, and let μ be a smooth, rapidly decreasing measure (that is, a Schwartz function multiplied by a Lebesgue measure) on V. Then the Fourier transform of μ is defined to be

$$\widehat{\mu}(l) = \int_V e^{i\langle l, X\rangle} d\mu(X).$$

Note $\widehat{\mu}$ is a Schwartz function on V^*. Given a tempered distribution D on V^*, its Fourier transform $\widehat{D}$ is a tempered, generalized function on V defined by

$$\langle \widehat{D}, \mu \rangle := \langle D, \widehat{\mu} \rangle.$$

Next, we recall Harish-Chandra's result on Fourier transforms of regular, semisimple orbits. If $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, define

$$\mathfrak{h}'' = \{X \in \mathfrak{h} |\ \alpha(X) \neq 0\ \forall \alpha \in \Delta_{\mathrm{real}}\}.$$

Here Δ_{real} denotes the real roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Lemma 2.1.2 (Harish-Chandra). *Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan, let $C \subset (\mathfrak{h}^*)' \subset \mathfrak{g}^*$ be a connected component of the set of regular elements in $\mathfrak{h}^*$, and let $\lambda \in C$ be regular, semisimple. Suppose $\mathfrak{h}_1 \subset \mathfrak{g}$ is a Cartan subalgebra, and $C_1 \subset \mathfrak{h}_1''$ is a connected component. Fix an element $g \in \operatorname{Int} \mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{h}_{\mathbb{C}} = \mathrm{Ad}_g(\mathfrak{h}_1)_{\mathbb{C}}$, and define $\lambda_1 \in (\mathfrak{h}_1)^*_{\mathbb{C}}$ by $\lambda_1 = \mathrm{Ad}_g^* \lambda$. Then*

$$\widehat{\mathcal{O}_\lambda}|_{C_1} = \frac{\sum_{w \in W_{\mathbb{C}}} a_w e^{iw\lambda_1}}{\pi}$$

for $a_w \in \mathbb{C}$ constants. Here $W_{\mathbb{C}}$ is the Weyl group of the roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $(\mathfrak{h}_1)_{\mathbb{C}}$, and π is a product of positive roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $(\mathfrak{h}_1)_{\mathbb{C}}$. Further, the constants a_w are independent of the choice of $\lambda \in C$.

Differentiating the above formula with respect to λ yields

$$\lim_{\lambda \in C,\ \lambda \to 0} \partial(\pi)|_\lambda \widehat{\mathcal{O}_\lambda}|_{C_1} = i^r \left(\sum \epsilon(w) a_w\right) \frac{\pi}{\pi}.$$

Observe that the coefficients a_w depend on a component $C \subset (\mathfrak{h}^*)'$ as well as a component $C_1 \subset (\mathfrak{h}_1)''$. For the remainder of this section, we fix $C \subset (\mathfrak{h}^*)'$ and we assume that $\mathfrak{h}$ is a fundamental Cartan subalgebra. To prove Theorem 2.1.1, we need only show the following lemma.

Lemma 2.1.3. *The following identity holds*

$$\sum \epsilon(w) a_w = (-1)^q |W(G, H)|.$$

Again, these coefficients a_w depend on a component $C_1 \subset (\mathfrak{h}_1)''$. In order to prove Lemma 2.1.3, we first prove it in the case where $\mathfrak{h}_1 = \mathfrak{h}$ using a result of Rossmann and Harish-Chandra descent. For our applications, it is important to give Berline-

Vergne's formulation [3] of Rossmann's result [18] (we also recommend the proof of Berline-Vergne).

Theorem 2.1.4 (Rossmann). *Let G be a reductive Lie group, and let H be a Cartan subgroup such that $H/Z(G)$ is compact. Let $\lambda \in C \subset (\mathfrak{h}^*)' = (Lie(H)^*)'$ be regular, semisimple, and choose positive roots $\Delta^+ \subset \Delta$ satisfying $\langle i\lambda, \alpha^\vee \rangle > 0$ for all $\alpha^\vee \in (\Delta^+)^\vee$. Then*

$$\widehat{\mathcal{O}_\lambda}|_{\mathfrak{h}'} = (-1)^q \frac{\sum_{w \in W(G,H)} \epsilon(w) e^{iw\lambda}}{\pi}$$

where π is the product of the positive roots.

This theorem implies Lemma 2.1.3 when $\mathfrak{h}_1 = \mathfrak{h}$ and G is of equal rank. Now, let G be an arbitrary reductive Lie group, and let $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}(G)$ be a fundamental Cartan subalgebra. Choose a Cartan involution θ such that $\mathfrak{h}$ is θ stable with decomposition $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. Then $M = Z_G(\mathfrak{a})$ is a reductive Lie group of equal rank. Harish-Chandra gave continuous maps [7]

$$\phi : \ \mathcal{S}(\mathfrak{g}^*) \to \mathcal{S}(\mathfrak{m}^*), \ \psi : \ \mathcal{SM}(\mathfrak{g}) \to \mathcal{SM}(\mathfrak{m})$$

well-defined up to a constant, where $\mathcal{S}(V)$ is the space of smooth, rapidly decreasing functions on a vector space V and $\mathcal{SM}(V)$ is the space of smooth, rapidly decreasing measures on a vector space V. Dualizing, we obtain maps

$$\phi^* : \ \mathrm{TD}(\mathfrak{m}^*) \to \mathrm{TD}(\mathfrak{g}^*), \ \psi^* = \mathrm{HC} : \ \mathrm{TGF}(\mathfrak{m}) \to \mathrm{TGF}(\mathfrak{g})$$

on tempered distributions and tempered generalized functions. We call the map on the right HC because it is Harish-Chandra's descent map. Thus far, these maps are only well-defined up to a constant; however, there is a nice way to normalize this constant. It follows from results of Rossmann [22] that one can fix the constant for ϕ^* so that $\phi^*(\mathcal{O}_\lambda^M) = \mathcal{O}_\lambda^G$ takes canonical measures on G-regular, semisimple coadjoint orbits to canonical measures on regular, semisimple coadjoint orbits. In [7], Harish-Chandra observes that ψ is (up to a constant) the Fourier transform of ϕ. Thus, we may require

$$\mathrm{HC}(\widehat{D}) = \widehat{\phi^*(D)}$$

for all $D \in \mathrm{TD}(\mathfrak{m}^*)$ and this precisely defines the map HC.

We fix this normalization. Arguments similar to the ones in [22] imply the following explicit formula for computing HC.

Lemma 2.1.5 (Harish-Chandra, Rossmann). *Let F be an M-invariant generalized function on $\mathfrak{m}$ that is given by integration against an analytic function on the set of regular, semisimple elements $\mathfrak{m}' \subset \mathfrak{m}$, which we also denote by F. Given $X \in \mathfrak{g}'$, let $\{Y_i\}_{i=1}^k$ be a set of representatives for the finite number of M-orbits in $\mathcal{O}_X^G \cap \mathfrak{m}$. Then $HC(F)$ is a G-invariant generalized function on $\mathfrak{g}$ that is given by integration against an analytic function on the set of regular, semisimple elements $\mathfrak{g}' \subset \mathfrak{g}$, which we also denote by $HC(F)$. Explicitly, we have*

$$HC(F)(X) = \sum_{i=1}^{k} F(Y_i) \left|\pi_{G/M}(Y_i)\right|^{-1}.$$

To define $|\pi_{G/M}(Y)|$, choose a Cartan $Y \in \mathfrak{h} \subset \mathfrak{m}$, let Δ_G (resp. Δ_M) be the roots of $\mathfrak{g}$ (resp. $\mathfrak{m}$) with respect to $\mathfrak{h}$, let Δ_G^+ be a choice of positive roots of Δ_G, and let $\Delta_M^+ = \Delta_G^+ \cap \Delta_M$. Then $|\pi_{G/M}(Y)| = |\prod_{\alpha \in \Delta_G^+ \setminus \Delta_M^+} \alpha(Y)|$. This definition is independent of the above choices.

Combining Theorem 2.1.4 and Lemma 2.1.5, we get the following corollary.

Corollary 2.1.6 (Rossmann). *Let G be a reductive Lie group with Cartan subgroup H, and let $q(G,H)$ be half the number of non-compact imaginary roots of G with respect to H. Let $\lambda \in \mathfrak{h}^* = Lie(H)^*$ be a regular element, and let $C_1 \subset \mathfrak{h}'$ be a connected component. Choose positive roots $\Delta^+ \subset \Delta$ satisfying*

(i) If $\alpha^\vee \in (\Delta^+)^\vee_{imag.}$, then $\langle i\lambda, \alpha^\vee \rangle > 0$.

(ii) If $\alpha \in \Delta^+_{real}$ and $X \in C_1$, then $\alpha(X) > 0$.

(iii) If α a complex root, then $\alpha \in \Delta^+$ iff $\overline{\alpha} \in \Delta^+$.

Then

$$\widehat{\mathcal{O}_\lambda^G}|_{C_1} = (-1)^{q(G,H)} \frac{\sum_{w \in W(G,H)} \epsilon_I(w) e^{iw\lambda}}{\pi}$$

where $W(G,H) = N_G(H)/H$, π *is the product of the positive roots, and* ϵ_I *is defined by*

$$w \cdot \pi_I = \epsilon_I(w)\pi_I, \ \ \pi_I = \prod_{\alpha \in \Delta^+_{imag}} \alpha.$$

Moreover, $\widehat{\mathcal{O}^G_\lambda}$ *is zero on Cartan subalgebras* $\mathfrak{h}$ *which are not conjugate to a Cartan subalgebra of* $Z_{\mathfrak{g}}(\lambda)$.

A version of this result containing a few typos can be found in [23]. Since $\epsilon_I = \epsilon$ on a fundamental Cartan $\mathfrak{h}$, this verifies Lemma 2.1.3 when $\mathfrak{h}_1 = \mathfrak{h}$. To finish the proof of Lemma 2.1.3, we use Harish-Chandra's matching conditions.

Theorem 2.1.7 (Harish-Chandra). *Let* $\mathfrak{h} \subset \mathfrak{g}$ *be a fundamental Cartan subalgebra, and let* $\mathfrak{h}_1$ *be another Cartan subalgebra. Let* $\lambda \in (\mathfrak{h}^*)'$, *choose* $g \in \operatorname{Int} \mathfrak{g}_{\mathbb{C}}$ *such that* $\operatorname{Ad}_g(\mathfrak{h}_1)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}}$, *and define* $\operatorname{Ad}^*_g \lambda = \lambda_1$. *Suppose* $\mathfrak{h}_2$ *is a third Cartan related to* $\mathfrak{h}_1$ *by a Cayley transform* c_α *via a noncompact, imaginary root* α *of* $\mathfrak{h}_1$, *and define* $\lambda_2 \in (\mathfrak{h}_2)^*_{\mathbb{C}}$ *such that* $c^*_\alpha \lambda_2 = \lambda_1$. *Let* $C_1 \subset \mathfrak{h}''_1$ *be a connected component containing an open subset of ker*(α), *and let* $C_2 \subset \mathfrak{h}''_2$ *be a connected component such that* $c_\alpha(C_1)$ *contains a wall of* $\overline{C_2}$. *Suppose*

$$\widehat{\mathcal{O}_\lambda}|_{C_1} = \frac{\sum a_w e^{iw\lambda_1}}{\pi_1}, \ \ \widehat{\mathcal{O}_\lambda}|_{C_2} = \frac{\sum b_w e^{iw\lambda_2}}{\pi_2}.$$

Then we have

$$\epsilon(w)a_w + \epsilon(s_\alpha w)a_{s_\alpha w} = \epsilon(w)b_w + \epsilon(s_\alpha w)b_{s_\alpha w}.$$

Here we identify the noncompact, imaginary root α *of* $\mathfrak{h}_1$ *with the corresponding real root of* $\mathfrak{h}_2$. *A product of positive roots of* $\mathfrak{g}$ *with respect to* $\mathfrak{h}_1$ *is denoted by* π_1, *and* π_2 *is the product of positive roots of* $\mathfrak{g}$ *with respect to* $\mathfrak{h}_2$ *satisfying* $c^*_\alpha \pi_2 = \pi_1$.

This theorem is Lemma 26 of [11] where Harish-Chandra remarks that it follows from Lemma 18 of [10]. Summing these relations over the entire Weyl group, we get

$$\sum \epsilon(w)a_w = \sum \epsilon(w)b_w.$$

Since any component of any Cartan can be related to a component of a fundamental Cartan via successive Cayley transforms, we deduce

$$\sum \epsilon(w) a_w = (-1)^q |W(G, H)|$$

whenever $\widehat{\mathcal{O}_\lambda}|_{C_1} = \frac{\sum a_w e^{iw\lambda_1}}{\pi}$ on any component $C_1 \subset \mathfrak{h}_1''$ for any Cartan $\mathfrak{h}_1$. This is the statement of Lemma 2.1.3. As we have already remarked, Theorem 2.1.1 follows.

2.2 Harish-Chandra's Limit Formula for Semisimple Orbits

In this section, we prove Harish-Chandra's limit formula for an arbitrary semisimple orbit. A group analogue of this result was proven with a different normalization of Haar measure on pages 33-34 of [12].

Theorem 2.2.1 (Harish-Chandra). *Let G be a reductive Lie group, let $\xi \in \mathfrak{g}^* = Lie(G)^*$ be a semisimple element, let $L = Z_G(\xi) \subset G$ be the corresponding reductive subgroup, and fix a fundamental Cartan subgroup $H \subset L$. Choose positive roots $\Delta_L^+ \subset \Delta_L = \Delta(\mathfrak{l}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ such that a complex root α is positive iff $\overline{\alpha}$ is positive, and put*

$$C = \{\lambda \in (\mathfrak{h}^*)' |\ \langle i\lambda, \alpha \rangle > 0 \text{ for all imaginary roots } \alpha \in \Delta^+\}.$$

Then

$$\lim_{\lambda \to \xi,\ \lambda \in C} \partial(\pi_L)|_\lambda \mathcal{O}_\lambda = i^{r(L)} (-1)^{q(L)} |W(L, H)| \mathcal{O}_\xi.$$

Here π_L is the product of the positive roots of L and $\mathfrak{h}^* = Lie(H)^* \subset \mathfrak{g}^* = Lie(G)^*$ is embedded in the usual way.

We prove the theorem by reducing to the case $\xi = 0$, which was proved in the last section. Let $d_{G/H}$ be a Haar measure on G/H, let $d_{G/L}$ be a Haar measure on G/L, and let $d_{L/H}$ be a Haar measure on L/H. Then it is a well-known fact (see for instance page 95 of [14]) that there exists a constant $c > 0$ such that

$$\int_{G/H} f(g \cdot H) d_{G/H} g = c \int_{G/L} \left(\int_{L/H} f(g \cdot lH) d_{L/H} l \right) d_{G/L} g$$

for $f \in C_c^\infty(G/H)$. Choosing a Haar measure on G/H (resp. G/L, L/H) is equivalent to choosing a top dimensional, alternating tensor $\eta_{G/H}$, well-defined up to sign, on $(\mathfrak{g}/\mathfrak{h})^*$ (resp. $\eta_{G/L}$, $\eta_{L/H}$ on $(\mathfrak{g}/\mathfrak{l})^*$, $(\mathfrak{l}/\mathfrak{h})^*$). The exact sequence

$$0 \to (\mathfrak{g}/\mathfrak{l})^* \to (\mathfrak{g}/\mathfrak{h})^* \to (\mathfrak{l}/\mathfrak{h})^* \to 0$$

gives rise to maps on alternating tensors. Abusing notation, we also write $\eta_{G/L}$ for the image of $\eta_{G/L}$ under the above map, and we also write $\eta_{L/H}$ for a preimage of $\eta_{L/H}$ under the above map. Then

$$\eta_{G/H} = \pm c(\eta_{G/L} \wedge \eta_{L/H}).$$

This can be proved by relating the multiplication gl on the group to addition on the Lie algebra and then applying Fubini's theorem.

To apply these remarks to our proof of the theorem, fix a Haar measure on the homogeneous space G/L by identifying $G/L \cong \mathcal{O}_\xi$ and using the canonical measure, fix a Haar measure on G/H by identifying $G/H \cong \mathcal{O}_\lambda$ for a fixed $\lambda \in C$, and fix a Haar measure on L/H by identifying $L/H \cong \mathcal{O}_\lambda^L$. Define $\eta_{G/H}$, $\eta_{G/L}$, and $\eta_{L/H}$ as above with respect to these measures. Then we get

$$\int_{G/H} f(g \cdot \lambda) dg = c_{\lambda,\xi} \int_{G/L} \left(\int_{L/H} f(gl \cdot \lambda) dl \right) dg.$$

Since ξ is fixed, writing c_λ instead of $c_{\lambda,\xi}$ from now on should not lead to any confusion.

Lemma 2.2.2. *Let Δ_G (resp. Δ_L) denote the roots of $\mathfrak{g}$ (resp. $\mathfrak{l}$) with respect to $\mathfrak{h}$. Let $\Delta_G^+ \subset \Delta_G$ be a choice of positive roots, and let $\Delta_L^+ = \Delta_G^+ \cap \Delta_L$. Then*

$$c_\lambda = \frac{\prod_{\alpha \in \Delta_G^+ \setminus \Delta_L^+} \langle \lambda, \alpha^\vee \rangle}{\prod_{\alpha \in \Delta_G^+ \setminus \Delta_L^+} \langle \xi, \alpha^\vee \rangle}$$

for $\lambda \in C$. *In particular,* $\lim_{\lambda \in C,\ \lambda \to \xi} c_\lambda = 1$.

Proof. Recall that $\eta_{G/H}$, $\eta_{G/L}$, and $\eta_{L/H}$ are top dimensional alternating tensors on $(\mathfrak{g}/\mathfrak{h})^*$, $(\mathfrak{g}/\mathfrak{l})^*$, and $(\mathfrak{l}/\mathfrak{h})^*$, well-defined up to a choice of sign. Extend these tensors complex linearly to $(\mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C})^*$, $(\mathfrak{g}_\mathbb{C}/\mathfrak{l}_\mathbb{C})^*$, $(\mathfrak{l}_\mathbb{C}/\mathfrak{h}_\mathbb{C})^*$ and denote them by η_λ^G, η_ξ^G, and η_λ^L. Note that we still have the identity $\eta_\lambda^G = \pm c_\lambda(\eta_\xi^G \wedge \eta_\lambda^L)$. Consider the root space decomposition

$$\mathfrak{g}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \bigoplus \left(\sum_{\alpha \in \Delta_G} (\mathfrak{g}_\mathbb{C})_\alpha \right).$$

For each $\alpha \in \Delta_G^+$, choose elements $X_\alpha \in (\mathfrak{g}_\mathbb{C})_\alpha$, $X_{-\alpha} \in (\mathfrak{g}_\mathbb{C})_{-\alpha}$, and $H_\alpha \in \mathfrak{h}_\mathbb{C}$ such that $\{X_\alpha, H_\alpha, X_{-\alpha}\}$ is an $\mathfrak{sl}_2$-triple. Then

$$\left[(2\pi)^m \eta_\lambda^G(\{X_\alpha\}_{\alpha \in \Delta_G})\right]^2 = \left(\frac{\omega_\lambda^m}{m!} (\{X_\alpha\}_{\alpha \in \Delta_G}) \right)^2 = \det\left(\omega_\lambda(X_\alpha, X_\beta)\right).$$

Here $m = \frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{h})$ and ω_λ is the Kostant-Kirillov symplectic form on $\mathcal{O}_\lambda^G$. Note that we need not order the tangent vectors $\{X_\alpha\}$ before applying the the square of the top dimensional alternating tensor η_λ^G to them since the value $\eta_\lambda^G(\{X_\alpha\}_{\alpha \in \Delta_G})^2$ is independent of this ordering. The second equality follows from explicitly expanding out $\omega_\lambda^m(\{X_\alpha\})$ into a sum with signs, squaring it, and identifying the result as the corresponding $2m$ by $2m$ determinant. Finally, we have

$$\det\left(\omega_\lambda(X_\alpha, X_\beta)\right) = \prod_{\alpha \in \Delta_G^+} \lambda(H_\alpha)^2.$$

This follows from the fact that $\omega_\lambda(X_\alpha, X_\beta) = \lambda[X_\alpha, X_\beta] \neq 0$ only if $\beta = -\alpha$ in which case we obtain $\lambda(H_\alpha)$. If $k = \frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{l})$ and $l = \frac{1}{2}(\dim \mathfrak{l} - \dim \mathfrak{h})$, then we similarly have

$$\left[(2\pi)^k \eta_\xi^G(\{X_\alpha\}_{\alpha \in \Delta_G \setminus \Delta_L})\right]^2 = \prod_{\alpha \in \Delta_G^+ \setminus \Delta_L^+} \xi(H_\alpha)^2$$

and

$$\left[(2\pi)^l \eta_\lambda^G(\{X_\alpha\}_{\alpha \in \Delta_L})\right]^2 = \prod_{\alpha \in \Delta_L^+} \lambda(H_\alpha)^2.$$

Combining the above formulas and the identity $\eta_\lambda^G = \pm c_\lambda(\eta_\xi^G \wedge \eta_\lambda^L)$ yields

$$c_\lambda^2 \prod_{\alpha\in\Delta_G^+\setminus\Delta_L^+} \xi(H_\alpha)^2 \prod_{\alpha\in\Delta_L^+} \lambda(H_\alpha)^2 = \prod_{\alpha\in\Delta_G^+} \lambda(H_\alpha)^2.$$

Solving for c_λ^2, taking the positive square root, and observing $H_\alpha = \alpha^\vee$ is the coroot, the lemma follows. $\square$

In the last lemma, we observed that c_λ is a constant multiple of the polynomial $\pi_{G/L}^\vee = \prod_{\alpha\in\Delta_G^+\setminus\Delta_L^+} \alpha^\vee$ in a neighborhood of ξ. Let $D(\mathfrak{h}^*)$ denote the Weyl algebra of polynomial coefficient differential operators on $\mathfrak{h}^*$. Given $D \in D(\mathfrak{h}^*)$, we may evaluate D at $\lambda \in \mathfrak{h}^*$ and get a distribution $D(\lambda)$ (ie. $D(\lambda)(f) := (Df)(\lambda)$). We write $D(\lambda) = 0$ if $D(\lambda)$ is the zero distribution.

Lemma 2.2.3. *The elements $\partial(\pi_L), \pi_{G/L}^\vee \in D(\mathfrak{h}^*)$ commute at the point ξ. More precisely,*

$$[\partial(\pi_L), \pi_{G/L}^\vee](\xi) = 0.$$

Proof. Suppose $S \subset \Delta_L^+$ is a subset, define $\pi_S = \prod_{\alpha\in S}\alpha$, and let $w \in \operatorname{Aut}\mathfrak{h}$ be a linear automorphism. Then for purely formal reasons,

$$\langle \partial(w\pi_S),\ w\pi_{G/L}^\vee\rangle(w\xi) = \langle \partial(\pi_S),\ \pi_{G/L}^\vee\rangle(\xi).$$

(If $D \in S(\mathfrak{h}^*)$ is a differential operator on $\mathfrak{h}^*$, $p \in S(\mathfrak{h})$ is a polynomial on $\mathfrak{h}^*$, and $\zeta \in \mathfrak{h}^*$ is a point, then $\langle D, p\rangle(\zeta) := (Dp)(\zeta)$ denotes differentiating the polynomials p by D and evaluating at ζ). Now suppose $w \in W_L$ where W_L is the Weyl group of root system Δ_L. Then $w\xi = \xi$ and $w\pi_{G/L}^\vee = \pi_{G/L}^\vee$. Hence,

$$\langle \partial(w\pi_S),\ \pi_{G/L}^\vee\rangle(\xi) = \langle \partial(\pi_S),\ \pi_{G/L}^\vee\rangle(\xi) \qquad (*)$$

for all $S \subset \Delta_L^+$ and all $w \in W_L$. Now define

$$wS := \{\alpha \in \Delta_L^+ \mid \alpha = \pm w\beta \text{ and } \beta \in S\}.$$

This defines an action of W_L on the set of subsets of Δ_L^+. Let W_S be the stabilizer of S in W_L. Then

$$\partial(\pi_L)\pi_{G/L}^\vee = \sum_{\substack{W_L-\text{orbits of} \\ \text{subsets } S\subset\Delta_L^+}} \frac{|W_S|}{|W_L|} \sum_{w\in W_L} \epsilon_L(w)\langle\partial(w\pi_S), \pi_{G/L}^\vee\rangle\partial(w\pi_{S^c}).$$

Here ϵ_L is the sign representation of W_L and S^c is the complement of S in Δ_L^+. Moreover, the the notation $\langle\partial(w\pi_S), \pi_{G/L}^\vee\rangle$ simply means that we differentiate the polynomial $\pi_{G/L}^\vee$ by $\partial(w\pi_S)$. Evaluating at ξ and applying (*), our sum becomes

$$\sum_{\substack{W_L-\text{orbits of} \\ \text{subsets } S\subset\Delta_L^+}} \frac{|W_S|}{|W_L|}\langle\partial(\pi_S), \pi_{G/L}^\vee\rangle(\xi) \;\; \partial\left(\sum_{w\in W_L} \epsilon_L(w)w\pi_{S^c}\right)\Bigg|_\xi.$$

Note that the polynomial $\sum\epsilon_L(w)w\pi_{S^c}$ is skew with respect to W_L. Thus, π_L must divide this polynomial. However, if $S \neq \emptyset$, then the degree of $\sum\epsilon_L(w)w\pi_{S^c}$ is less than the degree of π_L. Thus, our polynomial must be the zero polynomial if $S \neq \emptyset$. If $S = \emptyset$, then $\sum\epsilon_L(w)w\pi_{S^c} = |W_L|\pi_L$. Plugging this back into the above expression, we end up with $\pi_{G/L}^\vee(\xi)\partial(\pi_L)|_\xi$ as desired. $\square$

Now, we prove Theorem 2.2.1. If $f \in C_c^\infty(\mathfrak{g})$, then applying Lemmas 2.2.2 and 2.2.3 yields

$$\lim_{\lambda\in C,\ \lambda\to\xi} \partial(\pi_L)|_\lambda\langle\mathcal{O}_\lambda^G, f\rangle = \lim_{\lambda\in C,\ \lambda\to\xi} \partial(\pi_L)|_\lambda c_\lambda \int_{G/L}\langle\mathcal{O}_\lambda^L, g\cdot f\rangle dg$$

$$= \lim_{\lambda\in C,\ \lambda\to\xi} \partial(\pi_L)|_\lambda \int_{G/L}\langle\mathcal{O}_\lambda^L, g\cdot f\rangle dg = \int_{G/L} \lim_{\lambda\in C,\ \lambda\to\xi} \partial(\pi_L)|_\lambda\langle\mathcal{O}_\lambda^L, g\cdot f\rangle dg.$$

Applying Theorem 2.1.1, we have

$$\lim_{\lambda\in C,\ \lambda\to\xi} \partial(\pi_L)|_\lambda\langle\mathcal{O}_\lambda^L, g\cdot f\rangle = i^{r(L)}(-1)^{q(L)}|W(L,H)|f(g\cdot\xi).$$

Since we normalized the measure on $G/L \cong \mathcal{O}_\xi^G$ to be the canonical one, when we integrate both sides over G/L, we get

$$\lim_{\lambda \in C,\ \lambda \to \xi} \partial(\pi_L)|_\lambda \langle \mathcal{O}_\lambda^G, f\rangle = i^{r(L)}(-1)^{q(L)}|W(L,H)|\langle \mathcal{O}_\xi^G, f\rangle$$

as desired.

Chapter 3

Limit Formulas for Nilpotent Orbits

3.1 Applications of a Lemma of Rao, three Corollaries of Barbasch-Vogan, and a Limit Formula of Barbasch

We begin this section by recalling an unpublished lemma of Rao and three corollaries of Barbasch and Vogan. All of this material can be found on pages 46, 47, and 48 of [2]. However, unlike the previous treatment, we need to carefully keep track of certain constants for our applications. Thus, we provide updated statements, and for the convenience of the reader we provide sketches of updated proofs.

Identify $\mathfrak{g} \cong \mathfrak{g}^*$ via a G-equivariant isomorphism. Let $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{g}^* \cong \mathfrak{g}$, and let $\{X, H, Y\}$ be an $\mathfrak{sl}_2$-triple with nilpositive element X. Put $S_X = X + Z_{\mathfrak{g}}(Y)$. The map

$$\phi : G \times S_X \to \mathfrak{g}^*$$

given by $\phi : (g, \xi) \mapsto g \cdot \xi$ is a submersion. In particular, every orbit $\mathcal{O}_\nu \subset G \cdot S_X$ is transverse to S_X, and $G \cdot S_X \subset \mathfrak{g}^*$ is open.

Fix a Haar measure on G. This choice determines a Lebesgue measure on $\mathfrak{g} \cong \mathfrak{g}^*$.

If $\xi \in S_X$, then we have a direct sum decomposition

$$\mathfrak{g} = [\mathfrak{g}, X] \oplus Z_{\mathfrak{g}}(Y) \cong T_X \mathcal{O}_X \oplus T_\xi S_X.$$

We then obtain a Lebesgue measure on S_X as the 'quotient' of the Lebesgue measure on $\mathfrak{g}$ and the canonical measure on $\mathcal{O}_X \subset \mathfrak{g}^*$. Further, given $\nu \in \mathfrak{g}^*$, denote by $\mathcal{F}_\nu$ the fiber over ν under the map ϕ. If $g \cdot \xi = \nu$, then we have an exact sequence

$$0 \to T^*_\nu(G \cdot S_X) \to T^*_{(g,\xi)}(G \times S_X) \to T^*_{(g,\xi)}\mathcal{F}_\nu \to 0.$$

This exact sequence together with the above remarks and our choice of Haar measure on G determine a smooth measure on $\mathcal{F}_\nu$. Moreover, integration against these measures on the fibers of ϕ yields a continuous surjective map

$$\phi_* : \ C_c^\infty(G \times S_X) \longrightarrow C_c^\infty(G \cdot S_X).$$

Dualizing, we get an injective pullback map on distributions

$$\phi^* : \ D(G \cdot S_X) \to D(G \times S_X).$$

Now, we are ready to state Rao's lemma.

Lemma 3.1.1 (Rao). *If $\nu \in S_X$, then there exists a smooth measure $m_{\nu,X}$ on $\mathcal{O}_\nu \cap S_X$ such that*

$$\phi^*(\mathcal{O}_\nu) = m_G \otimes m_{\nu,X}.$$

Here m_G denotes the fixed choice of Haar measure on G. Although ϕ^* depends on this choice of Haar measure, $m_{\nu,X}$ does not.

One can write down $m_{\nu,X}$ by giving a top dimensional form on $\mathcal{O}_\nu \cap S_X$, well-defined up to sign. Essentially, we just divide the canonical measure on $\mathcal{O}_\nu$ by the canonical measure on $\mathcal{O}_X$. More precisely, the composition of the inclusion $[\mathfrak{g}, \nu] \hookrightarrow \mathfrak{g}$

and the projection defined by the decomposition $\mathfrak{g} = [\mathfrak{g}, X] \oplus Z_{\mathfrak{g}}(Y)$ yields a map

$$T_\nu \mathcal{O}_\nu \cong [\mathfrak{g}, \nu] \to [\mathfrak{g}, X] \cong T_X \mathcal{O}_X.$$

This map is a surjection because $\mathcal{O}_\nu$ is transverse to S_X. It pulls back to an exact sequence

$$0 \to T_X^* \mathcal{O}_X \to T_\nu^* \mathcal{O}_\nu \to T_\nu^*(\mathcal{O}_\nu \cap S_X) \to 0.$$

The canonical measures on $\mathcal{O}_\nu$ and $\mathcal{O}_X$ determine top dimensional alternating tensors up to sign on $T_\nu^* \mathcal{O}_\nu$ and $T_X^* \mathcal{O}_X$. Hence, our exact sequence gives a top dimensional, alternating tensor on $T_\nu^*(\mathcal{O}_\nu \cap S_X)$, well-defined up to sign.

We will need three corollaries of Barbasch and Vogan. Let $\mathcal{N} \subset \mathfrak{g}^*$ be the nilpotent cone. If $\nu \in \mathfrak{g}^*$, define

$$\mathcal{N}_\nu = \mathcal{N} \cap \overline{\cup_{t>0} \mathcal{O}_{t\nu}}.$$

Corollary 3.1.2 (Barbasch and Vogan). *We have four statements.*
(a) If $\mathcal{O}_X$ is a nilpotent orbit, then $\mathcal{O}_X \subset \mathcal{N}_\nu$ if, and only if $\mathcal{O}_\nu \cap S_X \neq \emptyset$.
(b) An orbit $\mathcal{O}_X \subset \mathcal{N}_\nu$ is open if, and only if $\mathcal{O}_\nu \cap S_X$ is precompact.
(c) If ν is semisimple, then $\mathcal{O}_X \subset \mathcal{N}_\nu$ is open if, and only if $\mathcal{O}_\nu \cap S_X$ is compact.
(d) Further, $\mathcal{O}_X \cap S_X = \{X\}$ for any nilpotent orbit $\mathcal{O}_X$.

We sketch a proof. Note $G \cdot S_X \subset \mathfrak{g}^*$ is an open subset containing $\mathcal{O}_X$; thus, $\mathcal{O}_X \subset \mathcal{N}_\nu$ iff $\mathcal{O}_{t\nu} \cap S_X \neq \emptyset$ for sufficiently small $t > 0$. However, if $\gamma_t = \exp(-\frac{1}{2}(\log(t))H)$, then

$$\mathcal{O}_{t\nu} \cap S_X = X + t\gamma_t(\mathcal{O}_\nu \cap S_X - X).$$

In particular, $\mathcal{O}_\nu \cap S_X \neq \emptyset$ iff $\mathcal{O}_{t\nu} \cap S_X \neq \emptyset$ for any $t > 0$. This verifies part (a).

For the second and third sentences, one shows that $\mathcal{O}_\nu \cap S_X$ bounded implies that $S_X \cap \mathcal{N}_\nu = \{X\}$ and $\mathcal{O}_\nu \cap S_X$ unbounded implies that $S_X \cap \mathcal{N}_\nu$ is unbounded. This follows from a straightforward calculation utilizing the ad_H-decomposition of $Z_{\mathfrak{g}}(Y)$ into eigenspaces with non-positive eigenvalues and the above relationship between $\mathcal{O}_\nu \cap S_X$, $\mathcal{O}_{t\nu} \cap S_X$, and γ_t. Using that $(G \cdot S_X) \cap \mathcal{N}$ is the union of nilpotent orbits containing $\mathcal{O}_X$ in their closures, parts (b) and (c) follow.

If we let $\nu = X$ in the last paragraph, we arrive at part (d).

Corollary 3.1.3 (Barbasch and Vogan). *Let* $n = \frac{1}{2}(\dim \mathcal{O}_\nu - \dim \mathcal{N}_\nu)$. *Then*

$$\lim_{t \to 0^+} t^{-n} \mathcal{O}_{t\nu} = \sum_{\substack{\mathcal{O}_X \subset \mathcal{N}_\nu \\ \dim \mathcal{O}_X = \dim \mathcal{N}_\nu}} \mathrm{vol}(\mathcal{O}_\nu \cap S_X) \mathcal{O}_X.$$

The volumes are computed with respect to the measures defined in Lemma 3.1.1. Moreover, the Fourier transform of the right hand side is the first non-zero term in the asymptotic expansion of the generalized function $\widehat{\mathcal{O}_\nu}$.

Again, we sketch a proof. Fix X, a nilpotent element with $\mathcal{O}_X \subset \mathcal{N}_\nu$ open, and let $m = \frac{1}{2}(\dim \mathcal{O}_\nu - \dim \mathcal{O}_X)$. We first show

$$\lim_{t \to 0^+} t^{-m} \mathcal{O}_{t\nu} = \mathrm{vol}(\mathcal{O}_\nu \cap S_X) \mathcal{O}_X$$

on the open set $G \cdot S_X$. By Rao's lemma, $\phi^*(\mathcal{O}_\nu) = m_G \otimes m_{\nu,X}$. Thus, it is enough to show

$$\lim_{t \to 0^+} t^{-m} m_G \otimes m_{t\nu,X} = \mathrm{vol}(\mathcal{O}_\nu \cap S_X) m_G \otimes \delta_X$$

if $\mathcal{O}_X \subset \mathcal{N}_\nu$ is an open orbit. Note that the support of the measure $m_{t\nu,X}$ is the precompact set $\mathcal{O}_{t\nu} \cap S_X$, and the precompact sets $\mathcal{O}_{t\nu} \cap S_X$ converge uniformly to δ_X by the above relationship between $\mathcal{O}_\nu \cap S_X$, $\mathcal{O}_{t\nu} \cap S_X$, and γ_t. Thus, it is enough to show

$$t^{-m} \mathrm{vol}(\mathcal{O}_{t\nu} \cap S_X) = \mathrm{vol}(\mathcal{O}_\nu \cap S_X).$$

This follows from a straightforward computation utilizing the above definition of the measures $m_{t\nu,X}$.

By Theorem 3.2 of [2], the distribution $\widehat{\mathcal{O}_\nu}$ has an asymptotic expansion at the origin

$$t^r \widehat{\mathcal{O}_{t\nu}} \sim t^l D_l + \cdots$$

where D_l is the leading term and r is the number of positive roots of G. If we show

$n = l - r$, then our limit will exist everywhere. If $n > l - r$, then the limit $\lim_{t\to 0^+} t^n \widehat{\mathcal{O}_{t\nu}}$ must be zero everywhere. However, we have seen that the Fourier transform of this limit is nonzero on $\mathcal{O}_X$ whenever $\mathcal{O}_X \subset \mathcal{N}_\nu$ is of maximal dimension.

The limit $\lim_{t\to 0^+} t^{-l+r}\mathcal{O}_{t\nu}$ must exist and be nonzero. However, if $n < l - r$, then the homogeneity degree of this invariant distribution and Corollary 3.9 of [2] imply that such a distribution would have to be supported on orbits of dimension greater than $\dim \mathcal{N}_\nu$. But, this is impossible since the limit $\lim_{t\to 0^+} t^{-l+r}\mathcal{O}_{t\nu}$, if it exists, is clearly supported in $\mathcal{N}_\nu$. Hence, $n = l - r$ and the limit $\lim_{t\to 0^+} t^{-n}\mathcal{O}_{t\nu}$ exists.

Now, let $k = \dim \mathcal{N}_\nu$, and let $\mathcal{N}_k$ be the union of nilpotent orbits of dimension at least k. We have shown that our desired limit formula holds on $\mathcal{N}_k$. However, in theory the limit could differ from $\sum_{\substack{\mathcal{O}_X \subset \mathcal{N}_\nu \\ \dim \mathcal{O}_X = \dim \mathcal{N}_\nu}} \operatorname{vol}(\mathcal{O}_\nu \cap S_X)\mathcal{O}_X$ by a distribution u supported on orbits of dimension less than $\mathcal{N}_\nu$. However, we deduce $u = 0$ from Corollary 3.9 of [2] after checking the homogeneity degree of the terms in our limit formula.

Corollary 3.1.4 (Barbasch and Vogan). *Suppose $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan, and let $C \subset (\mathfrak{h}^*)'$ be a connected component of the regular set. If $\nu, \lambda \in C$, then $\mathcal{N}_\nu = \mathcal{N}_\lambda$.*

When $n = \frac{1}{2}(\dim \mathcal{O}_\nu - \dim \mathcal{N}_\nu)$, we observe $\lim_{t\to 0^+} t^{-n}\mathcal{O}_{t\nu} = \lim_{t\to 0^+} \frac{1}{n!}\partial(\nu)^n|_{t\nu}\mathcal{O}_{t\nu}$. Then by Lemma 22 of [9],

$$\lim_{t\to 0^+} \partial(\nu)^n|_{t\nu}\mathcal{O}_{t\nu} = \lim_{t\to 0^+} \partial(\nu)^n|_{t\lambda}\mathcal{O}_{t\lambda}$$

if $\nu, \lambda \in C$. Clearly the support of $\lim_{t\to 0^+} \partial(\nu)^n|_{t\lambda}\mathcal{O}_{t\lambda}$ must be contained in $\mathcal{N}_\lambda$. Thus, the explicit formula for the limit on the left on $G \cdot S_X$ for open orbits $\mathcal{O}_X \subset \mathcal{N}_\nu$ in the proof of Corollary 3.1.3 implies $\mathcal{O}_X \subset \mathcal{N}_\lambda$ whenever $\mathcal{O}_X \subset \mathcal{N}_\nu$ is open. Thus, we deduce $\mathcal{N}_\nu \subset \mathcal{N}_\lambda$. By symmetry we have equality.

We record a special case of Corollary 3.1.3 because it will have useful applications for us.

Corollary 3.1.5. *If $\nu \in \mathfrak{g}^*$, then*

$$\lim_{t\to 0^+} \mathcal{O}_{t\nu} = \sum_{\substack{\mathcal{O}_X \ nilpotent \\ \mathcal{O}_\nu \cap S_X \ finite}} \#(\mathcal{O}_\nu \cap S_X)\mathcal{O}_X.$$

Now, we get to the applications. Suppose $\nu \in \mathfrak{g}^*$, and write

$$\lim_{t\to 0^+} \mathcal{O}_{t\nu} = \sum n_G(\mathcal{O},\nu)\mathcal{O}.$$

If $\mathcal{O}$ is an orbit occurring in the sum, then we let $\mathcal{O}_\mathbb{C} = \mathrm{Int}\mathfrak{g}_\mathbb{C} \cdot \mathcal{O}$ denote its complexification, and we denote by $n_{\mathrm{Int}\,\mathfrak{g}_\mathbb{C}}(\mathcal{O}_\mathbb{C},\nu)$ the coefficient in the corresponding limit formula of $\mathrm{Int}\mathfrak{g}_\mathbb{C}$-orbits.

Corollary 3.1.6. *The coefficients $n(\mathcal{O},\nu)$ are non-negative integers. Moreover,*

$$n_G(\mathcal{O},\nu) \le n_{\mathrm{Int}\,\mathfrak{g}_\mathbb{C}}(\mathcal{O}_\mathbb{C},\nu).$$

The coefficient $n_G(\mathcal{O},\nu)$ is a non-negative integer because it is the cardinality of a finite set by Corollary 3.1.5. Note $n_G(\mathcal{O},\nu)$ is the cardinality of the finite set

$$\mathcal{O}_\nu \cap (X + Z_\mathfrak{g}(Y))$$

while $n_{\mathrm{Int}\,\mathfrak{g}_\mathbb{C}}(\mathcal{O}_\mathbb{C},\nu)$ is the cardinality of the set $(\mathrm{Int}\mathfrak{g}_\mathbb{C}\cdot\mathcal{O}_\nu)\cap(X+Z_{\mathfrak{g}_\mathbb{C}}(Y))$. Since the former set is contained in the later set, we deduce $n_G(\mathcal{O},\nu) \le n_{\mathrm{Int}\,\mathfrak{g}_\mathbb{C}}(\mathcal{O}_\mathbb{C},\nu)$.

Corollary 3.1.7. *Let $\nu \in \mathfrak{g}^*$ be semisimple, let $L = Z_G(\nu)$, and suppose $\mathcal{O}$ is a nilpotent orbit with $n_G(\mathcal{O},\nu) \neq 0$. After conjugating by G, we may assume $\nu \in S_X$. There exists a maximal compact subgroup $K \subset G$ such that $Z_K\{X,H,Y\} \subset Z_G\{X,H,Y\}$ and $K \cap L \subset L$ are maximal compact subgroups. If K is such a group, then*

$$|Z_K(X)/Z_{K\cap L}(X)| \le n_G(\mathcal{O},\nu).$$

Proof. First, if $n(\mathcal{O},\nu) \neq 0$, then $Z_G\{X,H,Y\}/Z_L\{X,H,Y\}$ acts faithfully on the finite set $\mathcal{O}_\nu \cap S_X$ by Corollary 3.1.5. In particular, we have a chain of reductive groups

$$G \supset L \supset Z_G\{X,H,Y\}_0$$

where $Z_G\{X,H,Y\}_0$ is the identity component of $Z_G\{X,H,Y\}$. Recall that any compact subgroup of a reductive Lie group is contained in a maximal compact subgroup of a reductive Lie group. It follows from this fact that there exists a maximal compact subgroup $K \subset G$ such that

$$K \cap L \subset L, \quad K \cap Z_G\{X,H,Y\}_0 \subset Z_G\{X,H,Y\}_0$$

are maximally compact subgroups. But, it is not difficult to see that whenever $K \subset G$ is a maximal compact subgroup, we have $K \cap Z_G\{X,H,Y\}_0 \subset Z_G\{X,H,Y\}_0$ is maximally compact iff $K \cap Z_G\{X,H,Y\} \subset Z_G\{X,H,Y\}$ is maximally compact. This proves the first statement of the proposition.

Now, fix such a group K. Note that $Z_K\{X,H,Y\}$ acts on the finite set $\mathcal{O}_\nu \cap S_X$ with stabilizer $Z_{K\cap L}\{X,H,Y\}$. Thus, we deduce $|Z_K\{X,H,Y\}/Z_{K\cap L}\{X,H,Y\}| \leq n_G(\mathcal{O},\nu)$. Hence, to prove the corollary, it is enough to show that the injection

$$Z_K\{X,H,Y\}/Z_{K\cap L}\{X,H,Y\} \hookrightarrow Z_K(X)/Z_{K\cap L}(X)$$

is in fact a surjection.

To do this, we use two commutative diagrams. First, we have

$$\begin{array}{ccc} Z_K\{X,H,Y\}/Z_{K\cap L}\{X,H,Y\} & \longrightarrow & Z_G\{X,H,Y\}/Z_L\{X,H,Y\} \\ \downarrow & & \downarrow \\ Z_K(X)/Z_{K\cap L}(X) & \longrightarrow & Z_G(X)/ < Z_G(X)^0, Z_L(X) > \end{array}$$

where $Z_G(X)^0$ denotes the identity component of $Z_G(X)$ and $< Z_G(X)^0, Z_L(X) >$ denotes the group generated by $Z_G(X)^0$ and $Z_L(X)$. The top arrow is a surjection because the maximal compact subgroup $Z_K\{X,H,Y\}$ meets every component of the reductive Lie group $Z_G\{X,H,Y\}$. The arrow on the right is a surjection because every component of $Z_G(X)$ meets the Levi factor $Z_G\{X,H,Y\}$. Hence, to show that the arrow on the left is a surjection, it is enough to show that the bottom arrow is an injection.

To verify this last statement, we need some notation and a second commutative diagram. Find a real, reductive algebraic group $G_{\mathbb{R}}$ and a map $p : G \to G_{\mathbb{R}}$ with open image and finite kernel. Choose a maximal compact subgroup $K_{\mathbb{R}} \subset G_{\mathbb{R}}$ such that $p(K) \subset K_{\mathbb{R}}$, and choose a Levi subgroup $L_{\mathbb{R}} \subset G_{\mathbb{R}}$ such that $p : L \to L_{\mathbb{R}}$ has open image and finite kernel. Let $L_{\mathbb{C}}$ be the complexification of $L_{\mathbb{R}}$, and let $U \subset G_{\mathbb{C}}$ be a maximal compact subgroup with $K_{\mathbb{R}} = U \cap G_{\mathbb{R}}$. Choose a parabolic subgroup $P_{\mathbb{C}} \subset G_{\mathbb{C}}$ with Levi factor $L_{\mathbb{C}}$. Then we have the following commutative diagram.

$$\begin{array}{ccc} Z_K(X)/Z_{K\cap L}(X) & \longrightarrow & Z_G(X)/ < Z_G(X)^0, Z_L(X) > \\ \downarrow & & \downarrow \\ Z_U(X)/Z_{U\cap L_{\mathbb{C}}}(X) & \longrightarrow & Z_{G_{\mathbb{C}}}(X)/Z_{P_{\mathbb{C}}}(X) \end{array}$$

The left and bottom maps are easily seen to be injective; hence the top map also must be injective. The corollary follows. □

Next, we recall a proposition of Dan Barbasch [1], which provides an explicit formula for $n_{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}}(\mathcal{O}_{\mathbb{C}}, \nu)$. Let $\nu \in \mathfrak{g}_{\mathbb{C}}^*$ be a semisimple element, let $L_{\mathbb{C}} = Z_{\mathrm{Int}\mathfrak{g}_{\mathbb{C}}}(\nu)$, and let $\mathfrak{l}_{\mathbb{C}} \subset \mathfrak{p}_{\mathbb{C}}$ be a parabolic containing $\mathfrak{l}_{\mathbb{C}} = \mathrm{Lie}(L_{\mathbb{C}})$. Suppose $X \in (\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}_{\mathbb{C}})^*$ is a nilpotent element such that $\mathcal{O}_X^{\mathrm{Int}\mathfrak{g}_{\mathbb{C}}} \cap (\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}_{\mathbb{C}})^* \subset (\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}_{\mathbb{C}})^*$ is open.

Proposition 3.1.8 (Barbasch). *We have the limit formula*

$$\lim_{t \to 0^+} \mathcal{O}_{t\nu}^{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}} = |Z_{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}}(X)/Z_{P_{\mathbb{C}}}(X)|\, \mathcal{O}_X^{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}}$$

where $P_{\mathbb{C}} = N_{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}}(\mathfrak{p}_{\mathbb{C}})$.

In particular, if $Z_{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}}(X)$ is connected, then $\lim_{t\to 0^+} \mathcal{O}_{t\nu}^{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}} = \mathcal{O}_X^{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}}$. By a computation of Springer-Steinberg explained on page 88 of [6], this is true when $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{gl}(n, \mathbb{C})$. Moreover, every nilpotent coadjoint orbit for $\mathrm{GL}(n, \mathbb{C})$ can be written as such a limit by a result of Ozeki and Wakimoto explained in section 7.2 of [6]. Further, it also follows from results in 7.2 and Barbasch's limit formula that two limit formulas

$$\lim_{t\to 0^+} \mathcal{O}_{t\xi_1}^{\mathrm{GL}(n,\mathbb{C})}, \lim_{t\to 0^+} \mathcal{O}_{t\xi_2}^{\mathrm{GL}(n,\mathbb{C})}$$

yield the same nilpotent orbit if and only if $Z_{\mathrm{GL}(n,\mathbb{C})}(\xi_1)$ and $Z_{\mathrm{GL}(n,\mathbb{C})}(\xi_2)$ are conjugate. In the next corollary, we observe that these results also hold for $\mathrm{GL}(n,\mathbb{R})$.
To state it, we define the moment map. Let G be a reductive Lie group and let $\mathcal{P}$ be a conjugacy class of parabolic subgroups of G. Then

$$T^*\mathcal{P} = \{(\mathfrak{p},\xi) |\ \mathfrak{p} \in \mathcal{P},\ \xi \in (\mathfrak{g}/\mathfrak{p})^* \subset \mathfrak{g}^*\}$$

and the moment map is defined by $\mu(\mathfrak{p},\xi) = \xi$. (Of course, the moment map can be defined for any Hamiltonian action of a Lie group; however, we do not need the more general definition here).

Proposition 3.1.9. *There exists a bijection between conjugacy classes of Levi factors of parabolic subgroups of $GL(n,\mathbb{R})$ and nilpotent coadjoint orbits for $GL(n,\mathbb{R})$. Suppose $\mathcal{L}$ is a conjugacy class of Levi factors, and let $\mathcal{P}$ be a conjugacy class of parabolics containing $\mathcal{L}$. Then the orbit $\mathcal{O}_{\mathcal{L}}$ is the unique open, dense orbit in the image of the moment map of the real generalized flag variety*

$$\mathcal{O}_{\mathcal{L}} \subset \mu(T^*\mathcal{P}).$$

Alternately, we may choose $\xi \in \mathfrak{gl}(n,\mathbb{R})^*$ such that $Z_{GL(n,\mathbb{R})}(\xi) = L$. Then $\mathcal{O}_{\mathcal{L}}$ is also characterized by the limit formula

$$\lim_{t\to 0^+} \mathcal{O}_{t\xi} = \mathcal{O}_{\mathcal{L}}.$$

The first $\mathrm{GL}(n,\mathbb{R})$ statement follows immediately from the corresponding $\mathrm{GL}(n,\mathbb{C})$ statement together with the fact that every nilpotent coadjoint $\mathrm{GL}(n,\mathbb{C})$-orbit has an unique real form, and the fact that $\mathcal{O} \cap (\mathfrak{gl}(n,\mathbb{R}))/\mathfrak{p})^*$ is dense if and only if $(\mathrm{GL}(n,\mathbb{C}) \cdot \mathcal{O}) \cap (\mathfrak{gl}(n,\mathbb{C}))/\mathfrak{p}_{\mathbb{C}})^*$ is dense. It follows from Corollary 3.1.6 and the above $\mathrm{GL}(n,\mathbb{C})$ remarks that $\lim_{t\to 0^+} \mathcal{O}_{t\xi}$ is either zero or $\mathcal{O}_{\mathcal{L}}$. In the last two sections of this thesis, we will use the results of the first two sections to explicitly compute $\lim_{t\to 0^+} \widehat{\mathcal{O}_{t\xi}}$. We will observe that the answer is non-zero. This will complete the proof of the proposition and compute the Fourier transform of the nilpotent orbit $\mathcal{O}_{\mathcal{L}}$.

3.2 Limit Formulas for Even Nilpotent Orbits

In [4], Bozicevic proves the following limit formula for an even nilpotent orbit.

Proposition 3.2.1 (Rao, Bozicevic). *Suppose $\mathcal{O}_X$ is an even nilpotent orbit, let $\{X, H, Y\}$ be an $\mathfrak{sl}_2$-triple containing X, and let $Z = X - Y$. Then*

$$\lim_{t\to 0^+} \mathcal{O}_{tZ} = \mathcal{O}_X.$$

This formula was first proved by Rao in an unpublished paper. Bozicevic's formula has a coefficient in front of the $\mathcal{O}_X$. In fact, this coefficient is one. Bozicevic's proof involves deep results of Schmid and Vilonen. In this section, we show how this formula follows easily from the far more elementary results of the last section.

First, let $\mathfrak{p}_{\mathbb{C}}$ be the sum of non-negative eigenspaces for ad_H on $\mathfrak{g}_{\mathbb{C}}$. Then $\mathcal{O}_X^{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}} \cap (\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}_{\mathbb{C}})^* \subset (\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}_{\mathbb{C}})^*$ is open and we may apply Barbasch's result, Proposition 3.1.8. Further, a result of Barbasch-Vogan and Kostant explained on page 50 of [6] implies

$$Z_{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}}(X)/Z_{P_{\mathbb{C}}}(X) \cong Z_{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}}\{X, H, Y\}/Z_{P_{\mathbb{C}}}\{X, H, Y\}.$$

But, $Z_{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}}\{X, H, Y\} \subset Z_{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}}(H) \subset P_{\mathbb{C}}$. Hence, our coefficient is one and we have

$$\lim_{t\to 0^+} \mathcal{O}_{tZ}^{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}} = \mathcal{O}_X^{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}}.$$

Now, we need to prove a real version of this limit formula. By Corollary 3.1.6, we know that we must have $\lim_{t\to 0^+} \mathcal{O}_{tZ} = \sum \mathcal{O}_{X'}$ where the sum is over some subset of real forms of $\mathcal{O}_X^{\mathrm{Int}\,\mathfrak{g}_C}$. We know $\mathcal{O}_X$ must occur by Corollary 3.1.5 and the observation $Z = X - Y \in X + Z_{\mathfrak{g}}(Y)$. Now suppose $\mathcal{O}_{X'}$ is some other real form of $\mathcal{O}_X^{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}}$ occurring in our limit formula. Given an $\mathfrak{sl}_2$-triple $\{X', H', Y'\}$ containing X', we must have

$\mathcal{O}_Z \cap (X' + Z_{\mathfrak{g}}(Y')) \neq \emptyset$. But, $Z' = X' - Y' \in X' + Z_{\mathfrak{g}}(Y')$ and $\mathcal{O}_{Z'}^{\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}} \cap (X' + Z_{\mathfrak{g}_{\mathbb{C}}}(Y'))$ has one element by Corollary 3.1.5 and the above $\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}$-limit formula. Further, it was proven by Rao (unpublished) that $Z' = X' - Y' = X - Y = Z$ only if X and X' are conjugate (details of his elementary argument can be found on page 146 of [6]). Thus, we cannot have $Z \in X' + Z_{\mathfrak{g}}(Y')$ and no other real forms can occur in our limit formula. The proposition follows.

Chapter 4

Wave Front Cycles of Tempered Representations

4.1 Rossmann's Character Formula

In the next two sections, we use the limit formulas of section 3.1 to deduce two results concerning wave front cycles of tempered representations.

Let G be a real, reductive algebraic group, and let τ be an irreducible, tempered representation of G with character Θ_τ. Let θ_τ be the Lie algebra analogue of the character of τ. Rossmann associated to τ a finite union of regular, coadjoint orbits $\mathcal{O}_\tau \subset \mathfrak{g}^*$. He proved the following theorem [20], [21].

Theorem 4.1.1 (Rossmann). *As generalized functions, we have*

$$\widehat{\mathcal{O}_\tau} = \theta_\tau.$$

Here $\widehat{\mathcal{O}_\tau}$ denotes the Fourier transform of the canonical measure on $\mathcal{O}_\tau$.

If $\mathcal{O}_\tau = \mathcal{O}_\nu$ is a single orbit, then the leading term of the asymptotic expansion of $\widehat{\mathcal{O}_\tau} = \widehat{\mathcal{O}_\nu}$ is

$$\sum_{\substack{\mathcal{O}_X \subset \mathcal{N}_\nu \\ \dim \mathcal{O}_X = \dim \mathcal{N}_\nu}} \mathrm{vol}(\mathcal{O}_\tau \cap S_X)\widehat{\mathcal{O}_X}$$

by Corollary 3.1.3. More generally, suppose $\mathcal{O}_\tau = \cup \mathcal{O}_{\nu_i}$, and define $\mathrm{AS}(\tau) = \cup \mathcal{N}_{\nu_i}$. Then the leading term of $\widehat{\mathcal{O}_\tau}$ is a sum over the leading terms of the $\widehat{\mathcal{O}_{\nu_i}}$ of minimal degree. This is just the sum over $\mathrm{vol}(\mathcal{O}_\tau \cap S_X)\widehat{\mathcal{O}_X}$ where $\mathcal{O}_X \subset \mathrm{AS}(\tau)$ varies over orbits of maximal dimension. Hence, we have shown

$$\mathrm{WF}(\tau) = \sum_{\substack{\mathcal{O}_X \subset \mathrm{AS}(\tau) \\ \dim \mathcal{O}_X = \dim \mathrm{AS}(\tau)}} \mathrm{vol}(\mathcal{O}_\tau \cap S_X)\mathcal{O}_X.$$

We wish to show the following slightly stronger statement.

Theorem 4.1.2. *There exists a canonical measure on $\mathcal{O}_\tau \cap S_X$ such that*

$$WF(\tau) = \sum_{\mathcal{O}_\tau \cap S_X \ \textit{precompact}} \mathrm{vol}(\mathcal{O}_\tau \cap S_X)\mathcal{O}_X.$$

The sum is over nilpotent coadjoint orbits $\mathcal{O}_X$ such that $\mathcal{O}_\tau \cap S_X$ is precompact.

To deduce this theorem from our previous statement, we need only show that $\mathcal{O}_X \subset \mathrm{AS}(\tau)$ is of maximal dimension iff $\mathcal{O}_\tau \cap S_X$ is precompact and non-empty. If $\mathcal{O}_X \subset \mathrm{AS}(\tau)$ is of maximal dimension, then $\mathcal{O}_X \cap \mathcal{N}_{\nu_i}$ is open for every i and non-empty for at least one i. And by Corollary 3.1.2, $\mathcal{O}_{\nu_i} \cap S_X$ is precompact for every i and non-empty for at least one i. We conclude $\mathcal{O}_\tau \cap S_X$ is precompact and non-empty.

Conversely, suppose $\mathcal{O}_\tau \cap S_X$ is precompact and non-empty. Then $\mathcal{O}_{\nu_i} \cap S_X$ is precompact for every i and non-empty for some i. Corollary 3.1.2 tells us that $\mathcal{O}_X \subset \mathcal{N}_{\nu_i}$ is open for some i and $\mathcal{O}_X \cap \mathcal{N}_{\nu_i}$ is open for every i. Hence, we have that $\mathcal{O}_X \subset \mathrm{AS}(\tau)$ is open. But, by Theorem D of [24], $\mathrm{AS}(\tau)$ is the closure of the union of the orbits in $\mathrm{AS}(\tau)$ of maximal dimension. Thus, $\mathcal{O}_X \subset \mathrm{AS}(\tau)$ open implies that $\mathcal{O}_X \subset \mathrm{AS}(\tau)$ is of maximal dimension.

4.2 Noticed Nilpotents and Tempered Representations

In this section, we prove the following result. Recall the definition of the wave front cycle of an admissible representation from [2], [25].

Theorem 4.2.1. *Suppose $\mathcal{O}$ is an orbit contained in $WF(\tau)$ for a tempered representation τ, let $\nu \in \mathcal{O}$, and let L be a Levi factor of $Z_G(\nu)$. Then $L/Z(G)$ is compact.*

Before proving this theorem, we give an alternate way of stating it. We say $\mathcal{O}_X \subset \mathfrak{g}$ is a *noticed* nilpotent orbit if there is no proper Levi subalgebra meeting $\mathcal{O}_X$. This notion was first introduce in [17]. By Levi subalgebra, we mean the Levi factor of a real parabolic subalgebra of $\mathfrak{g}$. It is not difficult to show that any Levi factor L of $Z_G(X)$ is compact modulo $Z(G)$ iff $\mathcal{O}_X$ is a noticed nilpotent orbit.

Corollary 4.2.2. *If $\mathcal{O} \subset \mathfrak{g}^*$ is a nilpotent coadjoint orbit occurring in the wave front cycle of a tempered representation, then $\mathcal{O}$ is a noticed nilpotent orbit.*

Now we turn to the proof of Theorem 4.2.1. It is enough to prove the theorem when τ is an irreducible, tempered representation. Let $\mathcal{O}_X \subset \mathrm{AS}(\tau)$ be an open orbit, and identify $\mathfrak{g} \cong \mathfrak{g}^*$ via a G-equivariant isomorphism. Let $L \subset Z_G(X)$ be a Levi factor. Then there exists an $\mathfrak{sl}_2$-triple $\{X, H, Y\}$ such that $L = Z_G\{X, H, Y\}$. To prove the theorem, we must show

$$Z_G\{X, H, Y\}/Z(G) \text{ is compact.} \quad (*)$$

Now, if $\mathcal{O}_\tau = \cup \mathcal{O}_{\nu_i}$, then $\mathrm{AS}(\tau) = \cup \mathcal{N}_{\nu_i}$. Any open orbit in $\mathrm{AS}(\tau)$ must be open in some $\mathcal{N}_{\nu_i}$. Hence, it is enough to prove $(*)$ whenever $\mathcal{O}_X$ is open in $\mathcal{N}_\nu$ for a regular element $\nu \in \mathfrak{g}^*$.

Next, supplement A and supplement C of [19] imply that every regular orbit $\mathcal{O}_\nu$ can be written as a limit of regular semisimple orbits in the following sense. Let $\xi \in \mathfrak{g}^* \cong \mathfrak{g}$ be a semisimple element in the closure of $\mathcal{O}_\nu$, and let $\mathfrak{h} \subset \mathfrak{g}$ be a fundamental Cartan in $Z_{\mathfrak{g}}(\xi)$. Then there exists a connected component $C \subset (\mathfrak{h}^*)'$ such that

$$\lim_{\substack{\lambda \in C \\ \lambda \to \xi}} \mathcal{O}_\lambda = \mathcal{O}_\nu.$$

Hence, $\mathcal{N}_\nu = \mathcal{N}_\lambda$ for some regular semisimple element λ, and it is enough to prove $(*)$ for $\mathcal{O}_X$ open in $\mathcal{N}_\lambda$ with λ regular, semisimple.

Fix X and suppose $\mathcal{O}_X \subset \mathcal{N}_\lambda$ is open for some λ regular semisimple. By Corollary 3.1.4, $\mathcal{O}_\lambda \cap S_X$ is compact. Now, $L = Z_G\{X, H, Y\}$ acts on this space, and L must have at least one closed orbit (for instance, one can take an orbit of minimal dimension). Without loss of generality, we make it $L \cdot \lambda \subset \mathcal{O}_\lambda \cap S_X$. Choose a Cartan $\mathfrak{h}$ and a component of the regular set $C \subset (\mathfrak{h}^*)'$ such that $\lambda \in C$. If $\xi \in U_1 = G \cdot C$, then $\mathcal{O}_X \subset \mathcal{N}_\lambda = \mathcal{N}_\xi$ is open by Corollary 3.1.4. It then follows from Corollary 3.1.2 that $\mathcal{O}_\xi \cap S_X$ is compact for all ξ in the open set U_1. Define $U = U_1 \cap S_X$, an open subset of S_X.

Now, $L/Z_G(\lambda) \cong L \cdot \lambda \subset \mathcal{O}_\lambda \cap S_X$ is a closed subset of a compact set; hence, $L/Z_G(\lambda)$ is compact. Note that $Z_G(\lambda) \subset G$ is a Cartan since λ is regular, semisimple. Thus, $Z_L(\lambda) \subset Z_G(\lambda)$ is abelian and consists of semisimple elements. Hence, the connected component of the identity $Z_L(\lambda)_0$ must be contained in a Cartan B of L. We have seen that $L/Z_L(\lambda)$ is compact; hence, $L/Z_L(\lambda)_0$ is compact because $Z_L(\lambda)$ has finitely many components. This implies L/B is compact and finally

$$L/Z(L)$$

is compact since semisimple groups are compact iff they are compact modulo a Cartan.

Because the fibers of the projection

$$L/Z(G) \to L/Z(L)$$

are homeomorphic to $Z(L)/Z(G)$, to show that $L/Z(G)$ is compact, it is enough to show $Z(L)/Z(G)$ is compact.

The following lemma is the key step in proving that $Z(L)/Z(G)$ is compact.

Lemma 4.2.3. *Let $Z(\mathfrak{l})$ denote the center of $\mathfrak{l} = Lie(L)$, and let $Z(\mathfrak{g})$ denote the center of $\mathfrak{g}$. Then*

$$\bigcap_{\xi \in U} (Z(\mathfrak{l}) \cap Z_{\mathfrak{l}}(\xi)) = Z(\mathfrak{g}).$$

Proof. Clearly the right hand side is contained in the left hand side. To show the other direction, suppose $W \in (Z(\mathfrak{l}) \cap Z_{\mathfrak{l}}(\xi))$ for all $\xi \in U$. We will show $W \in Z(\mathfrak{g})$. Since $W \in \mathfrak{l}$, we know $X, H, Y \in Z_{\mathfrak{g}}(W)$. Further, $Z_{\mathfrak{g}}(W) \cap Z_{\mathfrak{g}}(Y) \subset Z_{\mathfrak{g}}(Y)$ is a vector subspace containing $U - X$ since $W \in Z_{\mathfrak{l}}(\xi)$ for $\xi \in U$ and $X \in Z_{\mathfrak{g}}(W)$. Since $U - X \subset Z_{\mathfrak{g}}(Y)$ is an open subset, we must have

$$Z_{\mathfrak{g}}(W) \supset Z_{\mathfrak{g}}(Y).$$

Now, view $\mathfrak{g}$ as a finite dimensional module for $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\} \cong \mathfrak{sl}_2\mathbb{R}$. Note $Z_{\mathfrak{g}}(W) \subset \mathfrak{g}$ is a subalgebra and a submodule for $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$ since $X, H, Y \in Z_{\mathfrak{g}}(W)$. But, the lowest weight vectors of each irreducible summand of $\mathfrak{g}$ are in $Z_{\mathfrak{g}}(W)$ since $Z_{\mathfrak{g}}(W) \supset Z_{\mathfrak{g}}(Y)$, and the lowest weight vectors of any finite dimensional $\mathfrak{sl}_2$ module generate the entire module. Thus, $Z_{\mathfrak{g}}(W) = \mathfrak{g}$ and $W \in Z(\mathfrak{g})$ as desired. $\square$

Before we get back to showing that $Z(L)/Z(G)$ is compact, we need two general remarks. First, suppose A is an abelian, real algebraic group, suppose $\phi : A \to \mathrm{Aut}(V)$ is a representation of A on a real vector space V, and suppose $S \subset V$ is a compact A-stable subset of V. Then A acts on S with compact orbits. This can be proved as follows. After complexifying the representation, we may diagonalize the action of the image $\phi(A)$ since it is abelian and consists of semisimple elements. Now, A must be isomorphic to a product of copies of S^1, $\mathbb{R}^\times$, and $\mathbb{C}^\times$. Using that every one dimensional character of these groups has either compact or unbounded image in $\mathbb{C}$, we deduce that every orbit of A on $V \otimes \mathbb{C}$ is either compact or unbounded. In particular, A must act on a compact $S \subset V$ with compact orbits.

Second, if A is an abelian, real algebraic group and A_1, A_2 are cocompact, algebraic, closed subgroups, then $A_1 \cap A_2$ is cocompact in A. This is because the fibers of the map

$$A/(A_1 \cap A_2) \to A/A_1$$

are homeomorphic to $A_1/(A_1 \cap A_2) \cong A_1A_2/A_2$, which is compact because it is a closed subset of A/A_2. More generally, if $A_1, \ldots, A_n$ is a finite collection of cocompact, algebraic, closed subgroups of a real, abelian algebraic group A, then

$$A/\cap_{i=1}^{n} A_i$$

is compact.

Now, back to the proof that $Z(L)/Z(G)$ is compact. By the first remark, $Z(L)$ acts on $\mathcal{O}_\xi \cap S_X$ with compact orbits for every $\xi \in U$. In particular, $Z(L)/(Z(L) \cap Z_L(\xi))$ is compact for all $\xi \in U$. In the above lemma, we showed

$$\bigcap_{\xi \in U} (Z(\mathfrak{l}) \cap Z_{\mathfrak{l}}(\xi)) = Z(\mathfrak{g}).$$

However, one can clearly choose $\xi_1, \ldots, \xi_k \in U$ such that the identity still holds when taking the intersection over this finite set. Then, by the second remark,

$$Z(L)/\bigcap_{i=1}^{k}(Z(L) \cap Z_L(\xi_i))$$

is compact. Since $\bigcap_{i=1}^{k}(Z(L) \cap Z_L(\xi_i))$ is a real algebraic group, it has a finite number of connected components and

$$Z(L)/(\bigcap_{i=1}^{k}(Z(L) \cap Z_L(\xi_i)))_0$$

is also compact where $(\bigcap_{i=1}^{k}(Z(L) \cap Z_L(\xi_i)))_0$ denotes the identity component. But, since $\bigcap_{i=1}^{k}(Z(L) \cap Z_L(\xi_i))$ and $Z(G)$ share a Lie algebra, $Z(L)/Z(G)$ is a quotient of

$$Z(L)/(\bigcap_{i=1}^{k}(Z(L) \cap Z_L(\xi_i)))_0.$$

Thus, $Z(L)/Z(G)$ is compact. This completes the proof of Theorem 4.2.1.

Chapter 5

Fourier Transforms of Nilpotent Coadjoint Orbits

5.1 Fourier Transforms of Semisimple Coadjoint Orbits for $\mathrm{GL}(n, \mathbb{R})$

In the next two sections, we use the limit formulas of sections 2.1, 2.2, and 3.1 to compute an explicit formula for the Fourier transform of a nilpotent coadjoint orbit of $\mathrm{GL}(n, \mathbb{R})$.

Let $G = \mathrm{GL}(n, \mathbb{R}) = \mathrm{GL}(2m + \delta, \mathbb{R})$ where $\delta = 0$ or 1, and let $\mathfrak{g} = \mathrm{Lie}(G)$. Fix a fundamental Cartan $\mathfrak{h}_0 \subset \mathfrak{g}$, and enumerate its imaginary roots

$$\{\alpha_1, \ldots, \alpha_m, -\alpha_1, \ldots, -\alpha_m\}.$$

Let $\mathfrak{h}_k$ be the Cartan obtained by applying Cayley transforms through the roots $\alpha_1, \ldots, \alpha_k$. Then $\mathfrak{h}_0, \ldots, \mathfrak{h}_m$ is a set of representatives of the conjugacy classes of Cartan subalgebras of $\mathfrak{g}$. In what follows, we will use these fixed Cayley transforms to identify $(\mathfrak{h}_k)_{\mathbb{C}} \cong (\mathfrak{h}_l)_{\mathbb{C}}$ (and all roots, coroots of $\mathfrak{h}_k$ with roots, coroots of $\mathfrak{h}_l$) without further comment.

Let $\Delta(\mathfrak{h}_l)$ (resp. $\Delta_{\text{imag.}}(\mathfrak{h}_l)$, $\Delta_{\text{real}}(\mathfrak{h}_l)$, $\Delta_{\text{cx.}}(\mathfrak{h}_l)$) denote the set of all (resp. imaginary, real, complex) roots of $\mathfrak{g}$ with respect to $\mathfrak{h}_l$. Choose a component $C_m \subset \mathfrak{h}'_m$, and define Δ^+ to be the set of roots α such that $\alpha(X) > 0$ for all $X \in C_m$. This fixes a choice of positive roots for $\mathfrak{g}$ with respect to $\mathfrak{h}_l$ for every l. Denote by $\Delta^+(\mathfrak{h}_l)$ (resp. $\Delta^+_{\text{imag.}}(\mathfrak{h}_l)$, $\Delta^+_{\text{real}}(\mathfrak{h}_l)$, $\Delta^+_{\text{cx.}}(\mathfrak{h}_l)$) the set of all (resp. imaginary, real, complex) positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}_l$. Now, choose a regular element $\lambda \in \mathfrak{h}^*_k$ satisfying:

(a) If $\alpha \in \Delta^+_{\text{imag.}}(\mathfrak{h}_k)$ is a positive, imaginary root of $\mathfrak{h}_k$, then

$$\langle \lambda, i\alpha^\vee \rangle < 0.$$

(b) If $\beta \in \Delta^+_{\text{real}}(\mathfrak{h}_k)$ is a real, positive root of $\mathfrak{h}_k$, then

$$\langle \lambda, \beta^\vee \rangle < 0.$$

Moreover, define

$$C_l(e) = \{X \in \mathfrak{h}'_l | \alpha(X) > 0 \ \forall \alpha \in \Delta^+_{\text{real}}(\mathfrak{h}_k)\},$$

and for every $u \in W_{\text{real}}(\mathfrak{h}_l)$, define

$$C_l(u) = u \cdot C_l(e).$$

Here $W_{\text{real}}(\mathfrak{h}_l)$ denotes the Weyl group of the real roots of $\mathfrak{g}$ with respect to $\mathfrak{h}_l$. Note $C_m(e) = C_m$.

Let $W_{\mathbb{C}}$ denote the complex Weyl group of $\mathfrak{g}_{\mathbb{C}}$ with respect to $(\mathfrak{h}_l)_{\mathbb{C}}$. For $k \leq l$, define a subset $W_{k,l} \subset W_{\mathbb{C}}$ to be the set of $w \in W_{\mathbb{C}}$ satisfying:

(i) If $\alpha \in \Delta_{\text{imag.}}(\mathfrak{h}_l)$, then $w^{-1}\alpha \in \Delta_{\text{imag.}}(\mathfrak{h}_k)$.

(ii) If $\alpha \in \Delta_{\text{cx.}}(\mathfrak{h}_l)$, then $w^{-1}\alpha \in \Delta_{\text{cx.}}(\mathfrak{h}_k)$.

(iii) If $\alpha \in \Delta^+_{\text{imag.}}(\mathfrak{h}_k)$ and $w\alpha \notin \Delta_{\text{imag.}}(\mathfrak{h}_l)$, then $w\alpha \in \Delta^+_{\text{real}}(\mathfrak{h}_l)$.

For each $w \in W_{k,l}$, denote by $N_{k,l}(w)$ the number of $\alpha \in \Delta^+_{\text{imag.}}(\mathfrak{h}_l)$ satisfying $w^{-1}\alpha \notin \Delta^+_{\text{imag.}}(\mathfrak{h}_k)$. Define

$$\epsilon_{k,l}(w) = (-1)^{N_{k,l}(w)}$$

for every $w \in W_{k,l}$.

Proposition 5.1.1. *If $\lambda \in \mathfrak{h}_k^*$ is regular, semisimple and $l \geq k$, then*

$$\widehat{\mathcal{O}_\lambda}|_{C_l(e)} = \frac{2^{l-k}\sum_{w\in W_{k,l}} \epsilon_{k,l}(w)e^{iw\lambda}}{\pi}.$$

If $l < k$, then $\widehat{\mathcal{O}_\lambda}$ vanishes on $\mathfrak{h}_l'$. Here $\pi = \prod_{\alpha\in\Delta^+}\alpha$ as usual.

Let θ be a Cartan involution fixing $\mathfrak{h}_k$, and decompose $\mathfrak{h}_k = \mathfrak{t} \oplus \mathfrak{a}$ where $\mathfrak{t}$ is the $+1$ eigenspace of θ and $\mathfrak{a}$ is the -1 eigenspace of θ. Set $M = Z_G(\mathfrak{a})$, and note

$$M \cong \mathrm{GL}(2,\mathbb{R})^{m-k} \times (\mathbb{R}^\times)^{2k+\delta}.$$

The identity component of M is $M_0 \cong \mathrm{GL}^+(2,\mathbb{R})^{m-k} \times (\mathbb{R}^\times_+)^{2k+\delta}$. We will compute $\widehat{\mathcal{O}^G_\lambda}$ by first computing $\widehat{\mathcal{O}^M_\lambda}$ and then applying Harish-Chandra descent.

Let $\Delta_M(\mathfrak{h}_l)$ be the set of roots of M with respect to $\mathfrak{h}_l$, and let π_M be the product of the roots of M with respect to $\mathfrak{h}_l$ that are positive for G. Let $W^M_{\mathbb{R}}(\mathfrak{h}_k)$ be the real Weyl group of M with respect to $\mathfrak{h}_k$, and fix $w \in W^M_{\mathbb{R}}(\mathfrak{h}_k)$. Note that $w\pi_M$ is the product of the roots α of M satisfying

$$\langle iw\lambda, \alpha^\vee\rangle < 0.$$

Then by Theorem 2.1.4, we have

$$\widehat{\mathcal{O}^{M_0}_{w\lambda}}|_{\mathfrak{h}_k'} = \frac{e^{iw\lambda}}{w\pi_M}.$$

Observe $\mathfrak{h}_l \subset \mathfrak{m}$, and put

$$C_l(e)_M = \{X \in \mathfrak{h}_l' |\ \alpha(X) > 0 \ \forall \text{ real roots } \alpha \in \Delta_M(\mathfrak{h}_l) \cap \Delta_G^+(\mathfrak{h}_l)\}.$$

If $u \in W^M_{\text{real}}(\mathfrak{h}_l)$, define

$$C_l(u)_M = u \cdot C_l(e)_M.$$

Now, decompose $w = w_r w_i$ into its components in the Weyl group of the real roots of $\mathfrak{h}_l$ and the Weyl group of the imaginary roots of $\mathfrak{h}_l$. Checking Harish-Chandra's matching conditions (Theorem 2.1.7) and using that $\widehat{\mathcal{O}^{M_0}_{w\lambda}}$ is tempered, we observe

$$\widehat{\mathcal{O}^{M_0}_{w\lambda}}|_{C_l(w_r)_M} = \frac{e^{iw\lambda}}{w\pi_M}.$$

Since w_r is in the real Weyl group of $\mathfrak{h}_l$ with respect to M_0 and the generalized function we are computing is M_0-invariant, we get

$$\widehat{\mathcal{O}^{M_0}_{w\lambda}}|_{C_l(e)_M} = \frac{\epsilon(w_i)e^{iw_i\lambda}}{\pi_M}.$$

Note that $\mathcal{O}^M_\lambda$ is the finite union of the orbits $\mathcal{O}^{M_0}_{w\lambda}$ where w ranges over the real Weyl group $W^M_{\mathbb{R}}(\mathfrak{h}_k)$. Hence,

$$\widehat{\mathcal{O}^M_\lambda}|_{C_l(e)_M} = \frac{2^{l-k}\sum_{w\in W^M_{\text{imag.}}(\mathfrak{h}_l)} \epsilon(w)e^{iw\lambda}}{\pi_M}$$

where $W^M_{\text{imag.}}(\mathfrak{h}_l)$ is the Weyl group of the imaginary roots of $\mathfrak{m}$ with respect to $\mathfrak{h}_l$.

Now, we can use Harish-Chandra descent (Lemma 2.1.5) to compute $\widehat{\mathcal{O}^G_\lambda}$. Given $X \in C_l(e)_M$, we must enumerate the M-orbits in $\mathcal{O}_X \cap \mathfrak{m}$. First, we choose representatives of the M-conjugacy classes of Cartans in $\mathfrak{m}$. For each ordered $(m-l)$-tuple $J = (j_1, \ldots, j_{m-l})$ with $1 \le j_1 < j_2 < \cdots < j_{m-l} \le m-k$, define $J^c = (r_1, \ldots, r_{l-k})$ to be the complementary indices among $1, \ldots, m-k$. Define $\mathfrak{h}_l^J$ to be the Cartan obtained from $\mathfrak{h}_k \subset \mathfrak{m}$ by applying the Cayley transforms associated to the roots α_{m-r_s} for $s = 1, \ldots, l-k$. One sees that the collection $\{\mathfrak{h}_l^J\}$ is a set of representatives for all M-conjugacy classes of Cartans in $\mathfrak{m}$ of imaginary rank $m-l$.

Now, every M-orbit in $\mathcal{O}_X \cap \mathfrak{m}$ must meet exactly one of the Cartans $\mathfrak{h}_l^J$. Thus, we have

$$(\mathcal{O}_X \cap \mathfrak{m})/M = \bigcup_J (\mathcal{O}_X \cap \mathfrak{h}_l^J)/W_{\mathbb{R}}^M(\mathfrak{h}_l^J)$$

where $W_{\mathbb{R}}^M(\mathfrak{h}_l^J)$ is the real Weyl group of M with respect to $\mathfrak{h}_l^J$. Moreover, for each Cartan $\mathfrak{h}_l^J$, the $W_{\mathbb{R}}^M(\mathfrak{h}_l^J)$-orbits on $\mathcal{O}_X \cap \mathfrak{h}_l^J$ are in bijection with the cosets $W_{\mathbb{R}}^M(\mathfrak{h}_l^J)\backslash W_{\mathbb{R}}^G(\mathfrak{h}_l^J)$. Thus, we have the formula

$$\widehat{\mathcal{O}_\lambda^G}(X) = \sum_J \sum_{u \in W_{\mathbb{R}}^M(\mathfrak{h}_l^J)\backslash W_{\mathbb{R}}^G(\mathfrak{h}_l^J)} \widehat{\mathcal{O}_\lambda^M}(uw_J X)|\pi_{G/M}(uw_J X)|^{-1}.$$

Here w_J is an element of G taking $\mathfrak{h}_l$ to $\mathfrak{h}_l^J$.

Note that we get isomorphisms $(\mathfrak{h}_k)_{\mathbb{C}} \cong (\mathfrak{h}_l)_{\mathbb{C}}$ and $(\mathfrak{h}_k)_{\mathbb{C}} \cong (\mathfrak{h}_l^J)_{\mathbb{C}}$ by applying successive Cayley transforms to $\mathfrak{h}_k$. Composing these isomorphisms with a complex Weyl group element that takes positive, non-compact imaginary (resp. real) roots of $\mathfrak{h}_l$ with respect to $\mathfrak{g}$ to positive, non-compact imaginary (resp. real) roots of $\mathfrak{h}_l^J$, we get a candidate for w_J. We will fix such a candidate for each J from now on.
Now, we have the formula

$$\widehat{\mathcal{O}_\lambda^M}|_{C_{l,J}(e)_M} = \frac{2^{l-k} \sum_{w \in W_{\text{imag.}}^M(\mathfrak{h}_l^J)} \epsilon(w) e^{iw\lambda}}{\pi_M^J}$$

for every J. Here π_M^J is the product of roots of $\mathfrak{h}_l^J$ with respect to $\mathfrak{m}$ that are positive for G, and $W_{\text{imag.}}^M(\mathfrak{h}_l^J)$ is the Weyl group of the imaginary roots of $\mathfrak{h}_l^J$ with respect to $\mathfrak{m}$. We define

$$C_{l,J}(e)_M = \{X \in (\mathfrak{h}_l^J)' | \alpha(X) > 0 \ \forall \text{ real roots } \alpha \in \Delta_M(\mathfrak{h}_l^J) \cap \Delta_G^+(\mathfrak{h}_l)\}$$

and more generally

$$C_{l,J}(u)_M = u \cdot C_{l,J}(e).$$

This formula is proved in the same way as the special case of $\mathfrak{h}_l^J = \mathfrak{h}_l$, which is proved above.

Partition $W_{k,l} = \bigsqcup_J W_{k,l}^J$ where $w \in W_{k,l}$ is in $W_{k,l}^J$ if $w \cdot \mathfrak{h}_l = \mathfrak{h}_l^J$. Every coset in $W_{\mathbb{R}}^M(\mathfrak{h}_l^J)\backslash W_{\mathbb{R}}^G(\mathfrak{h}_l^J)$ contains a unique representative u such that u^{-1} takes the positive real roots of $\mathfrak{h}_l^J$ with respect to $\mathfrak{m}$ to positive roots of $\mathfrak{h}_l^J$ with respect to $\mathfrak{g}$ and u^{-1} fixes the imaginary roots of $\mathfrak{h}_l^J$ with respect to $\mathfrak{m}$. When we sum over $W_{\mathbb{R}}^M\backslash W_{\mathbb{R}}^G$, we will really be summing over this set of representatives. Then

$$\sum_{u\in W_{\mathbb{R}}^M(\mathfrak{h}_l^J)\backslash W_{\mathbb{R}}^G(\mathfrak{h}_l^J)} \widehat{\mathcal{O}_\lambda^M}(uw_JX)|\pi_{G/M}^J(uw_JX)|^{-1} =$$

$$= 2^{l-k} \sum_{u\in W_{\mathbb{R}}^M(\mathfrak{h}_l^J)\backslash W_{\mathbb{R}}^G(\mathfrak{h}_l^J)} \sum_{w\in W_{\text{imag.}}^M(\mathfrak{h}_l^J)} \frac{\epsilon(w)e^{iw\lambda(uw_JX)}}{\pi_M^J(uw_JX)|\pi_{G/M}^J(uw_JX)|}$$

$$= 2^{l-k} \sum_{w\in W_{k,l}^J} \frac{\epsilon_{k,l}(w)e^{iw\lambda(X)}}{\pi_G(X)}.$$

The last equality follows from noticing that $\{w_J^{-1}u^{-1}w\}$ is really $W_{k,l}^J$ if w varies over $W_{\text{imag.}}^M(\mathfrak{h}_l^J)$ and u varies over our chosen set of representatives of $W_{\mathbb{R}}^M(\mathfrak{h}_l^J)\backslash W_{\mathbb{R}}^G(\mathfrak{h}_l^J)$. Further, we used

$$\epsilon(w) = \epsilon_{k,l}(w_J^{-1}u^{-1}w) \text{ and } \pi_M^J(uw_JX)|\pi_{G/M}^J(uw_JX)| = \pi_G(X).$$

Summing over all possible J, we get

$$\widehat{\mathcal{O}_\lambda}|_{C_l(e)} = 2^{l-k}\sum_J \sum_{w\in W_{k,l}^J} \frac{\epsilon_{k,l}(w)e^{iw\lambda}}{\pi} = \frac{2^{l-k}\sum_{w\in W_{k,l}}\epsilon_{k,l}(w)e^{iw\lambda}}{\pi}.$$

The vanishing of $\widehat{\mathcal{O}_\lambda}$ on the other Cartans follows from Harish-Chandra descent.

5.2 Fourier Transforms of Nilpotent Coadjoint Orbits for $\mathrm{GL}(n, \mathbb{R})$

Let $G = \mathrm{GL}(n, \mathbb{R})$, let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$, and let $\mathcal{O}_{\mathcal{L}}$ be as in Proposition 3.1.9. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and let H be the corresponding Cartan subgroup. Put

$$\mathfrak{h}'' = \{X \in \mathfrak{h} | \ \alpha(X) \neq 0 \ \forall \text{ real roots } \alpha\},$$

suppose $C \subset \mathfrak{h}''$ is a connected component, and put $C' = C \cap \mathfrak{h}'$. Choose positive roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ satisfying:

(i) For all positive real roots α and all $X \in C$, we have $\alpha(X) > 0$.

(ii) If α is a complex root, then α is positive iff $\overline{\alpha}$ is positive.

Suppose $L \in \mathcal{L}$ with $L \supset H$. Let $W(G, H)_L$ be the stabilizer of L in the real Weyl group of G with respect to H, and let $W(L, H)$ be the real Weyl group of L with respect to H. Note that the cardinality of the quotient

$$\left| \frac{W(G, H)_L}{W(L, H)} \right|$$

is independent of the choice of $L \in \mathcal{L}$ with $L \supset H$. Thus, we will denote the cardinality of this quotient by

$$\left| \frac{W(G, H)_{\mathcal{L}}}{W(\mathcal{L}, H)} \right|.$$

Theorem 5.2.1.

$$\widehat{\mathcal{O}_{\mathcal{L}}}|_{C'} = \left| \frac{W(G, H)_{\mathcal{L}}}{W(\mathcal{L}, H)} \right| \sum_{L \supset H, \ L \in \mathcal{L}} \frac{\pi_L}{\pi}.$$

Proof. By Proposition 3.1.9, we know

$$\lim_{t \to 0^+} \mathcal{O}_{t\xi}^G = \mathcal{O}_{\mathcal{L}} \text{ or } 0.$$

for any $\xi \in \mathfrak{g}^*$ such that $Z_G(\xi) \in \mathcal{L}$.

Fix such a $\xi \in \mathfrak{g}^*$ such that

$$L = Z_G(\xi) \cong \mathrm{GL}(q_1, \mathbb{R}) \times \cdots \times \mathrm{GL}(q_r, \mathbb{R}).$$

Then we can choose a fundamental Cartan and a labeling of roots in the last section so that $\mathfrak{h}_k \subset \mathfrak{l}$ is a fundamental Cartan and $\mathrm{Lie}(H) = \mathfrak{h}_l$ with $l \geq k$. Note

$$k = \sum \left\lfloor \frac{q_i}{2} \right\rfloor.$$

Choose positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}_l$ which satisfy the conditions of Theorem 5.2.1. This determines positive roots for $\mathfrak{g}$ with respect to $\mathfrak{h}_i$ for every i. Put

$$(C^*)' = \{\lambda \in (\mathfrak{h}^*)' | \ \langle i\lambda, \alpha^\vee \rangle < 0 \text{ for all } \alpha \in \Delta_L^+\}.$$

Then by Theorem 2.2.1,

$$\lim_{\lambda \to \xi, \ \lambda \in (C^*)'} \partial(\pi_L)|_\lambda \mathcal{O}_\lambda^G = i^{r(L,H)} |W(L,H)| \mathcal{O}_\xi^G.$$

And by Proposition 5.1.1,

$$\widehat{\mathcal{O}_\lambda^G}|_{C_l'(e)} = \frac{2^{l-k} \sum_{w \in W_{k,l}} \epsilon_{k,l}(w) e^{iw\lambda}}{\pi}.$$

Plugging this into the previous formula, we get

$$\widehat{\mathcal{O}_\xi^G}|_{C_l'(e)} = \frac{2^{l-k}}{|W(L,H)|} \frac{\sum_{w \in W_{k,l}} \epsilon_{k,l}(w) (w\pi_L) e^{iw\xi}}{\pi}.$$

Then

$$\widehat{\mathcal{O}_\mathcal{L}}|_{C_l'(e)} = \frac{2^{l-k}}{|W(L,H)|} \frac{\sum_{w \in W_{k,l}} \epsilon_{k,l}(w) (w\pi_L)}{\pi}$$

by Proposition 3.1.9.

Now, using parts (i) and (ii) of the definition of $W_{k,l}$, we deduce that if $w \in W_{k,l}$, then $w\pi_L = \pm \pi_{L'}$ for some Levi $L' \in \mathcal{L}$ with $L' \supset H$. Conversely, it is similarly not difficult to deduce that whenever $L' \in \mathcal{L}$ with $L' \supset H$, there exists $w \in W_{k,l}$ such that $w\pi_L = \pm\pi_{L'}$ from the definition of $W_{k,l}$. In fact, using part (iii) of the definition of $W_{k,l}$ together with condition (ii) for our choice of positive roots in Theorem 5.2.1, we see that $w\pi_L = \epsilon_{k,l}(w)\pi_{L'}$.

Combining these considerations, we get

$$\widehat{\mathcal{O}_{\mathcal{L}}}|_{C_l'(e)} = \frac{2^{l-k}}{|W(L,H)|} \sum_{L' \in \mathcal{L},\ L' \supset H} \#\{w \in W_{k,l} |\ wL = L'\} \frac{\pi_{L'}}{\pi}.$$

Finally, we use part (i) of the condition on our choice of positive roots, given at the beginning of this section together with the definitions of C' and $C_l'(e)$ to realize $C' = C_l'(e)$. And a simple counting argument shows

$$\#\{w \in W_{k,l} |\ wL = L'\} = \frac{|W(G,H)_L|}{2^{l-k}}.$$

The theorem follows.

$\square$

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