# Finding Optimal Strategies for Influencing Social Networks in Two Player Games

by

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## ABSTRACT

This thesis considers the problem of optimally influencing social networks in Afghanistan as part of ongoing counterinsurgency efforts. The social network is analyzed using a discrete time agent based model. Each agent has a belief [-0.5,0.5] and interacts stochastically pairwise with their neighbors. The network converges to a set of equilibrium beliefs in expectation. A 2-player game is formulated in which the players control a set of stubborn agents whose beliefs never change, and who wield significant influence in the network. Each player chooses how to connect their stubborn agents to maximally influence the network. Two different payoff functions are defined, and the pure Nash equilibrium strategy profiles are found in a series of test networks. Finding equilibrium strategy profiles can be difficult for large networks due to exponential increases in the strategy space but a simulated annealing heuristic is used to rapidly find equilibria using best response dynamics. We demonstrate through experimentation that the games formulated admit pure Nash equilibrium strategy profiles and that best response dynamics can be used to find them. We also test a scenario based on the author's experience in Afghanistan to show how non-symmetric equilibria can naturally emerge if each player weights the value of agents in the network differently.
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## TABLE OF CONTENTS

# List of Figures

# List of Tables

# List of Charts

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# 1 Introduction and Overview

Since 2001 the US military has been involved in operations in Iraq, Afghanistan, Somalia, and Libya. These conflicts are not conventional wars where armies fight each other with tanks and artillery, they are fought against insurgents with roadside bombs using guerilla tactics and propaganda campaigns. The US Army is increasingly involved in small conflicts against asymmetric foes that hide amongst the populace to protect themselves from the devastating weaponry of the US military. In order to effectively operate in these conflicts it is crucial for US forces to build ties with the people and win their support in order to deny the enemy the ability to hide amongst the populace.

A difficult question often faced by US Army commanders is how to optimally influence the populace. One of the major techniques used by Army commanders in Iraq and Afghanistan has been key leader engagement, where US forces regularly meet with local leadership to help them address their problems and win the support of the leader's constituents. One of the toughest questions that US commanders must answer is deciding which key leaders to engage. It is often unclear what impact talking to a leader may have. US forces do not have the time or resources to engage every leader available to them. Understanding the second and third order effects of talking to different leaders, and understanding which ones will lead to more effective influence in the populace is critical to ensure that US forces win the support of the populace. The problem motivation and further background is discussed in Chapter 2.

In this thesis a game theoretic approach is used to address the problem of choosing which key leaders to engage. To represent the social network we use a model initially formulated by Acemoglu et al[1] and expanded by Hung[2] in his thesis. The model is a discrete-time agent based model. Each agent has a scalar belief and interacts stochastically pairwise with their neighbors according to a set of influence parameters that are in input to the model. Agents in the model have different levels of influence, and there are also stubborn agents, which have extreme beliefs, and never change this belief. In our context stubborn agents represent the Taliban and US forces in the network. Both try to influence the rest of the network to adopt their extreme beliefs. These stubborn agents also wield the most influence in the network. The presence of stubborn agents and the stochastic interactions lead to the beliefs of the non-stubborn (mutable) agents being a random variable. Thus the mutable agent's beliefs do not converge to any specific set of beliefs. However, their beliefs do converge in expectation asymptotically, and this property is used to analyze the state of the network at some sufficiently large time in the future. This set of equilibrium beliefs turns out to be independent of the initial state of the network, and depends only on the influence parameters and topology of the network. The social networking model is discussed in greater detail in Sections 3.2 and 3.3.

On top of this social network architecture we formulate a perfect information symmetric two player game where a US player and Taliban player each try to find optimal strategies for influencing social networks. Strategies are defined as the choice of where to connect their stubborn agents to the rest of the network. The players want to connect their stubborn agents to the most influential agents in the networks, but finding the most influential agent is not always an easy task. We define a set of two payoff functions of the equilibrium beliefs of the network and then find the pure Nash equilibria using combinations of these payoff functions. The game and its payoff functions are defined in Section 3.4.

The game has pure Nash equilibria in all test networks used (see Section 4.1). These initial test cases give each player only one strategy choice (each player has one stubborn agent that can connect to only one agent in the network), and all of the equilibria are found by explicitly calculating the entire payoff matrix. However, if players have more than one strategy choice (if they can influence more than one agent at a time by creating multiple connections from their stubborn agents to the rest of the network), the strategy space increases exponentially, and the payoff matrix becomes difficult to enumerate.

To solve this problem a simulated annealing heuristic is used to conduct best response dynamics to search through the payoff matrices without fully enumerating them. The heuristic is defined in Section 3.5. This approach allows optimal best response strategies and pure Nash equilibria to be found in large networks with multiple connections allowed per player. The method also finds better strategies than Hung was able to find using the Knitro solver on a math program he formulated. It also finds better strategies than an experienced US Army officer who has dealt with this problem many times in combat solving this problem using US Army targeting doctrine. The simulated annealing heuristic is shown through experimentation to be able to find pure Nash equilibrium strategy profiles in a network of up to 519 agents, with each player making 6 connections to the network (resulting in $519^{12}$ possible strategy profiles) in a few hours (Matlab was used to implement the heuristic – faster times may be possible if it were implemented in a faster programming environment like Java or C++). The experimental results are shown in Sections 4.3 and 4.5.

Sensitivity analysis conducted in Section 4.4 also shows that the model is robust to changes in the influence parameters than affect how agents interact. Significant changes in these influence parameters do not change the pure Nash equilibria in many cases.

In the real world players often want to choose strategies with low variability. Especially when considering the problem of insurgencies, the stakes of losing the support of the populace are extremely high for both the US forces and the Taliban, and both players would like to be able to minimize the variance their strategies create in the beliefs of the network. Different strategy profiles used in influencing the network not only change the equilibrium beliefs, but also change the variance of the random variables corresponding to each agent's belief. In this thesis a potential candidate for characterizing the variance of the mean belief of the mutable agents is analyzed, but unfortunately does not work for our model (see Section 3.3 for a definition of the method and Section 4.2 for experimental results). However, in Section 4.5 we analyze a scenario based on real world experience that shows that if players have different objectives in influencing the network, pure Nash equilibria with low variance can be naturally achieved.

One weakness of the approach outlined here is that the state of the social network is only analyzed at equilibrium, which is independent of any transient state of the network. This means that players formulate strategies based without regard to the current beliefs of the network. This was done in order to make the problem tractable for our analysis methods because when players consider the current state of the network, the payoff matrix becomes exponentially large as the size of the network increases. This assumption makes the solutions less realistic than we would like, because in the real world US military forces and insurgents both care what people's current beliefs are when determining how to influence the populace.

Counterinsurgencies are not the only area where opposing sides are trying to influence social networks. Competing companies vying for customers both try to influence customers through advertising campaigns. Political candidates influencing voters are another natural example of players working against each other while influencing a social network. These are just two simple examples of the many social networks in the world that the game theoretic approach to influencing social networks can be applied to find optimal strategies.

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# 2 – Identifying Key Influencers to Win Popular Support in Counterinsurgency

As of May 2011, the US is currently involved in major military campaigns in Iraq, Afghanistan, and Libya. The aim of these operations is to establish functional democratic societies. Prior to these conflicts the US military was involved in lengthy peacekeeping and counterinsurgency operations in Haiti, Somalia, Bosnia, Macedonia, and Kosovo. As the US military looks to the future, it expects to continue to operate in these types of conflicts. In 2010 the United State Department of Defense conducted it's quadrennial defense review (QDR), discussing its planning priorities and expectations for the future. In this report they said the following:

> … the U.S. Armed Forces will continue to require capabilities to create a secure environment in fragile states in support of local authorities and, if necessary, to support civil authorities in providing essential government services, restoring emergency infrastructure, and supplying humanitarian relief… Nor are these types of operations a transitory or anomalous phenomenon in the security landscape. On the contrary, we must expect that for the indefinite future, violent extremist groups, with or without state sponsorship, will continue to foment instability and challenge U.S. and allied interests.
> *Excerpt from the 2010 Quadrennial Defense Review*[3]

For the US military to operate successfully in its missions, it will need to continue to improve and hone its counterinsurgency and peacekeeping skills.

## 2.1 – Insurgencies and Counterinsurgencies

The US Army defines an insurgency as "…an organized, protracted politico-military struggle designed to weaken the control and legitimacy of an established government, occupying power, or other political authority while increasing insurgent control[4]." Insurgents attempt to overthrow the government or existing power in many ways. When they conduct attacks against the government or civilians, the insurgents demonstrate their own strength while highlighting the inability of the government to maintain control. Insurgents also cause disruptions through aggressive information operations and propaganda campaigns to de-legitimize the government and spread their own ideology. Both of these are dimensions to how insurgents work to influence the population to gain popular support in their efforts to overthrow the government.

The US Army defines counterinsurgency as "…military, paramilitary, political, economic, psychological, and civic actions taken by a government to defeat insurgency[4]." Counterinsurgencies are undertaken by existing government forces or sometimes foreign soldiers to assist an ailing regime (as in the case of all US counterinsurgencies). Counterinsurgents try to simultaneously destroy the insurgents (literally and figuratively), while maintaining support among the populace. Their methods include traditional military tactics against insurgent strongholds such as bombings, artillery strikes, and raids. However, this also includes various public outreach efforts. A critical objective the counterinsurgent must achieve is to protect the people from the insurgent's attacks. The counterinsurgency can only be won with the support of the people, and not through killing insurgents alone[4].

## 2.2 – Popular Support in Insurgencies and Counterinsurgencies

In his book *The Logic of Violence in Civil War*, Kalyvas[5] defines popular support as either 1) a confluence of attitudes, preferences, and allegiances, or 2) a set of observable behaviors (attitudes and actions). Although these two sound similar, Kalyvas' research has shown that they are not the same. People with attitudes supporting an insurgency may not be willing to openly support them, and thus there are no observable behaviors. However, he does state that attitudes affect behaviors, and are thus targeted by both the insurgent and counterinsurgent.

For our analysis we subscribe to this idea that insurgents and counterinsurgents focus on changing people's attitudes, understanding that it may not lead immediately to action on the part of the individuals. However, the more that the population's attitudes support one side or the other, the more likely we are to see actions in terms of giving support and aid to one side or the other.

The methods that each side employs to win popular support can differ significantly. For insurgents, leaving threatening letters on doorsteps at night (night letters) are an easy way to intimidate people. Killings and assassinations against pro-government figures can be used to intimidate a populace that is not responding to propaganda or night letters. Spectacular attacks are effective at making evening headlines around the world and demonstrating the impotence of the government. As they gain control of areas, insurgents set up local governments to provide rule of law and basic services to the populace, showcasing the insurgents' ability to hold de facto control over the country.

Counterinsurgents use targeted operations against insurgents strongholds to degrade their ability to conduct operations. In addition to military operations, the counterinsurgent focuses on maintaining the support of the populace, and often undertakes activities to increase the quality of life in key areas. For example, increasing access to water or roads in insurgent strongholds can serve to win the populace's support back to the government.

To develop public support, both sides rely on direct contact with local leaders. In Afghanistan, Iraq, and other tribal societies, there are often tribal power structures operating outside of traditional government. In these tribal societies, millions of dollars used to build a new road may not be as effective at winning support as simply having tea and engaging with local elders over the course of a few months[6]. Consequently, the insurgent and counterinsurgent both focus heavily on identifying and influencing these key individuals.

Identifying the key influencers in the communities can be a difficult task, but lessons learned by the US Army in Iraq and Afghanistan help to address this. In tribal societies, most of the key people who wield power are well known among the populace. Consequently, focused patrols asking the right questions of several people in an area can usually determine who the power brokers are. Specifically, the Tactical Conflict Assessment Framework (TCAF) developed by the United States Agency for International Development (USAID) was implemented with success in Iraq and Afghanistan. TCAF consists of a series of questions used by patrolling soldiers to determine the needs and desires of the local populace are. (Appendix F contains a TCAF questionnaire). Given knowledge of the structure of the society and its key influencers, a significant problem still remains for both the insurgent and counterinsurgent. Who amongst the key influencers should be targeted? In the US Army, this analysis is referred to as non-lethal targeting.

## 2.3 – US Army Non-Lethal Targeting

The Army's methods for targeting local leaders have grown primarily from experience. In the US Army's two primary manuals on counterinsurgency[1], several pages cover how to set up a "targeting group", along with guidance as to what they should produce and their focus should be. At the company level in the Army the targeting group generally consists of the company Fire Support Officer, the Executive Officer, and the senior intelligence analyst. The Army states that the purpose of the targeting group is to "prioritize targets and determine the means of engaging them that best supports the commander's intent and operation plan. The focus of the targeting cell, in a counterinsurgency environment, is to target people, both the insurgents and the population.[7]" However, there is a lack of content on how they should go about selecting targets. In more than 200 pages of doctrine, the two primary field manuals on counterinsurgency for the Army (FM 3-24[4] and FM 3-24.2[7]) states only the following:

- Identify leaders who influence the people at the local, regional, and national levels.
- Win over passive or neutral people.
- Non-lethal targets include people like community leaders and those insurgents who should be engaged through outreach, negotiation, meetings, and other interaction.
- Meetings conducted by leaders with key communicators, civilian leaders, or others whose perceptions, decisions, and actions will affect mission accomplishment can be critical to mission success.
- Start easy… Don't try to crack the hardest nut first—don't go straight for the main insurgent stronghold, try to provoke a decisive showdown, or focus efforts on villages that support the insurgents. Instead, start from secure areas and work gradually outwards. Do this by extending your influence through the locals' own networks.

These two manuals combined spend more than 30 pages on targeting without discussing specifically how a unit should determine which local leaders to target. This is partly because targeting involves many local variables that are difficult to encapsulate in a manual. Cultural aspects and local history can have an enormous impact on power structures. Identifying these facts and determining not only who should be influenced, but also how this will impact society as a whole can be extremely difficult to predict.

Although doctrine explains that this analysis to identify influencers should be conducted, and warns about the difficulties of the task, it provides little concrete guidance or tools to aid the commander and staff in planning operations. How well a unit performs its non-lethal targeting is

[1] Field Manual 3-24: Counterinsurgency, and Field Manual 3-24.2: Tactics in Counterinsurgency

based largely on the experience and talents of the individuals performing both the targeting and analysis.

Most of the information leading to non-lethal targeting comes from patrols. The reason is that soldiers and leaders on patrols routinely ask questions to better understand the local culture, community structure, and environment. As a unit's understanding of the environment grows, it hears certain names repeated many times, indicating these individuals are local power brokers. However, simply because someone has power does not indicate that they should be targeted. To illustrate this, we give a short example from the author's experience.

> While serving as a company commander in Afghanistan in 2008, one of our lieutenants found out that there was a person with a small militia that was exercising control over a very unstable part of our area of operations. The area had been one of the most violent in our area, and then suddenly quieted down one day. We learned that this militia commander had personally interceded and killed several opposing tribal elders. We then also learned that the militia commander had been a mujahedeen commander in the Soviet-Afghan jihad, and owned several heavy weapons and anti-tank rockets, which are illegal to own in Afghanistan. It was troubling to learn that a well armed and trained militia force had been in our area of operations without our knowledge. We considered disarming them, but also thought about conducting an outreach to this very well armed and potentially dangerous individual to try to build ties with someone who had become a clear power broker in our area. After much deliberation and consideration, we decided that disarming him may simply lead to counterattacks, renewing the cycle of violence. We also decided not to engage him simply because we were uncertain of the tribal dynamics, and we didn't want to be seen as supporting him. We also did not know if we began to support him whether it might embolden him, or upset the tribes he had attacked. The resolution we came to was to simply wait and watch this commander and be happy the violence had stopped in the area. Whether this was the best decision is still unknown.

In addition to the difficulties in analyzing each person that a unit may want to target, US Army forces are quite constrained in their ability to conduct influencing operations. This stems from two issues: the low density of US troops in Afghanistan, and the requirement for troops to move in large groups for security. In Afghanistan it is not uncommon for a single infantry company of 140 soldiers to have an area of operations nearly 50% the size of Rhode Island (around 1500 square km), with anywhere from 60,000 to 100,000 Afghan citizens. Although the company has over a hundred soldiers, each patrol takes 20 or more soldiers. Additionally, the company must guard their bases, cook its food, fix its vehicles, and run an operations center. As a result, the company can sustain at most three patrols a day. The company has multiple requirements like escorting logistics convoys, visiting ongoing projects, delivering humanitarian assistance, and conducting maintenance each week. These may easily can consume half of a company's patrol capacity, leaving only a few missions for engaging local leaders. As a company commander in Afghanistan we could make at most three meetings with sub-governors (government leader for a district) or tribal elders each week. Each of our three lieutenants may have been able to make another two engagements each week. This means that for a company of over a hundred people, only about 10 local Afghans out of a population of tens of thousands spread over more than a thousand square kilometers can be regularly engaged and influenced. This constraint means that focusing on influencing the right people is critical to mission success.

The goal of this thesis is to develop a more refined model for the populace of a rural Afghan district, and develop decision aid tools to help commanders decide their non-lethal targets. Specifically it seeks to examine the $2^{nd}$ and $3^{rd}$ order effects of targeting one key individual over another, as well as the impact on the population's attitudes over time. If we had chosen to influence the militia commander in our area of Afghanistan, what might have happened? If I had a better understanding of the tribal dynamics at play there, then using the tools developed here I could have conducted an analysis to help decide whether engaging him, arresting him, or ignoring him would be the best solution. What might I have been able to accomplish with better analysis tools?

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# 3 – Modeling Approach and Formulation

Chapter 3 begins with an overview of previous work done in modeling societal beliefs. Next we discuss our model developed by Hung and Acemoglu et al. We then discuss analysis methods for determining expected belief propagation in the stochastic network, and discuss a potential method for characterizing the variance in the network's average belief over time. Finally we look at formulating a two player game on the network, in which the Taliban insurgents and the US forces are both trying to optimally influence the beliefs of the populace modeled by the network.

## 3.1 – Previous Work

Before discussing our methods we look at previous work done in the field leading to the development of our model.

### 3.1.1 – Social Science Literature on Opinion Leaders and Influential people

Katz and Lazerfeld[8] first hypothesized in 1955 that small numbers of influential opinion leaders play critical roles in determining societal consensus and public opinion. They stated that these key individuals wield significant influence over the people around them. Information from mass media and other external sources are filtered, dampened, and modified by these opinion leaders within the social network. They developed a 'two-step' model explaining the diffusion of messages from mass media through these opinion leaders to the general public. This model helped explain why some media campaigns did not affect all groups in the same way, and raised the idea of focusing media campaigns on opinion leaders to influence the rest of the populace. They also said that these opinion leaders are not necessarily public figures or traditional hierarchical leaders, but are often less visible individuals (like family members, co-workers, or neighbors).

Many people have since offered improvements to Katz and Lazerfeld's model to increase its utility, suggesting ways to help identify these opinion leaders in society. Weimann[9] postulated a new method for identifying the opinion leaders in society, saying that leaders tend to be in higher social strata than non-influential people. His research indicated that influence should be measured on a scale, as people cannot simply be categorized as 'influential' or 'non-influential'. Emerson[10] thought that although people with 'legitimate' authority exist in society, their power is not the key variable but rather their relationship with others. It is these connections between people that indicate how powerful an opinion leader is, and thus the amount of influence they wield differs with each of their neighbors. For example, a CEO of a major corporation has a lot of influence over their workers, but the CEO may have little sway in changing their own parents' minds. Lazer[11] observed that because people tend to associate with others like themselves, they form self-organizing networks consisting of like-minded people in closely connected communities, rather than waiting passively for other people to influence them.

### 3.1.2 – Opinion Dynamics Literature

While social scientists have formulated models for individual interactions and general processes for how groups reach consensus, other researchers have formulated mathematical tools for analyzing groups and beliefs. In 1964 Abelson[12] formulated a simple network model where interactions occur pairwise between all adjacent agents in the network. The result of each interaction is a function of the two individual's beliefs and each person's persuasiveness. These relationships are formulated as a series of differential equations whose solution yields a set of equilibrium beliefs (or societal consensus). He also identified a problem that these types of models have – over time they always lead to a consensus in belief which is not generally true in the real world. Degroot[13] later formulated a similar model using Markov chains in which each individual gives different weights to the opinions of their neighbors. Using Markov chain theory, the steady state probabilities of each individual's belief can be calculated (assuming the Markov chain is recurrent and aperiodic)[14]. Similar to Abelson, DeGroot's model leads to a consensus in belief where all individuals have identical beliefs. To address this problem in existing models, Friedkin and Johnson[15] created their theory of social influence which uses a system of linear equations to model how equilibrium is reached, but beliefs among actors are not necessarily identical in this equilibrium. Another model with non-consensus beliefs was given by Deffuant et al[16]. They proposed a model of confidence intervals for each agent in the network, wherein some agents have uncertain beliefs, and others are very certain in their beliefs. He studied the mixing of these beliefs and the establishment of equilibrium in the network. Later Deffuant extended the model by analyzing the effect of extreme agents, whose beliefs are completely certain and thus rarely change. He showed that depending on the network parameters and structure, extremists can have a polarizing effect on networks.

### 3.1.3 – Development of Our Model

More recently, Acemoglu et al[1] have proposed a stochastic agent-based model for analyzing dynamics beliefs in a network. Their model includes 'normal' and 'forceful' agents. Interactions between neighboring agents occur pairwise randomly according to a series of parameters, with forceful agents more likely to spread their beliefs than other agents. The belief of each agent is a random variable, but the beliefs of agents in the network eventually converge to a convex combination of the initial belief states. Acemoglu et al developed methods for characterizing the expectation of this set of beliefs that all agents converge to. Hung[2] made modifications to the model to include a set of 'very forceful' agents (who are even more likely to spread their beliefs than forceful agents), and a set of 'stubborn' agents each of whose beliefs never change. These stubborn agents also act as the most forceful types of agents in the model, propagating their beliefs throughout the network. In his model, the beliefs of agents are still a random variable, but they never converge to a single value due to the presence of 'stubborn' agents. Using the tools developed by Acemoglu et al, Hung showed that the random variables for each agent's belief have a well-defined first moment. Furthermore this first moment converges to an equilibrium value over time (meaning the expected state of each agent's belief converges to a fixed value over time). Finally, although the actual beliefs of agents in simulation don't converge to a fixed value (they converge only in expectation), beliefs do converge to a type of stochastic equilibrium in which the average of all agent beliefs oscillates around the expectation. Recently, Yildiz et al[17] showed that the variance of this oscillation around the expected mean belief can be tightly bounded in a model that has only binary beliefs. In Chapter 4 we conduct experimentation that shows this bound unfortunately has limited utility in our continuous belief model, as it fails to consistently predict the actual standard deviation in our model.

Hung's model maintains key features of Acemoglu et al's model and addresses several key limitations of previous social network models. First – the model uses the standard concept of pairwise interaction from the successful social science models. Second – the beliefs of agents do not converge to a set of fixed beliefs due to the model's randomness, which provides for more realism than many previous models. Third – although individual beliefs don't converge to a fixed value, they do converge in expectation, which allows us to analyze the expected spread of beliefs from stubborn agents to the general populace.

Beyond development of this social networking model, Hung's thesis makes two other important contributions. Hung created a network generator tool, which takes a series of simple inputs to generate homophilly based networks that resemble real tribal social networks in Afghanistan. The tool was the product of a significant amount of research in collaboration with the MIT Political Science Department, and will be used in this thesis to help rapidly create realistic Pashtun tribal networks for testing and analysis.

Hung also considered a network in which stubborn agents are attempting to maximally influence the network, and formulated a mathematical program to find an optimal strategy representing the connections that stubborn agents should make within the network to best influence people. This is a non-linear, non-convex, and mixed integer formulation. The solver used (Knitro) returned local optima even with significantly long run times (we know they are local because we are able to find better solutions using our Simulated Annealing algorithm in Chapter 4). However, the solutions Hung found to this formulation provide good policies for a set of stubborn agents to influence a network, and was an important first step in analyzing the problem of optimally influencing a stochastic social network.

## 3.2 – Network Model

The model initially formulated by Hung is designed to model a district in Afghanistan. A district generally consists of 10-20 villages of 100 or more households each. There were several reasons for this scope of analysis. First Hung's motivation was based on solving a problem faced by the US Army in Afghanistan of optimally influencing districts in Pashtun regions. Second, it was generally unknown at the time whether sufficient data could be reasonably gathered to model more than one district. Third, the optimization tools used by Hung did not scale well with the size of the network, and thus he could not analyze networks of more than a hundred agents sufficiently fast. Finally, it was believed that most rural Pashtun society beliefs within a district are highly independent of other districts because district boundaries follow tribal and social fault lines. Thus analysis of larger areas could be done a single district at a time.

In Chapter 4 we show that the basic network structure and tools developed here can be used to analyze much larger networks (hundreds of agents, corresponding to more than ten thousand individuals). However, most of the analysis stays within Hung's scope of multi-village and single district size since most of the data we have fits such networks.

For the reader's convenience all of the notation and definitions we introduce in Chapter 3 are also located in Appendix A.

## 3.2.1 – Agents in the Network

Each node in the network is called an agent, which represents a person (or persons). Arcs between agents represent relationships between them. The overall structure of the network is not dynamic, which means that once the network topology is set it will not change. One exception is the connections to and from stubborn agents that are controlled by the players as described later in the chapter.

### 3.2.1.1 – Types of Agents

Every agent in the model is characterized by a scalar belief between -0.5 and +0.5. This indicates how pro-US or pro-Taliban the agent is (see Figure 3.1).

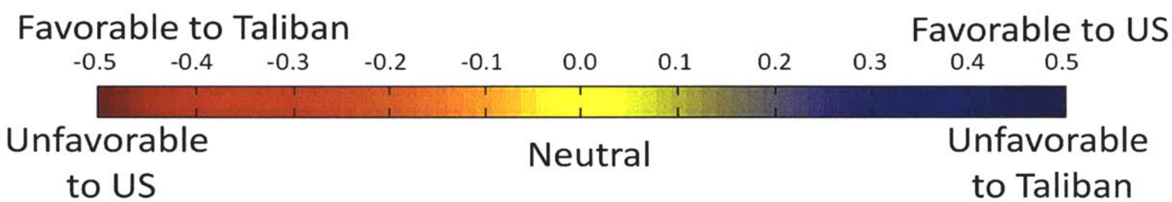


*Figure 3.1 – Belief Scale for Agents*

In addition to their individual beliefs, agents have a level of influence associated with them that can be one of four possible types:

**Regular**

A 'regular' agent represents a head of household. In rural Pashtun society, a household is generally defined as 2 or more generations of a family living together under the same roof, and can comprise up to 40 individuals[18]. Each of these has a head of household that is typically the eldest male. The head of household speaks for their family within village and tribal disputes. Although there is undoubtedly some disagreement on important issues within a household, because the head of the household is the only one voiced in important meetings, it is used as the smallest unit of analysis in our model. Thus each 'regular' agent in the network actually represents 20-40 individuals in a village.

### Influential

An 'influential' agent is a village leader that wields more influence than a normal head of house within their village. Examples include a wealthy merchant, a local religious figure, or a member of the district Shura (decision making/advising council in rural Pashtun society). 'Influential' agents are more likely to spread their beliefs to other agents than 'regular' agents are.

### Very Influential

A 'very influential' agent represents a district leader who wields influence outside of just their immediate village. Examples include district religious figures, government figures, or the district chief of police. 'Very Influential' agents have a higher probability of spreading their beliefs than 'Influential' or 'Regular' agents.

### Stubborn

'Stubborn' agents are individuals whose opinions never change. Their objective is to sway the population to their belief as much as possible. They are the key players in the model, and their beliefs and how they propagate in the network is the key part of the model analysis. Examples of 'stubborn' agents in our context are either Taliban insurgents or United States forces. It is highly unlikely that either of them is going to suddenly decide that the opposing side is right and then lay down their arms and quit. It is also assumed that 'stubborn' agents wield the most influence in the network and have the highest probabilities of influencing other agents.

All stubborn agents have extreme beliefs, and their belief is either -0.5, or +0.5. All other agent types have mutable beliefs that change over time as described below in Section 3.2.1.2. Agents belong to one or more of the following sets within the network:

$V_{TB}$ – Set of Stubborn agents with belief $-0.5$
$V_{US}$ – Set of Stubborn agents with belief $+0.5$
$V_S$ – Set of all Stubborn agents $= V_{TB} \cup V_{US}$

$V_R$ – Set of Regular Agents
$V_I$ – Set of Influential Agents

$V_V$ – Set of Very Influential Agents
$V_M$ – Set of mutable agents $= V_R \cup V_I \cup V_V$

$$V - \text{all agents} = V_M \cup V_S = V_R \cup V_I \cup V_V \cup V_{TB} \cup V_{US}$$

In Figure 3.2 we illustrate the ideas and definitions presented above.

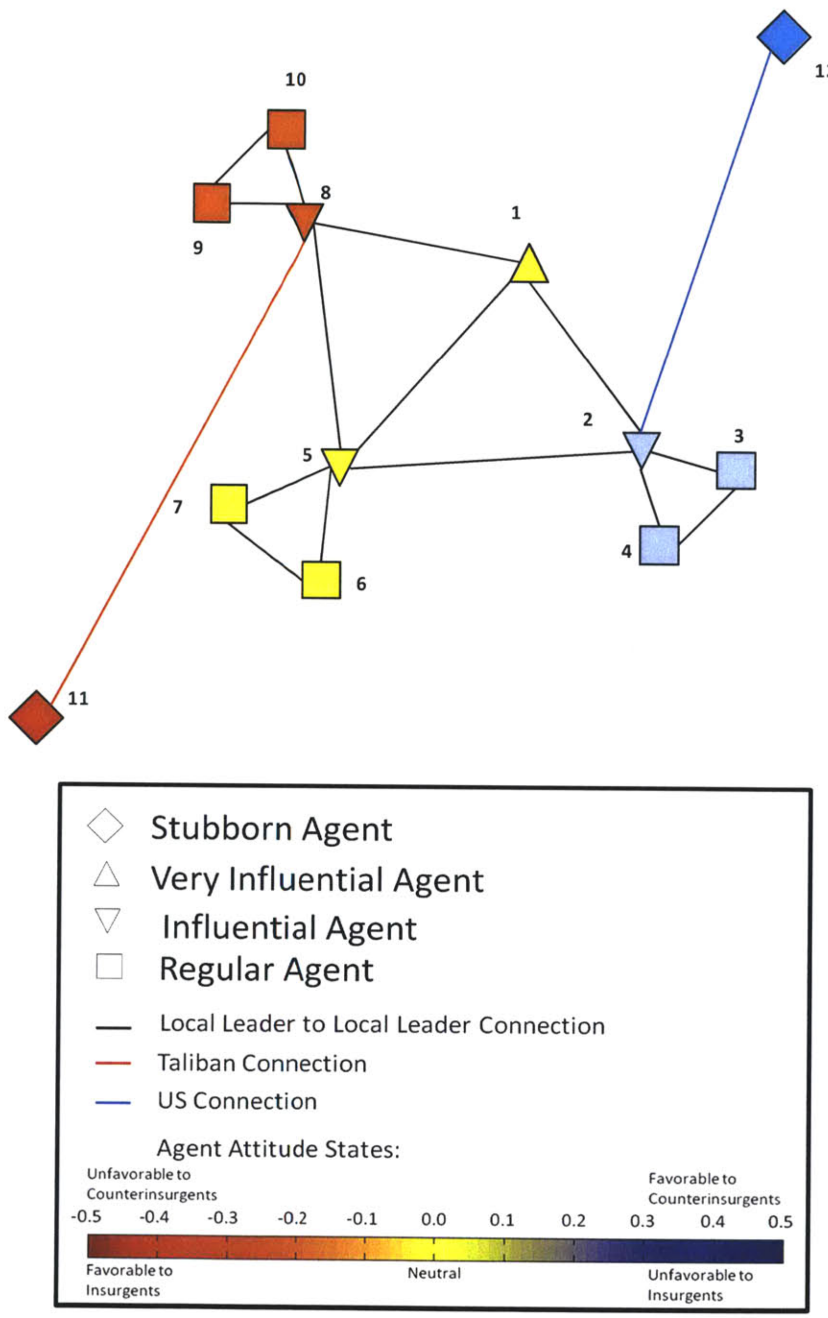


*Figure 3.2- Sample Network and Description of Network Symbology*

Table 3.1 shows the agent sets represented in the above network.

| Set: | $V_{TB}$ | $V_{US}$ | $V_S$ | $V_R$ | $V_I$ | $V_V$ | $V_M$ | V |
|---|---|---|---|---|---|---|---|---|
| Nodes in Set: | 11 | 12 | 11,12 | 3,4,6,7,9,10 | 2,5,8 | 1 | 1-10 | 1-12 |

*Table 3.1 – Node Set in Example Network*

Figure 3.2 shows a network of 3 villages with 2 households each. Every village has an 'influential' leader, and between the three villages there is a district leader that is 'very influential'. The two stubborn agents represent a US agent and a Taliban agent, each trying to sway different villages to side with them.

### 3.2.1.2 – Agent Interactions

The active agent is chosen according to a set of Poisson arrival processes. Every agent has an identical Poisson process with rate 1, and when an arrival occurs in an agent's process they become the active agent. Once active, the agent randomly selects a neighbor and interacts with them as described below. In our network we define time step t as the time of the t-th interaction on the network. With time steps servings as an index for interactions, we know exactly one arrival has occurred at each time step. We assume that all agents have identical Poisson processes (although this assumption could be relaxed), so conditioned on an arrival occurring somewhere in the network, the arrival is equally likely to have occurred in any agent's Poisson process.

$$\begin{aligned} &\mathrm{P(arrival\ occurred\ for\ agent\ i|an\ arrival\ occurred\ in\ the\ network)} \\ &\qquad = \mathrm{P(arrival\ occurred\ in\ agent\ j|an\ arrival\ occurred\ in\ the\ network)}\ \ \forall i,j \in V \end{aligned}$$

$$\text{therefore:}\quad \mathrm{P(arrival\ occurred\ in\ agent\ i|an\ arrival\ occurred\ in\ the\ network)} = \frac{1}{n}$$

$$\textit{where n is the number of agents in the network}$$

The belief of agent i at time step t is denoted as:

$$X_i(t) \in [-0.5, 0.5]$$

The beliefs of the n agents in the network at time t is denoted as an n x 1 vector:

$$X(t) = (X_1(t), X_2(t), \dots, X_n(t))$$

Once agent i has been selected, it then selects a random neighbor j. Hung's model allowed for a non-uniform distribution for the selection of neighbors, but we have omitted it here to simplify the model. One of the primary motivations for non-uniform interactions would be to better simulate 'closeness' among agents (e.g. people talk to their immediate family members and close friends more often than their village leader). Lacking any sufficient data to inform this distribution, we use a uniform distribution for our analysis.

Once a neighbor j has been selected to interact with, a pairwise interaction between agents i and j occurs in one of the following three ways:

**Averaging (β-type interaction):**

With probability $\beta_{ij}$, they reach a consensus equal to the average of their prior attitudes:

$$X_i(t+1) = X_j(t+1) = \frac{X_i(t) + X_j(t)}{2} \tag{3.1}$$

**Forceful (α-type interaction):**

With probability $\alpha_{ij}$, agent i 'forcefully' imparts $\left(1 - \varepsilon_{ij}\right)$ of its attitude on agent j:

$$\begin{aligned} X_i(t+1) &= X_i(t) \\ X_j(t+1) &= \varepsilon_{ij} \cdot X_j(t) + \left(1 - \varepsilon_{ij}\right) \cdot X_i(t) \end{aligned} \qquad 0 \le \varepsilon_{ij} \le 0.5 \tag{3.2}$$

$$\textit{where } \varepsilon_{ij} \textit{ represents the stubborness for agent } j \textit{ interacting with agent } i$$

In 'forceful' interactions the parameter $\varepsilon_{ij}$ is a rating of stubbornness for each agent. This parameter represents the amount of their own belief an agent will retain after being forcefully influenced by another agent. For simplicity this is assumed to be identical for all agent pairs.

$$\varepsilon_{ij} = \varepsilon \quad \forall i, j \in \{1,2, \ldots, n\}$$

**Identity (γ-type interaction):**

With probability $\gamma_{ij}$, both agents exhibit no change in attitude:

$$\begin{aligned} X_i(t+1) &= X_i(t) \\ X_j(t+1) &= X_j(t) \end{aligned} \tag{3.3}$$

$$\textit{where } \beta_{ij} + \alpha_{ij} + \gamma_{ij} = 1$$

We assume that within our model, all $\alpha_{ij}$, $\beta_{ij}$, $\gamma_{ij}$, are the same between different classes of agents. For example, all regular agents interacting with other regular agents have the same $\alpha_{ij}$, $\beta_{ij}$, $\gamma_{ij}$. Similarly, any time a stubborn agent i interacts with some influential agent j, those interactions have identical $\alpha_{ij}$, $\beta_{ij}$, $\gamma_{ij}$. However, if an influential agent i interacts with some stubborn agent j there is a different set of $\alpha_{ij}$, $\beta_{ij}$, $\gamma_{ij}$ (order matters). By assuming that $\alpha_{ij}$, $\beta_{ij}$, $\gamma_{ij}$, are the same between different classes of agents it simplifies the data requirements of the model. It is unclear whether the absence of this level of data has a large impact on the model or not. In Section 4.4 we show that inaccuracies in $\alpha_{ij}$, $\beta_{ij}$, $\gamma_{ij}$ do not have a significant impact on optimal strategies, but whether changing from a class-wise set of influence parameter to pair-wise influence parameters would change this is currently unknown. Even with this simplification there are still 16 different values of $\alpha_{ij}$, $\beta_{ij}$, and $\gamma_{ij}$. (4 types of agents that can each interact with 4 other types of agents).

### 3.2.2 – Belief Propagation with a Matrix-Vector Model

As mentioned previously, the belief state of all agents in the network at time t is represented by the n x 1 vector X(t). We now show that the stochastic evolution of these beliefs can be represented with a random matrix W:

$$X(t+1) = W * X(t) \tag{3.4}$$

Where W is a random stochastic matrix that can take on one of three different forms, based on whether an $\alpha$-type interaction, a $\beta$-type interaction, or a $\gamma$-type interaction occurs between agents i and j.

$$W = \begin{cases} A_{ij} & \text{if a } \beta - \text{type interaction occurs between agents i and j} \\ J_{ij} & \text{if an } \alpha - \text{type interaction occurs between agents i and j} \\ I & \text{if a } \gamma - \text{type interaction occurs between agents i and j} \end{cases} \tag{3.5}$$

If a $\gamma$-type interaction occurs, then W becomes the identity matrix – all agents retain their current beliefs. To better illustrate the other types of interactions we consider a network of 5 agents that are all connected to each other. Consider if agent 1 and agent 2 had a $\beta$-type interaction (they average their beliefs):

$$X(t+1) = \begin{bmatrix} \frac{X_1(t)+X_2(t)}{2} \\ \frac{X_1(t)+X_2(t)}{2} \\ X_3(t) \\ X_4(t) \\ X_5(t) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} * X(t)$$

$$\text{Thus W} = A_{12} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This can be also written as:

$$A_{12} = \mathrm{I} - \frac{\left(\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}\right)\left(\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}\right)'}{2} = \mathrm{I} - \frac{(\mathrm{e}_1 - \mathrm{e}_2)(\mathrm{e}_1 - \mathrm{e}_2)'}{2}$$

*DEFINITION: $e_i$ is a unit vector of length n, with entry i equal to 1 and all others equal to zero.*

In general when a β-type interaction occurs, W=$A_{ij}$, where $A_{ij}$ is:

$$\mathrm{A_{ij}} \equiv \mathrm{I} - \frac{(\mathrm{e_i}-\mathrm{e_j})(\mathrm{e_i}-\mathrm{e_j})'}{2} \qquad \forall \mathrm{i,j} \in \{1,2,\ldots \mathrm{n}\} \tag{3.6}$$

This only occurs if agent i is selected as the active agent, neighbor j is selected to interact with, and a β-type interaction occurs:

$$\mathrm{P(W = A_{ij}) = P(agent\ i\ selected) * P(neighbor\ j\ selected) * P(\beta - type\ interaction)}$$

Recall that the probability of any agent being selected as the active agent is equal:

$$\mathrm{P(agent\ i\ is\ selected)} = \frac{1}{\mathrm{n}} \qquad \forall \mathrm{i} \in \{1,2,\ldots \mathrm{n}\}$$

Once an agent is selected, it then chooses a neighbor uniformly:

$$\mathrm{P(any\ neighbor\ is\ chosen|agent\ i\ is\ selected)} = \frac{1}{|\mathrm{E_i}|} \qquad \forall \mathrm{i} \in \{1,2,\ldots \mathrm{n}\}$$

*DEFINITION: $E_i$ is the set of agents adjacent to agent i.*

Thus, the probability that W takes on value $A_{ij}$ is:

$$P(W = A_{ij}) = \frac{\beta_{ij}}{n * |E_i|} \qquad \forall i, j \in \{1,2, \dots n\} \tag{3.7}$$

Next we consider the structure of the matrix $J_{ij}$ for an α-type interaction for this same 5-agent network. Assume that agent 1 has forcefully influenced agent 2.

$$X(t+1) = \begin{bmatrix} X_1(t) \\ (1-\varepsilon)X_1(t) + \varepsilon X_2(t) \\ X_3(t) \\ X_4(t) \\ X_5(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-\varepsilon & \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} * X(t)$$

$$W = J_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-\varepsilon & \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -(1-\varepsilon) & 1-\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_{12} = I - (1-\varepsilon)e_2(e_2 - e_1)'$$

In general:

$$J_{ij} \equiv I - (1-\varepsilon)e_j(e_j - e_i)' \qquad \forall i, j \in \{1,2, \dots n\} \tag{3.8}$$

The probability that W takes on value $J_{ij}$ is:

$$P(W = J_{ij}) = P(\text{agent i selected}) * P(\text{neighbor j selected}) * P(\alpha - \text{type interaction})$$

$$P(W = J_{ij}) = \frac{\alpha_{ij}}{n * |E_i|} \qquad \forall i, i \in \{1,2, \dots n\} \tag{3.9}$$

We now combine (3.5), (3.6), (3.7), (3.8), and (3.9) to define the random matrix W as the following:

$$W = \begin{cases} A_{ij} \equiv I - \dfrac{(e_i - e_j)(e_i - e_j)'}{2} & \text{w.p.} \quad \dfrac{\beta_{ij}}{n * |E_i|} \\ J_{ij} \equiv I - (1-\epsilon)e_i(e_i - e_j)' & \text{w.p.} \quad \dfrac{\alpha_{ij}}{n * |E_i|} \\ I & \text{w.p.} \quad \dfrac{\gamma_{ij}}{n * |E_i|} \end{cases} \tag{3.10}$$

This illustrates how the stochastic process of pairwise interactions is modeled using vector-matrix notation. Another key insight is that although W is a random matrix, it is neither time dependent nor belief dependent. The possible realizations of W and the probabilities associated with each realization depend only on the network structure and influence parameters. Furthermore, although W is a random matrix, it is a random stochastic matrix, meaning all of its rows sum to one no matter what particular realization occurs.

## 3.3 – Network Analysis

We now discuss how this stochastic model of belief evolution can be analyzed. Specifically we concern ourselves with the expectation of the beliefs of the network, and discuss a potential method for characterizing the second moment of the average belief of the network.

### 3.3.1 – Expected State of Network

The time-independence and state-independence of W means that the expected state of the network can be calculated at any time in the future. The state of the network k steps into the future is given by:

$$X(t+k) = W * \ldots * W * W * X(t) \tag{3.11}$$

Taking the expectation of each side of (3.11) yields the expected state of the network at k steps in the future:

$$E[X(t+k)] = E[W * \ldots * W * W * X(t)] \tag{3.12}$$

Due to the independence of the W matrices from each other, the expectation of the product is equal to the product of the expectation.

$$E[X(t+k)] = E[W] * \ldots * E[W] * E[W] * E[X(t)] \tag{3.13}$$

*DEFINITION:* $\hat{\mathrm{W}} \equiv E[W]$

$$E[X(t+k)] = \hat{\mathrm{W}} * \ldots * \hat{\mathrm{W}} * \hat{\mathrm{W}} * \mathrm{X(t)} = \hat{\mathrm{W}}^{\mathrm{k}} * \mathrm{X(t)} \tag{3.14}$$

Finding the expected state of the network k steps into the future is done by multiplying our current beliefs by $\hat{W}^k$.

The expectation of the W matrix is found by taking the expectation of each element to yield:

$$\hat{W} = \begin{bmatrix} E[W_{11}] & \cdots & E[W_{1n}] \\ \vdots & \ddots & \vdots \\ E[W_{n1}] & \cdots & E[W_{nn}] \end{bmatrix} \tag{3.15}$$

To better organize the matrix $\hat{W}$ some structure is imposed in how the agents as ordered in the belief state vector X(t). This is done so that we can easily sub-partition $\hat{W}$ for computational reasons later. All mutable agents (non-stubborn) are numbered first (the first $|V_m|$ entries of X(t) are all mutable agents), followed by the stubborn agents with belief -0.5, and then stubborn agents with belief +0.5 are last. This results in the last $|V_S|$ entries of X(t) corresponding to stubborn agents, whose beliefs never change. This convention was followed in the example network shown previously (see Figure 3.2), and will be used throughout.

The (i,j) entry of W indicates how much of agent j's belief spreads agent i. This means the (i,j) entry of $\hat{W}$ is the expected amount of belief agent j imparts to agent i. The off-diagonal entries of $\hat{W}$ are calculated first. Recall from the types of interactions allowed and the definition of W that any off-diagonal entry of the matrix W can only take on 3 possible values: 0.5, 1-ε, and 0 (values of 1 and ε can only exist on diagonal entries); yielding the following equation:

$$\hat{W}_{ij} = 0.5 * P[W_{ij} = 0.5] + (1-\varepsilon) * P[W_{ij} = (1-\varepsilon)] + 0 * P[W_{ij} = 0]$$
$$\forall i, j \in \{1, 2, \dots n\}, i \neq j \tag{3.16}$$

The last term in (3.16) equals zero and is removed, yielding:

$$\hat{W}_{ij} = 0.5 * P[W_{ij} = 0.5] + (1-\varepsilon) * P[W_{ij} = (1-\varepsilon)] \tag{3.17}$$

The events which cause $W_{ij}$ to equal 0.5 or 1-ε are:

| Value | Event Required |
|---|---|
| 0.5 | Node j averages with node i, or node i averages with node j |
| 1-ε | Node j forcefully influences node i |

*Table 3.2 – Events and Values of $W_{ij}$*

These events occur with the following probabilities:

| Event | Probability Statement |
|---|---|
| Node j averages with node i | P(j averages with i)=P(node j is selected)*P(neighbor i is selected)*βji |
| Node i averages with node j | P(i averages with j)=P(node i is selected)*P(neighbor j is selected)*βij |
| Node j forcefully influences i | P(j forcefully influences i) = P(node j is selected)*P(neighbor i is selected)*αji |

*Table 3.3 – Events and Probabilities*

Substituting these individual probabilities yields the following:

$$P[W_{ij}(t) = 0.5] = P(\text{j averages with i}) + P(\text{i averages with j})$$

$$= P(\text{agent j is selected}) * P(\text{neighbor i is selected}) * \beta_{ji} + P(\text{agent i is selected}) * P(\text{neighbor j is selected}) * \beta_{ij}$$

Therefore:

$$P[W_{ij}(t) = 0.5] = \frac{1}{n} * \frac{1}{|E_j|} * \beta_{ji} + \frac{1}{n} * \frac{1}{|E_i|} * \beta_{ij} = \frac{1}{n}\left(\frac{\beta_{ji}}{|E_j|} + \frac{\beta_{ij}}{|E_i|}\right) \quad (3.18)$$

Now do the same for the (1-ε) case:

$$P[W_{ij}(t) = (1-\varepsilon)] = P(\text{j forcefully influences i})$$

$$= P(\text{j is selected}) * P(\text{neighbor i is selected}) * \alpha_{ji}$$

Therefore:

$$P[W_{ij}(t) = (1-\varepsilon)] = \frac{1}{n} * \frac{1}{|E_j|} * \alpha_{ji} = \frac{\alpha_{ji}}{n * |E_j|} \quad (3.19)$$

We now combine (3.13), (3.14), and (3.15) for finding off-diagonal entries of $\hat{W}$:

$$\hat{W}_{ij} = 0.5 * P[W_{ij}(t) = 0.5] + (1-\varepsilon) * P[W_{ij}(t) = (1-\varepsilon)]$$

$$\hat{W}_{ij} = 0.5 * \frac{1}{n}\left(\frac{\beta_{ji}}{|E_j|} + \frac{\beta_{ij}}{|E_i|}\right) + (1-\varepsilon) * \frac{\alpha_{ji}}{n * |E_j|}$$

$$\hat{W}_{ij} = \frac{\left(0.5 * \beta_{ji} + (1-\varepsilon) * \alpha_{ji}\right)}{n * |E_j|} + \frac{0.5 * \beta_{ji}}{n * |E_i|} \quad (3.20)$$

$$\forall i,j \text{ such that } i \neq j, \text{and nodes } i \text{ and } j \text{ are adjacent}$$

The diagonal entries of $\hat{W}$ are calculated using the fact that $\hat{W}$ is stochastic, which means its rows must sum to 1:

$$\hat{W}_{ii} = 1 - \sum_{j \neq i} \hat{W}_{ij} \quad \forall i, j \in \{1,2,\ldots,n\} \tag{3.21}$$

We could alternatively use a probabilistic calculation as done for the off-diagonal entries by calculating all the possible ways in which the diagonal entry would equal 1, 0.5, or ε, but it would yield the same result. Having shown how to calculate each entry of $\hat{W}$, we discuss some special structure in the matrix. The requirement that all of the stubborn agents are numbered last means the last $|V_S|$ rows of $\hat{W}$ represent the influence of the network on stubborn agents. Stubborn agents never change their beliefs, consequently the diagonal entries of the last $|V_S|$ rows of $\hat{W}$ are all 1 and all other entries in the row are 0 (because every stubborn agent retains all of their belief at every step). The first $|V_m|$ rows of $\hat{W}$ represent the expected effect of the network on each of the mutable agents. More specifically, within each of these rows, the first $|V_m|$ columns are the expected influence on agent i from all of the other mutable agents, and the last $|V_S|$ columns are the expected influence of agent i from the stubborn agents. Using this structure we partition $\hat{W}$ into 4 sub-matrices:

$$\hat{W} = \begin{vmatrix} B & D \\ 0 & I \end{vmatrix} \tag{3.22}$$

$$|B| = |V_M| x |V_M|, \qquad |D| = |V_M| x |V_S|, \quad |I| = |V_S| x |V_S|, \quad |0| = |V_S| x |V_M|$$

The matrix B is the expected effect of mutable agents on each other, matrix D is the expected effect of stubborn agents on mutable agents, and the zero and identity matrices represent the fact that stubborn agents never change their beliefs.

## 3.3.2 –Equilibrium State

To calculate the equilibrium state of the network, Equation 3.10 can be used to find the expected value of the network at k steps in the future, and then take the limit as k goes to infinity. The equilibrium belief vector is defined as $X^*$.

*DEFINITION: The Equilibrium belief of agent i is:*

$$\lim_{k \to \infty} E[X_i(t+k)]$$

*DEFINITION: The vector $X^*$ is the vector of the equilibrium agent beliefs.*

$$E[(X(t+k)|X(t)] = \hat{W}^k * X(t) \tag{3.23}$$

$$X^* = \lim_{k\to\infty} E[(X(t+k)|X(t)] = \lim_{k\to\infty} \hat{W}^k * X(t) \to \hat{W}^\infty * X(t) \tag{3.24}$$

A dual approach using Markov chain theory is used to calculate this infinite matrix product.

## 3.3.3 – Markov Chain Analysis

We consider $\hat{W}$ as the transition probabilities for a Markov chain. This is possible because $\hat{W}$ is a stochastic matrix (meaning all rows sum to 1), and can represent a series of transition probabilities of an n-state Markov chain. The chain has one state for every agent in the network, and the transition probabilities are the entries of $\hat{W}$ (e.g – the probability of moving from state 1 to state 2 in the Markov chain is $\hat{W}_{12}$). An example of this isomorphism is shown below in Figure 3.3.

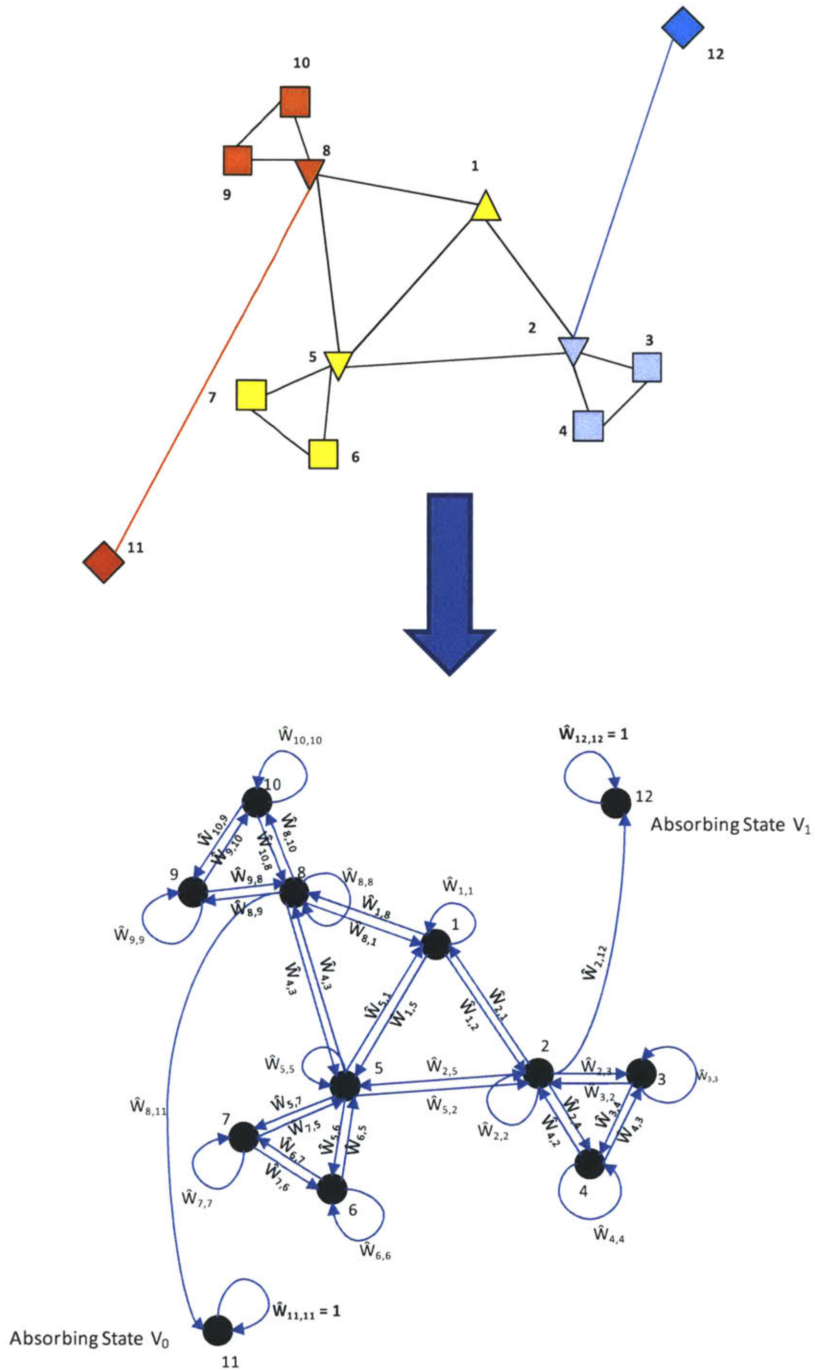


*Figure 3.3 – Illustration of Network to Markov chain isomorphism*

In the Markov chain, all of the stubborn agents are absorbing states. This is because in the $\hat{W}$ matrix, the diagonal entry of every stubborn agent is 1, which means that in the Markov chain they have a self-transition probability of 1. Thus once a random walk on the Markov chain enters one of these states they can never leave.

As shown in (3.24), the equilibrium beliefs $X^*$ are a function of $\hat{W}^\infty$. In a Markov chain with transition matrix $\hat{W}$, finding $\hat{W}^\infty$ is equivalent to finding the steady state probabilities of the Markov chain. This is the same as a random walk across the chain until being absorbed by one of the absorbing states for Markov chains with a set of absorbing classes. Using this fact, Yildiz et al have shown that finding the equilibrium state of a social network such as ours can be done by finding the absorption probabilities of a random walk on this Markov Chain (though we will not reiterate the proof here). His methodology is described below:

*DEFINITION: $\eta_{ij}$ is the probability that starting in state i on the Markov chain, a random walk ends in one of the absorbing states j corresponding to stubborn agent j in the network.*

- Start a random walk from state i in the Markov chain corresponding to agent i in the network
- Find the set of probabilities $\{\eta_{ij}\}$ $\forall j \in V_S$ (find $\eta_{ij}$ for each possible absorbing state j).
- Denote the expected equilibrium belief of agent i as $X_i^*$ with the following definition:

$$X_i^* = \sum_{j \in V_S} \eta_{ij} * X_j \tag{3.25}$$

*where $X_j$ is the belief of stubborn agent j corresponding to absorbing state j in the Markov chain*

This means that the equilibrium belief is independent of the starting beliefs of the mutable agents and in fact is only a function of the network topology (meaning the types of agents and their connections between each other). The topology and influence parameters affects the absorption probabilities, because it changes the values of $|E_i|$, $\alpha_{ij}$, $\beta_{ij}$, and $\gamma_{ij}$, which are used in calculating $\hat{W}$. To actually calculate the absorption probabilities for every node corresponding to a mutable agent we rely on the results of Yildiz et al, who showed that the probability of any mutable agent being absorbed by a stubborn node is the following:

$$\eta = (I - D)^{-1} * B$$

$$\textit{Where } \eta \textit{ is a } |V_M| x\ |V_S| \textit{ matrix, and the i,jth entry corresponds to } \eta_{ij} \tag{3.26}$$

$$X^* = \lim_{k \to \infty} E[X(t+k)] = \eta * X_S = (I - D)^{-1} * B * X_S \tag{3.27}$$

*$X_s$ is the vector of stubborn agent beliefs, which corresponds to the last $|V_S|$ entries of the belief vector $X(t)$, and D and B are the submatrices of $\hat{W}$ as previously defined.*

This gives a simple tool for calculating the equilibrium beliefs of all mutable agents in the network. This formula allows the equilibrium state vector of the network to be easily calculated but as previously discussed, the network only converges in expectation, the actual state vector does not converge to this:

$$X(t+k) \neq E[X(t+k)] \quad \text{w.h.p.} \tag{3.28}$$

$$\lim_{k\to\infty} X(t+k) \neq \lim_{k\to\infty} E[X(t+k)] \quad \text{w.p. } 1 \tag{3.29}$$

Although it is possible for the belief vector to actually equal the expected belief vector, the sample path required for this to occur over a long time interval has zero probability. The plot below in Figure 3.4 demonstrates this point. It shows the mean belief of the mutable agents over time for a simulation of 5000 interactions of the sample network. The average of the equilibrium beliefs of mutable agents is 0.3439. The actual average belief of the mutable agents starts at zero, and moves towards 0.3439, but then continues to oscillate around it, never settling into a perfect convergence.

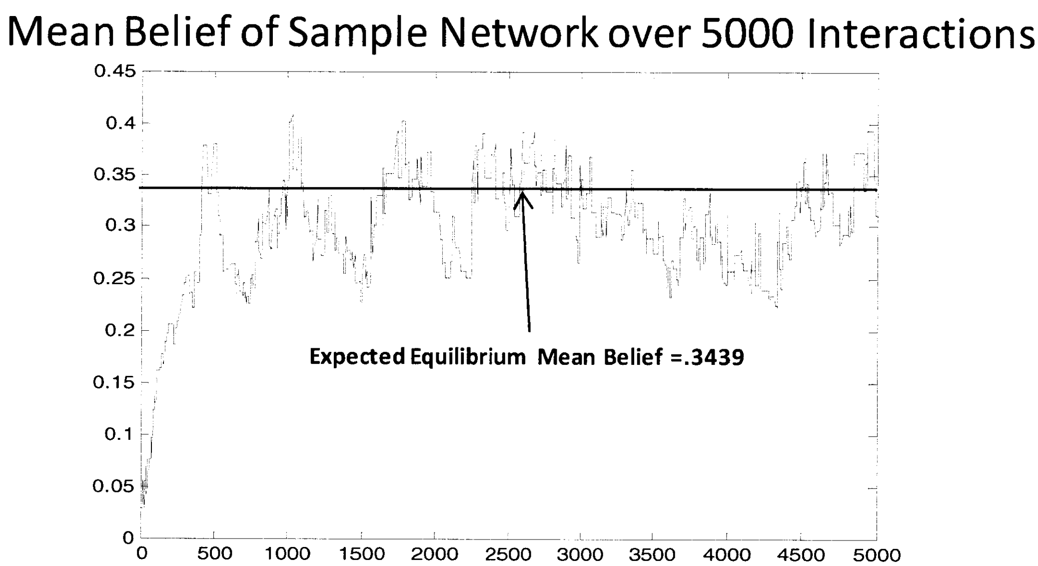


*Figure 3.4 – Mean Belief of Network over 5000 interactions*

*The Mean Belief of the network settles into a type of stochastic equilibrium, where it continues to oscillate around the equilibrium mean belief of .3439 over time.*

Being able to calculate the equilibrium is useful. It allows players controlling a set of stubborn agents to determine if a particular set of connections in the network will move the network more towards their stubborn agent beliefs. It would also be useful to measure the variance in this equilibrium. By variance in the equilibrium we mean a measure of how close the network beliefs actually match the equilibrium beliefs. The equilibrium belief of an agent may be .1, but in simulation that agent's beliefs can be anywhere in the interval [-0.5,0.5], and follows some unknown distribution associated with the network topology and influence parameters. The first moment of this distribution can be calculated efficiently using (3.27), but knowing something about its second moment would give greater insight into the behavior of the network over time. In the real world, someone trying to influence the network may be risk averse. They may prefer to influence a network in a way that may not have high equilibrium mean belief, but has a very high probability of actually reaching and staying at the equilibrium. High volatility in individual beliefs can lead to serious upheaval and unexpected consequences in societies, and the desire to minimize this variance motivates the next section.

### 3.3.4 – Analyzing Variance

Hung's work showed that the mean belief of a network generally converges within the equilibrium mean belief +/- $\delta$, where $\delta$ is some constant on the order of 0.15, as shown in Figure 3.4. However, the randomness of the model still causes individual beliefs, and the mean network belief, to continually change. It is even possible to move from having all equilibrium beliefs to having all extreme beliefs (either +0.5 or -0.5), although this is an extremely unlikely event.

Yildiz et al formulated a bound for the standard deviation of the equilibrium beliefs for a model in which agents have only a binary belief (either 0 or 1). We start with a quick review of prior notation and a few definitions required for his formula.

*Absorbing states $V_{TB}$ in the Markov chain correspond to stubborn agents with belief -0.5 in the network.*

*Absorbing states $V_{US}$ in the Markov chain correspond to stubborn agents with belief +0.5 in the network.*

*Let node $i \in V_M$ become a new absorbing state $V_1$ in the Markov chain*

*DEFINITION: $p_j^i$ is the probability that a random walk started at state j on the Markov chain will be absorbed by $V_I$ corresponding to mutable agent i*

Once we change node i to be an absorbing state in the Markov chain, there are 3 classes of absorbing states ($V_{TB}$, $V_{US}$, and $V_I$), which may actually consist of more than three states on the Markov chain ($V_{TB}$ and $V_{US}$ may contain several absorbing states if there are more than 2 stubborn agents in the network). To better show this transformation, Figure 3.5 shows the Markov chain corresponding to the network shown in Figure 3.4, in which node 1 has been converted into an absorbing state $V_1$:

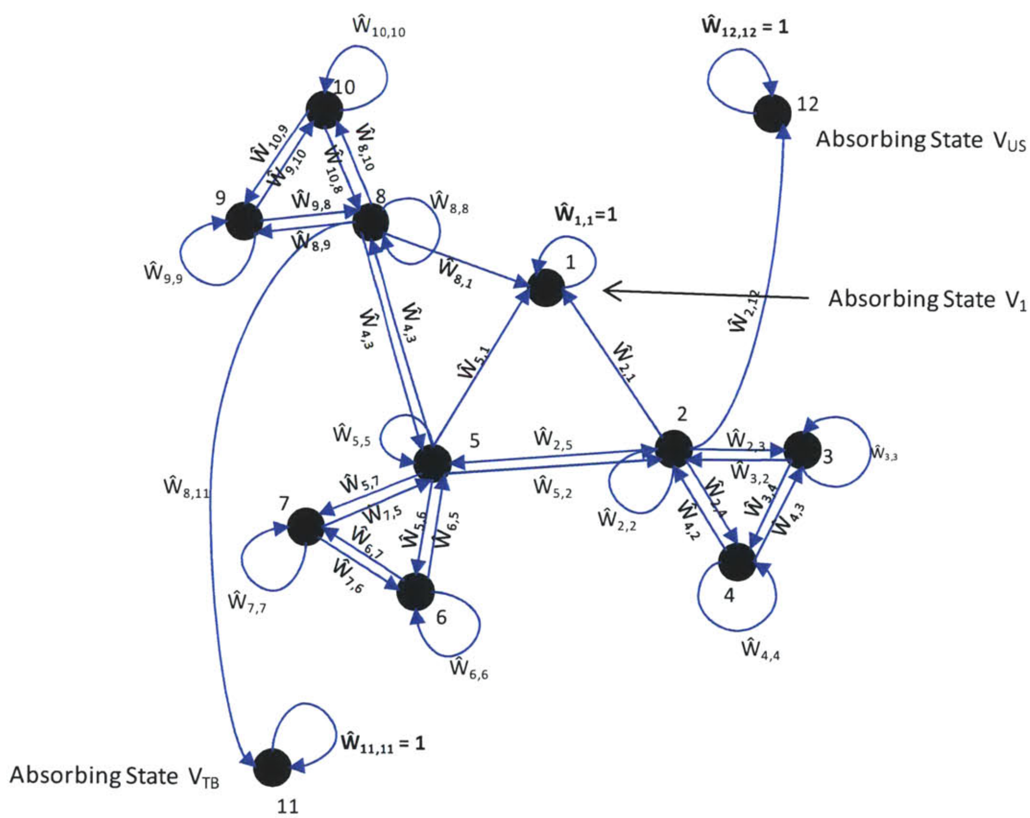


Figure 3.5 – Modifying the Markov chain to include new absorbing states

*State 1 is has been modified and is now an absorbing state*

After modifying the transition matrix for the Markov chain to reflect this new absorbing state, all of the $p_j^i$ can be efficiently calculated by simply raising this matrix to a very large power (essentially finding $\hat{W}^{\infty}$). This is not as efficient as using (3.27) to find the absorption probabilities in a single step, but in order to do this would require re-ordering of the $\hat{W}$ matrix to move the new absorbing state to the end of the belief vector. Our code is implemented in MATLAB, and it is generally faster to calculate $\hat{W}^{10^6}$ than it is to recalculate the entire $\hat{W}$ matrix and then use (3.27). This is because MATLAB is very efficient at matrix multiplications, but is slow to execute the for-loops we use to re-calculate a new $\hat{W}$ matrix. After finding $\hat{W}^{10^6}$ the i-th column contains all of the $p_j^i$. This is repeated for each mutable agent to calculate all possible $p_j^i$.

The last quantity required for Yildiz et al's bound on the standard deviation is:

$$r^i \equiv \sum_{j \in V_M, j \neq i} p_j^i \qquad \forall i \in V_M \tag{3.30}$$

$r^i$ is the sum of probabilities over all mutable agents being absorbed by node i (if it were an absorbing state). The interpretation for $r^i$ is it is a measure of how central and influential an agent is. The transition probabilities in the Markov chain represent pairwise influence, and thus the more influence an agent has, the higher the transition probabilities are, and the more likely random walks will end up in the absorbing state i.

Yildiz et al then derives the following bound on variance of equilibrium average belief of the network.

*DEFINITION: $p_j^{US}$ is the probability of a random walk starting at node j being absorbed by one of the states in $V_{US}$.*

$$p_j^{US} \equiv \sum_{i \in V_{US}} \eta_{ji}$$

$$\sigma(X^*) \leq \frac{1}{N^2} \sum_{j \in V_M} \left(p_j^{US} - (p_j^{US})^2\right) * (r^j)^2 \tag{3.31}$$

*where $\sigma(X^*)$ is the standard deviation of the equilibrium mean belief of the network*

Note $p_j^{US}$ is only the probability of being absorbed by $V_{US}$, and absorption probabilities for $V_{TB}$ are not considered in the bound. This is a consequence of the binary derivation done by Yildiz et al, in which the belief corresponding to $V_{TB}$ was 0, and thus did not affect the calculation of variance. Because this derivation is based off of a model of binary beliefs for all agents (either 0 or 1) the proof is not true in our model. However, we had hoped it would serve as a useful metric (if not a strict bound) in our model. Chapter 4 conducts the analysis on the bound, and unfortunately finds that the bound does not consistently predict standard deviation well in our model.

### 3.3.5 – Key Points

One of the most important facts is that the equilibrium belief is independent of the starting beliefs in the network. The only thing that affects the equilibrium is the topology, the influence parameters, and the beliefs of the stubborn agents (which are fixed in the beginning). Initial beliefs of the mutable agents do affect how quickly the network belief move to the equilibrium beliefs (convergence speed), but they do not affect the actual equilibrium beliefs themselves.

We have a method for analyzing the expected belief state of the network at any time in the future, and a candidate for analyzing the variance at that equilibrium. We do not have a tool for analyzing how quickly the beliefs converge to this equilibrium, or the variance at any time other than at equilibrium. Yildiz et al showed that in the worst case a network should converge to its equilibrium beliefs in $O(n^3 \log(n - |V_S|))$ time steps, but empirical analysis and simulations have shown that in general the networks converge much faster. Explicitly calculating the variance at any fixed number of time steps beyond 1 becomes exceedingly difficult, as the number of sample paths at each time step is $O(|E|)$ where E is the set of all arcs in the network. This means in order to calculate the exact variance 10 time steps in the future would require the calculation of $O(|E|^{10})$ possible realizations. Even for a small network with only 10 edges, this would require calculating $O(10^{10})$ realizations and summing across their probabilities to calculate the variance and it is generally not feasible. Alternatively it is possible to run many simulations of a network and collect sample variances, but in the game described below, the connections from stubborn agents to the mutable agents change. The number of associated topologies can be exponentially large, and thus calculating sample variances is not feasible.

## 3.4 – 2-Player Game Description

Now that we have formulated a model for the belief of a populace, we formulate a game in which 2 players compete to influence the populace.

### 3.4.1 – Players and Game Space

There are two players – the US (US Forces) and the TB (Taliban insurgents) each of which controls their corresponding set of stubborn agents – either $V_{US}$ or $V_{TB}$. Each player controls $\frac{|V_S|}{2}$ stubborn agents, each of who have $|E_S|$ connections they can make to the network:

$$|V_S| = \#\ of\ stubborn\ agents, each\ player\ controls\ half\ of\ them$$

$$|E_S| = \#\ of\ arcs\ from\ each\ stubborn\ agent\ to\ the\ network\ of\ mutable\ agents$$

$$|E_S|\ is\ the\ same\ value\ for\ each\ stubborn\ agent\ and\ is\ an\ input\ to\ the\ model$$

*Let C be the number of connections from a player's stubborn agents to the mutable agents in the network such that* $C = |E_S| * \frac{|V_S|}{2}$

A strategy is defined as the set of connections from the player's stubborn agents to the network. Each connection can be made to any mutable agent in the network. This is a perfect information, symmetric game – at any time, both players know the exact state of the network, they know the other player's strategy, and they can both choose the same set of possible strategies. The set of strategy options is defined as follows:

$$S -\ \text{Set of strategies available (same for both players)}$$
$$S_{US} -\ \text{Strategy chosen by the US player}$$
$$S_{TB} -\ \text{Strategy chosen by the TB player}$$

$$|S| = O(|V_M|^C)$$

In the example network presented earlier (see Figure 3.2), $C = 1$, $S_{US}=\{2\}$ , and $S_{TB}=\{8\}$ (the US player connects their stubborn agent to agent two, and the TB player connects their stubborn agent to agent eight).

All the games will be open loop one-step games, so each player makes a single strategy selection and then plays it forever. Once each player chooses a strategy the equilibrium state of the network determines the value of their payoff function (defined below).

### 3.4.2 – Payoff Functions

The payoff functions we examine are functions of the equilibrium state of the network. Recall that the equilibrium state of the network is defined as:

$$X^* = \lim_{k\to\infty} E[(X(t+k)|X(t)] = \lim_{k\to\infty} \hat{\mathrm{W}}^{\mathrm{k}} * X(t) \to \hat{\mathrm{W}}^{\infty} * X(t) = \eta * X_S$$

The equilibrium state of the network was shown in Section 3.3.3 to be independent of the current beliefs of the network, and depends only on the connections between agents, the influence parameters, and the beliefs of the stubborn agents.

This means that the payoff functions depend only on the strategies chosen by each player (which affects the connections between agents), and the payoff functions have a finite support. It also means that as a player chooses a strategy to maximize their payoff function, this strategy will be optimal regardless of the state of the network. This is not entirely realistic, as our motivation for this model is to better understand how US forces and insurgent forces influence public opinion to maintain popular support and in the real world, it is not accurate to assume that both sides will choose a strategy and maintain it forever. To understand why we make this assumption, consider if we maximized the mean belief at some finite time in the future (for example 50 time steps in the future), the current network state must be accounted for in calculating the this non-equilibrium state:

$$E[X(t+k)] = \hat{\mathrm{W}}^{\mathrm{k}} * \mathrm{X(t)}$$

The current state of the network matters for any finite k; only when we take the limit as k goes to infinite does the current state become irrelevant. This means that if we maximized payoff for the beliefs of the network at some finite time, the support of the payoff function must include not only the strategies of the players, but also the current beliefs of the network. This increases the support of the payoff function from a finite set of strategies, to an infinite set of all possible strategy-belief state combinations. Even if the belief states were discretized to make them finite, the support space for payoff functions would increase exponentially with the size of a network and the problem becomes intractable for the analysis methods presented in this thesis.

Given this difficulty, strategies are chosen using equilibrium payoff functions, understanding it is not a perfect fit, but hoping that it will lead to some insight and intuition we can apply to the real world problem. Each player's goal is to select a set of connections in the game that allows their stubborn agents to maximally influence the mutable agents. The definition of what maximal influence is may change depending on the payoff function selected.

Below three different payoff functions for the game are defined: Mean Belief, Agents Won, and Risk Averse Mean Belief. We now introduce a slightly modified notation for the matrix $\hat{W}$. We now write it as $\hat{W}(S_{US}, S_{TB})$ which denotes the fact that the matrix is a function of the strategies chosen by each player. That is, the topology of the network changes based on the connections associated with each strategy and thus $\hat{W}$ also changes.

Because $\hat{W}(S_{US}, S_{TB})$ changes with each player's strategy, we now redefine some of the network matrices and equations as functions of the strategies chosen by each player.

Original Equation 3.22:

$$\hat{W} = E[W] = \begin{vmatrix} D & B \\ 0 & I \end{vmatrix}$$

New Notation:

$$\hat{W}(S_{US}, S_{TB}) = E[W(S_{US}, S_{TB})] = \begin{vmatrix} D(S_{US}, S_{TB}) & B(S_{US}, S_{TB}) \\ 0 & I \end{vmatrix} \tag{3.32}$$

Original Equation 3.27:

$$X^* = \lim_{k \to \infty} E[X(k)] = \eta * X_S = (I - D)^{-1} * B * X_S$$

New Notation:

$$X^*(S_{US}, S_{TB}) = (I - D(S_{US}, S_{TB}))^{-1} * B(S_{US}, S_{TB}) * X_S \tag{3.33}$$

*where $X_S$ is a $|V_S|$ x 1 vector of the stubborn agent beliefs, and $X^*$ is the $|V_M|$ x 1 vector containing the equilibrium beliefs of each mutable agent.*

### 3.4.2.1 – Mean Belief

In the 'mean belief' payoff function, each player is trying to move the equilibrium mean belief towards their side. When the US player's stubborn agents have high influence, the mean belief of the mutable agent in the network moves towards +0.5. When the TB player's stubborn agents have high influence in the network, the mean belief of mutable agents moves towards $-0.5$. This means that the US player wants the network to have a high mean belief, and the TB player wants it to have low mean belief.

We define the payoff function as:

$$f_{US}(s_{US}, s_{TB}) = \frac{1}{|V_M|}\sum_{j=1}^{|V_M|} X_j^*(s_{US}, s_{TB}) = -f_{TB}(s_{US}, s_{TB}) \tag{3.34}$$

### 3.4.2.2 –Agents Won

If a mutable agent is being equally influenced by both the TB and US stubborn agents, then their equilibrium belief is zero (halfway between each stubborn agents' beliefs). If the US is influencing a mutable agent more than the TB player, then the mutable agent's belief will be greater than zero and if the TB is influencing them more, it will be less than zero. The 'agents won' payoff function awards 1 point to each player for mutable agent that they influence more than their opponent, and they lose 1 point for every mutable agent that their opponent influences more. If a mutable agent is equally influenced by both players stubborn agents, its equilibrium belief is zero, and both players get zero points.

| TB Points Earned | Mutable Agent Equilibrium Belief | US Points Earned |
|---|---|---|
| 1 | <0 | -1 |
| 0 | 0 | 0 |
| -1 | >0 | 1 |

Thus the maximum value of the payoff function is $|V_M|$, and the minimum value for each is $-|V_M|$. We denote this payoff function as $g(\cdot)$. The definition is:

$$1_{TB}(x)\begin{cases} -1 & \text{if } x > 1 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x < 0 \end{cases} \tag{3.35}$$

$$1_{US}(x)\begin{cases} 1 & \text{if } x > 1 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \tag{3.36}$$

$$g_i(S_{US}, S_{TB}) = \sum_{j=1}^{|V_M|} 1_i(X_j^*(S_{US}, S_{TB})) \quad i \in \{US, TB\} \tag{3.37}$$

### 3.4.2.3 – Risk Averse Mean Belief

The last payoff function is a variant of the 'mean belief' payoff function. In the 'mean belief' payoff function we cared only about equilibrium agent beliefs, disregarding the variability of the actual beliefs. Here variability is taken into account by calculating the standard deviation of the equilibrium solution using methods from Section 3.3. The payoff function is denoted as $h(\cdot)$. Its basic form will be:

$$h(S_{US}, S_{TB}) = \mathit{Mean\ Belief\ Payoff} - \lambda * \mathit{Expected\ Standard\ Deviation\ of\ Equilibrium\ Beliefs\ of\ Network}$$

*where $\lambda$ is a constant representing how much high standard deviation is penalized*

$$h_i(S_{US}, S_{TB}) = f_i(S_{US}, S_{TB}) - \lambda * \sigma\big(X^*(S_{US}, S_{TB})\big) \quad i \in \{US, TB\} \tag{3.38}$$

The standard deviation associated with the two strategies at equilibrium is found using Equation 3.31, and is a measure of how much variability we expect to see in the mean belief of the network over time. Although the result of this equation is simply a bound on the standard deviation in the binary model and not an explicit characterization of the variability at equilibrium, it is used here as a metric. The constant $\lambda$ is a penalty parameter which allows us to control the risk tolerance of players. Although in the real world, the parameter would likely be different for each player, here we make it identical for both to simplify analysis. Chapter 4 conducts further refinement and analysis of this payoff function and its tuning parameter. Two main questions are considered – "What are good values for the tuning parameter?" and "How useful is the bound on the standard deviation in actually characterizing the variability of different strategies?"

## 3.4.3 – Existence and Characteristics of Nash Equilibria in the Game

Now that we have formulated a game on the network model, the next question we hope to answer is whether equilibria exist in this game. Do pure Nash equilibria exist, and what do they look like? We do not care about mixed Nash equilibria, because it would mean a player has to randomly choose among a set of strategies. Telling a US commander in Afghanistan to randomly choose a strategy for winning the support of the populace is not useful operationally.

### 3.4.3.1 – Definition of Pure Nash Equilibrium

A strategy profile is a vector containing both players' strategies. In a game, a pure Nash equilibrium is a strategy profile from which neither player has any incentive to deviate on their own. More precisely, a pure Nash equilibrium is a strategy profile from which neither player can improve through a unilateral deviation in their strategy[19]. Although it may be possible to improve their payoffs if they both simultaneously change their strategies, a single change by one player is not sufficient. Consider the following payoff matrix:

| | | Player 1 | |
|---|---|---|---|
| | | A | B |
| Player 2 | A | **(1,2)** | (0,0) |
| | B | (0,0) | **(2,2)** |

*where (x,y) indicates that player 1 receives payoff x, and player 2 receives payoff y*

In this simple game, there are two Nash equilibria in which players choose identical strategies. Player 1 would prefer to play strategy B because that has the maximum payoff for them, however if player 2 has decided to play strategy A, then player 1 would also play strategy A, because a payoff of 1 is better than a payoff of 0. This demonstrates the principle that a Nash equilibrium is not necessarily the best payoff for a player, but rather, it is the best they can do given an opposing strategy. In a pure Nash equilibrium neither player can unilaterally deviate and result in an increased payoff. If a pure Nash equilibrium exists, it is equal to the maximum of its row and column in the payoff matrix.

Although there is no guarantee that our game has a pure Nash equilibria, computational experiments presented in Chapter 4 show empirical evidence indicating that they do always exist under the 'mean belief' and 'agents won' payoff functions.

#### 3.4.3.2 – Zero Sum vs. Non-Zero Sum Games

A zero-sum game means that if a player obtains some payoff, the other player's payoff must be the negative of this[19]. The way the 'mean belief' ($f_i(S_{US}, S_{TB})$) and 'agents won' ($g_i(S_{US}, S_{TB})$) payoff functions are defined they result in a zero sum game if both players use them.

$$f_{US}(S_{US}, S_{TB}) + f_{TB}(S_{US}, S_{TB}) = 0$$

$$g_{US}(S_{US}, S_{TB}) + g_{TB}(S_{US}, S_{TB}) = 0$$

Any other combinations of payoff functions $f(\cdot)$, $g(\cdot)$, or $h(\cdot)$ will not result in a zero sum game (example, if we use $f_{TB}(\cdot), g_{US}(\cdot)$, or $h_{TB}(\cdot), h_{US}(\cdot)$).

#### 3.4.3.3 – First-mover advantage

In a game where players do not choose their strategies simultaneously, one of the players may gain an advantage by choosing first. For example tic-tac-toe has a significant first mover advantage. By taking the middle square in tic-tac-toe on the first move, a player can ensure that they will never lose the game. Many games have first mover advantages (and sometimes disadvantages). If both players use the payoff functions $f(\cdot)$ or $g(\cdot)$, there is no first mover advantage. We show in Chapter 4 that first mover advantages can exist if players are using different pairs of payoff functions. The existence of first mover advantages can have a significant impact on game outcomes, and understanding when and why they exist can be important for players to maximize their payoff.

## 3.5 – Finding Optimal Strategies and Pure Nash Equilibria

First we examine the problem of a single player optimizing their strategy – given the network, and an opponent's strategy, what is my best response? We then discuss how we can either exhaustively enumerate the payoff matrix to find all of the pure Nash equilibria (defined in Section 3.5.2), or we can use best response dynamics to find a single pure Nash equilibrium. Best response dynamics optimizes each player's strategy individually, given the other player's current strategy (it finds their best response). For example, we fix the US player's strategy, and then optimize the TB strategy. This gives us the best response strategy for the TB. We then fix the TB player's strategy and optimize the US player's strategy against the TB strategy (find the US best response). This continues until a strategy profile is reached from which neither player can find a better strategy. Best response dynamics is not guaranteed to find pure Nash equilibrium strategy profiles unless the payoff matrix can be expressed as using convex function, but Chapter 4 shows that best response dynamics is able to find pure Nash equilibria very well in our game.

## 3.5.1 – Finding Optimal Strategies

An obvious question any player would want to ask is, "What is my best strategy?" Given a network, and an opponent's strategy, where should we connect our stubborn agents to have maximal influence in the network? Solving this problem can be very challenging for two reasons. First, if a player has several connections, the number of strategies increases exponentially with the number of connections, and it becomes difficult to efficiently search through the strategy space. Second, because the strategy sets are discrete, we cannot use any type of gradient descent or non-linear programming methods to find an optimal solution; any math program would have to be an integer program with a large number of binary decision variables.

Hung's work focused on developing such a math program to find strategies, but the formulation and solver used did not return optimal solutions. In this section we will discuss another approach that finds good solutions that are close to optimal (though there is bound of how close to optimal).

### 3.5.1.1 – Exhaustive Enumeration

**Description**

Exhaustive enumeration means that we evaluate every possible strategy available for a player, and then select the highest payoff strategy for that player.

**Benefits/ Disadvantages**

There are two advantages to this method. First – it guarantees optimality – we have searched every possible solution and chosen the best available payoff. Second – it shows if there are multiple optimal solutions. While this is easy to do for smaller networks, the problem becomes intractable very quickly. For example in a 20 agent-network in which a player has 3 connections we must calculate $20^3$=8000 potential strategies for the player in order to guarantee optimality. In general, finding a single player's optimal strategy given an opposing strategy takes $n^C$ calculations in our payoff function, where n is the size of the network, and C is the number of connections defined above. When considering each strategy, a new $\hat{W}$ matrix must be calculated (which take $O(n^2)$ calculations) and then inverted (which requires $O(n^3)$) for every calculation of a player's payoff[20]. This means that each strategy evaluated takes $O(n^3)$ calculations, and there are $n^C$ strategies that must be evaluated. This yields $O(n^3 * n^C) = O(n^{C+3})$ calculations required to use this method. This is polynomial for a fixed C, but as C increases, the time required for exhaustive enumeration increases exponentially. In general the method is very efficient for C=1, but for values of C that are larger than one this method is very slow.

#### 3.5.1.2 – Simulated Annealing

**Description**

Simulated Annealing is a neighborhood random search heuristic first described in 1983 by Kirkpatrick et al[21]. The heuristic is designed to simulate a system cooling (annealing). There is a system temperature that start at 1, and decreases geometrically with every iteration. The geometric decrease is defined as:

Temperature=$\alpha$*Temperature

where $\alpha$ is the cooling rate, (generally between 0.95 and 0.99)

At each step, the heuristic checks a random neighboring solution, and moves there if the solution is an improvement. If it is not, then with some probability that decreases as the temperature decreases, the heuristic may still move to it. As the temperature of the system cools, the heuristic will only take improving steps and settles into some local optimum, until finally the temperature reaches some minimum value and the algorithm is terminated.

The definition of a neighboring solution can mean many things; for our algorithm we define a neighboring solution as follows:

1). Choose a random stubborn agent, agent i.

2). Choose one of their connections to a neighboring agent j at random.

3). Choose a random agent k that is a neighbor of agent j.

The neighboring solution is the one in which connection i-j is replaced by i-k.

An example of this is demonstrated below in Figure 3.6.

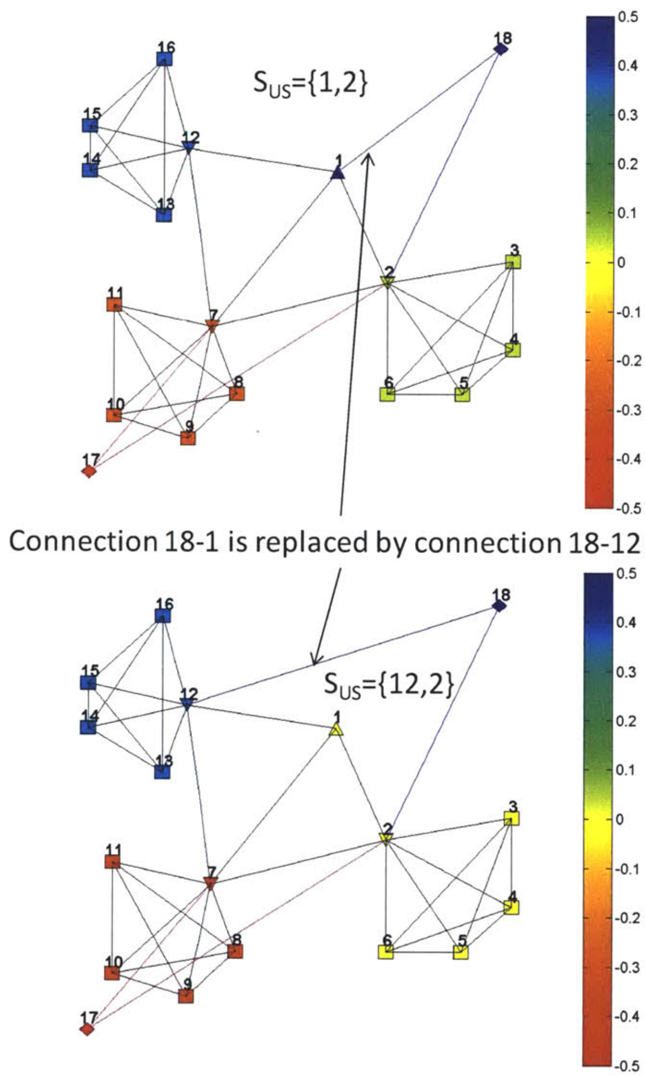


*Figure 3.6 – Illustration of Neighboring Solutions for Simulated Annealing*

If the neighboring solution increases our payoff, then it becomes the new solution. If it does not improve our payoff, then it may still be chosen probabilistically, as a function of the temperature of the system. The probabilistic function we use is:

$$P(Moving\ to\ nonimproving\ solution) = e^{\frac{New\ Solution\ Payoff - Current\ Solution\ Payoff}{Temperature}}$$

In the example from Figure 3.6, the neighboring strategy {12,2} yields less payoff than the original solution of {1,2} for the US player. The algorithm will move to this solution with probability:

$$P(Strategy\ \{1,2\}\ is\ discarded\ for\ \{12,2\}) = e^{\frac{f(\{1,2\})-f(\{12,2\})}{Temperature}}$$

A summary of the simulated annealing heuristic we use is below.

**Simulated Annealing Heuristic Summary**

Initial strategy is $S_0$, and the new strategy is $S_1$.

The payoff function is *f(•)*.

Initial Temperature =1.

1. Choose one of the player's stubborn agents at random, agent i.
2. Choose one of agent i's connections to some neighboring agent j at random.
3. Choose one of agent j's neighbors at random, agent k. Let strategy $S_1$ be the strategy where connection i-j is replaced by i-k.
4. If $f(S_0) < f(S_1)$ then change to strategy $S_1$
5. If $f(S_0) \geq f(S_1)$ then with probability $e^{\frac{f(S_1)-f(S_0)}{Temperature}}$ change to strategy $S_1$; otherwise retain strategy $S_0$.
6. Temperature=α*Temperature, where α is a cooling rate less than 1 *(0.97 in our code)*

Iterate until the temperature drops below the minimum temperature *(below $10^{-20}$ in our code).*

The algorithm also keeps track of the best solution seen thus far. Once the algorithm terminates, it returns the best solution seen thus far (which may or may not be the ending solution). The algorithm runs for $\frac{\log(\text{minimum temp})}{\log(\alpha)}$ iterations. With a cooling rate of 0.97 and minimum temperature of $10^{-20}$, this is ~1500 iterations. Each step of the algorithm calculates a new $\hat{W}$ matrix and inverts it, requiring $O(n^3)$ calculations. As shown, the heuristic runs through a fixed number of iterations, which does not scale with n.

**Benefits/Disadvantages**

One of the major advantages of simulated annealing is that it has been widely used; it is well understood and has been shown to generate high quality solutions in practice. It also runs very quickly compared to the solution times required for Hung's math program, and is relatively easy to implement. The major disadvantage of simulated annealing is the same as all heuristics – it has no guarantees on the quality of solutions but this is also true for Hung's optimization problem due to the non-convexity.

## 3.5.2– Finding Nash Equilibria

The previous section discussed how to find an optimal strategy given a fixed opposing strategy. In the game, a player's opponent cannot be counted on to play a fixed strategy throughout. One way to better understand the dynamics of how players behave in the game is to find and characterize the pure Nash equilibrium strategy profiles in the game.

Finding Nash equilibria within a payoff matrix is a straightforward thing to do. Any Nash equilibrium corresponds to cells that are both row and column maximums for each player (which means that no unilateral deviation can improve their payoff). The difficulty in this case becomes the problem of calculating the payoff matrix. If we had a 20 agent network, with 2 connections each, the payoff matrix would be $20^2$ x $20^2$ or 400 x 400 = 16000 entries. Given the fact that calculating all of these entries each takes $O(n^3)$ calculations, we need $O(n^C n^C n^3) = O(n^{2C+3})$, or $O(20^7) \sim O(10^9)$ calculations to fully enumerate the payoff matrix. Although Chapter 4 has experiments that exhaustively calculate the payoff matrix when C=1, this is not a tractable approach for finding Nash equilibria with more than one connection per player.

To deal with this problem we again use the simulated annealing heuristic. Each time simulated annealing is used for a player, it is the same as doing a row or column search for an optimal solution. Nash equilibria can be found by running simulated annealing for one player, and then for the other player, and then repeat this over and over until we converge to a stable set of strategies. If we converge to a pure Nash equilibria, then no unilateral deviation can produce an improved payoff for either player, which means simulated annealing will not be able to find any improvements.

This method of repeatedly optimizing each player's strategy while fixing the other one is called best response dynamics. There is no guarantee that best response dynamics will converge to a pure Nash equilibrium strategy profile, however in Chapter 4 we find that for our game on the payoff functions defined, best response dynamics generally does converge to pure Nash equilibrium strategy profiles.

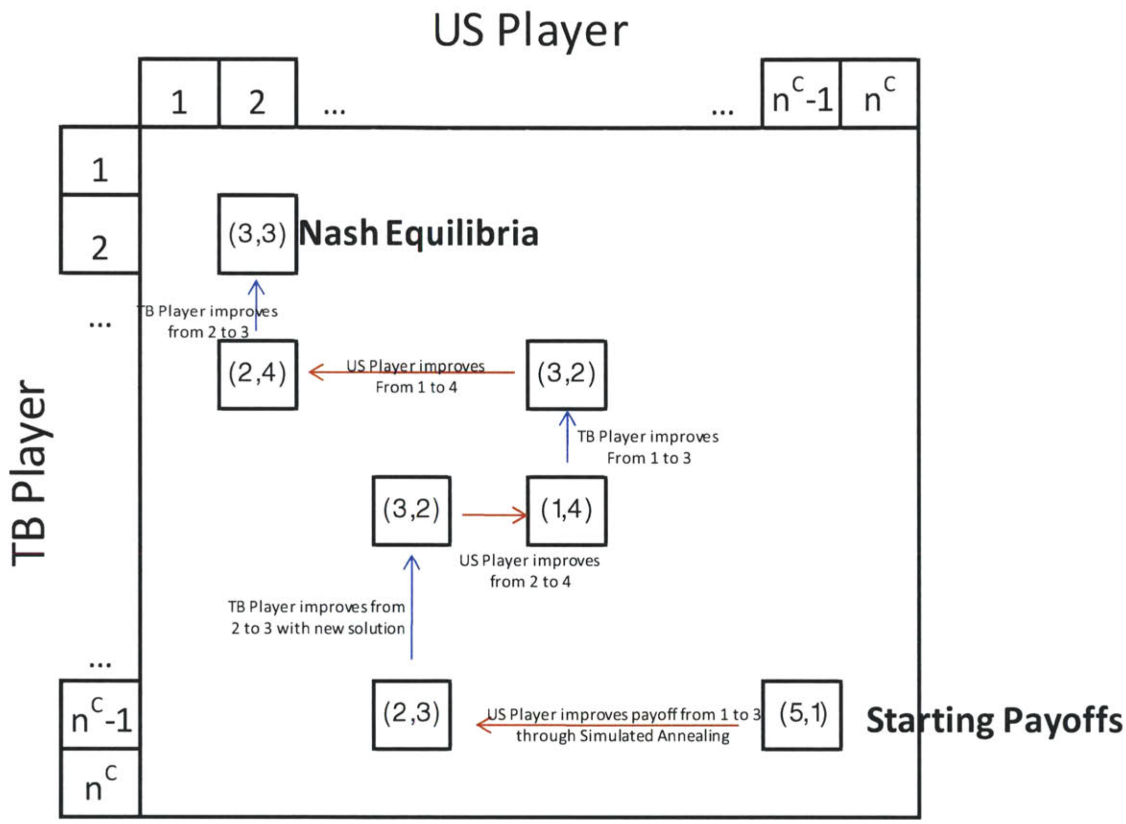


*Figure 3.7 – Notional Search through the Payoff Matrix via Simulated Annealing using Best Response Dynamics*

*Both players start at strategy $n^C$-1 and the TB player gets a payoff of 5 and the US player gets 1. Through simulated annealing the US player finds a strategy that increases their payoff from 1 to 3. The TB player then finds a strategy that increases their payoff from 2 to 3. The US and TB then continue to unilaterally improve their payoffs and eventually the Nash equilibrium strategy where both players use strategy 2 and get a payoff of 3 is found through best response dynamics.*

Figure 3.7 shows a notional payoff matrix for a zero-sum game, where simulated annealing conducts row and column searches until it reaches the estimated Nash equilibrium. We estimate that a Nash equilibrium is reached if, after several iterations for both players (we use 5 in our code), no improvements have been found. Short of calculating the entire row and column in the payoff matrix, it is unknown whether the resulting solution is a true Nash equilibrium, but

Chapter 4 provides strong evidence indicating the method works well in practice. The key advantage to this approach is that simulated annealing searches through the payoff matrix without fully enumerating it.

The simulated annealing method does have a significant disadvantage – it has difficulty in cases with multiple equilibria. It can find a pure Nash equilibrium if at least 1 exists, but this does not mean there are not others. Even if the algorithm is run several times with random starting solutions (a multi-start approach), it may converge to a single equilibrium repeatedly, having difficulty finding the other equilibria. A deeper discussion of this along with experimental results is presented in Chapter 4.

## 3.6 – Modeling Summary

We described a stochastic, pairwise-interaction model with stubborn agents whose beliefs never change. Although this stochastic network does not converge to a fixed set of beliefs, it does converge in expectation. This equilibrium can be easily calculated using an isomorphism to a Markov chain. Unfortunately the second moment of the network beliefs cannot be efficiently calculated but we examine (3.31) (the bound on standard deviation from Yildiz et al) in Chapter 4 to see whether it has any use as a metric. The stochastic network's equilibrium is independent of the starting beliefs, and depends only on the topology and influence parameters. More specifically it depends on the probabilities of stubborn nodes absorbing random walks on the associated Markov chain. We defined a 2-player game where each player controls a fixed number of connections from their set of stubborn agents to the rest of the network. For this game we defined 3 different payoff functions, where all of the payoff functions are measures of network properties at equilibrium. This makes the support space of the payoff functions tractable because the equilibrium state depends only on the connections made (strategies chosen). We defined a simulated annealing heuristic for finding near-optimal solutions for a single player, in addition to exhaustive enumeration. The simulated annealing heuristic can be used to rapidly find estimated pure Nash equilibria in the game using best response dynamics without fully calculating the payoff matrix.

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# 4 – Experiments and Model Analysis

In this chapter we conduct several experiments to understand the dynamics of the 2-player game and analyze the simulated annealing heuristic. The first set of experiments finds the Nash equilibria in a variety of networks. Next we examine the bound on standard deviation proposed by Yildiz et al discussed in Chapter 3 to see if it properly characterizes the standard deviation in this model. Third, the simulated annealing heuristic is analyzed to better understand its performance. Fourth we perform sensitivity analyses on the model with regard to the influence parameters. Finally we conduct an experiment showing how the game can naturally lead to non-symmetric low variance equilibria when players value agents differently.

## 4.0.1 – Data Parameters

In all of the experiments (except the sensitivity analysis section) we use the same set of influence parameters used by Hung. This is done so that we can compare our results to his. In addition, the numbers chosen by Hung were based on his research of Pashtun societal structure and thus were intended to be representative of an actual society. Furthermore, sensitivity analyses of these influence parameters conducted in Section 4.4 show surprising robustness to significant variations in the parameters, indicating that even if these parameters aren't entirely accurate, we can still have a high degree of confidence in the results.

The influence parameters developed by Hung for 'regular' agents are:

$$\begin{cases} \beta_{ij}=1.0, \ \alpha_{ij}=0.0, \ \gamma_{ij}=0.0, & \text{if } i,j \in V_R \\ \beta_{ij}=0.0, \ \alpha_{ij}=1.0, \ \gamma_{ij}=0.0, & \text{if } i \in V_I \cup V_V, j \in V_R \end{cases} \tag{4.1}$$

(4.1) means that a villager who is a 'regular' agent (head of household ($\in V_R$)) always reaches a pair-wise consensus with other regular agents, and always adopts a higher leader's attitude. Hung's parameters for 'influential' village leaders and district/regional leaders are:

$$\begin{cases} \beta_{ij}=0.1, \ \alpha_{ij}=0.0, \ \gamma_{ij}=0.9, & \text{if } i,j \in V_I \\ \beta_{ij}=0.1, \ \alpha_{ij}=0.0, \ \gamma_{ij}=0.9, & \text{if } i,j \in V_V \end{cases} \tag{4.2}$$

This second set of parameters in (4.2) implies that a village leader ($\in V_I$) or district/regional leader ($\in V_V$) reaches a pair-wise consensus with another leader of the same level with small probability (0.1), but would otherwise retain their own attitude.

$$\{\beta_{ij}=0.1,\ \alpha_{ij}=0.4,\ \gamma_{ij}=0.5,\ \text{ if } i \in V_V, j \in V_I \tag{4.3}$$

The parameters in (4.3) imply that an 'influential' village leader reaches a pair-wise consensus with a 'very influential' district/regional leader with small probability (0.1), and either adopts the district/regional leader's attitude (0.4) or retains his own attitude with greater probability (0.5).

$$\begin{cases} \beta_{ij}=0.0,\ \alpha_{ij}=1.0,\ \gamma_{ij}=0.0, & \text{if } i \in V_S, j \in V_M \\ \beta_{ij}=0.0,\ \alpha_{ij}=0.0,\ \gamma_{ij}=1.0, & \text{if } i,j \in V_S \end{cases} \tag{4.4}$$

This final set of parameters implies both the US and Taliban stubborn agents can always persuade any other agent to adopt their extreme attitude. This last assumption is justified by the observation that Taliban agents use armed propaganda and violence to persuade the population. Similarly, US forces have significant resources at their disposal to build projects and aid the local economies, and thus both wield significant influence.

### 4.0.2 – Experimental Setup

The computer used to perform all of the analysis is an HP Pavilion Notebook, with an AMD Turion II dual core Mobile M500 processor running at 2.2 GHz. The laptop has 8 GB of RAM, and is running 64 bit Windows 7 Home Premium. All of our code is written and run in 32-bit MATLAB version 8.8.0.346.

## 4.1 – Experiment Set 1: Characterizing Nash Equilibria

The purpose of this first set of experiments is twofold. First, it will verify when pure Nash equilibria exist in our game. Second, it will provide a set of Nash equilibria for us to compare other experiments against. For these experiments, we exhaustively enumerate the payoff matrix. As previously discussed in Chapter 3 computationally this can be very slow, but for a single connection per player (C=1), run times are reasonable. Furthermore, it is important for us to ensure optimality for this first set of experiments to provide a set of baseline results to later compare against heuristics to see whether they achieve optimality or not.

Three major classes of networks are analyzed along with two other specific networks created by Hung's network data generator. The three major classes are line networks (Figure 4.1), circle networks (Figure 4.9), and (fully) connected networks (Figure 4.12). Line networks and circle networks with both odd and even numbers of mutable agents are analyzed as well as different cases for each network type where there are different structures of 'regular', 'influential', and 'very influential' agents. Each case is selected to characterize a topology that is representative of properties found in segments of larger more complex networks. The full listing of networks is summarized in Table 4.1.

| Graph Type | # Mutable Agents | Description | Reason |
|---|---|---|---|
| Line | 9 | **Case A** - All agents are 'regular' | Base Case |
| | 10 | | |
| | 9 | **Case B** - 1 'Influential' agent in the middle, rest are 'regular' | How does the addition of an agent that is both influential and central effect the network? |
| | 10 | | |
| | 9 | **Case C** - 1 'Influential' agent on the end, rest are 'regular' | How does an influential but peripheral agent affect the network? |
| | 10 | | |
| Circle | 9 | **Case A** - All agents are 'regular' | Base Case |
| | 10 | | |
| | 9 | **Case B** - 1 'Influential' agent, rest are 'regular' | How does the addition of a single influential agent affect the network? |
| | 10 | | |
| | 9 | **Case C** - 2 'Influential' agents on opposite sides, rest are 'regular' | What happens when 2 influential agents are on opposite sides of a society? |
| | 10 | | |
| | 9 | **Case D** - 2 'Influential' agents adjacent, rest are 'regular' | What happens when both influential agents are highly connected? |
| | 10 | | |
| Fully Connected | 10 | **Case A** - All agents are 'regular' | Base Case |
| | 10 | **Case B** - 1 Influential agent, rest are 'regular' | Effect of influential agent in highly connected groups? |
| | 10 | **Case C** - 1 'Very Influential' agent, 2 'Influential' agents, rest are 'regular' | Effects of multiple types of agents in highly connected groups? |

*Table 4.1 – Network Test Case Descriptions*

For all of these cases each player has one stubborn agent with a single connection. The strategy for each player is defined by the choice of agent they connect the stubborn agent to. Each player has 9 or 10 options (depending on how many mutable agents are in each network). For the line networks and circle networks we test both with 9 and 10 mutable agents each in order to see if having odd or even degree (numbers of mutable agents) affects the pure Nash equilibria. All initial beliefs for mutable agents are 0 (i.e,– initially they are indifferent to supporting the Taliban or US Forces).

Three experiments for each network are performed – first the Nash equilibria are calculated when both players' payoff function is $f(\cdot)$ (mean belief); second, when both use the payoff function $g(\cdot)$ (agents won in equilibrium); and third, when the TB player uses $f(\cdot)$, and the US player uses $g(\cdot)$. The first two experiments are both zero sum games, but use different payoff functions. They will give some intuition on when and why different equilibria arise under different functions. The last experiment examines how the equilibria change when players have different payoff metrics, and whether non-symmetric equilibria and first mover advantages come into existence with different payoff functions.

Throughout the discussion of experimental results we refer to the 'centrality' of an agent in the network. There are several possible definitions of centrality. Here we use the term to mean the average number of steps required to get to any other agent in the network (where a step is moving to an adjacent agent). We also use the term 'strategy profile' throughout the chapter. For our game, a strategy profile is a vector containing the strategy of the TB player and the US player. Any pure Nash equilibrium has an associated strategy profile (although not every strategy profile is a pure Nash equilibrium).

## 4.1.1 – Line Networks

Figure 4.1 has diagrams of each of the cases of the even degree (10 mutable agent) line networks (diagrams of all of the networks along with their Nash equilibria are in Appendix B), to better illustrate the structure. The three cases allow us to examine the effect of an influential agent that is highly central versus one that is on the periphery of the network. It will help determine whether or not peripheral influential agents have a significant impact on pure Nash equilibria.

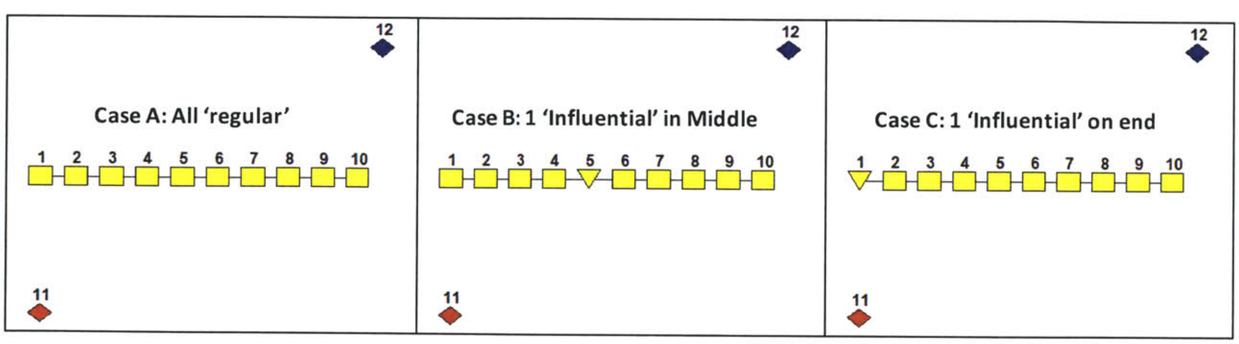



*Figure 4.1 – Even Line Network Test Cases*

#### 4.1.1.1 – Payoff Functions: Mean Belief vs. Mean Belief ($f(\cdot)$ vs. $f(\cdot)$)

Table 4.2 shows all of the pure Nash equilibria in the line network for 'mean belief' vs. 'mean belief' payoff functions. Each strategy profile corresponding to a pure Nash equilibrium is denoted as (x,y), where x is the strategy for the TB player and y is the strategy for the US player. Because both players have identical payoff functions, if strategy profile (x,y) is a pure Nash equilibrium, so is (y,x), although these matching strategy profiles are not listed in Table 4.2 for brevity.

| Pure Nash Equilibria for Line Networks | | |
|---|---|---|
| Payoff Functions - Mean Belief vs Mean Belief | | |
| Case | Even Degree | Odd Degree |
| A | (5,5), (5,6), (6,6) | (5,5) |
| B | (5,5) | (5,5) |
| C | (8,8) | (7,7) |

*Table 4.2 – Line Network Nash Equilibria for $f(\cdot)$ vs $f(\cdot)$*

*For each test case A, B, and C, the pure Nash equilibria are found by enumerating the entire payoff matrix when both players use the 'mean belief' payoff function This is done for even degree line networks (10 mutable agents) and odd degree line networks (9 mutable agents). Each pair (x,y) is a strategy profile corresponding to a pure Nash equilibrium where X is the TB player's strategy and Y is the US player's strategy. For these payoff functions if (x,y) is an equilibrium so is (y,x).*

For case A (all regular agents), the results are not surprising – when all agents are of the same type, centrality is the most important factor in determining Nash equilibria. On the even degree line network we do see multiple pure Nash equilibria corresponding to the two most central agents (agent 5 and 6 are equally central). Figure 4.2 shows two different pure Nash equilibrium strategy profiles on the line network with even degree and all regular agents.

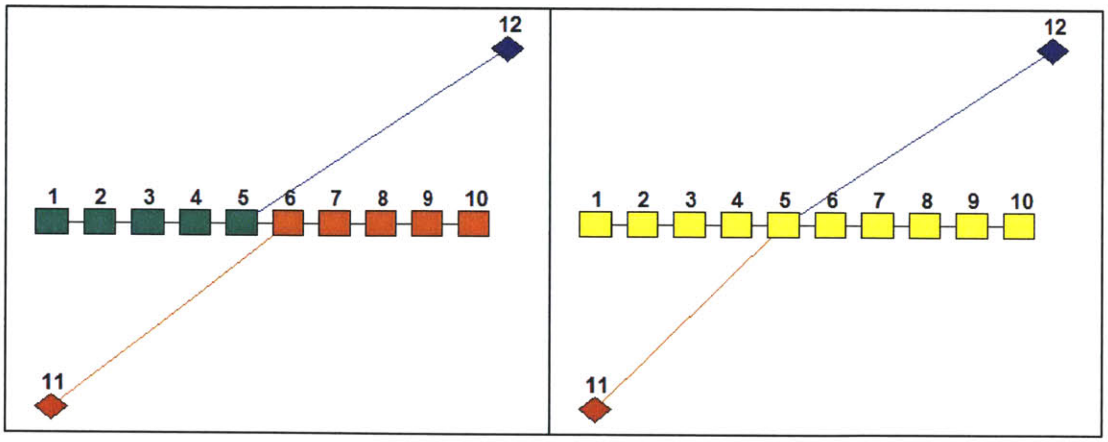



*Figure 4.2 – Even Line Network Equilibrium Beliefs with Different Strategy Profiles that are both Pure Nash Equilibria*

*The network on the left shows the equilibrium beliefs of each agent when the US player connects to 5 and the TB player connects to 6. Each player strongly influences 5 of the agents, but the mean belief of the network is zero. On the right both players connect to agent 5, and the equilibrium belief of every agent is zero, yielding a mean belief of zero. This shows 2 different strategy profiles that are both pure Nash equilibria and have identical payoffs for the players, but result in different equilibrium beliefs for the mutable agents.*

In Case B, both players connecting to the influential agent in the middle is the only pure Nash equilibrium strategy profile, whether there are even or odd numbers of agents. This is expected, because the middle agent is both highly central and the most influential agent in the network.

Case C has much more interesting equilibria due to two different phenomenon. First, with the current influence parameters, 'regular' agents cannot change the minds of influential agents (belief only flows down to regular agents, never up to more influential agents), which means that if no stubborn agent connects to an 'influential' agent, that agent will never change its belief, and effectively becomes a new type of stubborn agent with belief 0. Second, influence gets diffused over a long chain of agents. To better understand why connecting to the most influential agent is not an equilibrium strategy, we consider an example where the TB player connects to the influential agent (agent 1) under the naïve belief that this is the best strategy. Assuming complete information (players know the payoff matrix) and rationality, the US player would then connect to agent three and get an expected mean belief of 0.18 (see Figure 4.4). Figure 4.3 shows the equilibrium beliefs of each agent in the network if the TB player chooses to connect to agent one, and the US player connects to agent three.

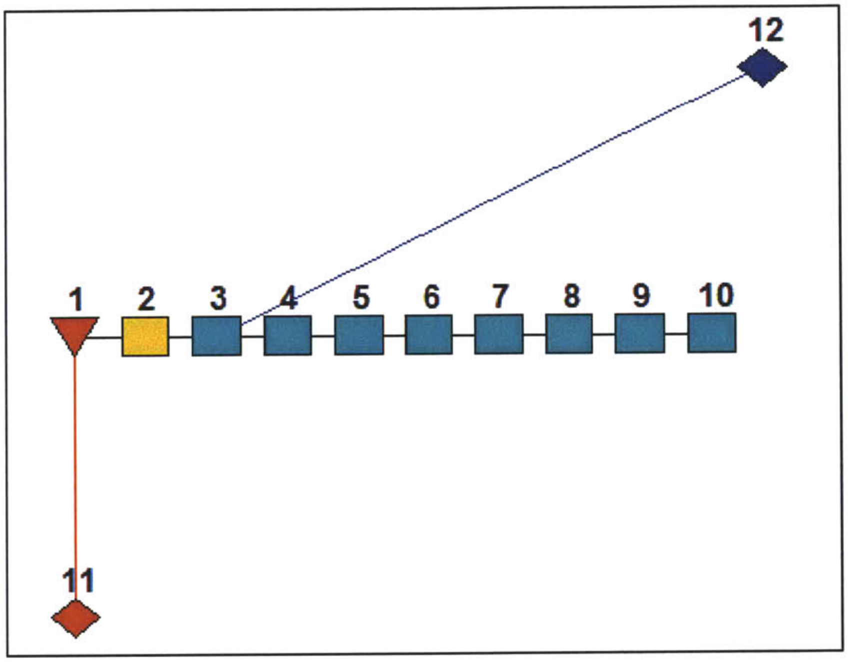



*Figure 4.3 –Even Line Network, Case C Equilibrium Beliefs for 1 vs. 3*

*Connecting to the most influential agent is not always the best strategy. If the TB player connects to the influential agent above, they influence agent 1 very strongly, but the US player is able to strongly influence agents 3-10 and cause the mean belief of the network to be 0.188, which yields a payoff of -.188 for the TB player.*

Figure 4.3 shows that although the TB influences agent one's belief to -0.5, the influence is rapidly diffused through the line network, and the US player strongly influences agent three and everything to the right of it. The reason that agent three is not completely won over to belief of +0.5 is that agent two moderates agent three's belief. This outcome is not good for the TB player, as Figure 4.4 shows that it yields -0.188 payoff (negative of US player). Assuming that the TB has full information and is rational, they would then connect to agent four. Each player then continues to move away from the influential agent until they both reach agent eight, which is the pure Nash equilibrium. Figure 4.4 shows the payoff matrix along with the set of strategy changes each player would make in order to maximize their payoff, eventually leading to the equilibrium strategy profile (8,8). Once they are at (8,8), any deviation by either player would strictly decrease their payoff (cell (8,8) is the maximum of the column and minimum of the row).

TB Player chooses to connect to the influential agent

| Strategy for US Player | Strategy for TB Player: 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.000 | -0.100 | -0.188 | -0.186 | -0.164 | -0.131 | -0.094 | -0.053 | -0.011 | 0.026 |
| 2 | 0.100 | 0.000 | -0.249 | -0.257 | -0.245 | -0.224 | -0.197 | -0.168 | -0.136 | -0.109 |
| 3 | 0.188 | 0.249 | 0.000 | -0.188 | -0.186 | -0.167 | -0.139 | -0.105 | -0.069 | -0.037 |
| 4 | 0.186 | 0.257 | 0.188 | 0.000 | -0.140 | -0.130 | -0.106 | -0.073 | -0.036 | -0.003 |
| 5 | 0.164 | 0.245 | 0.186 | 0.140 | 0.000 | -0.093 | -0.076 | -0.046 | -0.009 | 0.024 |
| 6 | 0.131 | 0.224 | 0.167 | 0.130 | 0.093 | 0.000 | -0.047 | -0.022 | 0.013 | 0.047 |
| 7 | 0.094 | 0.197 | 0.139 | 0.106 | 0.076 | 0.047 | 0.000 | -0.002 | 0.032 | 0.067 |
| 8 | 0.053 | 0.168 | 0.105 | 0.073 | 0.046 | 0.022 | 0.002 | 0.000 | 0.043 | 0.080 |
| 9 | 0.011 | 0.136 | 0.069 | 0.036 | 0.009 | -0.013 | -0.032 | -0.043 | 0.000 | 0.081 |
| 10 | -0.026 | 0.109 | 0.037 | 0.003 | -0.024 | -0.047 | -0.067 | -0.080 | -0.081 | 0.000 |

US Player responds by connecting to agent 3

*Figure 4.4 –Even Line Network, Case C Payoff Matrix Showing Convergence to Nash Equilibria*

*Though best response dynamics, even if the TB player starts with a sub-optimal strategy, the players can unilaterally search through the payoff matrix and get to the pure Nash equilibrium strategy profile of (8,8).*

These results at first may seem counterintuitive, but occur because regular agents cannot change the belief of influential leaders. For an odd-degree network the equilibrium is at agent seven instead of agent eight, but the principles causing this are identical.

This simple example also demonstrates best response dynamics. Each player takes the other's strategy as a fixed strategy, and then finds their best possible response to it. This is then repeated for the other player. This is continued, resulting in a series of row and column searches through the payoff matrix.

### 4.1.1.2 – Payoff Functions: Agents Won vs. Agents Won ($g(\cdot)$ vs. $g(\cdot)$)

In case A the results for $g(\cdot)$ vs. $g(\cdot)$ are the same as when using the 'mean belief' payoff functions ($f(\cdot)$ vs. $f(\cdot)$), which is not surprising. In the absence of influential agents, centrality among the regular agents remains the most important factor even with the new payoff function. Case B has similar results – players choose to connect to the influential agent that is also the most central in the network.

| Pure Nash Equilibria for Line Networks | | |
|---|---|---|
| Payoff Functions - Agents Won vs Agents Won | | |
| Case | Even Degree | Odd Degree |
| A | (5,5), (5,6), (6,6) | (5,5) |
| B | (5,5) | (5,5) |
| C | (6,6) | (5,5), (5,6), (6,6) |

*Table 4.3 –Line Network Nash Equilibria for g(·) vs g(·)*

*For each test case A, B, and C, the pure Nash equilibria are found by enumerating the entire payoff matrix when both players use the 'agents won' payoff function. This is done for even degree line networks (10 mutable agents) and odd degree line networks (9 mutable agents). Each pair (x,y) is a strategy profile corresponding to a pure Nash equilibrium where X is the TB player's strategy and Y is the US player's strategy. For these payoff functions if (x,y) is an equilibrium so is (y,x).*

In case C, for the even degree line network, each player connects to the most central agent and both get zero payoff. The odd degree line network yields more interesting results in Case C. Because regular agents cannot change the belief of influential agents it affects the equilibrium in a surprising way. To better understand this, Figure 4.5 shows the payoff matrix for the US player (the TB player's payoff is the negative of this matrix).

Expected Nodes Won by US Player

| Strategy for US Player | | Strategy for TB Player | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| | 1 | 0 | -7 | -5 | -5 | -3 | -3 | -1 | -1 | 1 |
| | 2 | 7 | 0 | -6 | -6 | -6 | -4 | -4 | -4 | -2 |
| | 3 | 5 | 6 | 0 | -4 | -4 | -2 | -2 | 0 | 0 |
| | 4 | 5 | 6 | 4 | 0 | -2 | -2 | 0 | 0 | 2 |
| | 5 | 3 | 6 | 4 | 2 | 0 | 0 | 0 | 2 | 2 |
| | 6 | 3 | 4 | 2 | 2 | 0 | 0 | 2 | 2 | 4 |
| | 7 | 1 | 4 | 2 | 0 | 0 | -2 | 0 | 4 | 4 |
| | 8 | 1 | 4 | 0 | 0 | -2 | -2 | -4 | 0 | 6 |
| | 9 | -1 | 2 | 0 | -2 | -2 | -4 | -4 | -6 | 0 |

Set of Equilibrium Payoffs

*Figure 4.5 – Line Network, Case C Payoff Matrix for g(·) vs g(·)*

*Above we see that if the TB players selects any strategy besides 5 or 6, then the US player can find a strategy with positive payoff – meaning that column 5 and 6 are the only columns with no positive entries. Because the TB player's payoff is the negative if the US player's payoff under these payoff functions, it means that if the TB player selected any strategy besides 5 or 6, then the US player's best response would yield negative payoff for the TB player.*

Figure 4.6 shows what the equilibrium beliefs of the network are under the pure Nash equilibrium strategy profile (5,6).

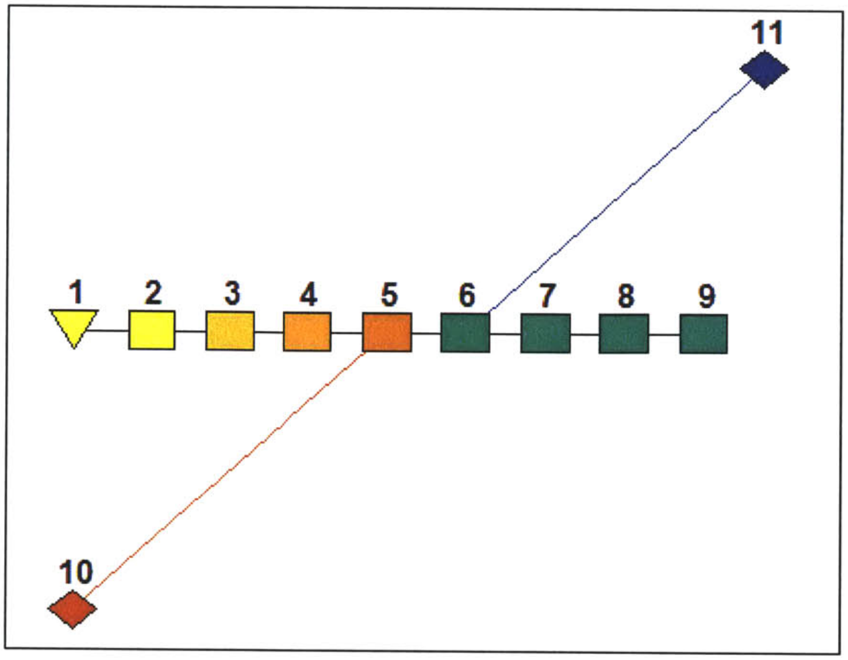



*Figure 4.6 – Line Network, Case C, Equilibrium Beliefs When Playing Nash Equilibrium Strategy*

*With the strategy profile (5,6), the TB player influences agent 2, 3, 4, and 5 enough to make them have negative equilibrium beliefs. The US player influences agents 6, 7, 8, and 9 such that their equilibrium beliefs are positive. Agent 1 maintains their initial belief of 0 because agent 2 cannot influence them – 'regular' agents cannot influence 'influential' agents. This means both players 'win' 4 agents and 'lose' 4 agents, yielding zero payoff for both under the 'agents won' payoff function.*

Under the payoff function of 'agents won', case C becomes equivalent to a line network with eight regular agents and the influential agent can be ignored in the network.

### 4.1.1.3 – Payoff Functions: Mean Belief vs. Agents Won ($f(\cdot)$ vs. $g(\cdot)$)

For this set of payoff functions it cannot be assumed that if (x,y) is an equilibrium that (y,x) is an equilibrium. The payoff functions for each player are different and there is no guarantee of symmetric equilibria.

| Pure Nash Equilibria for Line Networks | | |
|---|---|---|
| Payoff Functions - Mean Belief vs Agents Won | | |
| Case | Even Degree | Odd Degree |
| A | (5,5), (5,6), (6,5), (6,6) | (5,5) |
| B | (5,5), (5,6) | (5,5) |
| C | (7,6), (8,7) | (7,6),(6,5) |

*Table 4.4 – Line Network Nash Equilibria for $f(\cdot)$ vs $g(\cdot)$*

*For each test case A, B, and C, the pure Nash equilibria are found by enumerating the entire payoff matrix when the TB player uses the 'mean belief' payoff function and the US player uses the 'agents won' payoff function. This is done for even degree line networks (10 mutable agents) and odd degree line networks (9 mutable agents). Each pair (x,y) is a strategy profile corresponding to a pure Nash equilibrium where X is the TB player's strategy and Y is the US player's strategy.*

In case A both players connect to the most central agents. For the even degree case, any set of any agents five and six will yield a 'mean belief' of zero and hence zero payoff for the TB player. It also yields zero payoff for the the US player under the 'agents won' payoff function. Thus in case A both players get zero payoff in the any of the equilibria.

Case B has an interesting result for the even degree network. The TB player only wants to connect to agent five, but the US player can connect to agent 5 or 6 and still have a pure Nash equilibrium strategy profile. This is our first example of a non-symmetric equilibrium. It also demonstrates the game is not zero-sum, because if both connect to agent five then they both get zero payoff (mean belief of all agents is zero with identical strategies), but if the US player instead connects to agent 6 we see an increase in the TB player's payoff to 0.125 (see Figure 4.7), while the US player's payoff remains zero. This shows the game is non-zero sum, because the TB increases their payoff without decreasing the US payoff (which remains zero either way).

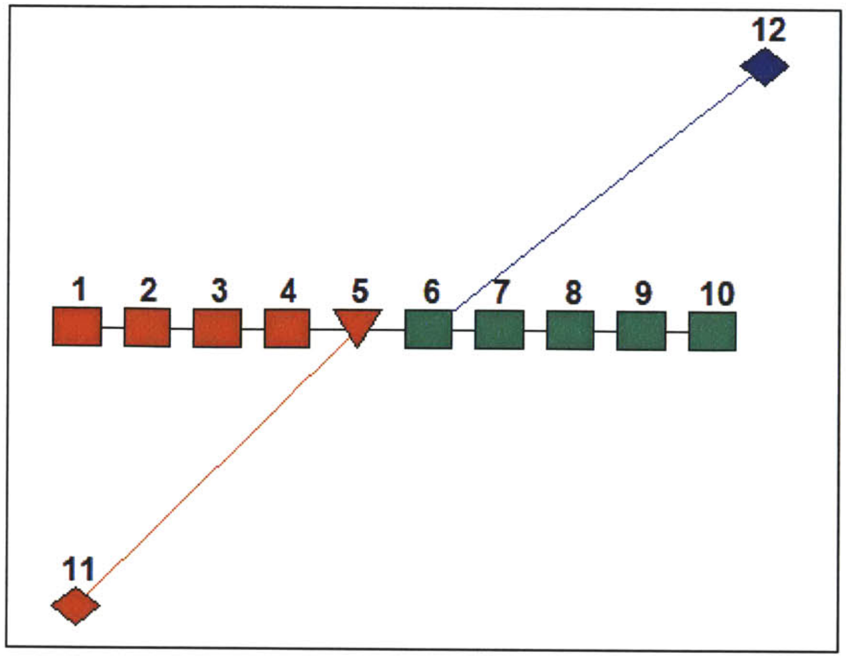



*Figure 4.7 – Line Network Case B Equilibrium Beliefs When Playing Nash Equilibrium Strategy*

*The TB player influences agents 1, 2, 3, 4, and 5 stronger than the US player influences agents 6, 7, 8, 9, and 10. The TB player using the 'mean belief' payoff function gets a payoff of 0.125. The US player using the 'agents won' payoff function 'win's 6-10, but 'loses' 1-5 yielding zero payoff.*

Case C also yields interesting results. There are two equilibria, with different payoffs for each player. Table 4.5 summarizes the payoffs for each player at the two equilibria in the even network (similar results apply for the odd degree case).

| Strategy Profile | TB Player Payoff | US Player Payoff |
|---|---|---|
| (7,6) | 0.0475 | 1 |
| (8,7) | 0.0002 | 3 |

*Table 4.5 – Player Payoffs at Equilibrium for Line Network, Case C, $f(\cdot)$ vs $g(\cdot)$*

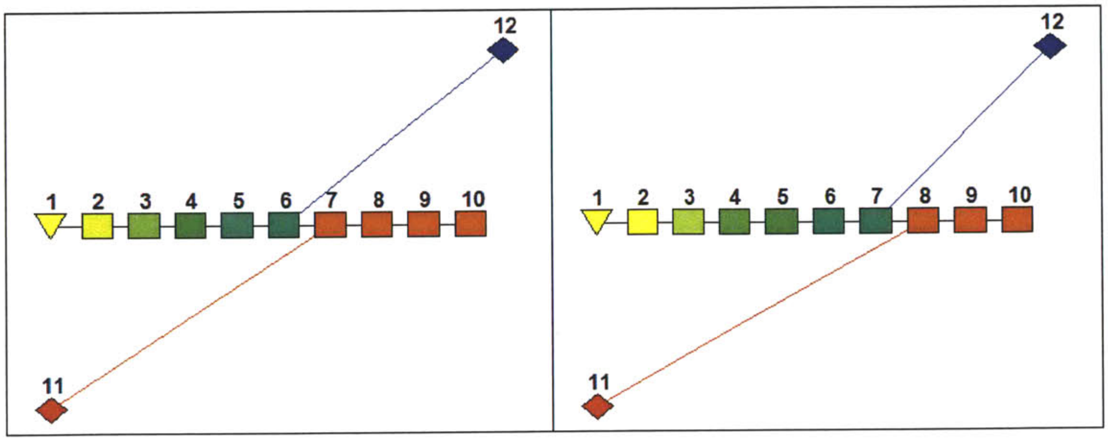



*Figure 4.8 – Depiction of Equilibrium Beliefs of Line Network, Case C, $f(\cdot)$ vs $g(\cdot)$ Using Equilibrium Strategies*

*If the US player gets to select which agent to connect to first, then they will select agent 7 because it yields greatest payoff for them. This would cause the TB player to select agent 8, as it is the best response for them given that the US player has selected agent 7. If the TB player selects first, then they would select agent 7, because it would maximize their payoff. However, once the TB player has chosen agent 7, the US player's best response becomes agent 6. This is an example of a first-mover advantage.*

Table 4.5 and Figure 4.8 show that having different payoff functions can induce a first mover advantage to the game.

### 4.1.1.4 – Line Network Experiment Summary

The first sets of experiments on the line networks have shown that with identical payoff functions it is a zero sum game with no first mover advantage. There are not significant differences between even and odd degree line networks. In some cases having a different degree can add or remove equilibria, but the principles that lead to the existence of the equilibria (generally centrality and influence) do not change whether the network has odd or even degree. When players use non-identical payoff functions the game can become non-zero sum, non-symmetric equilibria may exist, and a first mover advantage can be created.

We have also seen that the power of a single influential agent can be rapidly diffused over a line of regular agents, and effectively isolated from the rest of the network. This means that peripheral influential agents may have minimal effects on the rest of the network.

Finally, we see that pure Nash equilibria exist in all of the line networks, regardless of payoff functions.

## 4.1.2 – Circle Networks

The next set of experiments is conducted on circle networks of both odd and even degree (again 9 and 10 mutable agents each). The major difference between the circle networks and line networks is that every agent is equally central in a circle network. Four different cases are examined for the circle networks. Case A has all regular agents, Case B has a single influential agent, Case C has two influential agents on opposite sides of the circle, and Case D has influential agents adjacent to each other. This helps to examine the effect of highly central and connected influential agents versus influential agents controlling a small neighborhood of regular agents. Case C is an example wherein two groups interact with each other, but whose leaders have no direct communication with one another. Figure 4.9 has depictions of each of the different cases (all even degree).

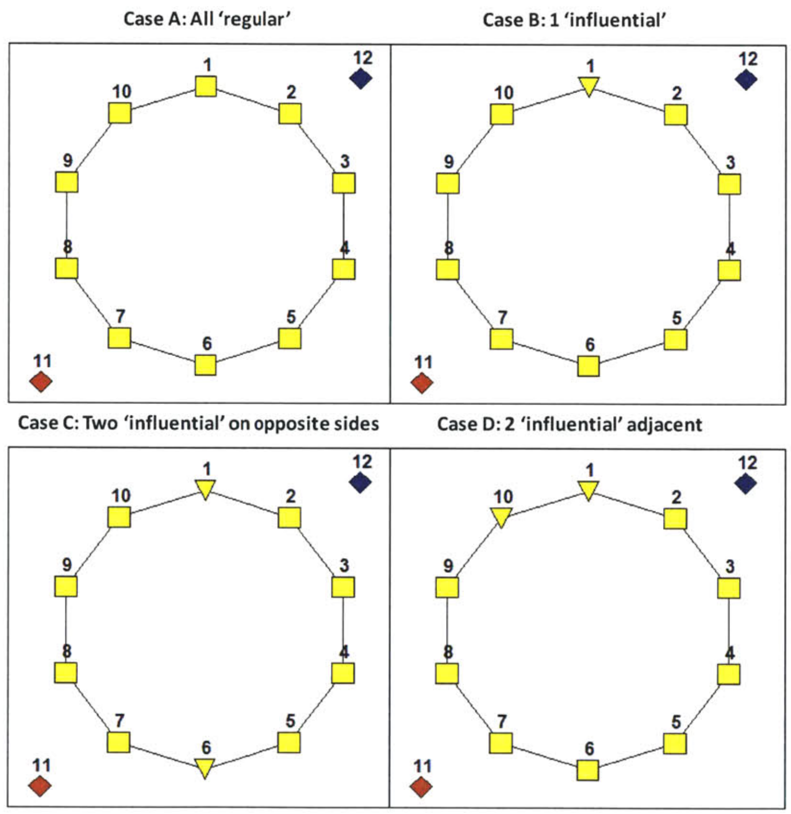



*Figure 4.9 – Even Circle Network Test Cases*

### 4.1.2.1 – Payoff Functions: Mean Belief vs. Mean Belief ($f(\cdot)$ vs. $f(\cdot)$)

For the circle networks with mean belief vs. mean belief, we see that the determining factor for the pure Nash equilibria is the level of influence of an agent. In Case A when there are no influential agents, any pair of agents is an equilibrium strategy profily. For all other cases the equilibria correspond to influential agents, as seen in Table 4.6.

| Pure Nash Equilibria for Circle Networks | | |
|---|---|---|
| Payoff Functions - Mean Belief vs Mean Belief | | |
| Case | Even Degree | Odd Degree |
| A | All pairs | All pairs |
| B | (1,1) | (1,1) |
| C | (1,1), (1,6), (6,6) | (1,1), (1,5), (5,5) |
| D | (1,1), (1,10), (10,10) | (1,1), (1,9), (9,9) |

*Table 4.6 – Circle Network Pure Nash Equilibria for $f(\cdot)$ vs $f(\cdot)$*

*For each test case A, B, C, and D, the pure Nash equilibria are found through enumerating the entire payoff matrix when both players use the 'mean belief' payoff function. This is done for even degree line networks (10 mutable agents) and odd degree line networks (9 mutable agents). Each pair (x,y) is a strategy profile corresponding to a pure Nash equilibrium where X is the TB player's strategy and Y is the US player's strategy. For these payoff functions if (x,y) is an equilibrium so is (y,x). All strategy profiles corresponding to pure Nash equilibria only connect to influential agents.*

When all agents are regular, any strategy profile is a pure Nash equilibrium. This is due the symmetry in the circle network with no influential agents. Figure 4.10 shows an example of this.

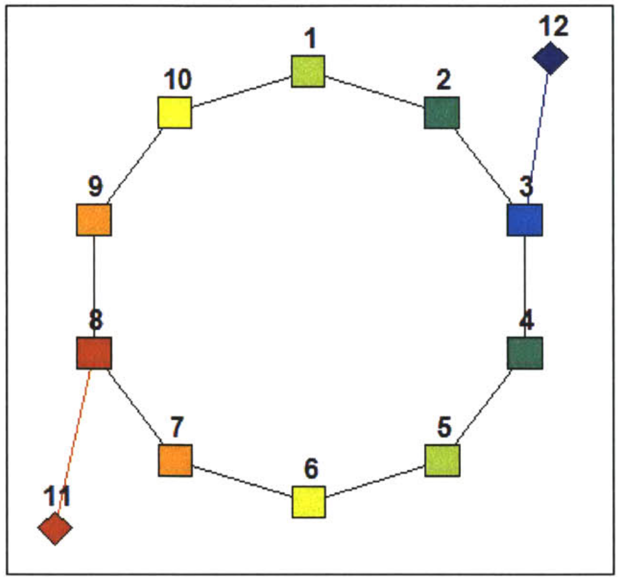



*Figure 4.10 – Circle Network, Case A Equilibrium Beliefs When Playing Nash Equilibrium Strategies*

*Each player connects to a regular agent. For every agent that they influence more than the other player, there is an agent opposite them that the other players influences identically. For example, agent 3 is won very strongly by the US player, but agent 8 is won equally strongly by the TB player. Thus the mean belief of the network is zero*

For cases B, C, and D, any strategy profile consisting of any combination of influential agents is a pure Nash equilibrium. This occurs because in a circle network all agents are equally central. Thus the positioning of influential agents is less important than in the line networks, and it is unsurprising that the forcefulness of an agent is the most important characteristic in determining pure Nash equilibria strategy profiles.

### 4.1.2.2 – Payoff Functions: Agents Won vs. Agents Won ($g(\cdot)$ vs. $g(\cdot)$)

For these payoff functions the results for Case A are identical to 'mean belief' vs. 'mean belief'. The symmetry of the circle network when all agents are regular makes any set of agents an equilibrium strategy profile, where each player 'wins' five agents and 'loses' five agents, yielding a payoff of 0 (or they have matching strategies and every agent has mean belief zero yielding zero payoff for both players).

| Pure Nash Equilibria for Circle Networks | | |
|---|---|---|
| Payoff Functions - Agents Won vs Agents Won | | |
| Case | Even Degree | Odd Degree |
| A | All pairs | All pairs |
| B | (1,1), (1,6), (6,6) | (1,1) |
| C | (1,1), (1,6), (6,6) | (1,1), (1,5), (5,5) |
| D | (1,1), (1,10), (10,10) | (1,1), (1,9), (9,9) |

*Table 4.7 – Circle Network Pure Nash Equilibria for g(·) vs g(·)*

*For each test case A, B, C, and D, the pure Nash equilibria are found by enumerating the entire payoff matrix when both players use the 'agents won' payoff function. This is done for even degree line networks (10 mutable agents) and odd degree line networks (9 mutable agents). Each pair (x,y) is a strategy profile corresponding to a pure Nash equilibrium where X is the TB player's strategy and Y is the US player's strategy. For these payoff functions if (x,y) is an equilibrium so is (y,x).*

Case B has a more interesting result. There is only 1 influential agent, but there is a pure Nash equilibrium involving agent six, which is not an influential agent. This is because agent six is exactly opposite of the influential agent.

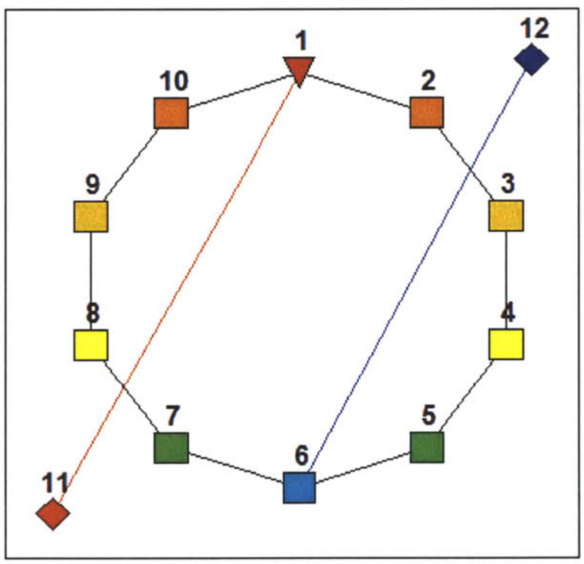



*Figure 4.11 – Circle Network, Case B Equilibrium Beliefs When Playing Nash Equilibrium Strategy Profile (1,6)*

*Although the TB player influences agents 9, 10, 1, 2, and 3 stronger than the US player influences 4, 5, 6, 7, and 8, they each influence 5 agents stronger than the other player. Under the 'agents won' payoff function this means they each get 5 points and lose 5 points, yielding a payoff of zero for both, even though the TB player wields more overall influence in the network equilibrium beliefs (the equilibrium mean belief of the network is negative)*

#### 4.1.2.3 – Payoff Functions: Mean Belief vs. Agents Won ($f(\cdot)$ vs. $g(\cdot)$)

For this combination of payoff functions there is no guarantee of symmetric equilibria, and the reader should not assume that if (x,y) is a pure Nash equilibrium then (y,x) is also an equilibrium unless it is listed in the table. Again for a circle network of all regular agents any strategy profile is a pure Nash equilibrium due to the symmetry.

| Pure Nash Equilibria for Circle Networks | | |
|---|---|---|
| Payoff Functions - Mean Belief vs Agents Won | | |
| Case | Even Degree | Odd Degree |
| A | All pairs | All pairs |
| B | (1,1), (1,6) | (1,1) |
| C | (1,1), (1,6), (6,1), (6,6) | (1,1), (1,5), (5,1), (5,5) |
| D | (1,1), (1,10), (10,1), (10,10) | (1,1), (1,9), (9,1), (9,9) |

*Table 4.8 – Circle Network Pure Nash Equilibria for $f(\cdot)$ vs $g(\cdot)$*

*For each test case A, B, C, and D, the pure Nash equilibria are found by enumerating the entire payoff matrix when the TB player used the 'mean belief' payoff function and the US player uses the 'agents won' payoff function. This is done for even degree line networks (10 mutable agents) and odd degree line networks (9 mutable agents). Each pair (x,y) is a strategy profile corresponding to a pure Nash equilibrium where X is the TB player's strategy and Y is the US player's strategy.*

Case B has a pure Nash equilibrium at (1,6). This is similar to what we saw in the $g(\cdot)$ vs $g(\cdot)$ case. The TB player wants to select agent one because it is the best strategy under the mean belief payoff function. The US player is ambivalent towards selecting either agent one or six, as both of them yield a payoff of zero for the US. This is another example of a non-zero sum equilibrium. If the US moves from strategy one to six, there is an increase in the TB's payoff with no change in the US payoff.

In case C and D any combination of influential agents yields a pure Nash equilibrium, as seen before. Because centrality is identical for all agents, influence is again the only determining characteristic in determining pure Nash equilibria.

#### 4.1.2.4 – Circle Network Experiment Summary

In the circle networks any strategy profile in a circle network with all regular agents is a pure Nash equilibrium because all agents are equally central in a circle network. This symmetry in circle networks makes the influence level of agents the most important factor in determining pure Nash equilibrium strategy profiles.

### 4.1.3 – Fully Connected Networks

The third set of network is fully connected (meaning all agents are connected to all other agents). These fully connected networks illustrate how equilibria are achieved in highly connected networks. We do not test both even and odd degree cases, because there is no significant difference for a fully connected network. No matter how many agents are in the network, all agents are adjacent to every other agent. Three different cases are tested – Case A is again all regular agents, Case B has a single influential agent, and Case C has 2 influential agents and one very influential agent. These different cases will allow us to see the impact of the introduction of a single influential agent, as well as the impact of multiple types of influential agents. We do not consider different locations for the influential agents because all agents in the network have identical positions (every agent is adjacent to every other agent). Figure 4.13 has diagrams of the three cases.

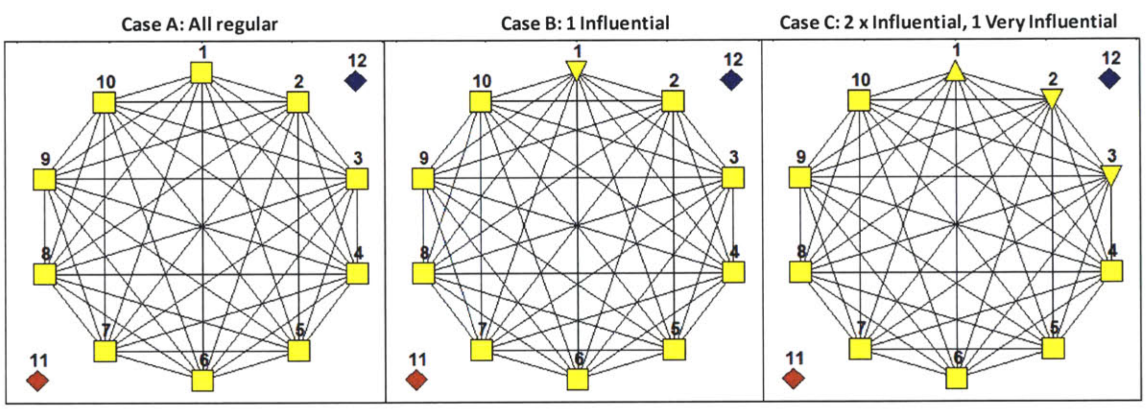



*Figure 4.12 – Fully Connected Network Test Cases*

In the fully connected networks the pure Nash equilibria are identical regardless of the payoff functions used.

| Nash Equilibria for Fully Connected Graphs | |
|---|---|
| Case | Any Payoff Functions |
| A | All pairs |
| B | (1,1) |
| C | (1,1) |

*Table 4.9 – Fully Connected Network Pure Nash Equilibria for All Payoff Functions*

*For each test case A, B ,and C the pure Nash equilibria are found by enumerating the entire payoff matrix when both players use different payoff functions. The equilibria turn out to be identical regardless of payoff function used. Each pair (x,y) is a strategy profile corresponding to a pure Nash equilibrium where X is the TB player's strategy and Y is the US player's strategy.*

These results are unsurprising. Every agent is equally central in a connected network, and thus when there are all regular agents, any strategy profile is a pure Nash equilibrium. When an influential agent is added, connecting to them should be the best strategy for both players to choose, since the influential agent communicates directly with every other agent. When a very influential agent is added the same phenomenon occurs, yielding the same results. In a fully connected network the influence of an agent is the only thing that matters. It is also unsurprising that this is independent of payoff function. The 'agents won' payoff function is related to the mean belief payoff (very high mean belief also indicates large numbers of agents won). Thus, if choosing one particular agent is a very strong strategy under one payoff function it is often a strong strategy under the other.

## 4.1.4 – Small Rural Pashtun Network

This network was formulated as a simple scenario by Hung for experimental results developed in his thesis. It is a 16 mutable agent network, with three villages. Each village has a single 'influential' leader, and there is a 'very influential' agent at the district level that influences each of the village leaders. Village leaders talk to adjacent village leaders, but the first and third village leaders (agents 2 and 12) do not communicate directly.

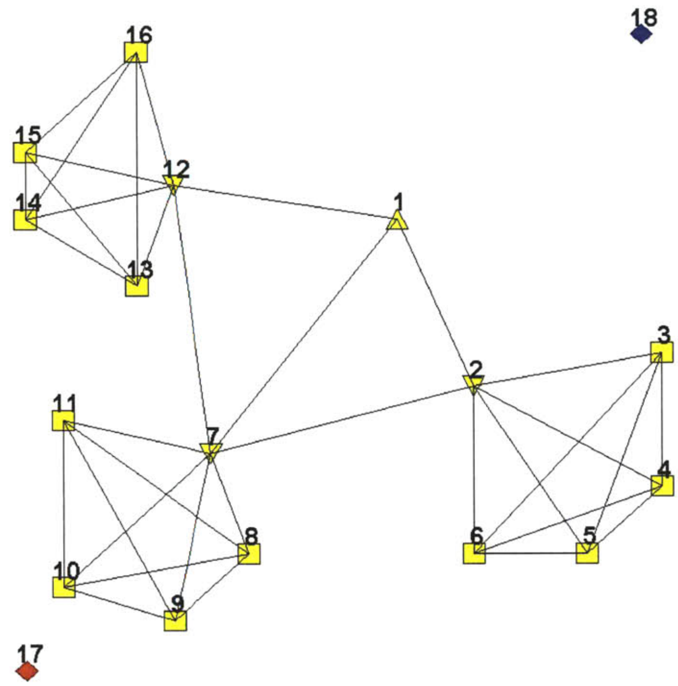



*Figure 4.13 – Small Rural Pashtun Network*

As in all of the previous Nash equilibria experiments, each player controls the connection of a single stubborn agent. This network differs from the three classes of networks previously analyzed in that it is larger, more complicated, and has characteristics of each type. There are small fully connected networks within each village, but connections between villages are much sparser as in the circle and line networks. It also has a perfect correlation between centrality and influence of agents. By this we mean that the most central agent is also the most influential, the second most central agent (agent 7) is one of the second most influential, etc. This mixture of both centrality and influence in the district leadership yields the following unsurprising results: no matter which payoff function is selected, (1,1) is the only pure Nash equilibrium strategy profile. This small rural Pashtun network has no non-symmetric equilibria or first-mover advantages independent of payoff functions used.

## 4.1.5 – Large Rural Pashtun Network

The last network analyzed in this set of experiments is a large rural Pashtun network, also developed by Hung in his thesis. This network is far more complex than any other network analyzed thus far. It has seven villages of differing sizes. Villages have different mixes of 'influential' and 'very influential' agents, with several district leaders operating in the network.

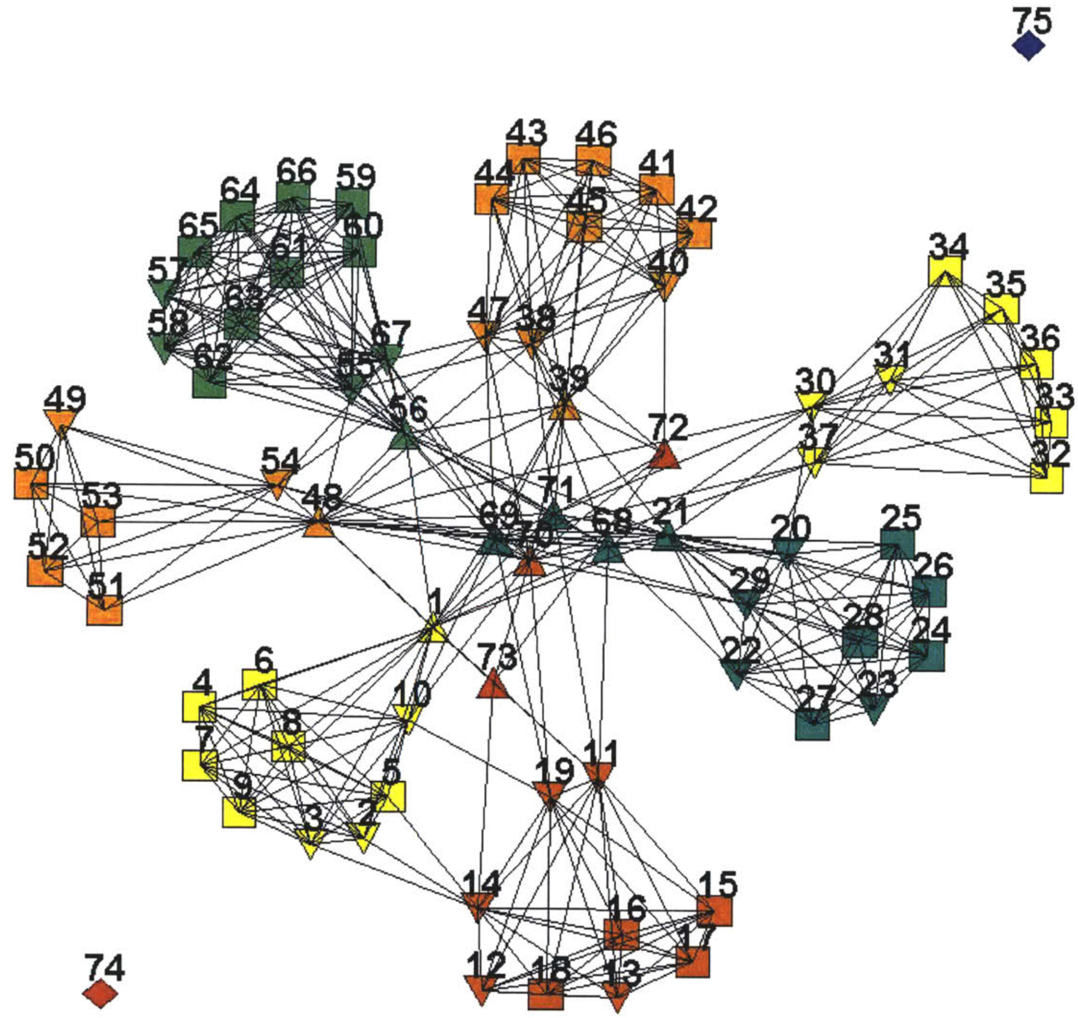



*Figure 4.14 – Large Rural Pashtun Network*

In this network, agents have different starting beliefs. The colors reflecting these initial beliefs in Figure 4.14 help visualize the different villages and associated leaders, but these colors do not represent any equilibrium beliefs.

Unlike most previous examples, determining which agents are most central and best to connect to is not obvious in any way. It turns out that agent 48 is the most central agent (mean distance of 1.68 to every other agent), with agent 70 being the second most central agent.

As in the small rural Pashtun network, we have a single pure Nash equilibrium for all sets of payoff functions – strategy profile (70,70). This is surprising because agent 70 is not the most central agent. To better understand this we looked at centrality among the 'very influential' agents only, thinking that although agent 48 is more central, perhaps agent 70 has more centrality among the district leadership. Both agents 48 and 70 are equally central among the 'very influential' agents, and also among all of the 'influential' agents. Clearly there is a more complex relationship between agent 70 and the rest of the network beyond centrality, highlighting the difficulty in determining these key influencers in social networks.

### 4.1.6 – Experiment Set 1 Summary

Every network analyzed had at least one pure Nash equilibrium. This provides evidence that our game always has pure Nash equilibria. This is useful; if mixed Nash equilibria were present it would mean that a player would have to randomly choose among a set of strategies, and telling a US commander to randomly select among a set of strategies is not useful operationally.

For all of the experiments discussed above the times taken to exhaustively calculate the payoff matrices and find the pure Nash equilibria are in Table 4.10.

| Network | Size | Time (sec) |
|---|---|---|
| Line Networks | 9-10 Agents | 0.14 |
| Circle Networks | 9-10 Agents | 0.14 |
| Fully Connect Networks | 10 Agents | 0.15 |
| Small Rural Pashtun Network | 16 Agents | 1.5 |
| Large Rural Pashtun Network | 73 Agents | 300 |

*Table 4.10 – Time to Find Pure Nash Equilibria through Exhaustive Enumeration*

This shows the computational challenge that exhaustive enumeration has with larger networks. The 9-10 agent networks run almost instantly, but a seven-fold increase from 10 to 73 agents results in 2000-fold increase in run time to find the equilibria.

Across all of the experiments we found that the strategy profiles correspdoning to pure Nash equilibria are a function of centrality of agents combined with their influence level. In sparse networks the centrality and topology of the network become the most important factors, but as we increase the number of connections between agents, the influence level starts to dominate in importance. Interesting results emerge in the cases between a sparse line network and a fully connected network where a combination of location in the network and influence level combine to make some agents more important than others. The larger and more complex the network becomes, the more difficult it is to determine equilibrium strategy profiles simply using centrality metrics.

We also find that using different payoff functions yields situations with non-symmetric equilibria, and can induce a first-mover advantage, although we did not see any examples of this in the larger networks. We believe this is mainly due to the high correlation between centrality and influence level in the rural Pashtun networks.

## 4.2 – Experiment Set 2: Analysis of The Bound on Standard Deviation

In this experiment we address the issue of variance in the model. There are two main reasons for this. First – in the real world most players would prefer to avoid a highly variable strategy. Second – penalizing high variance strategies (like using the 'risk averse mean belief' payoff function) will hopefully cause players to avoid the pure Nash equilibrium strategy profiles with identical strategies for both players (matching strategies) that we have seen throughout the previous set of experiments. In Iraq or Afghanistan it is important for commanders to show steady incremental improvement to maintain domestic political support. If a strategy has a 50-50 chance of leading to huge success, this is too much risk for an environment where quick success is less important than steady progress. Similarly a company planning an advertising campaign to introduce a new product to market would much rather start out by securing a base level of support in the market to establish a steady revenue stream than immediately try to win, at high risk, the entire network. Because most people influencing social networks have some level of risk aversion, we would also like to be able to mediate risk in the strategies chosen by players in our game.

Yildiz et al derived a bound on the standard deviation of the equilibrium mean belief in a model with only binary beliefs (all mutable agent beliefs are either 0 or 1); here we conduct an analysis to see how useful this bound is for predicting the standard deviation in our non-binary model. First the behavior of the bound is analyzed for different networks to see how the predicted standard deviation bound matches experimental standard deviation. Second the 'risk averse mean belief' payoff function is analyzed in the two player game to see what types of pure Nash equilibria occur, and whether it can result in low variance strategy profiles that perform well in mean belief.

### 4.2.1 – Experimental vs. Predicted Variance

The first experiment analyzes the line network used previously. We use the even degree line network with 10 regular agents because its behavior is simple and easy to understand. Throughout this section the term 'predicted standard deviation' is used to indicate the value produced by the bound on standard deviation yielded by (3.31). Although the number is a bound on standard deviation in a binary belief model the hope is to use this bound as a metric for variability in strategy profiles and is thus referred to as a predicted value throughout this analysis. We re-iterate the bound on standard deviation here for the reader's convenience.

*DEFINITION: $p_j^i$ is the probability that a random walk started at state j on the Markov chain will be absorbed by $V_I$ corresponding to mutable agent i*

$$r^i \equiv \sum_{j \in V_M, j \neq i} p_j^i \qquad \forall i \in V_M \tag{3.30}$$

*DEFINITION: $p_j^{US}$ is the probability of a random walk starting at node j being absorbed by one of the states in $V_{US}$.*

$$p_j^{US} = \sum_{i \in V_{US}} \eta_{ji}$$

$$\sigma(X^*) \leq \frac{1}{N^2} \sum_{j \in V_m} \left(p_j^{US} - (p_j^{US})^2\right) * (r^j)^2 \tag{3.31}$$

*where $\sigma(X^*)$ is the standard deviation of the mean equilibrium belief of the network*

#### 4.2.1.1 – Line Network

Several strategy profiles are simulated on the network, and the sample standard deviation of the mean belief of the network is calculated and compared to the predicted standard deviation. The goal is to determine how well the predicted standard deviation predicts the sample standard deviation. The strategy profiles are listed in Table 4.11, and were chosen to represent several cases in the network that would yield different variances. It includes strategy profiles that are expected to have high variability (e.g., like matching strategies), strategy profiles with low expected variability (e.g., when each player connects to opposing ends of the network), and several strategy profiles in between these two extremes. Each case listed shows the TB strategy vs. the US strategy, so 1 vs. 10 means that $S_{TB}$={1}, $S_{US}$={10}. Predicted Std Dev is the standard deviation predicted by (3.31). Experimental Std Dev is the standard deviation of the equilibrium mean belief of the network in simulation. Each simulation was run for 5000 interactions, and all of the experiments converged within the neighborhood of the network equilibrium mean belief within 1000 interactions (this was determined by visually inspecting the mean belief plot over time for each of the experiments). To ensure that our experimental standard deviation was actually the standard deviation of the network only around equilibrium, the first 1000 interactions were not used in calculating the standard deviation of the sample. This is shown in Figure 4.15 below.

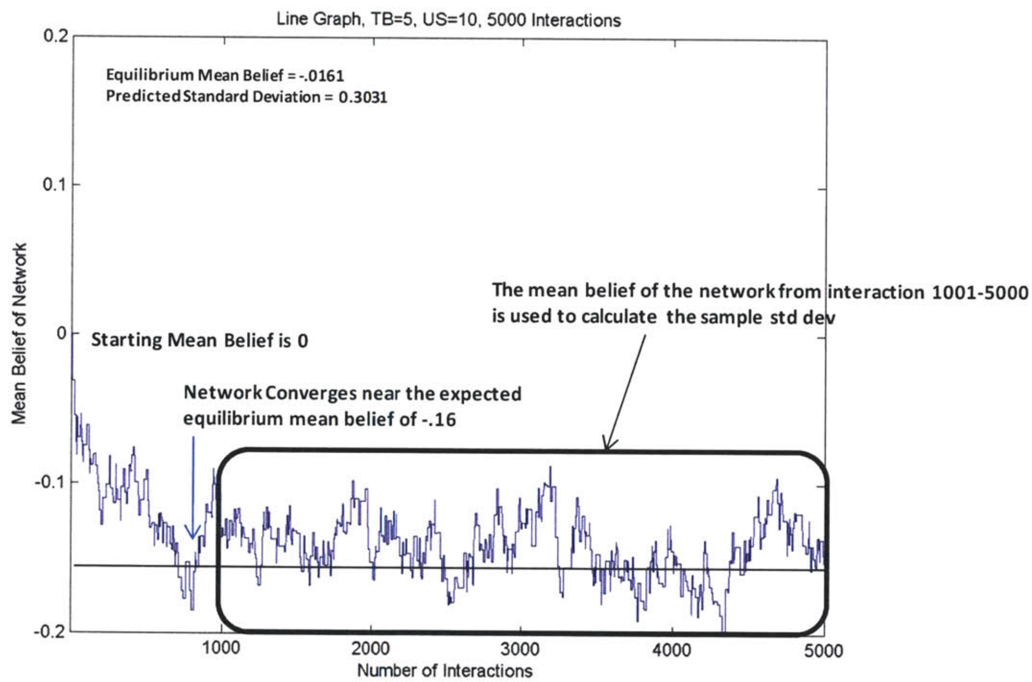



*Figure 4.15 – Simulated Mean Belief of Line Network, $S_{TB}$ = 5, $S_{US}$ = 10*

*The network moves towards the equilibrium mean belief and converges near it within 1000 interactions, and then oscillates around this value. To calculate the sample standard deviation of the equilibrium mean belief, only the mean belief from interaction 1001-5000 are used.*

The sample versus predicted values are in Table 4.11.

| Strategy Profile | Predicted Std Dev | Sample Std Dev |
|---|---|---|
| 5 vs 5 | 0.3230 | 0.0655 |
| 5 vs 6 | 0.1322 | 0.0351 |
| 5 vs 7 | 0.0944 | 0.0308 |
| 5 vs 8 | 0.0800 | 0.0272 |
| 5 vs 9 | 0.0811 | 0.0212 |
| 5 vs 10 | 0.0919 | 0.0233 |
| 4 vs 10 | 0.0977 | 0.0172 |
| 3 vs 10 | 0.1184 | 0.0181 |
| 2 vs 10 | 0.1529 | 0.0133 |
| 1 vs 10 | 0.1938 | 0.0139 |

*Table 4.11 – Predicted vs Sample Std Dev of Line Network*

The data is plotted in Chart 4.1 to better visualize it. The predicted standard deviation is plotted as the x coordinate and the sample standard deviation is plotted as the y coordinate for each strategy profile.

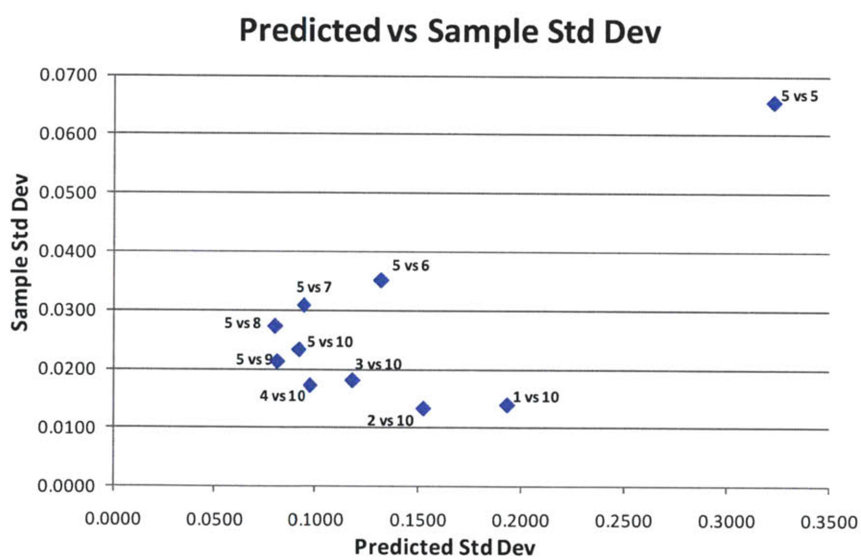


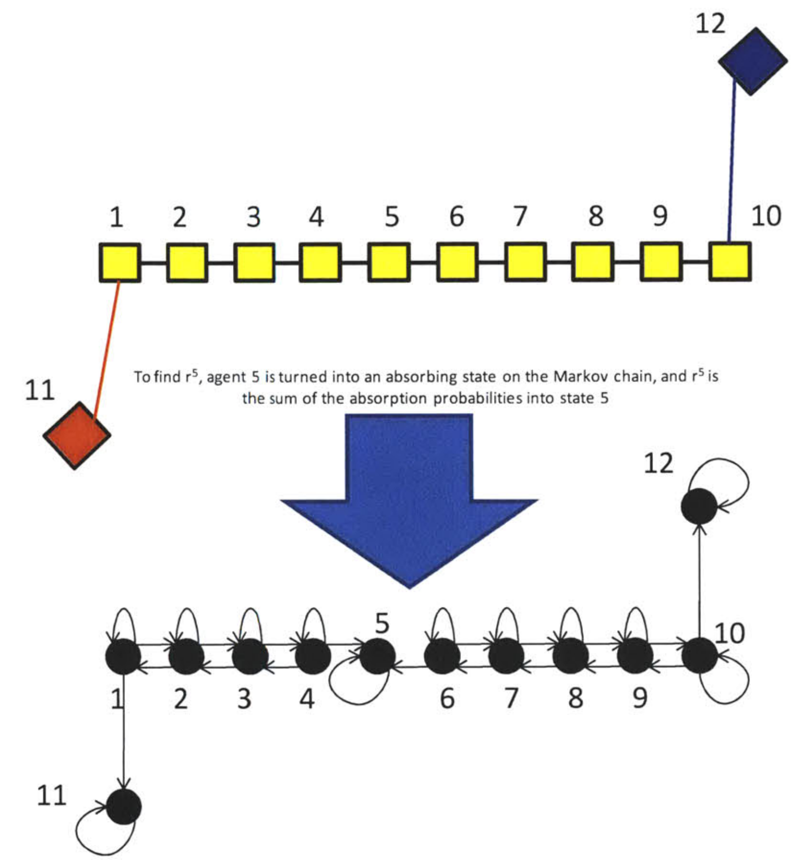

*Chart 4.1 – Predicted vs. Sample Std Dev for Line Network*

*The predicted vs sample standard deviation plot shows no useful correlation between the two, although matching strategies (5,5) is correctly predicted to have high standard deviation.*

A Kruskal-Wallis test was conducted to verify whether the predicted standard deviation had any statistical relation to the experimental standard deviation. In this test the null hypothesis is that the samples are related. The test returned a p-value of .0001, indicating that the null hypothesis is incorrect and that the samples are not related with any statistical significance.

The ideal plot would have a linear relationship, indicating that as the predicted standard deviation increases so does the experimental, but this is not evident. At first glance it is clear that one prediction is accurate – the strategy profile of matching strategies (5 vs. 5) is correctly predicted to have the highest standard deviation. After this prediction, the strategy profile of near-matching strategies (5 vs. 6) is predicted to have higher than normal standard deviation (it is the fourth largest prediction of the ten experiments). Unfortunately (3.31) also predicts very high standard deviation in strategy profiles that have low standard deviation. The most obvious case is 1 vs. 10, which is the second least variable of all the cases, yet is predicted to be the second highest.

The reason this occurs is due to how (3.31) calculates the bound on standard deviation. To illustrate this we look at the line network when players connect to the agents on the ends of the line network (1 vs. 10). In this case $r^i$ becomes large for nodes in the middle of the line network. For example consider $r^5$ – to find $r^5$ agent five is turned into an absorbing state, and the probabilities of absorption into node five are calculated for all of the remaining mutable agents. Agents 3, 4, 6, and 7 are more likely to be absorbed by agent five than to be absorbed by either of the stubborn agents, since they are closer to agent five than they are to agent 11 or 12.




| State on the Markov chain | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Absorption probability into State 5 | 0.112 | 0.334 | 0.556 | 0.778 | - | 0.818 | 0.637 | 0.455 | 0.273 | 0.092 |

| $r^5$ | 4.055 |
|---|---|

*Figure 4.16 – Example of Absorption Probabilities for $r^5$ on Even Degree Line Network*

When stubborn agent connections are on the extreme ends of the network, there are several central nodes that will have large values for $r^i$, and the bound on the standard deviation becomes large. This may be problematic, and is investigated further in our next experiments. The initial results indicate that the predicted standard deviation is large for strategy profiles that are highly variable, but it is unfortunately also large for strategy profiles that are on the edges of the network that in fact show very low variability in their equilibrium mean belief in simulation.

### 4.2.1.2 – Small Rural Pashtun Network

As with the line network, several strategy profiles are selected that represent different cases as described in the Table 4.12. The cases are listed in order from highest expected standard deviation to lowest.

| Strategy Profile | Reason |
|---|---|
| 1 vs 1 | Both choose 'Very Inf' |
| 1 vs 2 | TB has 'Very Inf', US has 'Inf' |
| 1 vs 3 | TB has 'Very Inf', US has 'Regular' |
| 12 vs 3 | TB has 'Inf', the other has 'Regular' |
| 13 vs 3 | Both Have 'Regular' |

*Table 4.12 – Description of Strategy Profile Test Combinations for Testing the Bound on Standard Deviation for the Small Rural Pashtun Network*

The predicted standard deviation is matched against the sample standard deviation from simulation. Again the sample standard deviation is calculated by excluding the first 1000 interactions to ensure that the standard deviation only measures the network behavior at equilibrium, and all experiments are run for 5000 interactions.

| Case | Predicted Std Dev | Sample Std Dev |
|---|---|---|
| 1 vs 1 | 0.2639 | 0.1489 |
| 1 vs 2 | 0.0313 | 0.0443 |
| 1 vs 3 | 0.0051 | 0.0374 |
| 12 vs 3 | 0.0051 | 0.0885 |
| 13 vs 3 | 0.0051 | 0.0296 |

*Table 4.13 – Predicted vs. Sample Std Dev for Small Rural Pashtun Network*

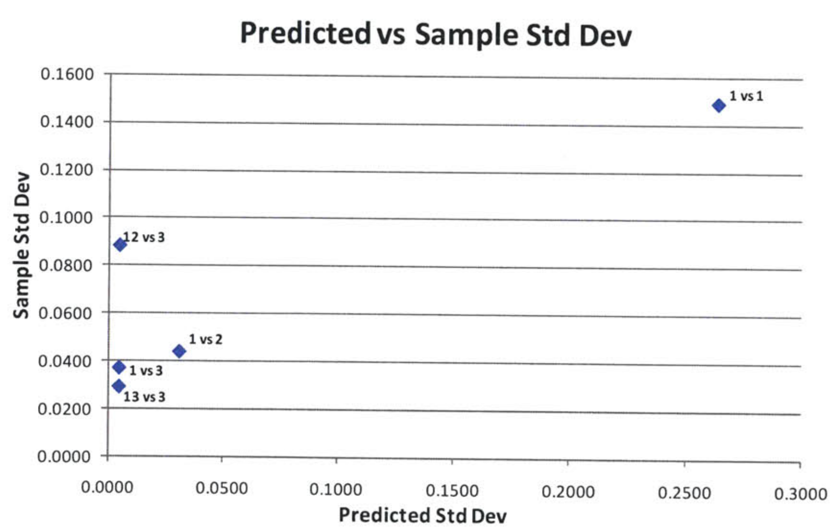



*Chart 4.2 – Predicted vs. Sample Std Dev for Small Rural Pashtun Network*

*No meaningful correlations are seen in the predicted vs sample values of standard deviation except for correctly predicting high standard deviation for matching strategies - (1,1).*

Again we performed a Kruskal-Wallis test was conducted to verify whether the predicted standard deviation had any statistical relation to the experimental standard deviation. The test returned a p-value of .7584, indicating that the null hypothesis is correct and that the samples are related. Given the previous results in the line network for the Kruskal-Wallis test we believe the positive results on this test may be due to a small sample size and not necessarily indicative that the predictions and experimental values are highly correlated.

The prediction correctly gives a high standard deviation to the worst case strategy profile of matching strategies. Beyond that are some strange results; 1 vs. 2 is predicted to be the second highest, but actually has relatively low standard deviation. All of the other 3 strategy profiles predicted to have identical standard deviation. All three of these strategy profiles have the US player connecting to a regular agent, but regular agents cannot spread any belief to influential agents. This has an enormous effect on the absorption probabilities on the Markov chain when calculating $r^i$ and results in significant error in predicted standard deviations.

The behavior of predicted versus actual standard deviation in the small rural Pashtun network works well in predicting the high variability of matching strategies, but otherwise exhibits poor results.

#### 4.2.1.3 – Large Rural Pashtun Network

Two different sets of strategy profiles are analyzed for the large rural Pashtun network. The first set starts with the Nash equilibrium strategy profile of 70 vs. 70, and the next four profiles move from this central winning strategy to 'influential' and then 'regular' agents. With these strategy changes there is a corresponding move from the center of the network to the periphery, and there should be a decrease in experimental standard deviation and hopefully a decrease in the predicted standard deviation. The second set has 10 strategy profiles of different combinations of 'very influential' agents in the network. The reason these combinations of agents are analyzed is in the hope that we may see some combinations of 'very influential' agents, that do well in 'mean belief', but also have low variance. If the bound correctly predicts the standard deviation in these cases it would hint at the existence of pure Nash equilibria strategy profiles with the desired properties (high mean belief and low standard deviation) under the 'risk averse mean belief' payoff function.

The first 5 strategy profiles on the large network yield the following:

| Strategy Profile | Predicted Std Dev | Sample Std Dev |
|---|---|---|
| 70 vs 70 | 0.8388 | 0.0414 |
| 48 vs 70 | 0.2490 | 0.0365 |
| 48 vs 29 | 0.3748 | 0.0203 |
| 49 vs 29 | 1.5431 | 0.0299 |
| 50 vs 28 | 0.1803 | 0.0057 |

*Table 4.14 – Predicted vs. Sample Std Dev for 1st Strategy Set on Large Rural Pashtun Network*

Again the prediction correctly penalizes matching strategies. There is an anomaly in the strategy profiles (49,29) though – the predicted standard deviation is 1.54. Considering that the maximum sample standard deviation possible in the network is only 0.5 (if the network mean belief moved from -0.5 to 0.5 at every step), this is troubling. The reason for this occurrence is that both 49 and 29 are 'regular' agents. The reducibility of the Markov chain and the fact that regular agents can only communicate to other regular agents in their village means that stubborn agents connected to regular agents cannot communicate with most of the network, and thus the values of $r^i$ will be very large (as seen previously). Overall we see that the prediction of standard deviation is not particularly accurate, but it does penalize matching strategies.

Next are ten strategy profiles of 'very influential' agents along with the strategy profile (70,70) for a total of eleven strategy profiles.

| Strategy Profile | Predicted Std Dev | Sample Std Dev |
|---|---|---|
| 70 vs 70 | 0.8388 | 0.0414 |
| 70 vs 21 | 0.2918 | 0.0434 |
| 48 vs 21 | 0.3748 | 0.0208 |
| 70 vs 56 | 0.2857 | 0.0524 |
| 21 vs 56 | 0.6529 | 0.0324 |
| 70 vs 72 | 0.3479 | 0.0489 |
| 70 vs 71 | 0.2652 | 0.0394 |
| 70 vs 69 | 0.2837 | 0.0396 |
| 21 vs 69 | 0.3724 | 0.0680 |
| 48 vs 71 | 0.3247 | 0.0615 |
| 48 vs 56 | 0.3467 | 0.0456 |

*Table 4.15 – Predicted vs. Sample Std Dev for 2nd Strategy Set on Large Rural Pashtun Network*

The data of predicted vs. sample standard deviation is plotted in Chart 4.3. Matching strategies (70 vs. 70) is once again predicted to be the worst, although it is not actually the most variable strategy profile. This is interesting, as in all previous networks a matching strategy always yielded the highest sample standard deviation. For these strategy profiles there is no useful relationship between the predicted and sample standard deviation. Previously the bound always properly predicted matching strategies as being the worst, but for the large rural Pashtun network this prediction is not correct.

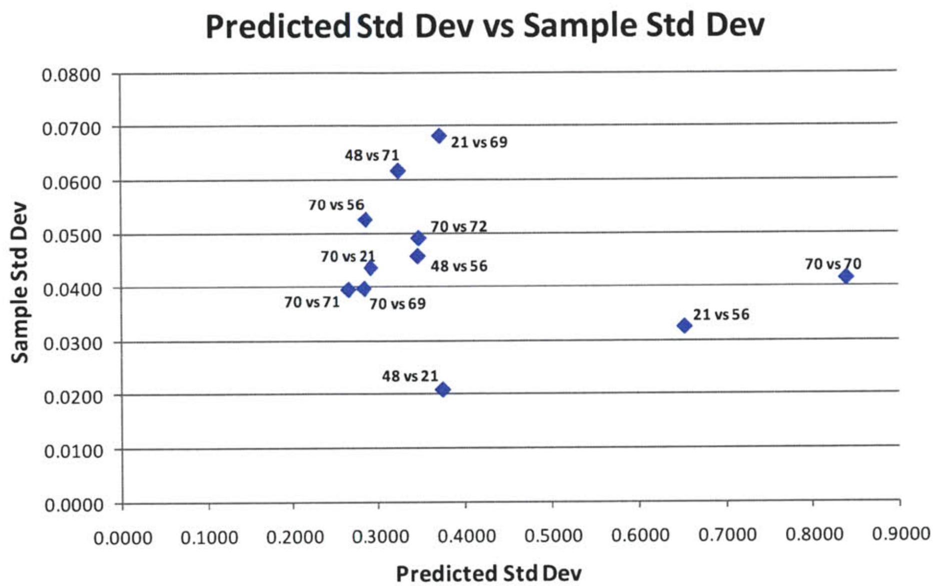



*Chart 4.3 – Predicted vs. Sample Std Dev for 2nd Strategy Set on Large Rural Pashtun Network*

*No correlations seen between predicted and sample standard deviation. Matching strategies (70,70) is incorrectly predicted to have the highest standard deviation.*

A Kruskal-Wallis test returned a p-value of .0007, indicating that the samples are not related with any statistically significance.

Thus far the results of the predicted variance bound seem poor. It penalizes matching strategies by predicting high standard deviation, but for other strategy profiles seems to be unable to make consistently useful predictions. Furthermore, we've just seen in the large rural Pashtun network that matching strategies is not always the most variable strategy profile.

### 4.2.1.4 – Modifying the Influence Parameters

We were concerned that the poor performance of the bound may be due to our selection of influence parameters. With 'regular' agents unable to influence 'influential' agents, it means the Markov chain is reducible and there are several states that are unreachable from other states, which has large effects on the absorption probabilities of random walks. Because (3.31) is a function of absorption probabilities, this could create problems. To investigate and account for this potential source of error, all of the above analyses were repeated for the small and large rural Pashtun networks (the line network has all regular agents and therefore this problem does not affect it because they all communicate in the Markov chain) with the following modification to the influence parameters:

Increase the chance of a β-type (averaging) interaction between 'regular' agents and 'influential' agents from 0 to 0.1, with interactions from 'influential' to 'regular' agents remaining the same (always forcefully influence them). 'Influential' agents still completely sway the minds of regular agents, but regular agents can occasionally influence their leaders by averaging with them. Making this change enables beliefs from a 'regular' agent to propagate to an 'influential' agent. This modification makes limited changes to the influence parameters, but fixes the problem of reducibility among the nodes corresponding to mutable agents in the Markov chain.

Chart 4.4 plots the predicted vs. sample standard deviation on the small rural Pashtun network with these changed parameters.

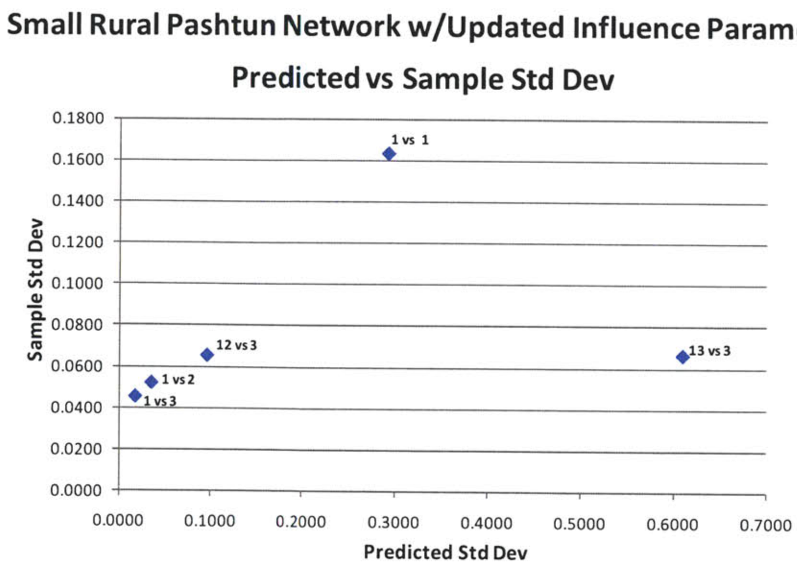



*Chart 4.4 – Predicted vs. Sample Std Dev w/Updated Influence Parameters for Small Rural Pashtun Network*

*Even with updated influence parameters to allow 'regular' agents to change the belief of 'influential' agents, the predicted vs sample standard deviation has no useful correlation.*

As before with the small network, matching strategies is correctly predicted to have large standard deviation, but strategy profile (13,3) is incorrectly predicted to have much larger standard deviation. In general there are not any useful predictions. The results for the large rural Pashtun network are shown in Chart 4.5.

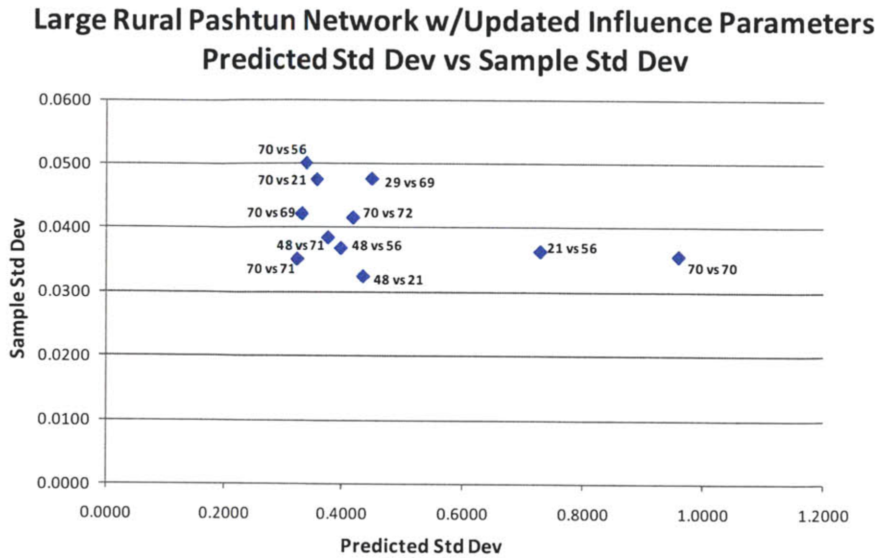



*Chart 4.5 – Predicted vs. Sample Std Dev w/Updated Influence Parameters for Large Rural Pashtun Network*

*Updated influence parameter to allow 'regular' agents to change the belief of 'influential' agents still does not lead to any useful correlations.*

In the large rural Pashtun network there is no correlation between predicted and sample standard deviation. Matching strategies is incorrectly predicted to be high variance. Strategy profiles with low standard deviation are predicted to be high, and the highest experimental standard deviation is predicted to be one of the lowest (70 vs. 56).

This analysis further indicates that (3.31) and the bound developed by Yildiz et al does not accurately predict the variability of the mean network equilibrium in our continuous belief model.

For the remainder of the analysis in this chapter, we revert back to the original set of influence parameters.

## 4.2.2 – Nash Equilibria in the 'Risk Averse Mean Belief' Payoff Function

Although the previous section indicated that the bound derived by Yildiz et al does not work well in our model, it does have a tendency to predict high variance for matching strategies. Thus, we conduct an analysis of the pure Nash equilibria in the 'risk averse mean belief' payoff function, as one of the goals of this payoff function is to penalize players just enough to deviate from matching strategies and move to another strategy profile. The main reason that we are concerned with eliminating matching strategies is that in the real world it is unlikely that both the US and Taliban would expend resources influencing all of the same people. Because the 'risk averse mean belief' payoff function depends on both the predicted standard deviation and the equilibrium mean belief, it is possible that these two components of the payoff function may have an interaction that makes the payoff function useful. We also conduct the analysis to ensure thoroughness prior to drawing a firm conclusion about whether there is any utility in the bound on standard deviation for our model.

The 'risk averse mean belief' payoff function $h(\cdot)$ is used for both the US and TB player.

$$\mathrm{h}(\mathrm{S_{US}}, \mathrm{S_{TB}}) = Mean\ Belief\ Payoff - \lambda * Predicted\ Standard\ Deviation\ of\ Network$$

*where $\lambda$ is a constant representing the penalty for high standard deviation*

$$\mathrm{h_i}(\mathrm{S_{US}}, \mathrm{S_{TB}}) = \mathrm{f_i}(\mathrm{S_{US}}, \mathrm{S_{TB}}) - \lambda * \sigma\left(X^*(\mathrm{S_{US}}, \mathrm{S_{TB}})\right) \quad \mathrm{i} \in \{\mathrm{US}, \mathrm{TB}\} \tag{3.38}$$

Initially the penalty parameter $\lambda$ is set to one. The networks used are the same as all of the networks used in Experiment 1 (see Section 4.1). They are reiterated below for the reader's convenience.

| Graph Type | # Mutable Agents | Description | Reason |
|---|---|---|---|
| Line | 9<br>10 | **Case A** - All agents are 'regular' | Base Case |
| | 9<br>10 | **Case B** - 1 'Influential' agent in the middle, rest are 'regular' | How does the addition of an agent that is both influential and central effect the network? |
| | 9<br>10 | **Case C** - 1 'Influential' agent on the end, rest are 'regular' | How does an influential but peripheral agent affect the network? |
| Circle | 9<br>10 | **Case A** - All agents are 'regular' | Base Case |
| | 9<br>10 | **Case B** - 1 'Influential' agent, rest are 'regular' | How does the addition of a single influential agent affect the network? |
| | 9<br>10 | **Case C** - 2 'Influential' agents on opposite sides, rest are 'regular' | What happens when 2 influential agents are on opposite sides of a society? |
| | 9<br>10 | **Case D** - 2 'Influential' agents adjacent, rest are 'regular' | What happens when both influential agents are highly connected? |
| Fully Connected | 10 | **Case A** - All agents are 'regular' | Base Case |
| | 10 | **Case B** - 1 Influential agent, rest are 'regular' | Effect of influential agent in highly connected groups? |
| | 10 | **Case C** - 1 'Very Influential' agent, 2 'Influential' agents, rest are 'regular' | Effects of multiple types of agents in highly connected groups? |

*Table 4.1 – Network Test Case Descriptions*

Because both players use the same payoff function, if strategy profile (x,y) is an equilibrium, so is (y,x), but the second equilibrium is omitted from the charts below for brevity.

### 4.2.2.1 – Line Networks

The pure Nash equilibria for the line networks are shown in Table 4.16.

| Nash Equilibria for Line Networks, $\lambda$=1 | | |
|---|---|---|
| Case | Even Degree | Odd Degree |
| A | (5,6) | (4,5), (5,6) |
| B | (5,7) | (5,5) |
| C | None | None |

*Table 4.16 – Line Network Pure Nash Equilibria for $h(\cdot)$ vs. $h(\cdot)$, $\lambda$=1*

*For each test case A, B ,and C the pure Nash equilibria are found through enumerating the entire payoff matrix when both players use the 'risk averse mean belief' payoff function. Each pair (x,y) is a strategy profile corresponding to a pure Nash equilibrium where X is the TB player's strategy and Y is the US player's strategy. For Case C there are no pure Nash equilibria.*

First it is interesting to note that with an influential agent on the end of the line (case C), there is no pure Nash equilibrium present. The other two cases yield encouraging results – the matching strategies have been largely removed. On the even degree line network when both payoff functions were just 'mean belief' two matching strategies ((5,5) and (6,6)) were present, but with the 'risk averse mean belief' payoff function there are only non-matching strategy profiles remaining. This is the goal of the 'risk averse mean belief' payoff function – penalize matching and highly variable strategy profiles and force players to move to other strategy profiles with identical (or almost identical) mean belief but lower variance (the previous experiments showed that 5 vs. 6 has half the sample standard deviation that a matching strategy does).

On the even line network with a single 'influential' agent in the middle (Case B), one player connects to the influential agent, and the other chooses the agent that is 2 agents away. This induces a significant first mover advantage.

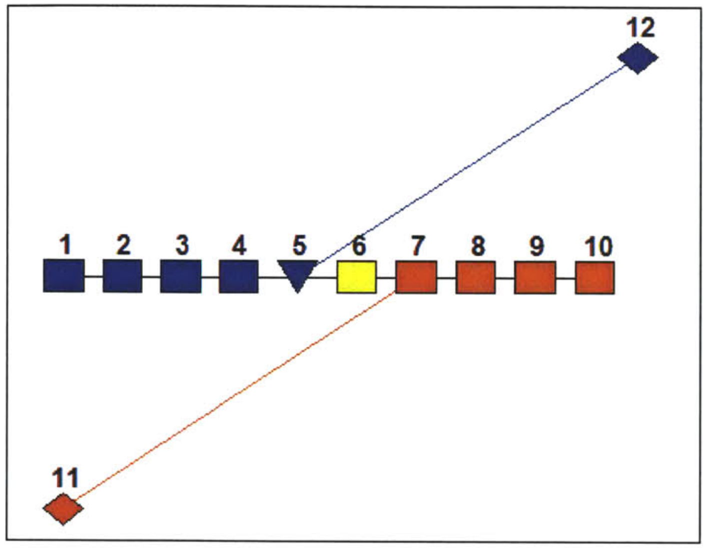



*Figure 4.17– Line Network, Case B Equilibrium Beliefs when Playing Nash Equilibrium Strategy under h(·) vs. h(·), λ=1*

*The US player connects to agent 5, which results in them influencing the network much stronger than the TB player. However, because the 'risk averse mean belief' payoff function penalizes high variance strategies the TB player does not want to match the US player by connecting to agent 5. They instead connect to agent 7. Although this means they influence the network less, there is much less variance in this strategy profile, and is the TB player's best response given that the US player connects to agent 5.*

It is interesting that in the case illustrated in Figure 4.17 the TB player's best response is not agent six. Choosing agent six would yield a higher mean belief in the network for them, but it also increases the variance and decreases the overall payoff function (it decreases from -.21 to -.26 for the TB player in the case shown in Figure 4.17).

### 4.2.2.2 – Circle Networks

The results for the circle network equilibria are:

| Nash Equilibria for Circle Networks λ=1 | | |
|---|---|---|
| Case | Even Degree | Odd Degree |
| A | All opposing pairs | All opposing pairs |
| B | (1,6) | (1,5), (1,6) |
| C | None | None |
| D | (1,10) | (1,9) |

*Table 4.17 – Circle Network Nash Equilibria for h(·) vs. h(·), λ=1*

*For each test case A, B, C, and D, the pure Nash equilibria are found by enumerating the entire payoff matrix when both players use the 'risk averse mean belief' payoff function. Each pair (x,y) is a strategy profile corresponding to a pure Nash equilibrium where X is the TB player's strategy and Y is the US player's strategy. For Case C there are no pure Nash equilibria.*

Again there is a case with no pure Nash equilibria (Case C is 2 'influential' agents on opposite sides of the circle). This is surprising; if each player chooses an influential agent on opposite sides of the circle it yields very low standard deviation, and zero mean belief. This is the type of strategy profile we hope that this payoff function would lead to. The reason it does not is because the bound on standard deviation incorrectly characterizes the strategy profile (1,5) as high-risk, heavily penalizing it and preventing it (or any other strategy profile) from being a pure Nash equilibrium.

In all of the other cases the behavior is what is wanted from the 'risk averse mean belief' payoff function – high variance matching strategies are no longer equilibria, and the payoff function instead rewards strategy profiles with connections on opposite sides of the circle (which are low variance).

In Case A all agents are regular and the risk averse payoff function removes all of the strategy profiles except the lowest variance ones, which are opposing pairs.

In Case B with a single influential agent the pure Nash equilibrium strategy profile is one player connecting to the 'influential' agent (number one), and the other player connects to the opposite agent. The player that connects to the 'influential' agent has a first mover advantage.

In Case D each player chooses one of the two adjacent 'influential' agents, but the matching strategies that previously existed have been removed.

#### 4.2.2.3 – Connected Networks

Next are the results from analyzing the fully connected networks.

| Nash Equilibria for Connected Networks, $\lambda$=1 | |
|---|---|
| Case | Equilibria |
| A | All non-matching pairs |
| B | (1,1) |
| C | (1,1) |

<u>*Table 4.18 – Connected Network Nash Equilibria for h(·) vs. h(·), $\lambda$=1*</u>

*For each test case A, B, and C, the pure Nash equilibria are found by enumerating the entire payoff matrix when both players use the 'risk averse mean belief' payoff function. Each pair (x,y) is a strategy profile corresponding to a pure Nash equilibrium where X is the TB player's strategy and Y is the US player's strategy.*

In Case A (all regular agents), any non-matching strategy profile is a pure Nash equilibrium. This is due to the symmetry of the connected network. Any non-matching strategy profile is identical to any other non-matching strategy profile in both variance and mean belief. The matching strategies have identical mean belief, but higher variance and thus are not equilibria.

In Case B and C both players choose to connect to the most influential agent present. Although this is a more variable strategy profile, the high connectedness of the network increases the variance of every strategy (any agent can influence any other agent at every step), and there is not sufficient difference in the variance to penalize matching strategies sufficiently to incentivize a player to deviate from this strategy profile.

#### 4.2.2.4 – Rural Pashtun Networks

It turns out that for both the small and large rural Pashtun networks, the pure Nash equilibrium strategy profiles under the 'risk averse mean belief' payoff function are identical to all of the previous payoff functions – (1,1) for the small network, and (70,70) for the large network.

The initial results for the 'risk averse mean belief' payoff function are encouraging. It successfully removes many undesirable matching strategies. For example, in networks with multiple equilibria that all have identical 'mean belief' payoff, only the least variable strategy profiles survive with the introduction of risk aversion, which is the intent. Unfortunately for our two networks that are based on a real societal structure – the small and large rural Pashtun networks – there is no change in the 'risk averse mean belief' from the 'mean belief' payoff – matching strategies is still the equilibrium strategy profile.

In order to further investigate the performance of the 'risk averse mean belief' payoff function on these networks we analyze the results from changing the penalty parameter, $\lambda$. The goal is to see what values of $\lambda$ are required for the equilibria to change from matching strategies to a new strategy profile, and what that profile is.

As seen in Table 4.19, as the penalty parameter is increased to 1.72, the equilibrium strategy profile for the small rural Pashtun network change from (1,1) to (1,7). Agent seven is the next most influential agent after agent one, so this is reasonable (and desirable) behavior. It induces a first mover advantage, and causes the players to avoid matching strategies with a strategy profile that is as close to being equal as possible. As $\lambda$ is increased further to 1.78, the equilibrium moves to strategy profiles (1,12) and (1,2). Agents 2 and 12 are the third most influential agents in the network after agent 1 and 7, but because they are more peripheral in the network they reduce variability, so again this is a reasonable strategy set. As $\lambda$ is increased past 6, the equilibria becomes agent 1 and any 'regular' agent. This is expected, as connecting to 'regular' agents reduces the variability of the network.

| Nash Equilibria for Small Rural Pashtun Network | |
|---|---|
| **Case** | **Equilibria** |
| $\lambda \leq 1.72$ | (1,1) |
| $1.72 \leq \lambda \leq 1.78$ | (1,7) |
| $1.78 \leq \lambda \leq 5.99$ | (1,2),(1,12) |
| $5.99 \leq \lambda$ | Agent 1 and any 'regular' agent |

*Table 4.19 – Small Rural Pashtun Network Pure Nash Equilibria for h(·) vs. h(·), varying $\lambda$*

*Depending on the values of the penalty parameter $\lambda$, there are different pure Nash equilibrium strategy profiles.*

However, it is a bit troubling that such a small change from 1.72 to 1.78 can cause the payoff function to change the equilibrium strategy profiles. If the payoff function requires such fine tuning to get the desired results it indicates that the function is not robust to changes in input parameters. Furthermore, if we have to perfectly tune the payoff function to some arbitrary value for each network then it is equivalent to simply selecting which strategy profiles 'look good' and loses any real predictive power.

Table 4.20 shows that the large rural Pashtun network does not change equilibrium strategy profiles until $\lambda$ increases past 6.48. Furthermore, once they change they move to significantly different strategies. The new equilibrium strategy profiles are agent 70, and any 'regular' agent from the village 2 (contains agents 11-19 and is at the bottom right of the network diagram in Figure 4.14).

| Nash Equilibria for Large Rural Pashtun Network | |
|---|---|
| **Case** | **Equilibria** |
| $\lambda \leq 6.47$ | (70,70) |
| $6.47 \leq \lambda$ | (70,15), (70,16), (70,17), (70,18) |

*Table 4.20 – Large Rural Pashtun Network Pure Nash Equilibria for h(·) vs. h(·), varying $\lambda$*

*Depending on the values of the penalty parameter $\lambda$, there are different equilibrium strategy profiles.*

Again, we see that for the large rural Pashtun network the payoff function requires significant tuning, and does not even yield reasonable or interesting equilibrium strategy profiles. This leads us to conclude that the 'risk averse mean belief' payoff function using the bound on standard deviation is not very useful for avoiding high risk strategies while still maintaining strategies that have strong 'mean belief' payoff.

### 4.2.3 – Experiment Set 2 Summary

Although some results are encouraging for the small networks, there is a fair amount of tuning required for the larger more complex networks. Furthermore, there is no guarantee of finding 'good' equilibrium strategy profiles. Experimental data indicate that the prediction of standard deviation in the network does not match simulation, and thus the payoff function is not correctly measuring the variability of the networks. The large rural Pashtun network is the most realistic one we have and for this network the results are very poor. Even on the small rural Pashtun network we see that depending on the choice of $\lambda$ we can get one of many possible equilibrium strategy profiles, and one of the equilibria (1,7), only works for a very small range of $\lambda$.

In conclusion, the bound on the standard deviation on the equilibrium mean belief of the network does not work well in our model. Consequently the 'risk averse mean belief' payoff function does not work well with this proxy for variability. Although the payoff function can be finely tuned to yield the desired results, the tuning required is different for each network, and is thus equivalent to changing the function every time to get an intuitively desirable result.

## 4.3 – Experiment Set 3: Analysis of Simulated Annealing Heuristic

In this section the simulated annealing heuristic is analyzed with regard to three questions:

-Can simulated annealing correctly characterize Nash equilibria using best response dynamics?

-How well does the run time for simulated annealing scale for larger networks?

-How does the simulated annealing algorithm perform compared to the solutions found using the Knitro solver for Hung's math program?

The first question is important to see how close to optimal the heuristic can perform. The answer to the second question will give anecdotal evidence of whether the heuristic can reasonably be applied to much larger and more realistic problems (hundreds or thousands of agents). Answering the last question allows us to verify if our heuristic can provide the same quality of solutions as Hung's math program or not.

### 4.3.1 – Can simulated annealing find Pure Nash Equilibria?

For this experiment, the simulated annealing heuristic is used to characterize pure Nash equilibrium strategy profiles through best response dynamics. It optimizes the US player's strategy against a fixed TB strategy. It then optimizes the TB strategy against the optimized US strategy. This continues until the simulated annealing heuristic gets to a solution that repeats itself five times for both players. This means once an equilibrium strategy profile is reached, simulated annealing is run five more times for each player before accepting it as an actual pure Nash equilibrium. This method will only find one pure Nash equilibrium strategy profile at a time. To determine whether multiple equilibria exist, the entire process is repeated 10 times, with a different random strategy profile as the starting point each time. This process is run on all of the same networks used in Experiment 1 (see Section 4.1). The resulting equilibrium strategy profiles and run times are recorded, and compared to the known equilibrium strategy profiles found previously.

In some cases the simulated annealing heuristic does not converge to a solution. There are instances where the heuristic cycles through multiple solutions repeatedly due to the existence nearby solutions of equal quality (multiple local optima); because the heuristic runs until it finds the same solution five times in a row this cycling prevents the heuristic from ever finding a solution. To prevent indefinite cycling, the heuristic is limited to 20 iterations – if both players have optimized their strategies through simulated annealing twenty times, and no set of strategies has been repeated at least five times in a row, the algorithm terminates with no solution. If the algorithm is able to find an equilibrium, it usually finds it within 2-3 iterations, and then repeats it five times, thus terminating at 20 iterations allows sufficient time for the algorithm to find a solution.

### 4.3.1.1 – Line Network Simulated Annealing Pure Nash Equilibria vs. Actual Pure Nash Equilibria

Table 4.21 shows the equilibria found by simulated annealing on the even degree line networks. At the bottom of the chart the actual equilibria found through exhaustive enumeration in Experiment Set 1 (see Section 4.1) are listed.

| Even Line Network Pure Equilibria Found by Simulated Annealing | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Case A: All Regular | | | Case B: 1 Inf in Middle | | | Case C: 1 Inf on End | | |
| US Strat | TB Strat | Time (sec) | US Strat | TB Strat | Time (sec) | US Strat | TB Strat | Time (sec) |
| 6 | 6 | 28 | 5 | 5 | 20 | 8 | 8 | 34 |
| 5 | 5 | 28 | 5 | 5 | 20 | 8 | 8 | 25 |
| 5 | 5 | 28 | 5 | 5 | 21 | 8 | 8 | 21 |
| 5 | 5 | 28 | 5 | 5 | 20 | 8 | 8 | 34 |
| 6 | 6 | 24 | 5 | 5 | 21 | 8 | 8 | 21 |
| 5 | 5 | 29 | 5 | 5 | 20 | 8 | 8 | 21 |
| 5 | 5 | 23 | 5 | 5 | 20 | 8 | 8 | 25 |
| 5 | 5 | 28 | 5 | 5 | 20 | 8 | 8 | 30 |
| 5 | 5 | 29 | 5 | 5 | 21 | 8 | 8 | 25 |
| 6 | 5 | 33 | 5 | 5 | 20 | 8 | 8 | 21 |
| Actual Nash Equilibria | | Time (sec) | Actual Nash Equilibria | | Time (sec) | Actual Nash Equilibria | | Time (sec) |
| (5,5), (5,6), (6,5), (6,6) | | 0.15 | (5,5) | | 0.16 | (8,8) | | 0.15 |

*Table 4.21 – Even Line Network Nash Equilibria found by Simulated Annealing*

*Each time it is run, simulated annealing finds a single pure Nash equilibrium using best response dynamics. By running it several times we are able to discover multiple equilibria. With only 1 connection in the network, simulated annealing runs slower than exhaustively enumerating the payoff function.*

In Case A there are 4 different equilibria. Simulated Annealing found (5,5) seven times, (6,6) twice, and (6,5) once. It did not find (5,6), but this is not a problem; we know that for identical payoff functions all equilibria are symmetric, so the strategy profile (6,5) can be inferred from (5,6). In Case B and C simulated annealing finds the correct equilibria every time.

For these small networks with only a single connection, the simulated annealing heuristic takes a much longer time to find the equilibria than exhaustive enumeration does (20 seconds vs. 0.15 seconds). This is expected and is not an issue of concern. The simulated annealing algorithm runs through several hundred iterations (based on the cooling rate and minimum temperature), independent of the number of connections. This means it runs slower when C=1, but as we show in Section 4.3.2, scales very well with increasing numbers of connections.

For brevity in the remaining cases just the summary information is shown instead of the output of every single run of the simulated annealing heuristic. Also, the times to exhaustively find the actual Nash equilibria are not shown, as they are all well under a second, and as previously discussed, the comparison of performance time on these small networks is not an issue we are concerned with yet.

In the odd degree line networks, there is only one equilibrium strategy profile for each case, and simulated annealing finds it correctly every time.

| Odd Line Networks | Actual Pure Nash Equilibrium | Pure Nash Equilibria found by Simulated Annealing | Mean Time (sec) |
|---|---|---|---|
| Case A: All Regular | (5,5) | (5,5) | 22 |
| Case B: 1 Inf in Middle | (5,5) | (5,5) | 18 |
| Case C: 1 Inf on End | (7,7) | (7,7) | 25 |

*Table 4.22 – Odd Degree Line Network Pure Nash Equilibria found by Simulated Annealing*

*Simulated Annealing finds the correct pure Nash equilibria correctly using best response dynamics.*

### 4.3.1.2 – Circle Network Simulated Annealing Nash Equilibria vs. Actual Nash Equilibria

Table 4.23 summarizes the results of the even degree circle networks. This set of networks presents a unique challenge for the simulated annealing heuristic – for Case A any pair of strategies is an equilibrium.

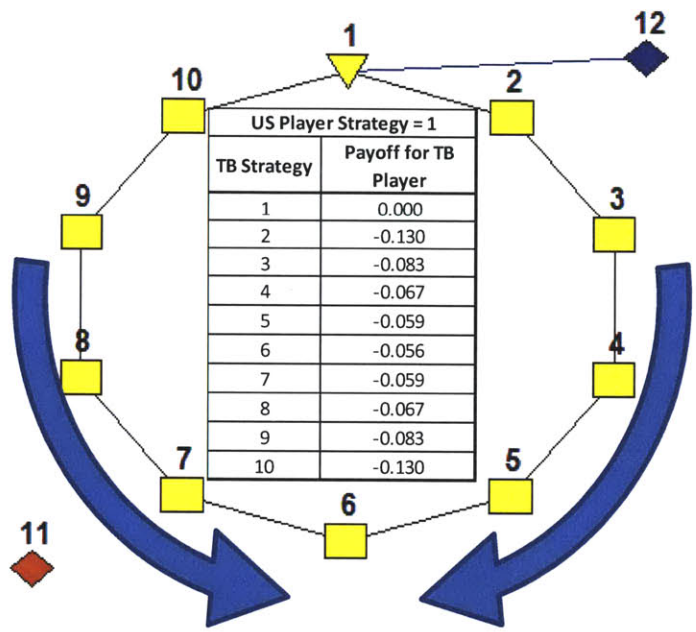
| Even Circle Networks | Actual Pure Nash Equilibrium | Pure Nash Equilibrium found by Simulated Annealing | Mean Time (sec) |
|---|---|---|---|
| Case A: All Regular | All pairs | (1,6), (2,4), (3,5), (3,6), (3,10), (4,4), (7,7), (7,9), (8,9), (9,10) | 23 |
| Case B: 1 Inf | (1,1) | (1,1), (1,6) | 27 |
| Case C: 2 Inf Opposite | (1,1), (1,6), (6,6) | (1,1), (1,6), (6,6) | 38 |
| Case D: 2 Inf Adjacent | (1,1), (1,10), (10,10) | (1,1), (1,10), (10,10) | 21 |

*Table 4.23 – Even Degree Circle Network Pure Nash Equilibria found by Simulated Annealing*

*Simulated Annealing finds the correct pure Nash equilibria correctly using best response dynamics. For Case A because any strategy profile is a pure Nash equilibrium, simulated annealing does not find all possible pure Nash equilibria.*

Because we only run the heuristic 10 times, it cannot find all 45 possible strategy profiles. However, the heuristic does converge and provide a solution on every run, and provides 10 different strategy profiles. If this were a much larger network that we did not know the actual answer for, and if simulated annealing yielded 10 distinct strategy profiles we could re-run the algorithm several times to see whether there are more equilibria or not. On the rural Pashtun networks that we test this will not be an issue, as there is only a single equilibrium strategy profile.

Case B also has a problem – it incorrectly returns (1,6) as an equilibria. It found this equilibrium in 2 of the 10 runs. This occurs because (1,6) is a local optimum in the simulated annealing heuristic. To demonstrate this, consider that the temperature has dropped low enough that we only take improving steps, the US player is connected to agent one, and the TB player's strategy is any agent three through eight. Looking at the payoff chart in Figure 4.18, every step of the algorithm will only allow moves towards agent six, because the only neighboring solutions with improvements move towards agent six. This is shown below.




| US Player Strategy = 1 | |
|---|---|
| TB Strategy | Payoff for TB Player |
| 1 | 0.000 |
| 2 | -0.130 |
| 3 | -0.083 |
| 4 | -0.067 |
| 5 | -0.059 |
| 6 | -0.056 |
| 7 | -0.059 |
| 8 | -0.067 |
| 9 | -0.083 |
| 10 | -0.130 |

*Figure 4.18 – Even Circle Network, Case B, Demonstrating why Simulated Annealing Converges to Local Optimum*

*When the temperature is low enough, simulated annealing takes only improving steps. This means that if the US player is connected to agent 1, and the TB player is connected to agent 3, 4, 5, 6, 7, 8, or 9, the only improving steps lead to agent 6, although this is not the optimum and is not a pure Nash equilibrium strategy profile.*

In cases C and D simulated annealing finds all 3 of the equilibria present with no incorrect solutions returned.

The results for the odd degree circle network are similar to those for the even degree circle network (see Table 4.24). In Case A simulated annealing finds 8 different equilibrium strategy profiles of the 36 possible (there were 2 repeats of the 10 found). Case B finds 2 incorrect strategy profiles for the same reason as in the even degree circle network (it finds (1,5) and (1,6)). Cases C and D return all correct solutions with no errors.

| Odd Circle Networks | Actual Pure Nash Equilibrium | Pure Nash Equilibrium found by Simulated Annealing | Mean Time (sec) |
|---|---|---|---|
| Case A: All Regular | All pairs | (1,3), (1,8), (2,5), (2,9), (4,5), (4,6), (4,7), (4,8) | 19 |
| Case B: 1 Inf | (1,1) | (1,1), (1,5), (1,6) | 23 |
| Case C: 2 Inf Opposite | (1,1), (1,5), (5,5) | (1,1), (1,5), (5,5) | 20 |
| Case D: 2 Inf Adjacent | (1,1), (1,9), (9,9) | (1,1), (1,9), (9,9) | 28 |

*Table 4.24 – Odd Degree Circle Network Pure Nash Equilibria found by Simulated Annealing*

*Simulated Annealing finds the correct pure Nash equilibria correctly using best response dynamics. For Case A because there any strategy profile is an equilibrium, simulated annealing does not find all possible pure Nash equilibria. In Case B it incorrectly returns 2 local optima as pure Nash equilibria.*

### 4.3.1.3 – Fully Connected Network Simulated Annealing Pure Nash Equilibria vs. Actual Pure Nash Equilibria

As seen in Table 4.25, the simulated annealing heuristic performed very well on the fully connected networks. For Case A, it found 8 different equilibrium strategy profiles (one of the equilibria was found twice, and another time it returned no solution due to cycling). The average run time in Case A was almost twice as long as for the other cases due to increased cycling of the algorithm with every strategy profile being an optimal solution.

In Case B and C, every iteration of the algorithm returned the correct equilibrium strategy profile of (1,1).

| **Connected Networks** | Actual Pure Nash Equilibrium | Pure Nash Equilibrium found by Simulated Annealing | Mean Time (sec) |
|---|---|---|---|
| Case A: All Regular | All pairs | (2,4), (2,5), (2,6), (2,7), (3,6), (4,7), (6,7), (7,9) | 42 |
| Case B: 1 Inf | (1,1) | (1,1) | 21 |
| Case C: 1 Inf +, 2 Inf | (1,1) | (1,1) | 23 |

*Table 4.25 – Connected Network Pure Nash Equilibria found by Simulated Annealing*

*Simulated Annealing finds the correct pure Nash equilibria correctly using best response dynamics. For Case A because there any strategy profile is an equilibrium, simulated annealing does not find all possible pure Nash equilibria.*

### 4.3.1.4 – Rural Pashtun Networks

The algorithm correctly found the pure Nash equilibria on both the small and large rural Pashtun networks every time they ran, although the run times were much larger than for the small 9 and 10 agent networks.

| Rural Pashtun Networks | Actual Pure Nash Equilibrium | Pure Nash Equilibrium found by Simulated Annealing | Mean Time (sec) |
|---|---|---|---|
| Small | (1,1) | (1,1) | 70 |
| Large | (70,70) | (70,70) | 1200 |

*Table 4.26 – Rural Pashtun Network Nash Equilibria found by Simulated Annealing*

*Simulated Annealing finds the correct pure Nash equilibrium strategy profiles correctly using best response dynamics.*

Recall that the exhaustive enumeration method took 1.5 seconds for the small rural Pashtun network and 300 for the large rural Pashtun network. We still have longer run times for simulated annealing, even on the larger networks.

For the 'mean belief' vs. 'mean belief' set of payoff functions, the simulated annealing algorithm works well in finding pure Nash equilibria. Although it does occasionally find incorrect equilibria, often they are easily verifiable. For example, in the circle network where strategy profiles (1,1) and (1,6) were returned as pure Nash equilibria, by simply checking the payoff of each player at each solution we can verify if they are both equilibria or not. If (1,1) has higher payoff for the US player than (1,6) does, it means the US player can unilaterally deviate to (1,1) and improve their payoff, thus (1,6) cannot be an equilibrium strategy profile, and is instead a local maximum returned by the algorithm.

In cases with only a single connection per player, exhaustive enumeration is much faster than simulated annealing. This occurs because the simulated annealing algorithm runs through the same number of iterations (recall the run time is a function of the cooling rate and minimum temperature, which were identical for all of these experiments), no matter the size of the network. To further compare the performance of simulated annealing vs. exhaustive enumeration, another experiment was conducted on the large rural Pashtun network. Each player was given 2 connections they could make from their stubborn agent to the mutable agents. This increases the number of strategies from 73 to 5184 ($73^2$), and exhaustive enumeration requires the calculation of a 5184 x 5184 payoff matrix. Exhaustively calculating this entire matrix for 'mean belief' vs. 'mean belief' took a bit over three days (~73.4 hours, or 264242 seconds). The only Nash equilibrium strategy profile was when both players connected to agent 48 and 70. Simulated annealing found the exact same strategy profile ten times in a row, with an average run time of only 1500 seconds. As seen in Table 4.27, when we have more than one connection, simulated annealing is the preferred solution method.

| Time to find Pure Nash Equilibria in Large Rural Pashtun Network | | |
|---|---|---|
| Method Used | 1 Connection | 2 Connections |
| Exhaustive Enumeration | 300 | 264242 |
| Simulated Annealing | 1200 | 1500 |

*Table 4.27 – Time to find Pure Nash Equilibria in Large Rural Pashtun Network with different numbers of connections*

*Exhaustive Enumeration finds pure Nash equilibria faster with only 1 connection available to each player, but with 2 connections, simulated annealing is much faster. This is because the run time in simulated annealing is insensitive to the number of connections, whereas exhaustive enumeration scales exponentially with the number of connections.*

### 4.3.2 – Simulated Annealing on Larger Networks

The next analysis is for a synthetic network of 519 agents. The network was randomly generated with a series of rules designed to create a network with properties similar to Hung's rural Pashtun networks. (The exact methodology is listed in Appendix C, along with a picture of the network).

This network is designed to generate something approximating the rural Pashtun networks. There are villages with 'influential' leaders, who talk to other village leaders, and at the district level several 'very influential' leaders communicate amongst themselves and the village leadership. It is not designed to be a perfect generator of actual rural Pashtun society, but rather something to generate networks with properties similar to Hung's that are much larger for testing the scalability of the simulated annealing algorithm.

In this large network, we give each player 2 stubborn agents, with 3 connections each. Simulated annealing optimizes a single player's strategy on this network in roughly 360 seconds. We ran simulated annealing on the network 10 times, and it returned the same pure Nash equilibrium strategy profile every time (matching strategies with both players connecting to agents 507, 508, 509, 510, 512, 513). The average run time was 8,358 seconds (about 2.3 hours).

The fact that simulated annealing returned the exact same solution every time, gives a high degree of confidence that this is a true pure Nash equilibrium in the network. There is of course no guarantee that this is a true Nash equilibrium, but calculating the true equilibria of this network exhaustively would take years with current computing power (the number of possible strategies is $519^6$ or about $10^{16}$). The simulated annealing algorithm gives a good solution with a high degree of confidence in reasonable time for this very large network.

### 4.3.3 – Comparison of Simulated Annealing Algorithm to Hung's Math Program

In his thesis, Hung performed several experiments to analyze the performance of his math program formulation for finding optimal strategies for a single player. He conducted experiments on both the small and large rural Pashtun networks, wherein there was a fixed TB strategy, and the US player used Hung's math program to find locally optimal solutions using different numbers of US stubborn agents, with different numbers of connections. His initial results had significant run times using the Knitro non-linear solver. When analyzing the large rural Pashtun network with 3 stubborn US agents with 3 connections each, the solver returned no solution after more than 14 hours (at which point Hung terminated the run). However, in the course of his experiments Hung found that by reducing the number of multi-start points, and increasing the integrality gap (his program is non-linear and mixed integer), he was able to get rapid results with relatively minor decreases in performance.

The specific test case that we use from Hung's thesis is the largest and most complex network for which we were able to obtain results (there was another larger network, but none of the solutions or run times were listed in his thesis). This test case uses the large rural Pashtun network of 73 mutable agents. The TB player has 3 stubborn agents, who each connect to the same four mutable agents (every TB agent connects to the same 4 mutable agents). Although it may not be realistic for the TB player to do this, it was how the experiment was designed by Hung in his thesis, and is thus repeated here to ensure we are using the same test case. The US player then has 3 stubborn agents, with 3 connections each. Figure 4.19 shows the initial setup of the network (prior to the US player choosing their strategy).

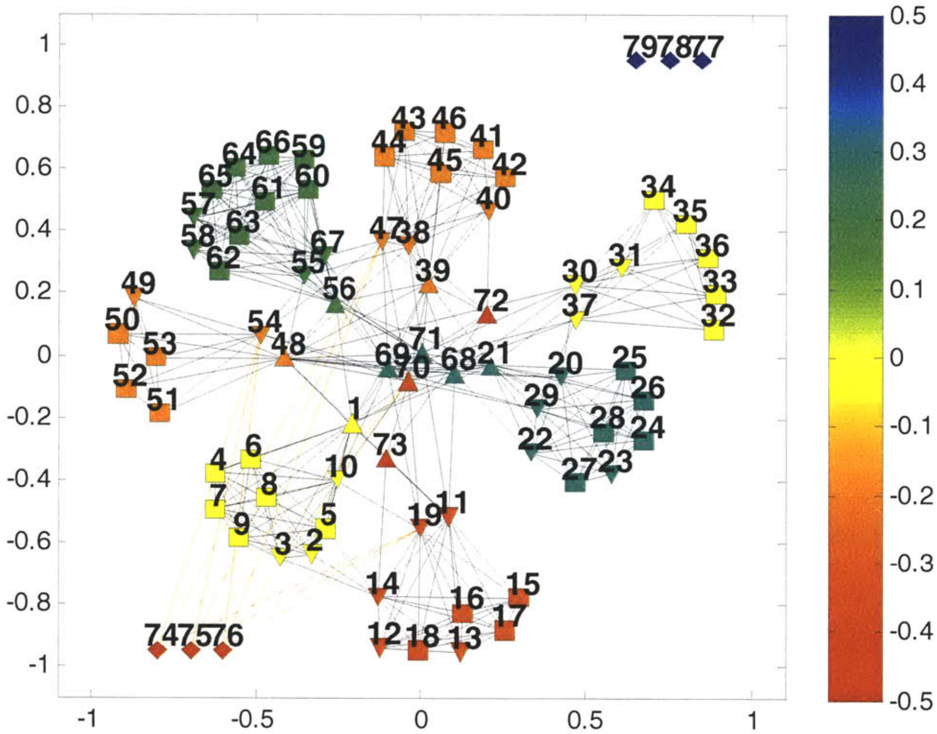



*Figure 4.19 – Large Rural Pashtun Test Case from Hung's Thesis*

Hung was able to find solutions in about a second (mean run time of .65 sec) that yielded an equilibrium mean belief of the network of 0.2295. After tuning the simulated annealing algorithm (by adjusting the cooling rate from .97 to .8) to operate in the same amount of time, it yielded solutions with an expected mean belief of 0.2866. These results are summarized in the Tables 4.28 and 4.29

| | Hung's Results | | |
|---|---|---|---|
| Run # | Time (sec) | US Strategy | Equilibrium Mean Belief |
| 1 | 2.65 | (19, 21, 29, 37, 39, 49, 67, 69, 70) | 0.2496 |
| 2 | 0.32 | (1, 12, 21, 49, 56, 68, 69, 71, 73) | 0.2407 |
| 3 | 0.27 | (1, 12, 13, 21, 49, 56, 68, 69, 71) | 0.2061 |
| 4 | 0.39 | (1, 12, 21, 39, 49, 56, 68, 69, 71) | 0.2517 |
| 5 | 0.23 | (1, 13, 21, 49, 56, 68, 69, 71, 73) | 0.2357 |
| 6 | 0.45 | (1, 13, 21, 40, 49, 56, 68, 69, 71) | 0.2167 |
| 7 | 0.29 | (2, 13, 21, 49, 56, 67, 68, 69, 71) | 0.2062 |

*Table 4.28 – Hung's Results on Large Rural Pashtun Network*

| | Simulated Annealing Results | | |
|---|---|---|---|
| Run # | Time (sec) | US Strategy | Equilibrium Mean Belief |
| 1 | 0.686 | (3, 14, 30, 48, 56, 69, 70, 70, 72) | 0.2775 |
| 2 | 0.624 | (1, 13, 22, 39, 48, 67, 69, 70, 70) | 0.2848 |
| 3 | 0.686 | (1, 19, 21, 37, 39, 48, 56, 69, 73) | 0.2852 |
| 4 | 0.748 | (1, 31, 39, 48, 68, 69, 70, 70, 73) | 0.2857 |
| 5 | 0.764 | (12, 21, 39, 40, 48, 67, 69, 70, 71) | 0.292 |
| 6 | 0.671 | (14, 21, 39, 48, 55, 56, 69, 70, 70) | 0.2875 |
| 7 | 0.733 | (1, 12, 21, 39, 48, 57, 69, 70, 70) | 0.2933 |

*Table 4.29 – Simulated Annealing Results on Large Rural Pashtun Network*

*Simulated annealing finds better solutions in the same amount of time than Hung's math program formulation*

If the simulated annealing algorithm is allowed to run around 10 seconds (this is with our original cooling parameter set to .97), it can provide solutions with a mean belief over 0.30. Thus we see that the simulated annealing algorithm can run in the same time as the best cases of Hung's math program while simultaneously providing better solutions. If Hung's math program is allowed to run for longer, there was essentially no increase in performance (less than 1% increase in optimality), but there were exponential increases in run time. If the simulated annealing algorithm is allowed to run longer by cooling at a slower rate, it can provide a 30% improvement over Hung's math program with only a 10-fold increase in run time. We've also already demonstrated that simulated annealing can analyze much larger networks in reasonable times in the previous experiment.

In Table 4.29, we see that simulated annealing has returned multiple solutions. With a higher cooling rate this still occurred, although the solutions were more similar to each other than those in Table 4.29. This occurs because the TB player is playing a highly sub-optimal strategy. When one player is player a sub-optimal strategy, there are many different solutions of almost equal quality available to the other player, and the chances of returning a local optima increase significantly. However, allowing it to cool slower does create more homogeneity in the solutions.

### 4.3.4 – Experiment 3 Summary

In this set of experiments we have shown that simulated annealing performs very well across all of the sample problems. In small networks is may be slower than an exhaustive search, but as the strategy options and size of the network increase, simulated annealing scales extremely well and can be used to analyze very large problems (500+ agents in a few hours). This is because simulated annealing runs in $O(n^3)$ time, whereas exhaustive enumeration is exponential in the number of connections. The solutions it provides are better than the solutions that the Knitro solver found for Hung's formulation. It is capable of finding pure Nash equilibrium strategy profiles efficiently in large networks. Although the algorithm may return local optima as false equilibria, by running the algorithm several times answers can be cross checked and verified, and a high degree of confidence can be gained in the solutions.

## 4.4 – Experiment Set 4: Sensitivity Analysis to Influence Parameters

In these experiments we analyze the impact that changes in the influence parameters, $\alpha$, $\beta$, and $\gamma$, have on pure Nash equilibria. Recall that these parameters control the interactions between agents:

$\alpha_{ij}$ = P(agent i forcefully imparting their belief to agent j | agent i and j interact)

$\beta_{ij}$ = P(agent i and agent j averaging beliefs | agent i and j interact)

$\gamma_{ij}$ = P(agent i and agent j retain their beliefs | agent i and j interact)

To test this we use 4 different sets of influence parameters. The first set is the control set and is the same set formulated by Hung and used throughout all of the previous analysis. Two others are extreme examples designed to test the robustness of the model, and the fourth set is a modification of the control set. Below we describe each of these data sets qualitatively, but for a listing of the exact parameters the reader can consult Appendix D.

The first extreme data set is representative of a consensus based heterarchical society, in which all mutable agents average with all other mutable agents. Everyone prefers to get along with each other, and thus always adopt middle of the road opinions instead of forcefully changing other's minds. Stubborn agents forcefully influence all mutable agents, but there is no difference between 'regular', 'influential', and 'very influential'. Every mutable agent averages with any other mutable agent they meet, regardless of agent type.

The second extreme data set is representative of a hierarchical society in which belief flows down from leaders. All mutable agents forcefully influence lower classes of agents, average within their own class, and have no influence on more influential classes of agents. Again, stubborn agents forcefully influence all classes of agents.

The last influence parameter data set is a modification of the control set, which includes small probabilities of lower agents influencing more influential types of agents. This is more representative of a western society in which there are influential leaders, but they also are influenced by their followers, instead of simply dictating beliefs to their constituents.

For each of these parameter sets, the pure Nash equilibrium strategy profiles are computed under the 'mean belief' vs. 'mean belief' set of payoff functions using exhaustive enumeration for all of the networks tested in Experiment Set 1 (see section 4.1).

The results of the line networks are in Table 4.30. Only the even line networks are shown, as the odd line networks have identical results (the reader can consult Appendix E to see all of them).

| Pure Nash Equilibria for Even Line Networks | | | | |
|---|---|---|---|---|
| Case | Control Set | Consensus Society | Heirarchical Society | Western Society |
| A | (5,5), (5,6), (6,6) | (5,5), (5,6), (6,6) | (5,5), (5,6), (6,6) | (5,5), (5,6), (6,6) |
| B | (5,5) | (5,5), (5,6), (6,6) | (5,5) | (5,5) |
| C | (8,8) | (5,5), (5,6), (6,6) | (8,8) | (5,5), (5,6), (6,6) |

*Table 4.30 – Nash Equilibria for Different Influence Parameters on Even Line Networks*

*The pure Nash equilibria were found for all of the line networks with each of the four different influence parameter sets. (x,y) is a strategy profile in which the TB player connects to agent x, and the US player connects to agent y.*

In Case A (all regular), the equilibrium strategy profiles are identical for all parameter sets. The consensus based society has the same equilibrium strategy profiles throughout, as it doesn't matter if there are any influential agents (all mutable agents operate the same). In case C, the control set and hierarchical set have different equilibria than the consensus and western society. This is because the influential agent on the end of the line acts as a stubborn agent with belief zero in these parameter sets (because regular agents cannot change the opinion of influential agents). Overall the line networks show good robustness to parameter changes.

| Pure Nash Equilibria for Even Circle Networks | | | | |
|---|---|---|---|---|
| Case | Control Set | Consensus Society | Heirarchical Society | Western Society |
| A | All pairs | All pairs | All pairs | All pairs |
| B | (1,1) | All pairs | (1,1) | (1,1) |
| C | (1,1), (1,6), (6,6) | All pairs | (1,1), (1,6), (6,6) | (1,1), (1,6), (6,6) |
| D | (1,1), (1,10), (10,10) | All pairs | (1,1), (1,10), (10,10) | (1,1), (1,10), (10,10) |

*Table 4.31 – Nash Equilibria for Different Influence Parameters on Even Circle Networks*

*The pure Nash equilibria were found for all of the circle networks with each of the four different influence parameter sets. (x,y) is a strategy profile in which the TB player connects to agent X, and the US player connects to agent Y.*

In the circle networks the major difference from the control set is for the consensus based society, where all pairs of agents are equilibria in all cases. All other modified parameter sets have pure Nash equilibria identical to the control set. Although the difference between the consensus society and the other sets is significant, the other Nash equilibria are subsets of the consensus based societies equilibria. This means if a player had incorrect information, thinking the influence parameters were the control set, hierarchical set, or western society set, and the game was being played in a consensus based society, the strategies would still be equilibria.

| Pure Nash Equilibria for Fully Connected Networks | | | | |
|---|---|---|---|---|
| Case | Control Set | Consensus Society | Heirarchical Society | Western Society |
| A | All pairs | All pairs | All pairs | All pairs |
| B | (1,1) | All pairs | (1,1) | (1,1) |
| C | (1,1) | All pairs | (1,1) | (1,1) |

*Table 4.32 – Nash Equilibria for Different Influence Parameters on Connected Networks*

*The pure Nash equilibria were found for all of the connected networks with each of the four different influence parameter sets. (x,y) is a strategy profile in which the TB player connects to agent X, and the US player connects to agent Y.*

In the fully connected networks we see results similar to those for the circle networks – any strategy profile is a pure Nash equilibrium in the consensus based society, and all of the other parameter sets yield identical equilibria.

| Pure Nash Equilibria for Rural Pashtun Networks | | | | |
|---|---|---|---|---|
| Case | Control Set | Consensus Society | Heirarchical Society | Western Society |
| Small | (1,1) | (7,7) | (1,1) | (1,1) |
| Large | (70,70) | (48,48) | (70,70) | (70,70) |

*Table 4.33 – Nash Equilibria for Different Influence Parameters on Rural Pashtun Networks*

*The pure Nash equilibria were found for the rural Pashtun networks with each of the four different influence parameter sets. (x,y) is a strategy profile in which the TB player connects to agent X, and the US player connects to agent Y. The equilibrium strategy profiles for the consensus based society are the 2nd best strategies for a player in the other societies, and the strategies in the equilibrium strategy profiles in the control set, hierarchical, and western societies are the 2nd best strategies available to a player in the consensus based society.*

For the rural Pashtun networks the equilibria again differ only for the consensus based society. However, it should be noted that the equilibrium strategies for the consensus based society are the 2nd best individual strategies for the other three data sets, and the equilibrium in the other three data sets is the 2nd best strategy for the consensus based society. This is true for both the large and small rural Pashtun networks. Thus even with significant error in the influence data parameters, the chosen strategy profiles are either optimal or near-optimal.

## 4.4.1 – Summary of Experiment Set 4

We have found that there is robustness in the pure Nash equilibria to influence parameter changes. The exception to this is for the consensus based society. The influence parameters for the consensus based society dictate that all types of mutable agents interact the same, and thus the model essentially has only 'regular' agents and 'stubborn' agents. If this were the case, then the model reduces to a much simpler model, with only 2 types of agents (regular and stubborn). It is unlikely that any society would be as uniform as this notional consensus based society, with all agents being equally influential. Furthermore, if a society were totally consensus based, but the player assumed otherwise, the resulting strategy profiles would still work well in the consensus based society. The only difficulty would be if an analysis assumed a consensus based society when it was not. As long as there are in fact differences between differing classes of agents, and the more influential agents are correctly identified, the equilibrium strategy profiles are extremely robust to the exact degree of influence represented in the parameter set. This indicates that an exhaustive analysis of a society need not be conducted to determine exactly how agents influence each other.

## 4.5 – Experiment Set 5: Real World Scenarios

For this set of experiments, we analyze two different scenarios to examine the utility of the methods developed in this thesis as potential decision support tools. First – we look at using the simulated annealing algorithm to provide a near optimal solution against a known TB strategy, and compare this to a strategy found using US Army counterinsurgency doctrine. Second – we look at a scenario where both the US and TB players use a weighted mean belief payoff function – with different weights for each player, to show an example of a real world scenario that creates non-symmetric equilibrium strategy profiles with low variance.

### 4.5.1 – Doctrine

The purpose of this experiment is to verify whether the simulated annealing heuristic can perform as well as an experienced US Army officer using doctrine to inform their decisions. We contacted a fellow US Army officer that has recently returned from Afghanistan, where he commanded a company and faced the problem of identifying key influencers in his area of operations. We set up the large rural Pashtun network with 3 stubborn agents, with 3 connections for each player as seen in Figure 4.20. The Taliban player's connections were set to the following strategy: (12,13,14,38,39,47,48,49,54).

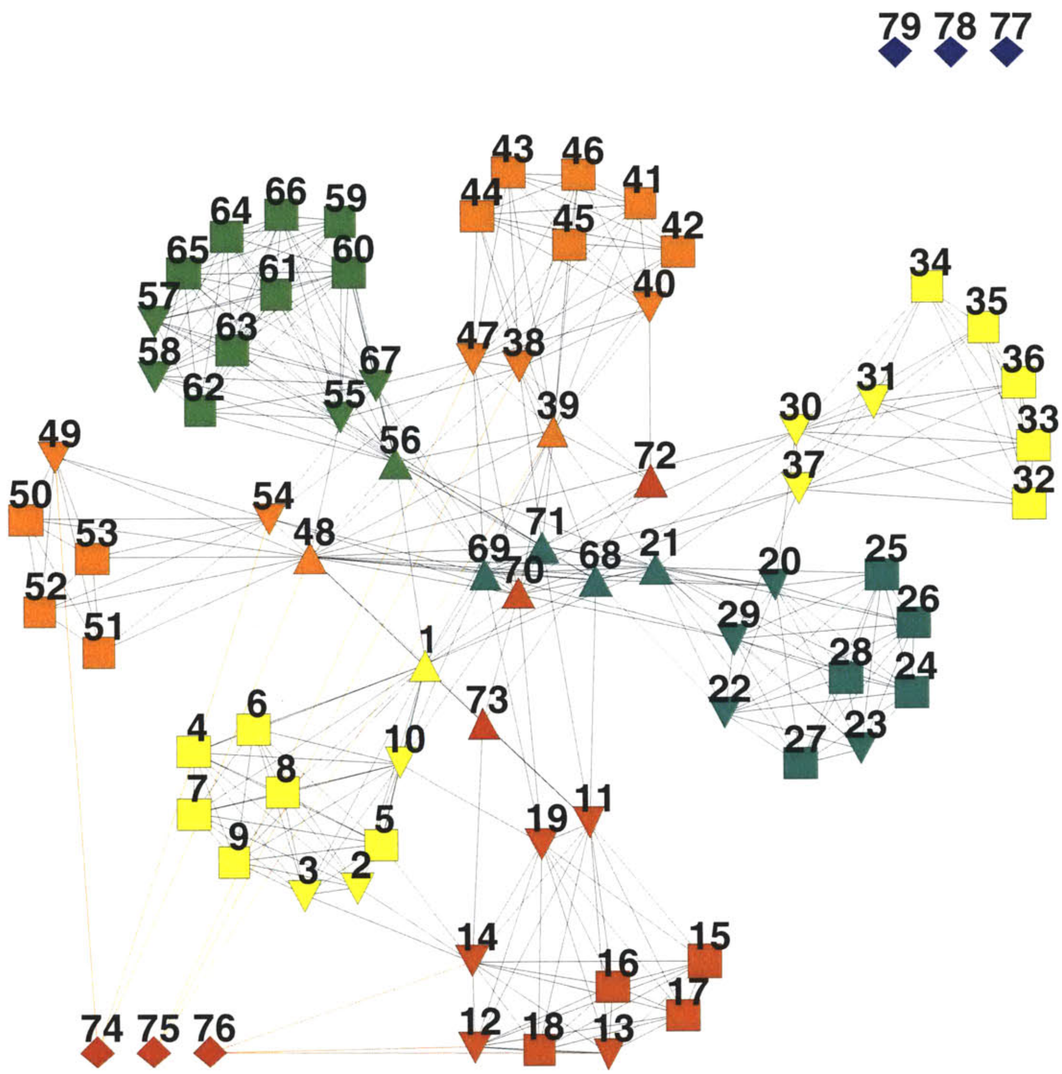



*Figure 4.20 – Large Rural Pashtun Network set up used for Experiment 5.1*

*Each player has 3 stubborn agents with 3 connections each. The TB player has a fixed strategy of (12,13,14,38,39,47,48,49,54). This scenario was then presented to an experienced combat leader from the US Army and they were asked to find an optimal solution using doctrine.*

This experienced US officer was then given the following set of doctrinal principles and asked to conduct an analysis of the network shown in Figure 4.19 to determine which mutable agents he would like to influence.

Doctrinal Principles:

- Identify leaders who influence the people at the local, regional, and national levels.
- Win over passive or neutral people.

- Nonlethal targets include people like community leaders and those insurgents who should be engaged through outreach, negotiation, meetings, and other interaction.
- Meetings conducted by leaders with key communicators, civilian leaders, or others whose perceptions, decisions, and actions will affect mission accomplishment can be critical to mission success.
- Start easy… Don't try to crack the hardest nut first—don't go straight for the main insurgent stronghold, try to provoke a decisive showdown, or focus efforts on villages that support the insurgents. Instead, start from secure areas and work gradually outwards. Do this by extending your influence through the locals' own networks.

The strategy he selected is shown in Table 4.34, and results in an equilibrium mean belief of 0.1268. Simulated annealing was run 10 times on the network from Figure 4.20 and found 7 different solutions. The average payoff for the US player was 0.1902, and the best was 0.1984.

| Doctrine - Informed Strategy | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| US Strategy | | | | | | | | | Payoff |
| 1 | 10 | 55 | 56 | 67 | 68 | 69 | 70 | 71 | 0.1268 |
| **Best Simulated Annealing Strategy** | | | | | | | | | |
| US Strategy | | | | | | | | | Payoff |
| 14 | 39 | 48 | 49 | 56 | 68 | 69 | 70 | 71 | 0.198350149 |

*Table 4.34 – Best Simulated Annealing Strategy vs Doctrine Based Strategy*

| Simulated Annealing Strategies | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| US Strat | | | | | | | | | Payoff | Time (sec) |
| 1 | 39 | 48 | 49 | 56 | 68 | 69 | 70 | 71 | 0.1954 | 92.5780 |
| 14 | 39 | 48 | 49 | 56 | 68 | 69 | 70 | 71 | 0.1984 | 91.7230 |
| 1 | 1 | 14 | 39 | 48 | 56 | 68 | 69 | 70 | 0.1773 | 91.9460 |
| 1 | 39 | 48 | 49 | 56 | 68 | 69 | 70 | 71 | 0.1954 | 93.1090 |
| 1 | 14 | 39 | 48 | 56 | 68 | 69 | 70 | 73 | 0.1914 | 92.2910 |
| 1 | 14 | 39 | 48 | 56 | 68 | 69 | 70 | 73 | 0.1914 | 92.1510 |
| 1 | 39 | 47 | 48 | 56 | 68 | 69 | 71 | 73 | 0.1870 | 93.0490 |
| 1 | 14 | 21 | 39 | 48 | 56 | 68 | 69 | 70 | 0.1920 | 92.0920 |
| 1 | 38 | 39 | 48 | 48 | 56 | 68 | 69 | 70 | 0.1834 | 92.4300 |
| 1 | 14 | 21 | 39 | 48 | 56 | 68 | 69 | 70 | 0.1920 | 92.4200 |
| | | | | | Average: | | | | 0.1902 | 92.3743 |

*Table 4.35 –Simulated Annealing Solution for Large Rural Pashtun Network, fixed TB Strategy*

These results show that the simulated annealing heuristic is able to find much better solutions under the 'mean belief' payoff function than even a highly experienced combat officer in the US Army that has dealt with this problem many times in Afghanistan. This provides evidence that simulated annealing finds high quality strategies.

### 4.5.2 – Weighted Mean Belief Payoff Functions

The motivation for this experiment is twofold– first is to demonstrate that although the 'risk averse mean belief' payoff function using a bound to approximate standard deviation was found not to be useful, the problem of matching strategies can be addressed if agents have different value for each player. The second reason is to verify that the model and analysis tools can be applied to and compared with real world scenarios. The author served as a company commander in Khowst and Paktika provinces in Afghanistan in 2008. While serving, our unit's primary motivation was to win the maximum number of people to the US side. Consequently most of our operations focused in the more heavily populated valleys that contained 80% or more of the provincial population. The Taliban on the other hand, were more focused on using parts of our area of operations as a route to the interior to the major cities of Gardez, Sharana, and Kabul. Thus, they cared only about maintaining supply routes through mountainous, sparsely populated villages. This often resulted in both of us settling into an equilibrium, where we maintained the majority of the population's support, and the Taliban maintained a minority of support in the key routes they needed. This experiment demonstrates this phenomenon using our model.

In this experiment, each player assigns a weight to each mutable agent in the network representing that player's value of winning the agent's attitude. Thus far, all of the 'mean belief' payoff analysis has used an identical set of weights (equal weights for all agents) for both players. Here each player assigns a different set of weights and thus has a different payoff function.

For this experiment we use the large rural Pashtun network, and each player has 3 stubborn agents, with 2 connections each. In this scenario, village 7 (agents 55-67) is notionally a mountainous village with only a small trail network connecting it to the rest of the valley, where villages 1-6 reside. The valley has a major road that moves through it that is important for US forces to use as a supply route. The US player is concerned with maintaining freedom of maneuver through the valley on this road, and thus wants to win the populace of these villages to the US side. The TB player utilizes the mountain trail network moving through village 7, and cares only about village 7. The US player evenly weights all agents in villages 1-6 with zero weight assigned to village 7 agents. The TB player evenly weights all agents in village 7, with zero weight assigned to villages 1-6.

The simulated annealing algorithm is run 10 times and finds the following equilibria:

| Simulated Annealing Nash Equilibria on Large Rural Pashtun Network; 3 Stubborn Agents w/2 Connections Each | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Run** | **US Strategy** | | | | | | **TB Strategy** | | | | | | **Time (sec)** |
| **1** | 1 | 48 | 68 | 69 | 70 | 71 | 55 | 56 | 56 | 58 | 67 | 67 | 524.3640035 |
| **2** | 39 | 48 | 56 | 68 | 69 | 70 | 55 | 56 | 56 | 57 | 58 | 67 | 646.6049921 |
| **3** | 1 | 21 | 48 | 68 | 69 | 70 | 55 | 56 | 56 | 57 | 58 | 67 | 703.7960015 |
| **4** | 1 | 48 | 68 | 69 | 70 | 71 | 55 | 55 | 56 | 56 | 58 | 67 | 530.3540032 |
| **5** | 1 | 21 | 48 | 69 | 70 | 71 | 55 | 55 | 56 | 56 | 67 | 71 | 1467.323001 |
| **6** | 1 | 39 | 48 | 68 | 69 | 70 | 55 | 55 | 56 | 56 | 67 | 67 | 530.7290062 |
| **7** | 1 | 21 | 48 | 68 | 69 | 70 | 55 | 55 | 56 | 56 | 67 | 67 | 406.1459966 |
| **8** | 1 | 21 | 48 | 68 | 69 | 70 | 55 | 55 | 56 | 56 | 57 | 67 | 463.2659946 |
| **9** | 39 | 48 | 56 | 68 | 69 | 70 | 55 | 56 | 56 | 57 | 58 | 67 | 967.1419982 |
| **10** | 1 | 39 | 48 | 68 | 69 | 70 | 55 | 55 | 56 | 56 | 67 | 67 | 739.7050001 |
| | | | | | | | Average Run Time | | | | | | 697.9429997 |

*Table 4.36 – Simulated Annealing Nash Equilibria on Large Rural Pashtun Network with Weighted Mean Belief Payoff Functions*

*Simulated Annealing returns different strategy profiles almost every time using a weighted mean belief payoff function, although all strategy profiles have good strategy options for both players. This demonstrates that simulated annealing can have problems with many local optima in the network.*

Unfortunately, there is no clear winner on which strategy sets are the true Nash equilibria. The algorithm has returned 5 different strategies for the US player, and 6 different strategies for the TB player. In order to better analyze this, we evaluate each of these strategies against each other, creating a 5 x 6 payoff matrix as seen in Table 4.37 and 4.38.

| US Strategies | | TB Strategies | |
|---|---|---|---|
| 1 | (1, 48, 68, 69, 70, 71) | 1 | (55, 56, 56, 58, 67, 67) |
| 2 | (39, 48, 56, 68, 69, 70) | 2 | (55, 56, 56, 57, 58, 67) |
| 3 | (1, 21, 48, 68, 69, 70) | 3 | (55, 55, 56, 56, 58, 67) |
| 4 | (1, 21, 48, 69, 70, 71) | 4 | (55, 55, 56, 56, 67, 71) |
| 5 | (1, 39, 48, 68, 69, 70) | 5 | (55, 55, 56, 56, 67, 67) |
| | | 6 | (55, 55, 56, 56, 57, 67) |

*Table 4.37 – Different Strategies Returned by Simulated Annealing on Large Rural Pashtun network with Weighted Mean Belief Payoff Functions*

| | | TB Strategy | | | | | |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 |
| US Strategy | 1 | 0.271, 0.41 | 0.278, 0.419 | 0.27, 0.409 | 0.222, 0.396 | 0.263, 0.395 | 0.27, 0.409 |
| | 2 | 0.258, 0.316 | 0.266, 0.358 | 0.26, 0.296 | 0.184, 0.232 | 0.253, 0.229 | 0.26, 0.296 |
| | 3 | 0.266, 0.413 | 0.273, 0.422 | 0.264, 0.412 | 0.205, 0.404 | 0.257, 0.398 | 0.264, 0.412 |
| | 4 | 0.264, 0.414 | 0.271, 0.422 | 0.263, 0.412 | 0.218, 0.401 | 0.256, 0.399 | 0.263, 0.412 |
| | 5 | 0.274, 0.412 | **0.281, 0.421** | 0.273, 0.411 | 0.212, 0.403 | 0.266, 0.397 | 0.273, 0.411 |

*Table 4.38 – Constructed Payoff Matrix of Different Strategies Returned by Simulated Annealing on Large Rural Pashtun Network with Weighted Mean Belief Payoff Functions*

*The payoff matrix shows a pure Nash equilibria amongst the strategies returned by simulated annealing that is a new strategy profile – the strategies used by the US and TB were not originally a strategy profile amongst the solutions returned by simulated annealing.*

This allows us to rule out several strategies, by allowing each player to use any of the strategies returned above to play the game. If a unilateral deviation from any of the strategies above to any other one strictly increases a player's payoff, the payoff matrix will show this. Although not all of the strategies are optimal, they are all strong strategies with very similar payoff. By finding the best strategy for each player, it will hopefully rule out several local optima and return a single Nash equilibrium in the game.

In this payoff matrix the only equilibrium strategy profile is:

$$S_{US}=\{1, 39, 48, 68, 69, 70\}$$

$$S_{TB}=\{55, 56, 56, 57, 58, 67\}$$

Payoff for US Player = 0.281, Payoff for TB Player = 0.421

This strategy profile was not returned by simulated annealing. This shows that although the strategy profiles returned by simulated annealing may not be true pure Nash equilibria, the strategies found can be used to provide a better strategy profile that is more likely to be a pure Nash equilibrium.

In this equilibrium strategy profile, each player chooses to influence different agents in the network. This shows a realistic example where each player has a different objective in influencing the network, which naturally leads to non-symmetric equilibrium strategy profiles.

We have also shown a useful method to employ when multiple equilibria are found through simulated annealing. By creating a payoff matrix of all the returned solutions and finding the Nash equilibria among them we can determine a better candidate for the true pure Nash equilibria. Multiple equilibria become more likely as the number of connections and stubborn agents increases because this exponentially increases the size of the strategy space that simulated annealing must search through, making it difficult to find global optima.

## 4.6 – Summary of Experiments

The first set of experiments showed that in sparse networks, centrality is very important in determining equilibria. The more connected the network becomes, the more the level of influence of an agent matters, and in fully connected networks influence level is the only thing that matters. When players use different payoff functions we see both non-symmetric equilibrium strategy profiles and first mover advantages, although most of the equilibria characterized were matching strategies. There is also experimental evidence that the game under these payoff functions always yields a pure Nash equilibrium.

We hoped that we could remove these matching strategies and also reduce the risk in a strategy profile using Yildiz et al's bound on standard deviation for a binary belief model. This did not work well for our model, and we currently have no useful method for accurately predicted risk and variance beyond simulation. However, in the real world, players often have different weighted mean beliefs, yielding non-matching equilibrium strategy profiles, as shown in experiment set 5.

Simulated annealing has proven to be a useful tool for finding optimal player strategies and characterizing Nash equilibria through best response dynamics in networks. For a single connection it is not an efficient as exhaustive enumeration, but as the number of players' connections increases beyond 1, simulated annealing performs several orders of magnitude faster. Unfortunately, as the number of connections increases it becomes more and more difficult for the algorithm to return global optima. However, combining all of the strategies returned by simulated annealing into a payoff matrix allows us to find the equilibrium strategy profile amongst the local optima returned by simulated annealing (though there is no guarantee it is a true pure Nash equilibrium). Simulated annealing is also able to return much better solutions than an experienced combat officer using doctrine, and also provides a better solutions than the Knitro solver found using Hung's math program.

The model shows robustness to changes in influence parameters as well. This is important as it indicates estimates of the $\alpha$, $\beta$, and $\gamma$ parameters do not have to be highly accurate.

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# 5 – Recommendations for Future Work

In this chapter we discuss several potential modifications and extensions to the social network model and 2-player game for future research. We first discuss reformulating the game as a stochastic game, which would consider the current state of the network in formulating strategy profiles. A second modification would be to make the game a partially observable game by restricting information on opposing player's strategies or the beliefs of the agents in the network. Another extension is to consider analyzing the social network and game on a dynamic network that changes over time. By increasing the range of strategy options to include media campaigns and more diffuse effects on the populace (like the goodwill generated by building a school) the game would be more realistic and better reflect the range of tools used to influence social networks. The final modification is to expand the possible interactions within the social networking model to include peer and group pressure effects.

## 5.1 – Stochastic Games

Stochastic games are played in a sequence of stages, in which each player receives some payoff. After each stage the game then evolves to a new stage stochastically, with the previous stage and the player's strategies affecting this evolution. This is a way to analyze the stochastic nature of the social network model more realistically. The network has a set of beliefs, and based off of how the player's choose their strategies, it affects the way in which the network beliefs can evolve.

Formulating the problem as a stochastic game could yield more interesting analysis by giving players the ability to respond to the network's beliefs as the evolve. For example, in Figure 5.1 we have a line network with a single central influential agent. It all of our previous analysis both players would want to connect to agent number three, as it is the most influential agent in the network. However, what if we considered the transient behavior of this network? The US player may instead want to connect to agent one in order to win it back more quickly than staying at agent three. However, if the US player leaves agent three, do they risk losing the rest of the network? Being able to analyze the short-term opportunity costs of engaging other leaders like this would be very useful for both commanders and staff, and could be analyzed more effectively through a stochastic game.

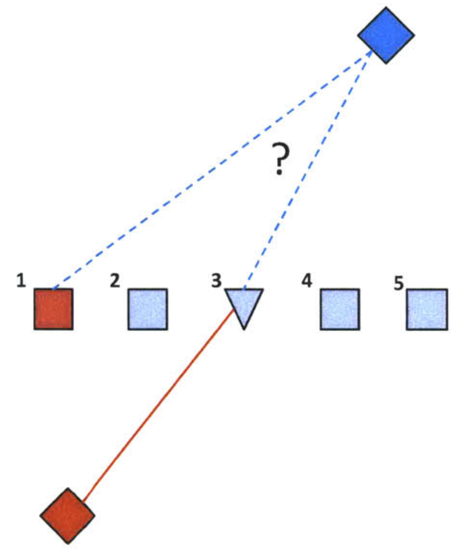


*Figure 5.1 – Line Network with US winning agents 2-5 and the TB winning Agent 1*

*Should the US player connect to agent 1 to win them back quickly, or if they do this will they lose the rest of the network to the TB player?*

To investigate this question we conducted some preliminary analysis using approximate dynamic programming. The initial results were encouraging, and showed that if one player has the ability to change strategies based off of the network state and the other player only played a single strategy that the player who considered the state of the network could influence the network far more effectively.

Unfortunately we were unable to analyze anything larger than around 16 agent networks, because the possible beliefs states of the network grow exponentially with the number of mutable agents (we approximated continuous beliefs with binary beliefs and thus the state space was $2^n$, which was still more than 65,000 possible belief states for a 16 agent network).

Formulating the problem as a stochastic game would be interesting, but would also be difficult to analyze. The current and future belief states would have to be considered to determine optimal strategies, and payoff functions would depend on the network state. This would mean the support space of the payoff function would grow exponentially in the size of the network.

## 5.2 – Partially Observable Games

In Afghanistan, neither US forces nor the Taliban have perfect intelligence. For example, it is easy for US forces to tell if a village or tribe supports the Taliban, but it is much harder to figure out if that means the Taliban are actively influencing the village, and if so who they are specifically influencing.

The game could be reformulated as a partially observable game. There are two major pieces of information that would make sense to have as partially observable – the opposing player's strategy and the network beliefs. Knowledge of the opposing player's strategy could be made partially observable by having player's infer the opposing strategies by observing the current state of the network. The best response strategy for each player depends on the opposing strategy (even in equilibrium payoff functions), which means that as the state of the network changes, it would change the player's belief of the opposing strategy. This would have the effect of making strategies state dependent.

It would not make sense to completely obscure knowledge of the network state, as both the US and the Taliban have many methods they use to gauge the support of the populace. However it would make sense to obscure parts of the information. In our game both players know the exact belief of every agent in the network. In reality neither the US nor the Taliban know the belief of every household in a district. Replacing this with some type of village or tribal aggregate beliefs would better reflect the true level of information both players usually have available in the real world.

## 5.3 – Evolving Networks

Our model assumes a static network, but over a long period of time this is not a reasonable assumption. Even in rural tribal societies tribal relations shift and leaders change. In societies where new leaders evolve, being able to recognize them and influence them early can be key to maintaining long term influence within a society. Furthermore, being able to find strategies that are robust to changes in society could be extremely valuable to players trying to insure themselves against an uncertain future.

## 5.4 – Other Strategy Options: Assassinations, Intimidation, Media Campaigns, and Reconstruction Projects

In our game the only strategy choices available were deciding who to engage within a society. While this is an important question, it is not the only method the US Army or insurgents use to influence networks. As discussed in Chapter 2, insurgents often use intimidation campaigns and assassinations to influence a populace. The US Army often uses broadcast media to distribute pro-government messages on radio and TV, in addition to spending millions of dollars on reconstruction projects. These media campaigns and reconstruction projects have diffuse effects in areas, and even though US soldiers may not be engaging any local leadership, these projects and media campaigns may serve to influence the populace. Similarly, insurgent intimidation campaigns can influence the populace without directly engaging individuals.

Additionally, allowing players to have more strategy options would introduce the problem of resource allocation to the game – should the US spend all of their time and effort influencing tribal leaders, or split it between reconstruction projects and influencing leaders? Being able to answer these tough questions would significantly increase the utility of these methods as a decision support tool for commanders.

## 5.5 – Peer Pressure and Group Dynamics

In a pro-Taliban village no matter how much US forces talk to and influence a single regular head of household (non-leader), it is unlikely that they can get this agent to support US forces while the rest of the village supports the Taliban. This is because of peer pressure that exists in groups. Although a group can be changed over time, overcoming peer pressure and group opinion dynamics greatly complicate the matter. We conducted some simple analysis where individuals adopt some small portion of their village mean belief at each time step. Adding this peer pressure effect did affect the transient behavior of the network, but it did not affect any of the long term equilibrium beliefs, and thus had no impact on our game. If the game were reformulated as a stochastic game this peer pressure effect would change the game.

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# 6 – Summary and Conclusions

In this thesis we consider the problem of finding optimal strategies for influencing social networks for US forces in Afghanistan. The social network is analyzed using a discrete time agent based model. Each agent has a scalar belief and interacts stochastically pairwise with their neighbors. In this model the beliefs of agents are random variables, but they approach an equilibrium belief in expectation. This convergence in expectation is used to analyze the future behavior of the network. This equilibrium belief is also independent of the agent's starting belief, and depends only on the network topology and influence parameters.

A 2-player game is formulated with different payoff functions on this social networking model where a US player and a Taliban player each control a set of stubborn agents whose beliefs never change. Each player then connects these stubborn agents to the rest of the network in a way that maximally influences the network beliefs according to each player's payoff function. Two primary payoff functions ('mean belief' and 'agents won') are defined, and depend only on the equilibrium beliefs of agents in the network. This means that the payoff depends only on how the players connect their stubborn agents to the network, and not on the current state of the network. Because the payoffs do not depend on the state of the network it significantly reduces the strategy space for each player, but it still increases exponentially in the number of connection that each player gets to make (there are $n^C$ possible strategy choices for each player, where C is the number of connections each player makes and n is the number of mutable agents in the network).

To understand the dynamics of the game we study pure Nash equilibria is a series of test networks to better understand the behavior of the model and the game. Two approaches are used – first is to exhaustively calculate the payoff matrix, which takes $O(n^3)$ calculations per strategy profile. With $n^{2C}$ strategy profiles, it requires $O(n^{3+2C})$ total calculations, and the calculation time is exponential in C. When C is one, this approach works well in our experiments, but for any value larger than 1 it rapidly increases in run time and is no longer tractable. The second approach is to use simulated annealing to individually optimize each player's best response against the other's current strategy. The simulated annealing heuristic uses best response dynamics to conduct a series of row and column searches through the payoff matrix without fully enumerating it. This approach takes $O(n^3)$ calculations and does not scale with the number of connections.

In analyzing pure Nash equilibria we find that in sparse networks, the location and centrality of agents become very important in determining the most influential agents (which are generally part of the equilibrium strategy profiles). As the number of connections increases, the level of influence of an agent (as defined by input parameters) becomes more important. The most interesting results emerge between a sparse and a highly connected network, where it becomes extremely difficult to predict the most influential agents. For identical payoff functions almost all of the pure Nash equilibria consist of identical strategies. This can be naturally avoided when players use weighted mean belief payoff functions, awarding different value to influencing different parts of the network.

The 2-player game in the social networking model presented here provide a way to analyze the behavior of popular support in insurgencies. Finding globally optimal best response strategies is a difficult problem in large networks, and characterizing the pure Nash equilibria is even more difficult. Fortunately we are able to use a simulated annealing heuristic to find the equilibria through best response dynamics. There is no guarantee that pure Nash equilibria exist, but we have shown evidence through experimentation that under the two main payoff functions of 'mean belief' and 'agents won' that the game admits pure Nash equilibria.

We were unable to characterize the variance in the social network, and hence the variance associated with strategy profiles. This is a problem that we leave for future research.

The game theoretic approach can also be extended to work in many other fields where multiple players are trying to influence social networks. For example, voter models where politicians compete for votes could be analyzed. Companies running advertising campaigns for competeing products are another example of players trying to influence social network. There are many more possible applications where players are trying to influence social networks in order to bring about a change in attitudes or adopt certain behaviors.

Modeling popular support in an insurgency is an extremely difficult task. Determining who to influence and how to influence them is rarely obvious, especially when an opponent is working against any efforts. The US Army has learned through trial and error how to influence Iraqis and Afghans, but there is still significant room for improvements. It is our hope that the tools and analysis methods discussed in this thesis help to continue the development of analysis techniques to aid the US Army as it looks to the future.

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# Appendix A – Notation and Definitions

## Agent Sets in the Network

$V_{TB}$ – Set of stubborn agents with belief $-.5$

$V_{US}$ – Set of stubborn agents with belief $+.5$

$V_s$ – Set of all stubborn agents $= V_0 \cup V_1$

$V_R$ – Set of Regular Agents

$V_I$ – Set of Influential Agents

$V_V$ – Set of Very Influential Agents

$V_m$ – Set of mutable agents $= V_R \cup V_I \cup V_V$

$V$ – all agents $= V_m \cup V_s = V_R \cup V_I \cup V_V \cup V_{TB} \cup V_{US}$

## Network Definitions and Notation

n: number of agents in the network  $n=|V|$

$X_i(t)$: Belief of agent i after interaction t, $X_i(t) \in [-.5, .5]$

X(t): vector of all beliefs of all agents in the network after interaction t
$X(t) = (X_1(t), X_2(t), \ldots., X_n(t)) \in \mathbb{R}^{n\times 1}$

$\beta_{ij}$: Probability of agent i having an averaging ($\beta$-type) interaction, conditioned on agent i interacting with agent j

$\alpha_{ij}$: Probability of agent i having a forceful ($\alpha$-type) interaction, conditioned on agent i interacting with agent j

$\gamma_{ij}$: Probability of agent i having an averaging ($\gamma$-type) interaction, conditioned on agent i interacting with agent j

$\varepsilon$: stubbornness rating of mutable agents – meaning the amount of their own belief an agent retains after being forcefully influenced by another agent

# Vector and Matrices Definitions and Notation

$e_i$ – A unit vector of length n (number of agents in the network), with entry I equal to 1, and all other entries equal to zero.

$E_i$ – The set of agents connected to agent i; $|E_i|$ = the degree (number of connections to other agents) of agent i

W – Random matrix that represents a single interaction on the network: X(t+1)=W*X(t)

$$W = \begin{cases} A_{ij} \equiv I - \frac{(e_i - e_j)(e_i - e_j)'}{2} & \text{with probability } \frac{\beta_{ij}}{n * |E_i|} \\ J_{ij} \equiv I - (1-\epsilon)e_i(e_i - e_j)' & \text{with probability } \frac{\alpha_{ij}}{n * |E_i|} \\ I & \text{with probability } \frac{\gamma_{ij}}{n * |E_i|} \end{cases}$$

$$\hat{W} \equiv E[W]$$

$$\hat{W} = \begin{bmatrix} B & D \\ 0 & I \end{bmatrix}$$

$$|B| = |V_m|x|V_m|, \quad |D| = |V_m|x|V_s|, \quad |I| = |V_s|x|V_s|, \quad |0| = |V_s|x|V_m|$$

# Network Equilibrium State Definitions and Notation

$X_i^*$ - equilibrium belief of agent i

$X^*$- vector of equilibrium beliefs of agents in the network

$$X^* = \lim_{k\to\infty} E[(X(t+k)|X(t)] = \lim_{k\to\infty} \hat{W}^k * X(t) \to \hat{W}^\infty * X(t)$$

$X_S$ - $|V_s|$ x 1 vector of stubborn agent beliefs; contains values of only +0.5 and -0.5; is also equal to the last $|V_s|$ entries of the belief vector X(t) or X* (stubborn agents are the same no matter what time).

# Markov Chain and Variances Analysis Definitions and Notation

$\eta_{ij}$: the probability that starting in state i on the Markov chain, a random walk ends in one of the absorbing states j.

$p_j^i$ : probability that a random walk started at state j on the Markov Chain is absorbed by state i

$r^i \equiv \sum_{j \in V_M, j \neq i} p_j^i \quad \forall i \in V_m$ $r^i$ can be thought of as a metric for centrality

*DEFINITION:* $p_j^{US}$ *is the probability of a random walk starting at node j being absorbed by one of the states in* $V_{US}$.

$$\boldsymbol{p_j^{US}} = \sum_{i \in V_{US}} \eta_{ji}$$

$\sigma(X^*)$ - The standard Deviation of the mean belief of the network around equilibrium – this is a measure of how much deviation we would expect to see in the actual mean belief of the network if it was simulated, versus the expected equilibrium mean belief we can calculate.

$$\sigma(X^*) \leq \frac{1}{N^2} \sum_{j \in V_m} \left(\boldsymbol{p_j^{US}} - (\boldsymbol{p_j^{US}})^2\right) * (r^j)^2$$

# 2-Player Game Definitions and Notation

First, we rewrite several previous quantities as functions of the strategies chosen by each player (strategies affect topology and change these quantities). They retain the same meaning as before, but we use this notation to indicate that they change with player's strategy choices.

$$\hat{W}(S_{US}, S_{TB}) = E[W(S_{US}, S_{TB})] = \begin{vmatrix} D(S_{US}, S_{TB}) & B(S_{US}, S_{TB}) \\ 0 & I \end{vmatrix}$$

$$X^*(S_{US}, S_{TB}) = (I - D(S_{US}, S_{TB}))^{-1} * B(S_{US}, S_{TB}) * X_S$$

S – Set of strategy options available (same for both players)

$S_{US}$ – Strategy chosen by the US player

$S_{TB}$ – Strategy chosen by the TB player

$|E_S|$ - the number of connections from each stubborn agent to the mutable agents in the network – identical for all stubborn agents.

C –the total number of connections each player has from their stubborn agents to the mutable agents in the network

$$C = |E_S| * \frac{|V_S|}{2}$$

Mean Belief payoff functions:

$$\mathrm{f_{US}(S_{US}, S_{TB})} = \frac{1}{|\mathrm{V_m}|} \sum_{j=1}^{|\mathrm{V_m}|} \mathrm{X_j^*(S_{US}, S_{TB})} = -\mathrm{f_{TB}(S_{US}, S_{TB})}$$

Agents Won Payoff Functions:

$$1_{\mathrm{TB}}(\mathrm{x}) \begin{cases} -1 & \text{if } \mathrm{x} > 1 \\ 0 & \text{if } \mathrm{x} = 0 \\ 1 & \text{if } \mathrm{x} < 0 \end{cases}$$

$$1_{\mathrm{US}}(\mathrm{x}) \begin{cases} 1 & \text{if } \mathrm{x} > 1 \\ 0 & \text{if } \mathrm{x} = 0 \\ -1 & \text{if } \mathrm{x} < 0 \end{cases}$$

$$\mathrm{g_i(S_{US}, S_{TB})} = \sum_{j=1}^{|\mathrm{V_m}|} 1_{\mathrm{i}} \left(\mathrm{X_j^*(S_{US}, S_{TB})}\right) \quad \mathrm{i} \in \{\mathrm{US}, \mathrm{TB}\}$$

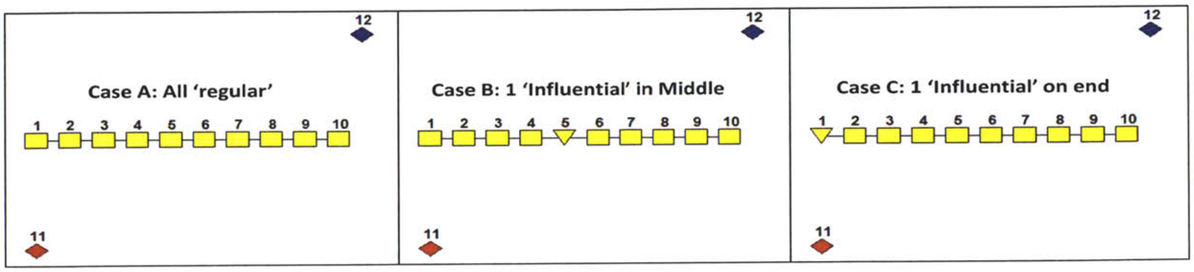
Risk Averse Mean Belief Payoff Function:

$$\mathrm{h_i(S_{US}, S_{TB})} = \mathrm{f_i(S_{US}, S_{TB})} - \lambda * \sigma\left(X^*(\mathrm{S_{US}, S_{TB}})\right) \quad \mathrm{i} \in \{\mathrm{US}, \mathrm{TB}\}$$

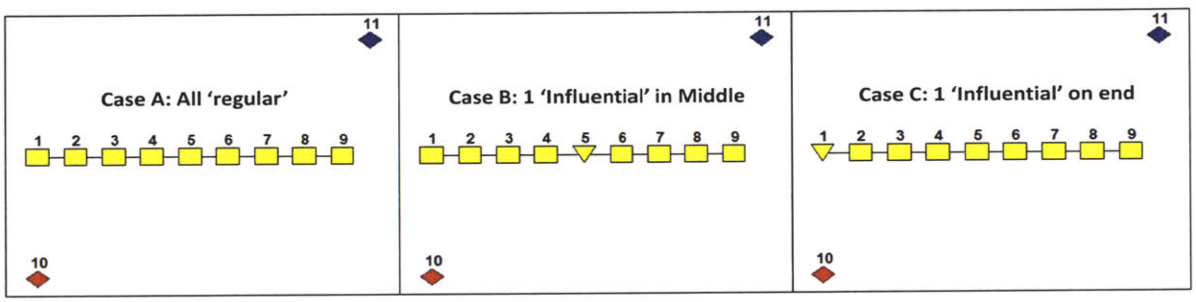
# Appendix B – Test Networks and Pure Nash Equilibria

## Line Networks

### Even Degree (10 Mutable Agents)




### Odd Degree (9 Mutable Agents)




### Pure Nash Equilibrium Strategy Profiles

| Nash Equilibria for Line Networks | | |
|---|---|---|
| Payoff Functions - Mean Belief vs Mean Belief | | |
| Case | Even Degree | Odd Degree |
| A | (5,5), (5,6), (6,6) | (5,5) |
| B | (5,5) | (5,5) |
| C | (8,8) | (7,7) |

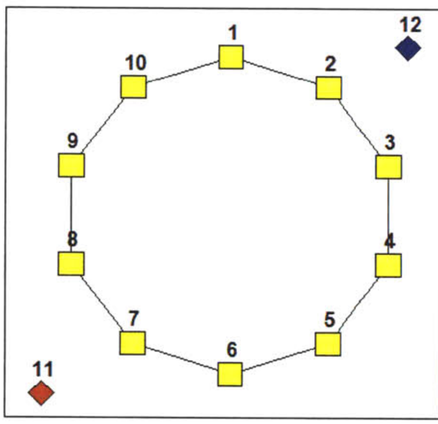
| Nash Equilibria for Line Networks | | |
|---|---|---|
| Payoff Functions - Nodes Won vs Nodes Won | | |
| Case | Even Degree | Odd Degree |
| A | (5,5), (5,6), (6,6) | (5,5) |
| B | (5,5) | (5,5) |
| C | (6,6) | (5,5), (5,6), (6,6) |

| Nash Equilibria for Line Networks<br>Payoff Functions - Mean Belief vs Nodes Won | | |
|---|---|---|
| Case | Even Degree | Odd Degree |
| A | (5,5), (5,6), (6,5), (6,6) | (5,5) |
| B | (5,5), (5,6) | (5,5) |
| C | (7,6), (8,7) | (7,6),(6,5) |

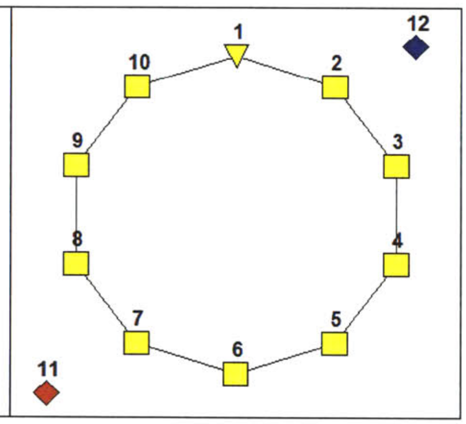
## Circle Networks

### Even Degree (10 Mutable Agents)

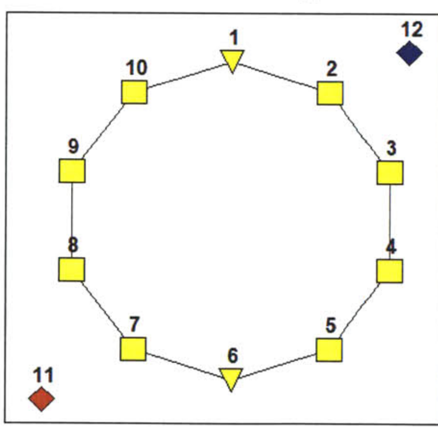
Case A: All 'regular'



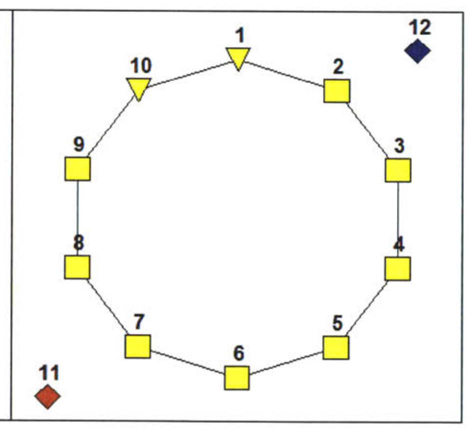

Case B: 1 'influential'




Case C: Two 'influential' on opposite sides

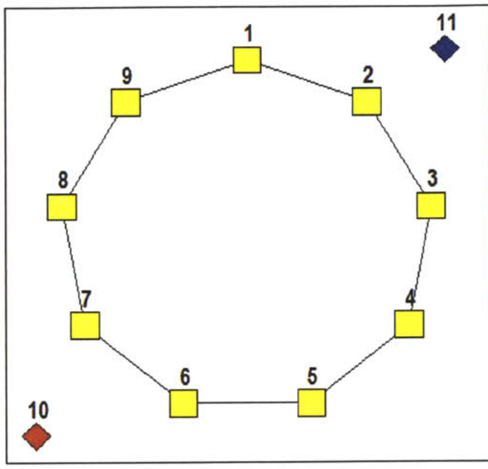



Case D: 2 'influential' adjacent

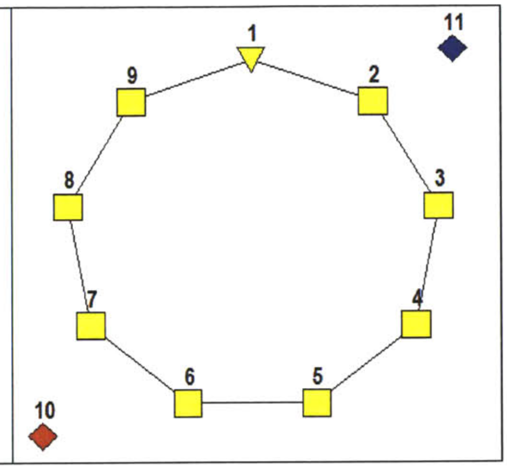



### Odd Degree (9 Mutable Agents)

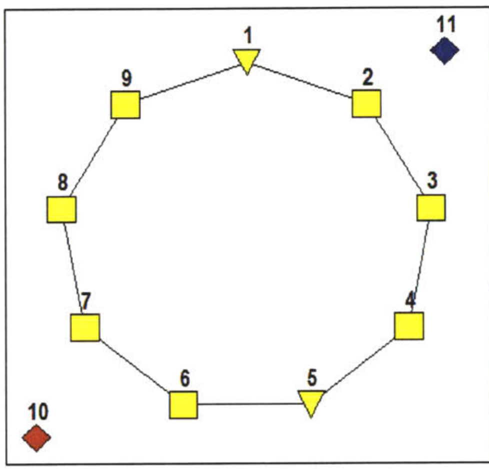
Case A: All 'regular'



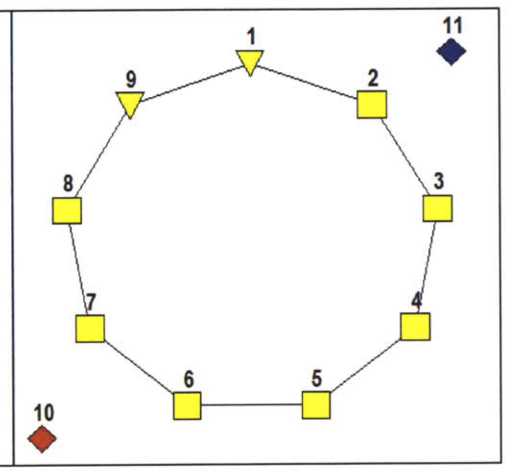

Case B: 1 'influential'




Case C: Two 'influential' on opposite sides




Case D: 2 'influential' adjacent



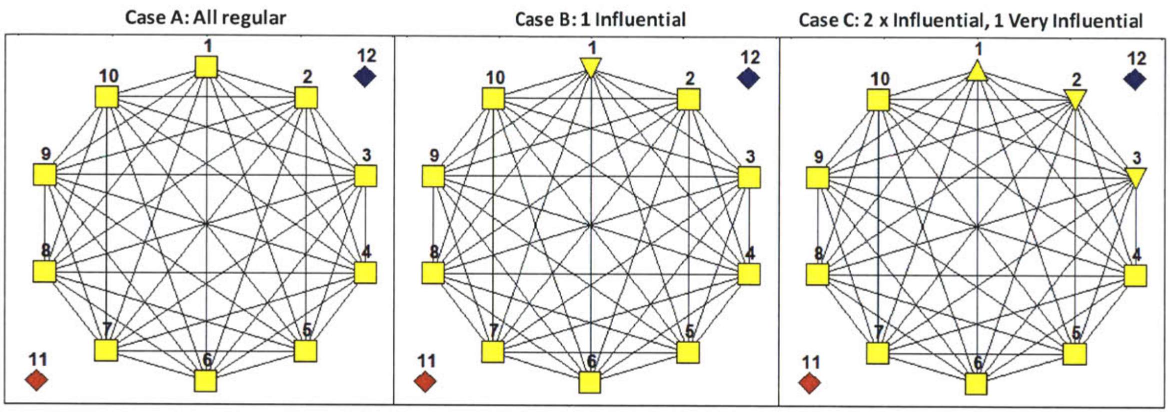

**Pure Nash Equilibrium Strategy Profiles**

| Nash Equilibria for Circle Networks | | |
|---|---|---|
| Payoff Functions - Mean Belief vs Mean Belief | | |
| Case | Even Degree | Odd Degree |
| A | All pairs | All pairs |
| B | (1,1) | (1,1) |
| C | (1,1), (1,6), (6,6) | (1,1), (1,5), (5,5) |
| D | (1,1), (1,10), (10,10) | (1,1), (1,9), (9,9) |

| Nash Equilibria for Circle Networks | | |
|---|---|---|
| Payoff Functions - Nodes Won vs Nodes Won | | |
| Case | Even Degree | Odd Degree |
| A | All pairs | All pairs |
| B | (1,1), (1,6), (6,6) | (1,1) |
| C | (1,1), (1,6), (6,6) | (1,1), (1,5), (5,5) |
| D | (1,1), (1,10), (10,10) | (1,1), (1,9), (9,9) |

| Nash Equilibria for Circle Networks<br>Payoff Functions - Mean Belief vs Nodes Won | | |
|---|---|---|
| Case | Even Degree | Odd Degree |
| A | All pairs | All pairs |
| B | (1,1), (1,6) | (1,1) |
| C | (1,1), (1,6), (6,1), (6,6) | (1,1), (1,5), (5,1), (5,5) |
| D | (1,1), (1,10), (10,1), (10,10) | (1,1), (1,9), (9,1), (9,9) |

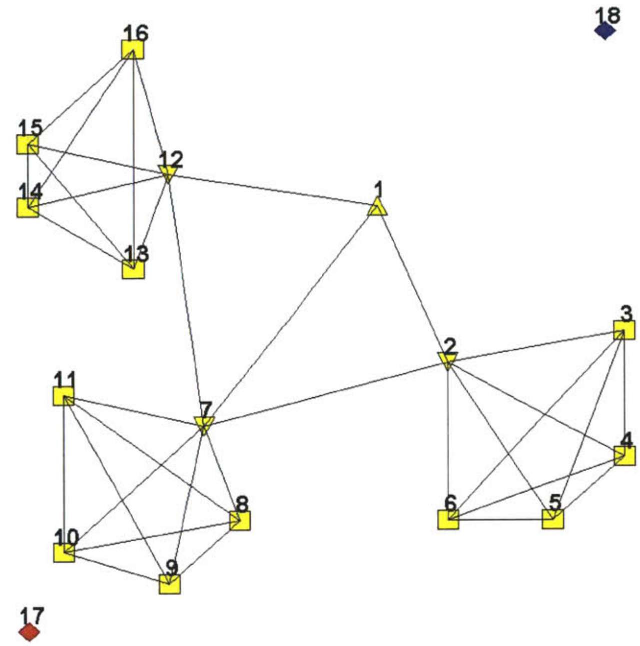
## Fully Connected Networks




### Pure Nash Equilibrium Strategy Profiles

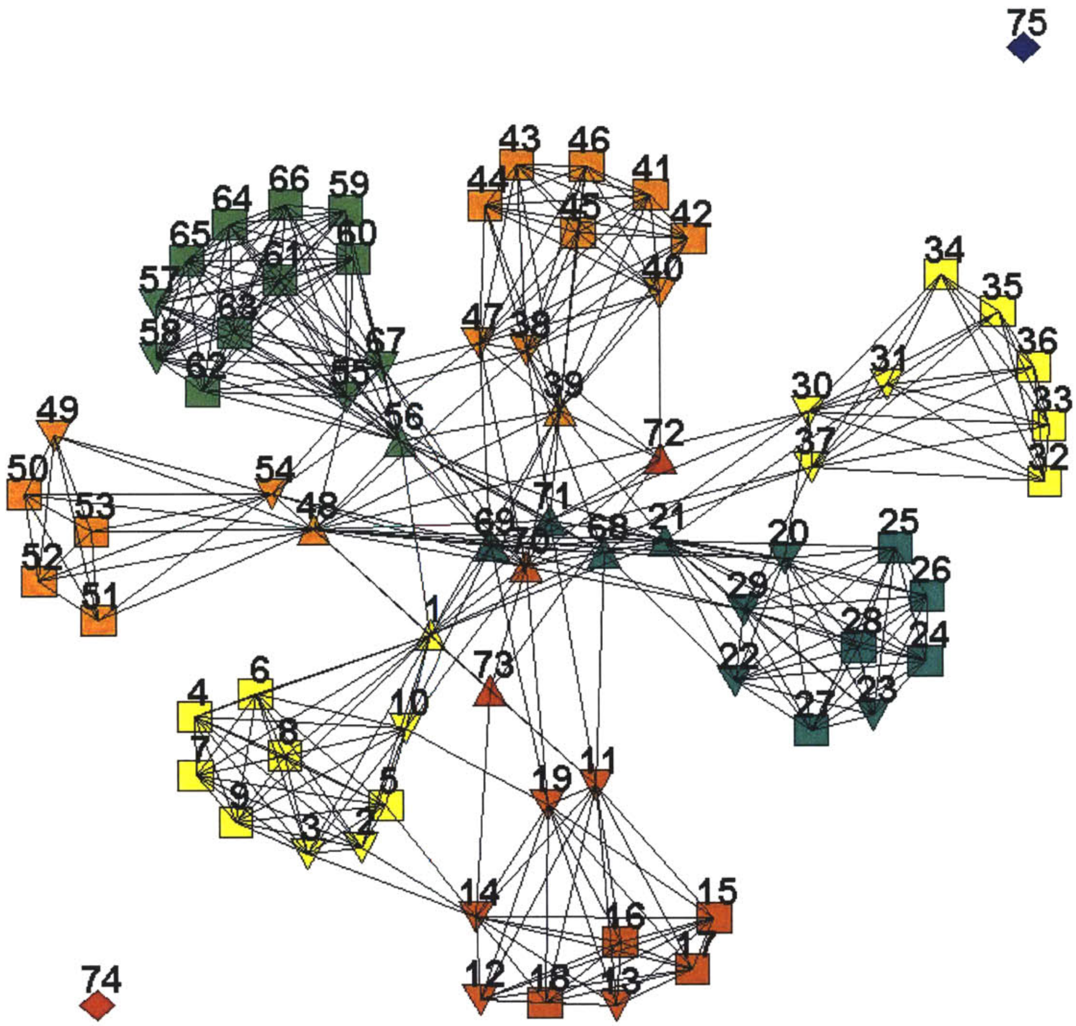
| Nash Equilibria for Fully Connected Networks | |
|---|---|
| Case | Any Payoff Functions |
| A | All pairs |
| B | (1,1) |
| C | (1,1) |

## Small Rural Pashtun Network




**For all combinations of payoff functions, the Pure Nash Equilibrium Strategy Profile for the small rural Pashtun Network is (1,1).**

## Large Rural Pashtun Network




**For all combinations of payoff functions, the Pure Nash Equilibrium Strategy Profile for the largerural Pashtun Network is (70,70).**

# Appendix C – Random Large Network Methodology and Equilibrium

**Methodology for Generating Large Random Networks:**

-Number of Villages = Uniform(10,20)

-Number of Regular Agents in Each Village = Uniform(20,30)

-Number of Influential Leaders in each Village = 0.2 * Number of Regular Agents

-Within a village all regular and influential agents are fully connected.

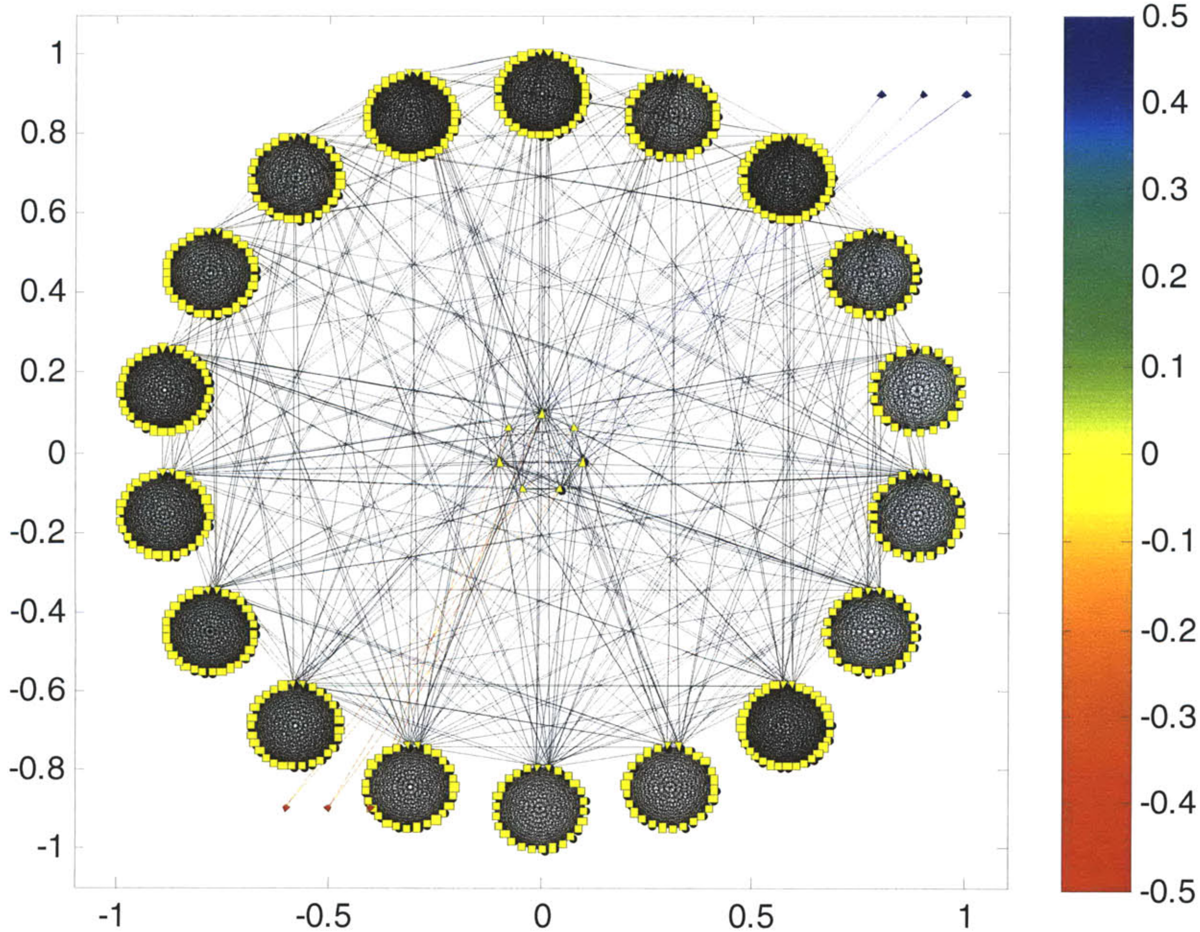
-Every Influential Leaders Randomly connect to 30% of the other Influential Leaders

-Number of Very Influential Agents = 0.2 * Total Number of Influential Village Leaders

-Very Influential Leaders randomly connect to 50% of the influential and very influential leaders in the network

This methodology creates a series of villages with a few different leaders, in which every person knows everyone else in the village.Villages leaders know about a third of other village leaders within their districts. At the district level are several very influential leaders that know 50% of the district and village leadership in the district.

The particular instance generated using this random network generator had 18 villages, with a total of 471 regular agents. There were 35 influential agents in the villages, and 7 very influential district leaders.


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5

# Appendix D – Listing of Influence Parameters in Experiment 4

Here we list the four different influence parameter sets used in Experiment 4. The influence parameters affect how each class of agents interacts with the other classes. The four classes of agents used in our model are:

Regular Agent

Influential Agent

Very Influential Agent

Stubborn Agent

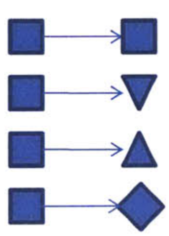
Each interaction can take on one of three types:

α is the probability of agent i forcefully influencing agent j

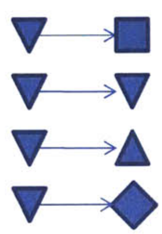
β is the probability of agent i averaging beliefs with agent j

γ is the probability of agent i having no impact on agent j

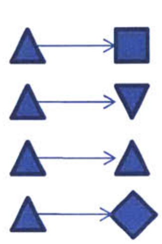
Control Set of Influence Parameters



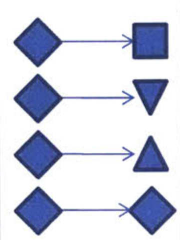
| | α | β | γ |
|---|---|---|---|
| | 0 | 1 | 0 |
| | 0 | 0 | 1 |
| | 0 | 0 | 1 |
| | 0 | 0 | 1 |



| | α | β | γ |
|---|---|---|---|
| | 1 | 0 | 0 |
| | 0 | 0.1 | 0.9 |
| | 0 | 0.1 | 0.9 |
| | 0 | 0 | 1 |



| | α | β | γ |
|---|---|---|---|
| | 1 | 0 | 0 |
| | 0.4 | 0.1 | 0.5 |
| | 0 | 0.1 | 0.9 |
| | 0 | 0 | 1 |



| | α | β | γ |
|---|---|---|---|
| | 1 | 0 | 0 |
| | 1 | 0 | 0 |
| | 1 | 0 | 0 |
| | 0 | 0 | 1 |

The Consensus society is one in which all agents always adopt middle of the road opinions and agree with each other. Every mutable agent always averages their belief with every other type of mutable agent.

Consensus Society Influence Parameters

| | α | β | γ |
|---|---|---|---|
| ■ → ■ | 0 | 1 | 0 |
| ■ → ▼ | 0 | 1 | 0 |
| ■ → ▲ | 0 | 1 | 0 |
| ■ → ◆ | 0 | 0 | 1 |

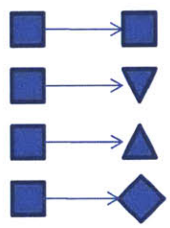
| | α | β | γ |
|---|---|---|---|
| ▼ → ■ | 0 | 1 | 0 |
| ▼ → ▼ | 0 | 1 | 0 |
| ▼ → ▲ | 0 | 1 | 0 |
| ▼ → ◆ | 0 | 0 | 1 |

| | α | β | γ |
|---|---|---|---|
| ▲ → ■ | 0 | 1 | 0 |
| ▲ → ▼ | 0 | 1 | 0 |
| ▲ → ▲ | 0 | 1 | 0 |
| ▲ → ◆ | 0 | 0 | 1 |

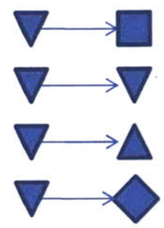
| | α | β | γ |
|---|---|---|---|
| ◆ → ■ | 1 | 0 | 0 |
| ◆ → ▼ | 1 | 0 | 0 |
| ◆ → ▲ | 1 | 0 | 0 |
| ◆ → ◆ | 0 | 0 | 1 |

The Heirarchical society is one in which belief is imposed upon less influential agents by more influential agents. All agents will forcefully influence any less influential type of agent every time they interact. Agents that are the same level of influence will average opinions every time they interact.

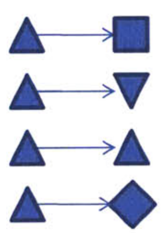
Heirarchical Society



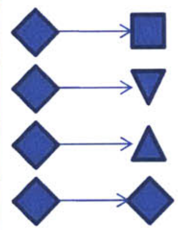
| α | β | γ |
|---|---|---|
| 0 | 1 | 0 |
| 0 | 0 | 1 |
| 0 | 0 | 1 |
| 0 | 0 | 1 |



| α | β | γ |
|---|---|---|
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |
| 0 | 0 | 1 |



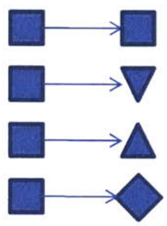
| α | β | γ |
|---|---|---|
| 1 | 0 | 0 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |



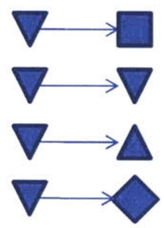
| α | β | γ |
|---|---|---|
| 1 | 0 | 0 |
| 1 | 0 | 0 |
| 1 | 0 | 0 |
| 0 | 0 | 1 |

The Western Society is a modification of the control set. It allows less influential agents to occasionally influence more influential agents. This is intended to model a western democratic society in which influential people don't have the ability to impose their beliefs on their constituents, and instead must sometimes respond to the demands of the masses.

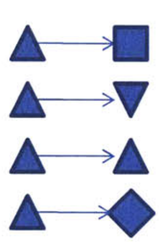
Western Society Influence Parameters



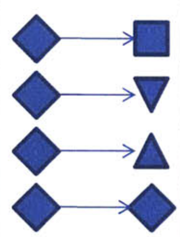
| α | β | γ |
|---|---|---|
| 0.1 | 0.9 | 0 |
| 0 | 0.1 | 0.9 |
| 0 | 0.1 | 0.9 |
| 0 | 0 | 1 |



| α | β | γ |
|---|---|---|
| 0.5 | 0.1 | 0.4 |
| 0.1 | 0.5 | 0.4 |
| 0 | 0.1 | 0.9 |
| 0 | 0 | 1 |



| α | β | γ |
|---|---|---|
| 0.9 | 0.1 | 0 |
| 0.5 | 0.1 | 0.4 |
| 0.1 | 0.5 | 0.4 |
| 0 | 0 | 1 |



| α | β | γ |
|---|---|---|
| 1 | 0 | 0 |
| 1 | 0 | 0 |
| 1 | 0 | 0 |
| 0 | 0 | 1 |

# Appendix E – Nash Equilibria for Experiment Set 4

## Line Network Pure Nash Equilibrium Strategy Profiles for Experiment Set 4

| Nash Equilibria for Even Line Networks | | | | |
|---|---|---|---|---|
| Payoff Functions - Mean Belief vs Mean Belief | | | | |
| Case | Control Set | Consensus Society | Heirarchical Society | Western Society |
| A | (5,5), (5,6), (6,6) | (5,5), (5,6), (6,6) | (5,5), (5,6), (6,6) | (5,5), (5,6), (6,6) |
| B | (5,5) | (5,5), (5,6), (6,6) | (5,5) | (5,5) |
| C | (8,8) | (5,5), (5,6), (6,6) | (8,8) | (5,5), (5,6), (6,6) |

| Nash Equilibria for Odd Line Networks | | | | |
|---|---|---|---|---|
| Payoff Functions - Mean Belief vs Mean Belief | | | | |
| Case | Control Set | Consensus Society | Heirarchical Society | Western Society |
| A | (5,5) | (5,5) | (5,5) | (5,5) |
| B | (5,5) | (5,5) | (5,5) | (5,5) |
| C | (7,7) | (5,5) | (7,7) | (5,5) |

## Circle Network Pure Nash Equilibrium Strategy Profiles for Experiment Set 4

| Nash Equilibria for Even Circle Networks | | | | |
|---|---|---|---|---|
| Payoff Functions - Mean Belief vs Mean Belief | | | | |
| Case | Control Set | Consensus Society | Heirarchical Society | Western Society |
| A | All pairs | All pairs | All pairs | All pairs |
| B | (1,1) | All pairs | (1,1) | (1,1) |
| C | (1,1), (1,6), (6,6) | All pairs | (1,1), (1,6), (6,6) | (1,1), (1,6), (6,6) |
| D | (1,1), (1,10), (10,10) | All pairs | (1,1), (1,10), (10,10) | (1,1), (1,10), (10,10) |

| Nash Equilibria for Odd Circle Networks | | | | |
|---|---|---|---|---|
| Payoff Functions - Mean Belief vs Mean Belief | | | | |
| Case | Control Set | Consensus Society | Heirarchical Society | Western Society |
| A | All pairs | All pairs | All pairs | All pairs |
| B | (1,1) | All pairs | (1,1) | (1,1) |
| C | (1,1), (1,5), (5,5) | All pairs | (1,1), (1,5), (5,5) | (1,1), (1,5), (5,5) |
| D | (1,1), (1,9), (9,9) | All pairs | (1,1), (1,9), (9,9) | (1,1), (1,9), (9,9) |

## Fully Connected Network Pure Nash Equilibrium Strategy Profiles for Experiment Set 4

| Nash Equilibria for Fully Connected Networks | | | | |
|---|---|---|---|---|
| Case | Control Set | Consensus Society | Heirarchical Society | Western Society |
| A | All pairs | All pairs | All pairs | All pairs |
| B | (1,1) | All pairs | (1,1) | (1,1) |
| C | (1,1) | All pairs | (1,1) | (1,1) |

**Rural Pashtun Network Pure Nash Equilibrium Strategy Profiles for Experiment Set 4**

| Nash Equilibria for Rural Pashtun Networks | | | | |
|---|---|---|---|---|
| Payoff Functions - Mean Belief vs Mean Belief | | | | |
| Case | Control Set | Consensus Society | Heirarchical Society | Western Society |
| Small | (1,1) | (7,7) | (1,1) | (1,1) |
| Large | (70,70) | (48,48) | (70,70) | (70,70) |

# Appendix F – Sample TCAF Questionnaire

**TACTICAL CONFLICT ASSESSMENT FRAMEWORK**
**GENERAL QUESTIONS – Initial Interview**
**KEY STAKEHOLDER INTERVIEW QUESTIONNAIRE**

| | | | |
|---|---|---|---|
| Q1. Province: | | Q2. MGRS/GPS Reference: | |
| Q3. District: | | Q4. Village: | |
| | | | |
| Q5. Last Name of Interviewer: | | Q6. Date of Interview (00/00/06) | |
| | | | |
| Person Interviewed | | | |
| Q7. First Name: | | Q8. Last Name: | |
| Q9. Title: | | Q9A. Tribe: | |
| Q10. Who is your Tribal Elder? | | | |

Q11. Organization/Affiliation (category)

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ☐ | 1 | District Government | ☐ | 2 | Local Government | ☐ | 3 | Tribal/Clan Leader | ☐ | 4 | Religious Leader |
| ☐ | 5 | Police | ☐ | 6 | Army | ☐ | 7 | Political Party | ☐ | 8 | Civil Society |
| ☐ | 9 | Health Sector | ☐ | 10 | Schools/ Education Sector | ☐ | 999 | Other: ______________ | | | |

Q12. How many people live in this village? ______________

Q13. How many households are there in this village? ______________

Q14. In the past 12 months, have more people moved in or out of this village?

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| ☐ (4) | More moved in | ☐ (3) | More moved out | ☐ (2) | The same number moved in as moved out | ☐ (1) | None moved in or out | ☐ (9) | I don't know *[DO NOT READ]* |

Q15. Why?

Q16. What are the major problems facing the residents of this village today? *[READ OPTIONS ONLY IF RESPONDANT IS UNABLE TO PROVIDE AN ANSWER; PLEASE PROBE FOR THREE MOST IMPORTANT SHOULD THE RESPONDANT LIST MORE THAN THREE]*

| | Worst Problem (15A) | Second Problem (15B) | Third Problem (15C) |
|---|---|---|---|
| A. Lack of food/potable water | 1 | 1 | 1 |
| B. Lack of paid work opportunities | 2 | 2 | 2 |
| C. Lack of shelter | 3 | 3 | 3 |
| D. Limited roads, sewage/sanitation, electric power | 4 | 4 | 4 |
| E. Access to health care | 5 | 5 | 5 |
| F. Access to education | 6 | 6 | 6 |
| G. Security | 7 | 7 | 7 |
| H. Discrimination | 8 | 8 | 8 |
| I. Government responsiveness to citizen concerns | 9 | 9 | 9 |
| J. Corruption | 10 | 10 | 10 |
| Other:______________ | | | |
| Other:______________ | | | |
| Other:______________ | | | |
| None | 0 | 0 | 0 |
| No further reply *[DO NOT READ]* | | 990 | 990 |
| I don't know *[DO NOT READ]* | 999 | | |

Q17. How much do you think the residents of this village trust each of the following institutions? *[READ OPTIONS; PROBE FOR STRENGTH OF ANSWER]*

| | Not at all | Just a little | Somewhat | A lot | I don't know *[DO NOT READ]* |
|---|---|---|---|---|---|
| A. The National Government | 0 | 1 | 2 | 3 | 9 |
| B. The Provincial Government | 0 | 1 | 2 | 3 | 9 |
| C. The municipal/local government | 0 | 1 | 2 | 3 | 9 |
| D. The courts | 0 | 1 | 2 | 3 | 9 |
| E. The local police | 0 | 1 | 2 | 3 | 9 |
| F. The national army | 0 | 1 | 2 | 3 | 9 |
| G. International forces | 0 | 1 | 2 | 3 | 9 |
| H. Tribal leaders | 0 | 1 | 2 | 3 | 9 |
| I. Religious leaders | 0 | 1 | 2 | 3 | 9 |
| J. Local social service organizations | 0 | 1 | 2 | 3 | 9 |
| K. Radio broadcasts | 0 | 1 | 2 | 3 | 9 |
| L. Newspapers | 0 | 1 | 2 | 3 | 9 |
| M. Television reports | 0 | 1 | 2 | 3 | 9 |

Q18. What are the most important things that can be done in the next six months to improve conditions for residents of this village? *[READ OPTIONS ONLY IF RESPONDANT UNABLE TO ANSWER; PLEASE PROBE FOR THREE MOST IMPORTANT TO RESPONDANT]*

| | Primary Response (44A) | Secondary Response (44B) | Tertiary Response (44C) |
|---|---|---|---|
| Increase food availability | 1 | 1 | 1 |
| Provide job/employment opportunities | 2 | 2 | 2 |
| Shelter construction | 3 | 3 | 3 |
| Property dispute adjudication | 4 | 4 | 4 |
| Reliable power supply | 5 | 5 | 5 |
| Potable water | 6 | 6 | 6 |
| Improve local roads | 7 | 7 | 7 |
| Waste management programs | 8 | 8 | 8 |
| Build schools | 9 | 9 | 9 |
| Teachers | 10 | 10 | 10 |
| School supplies | 11 | 11 | 11 |
| Health care facilities | 12 | 12 | 12 |
| Health care training | 13 | 13 | 13 |
| Medical supplies | 14 | 14 | 14 |
| Improve policing | 15 | 15 | 15 |
| Integrate militias into security structures | 16 | 16 | 16 |
| Increase or withdraw international forces | 17 | 17 | 17 |
| Improve rule of law | 18 | 18 | 18 |
| Decrease human rights violations | 19 | 19 | 19 |
| Anti-corruption/transparency programs | 20 | 20 | 20 |
| Citizen access to government representatives | 21 | 21 | 21 |
| Increase local government influence/activities | 22 | 22 | 22 |
| Other (1st response):______________________ | | | |
| Other (2nd response):______________________ | | | |
| Other (3rd response):______________________ | | | |
| Nothing / No problem | 0 | | |
| No further reply *[DO NOT READ]* | | 990 | 990 |
| Don't know *[DO NOT READ]* | 999 | | |