Competition in the Supply Option Market[1]

Submitted: July 17, 2006. Revised: July 19, 2007; March 27, 2008; October 8, 2008; December 30, 2008; March 20, 2009.

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Abstract

This paper develops a multi-attribute competition model for procurement of short life cycle products. In such an environment, the buyer installs dedicated production capacity at the suppliers before demand is realized. Final production orders are decided after demand materializes. Of course, the buyer is reluctant to bear all the capacity and inventory risk, and thus signs flexible contracts with several suppliers. We model the suppliers' offers as option contracts, where each supplier charges a reservation price per unit of capacity, and an execution price per unit of delivered supply. These two parameters illustrate the trade-off between total price and flexibility of a contract, which are both important to the buyer. We model the interaction between suppliers and the buyer as a game in which the suppliers are the leaders and the buyer is the follower. Specifically, suppliers compete to provide supply capacity to the buyer and the buyer optimizes its expected profit by selecting one or more suppliers. We characterize the suppliers' equilibria in pure strategies for a class of customer demand distributions. In particular, we show that this type of interaction gives rise to *cluster competition*. That is, in equilibrium suppliers tend to be clustered in small groups of two or three suppliers each, such that within the same group all suppliers use similar technologies and offer the same type of contract. Finally, we show that in equilibrium supply chain inefficiencies, i.e., the loss of profit due to competition, are at most 25% of the profit of a centralized supply chain.

1 Introduction

The introduction of new products is usually associated with uncertain sales forecasts. When a product life-cycle is short, firms usually have limited opportunities to postpone production decisions to the time accurate forecasts are available. For instance, it would be ideal to delay decisions on production quantities until the beginning of the selling season, when initial sales have been observed and thus sales expectations are more robust. However, if production lead times are long, this is obviously not possible, and thus firms must take *capacity and inventory risks* at product launch. In industries such as electronics or fashion retailing, managing these risks appropriately is critical for the long-term survival of firms.

One way to reduce the financial impact of these risks is to adjust supply costs with sales realizations. This is done through *flexible contracts* which allow scaling up production volume and costs with sales volume and revenues. The typical example of a flexible contract is an *option contract*. This contract is characterized by two parameters, a capacity reservation fee and an execution fee. For each unit of capacity installed by a supplier, the buyer pays in advance the reservation fee. When demand is realized, the buyer decides how many units to order, and pays the execution fee for each unit in the order.

[1]Research supported in part by the Center of eBusiness at MIT, ONR Contracts N00014-95-1-0232 and N00014-01-1-0146, NSF Contracts DMI-9732795, DMI-0085683 and DMI-0245352.

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Contracts similar to option contracts are common in industries such as textiles, plastics or semiconductors manufacturing. In some cases, such contracts are disguised under the name of *buy-back contracts*, which are equivalent to an option, as in newspaper or book distribution. Specifically, in a buy-back contract, the cost of the component and the amount of refund for returned items are specified; this corresponds to an option with reservation price equal to the total cost minus the refund, and an execution price equal to the refund. It is well known, see Pasternack [15], that buy-back contracts can be beneficial to both buyer and supplier, since they can coordinate the supply chain.

As shown by Martínez-de-Albéniz and Simchi-Levi [13], one way for a buyer to better manage its risks is to build a portfolio of option contracts, purchased from several suppliers. This allows the buyer to take advantage of the relative cost and flexibility of the different contracts. For instance, it can sign the contract that offers the lowest total cost for a portion of demand that is very likely to materialize; such a contract is typically associated with little flexibility. In addition, the buyer can add a more expensive but flexible contract to the portfolio, for the more volatile part of sales. An illustration of this strategy can be found in apparel retailing, where a retailer may place a large order overseas, at a low price, and at the same time, reserve some capacity locally and have the option of scaling up production if demand is high.

Evidently, this purchasing strategy can force changes in the way suppliers compete in the marketplace. Clearly, flexibility and price are the two attributes that the buyer cares about, and suppliers should take note of it. The objective of this paper is precisely to *analyze the suppliers' pricing strategy when they are competing through price and flexibility.* Specifically, our objective is to characterize what option contracts will be offered by the suppliers in a competitive equilibrium.

For this purpose, we focus on products with short life cycles and consider a single period model with many suppliers and a single buyer purchasing a single component. The sequence of events is as follows. First, each supplier offers an option contract to the buyer, with a given reservation and execution fee. After receiving all the competing bids, the buyer reserves capacity with some or all suppliers. Finally, after demand is realized, the buyer requests deliveries from each supplier, up to the installed capacity.

Of course, a supplier needs to take into account the competitors' bids when offering its preferred contract. A given supplier can thus undertake two main actions to become more competitive: either to lower its reservation price or to lower its execution price. The trade-off is clear. A supplier that charges mainly a reservation fee (and a small execution fee) competes with low overall price, but not necessarily flexibility. On the other hand, a supplier that charges mainly an execution fee (and a small reservation fee) typically emphasizes flexibility, but not overall price.

A supplier's bid also depends on its cost structure. In our model, we focus on two types of costs for each supplier.

- A *reservation cost* is associated with setting up the line and making preparations for production. We assume that this reservation cost has a linear or per-unit cost structure, corresponding to the acquisition of special machinery or specialized labor, that scales up with the level of capacity requested by the buyer.
- An *execution cost* is incurred when the supplier finalizes production and ships the components, after it receives the firm and final order from the buyer. This execution cost also has a per-unit cost structure, corresponding to labor, materials and logistics cost.

Different suppliers may have different costs for reserving capacity and delivering supply, depending on the type of technology (machinery) and their geographical location (labor, transportation). In addition, the cost structure of each firm may also be determined by its production strategy: a company that buys dedicated machines early on incurs most of the cost as a reservation cost; a company that leases these same machines later on has the ability to pass the corresponding cost as execution cost.

The supplier cost model is consistent with situations where the capacity installed under contract is *dedicated to the buyer*, and not shared with other firms. For instance, we are familiar with a large Taiwanese contract manufacturer that, upon signing a supply contract with a buyer, typically sets up a dedicated line for that buyer, in advance of the production season. The reservation cost for the dedicated line clearly increases with the capacity level. After demand is realized, the buyer, a PC manufacturer in this case, decides a final order quantity, up to the capacity, and this is produced and shipped by the contract manufacturer. The cost associated with production and shipping is the execution cost defined earlier. As a matter of fact, this PC manufacturer uses another supplier for the same component, with presumably a different cost structure.

Our objective in this paper is to understand how these suppliers compete. The model captures the multiple cost dimensions of the suppliers, i.e., reservation and execution costs, as well as the way they compete and differentiate their offerings, i.e., reservation and execution prices.

We describe the market equilibrium outcomes of the suppliers' option pricing game. We characterize the suppliers' equilibria in pure strategies for a class of customer demand distributions. Interestingly, this model is an extension of the traditional Bertrand price competition model to two dimensions. An important result in the Bertrand model (in one dimension) is that, in equilibrium, there is a unique supplier that captures all the orders. This supplier is the lowest cost supplier. The equilibrium price is between its cost and the cost of the second most competitive supplier. We show that this is not the case when two attributes are important to the buyer. Indeed, we demonstrate that in equilibrium, a variety of suppliers will coexist, and these suppliers offer different prices. We call this *cluster competition*, since suppliers tend to cluster in small groups of two or three suppliers each such that, within a group, all suppliers use similar technologies and offer the same type of contract.

Intuitively, our results imply that the best strategy for each supplier is to set a price very similar to some other supplier, while making sure that any share of capacity “stolen” from that supplier yields positive profit. Thus, the supplier does not simply undercut the other supplier, but instead skims carefully the type of capacity (e.g., with higher or lower probability of execution) that it wants to capture.

In addition, we show that in equilibrium, the supply chain inefficiencies, i.e., the loss of profit due to competition, are in general at most 25% of the first-best, i.e., the profit of a centralized supply chain, for a wide class of demand distributions. Finally, supplier competition through option contracts is particularly attractive to the buyer, since it may allocate more profit to the buyer than an Expected Vickrey-Clark-Groves (EVCG) mechanism, see Schummer and Vohra [18].

We start by reviewing the different streams of literature relevant to our research in Section 2. We then present the model in Section 3 and analyze the buyer's behavior in Section 4. We focus on the suppliers' game in Section 5, where we characterize best-response strategies and equilibria. Finally, we conclude with managerial insights in Section 6. All the proofs are presented in the appendix.

2 Literature review

Our starting point for this research is Martínez-de-Albéniz and Simchi-Levi [13]. In that paper, the authors develop a multi-period framework in which buyers optimize their purchasing strategy by carefully balancing price and flexibility. In particular, in their single period version, they provide a closed-form expression for the amounts of option capacities that a buyer purchases from a pool of suppliers. However, in their model the suppliers' bids are exogenous, i.e., there is no competition amongst them. In contrast, here we analyze the behavior of the suppliers when they compete across the two dimensions of price and flexibility captured respectively by reservation and execution fees.

We relate this research to the literature on supply contracts; for a review see Cachon [4] or Lariviere [11]. Some papers study buyer behavior under option contracts, e.g., Barnes-Schuster et al. [1] or Eppen and Iyer [7]. Most relevant to this work are papers that analyze the behavior of suppliers in offering options to a buyer, the prelude to introducing competition between suppliers. The existing literature usually models a sequential game à la Stackelberg, where a single buyer is the follower and a single supplier is the leader. Typically, competition in such models is introduced by a spot market. This spot market is the buyer's sourcing alternative and a potential client for the supplier. The focus is on (i) finding conditions for which both players are willing to sign a contract and (ii) determining option prices as the outcome of the negotiation process. Our paper moves from the traditional models of competition through dual sourcing, i.e., single supplier offering an option contract versus spot market, to a model of oligopolistic competition amongst suppliers offering different types of options.

The first publication in this stream of literature is by Wu et al. [24]. Motivated by electricity markets, they derive option prices as a function of the costs of the system, the spot price distribution and the buyer's utility. Later, Spinler et al. [19] and Golovachkina and Bradley [10] analyze similar models. A multi-sourcing version of this approach is presented in Wu and Kleindorfer [22]. Suppliers are characterized by their execution unit cost and their available capacity, and offer option contracts to the buyer. Wu and Kleindorfer derive Bertrand-like results, where competitive suppliers contract with the buyer up to their available capacity. Wu et al. [23] expand this model by proposing a capacity investment game between the suppliers, where, after installing capacity, the short-run price competition presented in Wu and Kleindorfer [22] takes place.

Interestingly, Wu and Kleindorfer [22] assume the same cost structure as ours, but suppliers make capacity investments before price competition, there is a random spot price and the buyer has a deterministic utility (which implies a deterministic demand function); in our model, on the other hand, capacity investments follow pricing decisions, there is no spot market but a random demand, and suppliers are uncapacitated. These differences lead to significantly different equilibrium characteristics. Specifically, Wu and Kleindorfer show that the buyer follows a greedy contracting rule, i.e., purchases capacity from the suppliers with the lowest overall price, up to capacity. Thus, the suppliers' equilibrium is such that all suppliers active in the contracting market offer an identical overall price. Potentially, if all suppliers have infinite capacity, Wu and Kleindorfer imply that a single supplier will be active in the contract market. Thus, deterministic utility leads to single sourcing, and multi-sourcing comes from having capacitated suppliers. In comparison, in our model uncertainty in demand leads to multi-sourcing, since multi-sourcing can manage demand uncertainty at a lower cost, compared to single sourcing.

Another related stream of the literature concentrates on analyzing multi-attribute auctions. This research follows the development of online auctions in B2B markets. Typically, the objective is to design the auction mechanism so as to reach an optimal outcome. An optimal outcome may be defined as social efficiency or profit maximization from the auctioneer's point of view (e.g., Myerson [14] in a one-dimensional auction). Usually, there is uncertainty in the suppliers' cost structure, and hence the design of the auction is done using probabilistic distributions of costs. In our paper, however, the costs are assumed to be deterministic and known to all the players; hence, our modeling approach is very different than the approach in this group of papers.

In this line of research, several authors have studied the winner determination problem, where a single supplier is awarded all orders. This differs from our formulation where all suppliers may potentially be selected for part of the procurement. For instance, Beil and Wein [2], following Che [5], present a multi-attribute Request For Quotation (RFQ) process where the buyer declares a scoring rule and chooses a winner among many suppliers, the one that obtains the highest score for the declared rule.

In a different direction, some research has been done on mechanism design where many bidders can be awarded orders at the same time. For instance, Schummer and Vohra [18] analyze a class of two-dimensional option auction mechanisms for a set of suppliers confronted with a single buyer. Their formulation is similar to our model but focuses on designing an efficient procurement mechanism where the suppliers have the incentive to truthfully reveal their costs. Because in their paper the contracting mechanism is different, suppliers compete but always end up bidding the true costs. In comparison, we analyze a model of supplier competition under complete information where suppliers are paid their bid prices.

3 Assumptions and Notation

Consider a single buyer purchasing a component that is used in the manufacturing of a final product. This component may be obtained from a number of suppliers. We make two assumptions regarding the selling price and the demand observed by the buyer.

Assumption 1 *The buyer sells to end customers at an exogenously fixed unit price p.*

Assumption 2 *The total customer demand D follows a distribution defined over an interval $[\underline{d}, \overline{d}] \subset [0, \infty]$. The c.d.f. of demand $F(\cdot)$ is strictly increasing in $[\underline{d}, \overline{d}]$. We assume that $F(\cdot)$ is a continuous and differentiable function over $(\underline{d}, \overline{d})$. Define $f(\cdot) = F'(\cdot)$ and $\overline{F}(\cdot) = 1 - F(\cdot)$.*

The buyer's objective is to maximize expected profit by optimally selecting the amount of capacity to reserve from each supplier.

We denote by N the number of suppliers in the market. The suppliers' cost structure consists of two parts. Each supplier incurs a fixed unit cost for reserving capacity, f_i, $i = 1, \ldots, N$ that can be seen as the unit cost of installing dedicated capacity in advance of production. In addition, suppliers pay a unit cost, c_i, $i = 1, \ldots, N$, for each unit executed by the buyer, which corresponds to the cost of finalizing the component plus transportation. These costs differ from supplier to supplier and may be explained by the use of different technologies or management practices. Without loss of generality, we assume that $c_1 \leq \ldots \leq c_N$.

Each supplier offers an option contract to the buyer. Such a contract is defined by two parameters, $v \geq 0$, the reservation price, and $w \geq 0$, the execution price. These values are determined by the supplier based on its cost structure as well as on whether the supplier emphasizes price or flexibility. Specifically, supplier i, $i = 1, \ldots, N$, takes position in the market by offering options at a reservation price v_i and an execution price w_i.

Given the suppliers' offerings, the buyer specifies the amount of capacity to reserve with each supplier[4]. At the time the buyer executes a contract with a supplier, it can purchase any amount up to the reserved capacity with the supplier. Thus, the profit of supplier i, $i = 1, \ldots, N$, is

[4]It can be shown that the suppliers have no incentive to build more than the amount specified in the contracts, since any additional capacity, for which no reservation fee is received, would fetch negative profit.

$(v_i - f_i)x_i + (w_i - c_i)q_i$ when the buyer reserves x_i units of capacity and executes q_i units, $0 \leq q_i \leq x_i$. The objective of the suppliers is to maximize their expected profit by selecting (w_i, v_i) optimally.

We analyze a two-stage model. In the first stage all suppliers submit bids that are defined by (w_i, v_i), $i = 1, \ldots, N$. At the same time, and based on these bids, the buyer decides on the amount of capacity to reserve with each supplier. In the second period, demand is realized and the buyer decides the amount to execute from each contract. If total capacity is not enough, unsatisfied demand is lost.

This is a game in which suppliers are first-movers and the buyer reacts myopically to the suppliers' bids. Thus, there are multiple players that compete knowing the reaction of the buyer. Suppliers have complete visibility on the buyer's decision making process, as well as on the demand distribution. Therefore, given any N pairs (w_i, v_i), $i = 1, \ldots, N$, each supplier can figure out the amount of capacity that the buyer would reserve with each individual supplier as well as the distribution of the amount of supply executed (requested) by the buyer.

We assume that suppliers submit sealed bids simultaneously. Thus, this is a one-shot game. We are interested in determining the equilibria of this game in pure strategies, i.e., the N-uples (w_i, v_i), $i = 1, \ldots, N$, where no supplier has an incentive to unilaterally change its bid.

Information-wise, we assume that the cost parameters of the suppliers are known to each other. Indeed, in practice, most firms have a rather precise idea on the type of technology used by each one of their competitors. This is a strong assumption, which is found as well in the asymmetric Bertrand or Hotelling models, for instance, see Vives [21].

4 The Buyer's Procurement Strategy

Martínez-de-Albéniz and Simchi-Levi [13] present a general framework for supply contracts in which portfolios of options can be analyzed and optimized. In this section, we review their framework in the context of a single period environment.

We consider a buyer facing N different options with terms (w_i, v_i), $i = 1, \ldots, N$. Martínez-de-Albéniz and Simchi-Levi show that the buyer's expected profit is concave in the quantity vector $x := (x_1, \ldots, x_N)$ purchased. Without loss of generality, we assume that $w_1 < \ldots < w_N < p$[5]. We define $w_{N+1} = p$, $v_{N+1} = 0$ and q_{N+1} the amount of lost sales, which creates an opportunity cost

[5]If two bids have the same execution price w, then we can discard the one with higher reservation price v, since it is never optimal to allocate any capacity to it.

of $w_{N+1}q_{N+1}$. We define also the total cost for a given demand realization d as

$$C(x,d) = \sum_{i=1}^{N+1} v_i x_i + \min_{q_1,\dots,q_{N+1}} \sum_{i=1}^{N+1} w_i q_i$$
$$\text{subject to } \begin{cases} 0 \le q_i \le x_i \; i = 1,\dots,N, \\ 0 \le q_{N+1}, \\ \sum_{i=1}^{N+1} q_i = d \end{cases}$$

Observe that the execution policy of the buyer (after demand is realized) is to use the option contracts with lower execution costs first. Hence, the buyer's profit is $\Pi(x,D) = pD - C(x,D)$. Thus, the expected profit is $\overline{\Pi}(x) = pE[D] - E[C(x,D)]$.

We denote $y_0 = 0$ and

$$y_i = x_1 + \ldots + x_i \text{ for } i = 1,\dots,N. \tag{1}$$

Then, letting $y := (y_1,\dots,y_N)$, $V(y) = \overline{\Pi}(x)$ satisfies for $i = 1,\dots,N$, see [13],

$$\frac{\partial V}{\partial y_i} = (v_{i+1} - v_i) + (w_{i+1} - w_i)Pr[D \ge y_i]. \tag{2}$$

Equation (2) thus provides the structure of the buyer's optimal portfolio, determined by the c.d.f. of customer demand. We can observe that the marginal value of increasing y_i while keeping the rest fixed (i.e., increasing x_i and decreasing x_{i+1}, in fact replacing capacity installed at $i+1$ by capacity at i) is equal to the decrease in reservation cost, $v_{i+1} - v_i$ per unit, plus the decrease in average execution cost, $(w_{i+1} - w_i)Pr[D \ge y_i]$ per unit. Under Assumption 2, when there are no identical bids from the suppliers, the profit is a strictly concave function of $(y_1,\dots,y_N)$ defined over the set

$$P = \left\{(y_1,\dots,y_N) \in \mathbb{R}^N \middle| 0 \le y_1 \le \ldots \le y_N\right\} \tag{3}$$

Strict concavity implies that the optimal solution is unique. Of course, when two or more suppliers submit identical bids, concavity is not strict, and the buyer may arbitrarily allocate capacity to any of the suppliers. If this occurs, we assume that the buyer randomly selects one of the bids, which will prevent suppliers from submitting identical bids. Instead, we allow that they submit *infinitely close bids*, i.e., bids that are different but infinitesimally close to each other. As a result, the buyer's allocation is unique. Thus, in the game analyzed in this paper, suppliers (leaders) know exactly how the buyer (follower) behaves.

To characterize the optimal portfolio, $(x_1^*,\dots,x_N^*)$, we need the following definitions.

Definition 1 *Supplier i is called* active *if $x_i^* > 0$. Otherwise, it is called* inactive.

Definition 2 *Given a set of t different pairs $\{(a_1,b_1),\dots,(a_t,b_t)\}$ with $a_1 \le \ldots \le a_t$, the* winning set *is the minimal subset $S = \{i_1,\dots,i_k\}$ of these points such that:*

(a) $a_{i_1} \leq \ldots \leq a_{i_k}$*;*

(b) for $1 \leq i < i_1$*,* $b_i - b_{i_1} \geq -(a_i - a_{i_1})$*;*

(c) for $j = 2, \ldots, k$*, for* $i_{j-1} < i < i_j$*,* $b_i - b_{i_j} \geq -\left(\frac{b_{i_{j-1}} - b_{i_j}}{a_{i_j} - a_{i_{j-1}}}\right)(a_i - a_{i_j})$*;*

(d) for $i_k \leq i \leq t$*,* $b_i \geq b_{i_k}$*.*

$i_1, \ldots, i_k$ *are called* winning points *among the* t *pairs. Also, the* lower envelope *is the curve* $Z^{(\mathbf{a},\mathbf{b})}(\cdot)$ *defined as follows*

$$Z^{(\mathbf{a},\mathbf{b})}(u) = \begin{cases} b_{i_1} - (u - a_{i_1}) & \text{for } u \leq a_{i_1} \\ b_{i_2} - \left(\frac{b_{i_1} - b_{i_2}}{a_{i_2} - a_{i_1}}\right)(u - a_{i_2}) & \text{for } a_{i_1} \leq u \leq a_{i_2} \\ \vdots & \\ b_{i_k} - \left(\frac{b_{i_{k-1}} - b_{i_k}}{a_{i_k} - a_{i_{k-1}}}\right)(u - a_{i_k}) & \text{for } a_{i_{k-1}} \leq u \leq a_{i_k} \\ b_{i_k} & \text{for } a_{i_k} \geq u, \end{cases} \tag{4}$$

These definitions, together with Equation (2), are used to characterize the optimal portfolio explicitly, as is done in the next proposition.

Proposition 1 *Supplier* i*,* $i = 1, \ldots, N$*, is active if and only if* i *is a winning point of* $\{(w_1, v_1), \ldots, (w_{N+1}, v_{N+1})\}$*.*

All the proofs are presented in the appendix.

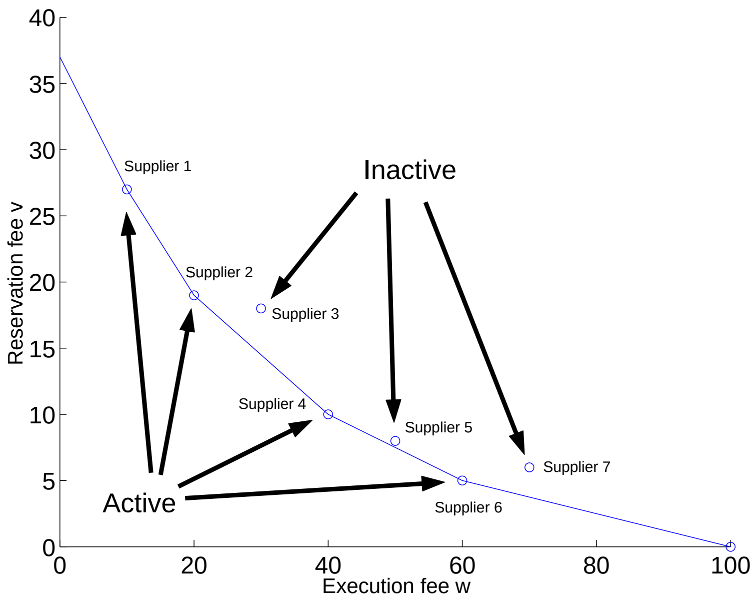

Figure 1: *Illustration of active and inactive bids.*

The winning points consisting of all active suppliers can be determined graphically, see Figure 1. Consider the supplier bids $(w_1, v_1) = (0, 37)$, $(w_2, v_2) = (10, 27)$, $(w_3, v_3) = (30, 18)$, $(w_4, v_4) = (40, 19)$, $(w_5, v_5) = (50, 8)$, $(w_6, v_6) = (60, 5)$ and $(w_7, v_7) = (70, 6)$. Plot the pairs (w_i, v_i) as well as $(w_8, v_8) = (p, 0) = (100, 0)$, as shown in Figure 1, with the w_i in the x-coordinate and the v_i in the y-coordinate. Determine the convex hull of the points, and in particular find the extreme points on the lower envelope as defined in Definition 2; these are the points $i_1 < \ldots < i_k$. For example, in Figure 1, there are $k = 5$ winning points: 1, 2, 4, 6 and 8. Thus, suppliers 1, 2, 4 and 6 are active, and 3, 5 and 7 are inactive.

Hence, the lower envelope is piecewise linear and convex. The segments have increasing slopes or equivalently decreasing negative slopes, that is,

$$1 > \frac{v_{i_1} - v_{i_2}}{w_{i_2} - w_{i_1}} > \ldots > \frac{v_{i_{k-1}} - v_{i_k}}{w_{i_k} - w_{i_{k-1}}} > 0.$$

The buyer's optimal strategy is to include only suppliers on the lower envelope that form segments with negative slopes between 0 and 1. This implies that $v_{i_1} + w_{i_1} < \ldots < v_{i_k} + w_{i_k}$ and $v_{i_1} > \ldots > v_{i_k}$. With these definitions, and recalling that $y_0 = 0$, the optimal portfolio is defined by

$$y_i^* = \begin{cases} \overline{F}^{-1}\left(\dfrac{v_{i_j} - v_{i_{j+1}}}{w_{i_{j+1}} - w_{i_j}}\right) & \text{if } i = i_j,\ j = 1, \ldots, k-1, \\ y_{i-1}^* \text{ for all others (and 0 if } i = 1). \end{cases}$$

Note that this extends the classical newsvendor critical fractile to multiple suppliers. The vector $\mathbf{x}^*$ follows directly from $\mathbf{y}^*$. In particular $x_i^* = 0$ for i different than $i_1, \ldots, i_k$.

Recalling Equation (2), this portfolio structure yields that the buyer reserves capacity with low execution cost w and high reservation cost v to cover the demand with higher realization probability; and uses capacity with high execution cost and low reservation cost to cover the right-tail of the demand.

5 The Behavior of Suppliers

In this section, we analyze the competitive interaction between the suppliers. For this purpose, we first describe the best bidding strategy of the supplier in response to competitors' bids; we then analyze the equilibria of the game.

Under some reasonable properties on the demand (Definition 3 and Theorem 1), and when all the suppliers are *efficient* (Definition 4), we characterize the structure of the (possibly multiple) equilibria. The main result is Theorem 3, which shows that any bid of supplier i either is either infinitely close to the one of $i-1$, in which case the bid falls in the segment connecting (c_{i-1}, f_{i-1}) to (c_i, f_i); or it is infinitely close to the bid of $i+1$, in which case it falls in the segment connecting (c_i, f_i) to (c_{i+1}, f_{i+1}). Thus, in equilibrium suppliers tend to mimic each other, and bid infinitely close to some competitor (due to Theorem 2). This can be interpreted as a multi-dimensional extension of Bertrand competition. However, unlike the one-dimensional case, when competing in two dimensions, a supplier is better off not pushing a competitor out of the market completely but rather placing a bid that is infinitely close to the competitor's bid[6]. Moreover, the equilibrium conditions imply that the suppliers will form clusters of two or three in the neighborhoods of their costs. An example of the clustering is provided in Figure 2. Three clusters are formed, depicted in the figure by the three solid dots representing the bid values. In each cluster, two suppliers place the same bid (suppliers 1 and 2; 3 and 4; and 5 and 6).

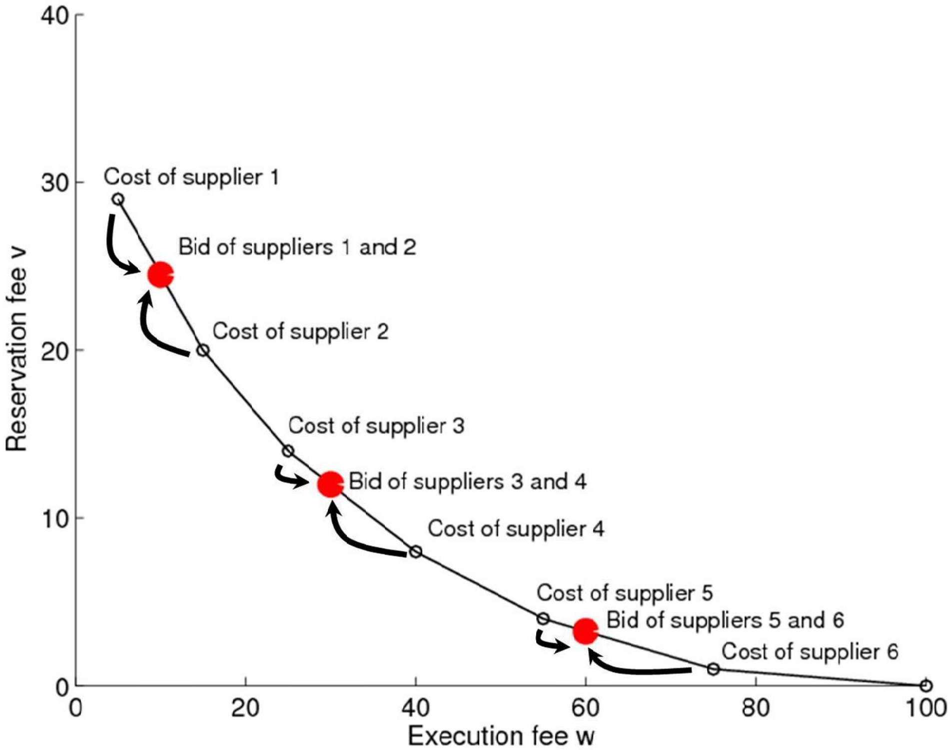

Figure 2: *Plot of costs and equilibrium bids for six different suppliers, for a* $[0, 1]$*-uniform demand.*

Theorem 3 also connects the lower convex hull of the bid pairs (w_i, v_i) to the lower convex hull of the cost pairs (c_i, f_i). Since the former is slightly above the latter, each one of the suppliers makes positive expected profits.

In addition, while efficient suppliers always participate in the equilibrium (Proposition 3), an inefficient supplier might be active or inactive in equilibrium, implying positive or zero profits (Theorem 6) depending on whether its cost pair is above or below the lower envelope of the equilibrium bids.

[6]To deal with bids that are infinitely close, but not equal, we use the concept of limit-equilibrium, and find it convenient to change variables from the bid-price pair (w, v) to the cumulative-capacity pair (y_-, y_+).

5.1 Supplier profit

Our first step is to understand how each supplier will set its reservation and execution price. Given that the buyer uses a portfolio approach as described in the previous section, it will maximize its expected profit, taking into account the behavior of other suppliers.

Consider the decision of a given supplier i who is confronted to the bids of other suppliers. Let $(\mathbf{w}^{other}, \mathbf{v}^{other})$ be the vector representing all other bids with the additional point $(p, 0)$. Given $(\mathbf{w}^{other}, \mathbf{v}^{other})$, we can identify the buyer's optimal procurement strategy. For simplicity and without loss of generality, we consider that only k of these bids are active when the (new) supplier is not present, and that they are indexed so that $w_1 \leq \ldots \leq w_k$ (the last one of the active suppliers is the dummy supplier with parameters $(p, 0)$). The buyer's best procurement strategy, excluding the bid of the new supplier for the moment, is to set

$$\begin{cases} y_j^{other} = \overline{F}^{-1}\left(\dfrac{v_j - v_{j+1}}{w_{j+1} - w_j}\right) & j = 1, \ldots, k-1, \\ y_k^{other} = \overline{F}^{-1}(0). \end{cases} \tag{5}$$

If the supplier now places a new bid (w, v), the buyer's optimal solution may change to take this bid into account. Of course, suppliers that were not active before are not going to be active with the new bid. However, it is entirely possible that some suppliers may become inactive when the new supplier enters with the bid (w, v). Finally, the new supplier may capture zero capacity if its bid makes it inactive. Clearly, in this case, if the supplier is inactive, we can withdraw it from the pool of bids and consequently the capacities allocated to the other suppliers remain unchanged. This happens when (w, v) is above the lower envelope which is described by the function $Z^{(\mathbf{w}^{other}, \mathbf{v}^{other})}(\cdot)$ in Definition 2. Thus, when $v \geq Z^{(\mathbf{w}^{other}, \mathbf{v}^{other})}(w)$, the new supplier is inactive and its profit is $\Pi = 0$. We define this bidding region which makes the supplier inactive as

$$A_{OUT} = \left\{(w, v) \in \mathbb{R}_+^2 | v \geq Z^{(\mathbf{w}^{other}, \mathbf{v}^{other})}(w)\right\}.$$

If the new supplier's bid is not in that region, then this supplier becomes active. Adding bid (w, v) to the rest of the bids may change the convex hull of the points in two different ways:

- The new supplier, i, becomes the first active supplier, i.e., there exist $h \in \{1, \ldots, k\}$ such that suppliers $i, h, \ldots, k-1$ are active and suppliers $1, \ldots, h-1$ are inactive. We define this region as A_{0h}.

$$A_{0h} = \left\{ (w, v) \in \mathbb{R}_+^2 \;\middle|\; \begin{array}{l} v - v_1 \leq -(w - w_1) \\ v - v_h \leq -\left(\dfrac{v_{h-1} - v_h}{w_h - w_{h-1}}\right)(w - w_h) \quad \text{(only if } h > 1) \\ v - v_h \geq -\left(\dfrac{v_h - v_{h+1}}{w_{h+1} - w_h}\right)(w - w_h) \end{array} \right\} \tag{6}$$

- The new supplier, i, is not the first active supplier, i.e., there exist $l \in \{1, \dots, k-1\}$ and $h \in \{1, \dots, k\}$, $h > l$, such that suppliers $1, \dots, l, i, h, \dots, k-1$ are active and $l+1, \dots, h-1$ inactive. We define this region as A_{lh}.

$$A_{lh} = \left\{ (w,v) \in \mathbb{R}_+^2 \;\middle|\; \begin{array}{l} v - v_l \geq -\left(\dfrac{v_{l-1} - v_l}{w_l - w_{l-1}}\right)(w - w_l) \\ (\text{or } v - v_1 \geq -(w - w_1) \text{ if } l = 1) \\ v - v_l \leq -\left(\dfrac{v_l - v_{l+1}}{w_{l+1} - w_l}\right)(w - w_l) \\ v - v_h \leq -\left(\dfrac{v_{h-1} - v_h}{w_h - w_{h-1}}\right)(w - w_h) \\ v - v_h \geq -\left(\dfrac{v_h - v_{h+1}}{w_{h+1} - w_h}\right)(w - w_h) \end{array} \right\} \tag{7}$$

These regions are illustrated in Figure 3. Intuitively, a bid in region A_{lh} implies that the new supplier forces suppliers $l+1, \dots, h-1$ out of the market, i.e., these suppliers receive zero capacity allocation.

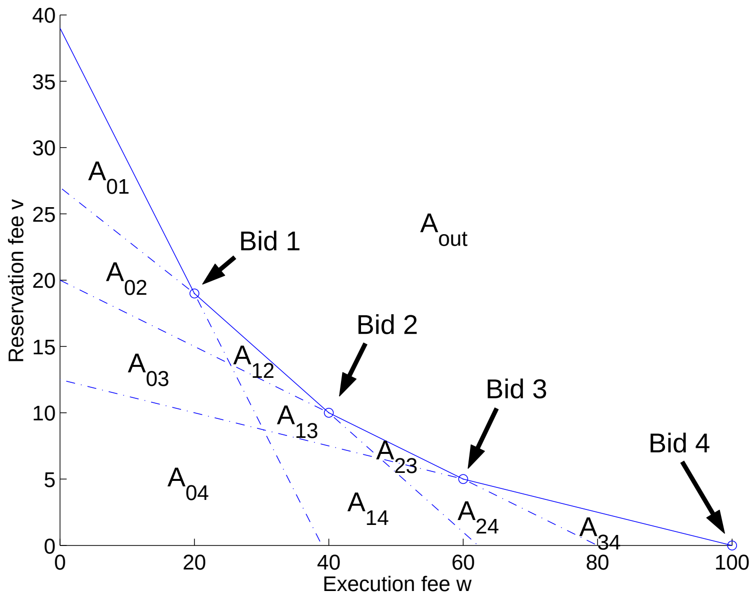

Figure 3: *Division of the bidding strategies in different regions.*

Consider that the supplier places a bid (w, v) in A_{lh}, $l > 0$. Following the buyer's optimal allocation (see Section 4), the capacity x allocated to this supplier is $x = y_+ - y_-$ where y_+ and y_- are given by the following set of equations[7].

$$\overline{F}(y_-) = \frac{v_l - v}{w - w_l} \text{ and } \overline{F}(y_+) = \frac{v - v_h}{w_h - w}. \tag{8}$$

[7]Since $(w, v) \in A_{lh}$, one must keep in mind that (y_-, y_+) is constrained. Specifically, $y_{l-1}^{other} \leq y_- \leq y_l^{other}$ (for $l > 0$), $y_{h-1}^{other} \leq y_+ \leq y_h^{other}$ and therefore $y_- \leq y_+$ (moreover, if $y_- = y_+$, supplier i is in A_{OUT} and is thus inactive).

The case of $l = 0$ is special, since $y_- = 0$ by construction. Hence, in that case,

$$y_- = 0 \text{ and } \overline{F}(y_+) = \frac{v - v_h}{w_h - w}.$$

The expected profit of supplier i is thus

$$\Pi = (v - f)(y_+ - y_-) + (w - c)\mathbb{E}\Big[\min\Big\{\max(D - y_-, 0), y_+ - y_-\Big\}\Big],$$

where the first part is the profit made on reserving capacity, and the second part is the expected execution profit (execution occurs for any demand D higher than y_-, up to the reserved capacity). Since $E\Big[\min\Big\{\max(D - y_-, 0), y_+ - y_-\Big\}\Big] = \int_{y_-}^{y_+}(u - y_-)f(u)du + (y_+ - y_-)\overline{F}(y_+)$, integration in parts yields

$$\Pi = (v - f)(y_+ - y_-) + (w - c)\int_{y_-}^{y_+}\overline{F}(u)du.$$

Using Equation (8), we can express (w, v) as a function of y_- and y_+ when $y_- < y_+$, since $f(\cdot) > 0$. Specifically,

$$\begin{aligned} v &= v_h + \overline{F}(y_+)\frac{-(v_l - v_h) + \overline{F}(y_-)(w_h - w_l)}{\overline{F}(y_-) - \overline{F}(y_+)} = v_l - \overline{F}(y_-)\frac{(v_l - v_h) - \overline{F}(y_+)(w_h - w_l)}{\overline{F}(y_-) - \overline{F}(y_+)} \\ w &= w_h - \frac{-(v_l - v_h) + \overline{F}(y_-)(w_h - w_l)}{\overline{F}(y_-) - \overline{F}(y_+)} = w_l + \frac{(v_l - v_h) - \overline{F}(y_+)(w_h - w_l)}{\overline{F}(y_-) - \overline{F}(y_+)}. \end{aligned} \tag{9}$$

This implies that we can express Π using y_- and y_+ instead of v and w. Within A_{lh}, $l > 0$,

$$\begin{aligned} \Pi(w, v) = J_{lh}(y_-, y_+) &= \left\{ \begin{array}{l} (v_h - f)(y_+ - y_-) + (w_h - c)\displaystyle\int_{y_-}^{y_+}\overline{F}(u)du \\ -\left[\dfrac{-(v_l - v_h) + \overline{F}(y_-)(w_h - w_l)}{\overline{F}(y_-) - \overline{F}(y_+)}\right]\displaystyle\int_{y_-}^{y_+}\Big(\overline{F}(u) - \overline{F}(y_+)\Big)du \end{array}\right. \\ &= \left\{ \begin{array}{l} (v_l - f)(y_+ - y_-) + (w_l - c)\displaystyle\int_{y_-}^{y_+}\overline{F}(u)du \\ -\left[\dfrac{(v_l - v_h) - \overline{F}(y_+)(w_h - w_l)}{\overline{F}(y_-) - \overline{F}(y_+)}\right]\displaystyle\int_{y_-}^{y_+}\Big(\overline{F}(y_-) - \overline{F}(u)\Big)du \end{array}\right. \end{aligned}$$

When $l = 0$, the transformation described in Equation (9) is not well defined, since different values of (w, v) yield the same $(y_- = 0, y_+)$. We observe that for a given y_+, $y_- = 0$, the profit with a bid $w = w_h - t$ and $v = v_h + \overline{F}(y_+)t$, $t \geq 0$, is,

$$\begin{aligned} \Pi &= (v_h + \overline{F}(y_+)t - f)y_+ + (w_h - t - c)\int_0^{y_+}\overline{F}(u)du \\ &= (v_h - f)y_+ + (w_h - c)\int_0^{y_+}\overline{F}(u)du - t\int_0^{y_+}\Big(\overline{F}(u) - \overline{F}(y_+)\Big)du. \end{aligned} \tag{10}$$

To maximize Π within A_{0h}, the supplier will select t as small as possible. This yields

$$\begin{aligned} v + w &= v_1 + w_1 \\ \frac{v - v_h}{w_h - w} &= \overline{F}(y_+). \end{aligned} \tag{11}$$

This justifies the extension of Equation (8) for $l = 0$. Consequently,

$$\Pi(w,v) = J_{0h}(y_+) \quad = \begin{cases} (v_h - f)y_+ + (w_h - c)\int_0^{y_+} \overline{F}(u)du \\ -\left[\dfrac{(v_h + w_h) - (v_1 + w_1)}{1 - \overline{F}(y_+)}\right] \int_0^{y_+} \Big(\overline{F}(u) - \overline{F}(y_+)\Big)du \end{cases}$$

Finally, the problem faced by the supplier is:

$$\sup_{(w,v)} \Pi(w,v) = \max\left(0, \max_{l=0,\ldots,k-1;h=l+1,\ldots,k} \left\{ \sup_{y_{l-1}^{other} \le y_- \le y_l^{other}; y_{h-1}^{other} \le y_+ \le y_h^{other}} J_{lh}(y_-, y_+) \right\}\right)$$

The optimization problem is defined as a supremum of profit, in terms of either (w,v) or (y_-, y_+). As we shall see later in the discussion after Theorem 2, optimization with respect to (w,v) does not always yield an optimal solution. Indeed, the supremum may be obtained by bidding arbitrarily close to another supplier, through an infinitely close bid. However, when using (y_-, y_+) as decision variables, an optimal solution is always obtained.

Since $\overline{F}(\cdot)$ is differentiable over $(\underline{d}, \overline{d})$, the expected profit is differentiable in (y_-, y_+):

$$\begin{aligned} \frac{\partial J_{lh}}{\partial y_-} &= (f - v_l) + (c - w_l)\overline{F}(y_-) + f(y_-)\left[\frac{(v_l - v_h) - \overline{F}(y_+)(w_h - w_l)}{(\overline{F}(y_-) - \overline{F}(y_+))^2}\right] \int_{y_-}^{y_+} \Big(\overline{F}(u) - \overline{F}(y_+)\Big)du \\ \frac{\partial J_{lh}}{\partial y_+} &= (v_h - f) + (w_h - c)\overline{F}(y_+) - f(y_+)\left[\frac{-(v_l - v_h) + \overline{F}(y_-)(w_h - w_l)}{(\overline{F}(y_-) - \overline{F}(y_+))^2}\right] \int_{y_-}^{y_+} \Big(\overline{F}(y_-) - \overline{F}(u)\Big)du \end{aligned} \tag{12}$$

Notice that $(v_l - v_h) - \overline{F}(y_+)(w_h - w_l) \ge 0$ and $-(v_l - v_h) + \overline{F}(y_-)(w_h - w_l) \ge 0$ hold; we shall use this observation later.

5.2 Border distributions

Maximizing J_{lh} on (y_-, y_+), such that $(w,v) \in A_{lh}$, may yield, in general, interior solutions or extreme solutions. It turns out that they are always extreme solutions for a class of customer demand distributions. First, we define the class of *border demand distributions.*

Definition 3 (border distribution) *A demand distribution is a border distribution if, for any supplier, for any region A_{lh}, defined by Equations (6) or (7), there is an optimal bid (w,v) that belongs to the border of the region.*

This definition implies that, when demand follows a border distribution, any supplier bidding in region A_{lh} will find an optimal bid on the boundary of region A_{lh}, for any cost parameters. For instance, for a supplier bidding in region A_{12} of Figure 3, there is an optimal bid on the boundary of A_{12} with either A_{02} or A_{13} or A_{OUT}.

The following theorem characterizes a class of distributions that are border distributions.

Theorem 1 *A log-concave demand distribution is a border distribution.*

The class of log-concave distributions, i.e., with $log(f)$ concave, includes distributions such as uniform, exponential, normal, etc. The proof of this result is presented in the appendix and can also be found in Martínez-de-Albéniz [12]. The idea behind the proof is that there are no interior local maxima of J_{lh} when the demand is log-concave. This can be easily verified for uniform demands (e.g., Figure 4 later), in which case J_{lh} is always a quadratic function, and it is never concave. As a result we can find a maximum of J_{lh} in the border of the region A_{lh}. Hence, for log-concave demand distributions, the suppliers will place their bids in the border of some region. This property allows us to determine their optimal bids.

5.3 Optimal bids

As we will soon see, it is of particular interest to examine the bidding strategy in A_{lh} where there is no active supplier between l and h. Notice that these are all the regions that share an edge with A_{OUT}. When we know that the optimal bid is in this region, we can characterize this optimal bid.

Consider supplier i bidding in such a region, A_{lh}, and define a_m to be the cumulative quantity captured by suppliers $1, \dots, l$ when i is absent, i.e.,

$$\overline{F}(a_m) = \frac{v_l - v_h}{w_h - w_l}. \tag{13}$$

The constraint of being in A_{lh} can be written as $a_l := y_{l-1}^{other} \leq y_- \leq a_m \leq y_+ \leq a_h := y_h^{other}$ where y_{l-1}^{other} and y_h^{other} are defined in Equation (5).

If the bid of the supplier does not make l or h inactive, we can derive useful properties. In this case, the optimal bid cannot be such that $y_- = a_l$ (because it makes l inactive) or $y_+ = a_h$ (h inactive). Therefore, since it is optimal to bid on the border of the region, it must be that $y_- = a_m$ or $y_+ = a_m$ is optimal. These imply that the supplier bids infinitely close to l or h.

In the first case, i.e., when $y_- = a_m$ is optimal, recall that $-(v_l - v_h) + \overline{F}(y_-)(w_h - w_l) = 0$ so from Equation (12), $\dfrac{\partial J_{lh}}{\partial y_+} = (v_h - f) + (w_h - c)\overline{F}(y_+)$ and therefore we must have that $c \leq w_h$ and $\overline{F}(y_+) = \dfrac{f - v_h}{w_h - c}$. Similarly, when $y_+ = a_m$ is optimal, $\dfrac{\partial J_{lh}}{\partial y_-} = (f - v_l) + (c - w_l)\overline{F}(y_-)$, hence $w_l \leq c$ and $\overline{F}(y_-) = \dfrac{v_l - f}{c - w_l}$. We summarize these results in the next theorem.

Theorem 2 *Given a border distribution, assume that, for a supplier with costs (c, f), the optimal bid belongs to some unique region A_{lh}, $l > 0$, where there is no active supplier between l and h. Define a_m as in Equation (13) and hence having $(w, v) \in A_{lh}$ is equivalent, for some a_l, a_h, to $a_l \leq y_- \leq a_m \leq y_+ \leq a_h$. Define z_l and z_h as follows,*

$$\overline{F}(z_l) = \frac{v_l - f}{c - w_l}, \tag{14}$$

$$\overline{F}(z_h) = \frac{f - v_h}{w_h - c}. \tag{15}$$

Then, one and only one case from the following is true.

(i) if $a_l \le z_l \le a_m$ *and* $z_h > a_h$*, then* $y_+^* = a_m$ *and* $y_-^* = z_l$ *(bid infinitely close to supplier* l*);*

(ii) if $a_l > z_l$ *and* $a_m \le z_h \le a_h$*, then* $y_-^* = a_m$ *and* $y_+^* = z_h$ *(bid infinitely close to supplier* h*);*

(iii) if $a_l \le z_l \le a_m \le z_h \le a_h$*, then* $y_+^* = a_m$ *and* $y_-^* = z_l$ *(bid infinitely close to supplier* l*) only if*

$$\frac{\int_{z_l}^{a_m} [\overline{F}(z_l) - \overline{F}(u)]du}{\overline{F}(z_l) - \overline{F}(a_m)} \ge \frac{\int_{a_m}^{z_h} [\overline{F}(u) - \overline{F}(z_h)]du}{\overline{F}(a_m) - \overline{F}(z_h)}; \tag{16}$$

and $y_-^* = a_m$ *and* $y_+^* = z_h$ *(bid infinitely close to supplier* h*) only if*

$$\frac{\int_{z_l}^{a_m} [\overline{F}(z_l) - \overline{F}(u)]du}{\overline{F}(z_l) - \overline{F}(a_m)} \le \frac{\int_{a_m}^{z_h} [\overline{F}(u) - \overline{F}(z_h)]du}{\overline{F}(a_m) - \overline{F}(z_h)}. \tag{17}$$

Intuitively, the theorem shows that there are two candidate optimal bids when we know that the optimal bid is in a given region A_{lh}, and there are no active suppliers in between l and h. First, the supplier may choose to bid infinitely close to l, in such a way so that $y_-^* = z_l$. Note that the optimal bid is not defined through (w, v), because of the allocation problem when two bids are equal[8]. Instead, the optimal bid is defined through (y_-, y_+). In practice, this means that a sequence of bids such that $w^\delta = w_l + \delta$ and $v^\delta = v_l - \overline{F}(z_l)\delta$ approaches the highest expected profit, when $\delta \to 0$ and positive. Alternatively, the supplier may choose to bid infinitely close to h, such that $y_+^* = z_h$. Similarly, a sequence of bids such that $w^\delta = w_h - \delta$ and $v^\delta = v_h + \overline{F}(z_l)\delta$ approaches the highest expected profit, when $\delta \to 0$ and positive. Note that in both of these bids, it is critical to set the value of y_-^* to z_l, or y_+^* to z_h, which determine the direction between the two similar bids. It is worth pointing out that these values z_l and z_h depend on the supplier's cost parameters (c, f).

Interestingly, no other situations are possible at optimality, since the demand distribution satisfies the border property and there is no optimal solution outside A_{lh} (thus discarding a strategy where $y_- = a_l$ or $y_+ = a_h$, where the solution would belong to some other $A_{l'h'}$).

Figure 4 illustrates the optimal bid decision discussed in the theorem. We depict the expected profit of a supplier with cost $c = 55$ and $f = 8$, competing against bids $(0, 55), (20, 35), (80, 2)$ and $(100, 0)$, with a buyer's demand that is uniformly distributed in [0,1]. Specifically, we calculate the expected profit obtained by bidding in A_{23}, i.e., between the second and third bids, $(20, 35)$ and $(80, 2)$, which implies $a_l = 0$, $a_m = 0.45$ and $a_h = 0.9$. The figure shows the iso-profit curves[9] as a function of (y_-, y_+), on the left-hand side, and of (w, v), on the right-hand side.

[8] Recall that bidding exactly the same as supplier l is not allowed, since it results in the buyer being indifferent between the two bids.

[9] Each curve corresponds to a certain level of profit. The bars at the bottom of the graphs show the profit level.

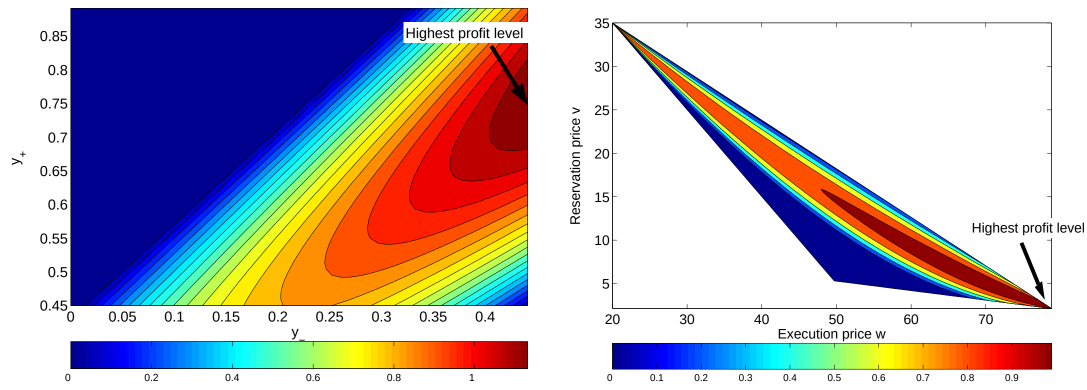

Figure 4: *Expected supplier profit (only non-negative values are shown, for better readability)*

As one can observe by searching for the highest profit level in the left-hand side figure, it is optimal to set $y_-^* = 0.45$ and $y_+^* = 0.76$. Note that this corresponds to the decision $y_- = a_m$ and $y_+ = z_h$. This decision achieves a higher profit than choosing $y_- = z_l$ and $y_+ = a_m$, as stated in Theorem 2.[10] It corresponds to placing a bid very close to $(w, v) = (80, 2)$. This is confirmed by observing the right-hand side figure, where indeed the highest profit level is found close to $(w, v) = (80, 2)$. Note however that the profit function is not defined at $(w, v) = (80, 2)$, which implies that the optimal bid should be $w^\delta = 80 - \delta, v^\delta = 2 + \overline{F}(0.76)\delta = 2 + 0.34\delta$, for small $\delta > 0$.

5.4 Equilibrium definition

We analyze now the equilibria of this game, in pure strategies. Consider first the following example. There are $N = 2$ suppliers with costs $(c_1, f_1) = (0, 60)$ and $(c_2, f_2) = (75, 5)$. The demand is uniformly distributed in $[0, 1]$. The selling price is $p = 100$, and hence the dummy supplier posts a bid $(p, 0)$. Consider the situation when both suppliers submit two bids that are very close to $(60, 12)$. Are these bids in equilibrium? In other words, does each supplier maximize its profit given the competitor's bid? As we demonstrate below, this is not the case.

Indeed, Figure 5 shows the profit functions of each supplier as a function of their bid (w, v). The upper figure represents the profit of supplier 1, with cost $(c_1, f_1) = (0, 60)$, with competing bids of $(60, 12)$ (supplier 2) and $(100, 0)$ (dummy supplier). The lower figure represents the profit of supplier 2, with cost $(c_2, f_2) = (75, 5)$, with competing bids of $(60, 12)$ (supplier 1) and $(100, 0)$ (dummy supplier). As one can see, both suppliers' profits are maximized by bidding infinitely close to $(w, v) = (60, 12)$. Since the profit maximizer is not well defined in (w, v), as seen in the previous

[10] Indeed, since $\overline{F}(u) = 1 - u$, Equation (15) implies that $z_h = 1 - \dfrac{8-2}{80-55} = 0.76$. Similarly, Equation (14) implies that $z_l = 1 - \dfrac{20-8}{55-20} = 0.2286$. Since $\dfrac{\int_{z_l}^{a_m} [\overline{F}(z_l) - \overline{F}(u)]du}{\overline{F}(z_l) - \overline{F}(a_m)} = \dfrac{(0.45-0.2286)^2/2}{0.45-0.2286} = 0.1107 < \dfrac{\int_{a_m}^{z_h} [\overline{F}(u) - \overline{F}(z_h)]du}{\overline{F}(a_m) - \overline{F}(z_h)} = \dfrac{(0.76-0.45)^2/2}{0.76-0.45} = 0.155$, Theorem 2 implies that the optimal bid is to set $(y_-, y_+) = (a_m, z_h)$.

section, supplier 1's profit function is maximized by setting $y^*_{1-} = 0$ and $y^*_{1+} = 0.2$, and supplier 2's profit function is maximized by $y^*_{2-} = 0.5333$ and $y^*_{2+} = 0.8$. Hence, this cannot be an equilibrium, since the suppliers do not agree on a capacity allocation. However, if the suppliers' bids would yield $y^*_{1+} = y^*_{2-}$, both suppliers would have no incentive to modify their bids.

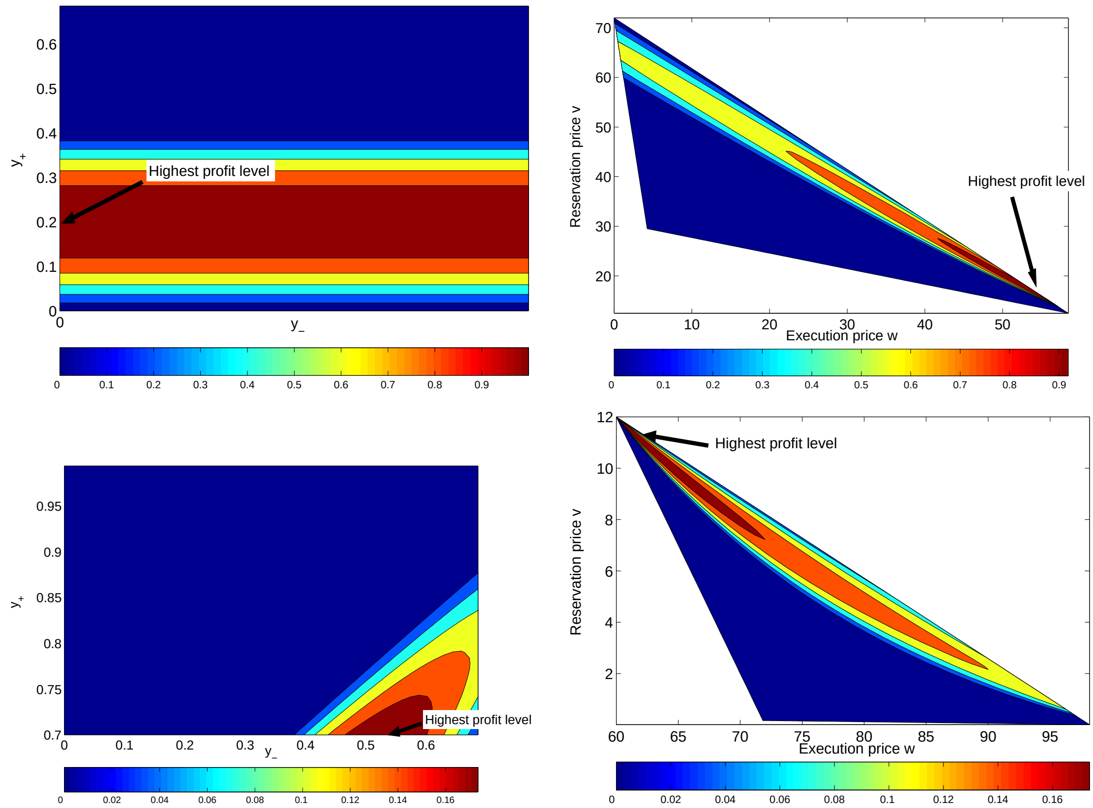

Figure 5: *Expected supplier profit for supplier 1 (top) and 2 (bottom)*

The example shows that the concept of equilibrium needs to be refined when two bids are identical[11]. To overcome this problem, we define the equilibrium for infinitely close bids. Using the optimality equations and (y_-, y_+), we can determine when a bid situation with ties is stable, as discussed for Figure 5. For this purpose, instead of Nash equilibrium, we consider the concept of ϵ-equilibrium as defined in Radner [17] or Fudenberg and Levine [9]. We say a set of pure strategies $(w_i, v_i)_{i=1,\ldots,N}$ is an ϵ-equilibrium of the bidding game when for each supplier

$$\Pi_i(w_i, v_i, \mathbf{w}_{-i}, \mathbf{v}_{-i}) \geq \sup_{(w,v)} \Pi_i(w, v, \mathbf{w}_{-i}, \mathbf{v}_{-i}) - \epsilon.$$

For $\epsilon \rightarrow 0$, we characterize the limit of ϵ-equilibria. In other words, we describe what sort of equilibria arises when suppliers choose bids that are very close to the optimum.

[11] With identical bids, the buyer's problem has multiple optimal solutions and hence it is not clear how demand is allocated to the two suppliers. It can be shown that in general, when the splitting is pre-determined exogenously, e.g., split the capacity and the allocation $50 - 50\%$, no equilibrium exists.

In what follows, we say that a set of pure strategies $(w_i, v_i)_{i=1,\ldots,N}$ is a *limit-equilibrium* of the bidding game when there exists, for each supplier i, for each ϵ, $(w_i^\epsilon, v_i^\epsilon)$, such that: (1) $(w_i^\epsilon, v_i^\epsilon) \to (w_i, v_i)$ when $\epsilon \to 0$, and (2) for each ϵ,

$$\Pi_i(w_i^\epsilon, v_i^\epsilon, \mathbf{w}_{-i}^\epsilon, \mathbf{v}_{-i}^\epsilon) \geq \sup_{(w,v)} \Pi_i(w, v, \mathbf{w}_{-i}^\epsilon, \mathbf{v}_{-i}^\epsilon) - \epsilon.$$

Essentially, this definition of equilibrium circumvents the continuity problem of the supplier profit function, and hence makes unnecessary the use of a rationing rule in case of a tie.

In this section, we provide necessary conditions for equilibrium. We do not analyze the existence of pure strategy equilibria, although these can be shown to exist under fairly general assumptions. Usual proof methods may not work because a given supplier's pay-off function is discontinuous, when its bid is equal to some other supplier's bid[12]. Fortunately, we are able to show existence by explicitly constructing an equilibrium, see Martínez-de-Albéniz [12] for the algorithmic details. An example of the algorithm is provided in Section 5.5.

Using the results of the previous section, we can characterize a crucial necessary condition for equilibrium, arising from Proposition 2.

Proposition 2 *Consider a border distribution. In every limit-equilibrium, if* $(w_i, v_i) = (w_j, v_j)$ *and both suppliers are active, then* (w_i, v_i) *belongs in the segment* $[(c_i, f_i); (c_j, f_j)]$. *That is, there is* $\theta \in [0, 1]$ *such that*

$$\begin{pmatrix} w_i \\ v_i \end{pmatrix} = \theta \begin{pmatrix} c_i \\ f_i \end{pmatrix} + (1 - \theta) \begin{pmatrix} c_j \\ f_j \end{pmatrix}.$$

This is a direct consequence of Theorem 2. Intuitively, if two suppliers, i and j, submit infinitely close bids and are at a limit-equilibrium, then it must be true that the quantities that they desire, their optimal y_{i+}^* and y_{j-}^*, must coincide, as seen in the discussion of Figure 5. This results on having the equilibrium bid in the cost segment $[(c_i, f_i); (c_j, f_j)]$.

5.5 Equilibria with efficient suppliers only

We start by defining the concept of efficiency which leads to a natural and desirable property of equilibria.

Definition 4 *We say that supplier i is efficient when* (c_i, f_i) *is a winning point in the set* $\{(c_1, f_1), \ldots, (c_N, f_N), (p, 0)\}$.

Proposition 3 *Assume that supplier i is efficient. Then, in every limit-equilibrium, this supplier is active, and* $\Pi_i \geq 0$.

[12]To our knowledge, the best result that we can hope for with a more general approach is existence of mixed-strategy equilibria, following Dasgupta and Maskin [6].

This implies that efficiency guarantees any supplier to be active in any equilibrium outcome. That is, the supplier will receive a positive share of capacity and make positive profit.

Proposition 4 *Given a border distribution, assume that all suppliers are efficient. Then, in every limit-equilibrium, for every pair* (i, j)*, if* $c_i < c_j$ *then* $w_i \leq w_j$*.*

Proposition 4 implies that if all suppliers are efficient, supplier i, $i = 1, \ldots, N$, bids in region $A_{i-1\ i+1}$ in every equilibrium. More importantly, this result confirms the intuition on the suppliers' bidding behavior. No supplier will bid an execution fee, w, lower than a competitor's execution fee if the competitor's execution cost is smaller. Put differently, the smaller a supplier's execution cost, c, the lower this supplier's execution bid, w.

Combining Theorem 2 and Propositions 2, 3 and 4 we can characterize strong necessary conditions on the equilibria.

Theorem 3 *For a border distribution, assume that all the suppliers are efficient. Define* $c_{N+1} = w_{N+1} = p, f_{N+1} = v_{N+1} = 0$*. Then, in every limit-equilibrium, supplier* i*,* $i = 2, \ldots, N$*, places its bid:*

- *either* $(w_i, v_i) = (w_{i-1}, v_{i-1})$*, and then this bid falls in the segment* $[(c_{i-1}, f_{i-1}); (c_i, f_i)]$ *(in the sense of Proposition 2);*
- *or* $(w_i, v_i) = (w_{i+1}, v_{i+1})$*, and then this bid falls in the segment* $[(c_i, f_i); (c_{i+1}, f_{i+1})]$*.*

When $i = 1$*, only the second case is possible, i.e.,* $(w_1, v_1) = (w_2, v_2)$*, and this common bid falls in the segment* $[(c_1, f_1); (c_2, f_2)]$*.*

The theorem builds on the optimal behavior of each supplier: when all suppliers are efficient (from Theorem 2), it is optimal for supplier i either to place a bid infinitely close to the bid of $i - 1$ or that of $i + 1$. This implies that supplier bids will be clustered in groups of two or three suppliers. This is true since according to the theorem, either two suppliers bid somewhere in the segment connecting their true cost parameters, or one supplier bids its true costs and two other suppliers place a similar bid to this one. Thus, in practice, one will observe less bids than the number of suppliers, roughly half of them. We call this cluster competition, since in equilibrium the market is divided into stable clusters.

The type of competition described in this result has some interesting properties. The most striking feature is that more than one supplier will be offering the same bid. One may then wonder whether any supplier in that position should instead reduce its bid a little bit so that it puts its rival out of the market. The answer is that all suppliers in the same cluster, i.e., offering the same bid, are better off staying in the cluster rather than trying to outbid the other members of the cluster. The intuition is provided by combining Equation (12), that implies that the optimal bid of i must be in region $A_{i-1,i+1}$ (otherwise first-order optimality conditions are not met), with Theorem 2. Indeed, in equilibrium each supplier has three alternatives.

1. Bid its true cost, $(w_i, v_i) = (c_i, f_i)$. Thus, in this case, the supplier makes zero profit, but reducing the bid would yield negative profit.
2. Place a bid such that $w_i < c_i$ and $v_i > f_i$, in which case, the other supplier in the cluster places a bid $w_{i-1} = w_i \geq c_{i-1}$ and $v_{i-1} = v_i \leq f_{i-1}$. Recall from Theorem 2 that it is then optimal to set $\bar{F}(y_{i-}) = \dfrac{v_{i-1} - f_i}{c_i - w_{i-1}}$, and thus supplier i has no interest in reducing y_{i-} and hence the capacity allocated to supplier $i-1$. In this situation, the supplier makes a profit on the reservation portion, and a loss if the total price is considered $(w_i + v_i < c_i + f_i)$. Hence, its profit is positive when a small fraction of the reserved capacity is executed, negative if most of it is executed, and positive in expectation.
3. Place a bid such that $w_i > c_i$ and $v_i < f_i$, in which case, the other supplier in the cluster places a bid $w_{i+1} = w_i \leq c_{i+1}$ and $v_{i+1} = v_i \geq f_{i+1}$. Again, from Theorem 2, supplier i has no interest in increasing y_{i+} and hence reducing the capacity of supplier $i+1$. Here the situation is the opposite of the previous case: the supplier makes a loss on the reservation portion, and a profit if the total price is considered $(w_i + v_i > c_i + f_i)$. Thus, while the expected profit is positive, it may be negative when a small fraction of the reserved capacity is executed.

The theorem also suggests that every supplier is competing directly with one of its rival suppliers, i.e., with the supplier who has the next smaller or the next larger execution cost c. An important insight from this observation is that, in equilibrium, each supplier's bid will be most sensitive to the bid of its closest competitor, and not to the rest of the bids. This implies that in equilibrium, competition is no longer done on a global basis (among all suppliers) but rather locally (between two or three competing suppliers).

In addition, Theorem 3 can be used to construct equilibria. If a limit-equilibrium exists, supplier i bids the same as $i-1$ or $i+1$. To construct an equilibrium, where supplier i bids the same as supplier $i-1$ (resp. $i+1$), in the segment $[(c_{i-1}, f_{i-1}); (c_i, f_i)]$ (resp. $[(c_i, f_i); (c_{i+1}, f_{i+1})]$), one must ensure that the supplier realizes a higher profit than bidding the same as $i+1$ (resp. $i-1$). Consider for example the case where $N = 3$, all suppliers are efficient, and $c_1 < c_2 < c_3 < c_4 = p$. When suppliers 1 and 2 bid their true cost (the dummy supplier 4 also bids its true cost), let $e(3)$ be the supplier whose bid supplier 3 prefers to imitate. That is, $e(3) = 2$ if supplier 3 is better off by placing a bid close to the cost of supplier 2; and $e(3) = 4$ if it is better to place it close to the cost of the dummy supplier 4. Then, if $e(3) = 2$, $(w_1, v_1) = (w_2, v_2) = (w_3, v_3) = (c_2, f_2)$ is an equilibrium; if $e(3) = 4$, $(w_1, v_1) = (w_2, v_2) = (c_2, f_2)$, $(w_3, v_3) = (p, 0)$ is an equilibrium. In fact, this approach allows one to construct one equilibrium, although, in general, many others exist. For example, when $e(3) = 2$, if $(w_1, v_1) = (w_2, v_2)$ is sufficiently close to (c_1, f_1) in the segment $[(c_1, f_1); (c_2, f_2)]$, then it is possible that $(w_3, v_3) = (p, 0)$ is also an equilibrium. This approach can be extended to arbitrary N, where $e(i)$ is determined for all i, and then suppliers are matched so that the proposed bids form an equilibrium. Again, it is then possible to construct one equilibrium, but with higher N many other different patterns may also provide equilibria: in one equilibrium, two suppliers might be in the same cluster, and in some other, in different clusters. A more detailed discussion can be found in Martínez-de-Albéniz [12].

Finally, as pointed out above, in general there are multiple equilibria. The theorem shows that all the possible equilibria belong to the lower envelope of the suppliers' true costs. Such equilibria should satisfy the optimality conditions in Equations (16) and (17). The following example (corresponding to Figure 5) illustrates the multiplicity of equilibria.

Example 1 *Assume that customer demand is uniformly distributed in* $[0,1]$*. Let* $N=2$ *and the true costs be* $(c_1, f_1) = (0, 60), (c_2, f_2) = (75, 5), p = 100$*. Both suppliers are efficient. For any* $w \in [50, 75]$*, the following bids form different equilibria:*

$$(w_1, v_1) = (w_2, v_2) = \left(w, 60 - \frac{55}{75}w\right), \quad y_1 = \frac{20}{75}, \quad y_2 = \frac{4}{15} + \frac{40}{3(100-w)}.$$

We should point out that in any of these equilibria, the buyer's expected profit is equal to $\dfrac{8(150-w)^2}{225(100-w)} \geq 64/9$*. On the other hand, an Expected Vickrey-Clark-Groves (EVCG) auction, which is supply-chain-efficient (see Schummer and Vohra [18] for details), would allocate supplier 1 a profit of* $32/3 - 8 = 8/3$*, supplier 2 a profit of* $32/3 - 8 = 8/3$ *and the buyer an expected profit of* $32/3 - 8/3 - 8/3 = 16/3 < 64/9$*. Thus, in this example, the first-price competition environment is preferred by the buyer to the supply-chain-efficient EVCG auction.*

The profit functions of suppliers, buyer and the entire supply chain as a function of w *(execution price of both suppliers in equilibrium) are plotted in Figure 6. As we can see, the profit of supplier 1 is increasing in* w*, whereas the profit of supplier 2 is decreasing in* w*. Thus, among all equilibria, there is no single one that both suppliers prefer: supplier 1 would always prefer an equilibrium with high* w*, while supplier 2 would prefer small* w*. Furthermore, the buyer's profit is increasing in* w *(but could be decreasing in other examples), and the total supply chain profit is increasing in* w*.*

Finally, to conclude this section, we provide a bound on the inefficiencies created by suppliers' competition. We define the total surplus as follows:

$$U = \text{ (PROFIT OF BUYER) } + \sum_{i=1}^{N} \text{ (PROFIT OF SUPPLIER } i\text{)}.$$

The payments between buyer and suppliers will cancel out, and this quantity will only capture the true revenue from customers minus the costs of production. Thus, we can express the total supply chain surplus as

$$\begin{aligned} U = \; & p \int_0^{y_N} \overline{F}(u)du - \sum_{i=1}^{N} f_i(y_i - y_{i-1}) - \sum_{i=1}^{N} c_i \int_{y_{i-1}}^{y_i} \overline{F}(u)du \\ = \; & \sum_{i=1}^{N} \Delta c_i \int_0^{y_i} [\overline{F}(u) - \overline{F}(y_i^*)]du. \end{aligned}$$

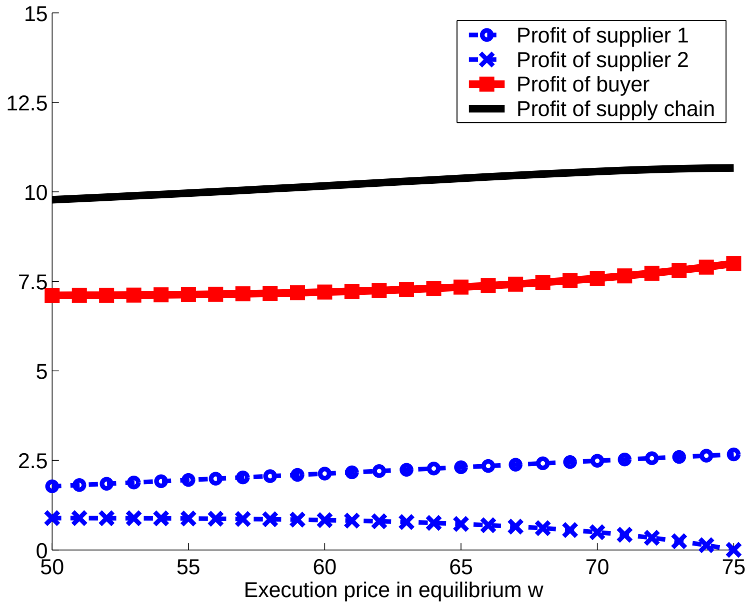

Figure 6: *Expected profit of suppliers, buyer and the entire supply chain in Example 1*

where $\overline{F}(y_i^*) = \dfrac{f_i - f_{i+1}}{c_{i+1} - c_i}$ and $\Delta c_i = c_{i+1} - c_i$. These quantities are well-defined when all the suppliers are efficient. The social surplus is maximized when $y_i = y_i^*$, $i = 1, \ldots, N$. In this case, the optimal surplus is

$$U^* = \sum_{i=1}^{N} \Delta c_i \int_0^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)] du.$$

When the suppliers compete, the allocation of capacities, y_i, $i = 1, \ldots, N$, is not necessarily efficient, in the sense that it is possible that $y_i \neq y_i^*$ for some i. The loss in surplus, due to the suppliers' competition, is equal to

$$\Delta U = \sum_{i=1}^{N} \Delta c_i \int_{y_i}^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)] du.$$

Theorem 4 *Given a border demand distribution and efficient suppliers, in every limit-equilibrium, the allocation of capacities obtains at least* 50% *of the optimal total surplus, i.e.,*

$$\frac{\Delta U}{U^*} \leq \frac{1}{2}.$$

This is the best bound available for general border distributions. However, this bound can be improved when we include additional conditions on the demand distribution, as shown next.

Theorem 5 *Given a log-concave demand distribution and efficient suppliers, in every limit-equilibrium, the allocation of capacities obtains at least* 75% *of the optimal total surplus, i.e.,*

$$\frac{\Delta U}{U^*} \leq \frac{1}{4}.$$

This bound is tight for two suppliers and uniform demand distribution.

The theorem thus implies that for uniform, exponential or normal demand distributions (which belong to the log-concave class) the loss of efficiency due to competition is no more than 25%. This bound is similar to others found in supply chain games: 25% for the single supplier, single buyer game (bilateral monopoly) with deterministic price-dependent demand; or $\frac{e-2}{e-1} \approx 41.8\%$ for the single supplier, single buyer game with stochastic (IGFR) demand, see Perakis and Roels [16]. However, the loss of efficiency here is due exclusively to supplier-supplier interactions, since, if all suppliers were integrated (or colluded), they would be able to extract all the supply chain profit[13].

5.6 Equilibria with inefficient suppliers

The previous results, characterizing equilibrium, are obtained under the assumption that all suppliers are efficient. We now investigate the case in which not all suppliers are efficient.

Interestingly, as we demonstrate below, it might happen that a non-efficient supplier is active at equilibrium. This occurs because bids are only partially linked to the true costs, and a non-efficient supplier may capture market share by positioning itself in a segment of the market with no, or low, competition.

Example 2 *Assume that customer demand is uniformly distributed in* $[0, 1]$*. Let* $N = 3$ *and the true costs be*

$$(c_1, f_1) = (0, 40), \quad (c_2, f_2) = (40, 20), \quad (c_3, f_3) = (70, 11), \quad p = 100.$$

Supplier 3 *is not efficient. If this was a centralized system, in which the true costs are considered, the allocation would be* $y_1^* = 0.5$*,* $y_2^* = 0.666$ *and* $y_3^* = 0.666$*, and hence the buyer would purchase capacities* $x_1^* = 0.5$*,* $x_2^* = 0.166$ *and* $x_3^* = 0$.

The following bids form an equilibrium:

$$(w_1, v_1) = (w_2, v_2) = (20, 30), \quad (w_3, v_3) = (100, 0), \quad y_1 = 0.5, \quad y_2 = 0.625, \quad y_3 = 0.633.$$

Thus, a non-efficient supplier captures capacity and makes positive profit.

The example suggests that the presence of inefficient suppliers can lead to counter-intuitive situations. The next theorem depicts the behavior of the suppliers at equilibrium.

Theorem 6 *For a border distribution, let* $\{(w_1, v_1), \ldots, (w_N, v_N), (p, 0)\}$ *be the bids of the suppliers in a equilibrium. Assume that supplier* i *is active. Then we must have that:*

(i) either there is $j = 1, \ldots, N + 1$ *such that supplier* j *is active,* $(w_i, v_i) = (w_j, v_j)$ *and moreover* (w_i, v_i) *belongs in the segment* $[(c_i, f_i); (c_j, f_j)]$ *(in the sense of Proposition 2);*

[13] For example, they could offer the contracts $w_n = p - \epsilon$ and $v_n = \overline{F}(y_n^*)\epsilon$, and for $i < n$, $w_i = w_{i+1} - \epsilon$, $v_i = v_{i+1} + \overline{F}(y_i^*)\epsilon$, for ϵ arbitrarily close to zero. As a result, if there is a single supplier, the inefficiency is equal to zero.

(ii) or there are $j,k = 1,\ldots,N+1$ such that supplier k is inactive, supplier j is active and $(w_i, v_i) = (w_k, v_k) + \theta(w_k - w_j, v_k - v_j)$ for some $\theta \geq 0$.

Note that the theorem still allows that an inefficient supplier is active in some equilibrium, as illustrated by supplier 3 in Example 2. In fact, in an equilibrium, an inefficient supplier will be active when its cost parameters are below the lower convex hull of the equilibrium bids; otherwise, it will be inactive.

Theorem 6 adds a new case to what was presented in Theorem 3. This new situation arises when an inactive supplier sets the price of some active supplier, case (ii). In that situation, the active supplier keeps the inefficient supplier out of the market by making its entry non-profitable. Another example of this phenomenon can be found in the *Bertrand model* with asymmetric players. There the equilibrium price is fixed by the second lowest cost supplier who is inactive. Also, although it is commonly argued that the only equilibrium in pure strategies is such that the most competitive producer captures all the market at a price equal to the second most competitive cost, as in Tirole [20] p.211, this equilibrium is not unique. As noted by Erlei [8], all the prices between the smallest and the second smallest costs are Nash equilibria of the system. This is true since an inefficient player can impact the market price by placing absurd bids knowing that it will not capture any market share. Typically, these bids below cost are discarded because they are dominated by the strategy of bidding at cost. In our model, it may also be possible to discard some of the equilibria proposed by Theorem 6, by eliminating for instance dominated strategies, but this presents significant challenges.

The next example illustrates case (ii) of the theorem.

Example 3 *Assume that customer demand is uniformly distributed in $[0,1]$. Let $N = 4$ and the true costs be*

$$(c_1, f_1) = (0, 40), \quad (c_2, f_2) = (40, 20), \quad (c_3, f_3) = (70, 6), \quad (c_4, f_4) = (80, 6), \quad p = 100.$$

Supplier 4 is not efficient but the rest are. The following bids form an equilibrium

$$\begin{array}{ll} (w_1, v_1) = (w_2, v_2) = (20, 30), & (w_3, v_3) = (88, 2.8), (w_4, v_4) = (80, 6) \\ y_1 = 0.5,\ y_2 = 0.6, & y_3 = 0.767,\ y_4 = 0.767. \end{array}$$

Supplier 4, while not capturing any capacity and making zero profit, determines the price of supplier 3, who is efficient and must react to the bid of supplier 4.

6 Discussion

In this research, we analyzed the procurement process between a single buyer and multiple suppliers. Suppliers compete on price and flexibility, two attributes that are important to the buyer. Specifically, each supplier offers a different option contract and the buyer reserves capacities at each supplier so as to maximize expected profit. We modeled the process as a single-shot game where suppliers submit an offer with a reservation and an execution fee. Under the assumption of the demand distribution having the border property, satisfied by any log-concave distribution, e.g., uniform, exponential or normal distributions, we characterized optimality conditions for suppliers' bids and provided necessary conditions for equilibrium bids.

Interestingly, equilibria in pure strategies give rise to what we call cluster competition. This provides several insights.

1 It pays to be efficient. *No matter how the competitors bid, when a supplier is efficient, it will capture orders from the buyer and will have a non-negative expected profit.*

In other words, being an efficient supplier means capturing market share, and no other supplier can push an efficient supplier out of business. Notice that our definition of efficiency allows having multiple efficient technologies, because the cost space is two-dimensional. This implies that an inefficient supplier may become efficient by reaching the efficient frontier defined by the lower envelope of the true costs of the other suppliers. Hence, this inefficient supplier does not necessarily have to change technology and copy the same exact cost as other suppliers; what is needed is a local improvement of its costs so as to move to the efficient frontier.

2 Suppliers compete with suppliers with similar cost structure. *When all suppliers are efficient, a supplier will compete against another supplier with similar technology, either the one with next lower or next higher execution cost.*

Indeed, in equilibrium, a supplier's bid is most sensitive to the bid of another supplier with similar technology. This leads to our third insight.

3 Competition preserves diversity and segments the market. *At a market equilibrium with efficient suppliers, the suppliers are clustered into small groups of no more than three suppliers and no less than two suppliers. All suppliers within each group offer the same option and share the order from the buyer.*

The market will thus be segmented by groups of similar technologies. Competition will diminish technological variety but will not eliminate it. This is in contrast to market behavior in the price-only competition. Thus, in our model, if at some point a supplier pushes its competitors in a given cluster out of the market, this supplier will increase its market share by moving to a different cluster.

4 Prices are directly related to true costs. *The equilibrium prices of the different options offered by the suppliers lie in the lower envelope of the costs of the system. That is, the reservation and execution equilibrium prices are linked to the true reservation and execution costs and no inflation of prices is stable.*

This insight shows the link between the costs of the system and the option prices available in the market. Specifically, if all suppliers are efficient, this implies a range of possible bids, each of which is along the lower envelope of the true suppliers' costs. However, many equilibria are possible, and hence it is not possible to predict the option prices.

5 Competition leads to a loss of supply chain profit. *While suppliers' prices are related to their true costs, the allocation of capacity can be quite different from the one achieved in a centralized system. However, our analysis indicates that the loss of system profit is no more than 50% of the maximum possible, and 25% for the class of log-concave distributions, a class that includes commonly used distributions such as the normal, uniform and exponential.*

Finally, this paper will be incomplete if we do not mention important extensions of our model. One possible direction is to allow buyers to purchase products at a spot market in addition to using the contracts signed with the suppliers. In such a model, suppliers and buyers negotiate contracts knowing that additional supply or demand are available in the spot market. Such a model would generalize not only the model in the current paper but also the models presented in Wu et al. [23]. Another extension would be to consider not only the suppliers' bids, but also their strategic technology choice, within a set of possible technologies. Alternatively, one could expand the multi-dimensional competition model to include other factors such as quality or lead time, where the optimal supply portfolio for the buyer would be found endogenously. All these extensions present significant technical challenges.

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Competition in the Supply Option Market

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Appendix: Proofs

Proposition 1

Proof. To obtain the optimal portfolio we maximize function $V(\cdot)$ over the feasible region P defined in Equation (3). From Equation (2), we observe that function $V(\cdot)$ is the sum of strictly concave functions of y_i, $i = 1, \dots, N$. Hence, it is strictly concave jointly in $(y_1, \dots, y_N)$. The feasible region is a polyhedral cone with non-empty interior. This implies that the Slater conditions hold for this problem and that the Karush-Kuhn-Tucker conditions are necessary and sufficient at optimality (see Bertsekas [3] for details).

Define for every constraint $y_{i-1} - y_i \leq 0, i = 1, \dots, N$, the associate Lagrange multiplier $\lambda_i \geq 0$. The KKT optimality conditions are, for $i = 1, \dots, N$, assuming $\lambda_{N+1} = 0$:

$$\begin{aligned} (v_{i+1} - v_i) + (w_{i+1} - w_i)\overline{F}(y_i) &= \lambda_{i+1} - \lambda_i \\ \lambda_i(y_{i-1} - y_i) &= 0 \\ y_{i-1} - y_i &\leq 0 \\ \lambda_i &\geq 0 \end{aligned} \tag{18}$$

Let $\{i_1, \dots, i_k\}$ be the winning set of $\{(w_1, v_1), \dots, (w_{N+1}, v_{N+1})\}$. Define $y_{i_1}, \dots, y_{i_{k-1}}$ such that

$$\overline{F}(y_{i_j}) = \frac{v_{i_j} - v_{i_{j+1}}}{w_{i_{j+1}} - w_{i_j}}$$

$\overline{F}(y_{i_k}) = 0$ and for the other variables $y_i = y_{i-1}$ (remember from Equation (1) that $a_l = 0$). Note that it can happen that $y_i = \infty$ for some i. Define also $\lambda_{i_1} = \ldots = \lambda_{i_k} = 0$ together with:

(i) for $1 \leq i < i_1$, $\lambda_i = (v_i - v_{i_1}) + (w_i - w_{i_1})$,

(ii) for $j = 1, \dots, k-1$, for $i_j < i < i_{j+1}$, $\lambda_i = (v_i - v_{i_j}) + (w_i - w_{i_j})\overline{F}(y_{i_j})$,

(iii) for $i_k < i$, $\lambda_i = (v_i - v_{i_k})$.

It is now sufficient to verify that this solution satisfies the KKT conditions, Equation (18). Evidently, the first three requirements in (18) are satisfied by construction. It remains to verify that $\lambda_i \geq 0$ for all $i = 1, \dots, N$. To see this, we analyze three different cases:

(i) for $1 \leq i < i_1$, $\lambda_i = (v_i - v_{i_1}) + (w_i - w_{i_1}) \geq 0$ from part (b) of Definition 2;

(ii) for $j = 1, \dots, k-1$, for $i_j < i < i_{j+1}$, $\lambda_i = (v_i - v_{i_j}) + (w_i - w_{i_j})\dfrac{v_{i_j} - v_{i_{j+1}}}{w_{i_{j+1}} - w_{i_j}} \geq 0$ from part (c);

(iii) for $i_k < i$, $\lambda_i = (v_i - v_{i_k}) \geq 0$ from part (d).

Finally, we see that no inactive point can be winning since this would imply that one of the inactive points is on the segment joining two other points. This would contradict the minimality of the winning set in Definition 2. ■

Proof of Theorem 1

Proof. It is easy to see that, in order to prove the result, one can show that the profit, as a function of (y_-, y_+), does not achieve a local interior maximum, and this holds for all cost and rival bids parameters. For this purpose, assume that for a given set of parameters, we have a strict local maximum of the profit function.

Take $a_m \geq 0$, define

$$\alpha_1 = \frac{v^l - f}{c - w^l},$$
$$\alpha_2 = \frac{f - v^h}{w^h - c},$$

and $\alpha_m = \overline{F}(a_m)$. Assume, with no loss of generality, that the feasible region is $0 \leq y_- \leq a_m \leq y_+$. Let (y_-, y_+) be a strict local maximum of the function

$$\Phi(y_-, y_+) = \frac{\alpha_1 - \alpha_m}{\alpha_1 - \alpha_2} \int_{y_-}^{y_+} \Big(\overline{F}(u) - \alpha_2\Big) du - \frac{\overline{F}(y_-) - \alpha_m}{\overline{F}(y_-) - \overline{F}(y_+)} \int_{y_-}^{y_+} \Big(\overline{F}(u) - \overline{F}(y_+)\Big) du,$$

which corresponds to the profit divided by $w_h - w_l$. Let $\alpha_- = \overline{F}(y_-)$ and $\alpha_+ = \overline{F}(y_+)$.

Since this is a strict interior maximum, the first order conditions are, after recombining the different terms,

$$0 = \frac{\partial \Phi}{\partial y_-} = -\frac{(\alpha_m - \alpha_2)(\alpha_1 - \alpha_-)}{(\alpha_1 - \alpha_2)} + f(y_-) \frac{(\alpha_m - \alpha_+)}{(\alpha_- - \alpha_+)^2} \Big\{ \int_{y_-}^{y_+} (\overline{F}(u) - \alpha_+) du \Big\} \tag{19}$$

$$0 = \frac{\partial \Phi}{\partial y_+} = \frac{(\alpha_1 - \alpha_m)(\alpha_+ - \alpha_2)}{(\alpha_1 - \alpha_2)} + f(y_+) \frac{(\alpha_- - \alpha_m)}{(\alpha_- - \alpha_+)^2} \Big\{ \int_{y_-}^{y_+} (\alpha_- - \overline{F}(u)) du \Big\}. \tag{20}$$

Let

$$A = \left(\frac{f(x)}{\overline{F}(x) - \overline{F}(y)} \right) \left(\frac{\displaystyle\int_x^y (\overline{F}(u) - \overline{F}(y)) du}{\overline{F}(x) - \overline{F}(y)} \right)$$

and

$$B = \left(\frac{f(y)}{\overline{F}(x) - \overline{F}(y)} \right) \left(\frac{\displaystyle\int_x^y (\overline{F}(x) - \overline{F}(u)) du}{\overline{F}(x) - \overline{F}(y)} \right).$$

The first order conditions are equivalent to

$$A = \frac{(\alpha_m - \alpha_2)(\alpha_1 - \alpha_-)}{(\alpha_1 - \alpha_2)(\alpha_m - \alpha_+)},$$
$$B = \frac{(\alpha_1 - \alpha_m)(\alpha_+ - \alpha_2)}{(\alpha_1 - \alpha_2)(\alpha_- - \alpha_m)},$$

or equivalently,

$$\begin{aligned}
\Big[\frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_-} - \frac{1-A}{A}\Big]\Big[\frac{\alpha_- - \alpha_+}{\alpha_+ - \alpha_2} - \frac{1-B}{B} + \frac{1}{B}\Big] &= 1 + \frac{1}{A}\frac{\alpha_- - \alpha_m}{\alpha_m - \alpha_+}, \\
\Big[\frac{\alpha_- - \alpha_+}{\alpha_+ - \alpha_2} - \frac{1-B}{B}\Big]\Big[\frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_-} - \frac{1-A}{A} + \frac{1}{A}\Big] &= 1 + \frac{1}{B}\frac{\alpha_m - \alpha_+}{\alpha_- - \alpha_m}.
\end{aligned} \tag{21}$$

We can see that when f is log-concave, then

$$\int_x^y (\overline{F}(x) - \overline{F}(u))du$$

is log-concave in y. This implies that $0 \leq B \leq 1$. Similarly, $0 \leq A \leq 1$ when f is log-concave. Thus under this assumption, $\alpha_- \geq \alpha_m \geq \alpha_+$ implies that the first order conditions can only be satisfied when $\alpha_+ - \alpha_2 \geq 0$ and $\alpha_1 - \alpha_- \geq 0$ or $\alpha_+ - \alpha_2 \leq 0$ and $\alpha_1 - \alpha_- \leq 0$.

Let

$$\begin{aligned}
a &= \frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_-} - \frac{1-A}{A} \\
b &= \frac{\alpha_- - \alpha_+}{\alpha_+ - \alpha_2} - \frac{1-B}{B} \\
c &= \frac{\alpha_- - \alpha_m}{\alpha_m - \alpha_+}.
\end{aligned}$$

Equation (21) can thus be expressed as

$$a(b + \frac{1}{B}) - 1 = \frac{1}{A}c,$$

$$b(a + \frac{1}{A}) - 1 = \frac{1}{B}\frac{1}{c}.$$

By multiplying these two equations, one obtains

$$\Big[a(b + \frac{1}{B}) - 1\Big]\Big[b(a + \frac{1}{A}) - 1\Big] = \frac{1}{AB},$$

or equivalently

$$\Big[ab - 1\Big]\Big[ab + \frac{a}{B} + \frac{b}{A} + \frac{1}{AB} - 1\Big] = 0.$$

We have two possible cases:

1. In the first case, we have $a, b \geq 0$, $ab = 1$, and thus $\alpha_1 \geq \alpha_- \geq \alpha_m \geq \alpha_+ \geq \alpha_2$. Thus Equation (21) becomes

$$\begin{aligned}
\frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_-} - \frac{1-A}{A} &= \frac{B}{A}\Big[\frac{\alpha_- - \alpha_m}{\alpha_m - \alpha_+}\Big], \\
\frac{\alpha_- - \alpha_+}{\alpha_+ - \alpha_2} - \frac{1-B}{B} &= \frac{A}{B}\Big[\frac{\alpha_m - \alpha_+}{\alpha_- - \alpha_m}\Big].
\end{aligned} \tag{22}$$

2. In the second case, we have $a, b \leq 0$, and $ab = 1 - \frac{a}{B} - \frac{b}{A} - \frac{1}{AB}$. Thus, Equation (21) becomes

$$\begin{aligned}
-\frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_-} - 1 &= \frac{\alpha_m - \alpha_+}{\alpha_- - \alpha_m}, \\
-\frac{\alpha_- - \alpha_+}{\alpha_+ - \alpha_2} - 1 &= \frac{\alpha_- - \alpha_m}{\alpha_m - \alpha_+}.
\end{aligned} \tag{23}$$

The second order condition for having an interior local maximum is that the Hessian of Φ is negative semi-definite. It is straightforward to see that the Hessian being negative semi-definite is equivalent to having that H, defined as follows, is negative semi-definite.

$$H = \begin{pmatrix} \frac{1}{A}\frac{\partial A}{\partial y_-} + \frac{f(y_-)}{\alpha_1 - \alpha_-} & \frac{1}{A}\frac{\partial A}{\partial y_+} + \frac{f(y_+)}{\alpha_m - \alpha_+} \\ -\frac{1}{B}\frac{\partial B}{\partial y_-} + \frac{f(y_-)}{\alpha_- - \alpha_m} & -\frac{1}{B}\frac{\partial B}{\partial y_+} - \frac{f(y_+)}{\alpha_+ - \alpha_2} \end{pmatrix}$$

We compute the quantities that define H in the following equations, evaluated at (y_-, y_+). We have

$$\begin{aligned}
\frac{1}{A}\frac{\partial A}{\partial y_-} &= \frac{f'(y_-)}{f(y_-)} + \Big[\frac{f(y_-)}{\overline{F}(y_-) - \overline{F}(y_+)}\Big]\Big[2 - \frac{1}{A}\Big], \\
\frac{1}{A}\frac{\partial A}{\partial y_+} &= \Big[\frac{f(y_+)}{\overline{F}(y_-) - \overline{F}(y_+)}\Big]\Big[1 + \frac{f(y_-)B}{f(y_+)A}\Big] - 2\Big[\frac{f(y_+)}{\overline{F}(y_-) - \overline{F}(y_+)}\Big] \\
\frac{1}{B}\frac{\partial B}{\partial y_-} &= -\Big[\frac{f(y_-)}{\overline{F}(y_-) - \overline{F}(y_+)}\Big]\Big[1 + \frac{f(y_+)A}{f(y_-)B}\Big] + 2\Big[\frac{f(y_-)}{\overline{F}(y_-) - \overline{F}(y_+)}\Big], \\
\frac{1}{B}\frac{\partial B}{\partial y_+} &= \frac{f'(y_+)}{f(y_+)} - \Big[\frac{f(y_+)}{\overline{F}(y_-) - \overline{F}(y_+)}\Big]\Big[2 - \frac{1}{B}\Big].
\end{aligned}$$

Thus, H can be expressed as

$$\begin{aligned}
H = &\begin{pmatrix} \frac{f'(y_-)}{f(y_-)} & 0 \\ 0 & -\frac{f'(y_+)}{f(y_+)} \end{pmatrix} \\
&+ \begin{pmatrix} \Big[\frac{f(y_-)}{\alpha_- - \alpha_+}\Big]\Big[2 - \frac{1}{A} - \frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_-}\Big] & \frac{B}{A}\Big[\frac{f(y_-)}{\alpha_- - \alpha_+}\Big] + \Big[\frac{f(y_+)}{\alpha_- - \alpha_+}\Big]\Big[\frac{\alpha_- - \alpha_m}{\alpha_m - \alpha_+}\Big] \\ \frac{A}{B}\Big[\frac{f(y_+)}{\alpha_- - \alpha_+}\Big] + \Big[\frac{f(y_-)}{\alpha_- - \alpha_+}\Big]\Big[\frac{\alpha_m - \alpha_+}{\alpha_- - \alpha_m}\Big] & \Big[\frac{f(y_+)}{\alpha_- - \alpha_+}\Big]\Big[2 - \frac{1}{B} - \frac{\alpha_- - \alpha_+}{\alpha_+ - \alpha_2}\Big] \end{pmatrix}
\end{aligned}$$

In case (2) defined above, we can take a look at H_{11}, using Equation (23):

$$\frac{f'(y_-)}{f(y_-)} + \Big[\frac{f(y_-)}{\alpha_- - \alpha_+}\Big]\Big[2 - \frac{1}{A} - \frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_-}\Big] \geq \frac{f'(y_-)}{f(y_-)} + \Big[\frac{f(y_-)}{\alpha_- - \alpha_+}\Big]\Big[3 - \frac{1}{A}\Big].$$

This quantity is the derivative of $\bar{A} = \dfrac{A}{\overline{F}(y_-) - \overline{F}(y_+)}$ with respect to y_-. We have that

$$\frac{1}{\bar{A}}\frac{\partial \bar{A}}{\partial y_-} = \begin{cases} \frac{f'(y_-)}{f(y_-)} + \frac{f(y_-)}{\overline{F}(y_-) - \overline{F}(y_+)} \\ + \left[\frac{f(y_-)}{\overline{F}(y_-) - \overline{F}(y_+)}\right]\left[2 - \frac{1}{A}\right]. \end{cases}$$

Since

$$\frac{1}{A}\frac{\partial A}{\partial y_-} = \frac{f'(y_-)}{f(y_-)} + \left[\frac{f(y_-)}{\overline{F}(y_-) - \overline{F}(y_+)}\right]\left[2 - \frac{1}{A}\right],$$

when A is increasing at y_-, then $\bar{A}$ also is. On the other hand, when A is decreasing at y_-, and since $A(y_+, y_+) = 1/2$, it must be that $A \geq 1/2$. Log-concavity of f implies that

$$\frac{f'(y_-)}{f(y_-)} + \frac{f(y_-)}{\overline{F}(y_-) - \overline{F}(y_+)} \geq 0$$

and thus $\bar{A}$ is again non-decreasing in y_-. Thus, $\dfrac{A}{\overline{F}(y_-) - \overline{F}(y_+)}$ is non-decreasing, which implies that H_{11} is non-negative. The same is true for H_{22}. Thus, in case (2), the matrix H cannot be negative semi-definite.

In case (1), using that $c = \dfrac{\alpha_- - \alpha_m}{\alpha_m - \alpha_+}$,

$$H = \begin{pmatrix} \dfrac{f'(y_-)}{f(y_-)} & 0 \\ 0 & -\dfrac{f'(y_+)}{f(y_+)} \end{pmatrix}$$
$$+ \begin{pmatrix} \Big[\dfrac{f(y_-)}{\alpha_- - \alpha_+}\Big]\Big[3 - \dfrac{2}{A} - \dfrac{Bc}{A}\Big] & \dfrac{B}{A}\Big[\dfrac{f(y_-)}{\alpha_- - \alpha_+}\Big] + c\Big[\dfrac{f(y_+)}{\alpha_- - \alpha_+}\Big] \\ \\ \dfrac{A}{B}\Big[\dfrac{f(y_+)}{\alpha_- - \alpha_+}\Big] + \dfrac{1}{c}\Big[\dfrac{f(y_-)}{\alpha_- - \alpha_+}\Big] & \Big[\dfrac{f(y_+)}{\alpha_- - \alpha_+}\Big]\Big[3 - \dfrac{2}{B} - \dfrac{A}{Bc}\Big] \end{pmatrix}$$

To get rid of c, we examine

$$
\begin{pmatrix} \frac{A}{f(y_-)} & \frac{B}{f(y_+)} \end{pmatrix} H \begin{pmatrix} \frac{A}{f(y_-)} \\ \\ \frac{B}{f(y_+)} \end{pmatrix}
$$

$$
= \begin{cases} \frac{f'(y_-)}{f(y_-)} \frac{A^2}{f(y_-)^2} \\ \\ -\frac{f'(y_+)}{f(y_+)} \frac{B^2}{f(y_+)^2} \\ \\ +\frac{2A(2A-1)}{f(y_-)(\alpha_- - \alpha_+)} + \frac{2B(2B-1)}{f(y_+)(\alpha_- - \alpha_+)} \end{cases}
$$

$$
= \frac{1}{(\alpha_- - \alpha_+)^2} \begin{cases} \frac{f'(y_-)}{f(y_-)} \Big\{ \int_{y_-}^{y_+} (\frac{\overline{F}(u) - \alpha_+}{\alpha_- - \alpha_+}) du \Big\}^2 \\ \\ -\frac{f'(y_+)}{f(y_+)} \Big\{ \int_{y_-}^{y_+} (\frac{\alpha_- - \overline{F}(u)}{\alpha_- - \alpha_+}) du \Big\}^2 \\ \\ +2(2A-1) \Big\{ \int_{y_-}^{y_+} (\frac{\overline{F}(u) - \alpha_+}{\alpha_- - \alpha_+}) du \Big\} \\ \\ +2(2B-1) \Big\{ \int_{y_-}^{y_+} (\frac{\alpha_- - \overline{F}(u)}{\alpha_- - \alpha_+}) du \Big\} \end{cases}
$$

Using that

$$
\Delta = y_+ - y_- = \int_{y_-}^{y_+} (\frac{\overline{F}(u) - \alpha_+}{\alpha_- - \alpha_+}) du + \int_{y_-}^{y_+} (\frac{\alpha_- - \overline{F}(u)}{\alpha_- - \alpha_+}) du,
$$

and defining

$$
z = \int_{y_-}^{y_+} (\frac{\overline{F}(u) - \alpha_+}{\alpha_- - \alpha_+}) du,
$$

we can express the terms in the last bracket as

$$
E = \frac{f'(y_-)}{f(y_-)} z^2 - \frac{f'(y_+)}{f(y_+)} (\Delta - z)^2 + 4 \frac{f(y_-)}{\alpha_- - \alpha_+} z^2 + 4 \frac{f(y_+)}{\alpha_- - \alpha_+} (\Delta - z)^2 - 2\Delta. \tag{24}
$$

By minimizing this expression in terms of z, we obtain a lower bound on this expression, i.e.

$$
\begin{aligned}
E \geq \quad & \Delta\left\{\frac{\Big[\frac{f'(y_-)}{f(y_-)}+4\frac{f(y_-)}{\alpha_--\alpha_+}\Big]\Big[-\frac{f'(y_+)}{f(y_+)}+4\frac{f(y_+)}{\alpha_--\alpha_+}\Big]\Delta}{\Big[\frac{f'(y_-)}{f(y_-)}+4\frac{f(y_-)}{\alpha_--\alpha_+}-\frac{f'(y_+)}{f(y_+)}+4\frac{f(y_+)}{\alpha_--\alpha_+}\Big]}-2\right\} \\
= \quad & \left\{\frac{1}{\frac{f'(y_-)}{f(y_-)}+4\frac{f(y_-)}{\alpha_--\alpha_+}-\frac{f'(y_+)}{f(y_+)}+4\frac{f(y_+)}{\alpha_--\alpha_+}}\right\} \\
& \left\{\left[\frac{f'(y_-)}{f(y_-)}\Delta+4\frac{f(y_-)\Delta}{\alpha_--\alpha_+}-2\right]\left[-\frac{f'(y_+)}{f(y_+)}\Delta+4\frac{f(y_+)\Delta}{\alpha_--\alpha_+}-2\right]-4\right\}
\end{aligned}
$$

We thus focus on the last term of the product,

$$
F=\left(\frac{f'(y_-)}{f(y_-)}\Delta+4\frac{f(y_-)\Delta}{\alpha_--\alpha_+}-2\right)\left(-\frac{f'(y_+)}{f(y_+)}\Delta+4\frac{f(y_+)\Delta}{\alpha_--\alpha_+}-2\right)-4. \tag{25}
$$

Over all log-concave distribution functions, F defined in Equation (25) is minimized when $\alpha_- - \alpha_+$ is maximized. This occurs when, after defining θ by $\beta_-\theta+\beta_+(1-\theta)=\beta_0$,

$$
f(t)=\begin{cases} f(y_-)e^{\beta_-(t-y_-)/\Delta} & \text{when } y_-\leq t\leq y_-+\theta\Delta \\ f(y_+)e^{-\beta_+(y_+-t)/\Delta} & \text{when } y_+-(1-\theta)\Delta\leq t\leq y_+ \end{cases}
$$

We thus know the structure of the worst-case log-concave distribution. By re-scaling the problem, F can be expressed using only β_-, β_+ (with $\beta_+\leq\beta_-$) and $\theta\in[0,1]$. To obtain the following expression, we scale Δ to 1 and the break-point value of the distribution $f(y_-+\theta\Delta)$ to 1. After defining, for $k\geq 0$,

$$
P_k(z)=\frac{e^z-1-z-\ldots-z^{k-1}/(k-1)!}{z^k}, \tag{26}
$$

we obtained the following scaled quantities

$$
\begin{aligned}
& f(y_-)=e^{-\beta_-\theta}=P_0\Big(-\beta_-\theta\Big) \\
& f(y_+)=e^{\beta_+(1-\theta)}=P_0\Big(\beta_+(1-\theta)\Big) \\
& \alpha_--\alpha_+=\frac{e^{-\beta_-\theta}-1}{-\beta_-}+\frac{e^{\beta_+(1-\theta)}-1}{\beta_+}=\theta P_1\Big(-\beta_-\theta\Big)+(1-\theta)P_1\Big(\beta_+(1-\theta)\Big).
\end{aligned}
$$

Define

$$
R_1(\beta_-,\beta_+)=\frac{P_0\Big(-\beta_-\theta\Big)}{\theta P_1\Big(-\beta_-\theta\Big)+(1-\theta)P_1\Big(\beta_+(1-\theta)\Big)},
$$

$$
G_1(\beta_-,\beta_+)=\beta_--2+4R_1(\beta_-,\beta_+)\geq 0,
$$

and

$$
R_2(\beta_-,\beta_+)=\frac{P_0\Big(\beta_+(1-\theta)\Big)}{\theta P_1\Big(-\beta_-\theta\Big)+(1-\theta)P_1\Big(\beta_+(1-\theta)\Big)},
$$

$$
G_2(\beta_-,\beta_+)=-\beta_+-2+4R_2(\beta_-,\beta_+)\geq 0.
$$

Thus, F can be expressed as $G_1G_2 - 4$.

Notice that, since

$$P_1(z)' = P_1(z) - P_2(z) = P_0(z)P_2(-z),$$

we have that

$$\begin{aligned}\frac{\partial R_1}{\partial \beta_-} &= R_1\Big(-\theta + \theta^2 P_2(\beta_-\theta)R_1\Big)\\ \frac{\partial R_1}{\partial \beta_+} &= R_1\Big(-(1-\theta)^2 P_2(-\beta_+(1-\theta))R_2\Big),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial R_2}{\partial \beta_-} &= R_2\Big(\theta^2 P_2(\beta_-\theta)R_1\Big)\\ \frac{\partial R_2}{\partial \beta_+} &= R_2\Big((1-\theta) - (1-\theta)^2 P_2(-\beta_+(1-\theta))R_2\Big),\end{aligned}$$

In order to obtain a lower bound on F, we examine two different cases: either the minimal value of F subject to $\theta \in [0,1]$ and $\beta_- \geq \beta_+$ is reached in an interior point, or it is reached at the border of the region, i.e., $\theta = 0$, $\theta = 1$ or $\beta_- = \beta_+$; in any of the latter cases, the distribution turns out to be an exponential distribution.

To analyze the first case, let's examine the critical points of F with respect to θ, β_- and β_+. That is, assume that

$$\frac{\partial F}{\partial \theta} = 0 = -\Big[4R_1\Big(\beta_- - (R_2 - R_1)\Big)\Big]G_2 + G_1\Big[4R_2\Big((R_2-R_1) - \beta_+\Big)\Big], \tag{27}$$

$$\frac{\partial F}{\partial \beta_-} = 0 = \Big[1 + 4R_1\theta\Big(-1 + \theta P_2(\beta_-\theta)R_1\Big)\Big]G_2 + G_1\Big[4R_1R_2\theta^2 P_2(\beta_-\theta)\Big], \tag{28}$$

and

$$\frac{\partial F}{\partial \beta_+} = 0 = \Big[-4R_1R_2(1-\theta)^2 P_2(-\beta_+(1-\theta))\Big]G_2 + G_1\Big[-1 + 4R_2(1-\theta)\Big(1-(1-\theta)P_2(-\beta_+(1-\theta))R_2\Big)\Big]. \tag{29}$$

Notice that we can express

$$\begin{aligned}R_2 - R_1 &= \frac{P_0\Big(\beta_+(1-\theta)\Big) - P_0\Big(-\beta_-\theta\Big)}{\theta P_1\Big(-\beta_-\theta\Big) + (1-\theta)P_1\Big(\beta_+(1-\theta)\Big)}\\ &= \beta_+ + \frac{\theta(\beta_- - \beta_+)P_1\Big(-\beta_-\theta\Big)}{\theta P_1\Big(-\beta_-\theta\Big) + (1-\theta)P_1\Big(\beta_+(1-\theta)\Big)}\\ &= \beta_- - \frac{(1-\theta)(\beta_- - \beta_+)P_1\Big(\beta_+(1-\theta)\Big)}{\theta P_1\Big(-\beta_-\theta\Big) + (1-\theta)P_1\Big(\beta_+(1-\theta)\Big)}\end{aligned}$$

Thus, Equation (27) can be rewritten as

$$\frac{R_1/G_1}{R_2/G_2} = \frac{(R_2 - R_1) - \beta_+}{\beta_- - (R_2 - R_2)} = \frac{\theta}{1-\theta} \frac{P_1\Big(-\beta_-\theta\Big)}{P_1\Big(\beta_+(1-\theta)\Big)}. \tag{30}$$

Equation (28) can be expressed as

$$4\theta\frac{R_1}{G_1} = \frac{1}{G_1} + 4\theta^2 P_2\Big(\beta_-\theta\Big) R_1 \Big[\frac{R_1}{G_1} + \frac{R_2}{G_2}\Big].$$

Using the inequality $(a+b)^2 \geq 4ab$ for any $a, b \in \mathbb{R}$, we obtain

$$16\theta^2\Big[\frac{R_1}{G_1}\Big]^2 \geq 16\theta^2 P_2\Big(\beta_-\theta\Big)\frac{R_1}{G_1}\Big[\frac{R_1}{G_1} + \frac{R_2}{G_2}\Big],$$

and hence

$$\frac{R_1}{G_1} \geq P_2\Big(\beta_-\theta\Big)\Big[\frac{R_1}{G_1} + \frac{R_2}{G_2}\Big], \tag{31}$$

Similarly, Equation (29) yields

$$\frac{R_2}{G_2} \geq P_2\Big(-\beta_+(1-\theta)\Big)\Big[\frac{R_1}{G_1} + \frac{R_2}{G_2}\Big], \tag{32}$$

Adding these two last equations, the term $R_1/G_1 + R_2/G_2$ cancels out, and thus

$$1 \geq P_2\Big(\beta_-\theta\Big) + P_2\Big(-\beta_+(1-\theta)\Big).$$

Notice that P_2 is convex, since $P_2'' = P_2 - 4P_3 + 6P_4 \geq 0$. We can therefore apply the convexity inequality

$$\frac{1}{2}P_2\Big(\beta_-\theta\Big) + \frac{1}{2}P_2\Big(-\beta_+(1-\theta)\Big) \geq P_2\Big(\frac{1}{2}\beta_-\theta - \frac{1}{2}\beta_+(1-\theta)\Big).$$

Thus, since $P_2(z) \leq 1/2$ if and only if $z \leq 0$, combining the two inequalities, we obtain that

$$\beta_-\theta \leq \beta_+(1-\theta)$$

Also, since $\beta_- \geq \beta_+$ by construction, we have that

$$\beta_-\theta \leq \beta_+(1-\theta) \leq \beta_-(1-\theta).$$

This implies, that for any critical point, we must have:

- either $\beta_- \geq \beta_+ \geq 0$ and $\theta \leq \dfrac{1}{2}$;
- or $0 \geq \beta_- \geq \beta_+$ and $\theta \geq \dfrac{1}{2}$.

When $\theta < 1/2$, Equation (31) yields

$$\frac{R_1}{G_1} \geq \frac{1}{2}\Big[\frac{R_1}{G_1} + \frac{R_2}{G_2}\Big],$$

and thus

$$\frac{R_1}{G_1} \geq \frac{R_2}{G_2}.$$

On the other hand, Equation (30) implies that, since $\theta/(1-\theta) < 1$, $P_1\Big(-\beta_-\theta\Big) \leq P_1(0) = 1$ and $P_1\Big(\beta_+(1-\theta)\Big) \geq P_1(0) = 1$,

$$\frac{R_1/G_1}{R_2/G_2} = \frac{\theta}{1-\theta}\frac{P_1\Big(-\beta_-\theta\Big)}{P_1\Big(\beta_+(1-\theta)\Big)} < 1.$$

This is a contradiction.

Similarly, when $\theta > 1/2$, we have again a contradiction using Equation (32) to show

$$\frac{R_1}{G_1} \leq \frac{R_2}{G_2}.$$

and Equation (30) to show

$$\frac{R_1/G_1}{R_2/G_2} > 1.$$

Thus, the only feasible case is $\theta = 1/2$ which implies $\beta_- = \beta_+ = 0$, and in this case $G_1 = G_2 = 2$, so that $F \geq 0$.

Thus, there are no critical points in the interior of $\theta \in [0,1]$ and $\beta_- \geq \beta_+$ such that $F < 0$.

The remaining case is when the minimum of F is reached at the border of the feasible set, i.e., the distribution is exponential, with parameter γ. In this case, we can express F as $g(\gamma)g(-\gamma) - 4$, with, for $z \in \mathbb{R}$,

$$g(z) = z - 2 + \frac{4}{P_1(z)}.$$

Note that $g(z) - g(-z) = -2z$ and

$$\begin{aligned} g(z) + g(-z) = & \ 4\frac{z(1-e^{-z}) + z(e^z - 1) - (e^z-1)(e^{-z}-1)}{(e^z-1)(e^{-z}-1)} \\ = & \ 4\left\{\frac{P_1(-z) + P_1(z)}{P_2(-z) + P_2(z)} - 1\right\}. \end{aligned}$$

Hence,

$$\begin{aligned} g(z)g(-z) - 4 = & \ \frac{1}{4}\Big(g(z) + g(-z)\Big)^2 - \frac{1}{4}\Big(g(z) - g(-z)\Big)^2 - 4 \\ = & \ 4\frac{\Big\{P_1(-z) + P_1(z)\Big\}^2}{\Big\{P_2(-z) + P_2(z)\Big\}^2} - 8\frac{\Big\{P_1(-z) + P_1(z)\Big\}}{\Big\{P_2(-z) + P_2(z)\Big\}} - z^2 \end{aligned}$$

After putting all terms under a common denominator, and observing that

$$\begin{aligned} &\Big\{P_1(-z) + P_1(z)\Big\}^2 = 4\Big\{P_2(-2z) + P_2(2z)\Big\}, \\ &\Big\{P_1(-z) + P_1(z)\Big\}\Big\{P_2(-z) + P_2(z)\Big\} = 2\Big\{P_1(-2z) + P_1(2z) - P_1(-z) - P_1(z)\Big\}/z^2, \\ &\Big\{P_2(-z) + P_2(z)\Big\}^2 = 4\Big\{P_2(-2z) + P_2(2z) - P_2(-z) - P_2(z)\Big\}, \end{aligned}$$

the numerator can be expressed as

$$12\Big\{P_2(-2z)+P_2(2z)\Big\}+4\Big\{P_2(-z)+P_2(z)\Big\}-16\Big\{P_1(-2z)+P_1(2z)\Big\}/z^2+16\Big\{P_1(-z)+P_1(z)\Big\}/z^2.$$

We can use a series expansion to show that this final term is non-negative:

$$2\sum_{k=0}^{\infty}\Big\{\frac{12\cdot 2^{2k}z^{2k}}{(2k+2)!}+\frac{4z^{2k}}{(2k+2)!}-\frac{16\cdot 2^{2k+2}z^{2k}}{(2k+3)!}+\frac{16z^{2k}}{(2k+3)!}\Big\}$$

The coefficient of z^{2k} in the series is

$$\frac{12(2k+3)4^k+4(2k+3)-64\cdot 4^k+16}{(2k+3)!}\geq 0,$$

for all $k\geq 0$. This shows that $F\geq 0$, in all cases.

This completes the proof, since we have found that the second order maximality condition cannot be satisfied. ■

Theorem 2

Proof. We have explained previously that under the assumptions of the proposition, either $y_+^*=a_m$ and $y_-^*=z_l$ or $y_+^*=z_h$ and $y_-^*=a_m$. Otherwise, it would be optimal to bid in some other region $A_{l'h'}$ in addition to A_{lh}. Since this is a contradiction to the hypothesis, it implies that the two possible optimal bids are either (w_l,v_l) or (w_h,v_h).

If $z_l>a_m$ or $z_h<a_m$, from Equation (12) it is clear that it is not optimal for the supplier to bid in this particular region A_{lh}, because it has an incentive to bid in A_{OUT} instead of A_{lh}. Similarly, if $z_l<a_l$ and $z_h>a_h$, neither one of the bids is admissible, and therefore there is an optimum outside A_{lh}. We can now partition the remaining possibilities into the three cases presented in the proposition.

In the two first cases, since only one of the two bids is admissible, it must be optimal. In the third case, it implies that $(c,f)\in A_{lh}$. Bidding (w_h,v_h) is better than (w_l,v_l) when

$$(v_h-f)(z_h-a_m)+(w_h-c)\int_{a_m}^{z_h}\overline{F}(u)du\geq(v_l-f)(a_m-z_l)+(w_l-c)\int_{z_l}^{a_m}\overline{F}(u)du.$$

Using Equations (14) and (15), this is equivalent to

$$(w_h-c)\int_{a_m}^{z_h}[\overline{F}(u)-\overline{F}(z_h)]du\geq(c-w_l)\int_{z_l}^{a_m}[\overline{F}(z_l)-\overline{F}(u)]du.$$

But also, we have that, similarly to Equation (9),

$$c\ =w_h-(w_h-w_l)\frac{\overline{F}(z_l)-\overline{F}(a_m)}{\overline{F}(z_l)-\overline{F}(z_h)}\ =w_l+(w_h-w_l)\frac{\overline{F}(a_m)-\overline{F}(z_h)}{\overline{F}(z_l)-\overline{F}(z_h)}.$$

Therefore, we can rewrite the previous condition as

$$\frac{\overline{F}(z_l) - \overline{F}(a_m)}{\overline{F}(z_l) - \overline{F}(z_h)} \int_{a_m}^{z_h} [\overline{F}(u) - \overline{F}(z_h)]du \geq \frac{\overline{F}(a_m) - \overline{F}(z_h)}{\overline{F}(z_l) - \overline{F}(z_h)} \int_{z_l}^{a_m} [\overline{F}(z_l) - \overline{F}(u)]du.$$

After simplifying this expression, we obtain Equations (16) and (17). ■

Proposition 2

Proof. Consider, in an equilibrium, that i and j submit infinitely close bids equal to (w, v). That is, for each $\epsilon > 0$, there are $(w_i^\epsilon, v_i^\epsilon), (w_j^\epsilon, v_j^\epsilon)$ that converge to (w, v) and such that

$$\Pi_i(w_i^\epsilon, v_i^\epsilon, \mathbf{w}_{-i}^\epsilon, \mathbf{v}_{-i}^\epsilon) \geq \sup_{(w,v)} \Pi_i(w, v, \mathbf{w}_{-i}^\epsilon, \mathbf{v}_{-i}^\epsilon) - \epsilon$$

and

$$\Pi_j(w_j^\epsilon, v_j^\epsilon, \mathbf{w}_{-j}^\epsilon, \mathbf{v}_{-j}^\epsilon) \geq \sup_{(w,v)} \Pi_i(w, v, \mathbf{w}_{-j}^\epsilon, \mathbf{v}_{-j}^\epsilon) - \epsilon.$$

Since in the limit $\epsilon \to 0$, both suppliers are making non-negative profits, then it must be true that either (1) for ϵ sufficiently small, $(w_i^\epsilon, v_i^\epsilon)$ must be in some region A_{lj}^i; or (2) for ϵ sufficiently small, $(w_i^\epsilon, v_i^\epsilon)$ must be in some region A_{jh}^i. Similarly, $(w_j^\epsilon, v_j^\epsilon)$ must be for ϵ small in some region $A_{l'i}^j$ or $A_{ih'}^j$. Indeed, if i (resp. j) bids in a different region, then i (resp. j) makes j (resp. i) inactive, which is ruled out by the assumption of the proposition.

We can now apply Theorem 2. Assume that i bids in A_{lj}^i. Then j must bid in $A_{ih'}^j$. By using the optimality equations (14) and (15), $y_{i-}^\epsilon \approx a_m = \overline{F}^{-1}\left(\frac{v_l - v}{w - w_l}\right)$ and $y_{i+}^\epsilon = \overline{F}^{-1}\left(\frac{v_i^\epsilon - v_j^\epsilon}{w_j^\epsilon - w_i^\epsilon}\right) \approx z_{ih} = \overline{F}^{-1}\left(\frac{f_i - v}{w - c_i}\right)$, and hence we have that the slope between (c_i, f_i) and $(w_i^\epsilon, v_i^\epsilon)$, and $(w_i^\epsilon, v_i^\epsilon)$ and $(w_j^\epsilon, v_j^\epsilon)$ must be very close. The same applies to the slope between (c_j, f_j) and $(w_j^\epsilon, v_j^\epsilon)$, and $(w_j^\epsilon, v_j^\epsilon)$ and $(w_i^\epsilon, v_i^\epsilon)$. Hence, taking the limit as $\epsilon \to 0$, it must be true that $(c_i, f_i), (w, v)$ and (c_j, f_j) are aligned.

Assume now the other possible case: i bids in A_{jh}^i then j bids in $A_{l'i}^j$. A similar analysis yields that $(c_j, f_j), (w, v)$ and (c_i, f_i) are aligned.

In both cases, we have that (w, v) belongs to the segment $[(c_i, f_i); (c_j, f_j)]$. ■

Proposition 3

Proof. Assume that supplier i is not active in an equilibrium of the game. It thus makes zero expected profit. Define the function $Z^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}(\cdot)$ as in Equation (4), the lower envelope made of all bids except i's. If $f_i < Z^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}(c_i)$, then by bidding $(c_i + \epsilon, f_i + \epsilon)$, supplier i achieves some positive profit for ϵ small enough. This contradicts the previous hypothesis and therefore we must have $f_i \geq Z^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}(c_i)$.

Construct the lower envelope $C^{-i}(\cdot)$ of the costs $(c_1, f_1), \dots, (c_{i-1}, f_{i-1}), (c_{i+1}, f_{i+1}), \dots, (c_N, f_N)$, $(p, 0)$. That is, $C^{-i}(\cdot) = Z^{(\mathbf{c}_{-i}, \mathbf{f}_{-i})}(\cdot)$. Assume that $C^{-i}(\cdot)$ is not a lower bound on the function $Z^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}(\cdot)$. This implies that there is an active bid (w_j, v_j) such that $v_j < C^{-i}(w_j)$. $j \neq i$ since i is not active and is not defining the function $Z^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}(\cdot)$. We claim that if supplier j bids in some region A_{lh}, it cannot be at equilibrium. Indeed, we can use Equation (12), in particular,

$$\frac{\partial J_{lh}}{\partial y_{j-}} \geq (f_j - v_l) + (c_j - w_l)\overline{F}(y_{j-}),$$

$$\frac{\partial J_{lh}}{\partial y_{j+}} \leq (v_h - f_j) + (w_h - c_j)\overline{F}(y_{j+}).$$

If this is an equilibrium, then j's bid in A_{lh} must be such that $(f_j - v_l) + (c_j - w_l)\overline{F}(y_{j-}) \leq 0$ and $(v_h - f_j) + (w_h - c_j)\overline{F}(y_{j+}) \geq 0$. This is equivalent to

$$f_j \leq v_l + (v_j - v_l)\frac{c_j - w_l}{w_j - w_l},$$

or if $l = 0$, $f_j + c_j \leq v_k + w_k$ for some k, and

$$f_j \leq v_h + (v_j - v_h)\frac{w_h - c_j}{w_h - w_j}.$$

But if all this feasible area is not strictly below $C^{-i}(\cdot)$, we can find some other supplier k bidding next to j that also satisfies $v_k < C^{-i}(w_k)$. By repeating the argument, we must find a third supplier l satisfying $v_l < C^{-i}(w_l)$ that is not j (so no cycling possible). When we reach the supplier with the smallest w or with the biggest w (the dummy supplier, $N+1$), we reach a contradiction: for the smallest w, we cannot find a different supplier satisfying the condition, for the dummy supplier $v_{N+1} = f_{N+1} = 0 = C^{-i}(c_{N+1}) = C^{-i}(w_{N+1}) = C^{-i}(p)$. Hence j cannot be in equilibrium, and this is a contradiction.

Therefore, the function $C^{-i}(\cdot)$ lies below the function $Z^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}(\cdot)$. This implies that i cannot be efficient, since it is not needed to define the function $C^{-i}(\cdot)$, and thus is not a winning point of $\{(c_1, f_1), \dots, (c_N, f_N), (p, 0)\}$. ■

Proposition 4

Proof. Using Proposition 3, we know that every supplier is active in equilibrium.

If the proposition was false, we could find suppliers i and j such that $c_i < c_j$ and $w_i > w_j$. We may furthermore assume without loss of generality that these are consecutive bidders, i.e. there is no bid (w, v) with $w_j < w < w_i$. To see this, assume that the active suppliers are indexed such that $w_1 \leq \ldots \leq w_t$ and in case of a tie, sorted by increasing execution cost c.

Select a pair (i, j) such that $i + 1 < j$ with $w_i < w_j$ and $c_i > c_j$. One of the following three cases is possible.

- The pair $(i, i+1)$ satisfies $w_i < w_{i+1}$ and $c_i > c_{i+1}$ and then $(i, i+1)$ are consecutive bidders.
- $w_i < w_{i+1}$ and $c_i \leq c_{i+1}$. Then, it is the pair $(i+1, j)$ that satisfies $c_{i+1} > c_j$ and $w_{i+1} < w_j$. Hence, we can iterate this argument until we find consecutive bidders i and j such that $c_i < c_j$ and $w_i > w_j$.
- $w_i = w_{i+1}$ but then, by construction, $c_i \leq c_{i+1}$. Hence, similarly to the previous case, we iterate the argument with the pair $(i+1, j)$.

Since $w_i > w_j$ and i and j are consecutive bidders, the bid of supplier j must be in the border of some region A_{li} (where there is no active supplier between l and i because if there was one it would not be active), where supplier l is active. Also, $w_i > w_j$ implies that $w_j = w_l$ and $v_j = v_l$ is optimal, from Theorem 2. But applying Proposition 2 yields that (w_j, v_j) belongs in the segment $[(c_l, f_l); (c_j, f_j)]$. Similarly, supplier i bids in some region $A_{jh}^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}$, with supplier h active and no active suppliers between j and h. With the same argument as before, we have that $(w_i, v_i) = (w_h, v_h)$ and this bid belongs in the segment $[(c_i, f_i); (c_h, f_h)]$.

Define,

$$\overline{F}(a_m) = \frac{v_j - v_i}{w_i - w_j},$$
$$\overline{F}(z_{jl}) = \frac{v_j - f_j}{c_j - w_j},$$
$$\overline{F}(z_{ih}) = \frac{v_i - f_i}{c_i - w_i},$$

and we have that $z_{jl} < a_m < z_{ih}$ because i and j are active.

We can also define

$$\overline{F}(z_{il}) = \frac{v_j - f_i}{c_i - w_j},$$
$$\overline{F}(z_{jh}) = \frac{f_j - v_i}{w_i - c_j}.$$

Since $c_i < c_j$, and supplier i is efficient, we must have that $f_i \leq f_j + (c_j - c_i)\overline{F}(z_{jl})$ because $\overline{F}(z_{jl})$ is the slope of the line joining (c_l, f_l) to (c_j, f_j). Similarly, $f_j \leq f_i - (c_j - c_i)\overline{F}(z_{ih})$. This implies that $z_{il} \leq z_{jl} < a_m < z_{ih} \leq z_{jh}$ as can be seen from Figure 7.

Finally, we apply Theorem 2. For this purpose, define the functions

$$\phi(z_l) = \frac{\int_{z_l}^{a_m} [\overline{F}(z_l) - \overline{F}(u)]du}{\overline{F}(z_l) - \overline{F}(a_m)}$$

and

$$\psi(z_h) = \frac{\int_{a_m}^{z_h} [\overline{F}(u) - \overline{F}(z_h)]du}{\overline{F}(a_m) - \overline{F}(y_2)}.$$

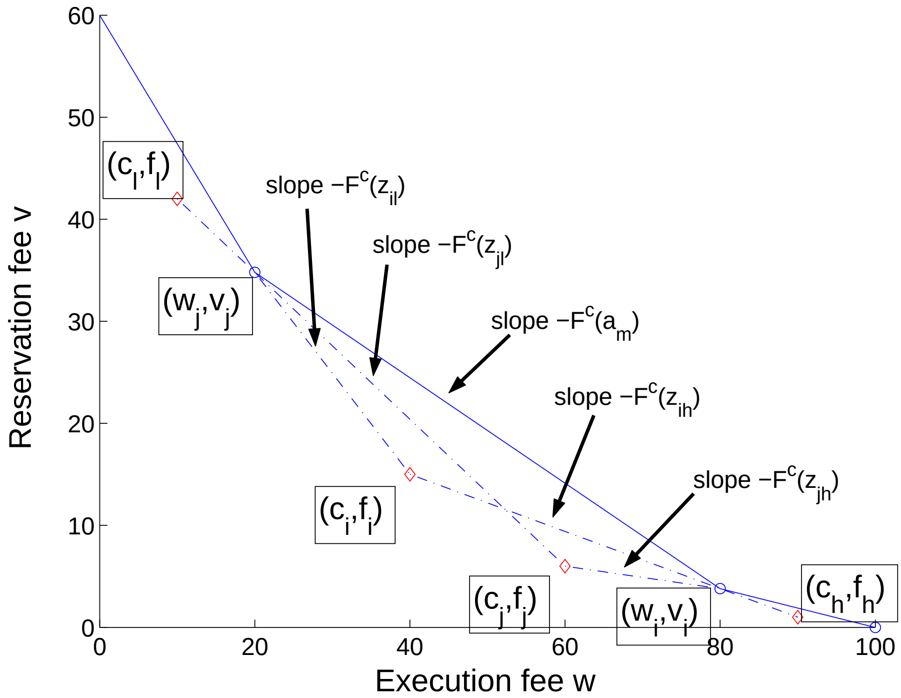

Figure 7: Geometric situation of costs (c_i, f_i) and (c_j, f_j) in region A_{lh}

Taking derivatives, we have, for $z_l > a_m$ and $z_h < a_m$,

$$\phi'(z_l) = -f(z_l)\frac{\displaystyle\int_{z_l}^{a_m}[\overline{F}(u) - \overline{F}(a_m)]du}{[\overline{F}(z_l) - \overline{F}(a_m)]^2} < 0$$

$$\psi'(z_h) = f(z_h)\frac{\displaystyle\int_{a_m}^{z_h}[\overline{F}(a_m) - \overline{F}(u)]du}{[\overline{F}(a_m) - \overline{F}(z_h)]^2} > 0.$$

Hence, $\phi(\cdot)$ is non-increasing and $\psi(\cdot)$ is non-decreasing.

We now apply the last case of Theorem 2. Since supplier i bids (w_i, v_i) and not (w_j, v_j), we have $\phi(z_{il}) \leq \psi(z_{ih})$. Similarly, for j, $\phi(z_{jl}) \geq \psi(z_{jh})$. $z_{il} \leq z_{jh} < a_m < z_{ih} \leq z_{jh}$ yields $\phi(z_{il}) \geq \phi(z_{jl}) \geq \psi(z_{jh}) \geq \psi(z_{ih})$, and hence $\phi(z_{il}) \leq \psi(z_{ih})$ implies that all inequalities are in fact equalities. Therefore $c_i = c_j$ which is a contradiction. ■

Theorem 3

Proof. Consider supplier $1 < i \leq N$. From Propositions 3 and 4, we know that at equilibrium it will be bidding in region $A_{i-1\ i+1}$ because otherwise one of the suppliers would be inactive or they would not be sorted in the correct order. Let $l = i - 1$ and $h = i + 1$. Supplier i will in particular bid in the border of this region, with $y_{i-} = a_m$ or $y_{i+} = a_m$ as established in Theorem 2. $y_{i-} = a_m$ is equivalent to saying that it is bidding $w_i = w_{i+1}$ and $v_i = v_{i+1}$ and $\overline{F}(y_{i+}) = \dfrac{f_i - v_{i+1}}{w_{i+1} - c_i}$. In this case, applying Proposition 2 yields that (v_i, w_i) belongs in the segment $[(c_i, f_i); (c_{i+1}, f_{i+1})]$. Similarly, $y_{i+} = a_m$ implies that (v_i, w_i) belongs in the segment $[(c_{i-1}, f_{i-1}); (c_i, f_i)]$, and this is of course possible only if $i > 1$. For $i = 1$, only the first case can occur, i.e., $w_1 = w_2$, $v_1 = v_2$, $y_{1-} = 0$ and y_{1+} such that $\overline{F}(y_{1+}) = \dfrac{f_1 - v_2}{w_2 - c_1}$. Again, Proposition 2 implies that (v_1, w_1) belongs in the segment $[(c_1, f_1); (c_2, f_2)]$. ■

Theorem 4

In the following proofs, let, for each demand distribution, for $x \le y$,

$$L(x,y) = \frac{\int_x^y [\overline{F}(x) - \overline{F}(u)]du}{\overline{F}(x) - \overline{F}(y)}, \tag{33}$$

and

$$R(x,y) = \frac{\int_x^y [\overline{F}(u) - \overline{F}(y)]du}{\overline{F}(x) - \overline{F}(y)} = (y-x) - L(x,y). \tag{34}$$

Observe that $L(x,y)$ is non-increasing in x and $R(x,y)$ is non-decreasing in y.

Proof. The loss in surplus occurs for every supplier i when $(w_i, v_i) = (w_{i-1}, v_{i-1})$ and $(w_i, v_i) \neq (w_{i+1}, v_{i+1})$. For all other cases, we have that $y_i = y_i^*$. We have two different possible cases.

(A) The market allocation is such that $y_i < y_i^*$.

(B) The market allocation is such that $y_i > y_i^*$.

In case (A), Equation (16) holds since $(w_i, v_i) = (w_{i-1}, v_{i-1})$. Therefore, using $z_l = y_{i-1}^*$, $a_m = y_i$ and $z_h \ge y_i^*$, and the notation of Equations (33) and (34),

$$L(y_{i-1}^*, y_i) \ge R(y_i, z_h).$$

Since the function $R(x,y)$ is non-decreasing in y and $z_h \ge y_i^*$, and $L(x,y)$ is non-increasing in x and $0 \le y_{i-1}^*$,

$$R(y_i, z_h) \ge R(y_i, y_i^*) \text{ and } L(0, y_i) \ge L(y_{i-1}^*, y_i).$$

Thus,

$$L(0, y_i) \ge R(y_i, y_i^*).$$

Examine now the loss created by supplier i.

$$\begin{aligned}
\int_0^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du = & \int_0^{y_i} [\overline{F}(u) - \overline{F}(y_i^*)]du + \int_{y_i}^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du \\
\ge & \Big(\overline{F}(y_i) - \overline{F}(y_i^*)\Big) y_i + \Big(\overline{F}(y_i) - \overline{F}(y_i^*)\Big) R(y_i, y_i^*) \\
\ge & \Big(\overline{F}(y_i) - \overline{F}(y_i^*)\Big)\Big(L(0, y_i) + R(y_i, y_i^*)\Big) \\
\ge & \Big(\overline{F}(y_i) - \overline{F}(y_i^*)\Big) 2R(y_i, y_i^*), \\
= & \; 2\int_{y_i}^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du.
\end{aligned}$$

Hence, we have that

$$\Delta c_i \int_{y_i}^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du \leq \frac{1}{2}\Delta c_i \int_0^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du.$$

In case (B), it must be that $i < N$. Since $(w_i, v_i) \neq (w_{i+1}, v_{i+1})$, Theorem 3 implies that $(w_{i+1}, v_{i+1}) = (w_{i+2}, v_{i+2})$, and this means that $w_i \leq c_i \leq c_{i+1} \leq w_{i+1} \leq c_{i+2}$, $y_{i+1} = y_{i+1}^*$ and $y_i \leq y_{i+1}^*$. We can now use Equation (17) for supplier $i+1$ in order to derive a bound on the loss. Here, $a_m = y_i$, $z_h = y_{i+1}^*$ and $z_l \leq y_i^*$,

$$R(y_i, y_{i+1}^*) \geq L(z_l, y_i).$$

This implies that

$$R(y_i, y_{i+1}^*) \geq L(y_i^*, y_i).$$

Now, note that

$$\begin{aligned}
(w_{i+1} - w_i)\overline{F}(y_i) &= (c_{i+1} - w_i)\overline{F}(z_l) + (w_{i+1} - c_{i+1})\overline{F}(y_{i+1}^*) \\
&\geq (c_{i+1} - w_i)\overline{F}(y_i^*) + (w_{i+1} - c_{i+1})\overline{F}(y_{i+1}^*),
\end{aligned}$$

where the inequality is justified by $c_{i+1} \geq w_i$ and $z_l \leq y_i^*$. This, together with $\Delta c_i \leq c_{i+1} - w_i$ and $w_{i+1} - c_{i+1} \leq \Delta c_{i+1}$, implies that

$$\begin{aligned}
\Delta c_i[\overline{F}(y_i^*) - \overline{F}(y_i)] &\leq (c_{i+1} - w_i)[\overline{F}(y_i^*) - \overline{F}(y_i)] \\
&\leq (w_{i+1} - c_{i+1})[\overline{F}(y_i) - \overline{F}(y_{i+1}^*)] \\
&\leq \Delta c_{i+1}[\overline{F}(y_i) - \overline{F}(y_{i+1}^*)].
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^{y_{i+1}^*} [\overline{F}(u) - \overline{F}(y_{i+1}^*)]du = &\int_0^{y_i} [\overline{F}(u) - \overline{F}(y_{i+1}^*)]du + \int_{y_i}^{y_{i+1}^*} [\overline{F}(u) - \overline{F}(y_{i+1}^*)]du \\
\geq &\Big(\overline{F}(y_i) - \overline{F}(y_{i+1}^*)\Big)\Big(L(y_i^*, y_i) + R(y_i, y_{i+1}^*)\Big) \\
\geq &\Big(\overline{F}(y_i) - \overline{F}(y_{i+1}^*)\Big)2L(y_i^*, y_i),
\end{aligned}$$

we have that

$$\Delta c_{i+1} \int_0^{y_{i+1}^*} [\overline{F}(u) - \overline{F}(y_{i+1}^*)]du \geq \quad \Delta c_i\Big(\overline{F}(y_i^*) - \overline{F}(y_i)\Big)2L(y_i^*, y_i),$$

and hence

$$\Delta c_i \int_{y_i}^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du \quad \leq \frac{1}{2}\Delta c_{i+1} \int_0^{y_{i+1}^*} [\overline{F}(u) - \overline{F}(y_{i+1}^*)]du.$$

Since $y_{i+1} = y_{i+1}^*$, this completes the proof of the bound for any border distribution. ■

Proof of Theorem 5

Let $\mathcal{F}$ be the set of log-concave distributions. We present two lemmas which, combined, provide the proof of the theorem.

Lemma 1 *When all suppliers are efficient, then in every equilibrium of the game*

$$\Delta U \leq 25\% U^*,$$

provided that for each $0 \leq x \leq y \leq z$*, such that* $L(x,y) \geq R(y,z)$*, where the functions* L *and* R *are defined in Equations (33) and (34) respectively,*

$$\Big[\overline{F}(y) - \overline{F}(z)\Big]\Big[2L(x,y) - R(x,y)\Big] - \Big[\overline{F}(x) - \overline{F}(y)\Big]R(x,y) \leq 0 \tag{35}$$

Proof. Assume that the condition defined in Equation (35) is satisfied for all $0 \leq x \leq y \leq z$, such that $L(x,y) \geq R(y,z)$.

The loss in surplus U occurs when $y_i \neq y_i^*$. That happens when a supplier i bids $(w_i, v_i) = (w_{i-1}, v_{i-1})$ and $(w_i, v_i) \neq (w_{i+1}, v_{i+1})$. When $(w_i, v_i) \neq (w_{i-1}, v_{i-1})$ and $(w_i, v_i) = (w_{i+1}, v_{i+1})$, $y_i = y_i^*$.

For the situation when loss is created, since bidding with supplier $i-1$ yields the maximum profit for supplier i, then by Theorem 2, we have that

$$L(y_{i-1}^*, y_i) \geq R(y_i, z_h), \tag{36}$$

where z_h is defined by $\overline{F}(z_h) = \dfrac{f_i - v_{i+1}}{w_{i+1} - c_i}$. We have two different possible cases.

(A) The market allocation is such that $y_i < y_i^*$.

(B) The market allocation is such that $y_i > y_i^*$.

In case (A), since $z_h \geq y_i^*$ and $y_{i-1}^* \geq 0$, Equation (36) yields that $L(0, y_i) \geq R(y_i, y_i^*)$.

We claim that in this case (A), we have

$$\int_{y_i}^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du \leq \frac{1}{4}\int_0^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du. \tag{37}$$

This is equivalent to saying that

$$\Big[\overline{F}(y_i) - \overline{F}(y_i^*)\Big]R(y_i, y_i^*) \leq \frac{1}{4}\left\{\begin{array}{l} \Big[1 - \overline{F}(y_i)\Big]R(0, y_i) \\ +\Big[\overline{F}(y_i) - \overline{F}(y_i^*)\Big]\Big[R(0, y_i) + L(0, y_i)\Big] \\ +\Big[\overline{F}(y_i) - \overline{F}(y_i^*)\Big]R(y_i, y_i^*) \end{array}\right\},$$

or put differently,

$$\Big[\overline{F}(y_i) - \overline{F}(y_i^*)\Big]\Big[3R(y_i, y_i^*) - R(0, y_i) - L(0, y_i)\Big] \leq \Big[1 - \overline{F}(y_i)\Big]R(0, y_i).$$

Since $R(y_i, y_i^*) \leq L(0, y_i)$, to prove Equation (37), it is sufficient to show that

$$\Big[\overline{F}(y_i) - \overline{F}(y_i^*)\Big]\Big[2L(0, y_i) - R(0, y_i)\Big] \leq \Big[1 - \overline{F}(y_i)\Big]R(0, y_i),$$

which is exactly the condition that we assumed true. This concludes the proof of case (A), i.e.,

$$\int_{y_i}^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du \leq \frac{1}{4}\int_0^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du.$$

In case (B), it must be that $i < N$. Since $(w_i, v_i) \neq (w_{i+1}, v_{i+1})$, Theorem 3 implies that $(w_{i+1}, v_{i+1}) = (w_{i+2}, v_{i+2})$, and this means that $w_i \leq c_i \leq c_{i+1} \leq w_{i+1} \leq c_{i+2}$, $y_{i+1} = y_{i+1}^*$ and $y_i \leq y_{i+1}^*$. We can now use Theorem 2, for supplier $i+1$, to yield

$$R(y_i, y_{i+1}^*) \geq L(z_l, y_i) \geq L(y_i^*, y_i),$$

where z_l is defined by $\overline{F}(z_l) = \dfrac{v_i - f_{i+1}}{c_{i+1} - w_i}$, which implies that $z_l \leq y_i^*$.

We claim that when $R(y_i, y_{i+1}^*) \geq L(y_i^*, y_i)$, then

$$\Delta c_i \int_{y_i^*}^{y_i} [\overline{F}(y_i^*) - \overline{F}(u)]du \leq \frac{1}{4}\Delta c_{i+1} \int_{y_i^*}^{y_{i+1}^*} [\overline{F}(u) - \overline{F}(y_{i+1}^*)]du. \tag{38}$$

Since the right-hand side is non-decreasing in y_{i+1}^*, it is sufficient to show that when $R(y_i, y_{i+1}^*) = L(y_i^*, y_i)$, Equation (38) is satisfied.

We must first note that

$$\begin{aligned}(w_{i+1} - w_i)\overline{F}(y_i) &= (c_{i+1} - w_i)\overline{F}(z_l) + (w_{i+1} - c_{i+1})\overline{F}(y_{i+1}^*) \\ &\geq (c_{i+1} - w_i)\overline{F}(y_i^*) + (w_{i+1} - c_{i+1})\overline{F}(y_{i+1}^*),\end{aligned}$$

where the inequality is justified by $c_{i+1} \geq w_i$ and $z_l \leq y_i^*$. This, together with $\Delta c_i \leq c_{i+1} - w_i$ and $w_{i+1} - c_{i+1} \leq \Delta c_{i+1}$, implies that

$$\begin{aligned}\Delta c_i\Big[\overline{F}(y_i^*) - \overline{F}(y_i)\Big] &\leq (c_{i+1} - w_i)\Big[\overline{F}(y_i^*) - \overline{F}(y_i)\Big] \\ &\leq (w_{i+1} - c_{i+1})\Big[\overline{F}(y_i) - \overline{F}(y_{i+1}^*)\Big] \\ &\leq \Delta c_{i+1}\Big[\overline{F}(y_i) - \overline{F}(y_{i+1}^*)\Big].\end{aligned}$$

Thus, in order to prove Equation (38), it is sufficient to show that

$$\frac{\int_{y_i^*}^{y_i} [\overline{F}(y_i^*) - \overline{F}(u)]du}{[\overline{F}(y_i^*) - \overline{F}(y_i)]} \leq \frac{1}{4}\frac{\int_{y_i^*}^{y_{i+1}^*} [\overline{F}(u) - \overline{F}(y_{i+1}^*)]du}{\overline{F}(y_i) - \overline{F}(y_{i+1}^*)},$$

or equivalently,

$$\Big[\overline{F}(y_i) - \overline{F}(y_{i+1}^*)\Big]L(y_i^*, y_i) \leq \frac{1}{4}\left\{\begin{array}{l}\Big[\overline{F}(y_i) - \overline{F}(y_{i+1}^*)\Big]R(y_i, y_{i+1}^*) \\ +\Big[\overline{F}(y_i) - \overline{F}(y_{i+1}^*)\Big]\Big[L(y_i^*, y_i) + R(y_i^*, y_i)\Big] \\ +\Big[\overline{F}(y_i^*) - \overline{F}(y_i)\Big]R(y_i^*, y_i)\end{array}\right\}.$$

Using that $L(y_i^*, y_i) = R(y_i, y_{i+1}^*)$, it is sufficient to show that

$$\Big[\overline{F}(y_i) - \overline{F}(y_{i+1}^*)\Big]\Big[2L(y_i^*, y_i) - R(y_i^*, y_i)\Big] \leq \Big[\overline{F}(y_i^*) - \overline{F}(y_i)\Big]R(y_i^*, y_i).$$

Again, using the condition defined in Equation (35), this is non-positive. This implies that in case (B), for all y_i^*, y_i, y_{i+1}^* such that $R(y_i, y_{i+1}^*) \geq L(y_i^*, y_i)$,

$$\Delta c_i \int_{y_i}^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du \leq \frac{1}{4}\Delta c_{i+1} \int_0^{y_{i+1}^*} [\overline{F}(u) - \overline{F}(y_{i+1}^*)]du.$$

Finally, putting together cases (A) and (B), we have

$$\begin{aligned} \Delta U = & \sum_{i=1}^{N} \Delta c_i \int_{y_i}^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du \leq \frac{1}{4}\sum_{i=1}^{N} \Delta c_i \int_0^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du \\ \leq & \frac{1}{4}U^*. \end{aligned}$$

■

Lemma 2 *When f is log-concave, then for each $0 \leq x \leq y \leq z$, such that $L(x, y) \geq R(y, z)$,*

$$\Big[\overline{F}(y) - \overline{F}(z)\Big]\Big[2L(x, y) - R(x, y)\Big] - \Big[\overline{F}(x) - \overline{F}(y)\Big]R(x, y) \leq 0.$$

Proof. Let's examine the worst-case scenario. For a fixed x, we claim that

$$\begin{aligned} &\sup_{x \leq y \leq z, F \in \mathcal{F}} \left\{ \begin{array}{c} \Big[\overline{F}(y) - \overline{F}(z)\Big]\Big[2L(x, y) - R(x, y)\Big] \\ -\Big[\overline{F}(x) - \overline{F}(y)\Big]R(x, y) \end{array} \right\} \leq 0 \\ &\text{s.t. } L(x, y) \leq R(y, z) \end{aligned} \tag{39}$$

Clearly, we only need to examine the case when $2L(x, y) \geq R(x, y)$. The objective is then maximized for the largest z feasible, given y and F. This implies that at the maximum, $L(x, y) = R(y, z)$, since $R(y, z)$ is non-decreasing in z.

Let's now examine the worst-case scenario in terms of distribution. We first need to define the following subclass of log-concave distributions.

Definition 5 *A distribution is truncated exponential on $I \subset \mathbb{R}_+$ if and only if there are β, K, a, b, $a < b$, such that for $t \in I$,*

$$f(t) = Ke^{\beta t}\mathbf{1}_{[a,b]}(t).$$

Without loss of generality, we can assume that $x = 0$, since for any other $x \geq 0$ we could prove the lemma with a shifted distribution.

Claim 1 *In the problem posed by Equation (39), given optimal $x = 0, y, z$, we claim that at the optimum f must be truncated exponential in $[y, z]$, with a rate equal to $f'(y)/f(y)$.*

Proof. We have two possible cases. Either $\overline{F}(z) = 0$ or not. If $\overline{F}(z) = 0$, $\overline{F}(y) > 0$, otherwise there is nothing to show. Assume that f is not truncated exponential. Define the distribution equal to f on $[0, y]$ and to the truncated exponential

$$g_\gamma(t) = f(y)e^{f'(y)/f(y)(t-y)}\mathbf{1}[y, \gamma]$$

on $[y, \infty)$. Since f is log-concave, then $f(t) \leq g_\gamma(t)$ for $y \leq t \leq \gamma$.

Define $\overline{G_\gamma}$ such that

$$\overline{G_\gamma}(t) = \overline{F}(y) - \int_y^t g_\gamma(u)du = \int_y^z f(u)du - \int_y^t g_\gamma(u)du$$

This is clearly increasing in γ. We have $\overline{G_y}(z) > 0$ and $\overline{G_z}(z) < 0$. We can thus find γ such that $\overline{G_\gamma}(z) = 0$, and hence for this particular γ,

$$\overline{F}(y) - \overline{F}(z) = \int_y^z g_\gamma(u)du.$$

Moreover, $L(0, y) = R(y, z) < \dfrac{\int_y^z [\overline{G_\gamma}(u) - \overline{G_\gamma}(z)]du}{\overline{G_\gamma}(y) - \overline{G_\gamma}(z)}$. This implies that for the log-concave distribution g_γ, we can decrease z to z' with $g_\gamma(z') > 0$, while still satisfying the feasibility constraint, thus increasing $\overline{F}(y) - \overline{F}(z)$ to a larger quantity $\overline{G_\gamma}(y) - \overline{G_\gamma}(z')$. Thus f cannot be the worst-case distribution. The only remaining possibility is that f is truncated exponential on $[y, \infty)$, with rate $f'(y)/f(y)$.

Finally, if $\overline{F}(z) > 0$, $f(z) > 0$ and $\overline{F}(y) > 0$. Assume that f is not exponential. Define, for $f'(y)/f(y) \geq \gamma \geq f'(z)/f(z)$, the distribution equal to f on $[0, y]$ and $[z, \infty)$, and to

$$g_\gamma(t) = \min\left\{f(y)e^{f'(y)/f(y)(t-y)}, f(z)e^{\gamma(t-z)}\right\}$$

on $[y, z]$. This is clearly log-concave. Fix γ such that

$$\overline{F}(y) - \overline{F}(z) = \int_y^z g_\gamma(u)du.$$

This implies that

$$\overline{G_\gamma}(t) = \overline{F}(z) + \int_t^z g_\gamma(u)du$$

is always greater than $\overline{F}(t)$. Thus $L(0, y) = R(y, z) \leq \dfrac{\int_y^z [\overline{G_\gamma}(u) - \overline{G_\gamma}(z)]du}{\overline{G_\gamma}(y) - \overline{G_\gamma}(z)}$. Hence, for the log-concave distribution g_γ, we can decrease z while still satisfying the feasibility constraint, thus increasing $\overline{F}(y) - \overline{F}(z)$ to a larger quantity. Thus f cannot be the worst-case distribution. The only possibility is that f is exponential, with rate $f'(y)/f(y)$.

In any case, we have showed that for the worst case distribution must be truncated exponential in $[y, z]$. ■

Claim 2 *In the problem posed by Equation (39), given optimal x, y, z, we claim that at the optimum f must be truncated exponential in $[x, y]$.*

Proof. Equation (39) can be rewritten as

$$\sup_{x \leq y \leq z, F \in \mathcal{F}} \quad \left\{ \begin{array}{c} \left[\overline{F}(y) - \overline{F}(z)\right] 2R(y,z) \\ -\left[\overline{F}(x) - \overline{F}(z)\right] R(x,y) \end{array} \right\} \leq 0$$

$$\text{s.t. } L(x,y) \leq R(y,z)$$

The proof is similar to the proof of the previous claim.

We have two cases to address: either $f(x) = 0$ or not. When $f(x) = 0$, assume that f is not truncated exponential. Define the distribution equal to f on $[y, \infty)$ and to the truncated exponential

$$g_\gamma(t) = f(y) e^{f'(y)/f(y)(t-y)} \mathbf{1}[\gamma, y]$$

on $[0, y]$. Since f is log-concave, then $f(t) \leq g_\gamma(t)$ for $\gamma \leq t \leq y$.

Define $\overline{G_\gamma}$ such that

$$\overline{G_\gamma}(t) = \overline{F}(y) + \int_t^y g_\gamma(u) du.$$

This is clearly decreasing in γ. We have $\overline{G_y}(x) < 1$ and $\overline{G_x}(x) > 1$. We can thus find γ such that $\overline{G_\gamma}(x) = 1$, and hence for this particular γ,

$$1 - \overline{F}(y) = \overline{F}(x) - \overline{F}(y) = \int_x^y g_\gamma(u) du.$$

Moreover, $R(y,z) = L(x,y) > \dfrac{\int_x^y [\overline{G_\gamma}(x) - \overline{G_\gamma}(u)] du}{\overline{G_\gamma}(x) - \overline{G_\gamma}(y)}$. This implies that for the log-concave distribution g_γ, we can increase x to x' with $g_\gamma(x') > 0$, while still satisfying the feasibility constraint, thus decreasing $\left[\overline{F}(x) - \overline{F}(z)\right] R(x,y)$ to a smaller quantity. This is true because $\overline{F}(x) = 1$ goes down to $\overline{G_\gamma}(x')$ and $R(x,y) = (y-x) - L(x,y) = (y-x) - R(y,z)$ goes down as well. Thus f cannot be the worst-case distribution. The only remaining possibility is that f is truncated exponential on $[x, y]$.

The last case to consider is that $f(x) > 0$. Assume that f is not exponential. Define, for $f'(x)/f(x) \geq \gamma \geq f'(y)/f(y)$, the distribution equal to f on $[0, x]$ and $[y, \infty)$, and to

$$g_\gamma(t) = \min\left\{ f(y) e^{f'(y)/f(y)(t-y)}, f(x) e^{\gamma(t-x)} \right\}$$

on $[x, y]$. This is clearly log-concave. Fix γ such that

$$\overline{F}(y) - \overline{F}(z) = \int_x^y g_\gamma(u) du.$$

This implies that

$$\overline{G_\gamma}(t) = \overline{F}(y) + \int_t^y g_\gamma(u) du$$

is always greater than $\overline{F}(t)$. Thus $R(y,z) = L(x,y) > \dfrac{\int_x^y [\overline{G_\gamma}(x) - \overline{G_\gamma}(u)]du}{\overline{G_\gamma}(x) - \overline{G_\gamma}(y)}$. Hence, for the log-concave distribution g_γ, we can increase x while still satisfying the feasibility constraint, thus decreasing $\Big[\overline{F}(x)-\overline{F}(z)\Big]R(x,y)$ to a smaller quantity. Thus f cannot be the worst-case distribution. The only possibility is that f is exponential.

In any case, we have showed that for the worst case distribution must be truncated exponential in $[x, y]$. ■

Having proved these two claims, we are ready to complete the proof. The worst-case is obtained for a truncated exponential distribution. We have three different cases to address:

(i) The rate is negative, i.e. $f(t) = Ke^{-\beta t}\mathbf{1}_{[a,b]}(t)$ for some parameters K, a, b, β with $a < b$ and $\beta > 0$.

(ii) The rate is positive, i.e. $f(t) = Ke^{\beta t}\mathbf{1}_{[a,b]}(t)$ for some parameters K, a, b, β with $a < b$ and $\beta > 0$.

(iii) The rate is zero, in which case the distribution is uniform.

We will start with the analysis of case (i). Hence, assume that $f(t) = Ke^{-\beta t}\mathbf{1}_{[a,b]}(t)$ for $a < b$ and $\beta > 0$. It is clear that for $x \le a < y < b$, $L(x, y) = L(a, y)$ and that for $a < x < b \le y$, $R(x, y) = R(x, b)$. Thus, we can without loss of generality consider the case where $a = 0 \le y_i \le y_i^* \le b$.

For this distribution, for all $a \le x \le y \le b$,

$$L(x,y) = \frac{(y-x)e^{-\beta x}}{e^{-\beta x} - e^{-\beta y}} - \frac{1}{\beta},$$

and

$$R(x,y) = \frac{1}{\beta} - \frac{(y-x)e^{-\beta y}}{e^{-\beta x} - e^{-\beta y}}.$$

Define the following functions

$$P_1(t) = \frac{e^t - 1}{t} \text{ and } P_2(t) = \frac{e^t - 1 - t}{t^2}.$$

It is easy to show that these are analytical functions on $\mathbb{R}$, infinitely differentiable, increasing and convex. Using this notation, we can express

$$L(x,y) = \frac{1}{\beta}\Big[\frac{1}{P_1(-\beta(y-x))} - 1\Big],$$

and

$$R(x,y) = \frac{1}{\beta}\Big[1 - \frac{1}{P_1(\beta(y-x))}\Big],$$

By writing $\Delta_1 = \beta(y-x)$ and $\Delta_2 = \beta(z-y)$, the constraint $L(x, y) = R(y, z)$ thus becomes

$$\frac{1}{P_1(-\Delta_1)} + \frac{1}{P_1(\Delta_2)} = 2,$$

On the other hand, the objective becomes

$$\begin{aligned}
&\Big[\overline{F}(y) - \overline{F}(z)\Big] 2R(y,z) - \Big[\overline{F}(x) - \overline{F}(z)\Big] R(x,y) \\
= \ & \frac{e^{-\beta y}}{\beta^2} \left[\begin{array}{l} (1 - e^{-\Delta_2}) \left(\dfrac{2}{P_1(-\Delta_1)} + \dfrac{1}{P_1(\Delta_1)} - 3 \right) \\ - (e^{\Delta_1} - 1) \left(1 - \dfrac{1}{P_1(\Delta_1)} \right) \end{array} \right]
\end{aligned}$$

We need to show that for $\Delta_1, \Delta_2 \geq 0$ satisfying the constraint, we have

$$\frac{(1 - e^{-\Delta_2})}{(e^{\Delta_1} - 1)} \left(\frac{2}{P_1(-\Delta_1)} + \frac{1}{P_1(\Delta_1)} - 3 \right) - \left(1 - \frac{1}{P_1(\Delta_1)} \right) \leq 0.$$

Notice first that since $1/P_1$ is convex, we have that $\Delta_2 \geq \Delta_1$. Note also that

$$1 - \frac{1}{P_1(\Delta_2)} = \frac{\Delta_2 P_2(\Delta_2)}{P_1(\Delta_2)} = \frac{(1 - e^{-\Delta_2}) P_2(\Delta_2)}{P_1(-\Delta_2) P_1(\Delta_2)},$$

and

$$\frac{1}{P_1(-\Delta_1)} - 1 = \frac{\Delta_1 P_2(-\Delta_1)}{P_1(-\Delta_1)} = \frac{(e^{\Delta_1} - 1) P_2(-\Delta_1)}{P_1(-\Delta_1) P_1(\Delta_1)}.$$

Finally, we remark that for all t,

$$\frac{P_2(t)}{P_1(-t) P_1(t)} = \frac{P_2(t)}{P_2(-t) + P_2(t)},$$

which is an increasing function, because P_2 is increasing.

The constraint on Δ_1, Δ_2, together with $\Delta_2 \geq \Delta_1$, thus implies that

$$\frac{(1 - e^{-\Delta_2}) P_2(\Delta_1)}{P_1(-\Delta_1) P_1(\Delta_1)} \leq \frac{(1 - e^{-\Delta_2}) P_2(\Delta_2)}{P_1(-\Delta_2) P_1(\Delta_2)} = \frac{(e^{\Delta_1} - 1) P_2(-\Delta_1)}{P_1(-\Delta_1) P_1(\Delta_1)}.$$

Thus $(1 - e^{-\Delta_2})/(e^{\Delta_1} - 1) \leq P_2(-\Delta_1)/P_2(\Delta_1)$. Hence it is sufficient to show that for all $t \geq 0$,

$$\frac{P_2(-t)}{P_2(t)} \Big(\frac{2}{P_1(-t)} + \frac{1}{P_1(t)} - 3 \Big) \leq 1 - \frac{1}{P_1(t)},$$

or equivalently, using that $P_1(-t) = e^{-t} P_1(t)$, $e^t = 1 + t P_1(t)$ and $P_1(t) = 1 + t P_2(t)$,

$$t^4 P_2(-t) \Big(2P_1(t) - 3P_2(t) \Big) \leq t^4 P_2(t)^2. \tag{40}$$

Since

$$\begin{aligned}
t^4 P_2(t)^2 = e^{2t} - 2e^t - 2te^t + 1 + 2t + t^2 &= \sum_{k=4}^{\infty} \frac{t^k}{k!} \Big[2^k - 2 - 2k \Big], \\
t^4 P_2(-t) P_1(t) = t(-e^t - e^{-t} + te^t + 2 - t) &= \sum_{k=4}^{\infty} \frac{t^k}{k!} \Big[k(k-1) - k(1 - (-1)^k) \Big], \\
t^4 P_2(-t) P_2(t) = -e^t - e^{-t} + te^t - te^{-t} + 2 - t^2 &= \sum_{k=4}^{\infty} \frac{t^k}{k!} \Big[(k-1)(1 + (-1)^k) \Big],
\end{aligned}$$

we have,

$$
\begin{aligned}
& -t^4 P_2(-t)\left(2P_1(t) - 3P_2(t)\right) + t^4 P_2(t)^2 \\
= \; & \sum_{k=4}^{\infty} \frac{t^k}{k!}\Big[2^k - 2 - 2k + 3(k-1)(1+(-1)^k) - 2k(k-1) + 2k(1-(-1)^k)\Big].
\end{aligned}
$$

The term under brackets is always non-negative for $k \geq 4$. Indeed, $1+(-1)^k \geq 0$ and $1-(-1)^k \geq 0$ for all k, and $2^k - 2 - 2k - 2k(k-1) = 2^k - 2 - 2k^2 \geq 0$ for $k \geq 7$. Moreover,

$$
2^k - 2 - 2k^2 + 3(k-1)(1+(-1)^k) + 2k(1-(-1)^k) = \begin{cases} 0 & \text{for } k = 4, \\ 0 & \text{for } k = 5, \\ 20 & \text{for } k = 6. \end{cases}
$$

This shows that Equation (40) is satisfied for $t \geq 0$. Thus, when F is a truncated exponential with negative rate, i.e. case (i),

$$
\begin{aligned}
& \sup_{x \leq y \leq z} \left\{ \begin{array}{c} \Big[\overline{F}(y) - \overline{F}(z)\Big]\Big[2L(x,y) - R(x,y)\Big] \\ -\Big[\overline{F}(x) - \overline{F}(y)\Big]R(x,y) \end{array} \right\} \leq 0 \\
& \text{s.t. } L(x,y) \geq R(y,z)
\end{aligned}
$$

Case (ii) can be analyzed similarly. In this case, using the same notation, $\Delta_1 = \beta(y-x)$ and $\Delta_2 = \beta(z-y)$, where β is now the positive rate of the exponential, the constraint is tight and hence equivalent to

$$
\frac{1}{P_1(\Delta_1)} + \frac{1}{P_1(-\Delta_2)} = 2,
$$

We must show now that

$$
\frac{(e^{\Delta_2} - 1)}{(1 - e^{-\Delta_1})}\Big(2P_1(-\Delta_1) - 3P_2(-\Delta_1)\Big) \leq P_2(-\Delta_1).
$$

Now, $\Delta_2 \leq \Delta_1$ and this implies that $(e^{\Delta_2} - 1)/(1 - e^{-\Delta_1}) \leq P_2(\Delta_1)/P_2(-\Delta_1)$. Hence, we must show that for all $t \geq 0$,

$$
P_2(t)\Big(2P_1(-t) - 3P_2(-t)\Big) \leq P_2(-t)^2,
$$

or equivalently, using that $P_2(-t)e^t = P_1(t) - P_2(t)$,

$$
t^4 e^t P_2(t)\Big(3P_2(t) - P_1(t)\Big) \leq t^4\Big(P_1(t) - P_2(t)\Big)^2.
$$

Since

$$t^4 P_2(t)^2 = \sum_{k=4}^{\infty} \frac{t^k}{k!}\Big[2^k - 2 - 2k\Big],$$

$$t^4 e^t P_2(t)^2 = \sum_{k=4}^{\infty} \frac{t^k}{k!}\Big[3^k - 2^{k+1} - k2^k + 1 + 2k + k(k-1)\Big],$$

$$t^4 P_2(t)P_1(t) = \sum_{k=4}^{\infty} \frac{t^k}{k!}\Big[k2^{k-1} - 2k - k(k-1)\Big],$$

$$t^4 e^t P_2(t)P_1(t) = \sum_{k=4}^{\infty} \frac{t^k}{k!}\Big[k3^{k-1} - k2^k - k(k-1)2^{k-2} + k + k(k-1)\Big],$$

$$t^4 P_1(t)^2 = \sum_{k=4}^{\infty} \frac{t^k}{k!}\Big[k(k-1)2^{k-2} - 2k(k-1)\Big],$$

we must show that

$$\sum_{k=4}^{\infty} \frac{t^k}{k!} \left[\begin{array}{l} k(k-1)\cdot 2^{k-2} - 2k(k-1) \\ -2k\cdot 2^{k-1} + 4k + 2k(k-1) \\ +2^k - 2 - 2k \\ +k\cdot 3^{k-1} - k\cdot 2^k - k(k-1)\cdot 2^{k-2} + k + k(k-1) \\ -3^{k+1} + 3\cdot 2^{k+1} + 3k\cdot 2^k - 3 - 6k - 3k(k-1) \end{array} \right] \geq 0.$$

The coefficients in the brackets are equal to

$$(k-9)3^{k-1} + (k+7)2^k + (-2k^2 - k - 5)$$

They are clearly non-negative for $k \geq 9$. For smaller values, we have

$$(k-9)3^{k-1} + (k+7)2^k + (-2k^2 - k - 5) = \begin{cases} 0 & \text{for } k = 4, \\ 0 & \text{for } k = 5, \\ 20 & \text{for } k = 6, \\ 224 & \text{for } k = 7 \\ 1512 & \text{for } k = 8. \end{cases}$$

Hence, for all $t \geq 0$,

$$t^4 e^t P_2(t)\Big(3P_2(t) - P_1(t)\Big) \leq t^4\Big(P_1(t) - P_2(t)\Big)^2,$$

and thus, when F is a truncated exponential with positive rate, i.e. case (ii),

$$\sup_{x \leq y \leq z} \left\{ \begin{array}{c} \Big[\overline{F}(y) - \overline{F}(z)\Big]\Big[2L(x,y) - R(x,y)\Big] \\ -\Big[\overline{F}(x) - \overline{F}(y)\Big]R(x,y) \end{array} \right\} \leq 0$$

$$\text{s.t. } L(x,y) \leq R(y,z)$$

Case (iii) is straightforward. When the distribution is uniform, $L(x,y) = R(x,y) = (y-x)/2$ for all $x \leq y$. Also the condition $L(x,y) \geq R(y,z)$ is equivalent to $z - y \leq y - x$. Thus,

$$\Big[\overline{F}(y) - \overline{F}(z)\Big]\Big[2L(x,y) - R(x,y)\Big] - \Big[\overline{F}(x) - \overline{F}(y)\Big]R(x,y) = \frac{1}{2}(y-x)(z+x-2y) \leq 0.$$

■

Theorem 6

Proof. Supplier i is active in this equilibrium. Since the demand follows a border distribution, supplier i bids in the boundary of some region A_{lh} constructed with all the bids except i's. If the bid (w_i, v_i) belongs to more than one region, choose A_{lh} with l and h active. We must consider two cases, either there is no supplier in the lower envelope between the bids of l and h, or there is one.

In the first case, there is j, j being l or h, such that j is active, and $(w_i, v_i) = (w_j, v_j)$, from Theorem 2. From Proposition 2, (w_i, v_i) belongs in the segment $[(c_i, f_i); (c_j, f_j)]$.

In the second case, there is one supplier, k on the lower envelope between l and h such that the bid (w_i, v_i) is in the border of A_{lh} and A_{lk} or A_{lh} and A_{kh}. k is thus inactive because of supplier i, and either the bids of l, k and i are aligned, or those of i, k and h. Such a situation is depicted in Figure 8. Hence, we find j, j being l or h, active, such that (w_i, v_i) is equal to $(w_j, v_j) + \theta(w_k - w_j, v_k - v_j)$ for some non-negative θ. ∎

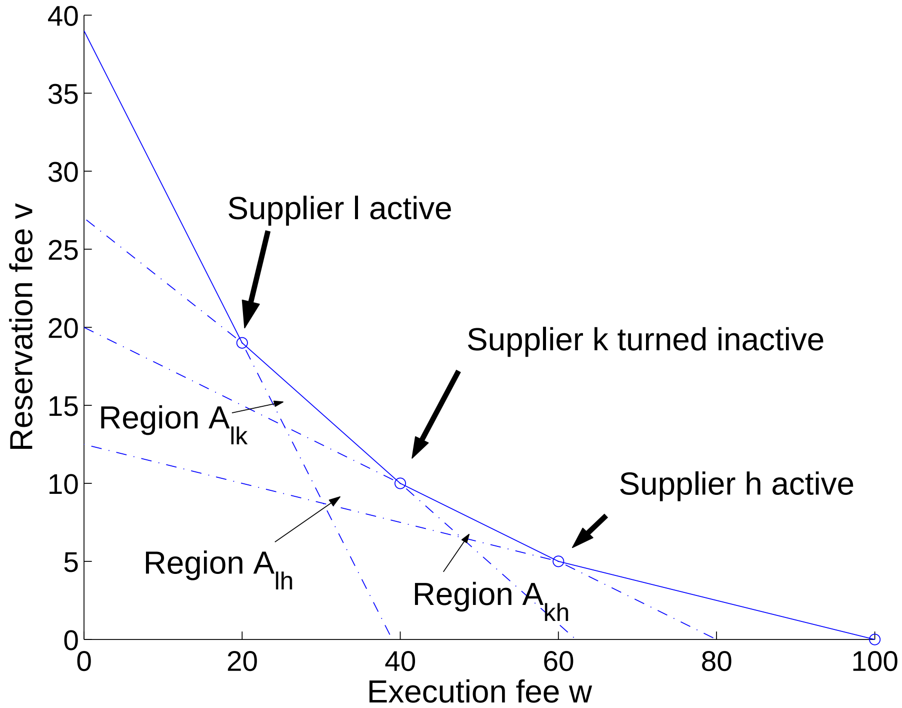

Figure 8: *Suppliers l and h are active and supplier k is turned inactive by supplier i's bid.*