A VARIATIONAL METHOD
FOR
APPROXIMATE SOLUTIONS TO LAMINAR FLOW PROBLEMS

by

JOHN MITCHELL REYNOLDS, III

B.S., Yale University (1951)

S.M., Massachusetts Institute of Technology (1957)

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

January, 1961

Signature of Author......................................
Department of Mechanical Engineering
January 26, 1961

Certified by......................................
Thesis Supervisor

Accepted by......................................
Chairman, Departmental Committee on Graduate Students

ABSTRACT

A VARIATIONAL METHOD FOR APPROXIMATE SOLUTIONS TO LAMINAR FLOW PROBLEMS

by

John Mitchell Reynolds, III

Submitted to the Department of Mechanical Engineering on January 26, 1961 in partial fulfillment of the requirements for the Degree of Doctor of Science.

The variational formulation of the Navier-Stokes and Energy equations is developed. It is found that, in the variational form, the physical velocities and temperature must be considered simultaneously with certain auxiliary variables which are not directly identified with the physical problems. The auxiliary variables are identified through the Euler equations and boundary integrals obtained by extremizing a Lagrange density in which the physical and auxiliary variables are mixed.

It is shown that, under certain broad restrictions, approximate solutions to problems of laminar fluid motion may be obtained through a computational procedure closely related to Galerkin's method.

One simple example and three more serious applications of the technique are presented. These three are: the Graetz problem of heat transfer from a constant temperature pipe to a laminar flow; the boundary layer over a semi-infinite flat plate; and the first approximation to the boundary layer over a flat plate.

Thesis Supervisor: Stephen H. Crandall

Title: Professor of Mechanical Engineering

ACKNOWLEDGMENTS

The financial support for this project was provided by the National Science Foundation under Grant G6573.

I am indebted to many members of the M.I.T. faculty for their support over the years spanned by this thesis. The stimulating and encouraging atmosphere which they provided is genuinely appreciated. In particular, I wish to express my thanks to Profs. R. J. Nickerson, P. Griffith, and K. R. Wadleigh.

To Prof. Stephen H. Crandall my deepest appreciation. As my teacher, he provided a compelling introduction to the potentialities of the Variational formulation and as thesis advisor, the essential ingredient of experienced intuition, without which the project could not have succeeded.

To my wife, Joan, and to Alexandra, Nancy, and Andrew, my inadequate thanks for their patience with the "iron curtain" which became an unnatural part of their lives.

BIOGRAPHY

The Author graduated from Yale University in 1951 whereupon he was employed by the Aircraft Gas Turbine Division of the General Electric Company. While with this division he spent one year in advanced engine design after which he joined the marketing section. The responsibility of this section was to wed customer requirements for advanced engines with the division's engineering and productive capacity to form a product having maximum production potential and reasonable technical risk.

In 1955 he resumed his studies at the Massachusetts Institute of Technology and was awarded an S.M. in 1957. Appointed Instructor in the Mechanical Engineering Department in 1959, he has remained in that capacity up to the present time.

TABLE OF CONTENTS

Section		Page
I	Introduction	1
II	The Variational Statement	3
II,A	The Auxiliary Variable Technique	4
II,B	The Laminar Flow Problem	12
II,C	Suggested Form of the Lagrangian	22
II,D	Comparison with the Method of Rosen and Herivel	27
II,E	The Energy Equation	28
II,F	Compressible Flow	30
II,G	Boundary Layer Flow	35
II,H	Working Form of the Lagrangian	40
III	Examples of Approximate Solutions	49
III,A	Formation of Couette Flow	49
III,A1	Galerkin's Method Applied to the Problem of the Formation of Couette Flow	62
III,B	The Incompressible Boundary Layer on a Flat Plate	64
III,C	First Order Approximation to the Boundary Layer on a Flat Plate	89
III,D	The Graetz Problem	104
Appendix	Bateman's Method	113
	Bibliography	115

I. INTRODUCTION

This investigation has been undertaken to develop a technique by which approximate solutions to the problems of fluid mechanics may be generated with a minimum of computational effort. More specifically, a computational framework is sought whereby reasonably accurate predictions of viscous fluid motions and energy transport rates may be obtained in cases for which no analytical solutions exist. Integral formulations of the problem are particularly well suited to this objective. Within the general framework of integral techniques which may be applied to the non-linear equations of motion of fluids one surmises that an algorithm based on the calculus of variations promises to yield results which are at least comparable to those derived by existing techniques.

The problem of applying Hamilton's Principle or some other statement based on the Calculus of Variations to fluid mechanics has interested a number of prominent investigators in the past and a considerable recent effort has produced a number of alternative approaches in recent years. Although a number of investigators have studied the general inviscid flow problem (refs. 1, 2, 3), the solution of the viscous flow problem is sufficiently more complicated to be treated entirely separately. The definitive work on the viscous problem may be considered as beginning with the demonstration by Helmholtz (ref. 4) of the fact that viscous dissipation is a minimum for motions of incompressible fluids in which accelerations may be neglected. Rayleigh (ref. 5) elaborated this somewhat by noting that the minimum exists even when there are appreciable accelerations provided that $\nabla^2 \vec{\omega} = 0$ where $\vec{\omega}$ is the vorticity vector. In 1929, C. B. Millikan (ref. 6) demonstrated that it is not possible to generate the Navier-Stokes equations as the extremizing conditions for an integral which contains only the fluid velocities and their derivatives. H. Bateman, noting Millikan's result succeeded in deriving the Navier-Stokes equations from a variation through the introduction of auxiliary variables (ref. 7), a method which he discussed more generally in ref. 8. Recently, H. Feschbach (ref. 9) has derived the governing equations for a number of dissipative physical phenomena by a method which is essentially the more general technique of Bateman. K. Washizu (ref. 10) has used the Feshbach technique in actually generating approximate solutions to a transient heat conduction problem.

One other recent line of development of the problem is worth noting. Herivel (ref. 11) and Rosen (ref. 12) have shown that it is possible to derive the Navier-Stokes equations from a restricted variational method in which the acceleration term is held constant during the variation. This interesting and apparently simpler approach to the problem is discussed in the body of this report. It is sufficient to note here that this method presents formidable computational difficulties when a Ritz-Rayleigh method is employed to generate approximate solutions.

II. THE VARIATIONAL STATEMENT

In spite of the fact that Millikan has demonstrated the impossibility of deriving the Navier-Stokes relations as the extremizing condition for some integral involving only the velocities and their derivatives, it is by no means certain that a useful variational statement of the problem cannot be discovered which involves other physical variables, e.g. thermodynamic properties, and possible restrictions on the variations other than the obviously necessary continuity restriction. Considerable effort has been expended in this study in an attempt to discover such a variational statement.

This effort, however, has yielded little of interest and it is the conclusion of this study that the use of the auxiliary variable technique is, apparently, the only rigorous means of generating the equations of fluid motion in a sufficiently general form to be of any real engineering use.

A. The Auxiliary Variable Technique

Suppose that it is desired to determine a definite integral over a region for which the condition that the integral be extremized (maximum, minimum, or saddle point) is that some function ϕ satisfy the equation

$$\mathcal{L}(\phi) = 0 \tag{1}$$

where $\mathcal{L}$ is some differential or integral operator. If $\phi = \phi(x_1, x_2, \cdots x_n)$ and the volume element under consideration is $dV = dx_1, dx_2 \cdots dx_n$, then it is apparent that the variational statement

$$\delta \int \psi \mathcal{L}(\phi)\, dV = 0 \tag{2}$$

will have equation (1) as an Euler equation. Here ψ is the "auxiliary variable", an arbitrary function which is not necessarily simply related to the variable of interest, ϕ .

Disregarding surface integrals, equation (2) may be written

$$\delta \iiint \{ \mathcal{L}(\phi)\, \delta\psi + \mathcal{M}(\psi)\, \delta\phi \} \, dv \tag{2a}$$

where $\mathcal{M}$ is some integral or differential operator. Therefore, it is seen that the variation of the auxiliary variable yields the desired Euler equation

$$\mathcal{L}(\phi) = 0$$

while the variation of the (physical) variable of interest yields an additional Euler equation

$$\mathcal{M}(\psi) = 0 \tag{3}$$

This technique, therefore, provides a means by which any differential equation may be considered an Euler equation resulting from the extremization of some integral. The difficulty in the general case is, of course, that a second Euler equation or set of Euler equations must also be satisfied by the auxiliary variable. For the case in which the desired equation (1) is linear, it can be shown (ref. 9) that ψ is the adjoint to ϕ and satisfies the equation and boundary conditions which are the adjoint of those satisfied by ϕ.

For linear physical problems it is therefore possible to identify the auxiliary variables as the adjoints of the physical variables. In the special case of self-adjoint operators, the physical and auxiliary variables are identical. In general, however, it is necessary to identify the auxiliary variable from the equation it must satisfy (e.g. equation (3) and the natural boundary conditions imposed by the variational process.

A simple example of the auxiliary variable technique will serve to clarify the method. Consider the case of the damped linear simple harmonic oscillator (ref. 10). The equation of motion is

$$m\ddot{x} + r\dot{x} + kx = 0 \tag{4}$$

where m , r , and k are the constants of the system.

From the above discussion, it is possible to derive equation (4) as an extremizing condition of the integral of the Lagrangian

$$L = y(m\ddot{x} + r\dot{x} + kx) \tag{5}$$

where y is an arbitrary auxiliary variable.

However, since it is desired to identify the auxiliary variable as closely as possible with the physical variable, a Lagrangian which is symmetrical in the two variables is desirable. Consequently, a more desirable Lagrangian is

$$L' = m\dot{x}\dot{y} - \frac{r}{2}(y\dot{x} - x\dot{y}) - kxy \tag{6}$$

It is apparent that expressions (5) and (6) are equivalent with regard to the Euler equations generated and differ only in the boundary integrals obtained. Expression (6) is more desirable in general, although not necessarily for all specific sets of boundary conditions, since the order of the highest derivative is reduced and the expression has a desirable symmetry in the two variables. The Euler equations generated from the Lagrangian (6) are

$$m\ddot{x} + r\dot{x} + kx = 0 \tag{4}$$

$$m\ddot{y} - r\dot{y} + ky = 0 \tag{7}$$

the boundary terms which result from the variation are

$$\left[\delta y\left(m\dot{x} + \frac{r}{2}x\right)\right]_{t_1}^{t_2} + \left[\delta x\left(m\dot{y} - \frac{r}{2}y\right)\right]_{t_1}^{t_2} \tag{8}$$

where t_1 and t_2 are the values of the independent variable t at the extremes of the interval of interest. The physical initial conditions

$$\begin{aligned} x(t_1) &= 0 \\ \dot{x}(t_1) &= 0 \end{aligned} \tag{9}$$

may be chosen to provide a specific problem for consideration. Then the expression (8) becomes

$$\left\{ \left(m\dot{x} + \frac{1}{2}x\right)\delta y + \left(m\dot{y} - \frac{1}{2}y\right)\delta x \right\}_{t=t_2} \tag{8a}$$

It is, of course, desirable that these boundary terms which were derived as a consequence of the statement

$$\delta \int_{t_1}^{t_2} L' \, dt = 0 \tag{10}$$

vanish. A convenient means of insuring that these vanish is supplied by the fact that the boundary conditions on the auxiliary (non-physical) problem have not been specified and may be chosen as

$$\begin{aligned} y(t_2) &= 0 \\ \dot{y}(t_2) &= 0 \end{aligned} \tag{11}$$

The auxiliary variable is then completely specified by equation (7) and the final conditions (11). That is, the auxiliary variable satisfies the time-reversed equation of the physical process with prescribed final conditions rather than initial conditions.

The procedure employed above to generate the Lagrangian for the harmonic oscillator may be generalized somewhat although it should be recalled that the most general statement of the auxiliary variable technique is equation (2). If attention is confined to systems described by differential equations with constant coefficients

$$\sum_{n=0}^{N} C_n \frac{\partial^n \phi}{\partial x_1^a \partial x_2^b \cdots \partial x_s^k} = 0 \tag{12}$$

where C_n

and $a + b + \cdots + k = n$

Then those terms for which n is even (including zero) may be generated from a Lagrangian of the form

$$L_{n_{even}} = -C_n \left(\frac{\partial^{n/2}\phi}{\partial x_r^A \partial x_q^B \cdots}\right)\left(\frac{\partial^{n/2}\psi}{\partial x_u^G \partial x_v^H \cdots}\right) \tag{13}$$

where ψ is the auxiliary variable and A , B , are not necessarily equal to a , b , , a fact which may be used to build symmetry into the expression. Also,

$$A + B + \cdots = G + H + \cdots = n/2$$

and r , q , u , v are indices in the set $1, 2, \cdots S$.

Those terms for which n is odd may be generated from a Lagrangian of the form

$$L_{n_{ODD}} = \frac{C_n}{2}\left\{\left(\frac{\partial^{\frac{n+1}{2}}\phi}{\partial x_s^{C}\,\partial x_t^{D}\cdots}\right)\left(\frac{\partial^{\frac{n-1}{2}}\psi}{\partial x_w^{J}\,\partial x_z^{K}\cdots}\right) - \left(\frac{\partial^{\frac{n-1}{2}}\phi}{\partial x_w^{J}\,\partial x_z^{K}\cdots}\right)\left(\frac{\partial^{\frac{n+1}{2}}\psi}{\partial x_s^{C}\,\partial x_t^{D}\cdots}\right)\right\} \tag{14}$$

where s , t , w , z are again indices contained in the original set 1 , 2 , $\cdots S$

and $C + D + \cdots = \frac{n+1}{2}$

while $J + K + \cdots = \frac{n-1}{2}$

In the more general case of variable coefficients, it is not always possible to form a symmetric Lagrangian, for example, if

$$\mathcal{L}(\phi) = \sum_{i} \frac{\partial^{m}}{\partial x_1^{a}\partial x_2^{b}\cdots\partial x_s^{t}}\left\{ P_i \; \frac{\partial^{n}\phi}{\partial x_1^{q}\partial x_2^{r}\cdots\partial x_s^{t}}\right\} \tag{15}$$

where $P_i = P_i(x_1, x_2, \ldots x_s)$

the form of the Lagrangian terms must be

$$L_i = (-)^{\frac{m+1}{2}} P_i \left(\frac{\partial^{n}\phi}{\partial x_1^{q}\partial x_2^{r}\cdots\partial x_s^{t}}\right)\left(\frac{\partial^{m}\psi}{\partial x_1^{a}\partial x_2^{b}\cdots\partial x_s^{t}}\right) \tag{16}$$

which can be made symmetric in ϕ and ψ only if $m = n$.

If these rules of formation are followed for all problems, those for which the auxiliary variable method is unnecessary will be detected from the fact that the Euler equation in the auxiliary variable is either identical to that for the physical variable or may be identified with some other relation pertinent to the physical system e.g. conservation of energy or mass.

The remainder of this report will be concerned with the application of the auxiliary variable technique to the more complicated relations governing momentum and energy transport in fluids.

B. The Laminar Flow Problem

Confining attention to incompressible flows for the present, the equation of fluid motion is

$$\frac{D\vec{V}}{Dt} + \frac{1}{\rho} \nabla p = \nu \nabla^2 \vec{V} \tag{17}$$

where the acceleration term is given by

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot (\nabla \vec{V}) = \frac{\partial \vec{V}}{\partial t} + \nabla \frac{V^2}{2} - \vec{V} \times (\nabla \times \vec{V})$$

The continuity equation for incompressible flow is

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{18}$$

The momentum equation (17) includes the nonlinear acceleration or inertia term. It is due to this term that the Navier-Stokes equations present such formidable resistance to exact solutions and, as might be anticipated, it is due to the inertia term that the variational form of the problem presents difficulties which necessitate appeal to the auxiliary variable technique. Since the momentum relation is comprised of three essentially separate terms, it is appropriate to consider them separately in devising the variational statement of the problem. Consequently, the component of the proposed Lagrangian which applies to the inertia term may be considered separately.

The familiar lack of uniqueness of the Lagrangian which generates a given Euler equation occurs in this nonlinear inertia term. That is, there is more than one Lagrangian which will yield $\frac{D\vec{v}}{Dt}$ as an Euler relation. The difference between the various Lagrangians, of course, Lies in the natural boundary conditions associated with the Euler expression. In self-adjoint problems this apparent lack of uniqueness in the Lagrangian is illusory, since in every case of interest the physical variables have a specific set of boundary conditions associated with them. When the auxiliary variable technique is employed, the system comprised of the physical and auxiliary problems is not completely defined since the boundary conditions on the auxiliary problem are arbitrary to a great extent. In general, it is desirable to attempt to cast the auxiliary problem into a form which corresponds to some physical problem or even the same physical problem as occurs in the physical system. This latter alternative would occur in self-adjoint systems. However, it is not generally possible to force the auxiliary system to correspond to a physical problem. In the general case, then, the only possibility which suggests itself is that the Lagrangian be chosen so that

a.) the complexity of the result is minimized, suggesting that the order of the highest derivative occuring in the Lagrangian be kept as low as possible, and

b.) the resulting auxiliary set of equations and boundary conditions be related as closely as possible to the physical system. This latter goal is realized by making the Lagrangian symmetrical in the real and auxiliary variables.

These goals were easily fulfilled in the simple harmonic oscillator example given above. The purpose here is to show that similar concepts are applicable to the nonlinear expression $\frac{D\vec{v}}{Dt}$.

The inertia term is written in terms of the velocity components as

$$\frac{D\vec{V}}{Dt} = \vec{i}\,(u_t + uu_x + vu_y + wu_z) + \vec{j}\,(v_t + uv_x + vv_y + wv_z) + \vec{k}\,(w_t + uw_x + vw_y + ww_z) \tag{19}$$

where subscripts denote partial differentiation with respect to the indicated variable.

Consider, for the present, the case of steady incompressible flow; since this flow exhibits the non-linearity of the more general flow. The auxiliary variable method indicates that a Lagrangian of the form

$$L = \vec{\eta} \cdot \frac{D\vec{V}}{Dt} \tag{20}$$

$$= \alpha(u_t + uu_x + vu_y + wu_z) + \beta(v_t + uv_x + wv_y + wv_z) + \gamma(w_t + uw_x + vw_y + ww_z)$$

is appropriate provided that only those functions are admitted to the problem which satisfy the continuity restriction

$$u_x + v_y + w_z = 0$$

Here $\vec{\eta} = \vec{i}\alpha + \vec{j}\beta + \vec{k}\gamma$ is the auxiliary variable corresponding to the physical velocity $\vec{V} = \vec{i}u + \vec{j}v + \vec{k}w$.

Extending the discussion of the auxiliary variable method applied to linear problems, one suspects that the symmetric Lagrangian

$$L^* = \frac{1}{2}\left\{ \vec{\eta} \cdot \frac{D\vec{V}}{Dt} - \vec{V} \cdot \frac{D\vec{\eta}}{Dt} \right\} \tag{21}$$

will also yield the desired Euler relation. Here the auxiliary variables are also required to satisfy a continuity relation, i.e.

$$\nabla \cdot \vec{\eta} = \alpha_x + \beta_y + \gamma_z = 0 \tag{22}$$

if the desured symmetry is to be preserved.

The fact that the variation

$$\delta \int L^* dx\, dy\, dz\, dt \tag{23}$$

does yield the desired Euler equations for the physical variable may be verified by applying the algorithm of the Calculus of Variations to equation (18). The resulting Euler equations are:

from variations of $\vec{\eta}$

$$\frac{D\vec{V}}{Dt} + \vec{V}(\nabla \cdot \vec{V}) = 0 \tag{24}$$

which is the desired expression if appeal is made to continuity. The Euler equation resulting from variations of $\vec{V}$ is

$$\frac{D\vec{\eta}}{Dt} + \frac{1}{2}\left[(\nabla\vec{\eta})\cdot\vec{V} - (\nabla\vec{V})\cdot\vec{\eta}\right] + \frac{\vec{\eta}}{2}(\nabla\cdot\vec{V}) = 0 \tag{25}$$

which must be satisfied by the auxiliary variable, $\vec{\eta}$, if the integral (23) is to be extremized.

It is seen, then, that the term of the Lagrangian corresponding to $D\vec{V}/Dt$ may be taken as expression (21). In addition, it should be noted that variations of either of the two components of (21) will also yield $D\vec{V}/Dt$ in the Euler equation. Consequently, there are at least three forms of the Lagrangian corresponding to $D\vec{V}/Dt$ in the Euler expression.

If the boundary integrals resulting from the indicated variations are included these are:

a) $$\delta\int\vec{\eta}\cdot\frac{D\vec{V}}{Dt}\,d\nu\,dt - \int dt\int_S(\vec{\eta}\cdot\delta\vec{V})\,\vec{V}\cdot d\vec{S} \tag{26}$$

b) $$-\delta\int\vec{V}\cdot\frac{D\vec{\eta}}{Dt}\,d\nu\,dt + \int dt\int_S(\vec{V}\cdot\delta\vec{\eta})\,\vec{V}\cdot d\vec{S} \tag{27}$$

c) $$\delta\int\frac{1}{2}\left\{\vec{\eta}\cdot\frac{D\vec{V}}{Dt} - \vec{V}\cdot\frac{D\vec{\eta}}{Dt}\right\}d\nu\,dt \tag{28}$$

$$+\frac{1}{2}\int dt\int_S\left\{\vec{V}\cdot\delta\vec{\eta} - \vec{\eta}\cdot\delta\vec{V}\right\}\vec{V}\cdot d\vec{S}$$

where $d\vec{s}$ refers to the surface element with the outward direction taken as positive.

These three forms are, of course, not exactly equivalent, but involve different boundary integrals. In each of these expressions boundary conditions need only be prescribed over those surfaces across which there exists a flow. On these surfaces, expression (a) requires that $\vec{n}$ vanish wherever $\vec{V}$ is not prescribed; expression (b) requires that be prescribed over all such surfaces; and expression (c) requires that $\vec{n}$ be prescribed over all surfaces across which there is a flow and vanish on all such surfaces on which the physical velocity is not prescribed.

The three alternative Lagrangians for the acceleration term which differ only in the natural boundary conditions imposed by the variation may each be applied to specific problems. The criterion of choice between them is simply which set of natural boundary conditions best reflects the nature of the particular problem at hand. Of the three, the third alternative (c) is symmetric in the two sets of variables and compares to the functions used by Morse and Feshbacle in their application of the auxiliary variable technique to linear dissipative systems. From a computational standpoint, however, the first and second alternatives, being simpler, are attractive, Between these two, the first (a) requires homogeneous boundary conditions for the auxiliary variable while the second (b) does not. This difference is often sufficient to determine the choice of the Lagrangian term since it is desirable to have the auxiliary problem be as closely related to the physical problem as possible.

The <u>viscous term</u> of the Navier-Stokes equations is, for flow with constant viscosity usually taken as (see, for example, ref. 13):

$$\mu \left\{ \nabla^2 \vec{V} + \frac{1}{3} \nabla (\nabla \cdot \vec{V}) \right\}$$

This may be derived from a variational statement by simply minimizing the dissipation function for the flow, e.g.

$$\delta \int \frac{1}{2} \Phi \, dv = 0 \tag{29}$$

where $\Phi = \mu \left\{ 2(u_x^2 + v_y^2 + w_z^2) + (v_x + u_y)^2 + (w_y + v_z)^2 + (u_z + w_x)^2 - \frac{2}{3}(\nabla \cdot \vec{V})^2 \right\}$

and the velocity is varied with suitable regard for continuity. For incompressible flows in two dimensions such a variation leads to the well known equation of motion for two dimensional "creeping" motion (inertia terms negligible) i.e.

$$\nabla^4 \psi = 0 \tag{30}$$

where ψ is the stream function defined by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{31}$$

This formal statement of the problem of "creeping" flow may be directly related to the Onsager (ref. 14) theory of minimum dissipation. The dissipation property of this flow has been established by Helmholtz and extended to the case for which $\nabla \times (\nabla^2 \vec{V}) = 0$ but $\frac{D\vec{V}}{Dt} \neq 0$ by Rayleigh (see ref. 15). These proofs do not involve the variational formation but rely on the positive definite character of the dissipation function.

Rayleigh's analysis may be put into the variational framework if it is noted that the following identity

$$\nabla \times (\vec{V} \times \operatorname{Curl} \vec{V}) = \mu \nabla \times (\nabla^2 \vec{V}) \tag{32}$$

results from taking the curl of the Navier-Stokes equation for incompressible flow. Rayleigh's restriction then implies that

$$\vec{V} \times \operatorname{Curl} \vec{V} = \nabla \zeta \tag{33}$$

where ς is a scalar. This being the case, it is apparent that the variational statement

$$\delta \iint \left\{ \vec{V} \cdot \nabla \phi + \frac{1}{2} \Phi \right\} dv = 0 \tag{34}$$

where $\phi = \frac{v^2}{2} - \frac{p}{\rho} + \varsigma$

will yield the equation of motion for this case as an Euler equation if admissible functions satisfy continuity.

For more general flows, however, one is forced to employ the auxiliary variable technique. It is suggested that a suitable term for generating the viscous component of the equation of motion is

$$\Phi' = \mu \Big\{ 2u_x\alpha_x + 2v_y\beta_y + 2w_z\gamma_z + (v_x + u_y)(\beta_x + \alpha_y) \tag{35}$$

$$+ (w_y + v_z)(\gamma_y + \beta_z) + (u_z + w_x)(\alpha_z + \gamma_x) \Big\}$$

for incompressible flow. This is simply a construct utilizing the form of the physical dissipation function. It is, however, symmetric in the real and auxiliary components and automatically reduces to the dissipation function for creeping flows since in that case the two Euler equations resulting from variations of $\bar{\Phi}'$ are identical and $\vec{\eta} = \vec{V}$.

One notes, however, that it is possible to determine alternative expressions for the Lagrangian of the viscous term just as it was in the case of the inertia term .

C. The Suggested Form

Keeping in mind the foregoing remarks on the lack of uniqueness of the Lagrangians for the various terms in the Navier-Stokes equations when the auxiliary variable technique is used, the following statement is suggested for the problem because of its symmetry and reducibility to known results in special cases. For incompressible flow, then

"For variations of the real and auxiliary velocities, $\vec{V}$ and $\vec{\eta}$, the condition that the following Lagrangian have an extremum is that the Navier-Stokes equations and a "similar" set in the auxiliary velocity be satisfied."

$$L = \int dx\,dy\,dz\,dt \Big\{ \tfrac{1}{2}\Big[\vec{\eta}\cdot\frac{D\vec{V}}{Dt} - \vec{V}\cdot\frac{D\vec{\eta}}{Dt}\Big] + \lambda_1 (\nabla\cdot\vec{V}) \tag{36}$$

$$+ \lambda_2 (\nabla\cdot\vec{\eta}) + \nu\Big[2u_x\alpha_x + 2v_y\beta_y + 2w_z\gamma_z + (v_x+u_y)(\beta_x+\alpha_y)$$

$$+ (u_z+w_x)(\alpha_z+\gamma_x) + (w_y+v_z)(\gamma_y+\beta_z)\Big]\Big\}$$

where $\vec{V} = \vec{i}u + \vec{j}v + \vec{k}w$

and $\vec{\pi} = \vec{i}\alpha + \vec{j}\beta + \vec{k}\gamma$

The result of setting $\delta L = 0$ is then

$$\int dx\,dy\,dz\,dt \Big\{ \delta\vec{\pi}\cdot\Big[\frac{D\vec{V}}{Dt} - \nu\nabla^2\vec{V} - \nabla\lambda_2\Big] \tag{37}$$
$$- \delta\vec{V}\cdot\Big[\frac{D\vec{\pi}}{Dt} + \frac{1}{2}(\nabla\vec{V})\cdot\vec{\pi} - \frac{1}{2}(\nabla\vec{\pi})\cdot\vec{V} + \nu\nabla^2\vec{\pi} + \nabla\lambda_1\Big]\Big\}$$
$$+ \int dx\,dy\,dz\ \frac{1}{2}\Big\{\vec{\pi}\cdot\delta\vec{V} - \vec{V}\cdot\delta\vec{\pi}\Big]_{t_1}^{t_2}$$
$$+ \int dt \int_S \Big\{\frac{1}{2}\big[\vec{\pi}\cdot\delta\vec{V} - \vec{V}\cdot\delta\vec{\pi}\big]\vec{V} + \lambda_1\delta\vec{V} + \lambda_2\delta\vec{\pi}$$
$$+ \nu\Big[\delta\vec{V}\cdot(\nabla\vec{\pi} + \vec{\pi}\nabla) + \delta\vec{\pi}(\nabla\vec{V} + \vec{V}\nabla)\Big]\Big\}\cdot d\vec{S} = 0$$

where S refers to the surface surrounding the volume in question and $\vec{S}$ is the outward normal to that surface. The initial and final times for the problem are t_1 and t_2 respectively.

Expression (36) differs from those considered previously in two respects. First, the partial time derivatives are included in the inertia terms; that is, a term

$$L_t = \frac{1}{2}\Big\{\vec{\pi}\cdot\frac{\partial\vec{V}}{\partial t} - \vec{V}\cdot\frac{\partial\vec{\pi}}{\partial t}\Big\}$$

has been included in the inertia term. In view of the preceding discussion of the auxiliary variable technique, it is seen that the form of this additional term follows directly from the discussion of the general linear problem (page 11). Second, the restriction that the functions admissible to the variation satisfy continuity has been dropped. This is accomplished through the familiar Lagrange-multiplier technique (see ref. 16) in which the arbitrary functions λ_1 and λ_2 are introduced into the Lagrangian to insure that variations of the real and auxiliary velocities satisfy continuity. In this connection, it is important to note that unless $\vec{V}$ and $\vec{n}$ are prescribed everywhere the Lagrange multipliers must be eliminated from the boundary terms or must be identified physically. The obvious identification is to let $\rho\lambda_2 = -p$ the pressure, and $\rho\lambda_1 = -\mathcal{P}$ an auxiliary "pressure." This relationship of pressure with the Lagrange multiplier enforcing incompressibility has already been noted by Sommerfeld (ref. 3).

Since the notation used above is quite compact, the nature of the Euler equations will be clearer if they are written out for two-dimensional steady incompressible flow.

For this case (employing the identification of π with p described above), the Euler equations are:

$$u u_x + v u_y + \frac{p_x}{\rho} - \nu \nabla^2 u = 0 \tag{38}$$

$$u v_x + v v_y + \frac{p_y}{\rho} - \nu \nabla^2 v = 0 \tag{39}$$

$$u \alpha_x + v \alpha_y - \frac{1}{2}\left(\alpha u_x + \beta v_x - u \alpha_x - v \beta_x\right) - \frac{\mathcal{P}_x}{\rho} + \nu \nabla^2 \alpha = 0 \tag{40}$$

$$u \beta_x + v \beta_y - \frac{1}{2}\left(\alpha u_y + \beta v_y - u \alpha_y - v \beta_y\right) - \frac{\mathcal{P}_y}{\rho} + \nu \nabla^2 \beta = 0 \tag{41}$$

It is seen that the equations in the auxiliary variable, while linear if u and v are presumed known, are very complex. These relations can be cast into the form derived by Bateman (see Appendix A) by the addition of a surface term.

In the two dimensional case, at least, a third alternative form for the auxiliary relations offers some advantage in that it is a simple matter to determine the physical conditions under which the auxiliary system corresponds to the time-reversed physical system. This would be the interpretation if the auxiliary equations are given by

$$\frac{D\vec{\pi}}{Dt} - \frac{\nabla \mathcal{P}}{\rho} + \nu \nabla^2 \vec{\pi} = 0 \tag{42}$$

This third form which differs from equations (39) and (40) by a boundary term is

$$u\alpha_x + v\alpha_y - \alpha u_x - \beta v_x - \frac{p_x}{\rho} + \nu\nabla^2\alpha = 0 \qquad (40a)$$

$$u\beta_x + v\beta_y - \alpha u_y - \beta v_y - \frac{p_y}{\rho} + \nu\nabla^2\beta = 0 \qquad (41a)$$

which can be put into the form of equation (41) if

$$\alpha u_x + \beta v_x = 0 \qquad (43)$$

$$\alpha u_y + \beta v_y = 0 \qquad (44)$$

This pair of relations has a non zero solution for α and β only if

$$u_x v_y - u_y v_x = 0 \qquad (45)$$

which is equivalent to the condition that

$$\nabla^2 p = 0 \qquad (46)$$

This may be verified by taking the divergence of the Navier-Stokes relation for the steady two dimensional incompressible flow under consideration. No such simple physical interpretation of the condition under which the auxiliary equations are the time reversed physical equations can be made for more general flows.

This restriction on the pressure is not satisfied by any but the most specialized flows. The only class of flows for which it holds generally true is for creeping flows for which, as discussed previously, the auxiliary variable technique is not necessary. However, there is one important exception, this restriction is met by the problem of the boundary layer over a semi-infinite flat plate with zero pressure gradient.

D. Comparison with the Method of Rosen and Herivel

In recent years a technique developed independently by Rosen (ref. 12) and Herivel (ref. 11) has received considerable attention. The essence of the method is to hold the inertia term $D\vec{v}/Dt$ constant during the variation. The Lagrangian for incompressible flow then takes the form

$$L = \frac{D}{Dt}\left(\frac{v^2}{2} + \frac{p}{\rho}\right) + \frac{\Phi}{2} \tag{47}$$

when Φ is again the dissipation function. This can be seen by inspection to yield the Navier-Stokes equations if $D\vec{v}/Dt$

in the first term $\left(i.e.\ \vec{V}\cdot\frac{D\vec{V}}{Dt}\right)$ is not varied. However, there are not a sufficient number of degrees of freedom present to permit the variation to be formally restricted by a Lagrange-multiplier term $\vec{\lambda}\cdot\delta\frac{D\vec{V}}{Dt}$ added to the Lagrangian. Therefore, only those functions for which $\delta\frac{D\vec{V}}{Dt}$ vanishes identically are admissible to the variation. In a computation for generating approximate solutions, such a restriction presents virtually insurmountable computational difficulties for all but the most specialized flows.

E. The Energy Equation

Since it is possible to generate any differential equation from a variational statement if the auxiliary variable technique is employed, it is possible to determine a Lagrangian whose Euler equation is the energy relation for incompressible viscous flows, i.e.

$$\rho C_v \frac{D\theta}{Dt} - k\nabla^2\theta + \Phi = 0 \tag{48}$$

where C_v is the specific heat at constant volume

k is the thermal conductivity

θ is the fluid temperature

Φ is the dissipation function

and C_v and k have been assumed to be constant.

If ς is the auxiliary variable then, for variations of ς and θ the Lagrange density

$$L = \frac{\rho C_v}{2}\left\{\varsigma \frac{D\theta}{Dt} - \theta\frac{D\varsigma}{Dt}\right\} + k(\nabla\theta\cdot\nabla\varsigma) + \bar{\Phi}(\theta+\varsigma) \tag{49}$$

yields the desired Euler equation.

The variation $\delta\int L\,dx\,dy\,dz\,dt = 0$ yields

$$\int dx\,dy\,dz\,dt\left\{\delta\varsigma\left[\rho C_v\frac{D\theta}{Dt} - k\nabla^2\theta + \bar{\Phi}\right]\right. \tag{50}$$

$$\left.-\delta\theta\left[\rho C_v\frac{D\varsigma}{Dt} + k\nabla^2\varsigma - \bar{\Phi}\right]\right\} + \int dx\,dy\,dz\,\frac{1}{2}\left\{\varsigma\delta\theta - \theta\delta\varsigma\right\}_{t_1}^{t_2}$$

$$+\int dt\int_S\left\{\frac{\rho C_v}{2}(\varsigma\delta\theta - \theta\delta\varsigma) - k\left[\delta\theta(\nabla\varsigma) + \delta\varsigma(\nabla\theta)\right]\right\}\cdot d\vec{S}$$

where the $t's$ have the same meaning as in the case of the equations of motion.

In this case, the auxiliary variable satisfies the Euler equation

$$\rho C_v\frac{D\varsigma}{Dt} + k\nabla^2\varsigma - \bar{\Phi} = 0 \tag{51}$$

This relation may be interpreted either as representing a time reversed temperature field, or as the temperature associated with a flow having negative viscosity (dissipation) and thermal conductivity (reversed heat flow at the boundaries).

F. Compressible Flow

Compressible flow is , of course, more general than has been considered above since density variations are admitted. The fact that the density is a function of the coordinates and has a variation indicates that another relation must be added to those of the incompressible case. Under this heading, the variational statement for the complete compressible, time-dependent, three dimensional flow will be derived. Discussion of the procedures will be omitted where the previous discussion of the incompressible or steady flows are applicable.

For compressible flow the continuity relation is given by

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{v}) = 0 \tag{52}$$

A suitable continuity restriction for the auxiliary system is

$$\frac{D^*\rho}{Dt} + \rho(\nabla \cdot \vec{n}) = 0 \tag{53}$$

where the fluid density in the auxiliary system is taken to be the same as that in the real system and

$$\frac{D^*\rho}{Dt} = \rho_t + u\rho_x + v\rho_y + w\rho_z \tag{54}$$

The equations of motion are given by

$$\rho\frac{Du}{Dt} = X - P_x + \mu\left\{\nabla^2 u + \frac{1}{3}(\nabla\cdot\vec{v})_x\right\} \tag{55}$$

$$\rho\frac{Dv}{Dt} = Y - P_y + \mu\left\{\nabla^2 v + \frac{1}{3}(\nabla\cdot\vec{v})_y\right\} \tag{56}$$

$$\rho\frac{Dw}{Dt} = Z - P_z + \mu\left\{\nabla^2 w + \frac{1}{3}(\nabla\cdot\vec{v})_z\right\} \tag{57}$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \tag{58}$$

A suitable Lagrangian for the system (55), (56), (57) with the continuity restrictions is (if the applied body force is given by the gradient of a scalar, i.e.

$$\begin{aligned} L = {} & \frac{\rho}{2}\left\{\vec{n}\cdot\frac{D\vec{v}}{Dt} - \vec{v}\cdot\frac{D\vec{n}}{Dt}\right\} + \lambda_1\left[\frac{D^*\rho}{Dt} + \rho(\nabla\cdot\vec{n})\right] \\ & + \lambda_2\left[\frac{D\rho}{Dt} + \rho(\nabla\cdot\vec{v})\right] + \mu\Big\{2(u_x\alpha_x + v_y\beta_y + w_z\gamma_z) \\ & + (v_x+u_y)(\beta_x+\alpha_y) + (u_y+v_z)(\gamma_y+\beta_z) + (u_z+w_x)(\alpha_z+\gamma_x) \\ & - \frac{2}{3}(u_x+v_y+w_z)(\alpha_x+\beta_y+\gamma_z)\Big\} \end{aligned} \tag{59}$$

Then the statement

$$\delta \int \mathcal{L}\, dx\, dy\, dz\, dt = 0 \tag{60}$$

has the equations of motion as three of its Euler equations if

$$\varphi \lambda_1 = \Omega - P \tag{61}$$

$$\nabla \Omega = \vec{F} = \vec{i}X + \vec{j}Y + \vec{k}Z$$

The additional Euler equations are

$$\rho \frac{D\alpha}{Dt} - \frac{\varphi}{2}\left(u\alpha_x + v\beta_x + w\gamma_x - \alpha u_x - \beta v_x - \gamma w_x\right) \tag{62}$$

$$+ \mu\left[\nabla^2 \alpha + \frac{1}{2}(\nabla \cdot \vec{n})_x\right] + (\varphi\lambda_2)_x = 0$$

$$\rho \frac{D\beta}{Dt} - \frac{\varphi}{2}\left(u\alpha_y + v\beta_y + w\gamma_y - \alpha u_y - \beta v_y - \gamma w_y\right) \tag{63}$$

$$+ \mu\left[\nabla^2 \beta + \frac{1}{2}(\nabla \cdot \vec{n})_y\right] + (\varphi\lambda_2)_y = 0 \tag{64}$$

$$\rho \frac{D\gamma}{Dt} - \frac{\varphi}{2}\left(u\alpha_z + v\beta_z + w\gamma_z - \alpha u_z - \beta v_z - \gamma w_z\right)$$

$$+ \mu\left[\nabla^2 \gamma + \frac{1}{2}(\nabla \cdot \vec{n})_z\right] + (\varphi\lambda_2)_z = 0$$

$$\frac{1}{2}\left(\vec{\eta}\cdot\frac{D\vec{v}}{Dt}-\vec{v}\cdot\frac{D\vec{\eta}}{Dt}\right)-\frac{D}{Dt}(\Lambda_1+\Lambda_2)=0 \tag{65}$$

where the last relation is a consequence of the variations of the density. This relation may be considered as a relation between the physical and auxiliary velocities. If the flow is steady $D\Lambda_1/Dt$ and $D\Lambda_2/Dt$ may be evaluated from the other six equations and equation (65) becomes

$$\vec{v}\cdot\left\{\frac{D}{Dt}(\vec{\eta}+\vec{v})+\nu\nabla^2(\vec{\eta}-\vec{v})\right\} \tag{66}$$

$$+\frac{\nu}{3}\frac{D}{Dt}\left[\nabla\cdot(\vec{\eta}-\vec{v})\right]=0$$

Therefore, in the absence of a body force and pressure gradient

since

$$\vec{v}\cdot\left\{\frac{D\vec{\eta}}{Dt}+\nu\left[\nabla^2\vec{\eta}+\frac{1}{3}\nabla(\nabla\cdot\vec{\eta})\right]\right\} \tag{67}$$

the auxiliary velocity satisfies. This relation indicates that under these conditions that component of the auxiliary equation which is parallel to the physical velocity, the "working" component, satisfies the time-reversed equation of motion.

Returning to the general problem, the integrated terms resulting from equation (60) are

$$\Pi = \int dx\,dy\,dz\,dt \left\{ \frac{\rho}{2}\left[\vec{n}\cdot\delta\vec{v} + \vec{v}\cdot\delta\vec{n}\right] + (\lambda_1 + \lambda_2)\delta\rho \right\}_{t_1}^{t_2} + \int dt \int_S \left\{ \frac{\rho}{2}\left[\vec{n}\cdot\delta\vec{v} - \vec{v}\cdot\delta\vec{n}\right]\vec{v} + \vec{v}(\lambda_1+\lambda_2)\delta\rho + \rho(\lambda_1\delta\vec{v} + \lambda_2\delta\vec{n}) + \mu\left[\delta\vec{n}\cdot(\nabla\vec{v}+\vec{v}\nabla) + \delta\vec{v}\cdot(\nabla\vec{n}+\vec{n}\nabla) - \frac{2}{3}\left[(\nabla\cdot\vec{v})\vec{n} + (\nabla\cdot\vec{n})\vec{v}\right]\right] \right\}\cdot d\vec{S} \tag{68}$$

where S is the surface of interest and $\vec{S}$ the outward normal to that surface.

The complete variatonal statement is then

$$\delta\left\{\int L_{59}\,dx\,dy\,dz\,dt - \Pi\right\} = 0 \tag{69}$$

where L_{59} is the Lagrangian for the general problem given by equation (59).

While considering the topic of compressible flows, it is appropriate to note that the use of the Galerlkin procedure (see ref. 15) for generating approximate solutions from the variational statement will, in general, be considerably less reliable than in the incompressible case since the velocity may exhibit discontinuities which cannot be approximated by a finite number of terms. For this reason, the application of the approximate technique to compressible flows will be the subject of a separate study.

G. Boundary Layer Flow

The Boundary Layer Equation (see ref. 18) may be derived as the extremum condition for the Lagrangian used to derive the Navier-Stokes equations by considering the orders of magnitude of the various terms involved.

At the outset, orders of magnitude must be assigned to the real and auxiliary variables. Based on experience in the solution of flow problems with the auxiliary variable technique, the auxiliary variables may be said to be closely related to their counterparts in the physical system. That is, α may be taken to be of the same order of magnitude as u and β of the same order of magnitude as v .

The boundary layer equation for steady, two-dimensional incompressible flow may then be derived as follows.

Let $\bar{u}$, $\bar{v}$, $\bar{\alpha}$, and $\bar{\beta}$ be the previously used dimensional variables and u , v , α , and β be the corresponding dimensionaless variables.

Following Prandtl, let

$$\alpha = \frac{\bar{\alpha}}{U}, \quad u = \frac{\bar{u}}{U}, \quad v = \frac{1}{\epsilon}\frac{\bar{v}}{U} = \frac{V}{\epsilon}, \quad \beta = \frac{1}{\epsilon}\frac{\bar{\beta}}{U} \tag{70}$$

$$x = \frac{\bar{x}}{L}, \quad y = \frac{\bar{y}}{\epsilon L}, \quad \text{and } p = \frac{\bar{p}}{\rho U^2}$$

$$\text{where } \epsilon^2 = \frac{\nu}{UL} = \frac{1}{Re} \tag{71}$$

For this discussion, it is convenient to choose the form given by equation (26) for that component of the Lagrange density which yields the inertia term in the Navier-Stokes equation. The statement for the Navier-Stokes equation in two dimensions is then

$$\delta \iint \Big\{ \bar{\alpha}(\bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}}) + \bar{\beta}(\bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}}) \tag{72}$$

$$+ \bar{\Pi}_1(\bar{\alpha}_{\bar{x}} + \bar{\beta}_{\bar{y}}) + \bar{\Pi}_2(\bar{u}_{\bar{x}} + \bar{v}_{\bar{y}}) + \nu\big[2\bar{u}_{\bar{x}}\bar{\alpha}_{\bar{x}}$$

$$+ 2\bar{v}_{\bar{y}}\bar{\beta}_{\bar{y}} + (\bar{v}_{\bar{x}} + \bar{u}_{\bar{y}})(\bar{\beta}_{\bar{x}} + \bar{\alpha}_{\bar{y}})\big]\Big\}\, dx\, dy = 0$$

which becomes for the dimensionless variables (80)

$$\delta \iint \Big\{ \alpha(u u_x + v u_y) + \alpha_y u_y + \lambda_1(\alpha_x + \beta_y) + \lambda_2(u_x + v_y)$$

$$+ \epsilon^2 \left[\beta(u v_x + v v_y) + 2 u_x \alpha_x + 2 v_y \beta_y + v_x \alpha_y + u_y \beta_x \right] \tag{73}$$

$$+ \epsilon^4 (v_x \beta_x) \Big\} \, dx\, dy = 0$$

In the boundary layer approximation the term $\epsilon = 1/R_e$ is considered small. That is the boundary layer approximation is valid at high Reynolds Numbers. Following the technique described by Kuo (ref. 17) the variables may be approximated as

$$u = u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \cdots \tag{74}$$

$$v = v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \cdots \tag{75}$$

$$\alpha = \alpha^{(0)} + \epsilon \alpha^{(1)} + \epsilon^2 \alpha^{(2)} + \cdots \tag{76}$$

$$\beta = \beta^{(0)} + \epsilon \beta^{(1)} + \epsilon^2 \beta^{(2)} + \cdots \tag{77}$$

where ϵ is used as a perturbation parameter. Inserting these expressions into equation (73) yields

$$
\begin{aligned}
\delta \iint \Big\{ & \alpha^{(0)} \left[u^{(0)} u_x^{(0)} + v^{(0)} u_y^{(0)} \right] + u_y^{(0)} \alpha_y^{(0)} \\
& + \lambda_1^{(0)} \left(\alpha_x^{(0)} + \beta_y^{(0)} \right) + \lambda_2^{(0)} \left(u_x^{(0)} + v_y^{(0)} \right) \\
& + \epsilon \Big[\alpha^{(1)} \left(u^{(0)} u_x^{(0)} + v^{(0)} u_y^{(0)} \right) + u_y^{(0)} \alpha_y^{(0)} \\
& + \lambda_1^{(0)} \left(\alpha_x^{(1)} + \beta_y^{(1)} \right) + \lambda_2^{(0)} \left(u_x^{(1)} + v_y^{(1)} \right) \\
& + \lambda_1^{(1)} \left(\alpha_x^{(0)} + \beta_y^{(0)} \right) + \lambda_2^{(1)} \left(u_x^{(0)} + v_y^{(0)} \right) \\
& + \alpha^{(0)} \left(u^{(1)} u_x^{(0)} + u^{(0)} u_x^{(1)} + v^{(0)} u_y^{(1)} + v^{(1)} u_y^{(0)} \right) + \alpha_y^{(0)} u_y^{(1)} \Big] \\
& + \epsilon^2 \Big[\qquad\qquad\qquad \Big] + \cdots \Big\} \, dx \, dy = 0
\end{aligned}
\tag{78}
$$

It can be seen that the Euler equation resulting from variations of $\alpha^{(0)}$ is

$$u^{(0)}u_x^{(0)} + v^{(0)}u_y^{(0)} - u_{yy}^{(0)} - \lambda_{1,x}^{(0)} \tag{79}$$
$$+ \epsilon\left[u^{(1)}u_x^{(0)} + u^{(0)}u_x^{(1)} + v^{(1)}u_y^{(0)} + v^{(0)}u_y^{(1)} - u_{yy}^{(1)} - \lambda_{1,x}^{(1)}\right]$$
$$+ \epsilon^2\left[\qquad\qquad\right] + \cdots = 0$$

The zero order terms constitute the boundary layer equation with the Lagrange multiplier ($\lambda_1^{(0)}$) being equated to the term $-\frac{1}{\rho}p_x^{(0)}$. The term of order ϵ is the first order boundary layer equation.

It is interesting to note that the first approximation in the physical system is the result of variations of the zero order term in the auxiliary variable. The variation of the first order auxiliary variable i.e. $\delta\alpha^{(1)}$ yields

$$\epsilon\left[u^{(0)}u_x^{(0)} + v^{(0)}u_y^{(0)} - u_{yy}^{(0)} - \lambda_{1,x}^{(0)}\right] + \epsilon^2\left[\quad\right] + \cdots = 0 \tag{80}$$

which is the zero order boundary layer equation. Consequently, it is not necessary to include the $\alpha^{(1)}$ terms in the Lagrangian designed to yield the first two approximations to the Navier-Stokes equation. In practice, the fact that the $u^{(0)}$ term may be considered without the $\alpha^{(1)}$ term is a great simplification of the computations involved in the Galerkin procedure. This fact will be illustrated in Example C.

A similar symmetry exists in the auxiliary system. That is, the zero order auxiliary equation is produced both by variations of $u^{(0)}$ in the zeroth approximation and by variations of $u^{(1)}$ in the first order approximation. This behavior of the auxiliary system is completely analogous to the occurence of the zero order physical equation in equations (79) and (80).

H. Working Form of the Lagrangian

The purpose of this study is to provide an integral technique for the approximate solution of problems in fluid mechanics through a variational statement. This is equivalent to attempting to generalize the Galerkin technique (see ref. 17) to this nonlinear problem. For linear self-adjoint systems, Galerkin's method is equivalent to the variational method and is used as a reference technique here because the specific form proposed by Galerkin is analogous to that obtained by the auxiliary variable method. From a purely mechanistic viewpoint, the Galerkin scheme is to use the individual trial functions as weighting functions to generate the approximate solutions to a given problem. For example, if the equation of motion is

$$\mathcal{L}(u) = 0 \tag{81}$$

when $\mathcal{L}$ is some differential operator and

$$u = u(x) \tag{82}$$

then the true solution may be approximated by

$$u^* = \sum_{i=1}^{N} C_i \phi_i \tag{83}$$

where the ϕ's are functions of x satisfying the essential boundary conditions.

The Galerkin method provides relations from which the constants may be evaluated as follows:

$$\int_{x_1}^{x_2} \phi_1 \mathcal{L}(u^*) dx = 0 \tag{84}$$

$$\cdots\cdots\cdots$$

$$\int_{x_1}^{x} \phi_N \mathcal{L}(u^*) dx = 0$$

One notes that this technique is equivalent to a variational method only if the problem is self-adjoint. For the general linear problem Galerkin's method must be modified to

$$\int_{x_1}^{x_2} \psi_1 \mathcal{L}(u^*)\,dx = 0, \text{ etc.} \tag{85}$$

where

$$\mathcal{L}^* = \sum_{i}^{M} c_i \psi_i \tag{86}$$

and $\mathcal{L}^*$ is the adjoint of u^* .

This statement may be generalized to include the non-linear, non-self-adjoint problems of fluid mechanics. Consider a problem to which the variational technique employing auxiliary variables has been applied. The resulting Euler equations, natural boundary conditions, and the physical boundary conditions often indicate that the form of the approximating functions must differ with regard to only one coordinate or independent variable. For such a problem the physical variable may be approximated by

$$u^*(x_1, x_2, \cdots) = \sum_i f_i(x_1)\, g_i(x_2, x_3 \cdots) \tag{87}$$

and the auxiliary variable by

$$\alpha^*(x_1, x_2 \cdots) = \sum_i h_i(x_1) \cdot g_i(x_2, x_3, \cdots) \tag{88}$$

If the g_i are specified functions, then

$$\delta u^* = \sum g_i \, \delta(f_i) \tag{89}$$

and

$$\delta \alpha^* = \sum g_i \, \delta(h_i) \tag{90}$$

and computations following the variational procedure will be of the form

$$\int g_1 \, \mathcal{L}(u^*) \, dx_2 \, dx_3 \cdots = 0 \tag{91}$$

$$\int g_2 \, \mathcal{L}(u^*) \, dx_2 \, dx_3 \cdots = 0$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\int g_N \, \mathcal{L}(u^*) \, dx_2 \, dx_3 \cdots = 0$$

from which N differential equations in the f_i will result. This extended form of the Galerkin method is illustrated in example A of this report.

The specific purpose of going through the variational procedure is to determine the weighting functions to be used in the Galerkin-type of computational procedure. In the general fluid problem it is not possible to identify the auxiliary variables with the physical variables in a simple manner. Bateman (ref. 7) has pointed out that the auxiliary equations are the adjoints of the perturbation equations for the fluid motion; a relationship which is too complex to be of real aid in identifying the weighting functions to be used for computational purposes. The precise form of the auxiliary variables to be used will therefore have to be tailored to each specific problem.

The emphasis on a physical or pseudo-physical interpretation for the auxiliary variables stems from the fact that as the Galerkin procedure is applied using successively more refined approximations to the velocities, the solution to the auxiliary systems must also be approached more closely. That is, since the Lagrangian involves both $\vec{V}$ and $\vec{\varkappa}$, if a close approximation is made to $\vec{V}$ but $\vec{n}$ is relatively far from a solution to the auxiliary problem, the integral will not be extremized. In the limit, a bona fide extremum is attained only by the exact solutions to both the physical and auxiliary problems. The degree of success attained with the variational technique is dependent on qualitative knowledge of the behavior of both the physical and auxiliary variables.

In reducing the problem to a computational procedure, one should not lose sight of the fact that the basis of the method is a variational one and relies upon the statement: "Of all the possible functions satisfying the essential boundary conditions of the problem, those which render the term $\int \mathcal{L}\, dv\, dt$ stationary will also be solutions of the Euler equation."

From a computational standpoint this requires that a complete set of functions satisfying certain boundary conditions be tested to determine what combination of these functions renders $\int \mathcal{L}\, dv\, dt$ stationary, that combination then also satisfies the Euler equation. In practice, a small number of functions are used in the approximate expression and it is necessary that the greatest care be taken to insure that it is possible for the particular functions chosen to describe the behavior of the physical variable being approximated. In other words, sufficient knowledge must be at hand so that the general behavior of the solution can be expressed by the terms used in the approximation.

The auxiliary variable which has been introduced into the present problem complicates this computational procedure for two reasons:

a.) variables in both the physical and auxiliary systems must be approximated, increasing the computational effort required to solve the problem.

b.) while it is often possible to anticipate important characteristics of the physical variables from purely physical reasoning, this aid in determining the approximating functions is not generally available for the auxiliary system.

When using the auxiliary variable technique it is, therefore important to relate the physical and auxiliary systems as closely as possible so that maximum insight may be applied to the approximations to the auxiliary variables. It is often possible to reduce the effort in applying the auxiliary variable technique to the same order of magnitude as that involved in more conventional methods. In fact, in many problems the computational form simply represents an extension of the Galerkin technique.

The question arises, "when is it possible to consider the approximate solution to be sufficiently accurate for engineering purposes?" This question has been considered often (ref. 20) with no generally applicable result. The often stated assumption is that for sufficiently regular problems an increased number of parameters in the approximation results in a more accurate approximation. For the case in which the integral can be proven to be minimized (or maximized) rather than merely stationary, the value of the integral for any given approximation yields a numerical guide to the excellence of the approximation and can therefore be used as a check on the assumption that an increase in the number of parameters improves the approximation. For the general case, it is possible, in theory at least, to compare the profiles generated with approximations of increased complexity and to consider the point at which the profile does not change appreciably with a further increase in complexity as a termination point. Quite aside from the obvious logical difficulty inherent in such an assumption, this method requires that the approximation include at least one more term than is required for an adequate description of the solution. This will, of course, significantly increase the computational effort involved.

An alternative method of evaluating the accuracy of the approximation is to use a "yardstick" which is separate from the variational formulation. For example, a least squares criterion could be used in which the value of $\int [\mathcal{E}(u)]^2 dv\, dt$ (where $\mathcal{E}(u) = 0$ is the equation of motion) would indicate the accuracy of the approximation.

III. Examples of Approximate Solutions:

A. Formation of Couette Flow

This is the simplest problem which includes any portion of the inertia terms although the non-linear terms are excluded. The governing equation is the diffusion equation , i.e.

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial y^2} = 0 \tag{A1}$$

with the boundary conditions

$$\begin{aligned} u &= 0 \quad at \quad t = 0 \\ u &= 0 \quad at \quad y = 0 \\ u &= v_0 \quad at \quad y = h \end{aligned} \tag{A2}$$

The formulation in terms of auxiliary variables may be obtained by specializing the Lagrangian for the Navier-Stokes equation.

$$L = \frac{1}{2}\left(\alpha u_t - u\alpha_t\right) + \nu\,\alpha_y u_y \tag{A3}$$

If possible, one chooses boundary conditions for the auxiliary variables which will cause the associated boundary integrals to vanish.

Taking the variation

$$\delta \iint L\,dy\,dt = \int_0^{\tau} dt \int_0^{h} dy \left\{ \delta\alpha\,(u_t - \nu u_{yy}) - \delta u\,(\alpha_t + \nu \alpha_{yy}) \right\}$$

$$+ \int_0^{h} dy \left[\tfrac{1}{2}(\alpha\,\delta u - u\,\delta\alpha) \right]_0^{\tau} + \nu \int_0^{\tau} dt \left[u_y\,\delta\alpha + \alpha_y\,\delta u \right]_0^{h} \tag{A4}$$

Considering the integrated terms which do not vanish because of the boundary conditions on u , i.e.

$$\int_0^{h} dy\,\tfrac{1}{2}\left\{ \alpha\,\delta u - u\,\delta\alpha \right\}_{t=\tau} + \nu \int_0^{\tau} dt\,\left(u_y\,\delta\alpha \right)_0^{h}$$

indicates that

$$\alpha = 0 \text{ at } t = \tau$$

$$\alpha = 0 \text{ at } y = 0 \tag{A5}$$

$$\alpha = v_0 \text{ at } y = h$$

insures the vanishing of these terms. Here α satisfies a final condition rather than an initial condition. This is in keeping with the interpretation of α as a time reversed velocity as indicated by the Euler equation in α.

Since it is desired to cast the problem into a computational form similar to that of the Galerkin method, it is desirable to choose forms for the approximations to u and α which differ in a single coordinate. A suitable approximation is

$$u = \sum_{1}^{N} \phi_n \left(\frac{y}{h}\right)^n \tag{A6}$$

where
$$\begin{aligned} &\phi_n = \phi_n(t) \\ &\sum_{1}^{N} \phi_n = V_0 \\ &\phi_N(0) = V_0 \\ &\phi_1(0) = \phi_2(0) = \cdots = \phi_{N-1}(0) = 0 \end{aligned} \tag{A7}$$

This approximation satisfies the boundary conditions at $y = 0, h$ and approximately satisfies the initial condition. The initial profile, $V_0 \left(\frac{y}{h}\right)^N$, will satisfy the initial condition exactly only for $N = \infty$. The approximation to the initial profile is, however, assumed to be adequate for large but finite values of N.

Since α has the same y dependence as u, let

$$\alpha = \sum_{1}^{N} \phi_n^* \left(\frac{y}{h} \right)^n \tag{A8}$$

where
$$\begin{aligned} \phi_n^* &= \phi_n^*(t) \\ \sum_{1}^{N} \phi_n^* &= v_0 \\ \phi_N^*(\tau) &= v_0 \\ \phi_1^*(\tau) = \phi_2^*(\tau) &= \cdots = \phi_{N-1}^*(\tau) = 0 \end{aligned} \tag{A9}$$

and, in general

$$\phi_n(t) \neq \phi_n^*(t)$$

then

$$\begin{aligned} \delta \int_0^{\tau} \int_0^h \Bigg\{ \sum_n \sum_p \Bigg[& \frac{1}{2} \left(\phi_n^* \phi_p' - \phi_p \phi_n^{*\prime} \right) \left(\frac{y}{h} \right)^{n+p} \\ & + \frac{\nu}{h^2} n p \, \phi_n^* \phi_p \left(\frac{y}{h} \right)^{n+p-2} \Bigg] \\ & + \lambda_a \left[\sum_{1}^{N} \phi_n^* - v_0 \right] + \lambda_b \left[\sum_{1}^{N} \phi_n - v_0 \right] \Bigg\} \, dy \, dt = 0 \end{aligned} \tag{A10}$$

or

$$\delta \int_0^{\tau} \int_0^{h} \Big\{ \sum_n \sum_p \Big[\phi_n^* \phi_p' \left(\frac{y}{h}\right)^{n+p} + \frac{\nu}{h^2} np \phi_n^* \phi_p \left(\frac{y}{h}\right)^{n+p-2} \tag{A11}$$

$$+ \lambda_a \Big[\sum \phi_n^* - v_0 \Big] + \lambda_b \Big[\sum \phi_n - v_0 \Big] \Big\} dt\, dy$$

$$+ \delta \int_0^{h} \frac{1}{2} \Big[\sum_n \sum_p \phi_p \phi_n^* \left(\frac{y}{h}\right)^{n+p} \Big]_0^{\tau} dy = 0$$

but $\phi_p \left(\frac{y}{h}\right)^p \simeq 0$ at $t = 0$

and $\phi_n^* \left(\frac{y}{h}\right)^n \simeq 0$ at $t = \tau$

Therefore, after performing the y integration these results

$$\delta \int_0^{\tau} dt \Big\{ \sum_n \sum_p \phi_n^* \left(\frac{\phi_p'}{n+p+1} + \frac{\nu}{h^2} \frac{np\phi_p}{n+p-1} \right) \tag{A12}$$

$$+ \lambda_a \Big[\sum \phi_n^* - v_0 \Big] + \lambda_b \Big[\sum \phi_p - v_0 \Big] \Big\} = 0$$

Now, since $\delta\phi_n^*$ is arbitrary, the Euler equations of interest are

$$\left\{ y_a + \sum_p \left(\frac{\phi_p'}{n+p+1} + \frac{\nu}{h^2}\phi_p \frac{np}{n+p-1} \right) \right\}_n = 0 \tag{A13}$$

The values of n and p to be chosen merit some discussion. In the first place, it is apparent that the initial condition of $u = 0$ everywhere but at the point $y = L$ where $u = v_0$ can be satisfied only approximately. If one assumes a three term approximation, i.e.

$$u = \phi_1 \frac{y}{h} + \phi_2 \left(\frac{y}{h}\right)^2 + \phi_m \left(\frac{y}{h}\right)^m \tag{A14}$$

then if $\phi_1(0) = \phi_2(0) = 0$ and $\phi_m(0) = v_0$ the initial and boundary conditions will be satisfied in the limit as $m \to \infty$. In practice, m is chosen so as to contribute something to the profile at time t. The closer the t of interest is to zero, the larger the m that is appropriate. Let us try two cases $m = 3$ and $m = 8$ for comparison.

The Euler equations are, then, three ordinary differential equations in

$$\phi_1(t),\ \phi_2(t),\ \text{and}\ \phi_m(t) \tag{A15}$$

for $n=1$: $\lambda_a + \frac{\phi_1'}{3} + \frac{\phi_2'}{4} + \frac{\phi_m'}{m+2} + \frac{\nu}{h^2}\left(\phi_1 + \phi_2 + \phi_3\right) = 0$

for $n=2$: $\lambda_a + \frac{\phi_1'}{4} + \frac{\phi_2'}{5} + \frac{\phi_m'}{m+3} + \frac{\nu}{h^2}\left(\phi_1 + \frac{4}{3}\phi_2 + \frac{2m}{m+1}\phi_m\right) = 0$

for $n=m$: $\lambda_a + \frac{\phi_1'}{m+2} + \frac{\phi_2'}{m+3} + \frac{\phi_m'}{2m+1} + \frac{\nu}{h^2}\left(\phi_1 + \frac{2m}{m+1}\phi_2 + \frac{m^2}{2m-1}\phi_m\right) = 0$

λ_a may be eliminated from these yielding

$$\frac{\phi_1'}{12} + \frac{\phi_2'}{20} + \frac{\phi_m'}{(m+2)(m+3)} - \frac{\nu}{h^2}\left(\frac{\phi_2}{3} + \frac{m-1}{m+1}\phi_m\right) = 0 \tag{A16}$$

$$\frac{m-2}{4(m+2)}\phi_1' + \frac{m-2}{5(m+3)}\phi_2' + \frac{m-2}{(m+3)(2m+1)}\phi_m'$$

$$- \frac{\nu}{h^2}\left[\frac{2(m-2)}{3(m+1)}\phi_2 + \frac{m^3-3m^2+2m}{(m+1)(2m-1)}\phi_m\right] = 0$$

but, from the boundary conditions

$$\phi_1 = V_0 - \phi_2 - \phi_m \tag{A17}$$

$$\therefore \quad \frac{\phi_2'}{30} + \phi_m'\left[\frac{1}{12} - \frac{1}{(m+2)(m+3)}\right] + \frac{\nu}{h^2}\left[\frac{\phi_2}{3} + \frac{m-1}{m+1}\phi_m\right] = 0$$

$$\phi_2' \frac{(m+7)(m-2)}{20(m+2)(m+3)} + \phi_m' \frac{(m-2)(2m^2+3m-5)}{4(m+2)(m+3)(2m+1)}$$

$$+ \frac{\nu}{h^2}\left[\frac{2(m-2)}{3(m+1)} + \frac{m^3-3m^2+2m}{(m+1)(2m-1)}\right] = 0$$

For $n = 3$, these are

$$\left(\frac{D}{30} + \frac{\nu}{3h^2}\right)\phi_2 + \left(\frac{D}{20} + \frac{\nu}{2h^2}\right)\phi_3 = 0 \tag{A18}$$

$$\left(\frac{D}{60} + \frac{\nu}{6h^2}\right)\phi_2 + \left(\frac{11}{420}D + \frac{3\nu}{10h^2}\right)\phi_3 = 0$$

where D is the operator $\frac{d}{dt}$.

Then $$\left(\frac{D}{420} + \frac{1}{10}\frac{\nu}{h^2}\right)\phi_3 = 0$$

and $$\phi_3 = c\,e^{-42\frac{\nu}{h^2}t}$$

but since $\phi_3(0) = v_0$

$$\phi_3 = v_0 e^{-42\frac{\nu}{h^2}t} \tag{A19}$$

the relations may then be solved for ϕ_2 where $\phi_2(0) = 0$

$$\phi_2 = \frac{3}{2} v_0\left[e^{-10\frac{\nu}{h^2}t} - e^{-42\frac{\nu}{h^2}t}\right] \tag{A20}$$

and, since $\phi_1 = v_0 - \phi_2 - \phi_3$

$$\phi_1 = v_0\left[1 - \frac{3}{2} e^{-10\frac{\nu}{h^2}t} + \frac{1}{2} e^{-42\frac{\nu}{h^2}t}\right] \tag{A21}$$

and finally

$$\frac{u}{v_0} = \frac{y}{h} - \frac{3}{2}\frac{y}{h}\left(1 - \frac{y}{h}\right) e^{-10\frac{\nu}{h^2}t} + \frac{y}{2h}\left(1 - 3\frac{y}{h} + 2\frac{y^2}{h^2}\right) e^{-42\frac{\nu}{h^2}t} \tag{A22}$$

for the case of $m = 3$.

For $m = 8$ the solution of equations (A17) is

$$\frac{u}{U_0} = \frac{y}{h} - \left(2.35\frac{y}{h} - 2.46\frac{y^2}{h^2} + 0.11\frac{y^8}{h^8}\right)e^{-9.95\frac{\nu}{h^2}t} \tag{A23}$$

$$+\left(1.35\frac{y}{h} - 2.46\frac{y^2}{h^2} + 1.11\frac{y^8}{h^8}\right)e^{-54.35\frac{\nu}{h^2}t}$$

The analytical solution to the same problem is given in ref. 21 as

$$\frac{u}{U_0} = \frac{y}{h} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-\nu n^2\pi^2 t/h^2}\sin\frac{n\pi y}{h} \tag{A24}$$

These three solutions are compared in figures 1 and 2 for different values of $\frac{\nu t}{h^2}$. From figure 1, it is seen that $m = 3$ does not yield satisfactory results for $\frac{\nu t}{h^2} = 0.1$ while the $m = 8$ solution lies within ten percent of the analytical result. Figure 2 compares the approximations at $\frac{\nu t}{h^2} = .05$. At this value of time $m = 8$ is no longer adequate but deviates from the analytical result by as much as thirty percent. Moreover, the $m = 8$ profile assumes negative values near the stationary wall indicating that the two-parameter system cannot follow a physically reasonable profile at the shorter time. Improvement of the approximate result in the small time range must come from an increased number of parameters and an increase in the value of m. The $m = 8$ approximation does, however, yield a value of the time constant of the starting motion which is in excellent agreement with the analytical result. That is, for large t the approximation predicts a time variation $e^{-9.95\frac{\nu}{h^2}t}$ while the exact solution gives $e^{-9.87\frac{\nu}{h^2}t}$.

From inspection of equation (A10), it is seen that the corresponding solutions in the auxiliary variable will be exactly the same as those for the physical velocity if t is replaced by $\tau - t$. Therefore, for $m = 3$

$$\frac{\alpha}{U_0} = \frac{y}{h} - \frac{3}{2}\frac{y}{h}\left(1-\frac{y}{h}\right)e^{-10\frac{\nu}{h^2}(\tau-t)} + \frac{y}{2h}\left(1-3\frac{y}{h}+2\frac{y^2}{h^2}\right)e^{-42\frac{\nu}{h^2}(\tau-t)} \qquad \text{(A25)}$$

as would be expected since α has been previously identified as the time-reversed velocity.

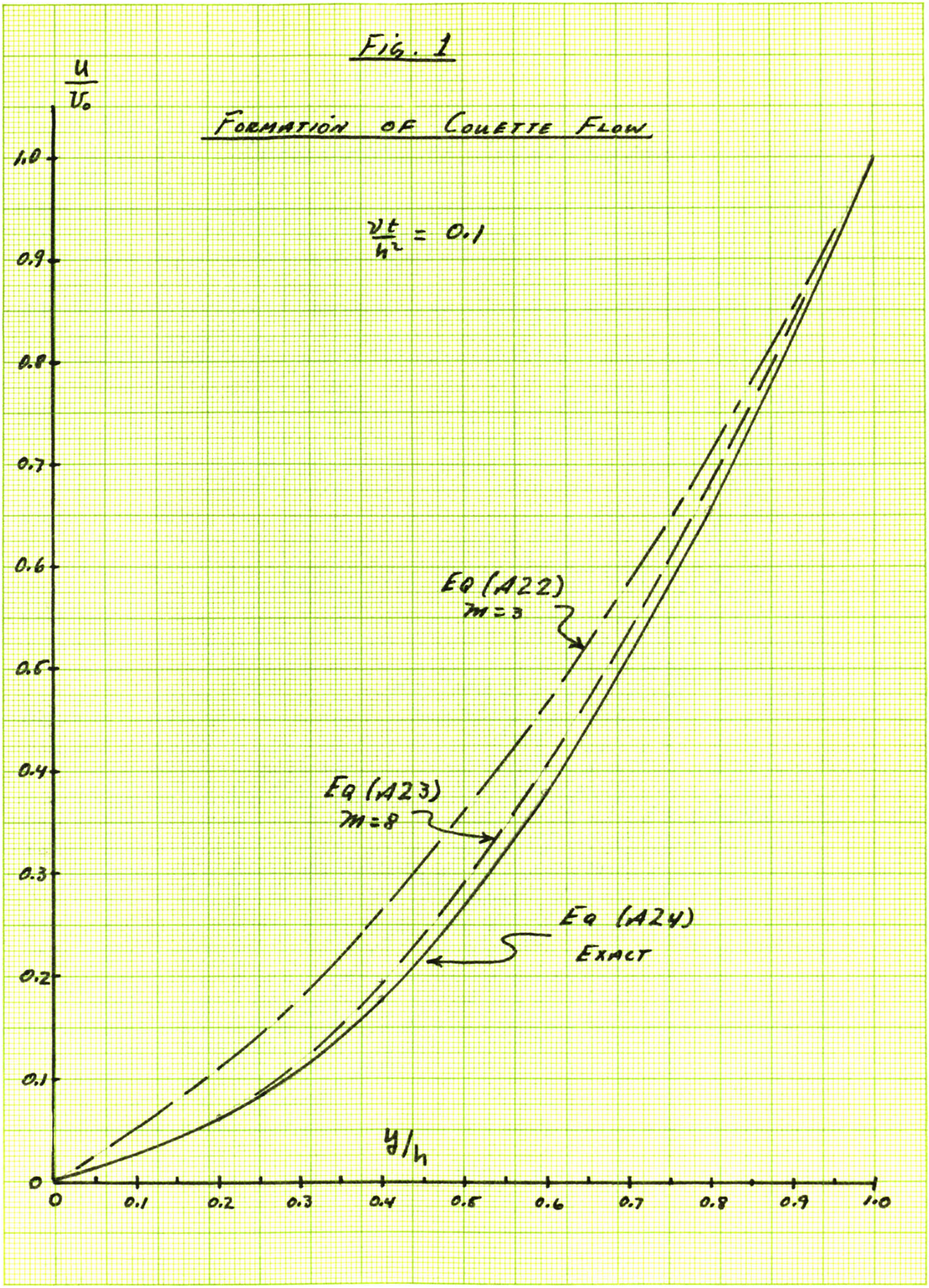
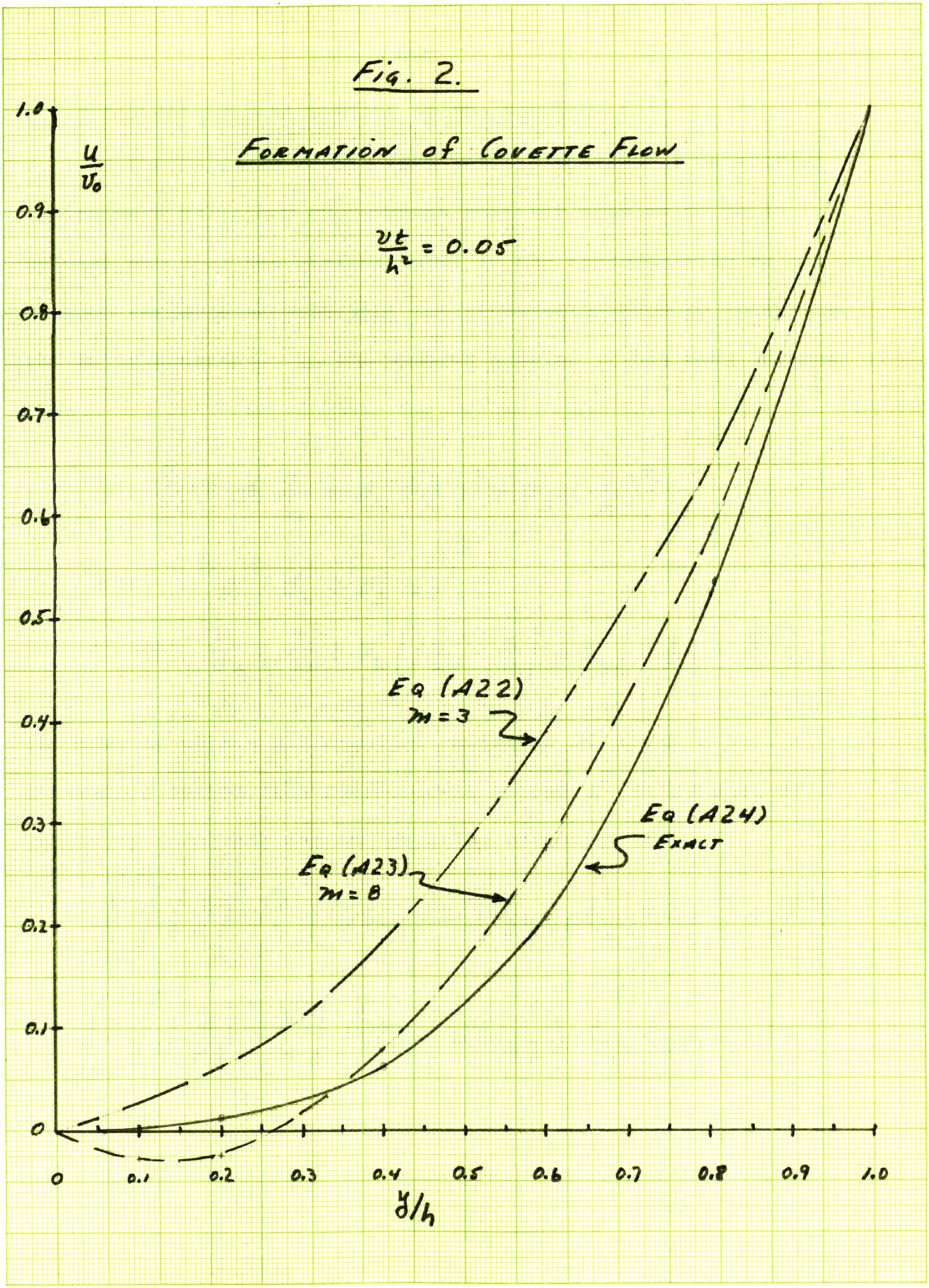


Fig. 1
Formation of Couette Flow
u/U₀
νt/h² = 0.1
Eq (A22)
m = 3
Eq (A23)
m = 8
Eq (A24)
Exact
y/h
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Fig. 2.


Formation of Couette Flow
$\frac{u}{U_0}$
$\frac{\nu t}{h^2} = 0.05$
Eq (A22)
m = 3
Eq (A23)
m = 8
Eq (A24)
Exact
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
y/h

1. Galerkin's Method Applied to the Problem

In the Galerkin method (see ref. 17 or 22) for generating approximate solutions one assumes that the problem is self-adjoint and uses the various terms in the approximation to the unknown variable as weighting functions with which to integrate the Euler equation. In the present problem it has been assumed that

$$\frac{u}{U_0} = \phi_1 \frac{y}{h} + \phi_2 \left(\frac{y}{h}\right)^2 + \phi_3 \left(\frac{y}{h}\right)^3$$

where $\phi_1 + \phi_2 + \phi_3 = 1$

so that $\frac{u}{U_0} = \frac{y}{h} + \phi_2\left[\left(\frac{y}{h}\right)^2 - \frac{y}{h}\right] + \phi_3\left[\left(\frac{y}{h}\right)^3 - \frac{y}{h}\right]$

in order to obtain differential relations in the ϕ's by Galerkin's method, the coefficients of the ϕ's are used as weighting functions for the equation of motion and

$$\int_0^h \left[\left(\frac{y}{h}\right)^2 - \frac{y}{h}\right] \left(u_t - \nu u_{yy}\right) dy = 0 \qquad \text{(A26)}$$

$$\int_0^h \left[\left(\frac{y}{h}\right)^3 - \frac{y}{h}\right] \left(u_t - \nu u_{yy}\right) dy = 0$$

An identical form can be obtained directly from the variational method. If only the solution for the physical velocity is considered, expression (A4) is

$$\int_0^h \int_0^\tau \left\{ \delta\alpha (u_t - \nu u_{yy}) \right\} dt\, dy = 0 \qquad \text{(A27)}$$

since the boundary integrals vanish for the approximations (A6) and (A8). In the three term $m = 3$ approximation

$$\phi_1^* = v_0 - \phi_2^* - \phi_3^*$$

Therefore,

$$\begin{aligned} \delta\alpha &= \delta\phi_1^* \left(\frac{y}{h}\right) + \delta\phi_2^* \left(\frac{y}{h}\right)^2 + \delta\phi_3^* \left(\frac{y}{h}\right)^3 \\ &= \delta\phi_2^* \left[\left(\frac{y}{h}\right)^2 - \frac{y}{h}\right] + \delta\phi_3^* \left[\left(\frac{y}{h}\right)^3 - \frac{y}{h}\right] \end{aligned} \qquad \text{(A28)}$$

and (A27) becomes

$$\int_0^\tau \delta\phi_2^* \left\{ \int_0^h \left[\left(\frac{y}{h}\right)^2 - \frac{y}{h}\right] (u_t - \nu u_{yy})\, dy \right\} dt \qquad \text{(A29)}$$

$$+ \int_0^\tau \delta\phi_3^* \left\{ \int_0^h \left[\left(\frac{y}{h}\right)^3 - \frac{y}{h}\right] (u_t - \nu u_{yy})\, dy \right\} dt$$

Since $\delta\phi_2^*$ and $\delta\phi_3^*$ are independent arbitrary functions, the Galerkin statement (A26) must be true.

B. The Incompressible Boundary Layer on a Flat Plate

This classic problem of Boundary Layer Theory (see ref. 18) is a popular testing area for approximate techniques. The simplest integral method for approximating the solution of this problem consists of approximating the velocity profile by a polynomial in terms of the variable y/Δ, where y is the height from the surface of the plate and $\Delta(x)$ is the boundary layer thickness and determining Δ by requiring the boundary layer equation to be satisfied on the average over the region of y/Δ from zero to unity. For example, the velocity profile may be approximated by

$$u = \sum_{1}^{N} a_n \left(\frac{y}{\Delta}\right)^n \tag{B1}$$

where the a_n are all constants determined by the boundary conditions on the problem. The function Δ is then determined by requiring that

$$\int_0^{\Delta} (u u_x + v u_y - \nu u_{yy})\, dy = 0 \tag{B2}$$

where v is determined from the stream function associated with u.

The method requires that sufficient boundary conditions be "invented" to provide relations among the a_n so that a different problem in the mathematical sense is being solved each time the number of coefficients is increased. This technique will be referred to as the "conventional" integral method hereafter.

Recently, there have appeared two papers (refs. 23 and 24) using a variational method to generate approximate solutions for this problem. In the first it was assumed that the principle of least dissipation is applicable to the problem and the coefficients in (B1) were determined by the variational technique. The boundary layer thickness, however, was not determined. In the second paper, the method of Herivel and Rosen (refs. 11 and 12) was applied in place of equation (B2) of the conventional method. That is, the coefficients of the approximation (B1) were determined by the boundary conditions as in the conventional method, but the boundary layer.thickness was determined by the Herivel-Rosen method rather than by (B2).

In none of these techniques are the boundary conditions held fixed and both the remaining coefficients of the approximation and the boundary layer thickness computed from a single scheme. The method discussed below will determine both the constants and the boundary layer thickness using a single set of boundary conditions.

As noted previously (p.35ff) the variational expression for the Navier-Stokes equations may be modified to yield the Boundary Layer equation through an argument based on the orders of magnitude of the various terms involved. The statement to the first order is (see equation 83) for the present problem

$$\delta \int_0^x \int_0^\Delta dx\,dy \left\{ \alpha(u u_x + v u_y) + \frac{\nu}{V_0} u_y \alpha_y + \lambda(u_x + v_y) \right\} = 0 \tag{B3}$$

where the velocities have been normalized with respect to the free-stream velocity, V_0 , but the coordinates are dimensional.

It is apparent from inspection of (B3) that the computational effort will be reduced if the Lagrange multiplier enforcing the continuity restriction is avoided. Consequently, it is advantageous to insist that the admissible functions satisfy continuity. In that case (B3) may be written in terms of the stream function as,

$$\delta\int_0^x\int_0^\Delta dx\,dy\left\{\alpha\left(\psi_y\psi_{xy}-\psi_x\psi_{yy}\right)+\frac{\nu}{\nu_0}\,\alpha_y\,\psi_{yy}\right\}=0 \tag{B4}$$

Before taking this variation, it is worth noting that at this point it is important to devise a form which will minimize the computational effort involved in determining the approximate solution. The result of the variation will be an Euler equation which is the boundary layer equation and another in the auxiliary variable, α . In addition, certain boundary terms involving both the real and auxiliary variables will result. The goal here is to choose the boundary conditions on the auxiliary variable in such a way that the real and auxiliary variables differ in their dependence on only one of the two coordinates involved as in Example A above. This cannot be accomplished in general; however, if such an interpretation is consistent with the auxiliary Euler equation and the natural boundary conditions, it may be done.

The variation of (B4) yields

$$
\begin{aligned}
&\int_0^{X}\!\!\int_0^{\Delta} dx\,dy \Big\{ \delta\alpha \left(\psi_y \psi_{xy} - \psi_x \psi_{yy} - \tfrac{\nu}{U_0}\, \psi_{yyy} \right) \\
&+ \delta\psi \Big[\tfrac{\partial}{\partial y} \left(u\alpha_x + v\alpha_y + \tfrac{\nu}{U_0}\, \alpha_{yy} \right) + \alpha_x \psi_{yy} \\
&- \alpha_y \psi_{xy} \Big] \Big\} + \int_0^{\Delta} dy \Big\{ \alpha \left(\psi_y \delta\psi_y - \psi_{yy}\, \delta\psi \right) \Big\}_0^{X} \\
&+ \int_0^{X} dx \Big\{ \delta\psi \left(\alpha \psi_{xy} + \alpha_y \psi_x - \alpha_x \psi_y - \tfrac{\nu}{U_0}\, \alpha_{yy} \right) \\
&\quad + \delta\psi_y \left(\tfrac{\nu}{U_0}\, \alpha_y - \alpha \psi_x \right) \Big\}_0^{\Delta} = 0
\end{aligned}
\qquad \text{(B5)}
$$

where X is some distance downstream of the leading edge at which the problem is terminated and $\Delta = \Delta(x)$ is the boundary layer thickness.

Considering the x integral first, the boundary conditions

$$
\begin{aligned}
u = v = 0 \quad &\text{at } y = 0 \\
u = U_0 \quad &\text{at } y = \Delta
\end{aligned}
\qquad \text{(B6)}
$$

insures that $\delta \psi_y$ vanishes at both limits and that $\delta \psi$ vanishes at the lower limit. There remains only the coefficient of $\delta \psi$ to be evaluated at the upper limit, that is the term

$$\left(\alpha \psi_{xy} + \alpha_y \psi_x - \alpha_x \psi_y - \frac{\nu}{v_0} \alpha_{yy} \right)_{y=\Delta} \tag{B7}$$

The physical nature of the problem indicates that

$$\begin{gathered} u_x = \psi_{xy} = 0 \text{ at } y = \Delta \\ u_y = 0 \text{ at } y = \Delta \end{gathered} \tag{B8}$$

the second and third terms of (B7) indicate that

$$\begin{gathered} \alpha_y = 0 \text{ at } y = \Delta \\ \alpha_x = 0 \text{ at } y = \Delta \end{gathered} \tag{B9}$$

are appropriate boundary conditions to impose on α .

The final term of (B7) will be allowed to remain non-vanishing for the present.

Considering the y integral of (B7), the boundary condition

$$\Delta = 0 \text{ at } x = 0 \tag{B10}$$

indicates that

$$\psi = u = 0 \text{ at } x = 0 \tag{B11}$$

Consequently, there remains only the integral at $x = X$. Here it is impossible to make any statement regarding the behavior of the physical variable and it is appropriate to introduce the auxiliary boundary condition

$$\alpha = 0 \text{ at } x = X \tag{B12}$$

Finally, it is noted that the Euler equations resulting from the statement (B5) are of such a nature as to permit the identical y dependence of the physical and auxiliary variables so that

$$\alpha = 0 \text{ at } y = 0$$
$$\alpha = v_0 \text{ at } y = \Delta \tag{B13}$$

Relations (B9), (B12), and (B13) constitute a set essential boundary conditions on the auxiliary variable α.

There remains one point to consider. Since it is desired to vary Δ as well as the coefficients of the approximations to the real and auxiliary variables, the upper limit of the y integral is to be varied. According to the theory of the Calculus of Variations (see ref. 25), the statements made above remain valid under this variation provided that the boundary term

$$\int_0^X dx \left\{ \delta\Delta \left[\alpha \left(u u_x + v u_y - \frac{\nu}{v_0} u_{yy} \right) \right] \right\}_0^\Delta \tag{B14}$$

is added. Equation B14 has not been written in terms of the stream function to facilitate its interpretation and use has been made of the fact that

$$\int \alpha_y u_y \, dx \, dy = - \int \alpha \, u_{yy} \, dx \, dy \tag{B15}$$

which in turn depends on the fact that

$$\int_0^X dy \left[\alpha_y u \right]_0^\Delta = 0 \tag{B16}$$

in view of boundary conditions (B6) and (B9).

Because of the physical boundary conditions (B6) and (B8), it is seen that the term (B14) reduces to

$$-\int_0^X dy \left[\frac{\nu}{U_0} \alpha u_{yy} \delta\Delta \right]_0^\Delta \tag{B17}$$

The variational statement then becomes

$$\delta \int_0^X \int_0^\Delta dx dy \left\{ \alpha (u u_x + v u_y - \frac{\nu}{U_0} u_{yy}) \right\} \tag{B18}$$

$$= \frac{\nu}{U_0} \int_0^X dy \left\{ \left[\alpha u_{yy} \delta\Delta \right]_0^\Delta + \left[\alpha_{yy} \delta\psi \right]_{y=\Delta} \right\}$$

where admissible functions satisfy continuity and the boundary conditions listed previously and variations of Δ are permissible.

The approximation

$$\frac{u}{U_0} = \sum_1^N a_n \sin \frac{n\pi y}{2\Delta} \qquad (n \text{ odd}) \tag{B19}$$

where the a_n are constants and

$$\sum_{1}^{N} (-)^{\frac{n-1}{2}} a_n = 1 \tag{B20}$$

satisfies the physical boundary conditions and qualitatively provides a good approximation to the anticipated form of the solution. Since the auxiliary variable satisfies conditions at the y boundaries which are identical to those satisfied by the real velocity, an appropriate assumption for the auxiliary variable is

$$\frac{\alpha}{v_0} = \sum_{1}^{M} \phi_n \sin\frac{n\pi y}{2\Delta} \qquad (n \text{ odd}) \tag{B21}$$

where the ϕ_n are functions of x only and

$$\sum_{1}^{M} (-)^{\frac{n-1}{2}} \phi_n = 1 \tag{B22}$$

Substituting these values into the integral to be varied in (B18) yields

$$\int_0^x\int_0^\Delta \Big\{ \sum_n \sum_p \sum_q a_n a_p \phi_q \frac{p\Delta'}{n\Delta} \cos\frac{p\pi y}{2\Delta} \sin\frac{q\pi y}{2\Delta} \left(\cos\frac{n\pi y}{2\Delta} - 1\right) \tag{B23}$$

$$+ \frac{\nu}{v_0} \sum_n \sum_q a_n \phi_q \left(\frac{n\pi}{2\Delta}\right)^2 \sin\frac{n\pi y}{2\Delta} \sin\frac{q\pi y}{2\Delta} \Big\} dx\,dy =$$

(B23) cont.

$$= \int_0^x \int_0^\Delta \Big\{ \sum_n \sum_p \sum_q a_n a_p \phi_q \left(\frac{p\Delta'}{n\pi}\right) \Big[\sin (q+p+n) \frac{\pi y}{2\Delta}$$

$$+ \sin (q+p-n) \frac{\pi y}{2\Delta} + \sin (q-p+n) \frac{\pi y}{2\Delta} + \sin (q-p-n) \frac{\pi y}{2\Delta}$$

$$- 2 \sin (q+p) \frac{\pi y}{2\Delta} - 2 \sin (q-p) \frac{\pi y}{2\Delta} \Big]$$

$$+ \frac{\nu}{V_0} \sum_n \sum_q a_n \phi_q \left(\frac{n^2\pi^2}{8\Delta^2}\right) \Big[\cos (q-p) \frac{\pi y}{2\Delta} - \cos (q+p) \frac{\pi y}{2\Delta} \Big] \Big\} dx\, dy$$

This expression may be integrated with respect to   . If it is recalled that the n, p, q's are all odd numbers and that a vanishing argument for the cosine results from a zero term of the integrand, the integrated form is

(B24)

$$\int_0^x dx \Big\{ \sum_n \sum_p \sum_q a_n a_p \phi_q \left(\frac{p\Delta'}{n\pi}\right) \Big[\frac{2q(q^2-p^2-n^2)}{[(q+p)^2-n^2][(q-p)^2-n^2]}$$

$$+ \left(\frac{1}{q+p} \cos (q+p) \frac{\pi y}{2\Delta} + \frac{1}{q-p} \cos (q-p) \frac{\pi y}{2\Delta} \right)_0^\Delta \Big]$$

$$+ \sum_q \frac{\nu}{V_0} a_q \phi_q \frac{q^2\pi^2}{8\Delta} \Big\}$$

In addition to (B24) a term may be included to enforce the restrictions (B20) and (B22). This term is

$$\int_0^x dt \left\{ \lambda_a \left[\sum_q (-)^{\frac{q-1}{2}} \phi_q - 1 \right] + \lambda_b \left[\sum_n (-)^{\frac{n-1}{2}} \phi_n - 1 \right] \right\} \Delta \qquad \text{(B25)}$$

where λ_a and λ_b are Lagrange multipliers.

In these expressions the terms to be varied are the a's, ϕ's and Δ. If variations of the ϕ 's are to furnish sufficient Euler equations to evaluate the a's and Δ, that is to evaluate the unknown parameters in the assumption for the physical variable, the approximation to the auxiliary variable must be carried out to one more term than that for the physical variable. If the approximation for u is terminated at the N th term, the variational statement (B18) becomes

$$\delta \int_0^x dt \left\{ \sum_n^N \sum_p^N \sum_q^{N+2} a_n a_p \phi_q \left(\frac{p\Delta'}{n\pi} \right) \left[\frac{2q\,(q^2 - p^2 - n^2)}{[(q+p)^2 - n^2][(q-p)^2 - n^2]} \right] + \qquad \text{(B26)}$$

(B26) cont.

$$+\left[\frac{1}{q+p}\cos(q+p)\frac{\pi y}{2\Delta}+\frac{1}{q-p}\cos(q-p)\frac{\pi y}{2\Delta}\right]_0^{\Delta}$$

$$+\sum_q^N \frac{\nu}{\nu_0} a_q \phi_q \frac{q^2\pi^2}{8\Delta}+\lambda_a\Delta\left[\sum_q^{N+2}(-)^{\frac{q+1}{2}}\phi_q-1\right]$$

$$+\lambda_b\Delta\left[\sum_n^N(-)^{\frac{n-1}{2}}a_n-1\right]\Big\}$$

$$+\frac{\nu}{\nu_0}\int_0^x\left\{\sum_q^{N+2}\sum_n^N(-)^{\frac{n+q-2}{2}}a_n\phi_q\left(\frac{n\pi}{2\Delta}\right)^2\delta\Delta\right\}dx$$

$$+\frac{\nu}{\nu_0}\int_0^x dx\left\{\sum_q^{N+2}\phi_q\left(\frac{q\pi}{2\Delta}\right)^2\sin\frac{q\pi y}{2\Delta}\left[\delta\sum_n^N a_n\frac{2\Delta}{n\pi}\left(1-\cos\frac{n\pi y}{2\Delta}\right)\right]\right\}_{y=\Delta}=0$$

This equation has been arranged in such a manner that only variations of ϕ must be taken to determine the approximation to the velocity u . Taking only this variation and not including terms which do not involve yields the appropriate Euler equations since the coefficient of $\delta\phi$ under the integral must vanish due to the arbitrary character of .

Thus, if N is taken to be 3, that is if the a's are taken to be a_1 and a_3 ; three Euler equations result from variations of ϕ . These equations and equation (B20) constitute a system of four equations in the four unknowns

a_1, a_3, Δ, and λ_a. If λ_a is eliminated, this set of equations becomes

$$\frac{2}{5}a_1^2 - \frac{4}{7}a_1a_3 - \frac{586}{315}a_3^2 = \frac{\nu\pi^3}{8U_0\Delta\Delta'}(a_1 + 9a_3) \quad \text{(B27)}$$

$$\frac{46}{105}a_1^2 - \frac{340}{189}a_1a_3 - \frac{1242}{385}a_3^2 = \frac{\nu\pi^3}{8U_0\Delta\Delta'}a_1 \quad \text{(B28)}$$

$$a_1 - a_3 = 1 \quad \text{(B29)}$$

The solutions of these equations are

$$a_1 = 0.575, 0.987, 1.151 \quad \text{(B30)}$$

and the corresponding Δ 's are $11.1, 4.20, 7.08$.

These three possibilities, all positive real numbers, brings to the fore the problem of uniqueness. Often, the engineer circumvents this problem by appealing to physical intuition. That is, if the problem possesses a unique physical solution and if the system of equations accurately describes the essentials of the physical phenomena, then the system of equations is expected to have a unique (real) solution. In the above development, however, the non-physical auxiliary problem has been introduced so that the mathematical system is not exclusively related to the physical problem. Consequently, the fact that the solution is not unique implies nothing regarding the physical problem only that there are three real solutions to the problem of minimizing the integral (B18).

It is often possible to eliminate solutions by appeal to their "physical reasonableness". That is, by comparing them qualitatively to the known or anticipated physical result. For example, the first of (B30) i.e., $a_1 = 0.575$ yields a negative shear stress at the plate and a negative component at the edge of the boundary layer. Since these results appear impossible from the physical standpoint, this solution may be disregarded as a solution to the physical problem. The other two solutions of (B30) are not eliminated so simply since both yield reasonable velocity profiles.

A formal means of choosing between the remaining two solutions would be to form the expression

$$\int_0^x dx \int_0^\Delta dy \left[u u_x + v u_y - \nu u_{yy} \right]^2 \tag{B31}$$

using each of the two results. The one which yields the smaller numerical value of the expression (B31) would then be taken as the better solution of the physical problem.

A more intuitive approach would be to recognize that, from a mathematical standpoint, the boundary layer thickness is that distance above the plate at which the velocity becomes exactly unity (i.e. $u = U_0$). Consequently, a very close approximation to the true velocity profile would yield a large value of the boundary layer thickness, and the profile which yields the larger boundary layer thickness (at the same level of approximation, of course) would be expected to be the better of the two. Note that the boundary layer thickness used here would actually have a value of infinity in the Blasius solution of the problem. The conventional boundary layer thickness is arbitrary since it is defined as that distance above the plate at which the velocity attains the arbitrary value of 0.99 U_0 .

Based on either of the above approaches the result of the computations is that

$$\frac{u}{U_0} = 1.151 \sin \frac{\pi y}{2\Delta} + 0.151 \sin \frac{3\pi y}{2\Delta} \qquad \text{(B32)}$$

where

$$\Delta = 7.08 \sqrt{\frac{\nu x}{U_0}} \tag{B33}$$

The two profiles associated with the physically plausible solutions (B30) are shown in Fig. 3. Figure 4 shows the first approximation with the variational method along with the a_1 = 1.151 second approximation profile and the Blasius profile. The first approximation is

$$\frac{u}{U_0} = \sin \frac{\pi y}{2\Delta} \tag{B34}$$

where Δ is evaluated by requiring that the integral in (B18) be stationary. That is

$$\Delta = 4.41 \sqrt{\frac{\nu x}{U_0}} \tag{B35}$$

It is seen that the profile corresponding to a_1 = 1.151 provides a reasonable approximation to the Blasius solution somewhat beyond the computed value of the boundary layer thickness. In fact, the profile remains above the value $\frac{u}{U_0}$ = 0.99 to a value of η = 9.50 rather than η = 7.08 which is the edge of the boundary layer defined by Δ .

Since the curves of figure 4 do not provide sufficient spread for the relative behavior of the various profiles to be seen in any detail, values of u/v_o are presented in Table B1.

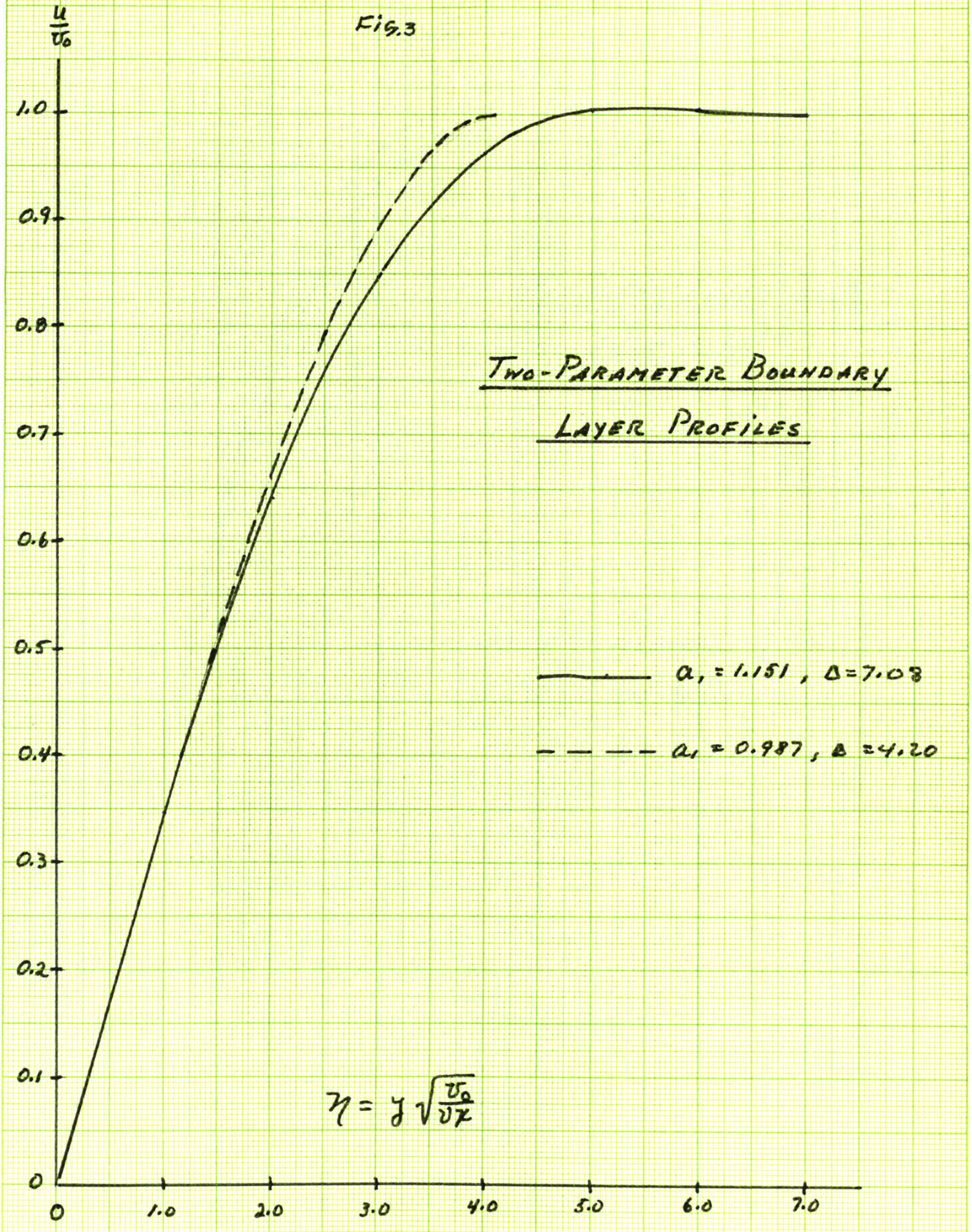


Fig. 3
Two-Parameter Boundary Layer Profiles
$\frac{u}{v_0}$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
$a_1 = 1.151$, $\Delta = 7.08$
$a_1 = 0.987$, $\Delta = 4.20$
$\eta = y\sqrt{\frac{v_0}{\nu x}}$

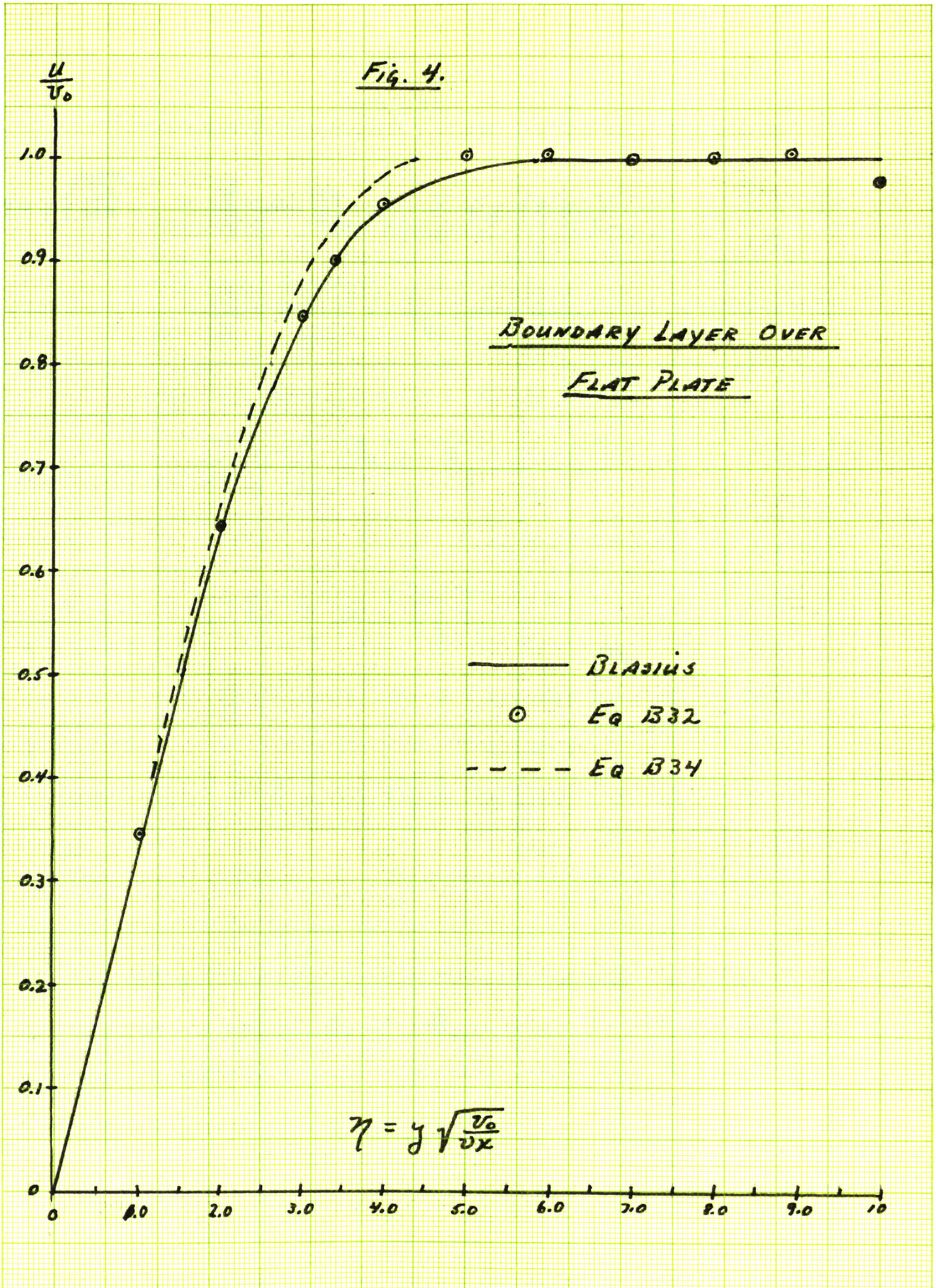


Fig. 4.
$\frac{u}{U_0}$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10
BOUNDARY LAYER OVER FLAT PLATE
BLASIUS
EQ B32
EQ B34
$\eta = y\sqrt{\frac{U_0}{\nu x}}$

TABLE B1

$\eta = y\sqrt{\frac{U_0}{\nu x}}$	$\frac{u}{U_0} = \sin\frac{\pi y}{2\Delta_1}$	Eq. (B32)	Blasius
1	0.349	0.347	0.330
2	0.655	0.641	0.630
3	0.878	0.848	0.846
4	0.990	0.963	0.956
4.4	1.000 (Δ_1)	-----	-----
5	(.977)	1.003	0.992
6		1.005	0.999
7		1.000	1.000
7.08		1.000 (Δ_2)	-----
8		(1.004)	1.000
9		(1.006)	1.000
9.5		(0.993)	1.000
10		(0.974)	1.000

In Table B1 the profiles should be terminated at the edge of the boundary layers Δ_1 and Δ_2. However, the profiles are extended to illustrate their behavior until they deviate from the exact solution by one percent.

The boundary layer parameters which are of interest in comparing these profiles are:

1.) Displacement Thickness (Δ^*) defined by

$$\frac{\Delta^*}{\Delta} = \int_0^1 \left(1 - \frac{u}{U_0}\right) d\left(\frac{y}{\Delta}\right) \tag{B36}$$

2.) Momentum Thickness (θ) defined by

$$\frac{\theta}{\Delta} = \int_0^1 \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) d\left(\frac{y}{\Delta}\right) \tag{B37}$$

3.) The Plate Shear Stress (τ_0) defined by

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{B38}$$

or (τ_{oi}) which may be computed from the integral (B37)

$$\tau_{oi} = \rho U_0^2 \frac{d\theta}{dx} \tag{B39}$$

The values of these boundary layer parameters are given in Table (B2) below.

TABLE B2

Velocity Distribution	$\Delta^*\sqrt{\frac{U_0}{\nu x}}$	$\theta\sqrt{\frac{U_0}{\nu x}}$	$\frac{\tau_0}{\mu U_0}\sqrt{\frac{\nu x}{U_0}}$	$\frac{\tau_{0i}}{\mu U_0}\sqrt{\frac{\nu x}{U_0}}$
$\frac{u}{U_0} = \sin\frac{\pi y}{2\Delta}$	1.59	0.612	0.357	0.306
$\frac{u}{U_0} = 1.151\sin\frac{\pi y}{2\Delta} + .151\sin\frac{3\pi y}{2\Delta}$	1.66	0.665	0.356	0.333
Blasius Profile	1.729	0.664	0.332	0.332

The values of Table B2 indicate an adequate approximation has been attained for most practical purposes. It is to be expected that the wall shear stress computed from the slope of the profile at the wall would be inferior to that computed by the integral method since the profile is generated from an integral statement. The displacement thickness is low by a matter of 4% which leaves something to be desired although not enough to warrant the lengthy computations which would be involved were a third approximation to be made.

Satisfaction of the boundary conditions on the auxiliary velocity at $x = X$ is possible if, for the two term approximations,

$$\phi_1 = \lim_{A \to \infty} B\left[1 - \left(\frac{x}{X}\right)^A\right] \tag{B40}$$

and $$\phi_3 = \lim_{A \to \infty} (B-1)\left[1 - \left(\frac{x}{X}\right)^A\right]$$

where A and B are constants. The feasibility of this assumption is tested by determining B. If B is finite, the assumption yields a reasonable result. The first order assumption for the physical velocity is

$$u = \sin \frac{\pi y}{2\Delta} \tag{B41}$$

where $$\Delta = c\sqrt{\frac{\nu x}{U_0}} \quad , \quad (c = 4.41)$$

and c is the (constant) parameter to be varied in taking δu. If (B40_ and (B41) are inserted into equation (B26) then,

$$\int_0^1 \sqrt{\frac{\nu X}{U_0}}\, \delta c \left\{ \frac{A}{16\pi}(6B-1)\left(\frac{x}{X}\right)^{A-1/2} \right.$$

$$\left. -\frac{1}{c^2}\left[\frac{\pi^2}{8}(B-2) + \frac{\pi}{2}(8B-9)\right]\left[\left(\frac{x}{X}\right)^{-1/2} - \left(\frac{x}{X}\right)^{A-1/2}\right] \right\} d\left(\frac{x}{X}\right) = 0$$

which when integrated yields

$$\left[2.803\,\frac{B}{C^2} - 16.603\,\frac{B-1}{C^2}\right]\left(2 - \frac{1}{A+1/2}\right) - \left[\frac{AB}{3\pi} + \frac{A(B-1)}{15\pi}\right]\frac{1}{A+1/2} = 0 \tag{B42}$$

Taking the limit of (B42) as A approaches infinity results in an equation in B whose solution is

$$B = 1.12 \tag{B43}$$

Therefore,

$$\alpha = \lim_{A \to \infty}\left[1 - \left(\frac{x}{X}\right)^A\right]\left\{1.12 \sin\frac{\pi y}{2\Delta} + 0.12 \sin\frac{3\pi y}{2\Delta}\right\} \tag{B44}$$

a reasonable result. In fact, for all $x < X$ this result indicates that the auxiliary velocity closely resembles the physical velocity. A true time-reversal that is, reversed dependence on x , is not exhibited here due to the fact that the auxiliary variable was forced to have the same boundary layer shape as the physical velocity.

First Order Approximation to the Boundary Layer on a Flat Plate at Zero Incidence.

The relation describing the flow over a flat plate is, of course, the Navier-Stokes equation. However, due to the difficulties in solving this equation in all its splendor certain approximations are commonly made which are supposed to be valid in various physical flow regimes. The present problem is usually considered from the point of view of the Prandtl boundary layer approximation which is valid for large values of the Reynolds modulus. However, considerable effort has been expended in studying the problem from the point of view of the Oseen approximation which is valid at low values of the Reynolds Modulus. Little has been reported in the flow region between these extremes, however, Kuo (ref. 19) has ingeneously determined the solution of the first order boundary layer approximation to the flow over a finite flat plate. Since this problem is sufficiently new to merit interest, this section will be devoted to a demonstration of the variational method applied to the first order boundary layer over a flat plate with zero pressure gradient.

Before considering the details of the first order solution, it is important to establish the fact that the first order problem is of the boundary layer type with its attendant simplifications. To establish this consider the first order term of equation (78).

$$\delta\int_0^X\int_0^Y dx\,dy\ \epsilon\Big\{\alpha^{(1)}\left(u^{(0)}u_x^{(0)}+v^{(0)}u_y^{(0)}\right)+\alpha_y^{(1)}u_y^{(0)} \tag{C1}$$

$$-\mathcal{P}^{(1)}\left(u_x^{(0)}+v_y^{(0)}\right)-p^{(0)}\left(\alpha_x^{(1)}+\beta_y^{(1)}\right)$$

$$+\alpha^{(0)}\left(u^{(0)}u_x^{(1)}+u^{(1)}u_x^{(0)}+v^{(1)}u_y^{(0)}+v^{(0)}u_y^{(1)}\right)+\alpha_y^{(0)}u_y^{(1)}$$

$$-\mathcal{P}^{(0)}\left(u_x^{(1)}+v_y^{(1)}\right)-p^{(1)}\left(\alpha_x^{(0)}+\beta_y^{(0)}\right)\Big\}=0$$

At this level of approximation the zero order velocities are known and the variations yield the following Euler equations in the physical variables

For $\delta\alpha^{(1)}$: $u^{(0)}u_x^{(0)}+v^{(0)}u_y^{(0)}-u_{yy}^{(0)}+p_x^{(0)}=0$ (C2)

For $\delta\beta^{(1)}$: $p_y^{(0)}=0$ (C3)

For $\delta\alpha^{(0)}$: $$u^{(0)}u_x^{(1)} + u^{(1)}u_x^{(0)} + v^{(1)}u_y^{(0)} + v^{(0)}u_y^{(1)} \tag{C4}$$
$$- u_{yy}^{(1)} + p_x^{(1)} = 0$$

For $\delta\beta^{(0)}$: $$p_y^{(1)} = 0 \tag{C5}$$

Equations (C2) and (C3) are automatically satisfied by the zero order solution. Equation (C4) is the governing equation of the first order solution. Equation (C5) gives the important result that the first order pressure does not vary in the y direction, consequently the pressure in the first order problem is determined by the external (potential flow). One concludes that, since a first order potential flow is required and the orders of various derivatives are the same as for the Prandtl equation (C2), the problem remains of the boundary layer type in the first order approximation. A suitable potential flow must then be determined to provide the boundary conditions at the first order boundary layer.

Following Kuo, one notes that at the edge of the zero order boundary layer

$$U^{(0)} = 1 \tag{C6}$$
$$V^{(0)} = \frac{\bar{v}^{(0)}}{U_\infty} = \epsilon \frac{K}{2\sqrt{x}}$$

Where the variables are those of expression (70) and K = 1.73 for the Blasius solution. If the velocity field in the potential region is expanded in terms of thickness parameter ϵ , then in the potential region

$$U = 1 + \epsilon U^{(1)}(x,y) + \cdots$$
$$V = 0 + \epsilon V^{(1)}(x,y) + \cdots \tag{C7}$$

From (C6) it is seen that

$$V^{(1)} = \frac{K}{2\sqrt{x}} \tag{C8}$$

The first order potential flow is then given as that for which $V = V^{(1)}$ at the plate surface and the velocities vanish at infinity. Such a flow results from a line source at the surface of the plate with the source strength varying as $1/\sqrt{s}$ where s is the dimensionless distance along the plate. The velocity potential for the flow is then

$$d\phi = \frac{K}{2\pi} \ln(z-s) \frac{ds}{\sqrt{s}}$$

where z denotes $x + iY$. The first order velocities are then

(C9)

$$U^{(1)} - i V^{(1)} = \frac{K}{2\pi} \int_0^1 \frac{ds}{\sqrt{s}(z-s)} \tag{C9}$$

$$= -\frac{iK}{2\sqrt{z}} + \frac{K}{2\pi\sqrt{z}} \ln \frac{1+\sqrt{z}}{1-\sqrt{z}}$$

The appropriate boundary condition for the first order boundary solution is then

$$\text{at } y = \Delta^{(1)}, \quad u^{(1)} = \frac{K}{2\pi\sqrt{x}} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} \tag{C10}$$

The solution to the first order equation (C4) for flow over a finite flat plate (boundary condition C10) was obtained by Kuo. The solution of this problem by means of the present variational technique is greatly complicated (as it is analytically) by the fact that the potential flow exhibits a logarithmic singularity at the trailing edge (x = 1). The pressure gradient then, is also singular at the trailing edge. It is found that, under these conditions, the variational technique yields differential equations in the parameters of the assumed solution which are as involved as those solved by Kuo.

These computational difficulties, coupled with the fact that the trailing edge condition imposed by (C10) does not seem physically reasonable when applied to viscous flows, led to the rejection of this problem in favor of the first approximation to the boundary layer over a semi-infinite flat plate. For this problem the trailing edge condition, of course, vanishes.

The boundary condition for the semi-infinite plate is determined by taking the limit of (C10) as the plate length approaches infinity, i.e.

$$U_\infty^{(1)} = \lim_{L \to \infty} \frac{K}{2\pi} \sqrt{\frac{L}{x}} \ln \frac{\sqrt{L} + \sqrt{x}}{\sqrt{L} - \sqrt{x}} \tag{C11}$$

Thus the boundary condition for the semi-infinite plate is

$$\text{at } y = \Delta^{(1)}, \quad u^{(1)} = \frac{K}{\pi} \tag{C12}$$

The other physical boundary conditions follow those of the zero order problem

$$\begin{aligned} &\text{at } y = \Delta^{(1)}, && u_y^{(1)} = 0 \\ & && u_x^{(1)} = 0 \\ &\text{at } y = 0, && u^{(1)} = v^{(1)} = 0 \\ &\text{at } x = 0, && u^{(1)} = 0 \end{aligned} \tag{C13}$$

Since $u^{(1)}_y = 0$ at the edge of the boundary layer, the variational statement in terms of the stream function is, from (C1)

$$\delta \int_0^x dx \int_0^{\Delta^{(1)}} dy \;\; \mathcal{L}^{(0)} \Big(u^{(0)} \psi^{(1)}_{xy} + \psi^{(1)}_y u^{(0)}_x$$

$$+ v^{(0)} \psi^{(1)}_{yy} - \psi^{(1)}_x u^{(0)}_y - \psi^{(1)}_{yyy} \Big) = 0 \qquad \text{(C14)}$$

which is seen by inspection to require that $\mathcal{L}^{(0)}$ satisfy the same boundary conditions as in the zero order case.

Considering $u^{(0)}$ the interesting result in this computation is to be the shear stress at the surface of the plate since there exists no solution of the Blasius type by which velocity profiles may be compared. The first and second approximations to the zero order solution are seen to differ insignificantly in the vicinity of the plate, and, consequently, the first approximation is chosen to simplify the computation. So here

$$u^{(0)} = \sin \frac{\pi y}{2\Delta^{(0)}} \qquad \text{(C15)}$$

where $\Delta^{(0)} = 4.4 / \sqrt{x}$

The form (C15) is, however, inconvenient for the present problem due to the fact that there are two boundary layer thicknesses involved, $\Delta^{(0)}$ and Δ'' . Consequently, should it become necessary to include both of these in the integral to be varied, trancendental expressions in will result. To avoid this potential difficulty (C15) may be accurately approximated as

$$u^{(0)} = 1.57 \frac{y}{\Delta^{(0)}} - 0.64 \frac{y^3}{\Delta^{(0)3}} + 0.07 \frac{y^5}{\Delta^{(0)5}} \qquad \text{(C15a)}$$

and $\alpha^{(0)}$ as

$$\alpha^{(0)} = \phi_1 \frac{y}{\Delta^{(0)}} + \phi_3 \frac{y^3}{\Delta^{(0)3}} + \phi_5 \frac{y^5}{\Delta^{(0)5}} \qquad \text{(C16)}$$

where the ϕ 's are functions of γ . The restrictions

$$\phi_1 + \phi_3 + \phi_5 = 1 \qquad \text{(C17)}$$

and $\qquad \phi_1 + 3\phi_3 + 5\phi_5 = 0$

insure that $\alpha^{(0)}$ goes to unity and $\alpha^{(0)}_y$ vanishes at the edge of the zero order boundary layer.

The rather unimaginative assumption

$$\frac{u^{(1)}}{V^{(1)}} = a\frac{y}{\Delta^{(1)}} + b\frac{y^2}{\Delta^{(1)2}} + c\frac{y^3}{\Delta^{(1)3}} \tag{C18}$$

may now be made for the first order solution. Here, the restrictions

$$\begin{aligned} a+b+c&=1 \\ a+2b+3c&=0 \end{aligned} \tag{C19}$$

insure that the essential boundary conditions are satisfied.

At this point the question of the two different boundary layer thicknesses must be considered. Since the integration in the variational method is to be performed over the entire region of interest, the upper limit of integration must always be the greater of $\Delta^{(0)}$ and $\Delta^{(1)}$. This question regarding the upper limit coupled with the discontinuous nature of the velocities at their respective Δ 's greatly complicates the computations involved. Consequently, it would be desirable to consider $\Delta^{(1)} = \Delta^{(0)}$ as a first approximation. Inspection of the first order boundary layer equation indicates this to be a reasonable approximation. Since the zero order velocities are the coefficients in this equation, it does not seem reasonable to expect the first order solution to approach a constant value while the coefficients in the equation are varying appreciably, i.e. for $y < \Delta^{(0)}$. On the other hand, physical reasoning indicates that the potential flow solution from which the first order boundary condition is obtained would be expected to occur soon after the value $y = \Delta^{(0)}$ is reached. Consequently, it seems improbable that $\Delta^{(1)}$ can exceed $\Delta^{(0)}$ significantly.

The test of such an assumption is, of course, the physical reasonableness of the results obtained by incorporating it into the computations.

Proceeding, then, with $\Delta^{(1)} = \Delta^{(0)}$ and taking

$$u^{(0)} = \sum b_p \left(\frac{y}{\Delta}\right)^p \tag{C20}$$

$$\alpha^{(0)} = \sum \phi_q \left(\frac{y}{\Delta}\right)^q$$

$$u^{(1)} = \sum a_n \left(\frac{y}{\Delta}\right)^n$$

equation (C14) becomes

$$\delta \int_0^\infty dx \int_0^\Delta dy \left\{ \left[\sum_n \sum_p \sum_q a_n b_p \phi_q \Delta' \left(\frac{p}{n+1} + \frac{n}{p+1}\right) \frac{y^{n+p+q}}{\Delta^{n+p+q+1}} \right. \right. \tag{C21}$$

$$\left. + \lambda_a \left[\sum \phi_q - 1\right] + \lambda_b \left[\sum q\phi_q\right] + \sum_n \sum_q n(n-1) a_n \phi_q \frac{y^{n+q-2}}{\Delta^{n+q}} \right\} = 0$$

where λ_a and λ_b enforce the relationships between the ϕ 's given in (C17).

After performing the y integration this is

$$\delta\int_0^\infty d\psi\left\{\sum_n\sum_p\sum_q \frac{a_n b_p \phi_q \Delta'}{n+p+q+1}\left(\frac{p}{n+1}+\frac{n}{p+1}\right)\right. \tag{C22}$$

$$\left.+\lambda_a\left[\sum\phi_q-1\right]+\lambda_b\left(\sum\phi_q\right)+\sum_n\sum_q \frac{n(n-1)}{n+q-1}\,\frac{a_n\phi_q}{\Delta}\right\}=0$$

Taking the variations of the ϕ 's, i.e. ϕ_1 , ϕ_3 , ϕ_5 for assumption (C16) and eliminating λ_a and λ_b results in a single algebraic equation for the undetermined coefficient in assumption (C18). This expression is

$$\sum_{n=1}^{3}\sum_{p=1}^{3} a_n b_p \Delta'\left(\frac{p}{n+1}+\frac{n}{p+1}\right)\left(\frac{1}{n+p+2}+\frac{1}{n+p+6}-\frac{2}{n+p+4}\right) \tag{C23}$$

$$+\sum_{n=1}^{3}(n-1)\frac{a_n}{\Delta}\left(1+\frac{n}{n+4}-\frac{2n}{n+2}\right)=0$$

Solution of (C23) and equations (C19) yields

$$a_1 = 3,\ a_2 = -3,\ a_3 = 1 \qquad \text{approximately} \tag{C24}$$

The approximate solution for $u^{(1)}$ is

$$u^{(1)} = \frac{K^{(0)}}{\pi}\left[3\frac{y}{\Delta^{(0)}} - 3\left(\frac{y}{\Delta^{(0)}}\right)^2 + \left(\frac{y}{\Delta^{(0)}}\right)^3\right] \tag{C25}$$

where $\quad v^{(0)} = \dfrac{\epsilon K^{(0)}}{2\sqrt{x}} \quad at \quad y = \Delta^{(0)}$

Evaluating $K^{(0)}$ from (C15a) gives

$$K^{(0)} = 1.614 \tag{C26}$$

which compares with the value of 1.73 for the exact (Blasius) solution of the zero order problem.

The drag coefficient for the plate for both the zero and first solutions may be evaluated from the velocity

$$\bar{u} = \bar{u}^{(0)} + \frac{\bar{u}^{(1)}}{\sqrt{Re}} \tag{C27}$$

The drag coefficient is evaluated from the expression

$$C_D = \frac{2}{\rho L V_0^2}\int_0^L \mu \left(\frac{\partial \bar{u}}{\partial y}\right)_{y=0} d\bar{x}$$

Inserting (C27) into this expression yields

$$C_D = \frac{1.42}{\sqrt{\mathcal{R}e}} + \frac{1.36}{\mathcal{R}e} \tag{C28}$$

This prediction of C_D is shown in figure 5. It is seen that the predicted values of C_D effectively joins those of the Blasius solution and those of the Oseen solution to the flat plate problem. The curve shown for the Oseen solution is that of Piercy and Winny (ref. 26) whose solution of the problem has recently been corroborated by Tomotika and Yosinobu (ref. 27).

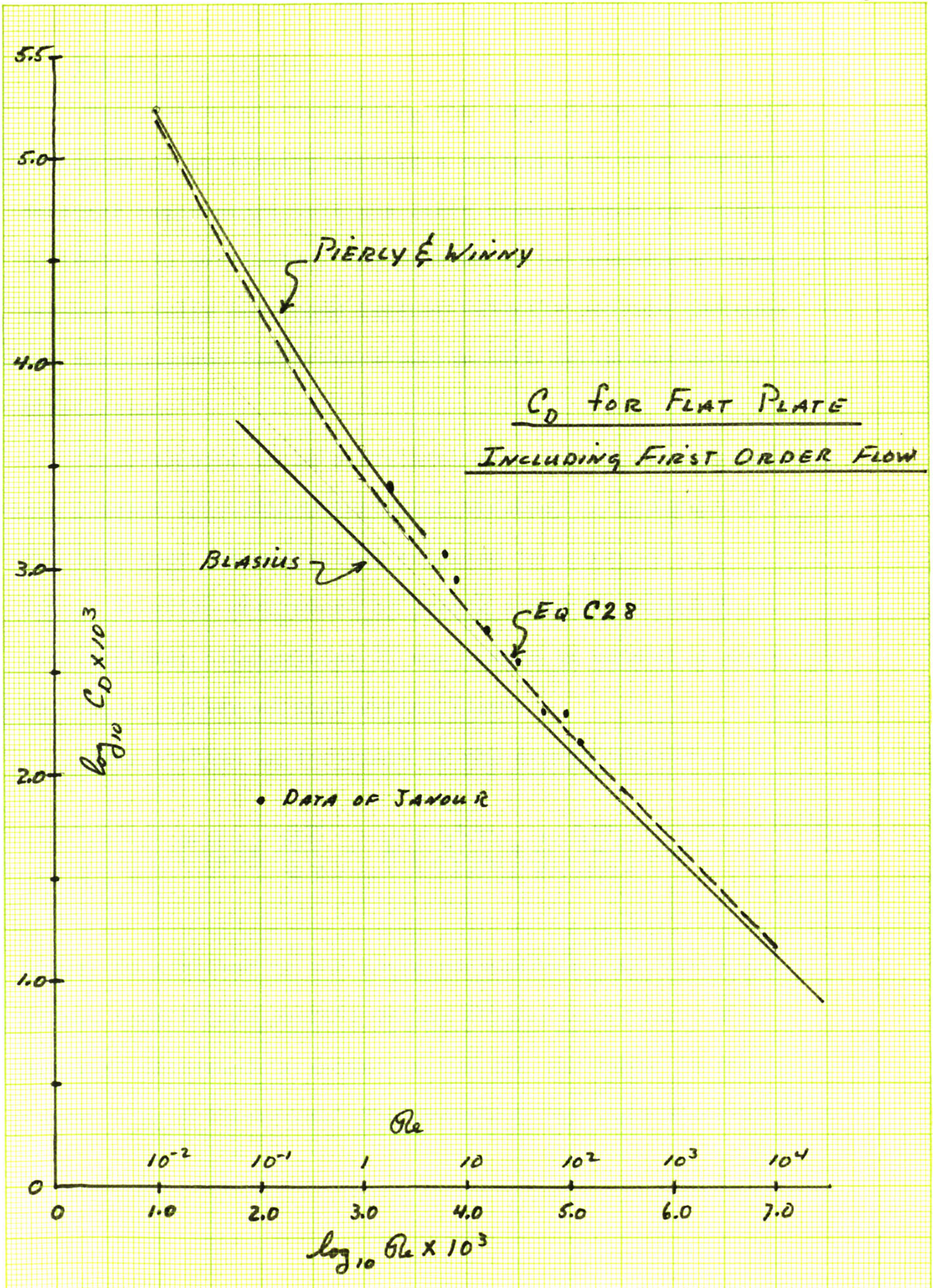
The fact that the first order solution should, in fact, join the Blasius and Oseen solutions has been pointed out by Kuo. Kuo showed that higher approximations to the boundary layer problem will not change the values of shear stress at the plate surface.

It should be pointed out that Kuo's solution for the finite flat plate gives

$$C_D = \frac{1.33}{\sqrt{\mathcal{R}e}} + \frac{4.12}{\mathcal{R}e}$$

which is greater than the Oseen solution at $\mathcal{R}e = 10^{-2}$ by approximately 300%. However, Kuo's result does follow some data taken by Janour (ref. 28) extremely well. The present solution (C28) lies some 30% below the Janour data at $\mathcal{R}e = 10$, the lowest value attained by Janour. This is, of course, the maximum deviation from the data taken.

The difference between Kuo's result and the present solution stems from Kuo's predicted singularity in pressure and velocity at the trailing edge. Since this boundary condition appears physically questionable, and the present solution does join two known solutions, relation (C28) constitutes an acceptable approximate solution to the problem in the absence of experimental data in the very low Reynolds number region



C_D FOR FLAT PLATE
INCLUDING FIRST ORDER FLOW
PIERLY & WINNY
BLASIUS
EQ C28
• DATA OF JANOUR
$\log_{10} C_D \times 10^3$
5.5
5.0
4.0
3.0
2.0
1.0
0
Re
10^{-2}
10^{-1}
1
10
10^2
10^3
10^4
0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
$\log_{10} Re \times 10^3$

D. The Graetz Problem

The problem of determining the temperature of a liquid stream which is flowing through a cylindrical tube having a constant temperature wall was first investigated by Graetz in 1885. In particular, the problem to be considered here assumes a fully developed laminar velocity profile throughout the constant temperature section and neglects viscous dissipation.

This appears to be a fruitful area for application of the present approximate technique because, in addition to the fact that the analytical solution is involved, the series solution determined by Graetz presents formidable computational difficulties if one wishes to extend it into the region of normal engineering flows.

This problem is described by the energy equation

$$\rho c_p u \frac{\partial T}{\partial x} - k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) = 0 \tag{D1}$$

where, in addition to the above assumptions, the curvature of the temperature profile in the stream direction (x) has been neglected in comparison to the curvature in the normal (r) direction. The velocity, u , in equation (D1) is taken as that corresponding to fully developed laminar flow, i.e.

$$u = 2u_m\left(1 - \frac{r^2}{R^2}\right) \tag{D2}$$

where u_m is the mean flow velocity and R is the pipe radius.

The transformation of the Lagrangian for the energy equation to cylindrical coordinates is straight forward yielding

$$\delta \int_0^X \int_0^R \left\{ \frac{u}{2}\left(T_x \Upsilon - \Upsilon_x T\right) + \frac{k}{\rho c_p} T_r \Upsilon_r \right\} r\,dr\,dx = 0 \tag{D3}$$

where Υ is the auxiliary temperature.

Performing the indicated variation yields

$$\begin{aligned} \int_0^X \int_0^R \Big\{ \delta\Upsilon\Big[u T_x - \frac{k}{\rho c_p}\Big(T_{rr} - \frac{T_r}{r}\Big)\Big] \\ - \delta T\Big[u \Upsilon_x + \frac{k}{\rho c_p}\Big(\Upsilon_{rr} - \frac{\Upsilon_r}{r}\Big)\Big] \Big\} r\,dr\,dx \\ + \int_0^R u\Big[\Upsilon(\delta T) - T(\delta \Upsilon)\Big]_0^X r\,dr \\ + \frac{k}{\rho c_p}\int_0^X \Big[r\left(T_r \delta\Upsilon + \Upsilon_r \delta T\right)\Big]_0^R dx \end{aligned} \tag{D4}$$

If T in (D4) is replaced by

$$\theta = T - T_{x=0}$$

and Υ by

$$\theta^* = \Upsilon - \Upsilon_{x=x}$$

the first boundary integral of (D4) will vanish. The second boundary integral vanishes automatically since θ is prescribed on the wall and θ^* must also be prescribed there.

The boundary condition at the inlet section is similar to the initial condition encountered in problem A above and will be handled in a similar manner. Let us assume that

$$\theta = \theta_w + \sum_{1}^{N} f_n \left(1 - \frac{r^{2n}}{R^{2n}}\right) \tag{D5}$$

where $f_n = f_n(x)$

In order to satisfy the boundary conditions as well as possible within this assumption let

$$f_1(0) = f_2(0) = \cdots = f_{N-1}(0) = 0 \tag{D6}$$

and $f_N(0) = -\theta_w$

Then the larger the value of N taken, the greater will be the accuracy in fitting the boundary condition at the inlet to the constant temperature section.

Similarly let

$$\theta^* = \theta_w^* + \sum_{1}^{N} \xi_n^* \left(1 - \frac{r^{2n}}{R^{2n}}\right) \tag{D7}$$

where $\xi_N^*(x) = -\theta_w$

For purposes of determining the physical temperature, θ, equation (D3) may be written

$$\delta \int_0^R \int_0^X \left\{ 2u_m \theta_x \theta^* + \frac{k}{\rho C_p} \theta_r \theta_r^* \right\} r\,dr\,dx \tag{D3a}$$

which becomes after combining with (D5) and (D7) but dropping the θ_w terms since they are constant and not varied

$$\delta \int_0^1 \int_0^X \Big\{ 2u_m \sum_n \sum_p R^2 \xi_n' \xi_p^* \left(y - y^{2p+1} - y^{2n+1} + y^{2(n+p-1)} - y^3 + y^{2p+3} + y^{2n+3} - y^{2n+2p+3} \right) + \frac{k}{\rho C_p} \sum_n \sum_p \xi_n \xi_p^* y^{2n+2p-1} \Big\} dy\,dx = 0 \tag{D8}$$

where $y = \frac{r}{R}$

Performing the y integration and taking the variations of the auxiliary variables yields the following three equations for $n = 1, 2, 3$

(D9)

$$\sum_{1}^{3}\left\{\xi_n'\left[\frac{1}{3}+\frac{2}{n+2}-\frac{1}{n+3}-\frac{1}{n+1}\right]+\Gamma\xi_n\frac{n}{n+1}\right\}=0$$

$$\sum_{1}^{3}\left\{\xi_n'\left[\frac{5}{12}-\frac{1}{n+1}+\frac{1}{n+3}+\frac{1}{n+2}-\frac{1}{n+4}\right]+\Gamma\xi_n\frac{4n}{n+2}\right\}=0$$

$$\sum_{1}^{3}\left\{\xi_n'\left[\frac{9}{20}-\frac{1}{n+1}+\frac{1}{n+4}+\frac{1}{n+2}-\frac{1}{n+5}\right]+\Gamma\xi_n\frac{6n}{n+3}\right\}=0$$

where $\Gamma = \dfrac{k}{\rho C_p U_m R^2}$

The first two approximations to the solution, i.e. n = 1, 2 and n = 1, 2, 3, are contained in (D9). These solutions are for $n = 1, 2$

$$\xi_1 = -2.05\,\theta_w \left[e^{-3.67\Gamma x} - e^{-36.3\Gamma x} \right] \tag{D10}$$

$$\xi_2 = \theta_w \left[0.729\, e^{-3.67\Gamma x} - 1.729\, e^{-36.3\Gamma x} \right]$$

and for $n = 1, 2, 3$

$$\xi_1 = -\theta_w \left[1.22\, e^{-3.67\Gamma x} - 4.56\, e^{-27.0\Gamma x} + 3.34\, e^{-139\Gamma x} \right] \tag{D11}$$

$$\xi_2 = -\theta_w \left[0.98\, e^{-3.67\Gamma x} + 6.78\, e^{-27.0\Gamma x} - 7.76\, e^{-139\Gamma x} \right]$$

$$\xi_2 = -\theta_w \left[-0.95\, e^{-3.67\Gamma x} - 2.79\, e^{-27.0\Gamma x} + 4.73\, e^{-139\Gamma x} \right]$$

Computing the mean temperature from

$$\theta_m = \frac{1}{U_m \pi R^2} \int_0^R u\,\theta\,(2\pi)\, r\, dr$$

yields, finally

for $n = 1, 2$ (D12)

$$\frac{\theta_m}{\theta_w} = 1 - 0.759\, e^{-11.5/G_x} - 0.074\, e^{-114/G_x}$$

and for $n = 1, 2, 3$

$$\frac{\theta_m}{\theta_w} = 1 - 0.783\, e^{-11.5/G_x} - 0.101\, e^{-84.6/G_x} - 0.016\, e^{-436/G_x} \qquad \text{(D13)}$$

which may be compared with the Graetz solution

$$\frac{\theta_m}{\theta_w} \doteq 1 - 0.820\, e^{-11.5/G_x} - 0.0972\, e^{-69.5/G_x} - 0.0135\, e^{-166/G_x} \qquad \text{(D14)}$$

where

$$G_x = \frac{w C_p}{k L} = \pi \frac{\rho R^2 C_p u_m}{k x}$$

These relations are compared in figure 6 in which it is seen that expressions (D12) and (D13) are significantly in error in the region near G_x = 100. Beyond this value of the Graetz modulus, expression (D14) also deviates from observed values of θ_m/θ_w . This deviation reflects the fact that the value of the predicted mean temperatures remains greater than zero at $x = 0$ or $G_x = \infty$. This deviation does not, of course, reflect on the accuracy of the Graetz solution but is due to the limited number of terms which can be computed without the use of a digital computer.

Extension of the present method to n = 4 would involve a significant computational effort since the sixteen coefficients involved in the solution of equations (D9) where $G_x = \dfrac{w c_p}{k x}$, $w = \pi R^2 \rho u_m$

These relations are compared in figure 6 in which it is seen that expressions (D12) and (D13) are significantly in error in the region near $G_x = 100$. Beyond this value of the Graetz modulus, expression (D14) also deviates from observed values of θ_m/θ_w . This deviation reflects the fact that the value of the predicted mean temperatures remains greater than zero at $x = 0$ or $G_x = \infty$. This deviation does not, of course, reflect on the accuracy of the Graetz solution but is due to the limited number of terms which can be computed without the use of a digital computer.

Extension of the present method to $\eta = 4$ would involve a significant computational effort since the sixteen coefficients involved in the solution of equations (D9) would have to be determined. An alternative approach to the form of the approximation (D5) appears in order. One alternative which would accurately reflect the boundary condition at $x = 0$ would be

$$\theta = \lim_{m \to \infty} \left\{ \theta_w - \theta_w \left(1 - \frac{r^m}{R^m}\right) + \sum_n \xi_n \left(1 - \frac{r^{2n}}{R^{2n}}\right) \right\} \qquad \text{(D15)}$$

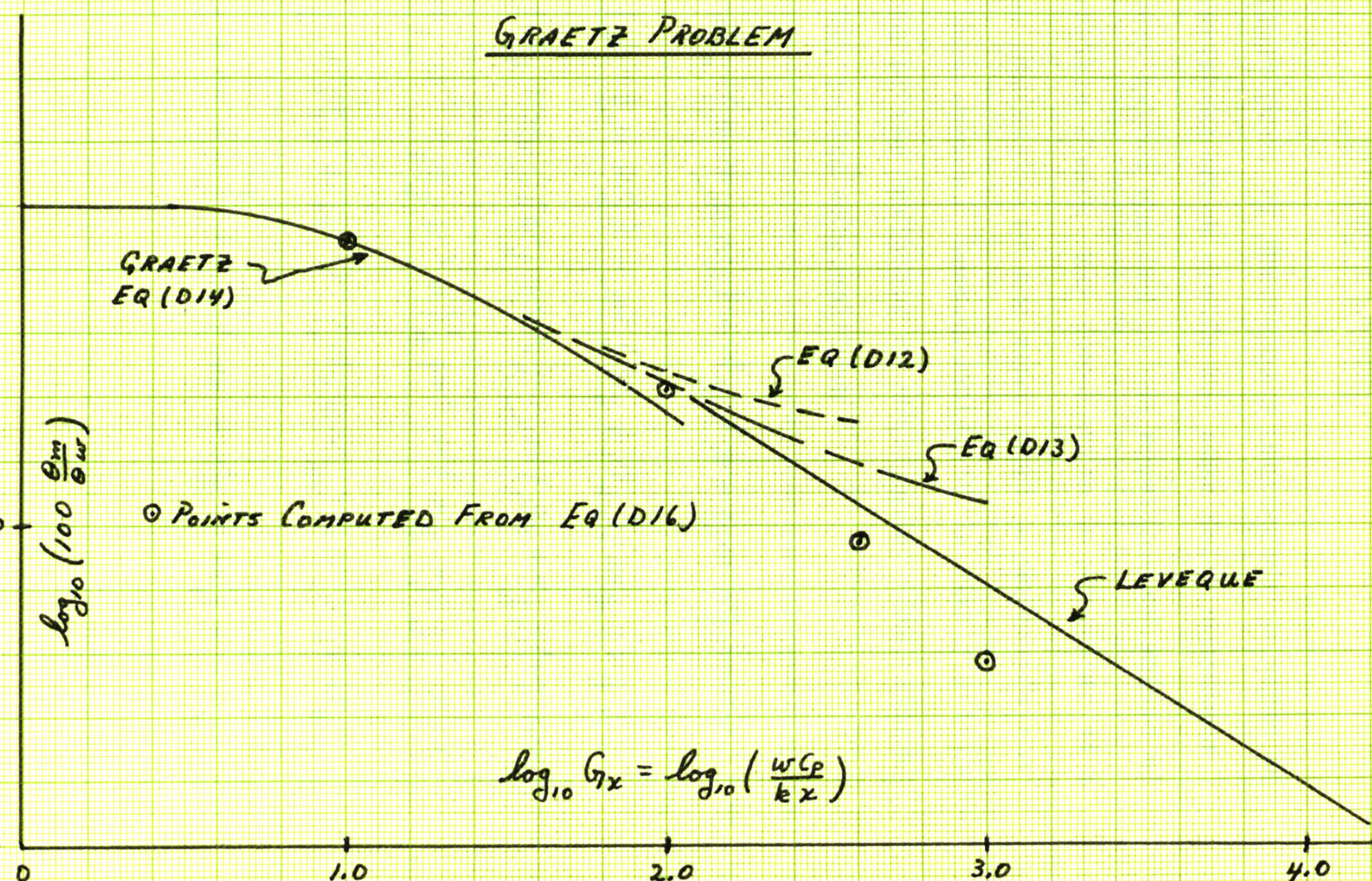
where all the f_n vanish at $x = 0$. Such an assumption loses physical meaning if the temperature gradient at the wall is considered, but may yield meaningful values of the mean temperature. Performing the variation (D3a) yields

$$\frac{\theta_m}{\theta_w} = 1 - 0.73\, e^{-11.5/G_x} - 0.27\, e^{-114/G_x} \tag{D16}$$

for $n = 1, 2$

From figure 6 it is seen that this approximation provides reasonably good agreement with the Graetz solution over its range of validity ($G_x \leq 100$). The extension of the Graetz solution by Leveque (see ref. 20) provides a means of evaluating the solution (D16) in the range . The Leveque curve was not developed as a solution to (D1) for this problem but was determined from a solution for flow over a flat plate. This extension has been confirmed experimentally and hence it provides a criterion for comparison of (D16) with experiment. It is seen that the simple approximation (D16) lies within 15% of the Graetz-Leveque curve in the range $G_x \leq 400$.

Fig. 6


GRAETZ PROBLEM
GRAETZ EQ (D14)
EQ (D12)
EQ (D13)
LEVEQUE
⊙ POINTS COMPUTED FROM EQ (D16)
$\log_{10}\left(100\,\frac{\theta_m}{\theta_w}\right)$
$\log_{10} G_x = \log_{10}\left(\frac{wC_p}{kx}\right)$
0
1.0
2.0
3.0
4.0

APPENDIX A

BATEMAN'S METHOD

H. Bateman (ref. 7) developed a Lagrangian in terms of auxiliary variables which yields the same Euler equations as the present method applied to the equations of motion of an incompressible fluid. His formulation is presented as a fact without indication of the means used to determine the particular form chosen. Consequently, it is simply presented here in terms of the variables of this paper. The Bateman Lagrangian is, in its original form

$$
\begin{aligned}
L_B = {} & \alpha_x (p - 2\mu u_x) + \beta_y (p - 2\mu v_y) \\
& + \gamma_z (p - 2\mu w_z) - \mu(\gamma_y + \beta_z)(w_y + v_z) \\
& - \mu(\alpha_z + \gamma_x)(u_z + w_x) - \mu(\beta_x + \alpha_y)(v_x + u_y) \\
& + \rho\big[\alpha_x u^2 + \beta_y v^2 + \gamma_z w^2 + (\gamma_y + \beta_z) v w \\
& + (\alpha_z + \gamma_x) u w + (\beta_x + \alpha_y) u v + u \alpha_t \\
& + v \beta_t + w \gamma_t\big] + \mathcal{P}(u_x + v_y + w_z)
\end{aligned}
$$

which may be written as

$$L_B = \rho \vec{V} \cdot \frac{D\vec{r}}{Dt} - \mu \left\{ 2 u_x \alpha_x + 2 v_y \beta_y + 2 w_z \gamma_z \right.$$

$$+ (\beta_x + \alpha_y)(v_x + u_y) + (\alpha_z + \gamma_x)(u_z + w_x)$$

$$+ (\gamma_y + \beta_z)(w_y + v_z) + p(\nabla \cdot \vec{r}) + \mathcal{P}(\nabla \cdot \vec{V})$$

This is seen to be identical to the Lagrangian proposed in this report. It is surprising that, in a field in which analyses are so difficult, this result has not been exploited during the thirty years since its development by Bateman.

REFERENCES

1. Lush, P.E., Cherry, T.M., "The Variational Method in Hydrodynamics," Quarterly Journal of Mechanics and Applied Mathematics, V9, 1958.

2. Bateman, H., "The Partial Differential Equations of Mathematical Physics," Cambridge University Press, 1959, Section 2.52.

3. Sommerfield, A., "Mechanics of Deformable Bodies," Academic Press, Inc., New York, 1950, Section III.12.

4. Helmholtz, H.v., "Collected Works," V1, 1869, P. 223. See also ref. 15.

5. Rayleigh, Lord, "Collected Works," V1, 1913, p. 187. See also ref. 15.

6. Millikan, C.B., "On the Steady Motion of Viscous, Incompressible Fluids, with Particular Reference to a Variational Principle," Philisophical Magazine, V7, 1929.

7. Dryden, H.L., Murnagham, F.P., Bateman, H., "Hydrodynamics," Dover Publications, New York, 1956, P. 168, 224.

8. Bateman, H., "On Dissipative Systems and Related Variational Principles," Physical Revue, V38, P. 815, August, 1931.

9. Morse, P.M. and Feshbach, H., "Methods of Theoretical Physics," McGraw-Hill Book Company, New York, 1953, Chapter 3.

10. Washizeu, K., "Application of the Variational Method to Transient Heat Conduction Problems," Tech. Report 25-17, Aeroelastic and Structures Research Laboratory, Massachusetts Institute of Technology, March, 1955.

11. Herivel, J.W., "A General Variational Principle for Dissipative Systems," Proceedings of the Royal Irish Academy, V56, Section A, 1954.

12. Rosen, P., "On Variational Principles for Irreversible Processes," Journal of Chemical Physics, V21, No. 7, 1953.

13. Kuethe, A.M., Shetzer, J.D., "Foundations of Aerodynamics," John Wiley and Sons, Inc., 1950. Appendix B.

14. Onsager, L., "Reciprocal Relations in Irreversible Processes, I," Physical Review, V37, 1931, P. 405.

15. Lamb, H., "Hydrodynamics," Dover Publications, New York, 1945, Art. 344.

16. Hildebrand, F.B., "Methods of Applied Mathematics," Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1952, Chapter 2.

17. Crandall, S.H., "Engineering Analysis," McGraw-Hill Book Company, New York, 1956.

18. Schlichting, H., "Boundary Layer Theory," McGraw-Hill Book Company, New York, 1955, Chapter VII.

19. Kuo, Y.H., "On the Flow of an Incompressible Viscous Fluid Past a Flat Plate at Moderate Reynolds Numbers," Journal of Mathematics and Physics, V32, 1953.

20. Jakob, M., "Heat Transfer," John Wiley and Sons, Inc., New York, 1949, Vol. 1, pp. 451-464.

21. Carlsaw, H.S., Jaeger, S.C., "Conduction of Heat in Solids," Second Edition, Oxford University Press, London, 1959, p. 313.

22. Fox, C., "An Introduction to the Calculus of Variations," Oxford University Press, 1954.

23. Lieber, P., Koon-Sang Wan, "A Principle of Minimun Dissipation for Real Fluids," Ninth International Congress of Applied Mechanics, VI, 1957.

24. Lieber, P., Koon-Sang Wan, "A Principle of Virtual Displacement for Real Fluids," Ninth International Congress of Applied Mechanics, VI, 1957.

25. Forsyth, A.R., "Calculus of Variations," Dover Publications, 1960.

26. Piercy, N.A.V., Winny, H.F., "Skin Friction of Flat Plates," Proceedings of the Royal Society (London), A140, 1933.

27. Tomotika, S., Yosinobu, H., "The Flow of a Viscous Liquid Past a Flat Plate of Small Reynolds Numbers," Journal of Mathematics and Physics, Vol. 35, 1956.

28. Janour, Z., "Resistance of a Plate in Parallel Flow at Low Reynolds Numbers," NACA TM 1316, 1951.