SCHUBERT CALCULUS IN GENERALIZED COHOMOLOGY

by

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1.Introduction. Associated to a compact Lie group G and a maximal torus T is the generalized flag variety G/T. The study of geometry and topology of the flag variety is one of the central issues in Lie theory. One aspect of this study is the intersection theory of algebraic cycles on G/T classically known as the Schubert Calculus.

The Schubert Calculus was studied in terms of rational cohomology by Borel, Bott, Kostant and Bernstein-Gel'fand-Gel'fand and in terms of complex K-theory by Demazure and Kostant-Kumar. Their work focused predominantly on the algebro-geometric properties of the flag variety.

Here, on the other hand, we take the topological viewpoint. As a consequence we are able to extend the classical results to complex-oriented multiplicative cohomology theories. Examples of such are ordinary cohomology, K-theory, complex cobordism and elliptic cohomology.

The flag variety is a smooth algebraic variety over the field of complex numbers. It is endowed with the structure of a CW-complex by the Bruhat cellular decomposition. The cells X_w (which we shall call Schubert cells) are indexed by elements w of the Weyl group W. The dimension of the cell X_w is equal to two times the length of the corresponding word. The Schubert cells form a free basis for the cohomology of G/T and the Classical Schubert calculus describes the multiplicative structure of $H^*(G/T;\mathbf{Z})$ in terms of this basis.

The closure $\overline{X}_w$ of a Bruhat cell X_w is an algebraic subvariety of G/T (possibly singular) which is called a Schubert variety. Let i_w denote the inclusion of the Schubert variety $\overline{X}_w$. Then $i_{w*}\mathcal{O}_{\overline{X}_w}$ is a coherent sheaf on G/T and defines an element of $K_{coh}(G/T)$ - the Grothendieck group of coherent sheaves on G/T. By a theorem of Grothendieck, $K_{coh}(G/T)$ coincides with $K(G/T)$, the complex K-theory of the flag variety. In this way Bruhat cells determine classes in $K^*(G/T)$.

In general the cells X_w do not determine classes in $h^*(G/T)$ for an arbitrary generalized cohomology theory. However, for a multiplicative cohomology theory h^* with complex orientation we construct a family of elements in $h^*(G/T)$ which

- generalize the classes determined by X_w (in ordinary cohomology and K-theory);
- generate $h^*(G/T)$ as a module over the ring of coefficients h^*;
- arise from classical geometric objects.

These elements descend to $h^*(G/T)$ from $MU^*(G/T)$ under the complex orientation $MU \to h$. In $MU^*(G/T)$ they are represented by the Bott-Samelson resolutions of singularities.

An algebra generated by operators which are in one to one correspondence with the simple reflections in the Weyl group acts on $h^*(G/T)$. In fact any resolution class arises by applying a product of these operators to the resolution of the zero dimensional Schubert variety. These operators were considered by Bernstein-Gel'fand-Gel'fand, Demazure and Kostant-Kumar. Using a geometric description of the operators we define analogous operators on $h^*(G/T)$. The formula of Brumfiel-Madsen for transfer in bundles of homogeneous spaces yeilds an expression for the operators in terms of the Euler classes of line bundles on G/T. The formulas which we obtain generalize those known in the classical cases.

The classical operators satisfy braid relations (the relations between two simple reflections). This fact is responsible for existence of Schubert cycles in ordinary cohomology and K-theory. We show that these are essentially the only cases. More precisely we show that the braid relations are satisfied only for a generalized cohomology theory with the associated formal group law of ordinary cohomology or K-theory.

The operators enable us to express the resolution classes in terms of the characteristic classes. We also describe a recursive procedure for calculation of products of resolution classes with the Euler classes of line bundles which leads to a procedure for calculation of products of resolution classes. The algorithm yields Schubert calculus for any complex oriented cohomology theory but differs from known algorithms even in the classical cases.

This paper is organized as follows. After recalling some facts about multiplicative cohomology theories with complex orientation we give a short account of Quillen's geometric interpretation of complex cobordism. In the following sections we use this description of MU to define a family of classes in $MU^*(G/T)$ and give a procedure for calculating cup and cap products. In the section on operators we give our definition of the latter, derive the formula for the operators, and treat satisfiability of braid relations. The resulting formulas describe Schubert Calculus for any multiplicative cohomology theory with complex orientation.

2.Complex Oriented Cohomology Theories. In this section we shall recall the basic properties of multiplicative generalized cohomology theories with complex orientation.

A generalized cohomology theory h is a functor from topological spaces to graded Abelian groups satisfying Eilenberg-Steenrod axioms except for the dimension axiom [Dy]. A generalized cohomology theory h is called multiplicative if it is endowed with

1. an associative and graded commutative cup product (i.e. for every space X, $h^*(X)$ is a ring (with unit) and a map of spaces induces a ring homomorphism);
2. a coherent choice of the generators for $h^*(\mathbf{S^n})$ (for all n) over the ring of coefficients so that the suspension isomorphisms and multiplication by these generators coincide.

Let x be a point in X and $E \to X$ be a (real) vector bundle of rank n. The inclusion of the fiber of E over the point x

$$\mathbf{R}^n \to E$$

induces a map of Thom spaces

$$\mathbf{S^n} \to X^E.$$

An element in $h^*(X^E)$ which restricts to the generator of $h^n(\mathbf{S^n})$ under the induced map for all points x in X is called the Thom class of E and will be denoted U_E. A bundle which has a Thom class in h is said to be orientable for the theory h.

Example: The complex cobordism theory MU is a multiplicative cohomology theory. For a finite complex X an element of $MU^{2q}(X)$ is represented by a map

$$\Sigma^{2n} X \to MU(n+q)$$

for a large positive integer n. A complex vector bundle $E \to X$ of (complex) rank q has a classifying map

$$X \to BU(q)$$

inducing the map of Thom spaces

$$X^E \to MU(q)$$

which represents the Thom class U_E in $MU^{2q}(X^E)$.

The following properties of a multiplicative cohomology theory h are equivalent:

1. complex vector bundles are oriented for h;
2. the restriction map
$$h^*(BU(1)) \to h^*(\mathbf{S^2})$$
is surjective;
3. there is a natural transformation of multiplicative cohomology theories
$$MU \to h$$
(also called the complex orientation or the Thom class).

A cohomology theory possessing these properties is called complex oriented (or endowed with a complex orientation given by the map in 3). In a complex oriented cohomology theory the Euler class and the Chern classes of complex vector bundles are defined and satisfy the usual properties.

A complex orientation of a proper map of (smooth) manifolds
$$f : M \to N$$
is a factorization
$$M \overset{i}{\hookrightarrow} E \xrightarrow{\pi} N$$
where E is a complex vector bundle and i is an embedding with a stably complex normal bundle. A compact manifold is said to be complex oriented if and only if the map to a point is complex oriented (equivalently the tangent bundle is stably complex). A great number of examples of complex oriented maps comes from analytic maps between complex manifolds.

Let h be a complex oriented theory. A proper complex oriented map
$$f : M \to N$$
induces the Gysin homomorphism
$$f_* : h^*(M) \to h^{*+k}(N)$$
of degree $k = dimN - dimM$.

With proper choices of complex orientations made the Gysin homomorphism enjoys the following properties:

1. The projection formula holds (i.e. with notations as above f_* is a map of $h^*(N)$ modules).

2. Naturality (i.e. $(f \circ g)_* = f_* \circ g_*$).

3. Base change. Consider a Cartesian square

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ \theta \downarrow & & \downarrow \tau \\ X & \xrightarrow{f} & Z \end{array}$$

   Then $f^* \circ \tau_* = \theta_* \circ g^*$.

To a cohomology theory h with complex orientation corresponds a formal group law defined over the coefficient ring h^*, i.e. a power series $\mathcal{F}$ over h^* in two variables, say X and Y, satisfying

1. $\mathcal{F}(X,Y) = \mathcal{F}(Y,X)$;

2. $\mathcal{F}(X,0) = X$;

3. $\mathcal{F}(X, \mathcal{F}(Y,Z)) = \mathcal{F}(\mathcal{F}(X,Y), Z)$.

The group law expresses the Euler class of the tensor product of two complex line bundles in terms of the Euler classes of each, i.e.

$$\chi(L_1 \otimes L_2) = \mathcal{F}(\chi(L_1), \chi(L_2)).$$

For example the group law of ordinary cohomology is additive, i.e $\mathcal{F}(X,Y) = X + Y$, and the group law of complex K-theory is multiplicative, i.e. $\mathcal{F}(X,Y) = X + Y + uXY$, where u is a unit. As Quillen showed in [Q] the group law associated with complex cobordism is the universal group law.

3. Geometric interpretation of complex cobordism. In this section we shall recall a geometric definition of the generalized cohomology theory of complex cobordism and of the corresponding homology theory of complex bordism and describe the associated notions (e.g. the induced maps, products, duality) in terms of these definitions.

We shall define the (co)bordism groups for a smooth manifold (not necessarily compact). A finite CW complex is homotopy equivalent to an open subset of Euclidean space. All maps will be assumed to be smooth.

Complex bordism theory MU was originally defined by geometric means as bordism classes of maps of stably complex manifolds. More precisely for a space X the underlying set of the group $MU_q(X)$ is the set of equivalence classes of maps

$$M \xrightarrow{f} X.$$

M is closed, stably almost complex manifold. This means that M is a compact smooth manifold without boundary of dimension q with $\mathbf{T}M$ stably complex. Two such maps (M,f) and (N,g) are considered equivalent (bordant) if and only if their disjoint union extends to a map

$$W \to X$$

of a compact stably almost complex manifold W of dimension $q+1$, whose boundary is the union of M and N. It is also required that the stably almost complex structures on M and N induced by the embedding into W and the original ones are equivalent. The Abelian group structure on $MU_*(X)$ is given by the operation of disjoint union. By a theorem of Rene Thom the resulting groups coincide with those obtained by a homotopy-theoretic construction [S].

The dual cohomology theory MU called complex cobordism was given a geometric description by D.Quillen [Q]. We present an outline of his construction below.

For a manifold X of dimension n an element in $MU^{n-q}(X)$ is represented by a proper map

$$M \to X$$

of a (not necessarily compact) manifold M of dimension q together with an equivalence class of complex orientations. Two such are equivalent if and only if they are bordant as maps with complex orientations.

A map of spaces

$$X \xrightarrow{g} Y$$

induces a map

$$g_* : MU_*(X) \to MU_*(Y)$$

in bordism by composing: $g_*[(M,f)] = [(M, g\circ f)]$. It also induces a map

$$g^* : MU^*(Y) \to MU^*(X)$$

in cobordism by the following construction. Let (M,f) represent an element in $MU^*(Y)$. The map f can be chosen to be transverse to g. Then $g^*[(M,f)]$ is represented by the left vertical arrow in the diagram

$$\begin{array}{ccc} M\times_Y X & \longrightarrow & M \\ \downarrow & & \downarrow f \\ X & \xrightarrow{g} & Y \end{array}$$

Complex cobordism is a multiplicative cohomology theory. The external product

$$MU^*(X)\otimes MU^*(Y) \to MU^*(X\times Y)$$

is defined by taking the Cartesian product of maps. For a space X, $MU^*(X)$ is a (graded) commutative ring with unit under cup product defined by

$$[(M,f)][(N,g)] = \Delta^*[(M\times N, f\times g)],$$

where Δ is the diagonal embedding of X into $X\times X$. In other words the cup product is represented by the fiber product (i.e. by the "geometric intersection of cycles"). The unit element is represented by the identity map.

There is a cap product pairing

$$MU_*(X)\otimes MU^*(X) \to MU_*(X)$$

given by the fiber product of maps. Let

$$M \xrightarrow{f} X$$

represent a bordism class and let

$$N \xrightarrow{g} X$$

represent a cobordism class and let these representatives be chosen so that they are transverse to each other. Then the cap product of $[(M,f)]$ and $[(N,g)]$ is represented by the fiber product of maps. Since the map g has a complex orientation and N is stably almost complex, $M \times_X N$ inherits a stably almost complex structure.

As was shown in Section 2 MU is complex oriented by the identity transformation. In fact a stably almost complex manifold X has a fundamental class in $MU_*(X)$ represented by the identity map. Poincare duality can be described as follows. Let (M, f) represent an element in $MU^*(X)$. The complex orientation of f and the stably almost complex structure on X induce a stably almost complex structure on M. With this structure (M, f) represents an element in $MU_*(X)$ of complementary (with respect to the dimension of X) degree which is the Poincare dual.

For a proper complex oriented map

$$X \xrightarrow{f} Y$$

the Gysin map

$$f_* : MU^*(X) \to MU^*(Y)$$

is defined by composing with f. In presence of Poincare duality this is the adjoint to the induced map in bordism.

Stably complex vector bundles are oriented in complex cobordism. For a vector bundle $E \to X$ with the zero section ζ the Thom class U_E in $MU^*(X^E)$ is defined by $U_E = \zeta_*(1)$, where X_E denotes the Thom space of E and ζ denotes the zero section. The Euler class $\chi(E)$ in $MU^*(X)$ is defined by $\chi(E) = \zeta^*\zeta_*(1)$. In other words the Euler class is represented by the inclusion of the submanifold of zeros of a generic section.

4.Geometry and toplogy of the flag variety. In what follows G is a compact simply-connected Lie group. We shall fix a maximal torus $T \subset G$. The complexified Lie algebras of G and T will be denoted by $\mathbf{g}$ and $\mathbf{t}$ respectively. Let $\mathcal{R}$ denote the set of roots of G and $\mathcal{R}^+$ the set of positive roots with the corresponding simple roots $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. The compact group G can be embedded in a complex group $G_{\mathbf{C}}$ with Lie algebra $\mathbf{g}$. In $G_{\mathbf{C}}$ we choose the Borel subgroup B containing T and having $\mathcal{R}^+$ for the roots. We shall denote its Lie algebra by $\mathbf{b}$.

The inclusion

$$(G,T) \hookrightarrow (G_{\mathbf{C}}, B)$$

induces an isomorphism

$$G/T \to G_{\mathbf{C}}/B$$

and a complex structure on G/T (see [BH]). In fact G/T is a smooth projective algebraic variety over the complex numbers. Since the flag manifold is a homogeneous space we have the identification

$$\mathbf{T}G/T = G \times_T \mathbf{g}/\mathbf{b}.$$

As a representation of T the Lie algebra $\mathbf{g}$ decomposes as

$$\mathbf{g} = \mathbf{b} \oplus \bigoplus_{\alpha \in \mathcal{R}^+} \mathbf{C}_{-\alpha}$$

where $\mathbf{C}_{-\alpha}$ denotes the character of T defined by the root $-\alpha$. Therefore the tangent bundle of the flag manifold decomposes into a sum of line bundles

$$\mathbf{T}G/T = \bigoplus_{\alpha \in \mathcal{R}^+} L(-\alpha).$$

Here and in what follows the line bundle $L(\lambda)$ is defined by

$$L(\lambda) = G \times_T \mathbf{C}_\lambda$$

for a weight λ of $\mathbf{t}$.

Let W denote the Weyl group of G. The flag variety can be decomposed into a union of B-orbits (under the left B-action):

$$G_{\mathbf{C}}/B = \bigcup_{w \in W} X_w$$

where $X_w = BwB$ and $dim X_w = 2l(w)$ ($l(w)$ denotes the length of the word w) ([BGG]). This decomposition gives G/T a structure of a CW complex. The closure $\overline{X}_w$ of the cell X_w is an algebraic subvariety of G/T (usually singular) called a Schubert variety. Its boundary is a union of cells corresponding to shorter words in the Weyl group (in fact to all the words which are smaller then w in Bruhat ordering). This means that it is made up of cells of codimension at least two. Consequently $\overline{X}_w$ defines a cycle on G/T of dimension $2l(w)$. Therefore the integral colomology of the flag variety is free with basis $\overline{X}_w$, $w \in W$ and is concentrated in even degrees.

Since MU^* is also evenly graded the Atiyah-Hirzebruch spectral sequence

$$H^*(G/T; MU^*) \Rightarrow MU^*(G/T)$$

collapses at the E_2 term and there is an isomorphism of abelian groups

$$MU^*(G/T) \cong H^*(G/T) \otimes MU^*.$$

Let h be a multiplicative complex oriented cohomology theory. Then the orientation $MU \to h$ induces a map of corresponding Atiyah-Hirzebruch spectral sequences which commutes with the differentials and products. The fact that the spectral sequence for MU collapses at E_2 implies that the spectral sequence for h collapses also. Consequently $h^*(G/T)$ is a free module over the coefficient ring h^* on a basis of elements of even degree. It is also clear that the image of $MU^*(G/T)$ in $h^*(G/T)$ generates $h^*(G/T)$ over the ring of coefficients. It follows that we can restrict our attention to complex cobordism. The results for other cohomology theories are obtained by specialization using the Thom class map.

The Schubert cycles $\overline{X}_w$ which generate the associated graded group $H^*(G/T) \otimes MU^*$ of $MU^*(G/T)$ do not have canonical liftings to cobordism classes. However, we can use the resolutions of singularities of Schubert varieties to represent liftings. The canonical family of such cobordism classes is provided by Bott-Samelson resolutions of singularities of Schubert varieties (also known as the canonical resolutions). They appeared in the context of Schubert calculus in the work of Bott-Samelson [BS], Demazure [D], and Arabia [A].

The Weyl group W of G is generated by reflections $s_1, ..., s_l$ which are in one to one correspondence with the simple roots. We shall also consider subgroups $H_1, ..., H_l$ of G of maximal rank. The subgroup H_k can be described as the subgroup of maximal rank with roots α_k and $-\alpha_k$. It is a product of T with a copy of $SU(2)$ in G.

We give a brief account of the construction and the properties of the Bott-Samelson resolutions below. A detailed study is presented in the paper by Bott and Samelson in Chapter 3.

Let $I = (i_1, \ldots, i_n)$ be a multiindex with $1 \leq i_j \leq l$. The space R_I of the resolution is given by

$$R_I = H_{i_1} \times_T H_{i_2} \times_T \cdots \times_T H_{i_n}/T$$

where T acts by multiplication on the right on H_{i_k} and by multiplication on the left (by the inverse) on $H_{i_{k+1}}$. There is a map

$$R_I \xrightarrow{r_I} G/T$$

induced by group multiplication:

$$(h_1, \ldots, h_n) \mapsto h_1 \cdots h_n \cdot T.$$

Suppose the multiindex I indexes a decomposition of a word $w \in W$ into a product of simple reflections

$$w = s_{i_1} \cdot s_{i_2} \cdots s_{i_n}$$

with $l(w) = n$ (such a decomposition is called reduced). Then (R_I, r_I) is a resolution of singularities of $\overline{X}_w$ i.e. the map r_I, is proper and birational ([D]).

The spaces R_I are towers of $\mathbb{C}P^1$ bundles. For example for a multiindex of length one we have

$$R_{(i_j)} = H_{i_j}/T \cong \mathbb{C}P^1.$$

In general let I be as before and let $J = (i_1, \ldots, i_{n-1})$. There is a natural projection

$$R_I \rightarrow R_J$$

which is a projective line bundle. In fact R_I is the projectivisation of a complex rank two vector bundle on R_J. This makes $h^*(R_I)$ easily computable for any complex oriented theory h. For example $h^*(R_I)$ is generated (as an algebra over the coefficients) by the degree two part. The latter fact plays an important role in what follows.

The resolution R_I is a complex manifold and the map r_I is a holomorphic map, and, therefore, naturally complex oriented. The pair (R_I, r_I) represents a complex cobordism class which we shall call a resolution class and denote Z_I in what follows. By Z_e we shall denote the cobordism class represented by the inclusion of the zero dimensional cell which corresponds to the empty multiindex.

The resolution class corresponding to a reduced decomposition of $w \in W$ descends to the class of the Schubert variety $\overline{X}_w$ in ordinary cohomology and complex K-theory. If the multiindex I indexes a nonreduced product of reflections the corresponding resolution class descends to zero. The cobordism classes corresponding to different choices of reduced decompositions are different and, in fact, usually descend to different classes under complex orientation. This follows from results in [BE2] and will be discussed in more detail in the following sections.

There is an ample supply of complex line bundles on G/T which account for all of its rational cohomology. These line bundles are associated to the characters of T. They are induced by the map

$$G/T \to BT$$

classifying the principal bundle

$$G \to G/T.$$

The induced homomorphism in cohomology is called the "characteristic homomorphism"

$$\chi : H^*(BT) \to H^*(G/T).$$

The cohomology ring of BT with complex coefficients can be naturally identified with the completion of the ring of complex valued polynomial functions on the complexified Lie algebra of T. The linear functionals are Lie algebra characters which exponentiate to characters on T. Under the characteristic homomorphism they are mapped to Euler classes of line bundles on G/T associated to the corresponding characters of T.

The characteristic homomorphism is surjective in rational cohomology. Consequently it is surjective in any complex oriented theory whose coefficients form an algebra over the rational numbers. In that case there is a short exact sequence

$$0 \to (h^*(BT)^W_+) \to h^*(BT) \to h^*(G/T) \to 0$$

which identifies the kernel of the characteristic homomorphism with the ideal in $h^*(BT)$ generated by the Weyl group invariants of positive degree.

5.Operators on $MU^*(G/T)$. In this section we define operators A_i which generate an algebra acting on $MU^*(G/T)$. This action makes $MU^*(G/T)$ into a cyclic module generated by the class Z_e represented by the inclusion of the zero dimensional cell. The operators allow us to express the resolution classes in terms of the image of the characteristic homomorphism. They are used in conjunction with the cap product formula to describe the multiplicative structure of $MU^*(G/T)$.

For each $i = 1, \ldots, l$ there is a holomorphic fiber bundle

$$\pi_i : G/T \to G/H_i$$

with fiber isomorphic to $\mathbf{C}P^1$. Using these we define a family of operators A_i (MU^*-module endomorphisms) on $MU^*(G/T)$ by

$$A_i = \pi_i^* \circ \pi_{i*} : MU^*(G/T) \to MU^{*-2}(G/T).$$

These operators appear in the work of Bernstein-Gelfand-Gelfand, Kostant-Kumar, and Demazure and play an essential role in Schubert calculus. This largely stems from their action on the resolution classes as described in the following proposition.

Proposition 1. Let $I = (i_1, \ldots, i_n)$ and $J = (i_1, \ldots, i_{n+1})$. Then

1. $A_{i_{n+1}} Z_I = Z_J$
2. $A_i^2 = A_i(1)A_i$

Proof: 1.From the description of MU given earlier we find that $\pi i_{n+1} * Z_I$ is represented by the map $(R_I, \pi_{i_{n+1}} \circ r_I)$. Since $\pi_{i_{n+1}}$ is a submersion it is transverse to $\pi_{i_{n+1}} \circ r_I$. Therefore $A_{i_{n+1}} Z_I$ is represented by the left vertical arrow in the diagram

$$\begin{array}{ccc} H_{i_1} \times_T \cdots \times_T H_{i_n}/T \times_{G/H_{i_{n+1}}} G/T & \longrightarrow & R_I \\ \downarrow & & \downarrow \pi_{i_{n+1}} \\ G/T & \xrightarrow{\pi_{i_{n+1}}} & G/H_{i_{n+1}} \end{array}$$

There is an isomorphism

$$\begin{array}{ccc} H_{i_1} \times_T \cdots \times_T H_{i_{n+1}}/T & \xrightarrow{\sim} & H_{i_1} \times_T \cdots \times_T H_{i_n}/T \times_{G/H_{i_{n+1}}} G/T \\ (h_{i_1}, \ldots, h_{i_{n+1}}) & \mapsto & (h_{i_1}, \ldots, h_{i_1} \cdot h_{i_2} \cdots h_{i_{n+1}}) \end{array}$$

The inverse is given by

$$(h_{i_1}, \ldots, h_{i_n}, g) \mapsto (h_{i_1}, \ldots, h_{i_n}, (h_{i_1} h_{i_2} \cdots h_{i_n})^{-1} g).$$

2. This follows directly from the projection formula.

Corollary 1: $Z_I = A_{i_1} \circ \cdots \circ A_{i_n} Z_e$, where Z_e denotes the class represented by the inclusion of the zero dimensional Schubert variety.
Corollary 2: The operators A_i acting on ordinary cohomology satisfy $A_i^2 = 0$.
Proof: This follows from the observation that $A_i(1)$ is an element of degree -2 of the ring of coefficients.

The operators defined above allow us to relate the classes of the form Z_I to the characteristic classes of line bundles on G/T as follows. For a complex vector bundle E let $\chi(E)$ denote the Euler class of E (in MU). As we explained in the preceeding section, the tangent bundle of the flag variety decomposes as a sum of line bundles

$$\mathbf{T}G/T = \bigoplus_{\alpha \in \mathcal{R}^+} L(-\alpha).$$

The Euler characteristic of the flag variety is equal to the order of the Weyl group (as follows from the description of the cellular structure of G/T). Therefore we have the following equalities

$$|W| Z_e = \chi(\mathbf{T}G/T) = \prod_{\alpha \in \mathcal{R}^+} \chi(L(-\alpha)).$$

The second one follows from the Whitney sum formula. The first one follows from the fact that there is a vector field with the number of simple zeros equal to the Euler characteristic. Since the flag manifold is path connected all points define the same cobordism class.

As follows from Proposition 1 other resolution classes are generated from Z_e by repeated application of operators A_i. That is if $I = (i_1, \ldots, i_n)$

$$Z_I = A_{i_1} \circ \cdots \circ A_{i_n} Z_e.$$

Using the previously obtained expression for Z_e we arrive at

$$Z_I = \frac{1}{|W|} A_{i_n} \cdots A_{i_1} \prod_{\alpha \in \mathcal{R}^+} \chi(L(-\alpha_j)).$$

The operators A_i have natural liftings to operators C_i on $MU^*(BT)$. Let

$$p_i : BT \to BH_i$$

be the $\mathbf{C}P^1$ fiber bundle induced by the inclusion $T \to H_i$. Then define

$$C_i = p_i^* \circ p_{i*} : MU^j(BT) \to MU^{j-2}(BT)$$

Naturality properties imply that the characteristic homomorphism intertwines the actions of C_i and A_i.

The operators C_i acting on a suitable localization of $MU^*(BT)$ are given by the formula of the Proposition 2 below. The formula is derived in [BE2] using homotopy theoretic considerations.

Proposition 2:

$$C_i = (1 + s_i)\frac{1}{\chi(L(-\alpha_i))}$$

Proof: We apply the formula of Brumfiel-Madsen for transfer in bundles of homogeneous spaces with compact structure group (see e.g. [BM]) to the map p_i and use the relationship between transfer and the Gysin homomorphism for smooth fibre bundles (see [BG]). See [BE2] fo details.

The operators discussed above are in fact defined for any complex oriented cohomology theory and are given by the same formula where the Euler class should be interpreted as the Euler class in that theory. The examples below show that in the classical cases the familiar formulas are recovered.

Examples:

1. (Ordinary cohomology.) The cohomology ring of BT is naturally isomorphic to the ring of polynomial functions on the complexified Lie algebra of T. The latter is generated by the weights which are assigned degree two. The isomorphism is established by mapping an integral weight to the Euler class of the dual of the line bundle on BT associated to the corresponding character of T. It is customary to identify the two rings and to set symbolically $\chi(L(\lambda)) = \lambda$. In this notation the operator C_i is given by the following formula:

$$C_i = (1 + s_i)\frac{1}{-\alpha_i}$$

which appears in [BGG] and [KK1].

2. (Complex K-theory.) K-theory of BT is naturally isomorphic to a completion of the representation ring of T. Under this isomorphism a character is identified with the dual of the associated line bundle. Recall that the Euler class of a line bundle L is given by $1 - L$. Denoting by e^λ the character corresponding to the weight λ we obtain the following formula for the operator C_i in K-theory:

$$C_i = (1 + s_i)\frac{1}{1 - e^{-\alpha_i}}$$

which is used for example in [KK2].

The operators in the examples above satisfy braid relations as can be checked directly. Recall that the braid relations are the relations, satisfied by the simple reflections in the Weyl group. This implies that an ordered product of the operators indexed by a reduced decomposition of an element of the Weyl group is in fact independent of the choice of reduced decomposition and depends only on the Weyl group element itself.
Corollary 3: For a multiindex $I = (i_1, \ldots, i_n)$ consider the decomposition of a word $w = s_{i_1} \cdots s_{i_n}$ in the Weyl group. If the decomposition is not reduced then the corresponding product of operators acting on ordinary cohomology is zero.
Proof: Since the operators acting on ordinary cohomology satisfy braid relations a product corresponding to a reduced decompositions is equal (by repeated application of braid relations) to one involving a square of an operator. The claim follows now from Corollary 2.
Corollary 4: The image of class Z_I in ordinary cohomology is zero if and only if I indexes a reduced decomposition.
Proof: This follows directly from Corollaries 2 and 3.

The following theorem show that these are essentially the only two cases in which it is true.

Theorem:([BE2]) Let G be a compact connected Lie group with at least two nonorthogonal roots and let h be a multiplicative cohomology theory with complex orientation with torsion-free coefficient ring h^*. Then the operators C_i for $i = 1, \ldots, l$ defined above satisfy the braid relations if and only if the formal group law associated with h is polynomial.
Proof: For complete proof see [BE2]. Here we present a rough outline. Since the braid relations involve only pairs on nonorthogonal roots the question reduces to the rank two simple groups and we proceed by examining cases. For example suppose we

have the identity

$$C_1C_2C_1 = C_2C_1C_2.$$

This is the case $SU(3)$. Using the formula for the operators of the preceding proposition we expand both sides into a linear combination of products of reflections with coefficients in the fraction field $\mathcal{Q}$ of $h^*(BT)$. In doing so it is important to remember that reflections act on Euler classes by acting on the roots. Since the elements of the Weyl group act on $\mathcal{Q}$ in a linearly independent fashion the identity on the operators is checked by equating the coefficients in the expansion. This leads to the functional equation

$$g(\alpha_1)g(\alpha_2) + g(-\alpha_1)g(\alpha_1+\alpha_2) = g(\alpha_1)g(-\alpha_1)$$

where $g(\alpha)$ stands for $\chi(L(-\alpha_i))$. Such an equation implies vanishing of coefficients of the formal group law. It turns out that only the additive and the multiplicative group laws satisfy it. The argument relies on the technical results of [BE1] and their extension in [G].

Remark: It should be pointed out that the theorem does not imply the result for the operators on $h^*(G/T)$. One must show that the difference of the two sides of the braid relation acting on a resolution class lies in the kernel of the characteristic homomorphism. This can be done in interesting cases (e.g. elliptic cohomology) using the Schubert calculus developed in the next section.

6.Cap product formula and Schubert calculus. In the preceding section we developed a method for expressing resolution classes in terms of the image of the characteristic homomorphism. We shall presently describe a method for computing the products of resolution classes with characteristic classes of line bundles on G/T. Combined with the results of the previous section this yields a method for computing products of resolution classes.

Let $I = (i_1, \ldots, i_n)$ be a multiindex with $1 \le i_j \le l$. Recall that the cobordism class $Z_I \in MU^*(G/T)$ is represented by the map

$$r_I : R_I \to G/T.$$

Let $L(\lambda)$ be the line bundle associated to a weight λ. It has the Euler class $\chi(L(\lambda))$ in $MU^*(G/T)$. The cap product formula expresses the product $Z_I\chi(L(\lambda))$ in terms of other resolution classes.

Lemma 1.

$$Z_I\chi(L(\lambda)) = r_{I*}\chi(r_I^*L(\lambda)).$$

Proof: The equality is a direct consequence of the projection formula for the Gysin homomorphism. We can rewrite the right hand side as

$$r_{I*}\chi(r_I^*L(\lambda)) = r_{I*}r_I^*\chi(L(\lambda)) = r_{I*}(1)\chi(L(\lambda) = Z_I\chi(L(\lambda)).$$

Lemma 1 shows that the cap product formula comes from a formula for Euler classes in $MU^*(R_I)$. We shall be dealing primarily with the space R_I. The line bundle $r_I^*L(\lambda)$ will be simply by $L(\lambda)$. The space R_I has the advantages of being much more simple in structure than the flag variety. As was pointed out in Section 4 it's integral cohomology is generated (as an algebra over the ring of coefficients) by classes of degree two, and the same is true about complex cobordism.

In order to perform calculations in $MU^*(R_I)$ we need to establish some notation. Given a multiindex $I = (i_1, \ldots, i_n)$ we define new multiindecies $I_{<k}, I_{>k}, I^k$ by:

$$\begin{aligned} I_{<k} &= (i_1, \ldots, i_{k-1}) \\ I_{>k} &= (i_{k+1}, \ldots, i_n) \\ I^k &= (i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n). \end{aligned}$$

There is a natural projection

$$\pi_{<k} : R_I \to R_{I_{<k}}$$

given by

$$(h_{i_1}, \ldots, h_{i_n}) \mapsto (h_{i_1}, \ldots, h_{i_{k-1}}).$$

A subindex J of I of length k is determined by a one to one order preserving map

$$\sigma : \{1, \ldots, k\} \to \{1, \ldots, n\}$$

by the rule $j_m = i_{\sigma(m)}$. For the subindex J of I of length k there is a natural embedding

$$i_{J,I} : R_J \to R_I$$

defined by converting a k-tuple $(h_{j_1}, \ldots, h_{j_k})$ to the n-tuple having h_{j_m} in the $i_{\sigma(m)}$-th slot for $1 \leq m \leq k$ and the identity element elsewhere. Observe that $r_J = r_I \circ i_{J,I}$.

A pair $(R_J, i_{J,I})$ represents an element of $MU^*(R_I)$. We shall denote it as well as the integral cohomology class it determines by $[R_J]$. In view of the observation above we have

Lemma 2. $r_{I*}[R_J] = Z_J$

In view of the lemma the classes of the form $[R_J]$ are precisely the classes in terms of which we want to obtain the expression for $\chi(L(\lambda))$.

A complex line bundle is determined up to isomorphism by its first Chern class $c_1(L)$ in integral cohomology. The group of line bundles on R_I denoted $Pic(R_I)$ is isomorphic to $H^2(R_I; \mathbf{Z})$, which is free and has a convinient basis of elements, whose liftings to $MU^*(R_I)$ can be chosen to be $[R_{I^k}]$.

The isomorphism

$$Pic(R_I) \to H^2(R_I; \mathbf{Z})$$

is given by the first Chern class. The target group is free with basis consisting of classes $[R_{I^k}]$ with $1 \leq k \leq n$. Therefore we can choose a basis for $Pic(R_I)$ consisting of line bundles L_k (where $1 \leq k \leq n$, satisfying

$$c_1(L_k) = [R_{I^k}].$$

In fact the line bundles L_k can be constructed to satisfy

$$\chi(L_k) = [R_{I^k}]$$

in complex cobordism. One only needs to make sure that L_k has a section vanishing precisely on R_{I^k}. This basis is useful in calculation of Euler class for it yields the answer in terms of classes $[R_{I^k}]$. In case of ordinary cohomology the answer is just a linear combination, but in general will be a sum of products of classes $[R_{I^k}]$ computed according to the group law associated with the cohomology theory.

Proposition 3. Let λ be a weight. Let $I = (i_1, ..., i_n)$ be a multiindex and let s_I denote the corresponding product of reflections. Then the line bundle $L(\lambda)$ on R_I decomposes as

$$L(\lambda) = \bigotimes_{k=1}^{n} L_k^{<s_{I_{>k}}\lambda, \alpha_{i_k}>}$$

Proof: Let

$$L(\lambda) = \bigotimes_{k=1}^{n} L_k^{\otimes a_k}$$

be the decomposition of the line bundle $L(\lambda)$ in terms of the basis consisting of the line bundles L_k described above. To compute the exponents a_k we use ordinary cohomology.

Taking the first Chern class of both sides of the above formula we obtain

$$c_1(L(\lambda)) = \sum_{k=1}^{n} a_k [R_{I^k}].$$

Poincare duality implies that the coefficients a_k are exactly the solutions of the system of linear equations (over the integers) obtained by multiplying both sides of the above by the classes $[R_{i_j}]$ for $1 \leq j \leq n$:

$$c_1(L(\lambda))[R_{i_j}] = \sum_{k=1}^{n} a_k [R_{I^k}][R_{i_j}].$$

The intersection numbers which appear in the equations are determined in the Lemmas 3 and 4 following the proposition giving

$$< \lambda, \alpha_{i_j} >= a_j + \sum_{k=j+1}^{n} a_k < \alpha_{i_k}, \alpha_{i_j} > .$$

Lemma 5 shows that the same system of equations is satisfied by $< s_{I_{>k}}\lambda, \alpha_{i_k} >$. Therefore the a_k's are given by

$$a_k =< s_{I_{>k}}\lambda, \alpha_{i_k} > .$$

Remark: Similar calculations appear in [BS] Chap. 3.

Lemma 3. $c_1(\pi^*_{<k} r^*_{I_{<k}} L(\lambda))[R_{i_j}] = - < \lambda, \alpha_{i_j} >$ if $j < k$ and is zero otherwise.
Proof: If $j < k$, then

$$r_{I_{<k}} \circ \pi_{<k} \circ i_{(i_j),I} = r_{i_j}.$$

If $j \geq k$, then $r_{I_{<k}} \circ \pi_{<k} \circ i_{(i_j),I}$ is a constant map.

Using the projection formula as in Lemma 1 it is easy to see that $c_1(\pi^*_{<k} r^*_{I_{<k}} L(\lambda))[R_{i_j}]$ is equal to the degree of the line bundle $i^*_{(i_j),I} \pi^*_{<k} r^*_{I_{<k}} L(\lambda)$ on R_{i_j}. If $j \geq k$, then the line bundle $i^*_{(i_j),I} \pi^*_{<k} r^*_{I_{<k}} L(\lambda)$ is trivial, hence of degree zero. If $j < k$ consider the line bundle $r^*_{i_j} L(\lambda)$ on R_{i_j}. By Proposition 1 the map

$$r_{i_j} : R_{i_j} \longrightarrow G/T$$

can be identified with the inclusion

$$i : H_{i_j}/T \hookrightarrow G/T.$$

Under the identification of H_{i_j}/T with the projectivization of the two dimensional representation of H_{i_j} with highest weight λ_{i_j} (the dual fundamental weight) the line bundle $i^* L(\lambda_{i_j})$ corresponds to $\mathcal{O}_{CP^1}(-1)$ - the tautological line bundle of degree -1. A line bundle associated to a weight λ is a tensor power of $i^* L(\lambda_{i_j})$ with exponent the multiplicity of λ_{i_j} in λ which is precisely $< \lambda, \alpha_{i_j} >$.

Lemma 4. The following integral cohomology intersection numbers arize:
if $j > k$ then $[R_{I^k}][R_{i_j}] = 0$;
if $j = k$ then $[R_{I^k}][R_{i_j}] = 1$;
if $j < k$ then $[R_{I^k}][R_{i_j}] = < \alpha_{i_k}, \alpha_{i_j} >$.
Proof: In the second case we have a transverse intersection at R_e (one point) so $[R_{I^k}][R_{i_j}] = 1$. In the remaining cases R_{i_j} is contained in R_{I^k}. The normal bundle of R_{I^k} in R_I is easily seen to be $\pi^*_{<k} r^*_{I_{<k}} L(-\alpha_{i_k})$. Then the intersection number can be computed by

$$[R_{I^k}][R_{i_j}] = c_1(\pi^*_{<k} r^*_{I_{<k}} L(-\alpha_{i_k}))[R_{i_j}]$$

in which case the answer follows from Lemma 1.

Lemma 5. Let $I = (i_1, ..., i_n)$ be a multiindex with $1 \leq i_j \leq l$, and let s_I be the corresponding product of simple reflections. Then for any weight λ the following identity holds:

$$\lambda = s_I\lambda + \sum_{k=1}^{n} < s_{I_{>k}}\lambda, \alpha_{i_k} > \alpha_{i_k}.$$

Proof: The identity is easily verified by expanding $s_I\lambda$.

As an immediate corollary of the proposition we obtain the cap product formula in ordinary cohomology.

Theorem: Let λ be a weight and let $L(\lambda)$ denote the associated line bundle on G/T. Let $I = (i_1, ..., i_n)$ be a multiindex indexing a reduced decomposition of $w = s_I$ in the Weyl group (i.e. $l(w) = n$). Then the cap product of $c_1(L(\lambda))$ with the homology class of the Shubert variety $\overline{X_w}$ is given by the formula

$$c_1(L(\lambda)) \cap \overline{X_w} = \sum_k < \lambda, s_{I_{>k}}^{-1}\alpha_{i_k} > \overline{X_{s_{I^k}}}$$

where k satisfies $1 \leq k \leq n$ and $l(s_{I^k}) = l(w) - 1$.
Proof: Take the resolution R_I of the Schubert variety $\overline{X_w}$ given by the multiindex I and let $L(\lambda)$ also denote the pullback of the line bundle to R_I. Then according to the proposition $L(\lambda)$ decomposes as

$$L(\lambda) = \bigotimes_{k=1}^{n} L_k^{<\lambda, s_{I_{>k}}^{-1}\alpha_{i_k}>}.$$

Computing the first Chern class of both sides we obtain

$$c_1(L(\lambda)) = \sum_{k=1}^{n} < \lambda, s_{I_{>k}}^{-1}\alpha_{i_k} > [R_{I^k}].$$

Now, since the expression on the right hand side does not contain any products, pushing it forward by the Gysin map r_{I*} yields

$$r_{I*}c_1(L(\lambda)) = \sum_{k=1}^{n} < \lambda, s_{I_{>k}}^{-1}\alpha_{i_k} > Z_{I^k}.$$

Here Z_I denotes the class in ordinary homology which arises from the class Z_I in complex bordism represented by the resolution (R_I, r_I). The left hand side is equal to $c_1(L(\lambda))Z_I$ by Lemma 1.

As was noted earlier (see corollary to Proposition 1) under the Thom class map the bordism class Z_I descends to $\overline{X_{s_I}}$ if the multiindex I indexes a reduced decomposition of s_I and to zero otherwise. Thus we obtain the cap product formula.

We shall now present a procedure for calculation of products $\chi(L(\lambda))Z_I$ in complex cobordism. The method outlined in the proof of the cup product formula in ordinary cohomology breaks down when followed in cobordism because the group law in cobordism is not additive like in ordinary cohomology. Consequently when we compute the Euler class of

$$L(\lambda) = \bigotimes_{k=1}^{n} L_k^{<\lambda, s_{I>k}^{-1}\alpha_{i_k}>}$$

we obtain an expression for $\chi(L(\lambda))$ containing products of classes of the form $[R_{I^k}]$. The Gysin map r_{I*} is not a ring homomorphism and we have to compute the products before applying it. Since the products can contain more than two factors we will have to compute products of the form $[R_{I^k}][R_J]$ where J is a submultiindex of I. We shall reduce the computation of products to the computation of the Euler classes of line bundles on R_J. The latter are computed using the formula of the proposition and the group law. Notice that R_J is of strictly lower dimension than R_I. Although every application of the group law yields new products the dimension of the ambient space decreases. This means that the group law will eventually give product free result to which a pushforward can be applied.

It remains to reduce the computation of $[R_{I^k}][R_J]$ to a calculation of characteristic classes. Two cases arise, namely either $I^k \cup J = I$ or J is a subindex of I^k. If $I^k \cup J = I$ the intersection is transverse and the homological intersection coincides with the set theoretic one, i.e.

$$[R_{I^k}][R_J] = [R_{I^k \cap J}].$$

Otherwise we have $J \subset I^k$ and

$$R_J \overset{i_{J,I^k}}{\hookrightarrow} R_{I^k} \overset{i_{I^k,I}}{\hookrightarrow} R_I \overset{r_I}{\to} G/T.$$

Let ν denote the normal bundle of R_{I^k} in R_I. Then

$$[R_J][R_{I^k}] = i_{J,I^k *}\chi(i^*_{J,I^k}\nu).$$

We next observe that

$$i^*_{J,I^k}\nu = i^*_{J,I^k}\pi^*_{<k} r^*_{I_{<k}} L(-\alpha_{i_k}).$$

The following cases arise. If $J \subset I_{>k}$ then R_J is contained in the fiber of $\pi_{<k}$. In that case $i^*_{J,I^k}\nu$ is trivial and the intersection is equal to zero. If $J \subset I_{<k}$ then

$$\pi_{<k} \circ i_{J,I^k} = i_{J,I_{<k}}$$

and

$$i^*_{J,I^k}\nu = r^*_J L(-\alpha_{i_k}).$$

Finally if neither of the above is the case there is an index m such that $J_{<m} \subset I_{<k}$ and $J_{>m} \subset I_{>k}$. Then

$$\pi_{<k} \circ i_{J,I^k} = i_{J_{<m},I_{<k}} \circ \pi_{<m}$$

and for the line bundle we have

$$i^*_{J,I^k}\nu = \pi^*_{<m} i^*_{J_{<m},I_{<k}} r^*_{I_{<k}} L(-\alpha_{i_k}) = \pi^*_{<m} r^*_{J_{<m}} L(-\alpha_{i_k}).$$

This completes the discussion of the cap product formula in complex cobordism.

We now have the means to express a resolution class Z_I in terms of the characteristic classes (using the operators A_i) and to calculate the products of resolution classes with characteristic classes (using the cap product "formula" described above). These combined yield a method for calculation of products of resolution classes in terms of resolution classes. We summarize all of the preceding discussion in a theorem.

Theorem: Let G be a compact connected Lie group with maximal torus T. Let h be a multiplicative cohomology theory with complex orientation. Then there is an algorithmic procedure for computing products in $h^*(G/T)$ in the set of generators of the form Z_I. The procedure depends only on the root system data and the formal group law associated with the cohomology theory.

Calculation. In this section we present the results of the calculations in the case of the group $G = SU(3)$. The flag variety is (complex) tree dimensional and has six cells.

There are two simple roots α_1 and α_2. The positive roots include the simple roots and their sum. The Weyl group is generated by two reflections s_1 and s_2 satisfying $s_1 s_2 s_1 = s_2 s_1 s_2$. Below we list the resolution classes in terms of the characteristic classes for all reduced decompositions of the elements of the Weyl group. To simplify notation we shall write $\chi(\lambda)$ for $\chi(L(\lambda))$.

$$Z_e = \tfrac{1}{6}\chi(-\alpha_1)\chi(-\alpha_2)\chi(-\alpha_1-\alpha_2)$$

$$Z_{s_1} = \tfrac{1}{3}\chi(-\alpha_1)\chi(-\alpha_1-\alpha_2)$$

$$Z_{s_2} = \tfrac{1}{3}\chi(-\alpha_2)\chi(-\alpha_1-\alpha_2)$$

$$Z_{s_1 s_2} = \tfrac{1}{3}(\chi(-\alpha_1) - \chi(-\alpha_1-\alpha_2))$$

$$Z_{s_2 s_1} = \tfrac{1}{3}(\chi(-\alpha_2) - \chi(-\alpha_1-\alpha_2))$$

$$Z_{s_1 s_2 s_1} = \tfrac{1}{3}\left(\tfrac{\chi(-\alpha_1-\alpha_2)}{\chi(-\alpha_1)} + \tfrac{\chi(-\alpha_2)}{\chi(\alpha_1)} + 2\right)$$

$$Z_{s_2 s_1 s_2} = \tfrac{1}{3}\left(\tfrac{\chi(-\alpha_1-\alpha_2)}{\chi(-\alpha_2)} + \tfrac{\chi(-\alpha_1)}{\chi(\alpha_2)} + 2\right)$$

For dimensional reasons the following products vanish:

$$Z_e Z_{s_1} = Z_e Z_{s_2} = Z_e Z_{s_1 s_2} = Z_e Z_{s_2 s_1} = 0.$$

Let a_{ij} denote the coefficient of the monomial $X^i Y^j$ in the universal formal group law so that

$$F(X,Y) = \sum_{i,j\geq 1} a_{ij} X^i Y^j.$$

Commutativity implies that $a_{ij} = a_{ji}$.

The remaining products are:

$$Z_{s_1} Z_{s_1 s_2} = Z_{s_2} Z_{s_2 s_1} = Z_e$$

$$Z_{s_1} Z_{s_2 s_1} = Z_{s_2} Z_{s_1 s_2} = 0$$

$$Z^2_{s_1 s_2} = Z_{s_2}$$

$$Z^2_{s_2 s_1} = Z_{s_1}$$

$$Z_{s_1 s_2} Z_{s_2 s_1} = Z_{s_1} + Z_{s_2} + a_{11} Z_e$$

$$Z_I Z_{s_1 s_2 s_1} = Z_I for I = e, 1, 2, 21$$

$$Z_{s_1 s_2} Z_{s_1 s_2 s_1} = Z_{s_1 s_2} + a_{12} Z_e$$

$$Z^2_{s_1 s_2 s_1} = Z_{s_1 s_2 s_1} + a_{12} Z_{s_1}$$

References:

[A] A. Arabia, Cohomologie T-equivariante de la variete de drapeaux d'un groupe de Kač-Moody, preprint (1987).

[BG] J.C. Becker, D.H. Gottlieb, The transfer map and fiber bundles, *Topology* **14** (1975), 1-12.

[BGG] Bernstein, I.N., Gel'fand, I.M., Gel'fand, S.I., Schubert cells and the cohomology of spaces G/P, *Russian Math Surveys* **28** (1973), 1-26.

[BE1] P. Bressler, S. Evens, On certain Hecke rings, *Proc. Natl. Acad. Sci. USA* **84** (1987), 624-625.

[BE2] P. Bressler, S. Evens, The Schubert calculus, braid relations, and generalized cohomology, *Trans. AMS*, to appear.

[BH] A. Borel, F. Hirzebruch, Characteristic classes and homogeneous spaces I, *Amer. J. Math.* **80** (1958), 458-538.

[BM] G. Brumfiel, I. Madsen, Evaluation of the transfer and the universal surgery classes,*Inventiones math.* **32** (1976),133-169.

[BS] R. Bott, H. Samelson, Application of the theory of Morse to symmetric spaces, *Amer. J. Math.* **80** (1958), 964-1029.

[D] M. Demazure, Desingularization des varietes de Schubert, *Ann. Ec. Norm. sup.*(**4**) **7** (1974), 53-58.

[Dy] E. Dyer, Cohomology Theories, W.A.Benjamin, 1969.

[G] E. Gutkin, Representations of Hecke algebras,*Trans. Amer. Math. Soc.* **307**, 1988.

[KK1] B. Kostant, S. Kumar, The nil Hecke ring and cohomology of G/P for a Kac-Moody group G, *Advances in Math* **62** (1986), 187-237.

[KK2] B. Kostant, S. Kumar, T-equivariant K-theory of generalized flag varieties, *Proc. Natl. Acad. Sci. USA* **74** (1987), 4351-4354.

[Q] D.G. Quillen, Elementary proofs of some results of cobordism theory using Steenrod operations, *Advances in Math* **7** (1971), 29-56.

[S] R. Stong, Notes on cobordism theory, Princeton University Press, 1968.