HODGE COHOMOLOGY
OF
NEGATIVELY CURVED MANIFOLDS

by

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S.B., Massachusetts Institute of Technology
(1982)

SUBMITTED TO THE DEPARTMENT OF
MATHEMATICS IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

JUNE, 1986

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Submitted to the Department of Mathematics on May 2, 1986 in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in Mathematics.

ABSTRACT

The Hodge Laplacian acting on differential forms is examined for a certain class of complete Riemannian manifolds of dimension n. This class consists of the interiors of compact manifolds with boundary, each endowed with a 'conformally compact' metric. Such a metric, by definition, is of the form $g = \rho^{-2}h$ where h is any smooth nonsingular metric and ρ is a defining function for the boundary. These manifolds are all negatively curved near infinity; examples include the hyperbolic space $\mathbb{H}^n$ and those of its quotients which have no cusps.

A parametrix is constructed for the Laplacian acting on the space of L^2 k-forms for all degrees $k \neq \frac{1}{2}n$, $\frac{1}{2}(n\pm1)$. Thus, in these degrees the Laplacian is Fredholm, and in particular its null space is finite dimensional. The space of L^2 harmonic k-forms is then identified with the relative and absolute deRham cohomology of the manifold when $k < \frac{1}{2}(n-1)$ and $k > \frac{1}{2}(n+1)$, respectively. It is also shown that the range of the Laplacian is closed when $k = n/2$, although its null space is of infinite dimension.

The parametrix construction is microlocal: the Laplacian should be thought of here as a degenerate elliptic operator on a compact manifold, and a space of pseudodifferential operators large enough to contain its Green operator is defined and studied. A fairly complete calculus, including L^2 continuity properties, is developed along the way.

Thesis Supervisor: Richard B. Melrose

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Acknowledgements

Among the many people who have bettered my life in some way during the past several years I wish especially to thank my family, Chet Roys, John Cohn, Diane Mariano, Joanne Murray, Dr. Michael Smith, Jimmy Page, Rick Kreminski, Chris Wendel and John Ostlund. For the mathematics they have taught me I am grateful to Woody Lichtenstein, Ted Shifrin, Victor Guillemin, Gunther Uhlmann and particularly David Jerison. Thanks also to Viola Wiley for a magnificent typing job.

Finally, well deserving a paragraph of his own is Richard Melrose. No amount of gratitude on my part would suffice for all that he has done on my behalf.

"Don't brood too much on the superiority of the unseen to the seen. It's true, but to brood on it is mediaeval."

— from Howards End

by E. M. Forster

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Introduction

The many recent advances in the analysis of partial differential equations notwithstanding, the study of general linear elliptic operators on unbounded domains remains as yet inchoate. The lack of a systematic theory is due no doubt to the fact that the geometry of the domain and the asymptotic degeneracies of the equation each can radically affect the analytic nature of the operator. A natural class of problems in which the geometry and analysis are rather intimately intertwined is in the study of Laplace's operator on a complete Riemannian manifold, acting either on functions or differential forms. Even this case is far from well understood.

A typical question here is whether, on a complete non-compact Riemannian manifold M, there exist functions, or k-forms, which are both harmonic and square-summable (or bounded). As was widely suspected from the classical separation of Riemann surfaces into those which do or do not admit bounded harmonic functions, the operative feature is the curvature. (Of course, the L^2 class is not well defined in the conformal geometry of Riemann surfaces, whereas boundedness is an invariant concept, so long as attention is restricted to functions.) The first general theorem of this type in higher dimensions is due to Yau

[28], and asserts the nonexistence of nonconstant bounded harmonic functions on complete manifolds of nonnegative Ricci curvature.

Negative curvature produces the opposite effect, as may be discerned from the harmonic analysis of the hyperbolic space $\mathbb{H}^n$. Although this space has no L^2 harmonic functions, it admits many bounded harmonic functions--in fact, by a Poisson-type representation theorem, there is one corresponding to each continuous function on $S^{n-1} = \partial\mathbb{H}^n$ (and others with less regular boundary values).

Much attention over the past several years has focused on the Laplacian of complete negatively curved manifolds, and some quite general results have been obtained. Let us discuss some of these, first for the Laplacian on functions. Here the concern is only with bounded harmonic functions since there are no nontrivial L^2 ones. In fact, if $u \in L^2 \cap \text{dom } \Delta$ is harmonic, an integration by parts—which relies on the completeness of M to ensure the lack of a boundary term—implies that $du = 0$. Thus u is constant, and must vanish if the volume of M is infinite.

If M is simply connected and has all sectional curvatures K_M negative, then the theorem of Cartan-Hadamard asserts that $\exp_p : T_pM \longrightarrow M$ is a diffeomorphism for each $p \in M$. It is possible to define a sphere at infinity, S_∞, for such manifolds, cf. Eberlein-O'Neill

[15]. Each point $x \in S_\infty$ is identified with a set of geodesics all asymptotic to one another, and these are said to converge to x. S_∞ itself is the aggregate of such mutually asymptotic classes of geodesics. There is a natural topology which makes $\bar{M} = M \cup S_\infty$ into a compact topological manifold with boundary (in fact, a disc). The asymptotic Dirichlet problem may then be posed: for any given continuous function f on S_∞, find a function u on M which is harmonic and assumes the boundary values f asymptotically. It should be remarked that it is insufficient merely to require

$$\lim_{t\to\infty} u(\gamma(t)) = f(x)$$

for any geodesic $\gamma(t)$ converging to $x \in S_\infty$; instead, the convergence must occur in the topology of $\bar{M}$, which is stronger than this 'radial' convergence.

This asymptotic Dirichlet problem was posed by Choi in [7] and partially solved. For various technical reasons one must assume that the sectional curvatures are bounded away from zero: $K_M \le -a^2 < 0$. Choi proved the result with this hypothesis, but also assuming a convexity condition at infinity—namely that for two distinct points $x, y \in S_\infty$ there are disjoint geodesically convex neighbourhoods U_x, U_y with $x \in U_x$, $y \in U_y$. This extra condition is unnecessary when M is a surface. Mike Anderson finished the proof [1] by showing that if there is a two-sided bound on curvature, $-b^2 \leq K_M \leq -a^2 < 0$, then M satisfies the convexity condition. Sullivan independently proved the result [27] using probabilistic methods.

This positive resolution has the consequence that M carries many bounded harmonic functions. In fact, the asymptotic maximum principle of Yau asserts that, in this situation, the infimum and supremum of u are attained on S_∞, hence coincide with those of f. At this juncture it is reasonable to seek a Poisson-type representation formula. This was accomplished in Anderson-Schoen [3].

Their paper also addresses a quite interesting and relevant question: namely, to what extent may $\bar{M}$ be considered a smooth manifold? Inasmuch as there is a homeomorphism between the unit sphere S_pM in the tangent space at any point and S_∞, one may form the composite map

$$T_{pq} : S_pM \longrightarrow S_\infty \longrightarrow S_qM$$

and study its regularity. Anderson and Schoen prove that this map is Hölder continuous with exponent α, with $\alpha = a/b$ depending only on the curvature bounds. Little else is known one way or the other about this question, though it seems likely that T_{pq} is rarely C^∞. We note, however, the result of Fefferman [16] concerning the Bergman metric in the interior of a smoothly bounded strictly pseudoconvex domain. This metric has asymptotically constant holomorphic sectional curvatures, and he proves that each T_{pq} (which is only defined locally) is C^∞; in fact, the map from S_pM to the boundary of the domain is C^∞.

Finally, on a spectral note, it is also known that the spectrum of the Laplacian (on functions) is contained in the interval $[(n-1)^2a^2/4, \infty)$, cf. McKean [21]. In addition, if M (is simply connected and) has sectional curvatures tending uniformly to $-\infty$, its Laplacian has pure point spectrum, cf. Donnelly-Li [13].

Much less is known about the Laplacian acting on differential forms. Of course, when M is compact the Hodge theorem provides topological meaning to the dimension of the space of harmonic forms in each degree. When M is complete but noncompact, it is natural to allow the Laplacian to act on

$$L^2\Omega^k = \{\omega \in \Omega^k = \Gamma(\Lambda^k M) \ : \ \int |\omega|^2 < \infty\}$$

the space of L^2 k-forms over M. The subspace of harmonic forms is denoted

$$\mathcal{H}^k = \{\omega \in L^2\Omega^k \cap \operatorname{dom} \Delta \ : \ \Delta\omega = 0\}.$$

Recall that $\omega \in \operatorname{dom} \Delta$ means $d\omega \in L^2\Omega^{k+1}$, $\delta\omega \in L^2\Omega^{k-1}$, and $\delta d\omega$, $d\delta\omega \in L^2\Omega^k$. The (by now) classical theorem of Andreotti-Vesentini [4] allows one to integrate by parts, as on a compact manifold, to deduce that

$$\mathcal{H}^k = \{\omega \in L^2\Omega^k : d\omega = \delta\omega = 0\}$$

provided M is complete. Notice that this implies $\omega \in \text{dom}\,\Delta$ if $\omega \in L^2\Omega^k$ and $\Delta\omega = 0$ pointwise. We shall call $\mathcal{H}^k$ the Hodge cohomology space of degree k.

Another similar, but perhaps more widely studied, space is the L^2 cohomology space of degree k:

$$L^2H^k = \{\omega \in L^2\Omega^k : d\omega = 0\}/d\{\eta \in L^2\Omega^{k-1} : d\eta \in L^2\Omega^k\}.$$

Since each $\omega \in \mathcal{H}^k$ is closed, there is a natural map $\mathcal{H}^k \longrightarrow L^2H^k$ for each k. However, unlike the compact setting, the two spaces are rarely isomorphic. A gauge of their relationship is afforded by Kodaira's weak Hodge decomposition [9]:

$$L^2\Omega^k = \overline{d\{L^2\Omega^{k-1} \cap \text{dom}\, d\}} \oplus \overline{\delta\{L^2\Omega^k \cap \text{dom}\,\delta\}} \oplus \mathcal{H}^k$$

which is valid quite generally. The summands here are pair-wise orthogonal. From this it is easy to see that

$$\mathcal{H}^k \simeq \{\omega \in L^2\Omega^k : d\omega = 0\}/\overline{d\{L^2\Omega^{k-1} \cap \text{dom}\, d\}}$$

whence the map $\mathcal{H}^k \longrightarrow L^2\mathcal{H}^k$ is always injective. Furthermore, the two spaces coincide precisely when the range of d is closed.

The L^2 cohomology will rarely be mentioned again. The bulk of this dissertation is devoted to the computation of $\mathcal{H}^k$ for a certain class of manifolds. It has been suspected for quite a while that if M is simply connected, and its sectional curvatures satisfy $-b^2 \le K_M \le -a^2 < 0$, then most of the harmonic spaces $\mathcal{H}^k$ should be trivial.

It is straightforward to do the necessary calculations for the hyperbolic space $\mathbb{H}^n$, or in fact for any complete rotationally symmetric manifold (which need not satisfy the curvature constraints), see Dodziuk [10], to arrive at the following conclusion: if in polar coordinates the metric of such a manifold has the form $ds^2 = dr^2 + f(r)^2 d\theta^2$, $r \in \mathbb{R}^+$, $\theta \in S^{n-1}$, then

$$\dim \mathcal{H}^k = \begin{cases} 1 & \text{if } k = 0, n \text{ and } \int_0^\infty f(r)^{n-1} dr < \infty \\ \infty & \text{if } k = n/2 \text{ and } \int_1^\infty \frac{dr}{f(r)} < \infty \\ 0 & \text{otherwise.} \end{cases}$$

The first line simply asserts that constant functions are in L^2 iff the volume of M is finite. The second is more interesting; it reflects the fact that both the kernel of the coboundary operator and the L^2 norm are conformal invariants on $\Omega^{n/2}$. The finiteness of the integral means that M is conformally equivalent to a Euclidean disc of finite radius: introduce $R(r) = \exp \int_1^r \frac{dt}{f(t)}$ as the new radial variable. Then one only need note that there certainly is an infinite dimensional family of smooth harmonic forms on such a disc.

It is striking that in this proposition, the important geometric feature is the growth of the metric rather than the curvature. This is unexpected since the spaces $\mathcal{H}^k$ implicitly depend on the derivatives of the metric, by way of the coefficients of the Laplacian. If g' is uniformly equivalent to g, $c_1 g \leq g' \leq c_2 g$, then its Laplacian has domain different than that for g unless g' is uniformly C^2 close to g. In contradistinction, the spaces L^2H^k are patently stable under such a C^0 perturbation of the metric.

Further support for the conjectured vanishing of Hodge cohomology outside of the middle degree(s) was given by the discovery of Donnelly-Xavier [14] that, so long as the curvature is (negative and) tightly enough pinched, $|a/b-1| < \epsilon(n,k)$, then $\mathcal{H}^k = 0$ for all k such that $|k-n/2| > 1/2$. Donnelly-Fefferman [12] extended the method to show that for the Bergman metric of a strictly pseudoconvex domain in $\mathbb{C}^n$, where $\mathcal{H}^k$ splits as $\oplus_{p+q=k} \mathcal{H}^{p,q}$, then

$$\dim \mathcal{H}^{p,q} = \begin{cases} 0 & p+q \neq n \\ \infty & p+q = n \end{cases} .$$

Recall that the Bergman metric has holomorphic sectional curvatures asymptotically constant at the boundary of the domain.

It was thus quite surprising when Mike Anderson [2] constructed counterexamples to the conjecture; there exist simply connected manifolds with $-b^2 \leq K_M \leq -a^2 < 0$ for which $\mathcal{H}^k$ is infinite dimensional for any given k. Furthermore, it is seen that the curvature pinching ratio of Donnelly-Xavier is sharp.

Many questions are thereby raised dealing with how one might recognize the negatively curved manifolds with suitable vanishing of the harmonic spaces. It appears that the key to this problem lies in the (intrinsic) regularity of the sphere at infinity. The characterization of those manifolds for which S_∞ is smoothly attached is very likely quite difficult, so a first step might be to a priori assume some regularity at infinity. Among the many forms this assumption could take, we pursue the course of considering manifolds with metric differing from that of the interior of a compact Riemannian manifold with boundary by a singular conformal factor; these we call conformally compact.

More explicitly, let M be a compact manifold with boundary, h a metric nondegenerate and smooth up to ∂M, and ρ a function defining ∂M (so that $d\rho|_{\partial M} \neq 0$). Consider the metric $g = \rho^{-2}h$. The singular factor ρ^{-2} has the effect of pushing ∂M to infinity. It is not hard to see that, like the Poincaré metric on the unit ball after which it is patterned, this metric is complete and, more importantly, negatively curved near ∂M. There are other ways to produce metrics with 'inverse-square asymptotics' and the correct geometry. In particular, the Bergman metric is obtained as the complex Hessian of

$$c \log(\phi_1 \rho^{-n-1} + \phi_2 \log \rho)$$

with ϕ_1, ϕ_2 smooth, ϕ_1 nonvanishing, and ρ the distance to the boundary. (For the unit ball, $\phi_1 = 1$ and $\phi_2 = 0$.) But, were it to be written as $\rho^{-2}h$, the metric h then partially degenerates at ∂M —in fact, precisely in the directions of the CR bundle in $T\partial M$.

Our principal result is the

<u>Theorem</u> (4.8): For the metric $g = \rho^{-2}h$ on M, as described above, there are natural isomorphisms

$$\mathcal{H}^k \simeq H^k(M, \partial M) \qquad k < \frac{n-1}{2}$$

$$\mathcal{H}^k \simeq H^k(M) \qquad k > \frac{n+1}{2}$$

obtained by sending a (necessarily closed) form $\omega \in \mathcal{H}^k$ to its de Rham class. By virtue of conformal invariance, $\mathcal{H}^k$ is infinite dimensional when $k = n/2$.

This is proved by constructing a parametrix for Δ_g, i.e. a pseudodifferential operator E such that

$$\Delta_g E = I - Q, \quad E^* \Delta_g = I - Q^*$$

where Q and its adjoint Q^* are compact and smoothing on $L^2(dg)$. From a fairly explicit knowledge of the Schwartz kernels of Q and Q^*, it is possible to deduce the boundary asymptotics of elements of $\mathcal{H}^k$. This, in turn, allows the isomorphisms of the theorem to be proved. A closely related 'Poisson' operator solves the asymptotic Dirichlet problem, but due to constraints of time and space this will not be developed here.

The main difficulty in the construction of E arises from the fact that the principal symbol of Δ_g degenerates quadratically at ∂M. The Schwartz kernel of E must therefore have a fairly complicated singularity at the corner of $M \times M$. The systematic introduction of singular coordinates at this corner allows one to adequately 'resolve' this singularity. In effect, the degeneracies are transferred from the Schwartz kernel of the operator in question to the geometry of the new 'blown up' manifold which replaces $M \times M$. This type of construction was first developed by Melrose [22] to deal with differential operators degenerating somewhat less thoroughly than the ones considered here. Melrose-Mendoza [23] studies that class of operators further, in particular proving Fredholm properties for the elliptic elements. Both that work and the present one are part of a more general construction, to be contained in the, as yet mythical, Melrose-Mendoza [24]-- from which inspiration for the following pages comes.

The interest in these methods as presented here lies perhaps in the ease of their applicability to a natural geometric question (albeit in a somewhat artificial setting). It seems likely that similar techniques will prove effective in other related geometric problems. For instance, Mazzeo-Melrose [20] employs essentially the same construction as here to deduce the meromorphic extension of the resolvent and Eisenstein series for certain quotients of $\mathbb{H}^n$.

This paper is organized as follows. The first chapter examines conformally compact manifolds from a differential geometric point of view. Their curvature is seen to have fairly simple behaviour, and in particular is negative at infinity. Asymptotics of the geodesics are also studied, and it is shown that in a special case the ideal boundary has an intrinsic regularity structure. In the second chapter, the V_0 operator calculus is developed, including L^2 continuity of its elements, and the parametrix construction is outlined. In Chapter 3, a certain model for Δ_g is analyzed. In these circumstances this model is nothing but the (constant curvature) hyperbolic Laplacian; however we require some rather unusual mapping properties of it. Finally, the hard work now complete, the actual construction and proof of the Main Theorem are contained in the brief last chapter.

In conclusion we recall an old conjecture of Heinz Hopf's, for it has stimulated much of the work directed toward the computation of $\mathcal{H}^k$ on negatively curved manifolds. One formulation of the classical uniformization theorem classifies surfaces by which constant curvature metric they admit. Motivated both by this and the higher-dimensional Gauss-Bonnet Theorem, Hopf proposed the generalization:

Conjecture: If M^{2m} is compact and admits a metric with strictly negative sectional curvatures, then $(-1)^m\chi(M) > 0$.

The corresponding query for compact flat manifolds is trivial, and the one for compact manifolds of positive curvature is likewise unresolved. The proposed methods of proof are quite different for the positive and negative cases; in the positive case much work has been devoted toward showing that the Gauss-Bonnet integrand—the Pfaffian—is pointwise positive. This was shown to be a hopeless task by Geroch [17] (who gives a local counter-example) and Bourgouignon-Karcher [6] (who give a global one).

Following a similar pattern of history, Anderson's counterexample [2] likewise dashed hopes that the negative case could be resolved by settling the hopefully less complicated issue of determining the L^2 harmonic spaces on complete noncompact negatively curved manifolds. In fact, Singer had suggested that combining Atiyah's index theorem for covers [5] with suitable vanishing of the Hodge cohomology would settle Hopf's conjecture. This index theorem equates the alternating sum of the Betti numbers of a compact manifold M, which of course is just the Euler characteristic, with the alternating sum of certain L^2 Betti numbers on the universal cover $\tilde{M}$:

$$\chi(M) = \sum (-1)^k b_k(M) = \sum (-1)^k b_k^{\Gamma}(\tilde{M})$$

$\Gamma = \pi_1(M)$. These b_k^{Γ} are nothing but the (normalized) dimension of the $\mathcal{H}^k$. Thus, since if the curvature is negative $\tilde{M}$ is simply connected, if one could show that $\mathcal{H}^k = 0$ for $k \neq m = \frac{1}{2} \dim M$ for the negatively curved pullback metric on $\tilde{M}$, it follows that

$$\chi(M) = \sum (-1)^k b_k^\Gamma(\tilde{M}) = (-1)^m b_m^\Gamma.$$

The b_k^Γ are always nonnegative, and it is not hard to show that b_m^Γ is strictly positive, so the conjecture is proved.

This method of proof does work when the curvature is close to being constant—and in particular when M is hyperbolic—by virtue of Donnelly-Xavier's work. We remark also that Anderson's counterexamples are not universal covers of compact manifolds since their isometry groups have no cocompact subgroups. Thus it may well still be true that such periodic covering manifolds have vanishing Hodge cohomology outside the middle degree, but to prove this would likely be quite difficult. It is, I think, a question that warrants attention.

Chapter 1. Conformally Compact Metrics

A. Definitions

The interior of a compact manifold with boundary M may be endowed with a complete Riemannian metric, and thus becomes a complete open manifold. The general effect is that ∂M is placed at infinity; geodesics take an infinite time to reach it. The seminal example, and model, for us is the Poincaré metric on the unit ball B^n. In local coordinates near the boundary the components of the metric become arbitrarily large. The rate of this blow-up is essentially fixed by balancing the requirements that the metric be complete and that its sectional curvatures be negative and bounded away from zero near ∂M. Explicitly, suppose

(1.1) h is a nondegenerate smooth metric on the (closed) manifold M.
ρ is a defining function for ∂M: $\rho \geq 0$, $\rho^{-1}(0) = \partial M$, and $d\rho$ is nonvanishing on $N^*\partial M\backslash 0$

and set

$$g = \rho^{-\sigma} h, \qquad \sigma > 0.$$

Then, from computations in sections B and C of this chapter it will follow that g is complete for $\sigma \geq 2$, whereas its sectional curvatures are bounded above by a negative constant only when $\sigma \leq 2$. In fact the curvature diverges to negative infinity if σ is strictly less than 2.

This thesis focuses on a study of the analytic properties of

$$(1.2) \qquad g = \rho^{-2}h$$

which is both complete and negatively curved near ∂M, and in particular of its Laplacian. Let us however mention that there are other examples of metrics of this general form which are both complete and negatively curved, but which are not "conformally compact"—i.e. as in (1.2) with ρ and h satisfying the hypotheses of (1.1). For example, the Bergman metric on the unit ball in $\mathbb{C}^n$ has the correct geometry (and there is a vanishing theorem for its Hodge cohomology), but in Cartesian coordinates it blows up as $(1-|z|^2)^{-1}$ in directions converging to the complex subbundle of $T\partial B^n$ and as $(1-|z|^2)^{-2}$ only in the other two directions. It would be quite interesting to understand the ways ρ and h could degenerate so that $\rho^{-2}h$ still is complete and of bounded geometry.

This chapter is devoted to a discussion of the geometry of (M,g), g as in (1.2). In particular, we examine the behaviour of diverging geodesics fairly thoroughly, as well as asymptotic properties of the curvature tensor. All computations use ρ and h routinely, but note that, in addition, $g = (\phi\rho)^{-2}\phi^2 h$ for any strictly positive $\phi \in C^\infty(M)$. All formulae for the geometric quantities of g must therefore be invariant under such a transformation.

B. Geodesics at Infinity

Introduce coordinates $(z^1,\cdots,z^n)$ near a point $p \in \partial M$ such that $z^n = 0$ on the boundary and $\partial/\partial z^n$ is the inward pointing h-unit normal. $z' = (z^1,\cdots,z^{n-1})$ are then coordinates on ∂M. It is more convenient to work with the co-geodesic flow, the equations for which, in terms of z and the dual coordinates $\xi_1,\cdots,\xi_n$ are

$$\dot{z}^i = \rho^2 h^{ij}\xi_j$$

$$\dot{\xi}_i = -\rho \frac{\partial\rho}{\partial z^i} h^{pq}\xi_p\xi_q - \frac{1}{2}\rho^2 \frac{\partial h^{pq}}{\partial z^i}\xi_p\xi_q \tag{1.3}$$

$$= -\frac{1}{\rho}\frac{\partial\rho}{\partial z^i} - \frac{1}{2}\rho^2 \frac{\partial h^{pq}}{\partial z^i}\xi_p\xi_q .$$

Here all indices vary between 1 and n, the summation convention on repeated indices is used, and attention is restricted to the energy surface

$$\rho^2 h^{ij}\xi_i\xi_j = 1. \tag{1.4}$$

Notice that (1.4) implies

$$C_1/\rho \le |\xi| \le C_2/\rho, \qquad |\xi|^2 = \xi_1^2 + \cdots + \xi_n^2. \tag{1.5}$$

C_1, C_2 depend only on the largest and largest eigenvalues of the matrix h^{ij}.

Now consider a maximally extended geodesic $\gamma(t) = (z^1(t),\cdots,z^n(t))$ and suppose

$$\rho(\gamma(0)) < \epsilon, \qquad \dot{z}^n(0) < 0$$

where ϵ is sufficiently small. γ is, of course, the projection of a Hamiltonian curve $(z^1,\cdots,z^n,\xi_1,\cdots,\xi_n)$ which solves (1.3). By virtue of the choice of coordinates,

$$\partial\rho/\partial z^i = a_i(z)\rho(z), \quad i = 1,\cdots,n-1, \quad \partial\rho/\partial z^n > 0$$

for some C^∞ functions a_i, and in particular $\rho^{-1}\partial\rho/\partial z^i$ is smooth up to ∂M. Using (1.5) in (1.3) one concludes

$$\begin{aligned} &\dot{\xi}_i = O(1) \qquad i = 1,\cdots,n-1 \\ &-k_1/\rho \le \dot{\xi}_n \le -k_2/\rho \end{aligned} \tag{1.6}$$

where k_1, k_2 depend only on ϵ. We first prove

(1.7) Lemma: If $\gamma(t)$ is a geodesic as described above, and $\epsilon > 0$ is sufficiently small, then $\dot{z}^n(t) < 0$ for all $t > 0$.

Proof: By (1.3) $\dot{z}^n = \rho^2 h^{ni}\xi_i = \rho^2(h^{n\alpha}\xi_\alpha + h^{nn}\xi_n)$ where the sum in α is from 1 to n-1. Set $F(t) = h^{n\alpha}\xi_\alpha + h^{nn}\xi_n$; then $F(0) < 0$ by assumption, and we compute

$$F'(t) = h^{n\alpha}\dot{\xi}_\alpha + h^{nn}\dot{\xi}_n + \frac{\partial h^{n\alpha}}{\partial z^i}\dot{z}^i\xi_\alpha + \frac{\partial h^{nn}}{\partial z^i}\dot{z}^i\xi_n .$$

By (1.5) and (1.3), $\dot{z}^i = O(\rho)$ so $\dot{z}^i\xi_\alpha$, $\dot{z}^i\xi_n$ are $O(1)$. Thus

$$F'(t) = h^{nn}\dot{\xi}_n + O(1) \leq h^{nn}\dot{\xi}_n + C$$

where C again depends only on the metric but not on γ. By applying the bound (1.6) and assuming ϵ is small enough we may ensure that

$$\dot{\xi}_n(t) < -K \quad \text{if} \quad z^n(t) < z^n(0)$$

where K is chosen so that $-\frac{1}{2}K + C < -1$. Now set

$$t_0 = \inf\{t > 0 : \dot{z}^n(t) \geq 0\}.$$

If $0 < t < t_0$ then $z^n(t) < z^n(0)$, so $\dot{\xi}_n(t) < -K$. Finally, if $h^{nn} > 1/2$ in $\{z : \rho(z) < \epsilon\}$, we conclude that

$$F'(t) \leq h^{nn}\dot{\xi}_n + C \leq -\tfrac{1}{2}K + C < -1$$

for all $0 < t < t_0$. Hence $F(t_0) < F(0) < 0$, i.e. $\dot{z}^n(t_0) < 0$, a contradiction.

(1.8) Proposition: g is a complete metric. In particular, if $\gamma(t)$ is a geodesic with $\rho(\gamma(0)) < \epsilon$, $\dot{z}^n(0) < 0$, and ϵ is small enough, then γ can be extended to infinite length.

Proof: First observe that the function z^n of our coordinate system may be defined on a collar neighbourhood of the full boundary. On this neighbourhood there is a product metric

$$k = (z^n)^{-2}(h|_{\partial M} + (dz^n)^2).$$

Clearly for some c, $g \geq ck$. This implies that

$$1 = |\gamma'(t)|_g \geq c|\dot{z}^n|/z^n = -c\dot{z}^n/z^n$$

$$\Rightarrow t = \int_0^t |\gamma'(\tau)|_g d\tau \geq \int_0^t (-c)\frac{\dot{z}^n(\tau)}{z^n(\tau)}\, d\tau$$

which by Lemma (1.7) reduces to

$$t \geq c \int_{z^n(t)}^{z^n(0)} \frac{dz^n}{z^n} = c \log(z^n(0)/z^n(t)).$$

In other words, t tends to infinity as $z^n(t)$ tends to 0, and consequently γ has infinite length.

Notice that completeness still obtains if $(z^n)^{-2}$ is replaced by $(z^n)^{-\sigma}$, $\sigma > 2$, in k and the rest of the proof is modified accordingly. Now, as might be hoped, it is also true that geodesics approach definite points of the boundary. More is true:

(1.9) Proposition: $\gamma(t)$ tends to a definite point of the boundary as $t \longrightarrow \infty$. Furthermore, taking z^n as the new variable, the reparametrized geodesic is tangent to the h-unit normal $\partial/\partial z^n$ at the boundary.

Proof: Since $\dot{\xi}_n \longrightarrow -\infty$, we may use ξ_n as a nonsingular parameter, and then

$$\partial z^i/\partial \xi_n = \rho^2 h^{ij}\xi_j/(-2\,\frac{\rho_n}{\rho} + O(1))$$

$$= \rho^3 h^{ij}\xi_j/(-2\rho_n + O(\rho)) = O(\rho^2),$$

$i = 1,\cdots,n-1$, $\rho_n = \partial\rho/\partial z^n$. From (1.5) it follows that

$$|\partial z^i/\partial \xi_n| \le C\xi_n^{-2}.$$

Hence $\gamma(t)$ tends to $(z_\infty^1,\cdots,z_\infty^{n-1},0)$, where

$$z_\infty^i = z_0^i + \int_0^\infty \dot z^i(t)\partial t$$

$$= z^i(0) + \int_{\xi_n(0)}^{-\infty} (\partial z^i/\partial \xi_n)\partial \xi_n$$

is well-defined since the integral converges.

Next, (1.5) and (1.6) imply

$$\xi_n \ge -C/\rho, \quad \dot\xi_n \le -k_2/\rho$$

$$\Rightarrow \partial\xi_n/\partial t \le k\xi_n, \quad k > 0.$$

Recalling that $\xi_n < 0$, this differential inequality integrates to

$$\xi_n(t) \le -Ce^{kt}.$$

On the other hand, $\dot{\xi}_\alpha = O(1)$, $\alpha = 1,\cdots,n-1$, and thus

$$|\xi_\alpha(t)| \le Ct.$$

Finally,

$$dz^i/dz^n = \dot{z}^i/\dot{z}^n = \rho^2 h^{ij}\xi_j/\rho^2 h^{nj}\xi_j$$

$$= \frac{h^{i\alpha}(\xi_\alpha/\xi_n) + h^{in}}{h^{n\alpha}(\xi_\alpha/\xi_n) + h^{nn}} .$$

By assumption $h^{in} \longrightarrow 0$, $h^{nn} \longrightarrow 1$, and by the estimates above, $\xi_\alpha/\xi_n \longrightarrow 0$. We conclude that $dz^i/dz^n \longrightarrow 0$.

C. Curvature Asymptotics

The conformally compact metric g of (1.2) is patently modelled on the Poincaré metric on the unit ball in $\mathbb{R}^n$. The obvious desire is that much of the well understood hyperbolic geometry (and analysis) will be reflected in this more general setting. Indeed, this is the case with geodesic behaviour at infinity, as we have demonstrated in the last section. It is also true that the sectional curvatures of g are negative when ρ is small;

in fact there is a well defined limiting curvature tensor for which ∂M is isotropic.

(1.10) Proposition: Let $\gamma(t)$ be a geodesic which approaches $p \in \partial M$. The sectional curvatures in any direction at $\gamma(t)$ tend to $-(\partial\rho/\partial z^n)^2(p)$ as $t \longrightarrow \infty$. Here, as usual, $\partial/\partial z^n$ is the h-unit normal to ∂M at p.

Remark: Since $\rho_n \neq 0$ on ∂M, this proposition implies that the sectional curvatures are bounded between two negative constants in some collar neighbourhood of the boundary. Also, recall from the end of section A that we may replace ρ by $\phi\rho$, h by $\phi^2 h$, for any strictly positive function ϕ without altering g. The quantity $\partial\rho/\partial z^n$ is invariant under this transformation at $\rho = 0$. Indeed, the new ϕ^2h-unit normal is $\phi^{-1}\partial/\partial z^n$ and

$$\phi^{-1}\partial_{z^n}(\phi\rho) = \rho_n + \phi^{-1}\partial\phi/\partial z^n \rho = \rho_n \quad \text{at} \quad \rho = 0.$$

Proof: For the duration of the proof only, we use coordinates $(z^1,\cdots,z^n)$ which are normal with respect to h, for which p corresponds to 0, and such that $\partial/\partial z^n$ is orthogonal to ∂M at p. Thus $\partial\rho/\partial z^i(p) = 0$, $i = 1,\cdots,n-1$, $\partial\rho/\partial z^n(p) > 0$. and

$$h_{ij} = \delta_{ij} + O(|z|).$$

Set $u = \log \rho$, with subscripts on u referring to z^i derivatives, and denote by $\tilde{\Gamma}^k_{ij}$ the Christoffel symbols of h_{ij}. Notice that $\tilde{\Gamma}^k_{ij} = O(|z|)$. The Christoffel symbols of g_{ij} are

$$\Gamma^k_{ij} = \tilde{\Gamma}^k_{ij} - u_i\delta^k_j - u_j\delta^k_i + u_\ell h^{\ell k} h_{ij}. \tag{1.11}$$

Insert these into the formula expressing the components of the curvature tensor in terms of the Γ's and their derivatives. A straightforward but messy calculation shows that

$$R_{ijij} = \rho^{-2}\{\tilde{R}_{ijij} + u_{ii} - 2u_{ij}\delta_{ij} + u_{jj}$$

$$+ |\nabla u|^2(\delta_{ij}-1) + u_i^2 - 2u_iu_j\delta_{ij} + u_j^2 + O(|x|^2)\}.$$

Here R, $\tilde{R}$ are components of the curvature tensor for g, respectively h. Substitute

$$u_i = \rho_i/\rho, \quad u_{ii} = \rho\rho_{ii}/\rho^2 - \rho_i^2/\rho^2$$

and assume $i \neq j$ to get

$$R_{ijij} = \rho^{-2}\tilde{R}_{ijij} + \rho^{-4}\{\rho(\rho_{ii}+\rho_{jj}) - |\nabla\rho|^2 + \rho^2 O(|z|^2)\}.$$

Therefore, the sectional curvature in the $i-j$ plane is

$$\frac{R_{ijij}}{\rho^{-4}(h_{ii}h_{jj}-h_{ij}^2)} \longrightarrow -|\nabla\rho|^2 = -\left[\frac{\partial\rho}{\partial z^n}\right]^2$$

as $t \longrightarrow \infty$.

It is easy to follow through this proof using the metric $\rho^{-\sigma}h$, for any $\sigma > 0$, instead. Indeed, the new Γ^k_{ij} are obtained from those for $\sigma = 2$ by replacing u with $\frac{1}{2}\sigma u$. The sectional curvature in the $i-j$ plane now becomes

$$-\frac{\sigma^2}{4}\rho^{\sigma-2}|\nabla\rho|^2 + \frac{\sigma}{2}(\frac{\sigma}{2}-1)\rho^{\sigma-2}(\rho_i^2+\rho_j^2) + \frac{\sigma}{2}\rho^{\sigma-1}(\rho_{ii}+\rho_{jj}) + O(\rho^{\sigma}).$$

This substantiates the claim of section A that only when $\sigma = 2$ is the curvature bounded between two negative constants near ∂M.

D. Boundary Regularity

By their very definition, conformally compact manifolds have C^{∞} compactifications, but it is quite unclear to what extent this regularity is intrinsic to the metric g. In the present section we partially resolve this issue.

Geometric regularity of an 'ideal' boundary is naturally gauged by the asymptotic properties of diverging geodesics. In this particular setting, let $p \in M$ and $\gamma(t)$ be a geodesic ray emanating from p and tending toward the boundary. Define

$$\pi_p : w \longrightarrow \lim_{t\to\infty} \exp_p(tw).$$

This maps a neighbourhood of $\gamma'(0)$ in S_pM (the unit sphere in T_pM) to ∂M. Since g has negative curvature bounded away from zero in the cone around $\gamma(t)$, the results of Eberlein-O'Neill [15] imply that π_p is a local homeomorphism. To impose an intrinsic smoothness structure on ∂M it is sufficient to show that the composition maps $\pi_q^{-1}\pi_p : S_pM \longrightarrow S_qM$ all possess a certain fixed regularity. Anderson-Schoen [3] prove quite generally that each such map is C^α, where the Hölder exponent α depends only on the upper and lower curvature bounds. Here, however, we are able to strengthen this considerably, but only with the strong proviso that the limiting sectional curvature function on ∂M $-(\partial\rho/\partial z^n)^2$ is constant in a neighbourhood of $\pi_p(\gamma'(0)) = \gamma_\infty$. This bears marked resemblance to Fefferman's analogous result [16] for the Bergman metric of a strictly pseudoconvex domain, in which case the curvature is asymptotically constant along the whole boundary. I do not yet understand how essential this hypothesis is for conformally compact metrics.

(1.12) Proposition: Suppose, for the metric $g = \rho^{-2}h$, that $-(\partial\rho/\partial z^n)^2$ is constant on a neighbourhood of

$\pi_p(\gamma'(0)) = \gamma_\infty$ in ∂M. Then π_p maps a neighbourhood of $\gamma'(0) \in S_pM$ diffeomorphically to a neighbourhood of γ_∞ in ∂M.

<u>Proof</u>: The technique, inspired by [16], is to reparametrize the geodesic equations (1.3) so as to obtain a nonsingular system on a finite interval. It is convenient to choose coordinates so that $h^{nn} = 1$, $h^{\alpha n} = 0$. Here and in the following, we assume all Greek indices $\alpha, \beta, \gamma, \cdots$ vary between 1 and n, while $i, j, \cdots$ take the values $1, \cdots, n$. By renormalizing we also assume $\rho(\gamma(0)) = 1$.

Introduce new functions $v^1, \cdots, v^n$ by

(1.13) $$\dot{z}^\alpha = \rho^2 v^\alpha, \quad \dot{z}^n = \rho v^n$$

so that

(1.14) $$v^\alpha = h^{\alpha\beta}\xi_\beta, \quad v^n = \rho\xi_n.$$

From the relationship (1.4) we have

$$\rho^2 h_{\alpha\beta} v^\alpha v^\beta + (v^n)^2 = 1.$$

By differentiating (1.14) and using this last equation, a bit of algebra shows that the v^i satisfy

$$\dot{v}^{\alpha} = \rho^2 h_{\beta\delta} \frac{\partial h^{\alpha\beta}}{\partial z^{\gamma}} v^{\gamma} v^{\delta} + \rho h_{\beta\gamma} \frac{\partial h^{\alpha\beta}}{\partial z^{n}} v^{\gamma} v^{n}$$

(1.15)

$$- \frac{h^{\alpha\beta}\rho_{\beta}}{\rho} - \frac{\rho^2}{2} h_{\mu\sigma} h_{\upsilon\tau} h^{\alpha\beta} \frac{\partial h^{\mu\upsilon}}{\partial z^{\beta}} v^{\sigma} v^{\tau}$$

$$\dot{v}^{n} = \rho\rho_{\alpha} v^{\alpha} v^{n} - \rho^2 \rho_{n} h_{\alpha\beta} v^{\alpha} v^{\beta}$$

$$- \frac{1}{2} \rho^3 h_{\mu\alpha} h_{\upsilon\beta} \frac{\partial h^{\mu\upsilon}}{\partial z^{n}} v^{\alpha} v^{\beta}.$$

Now introduce z^n as the new parameter. The resulting equations of motion are

$$dz^{\alpha}/dz^{n} = \rho v^{\alpha}/v^{n}$$

(1.16)

$$dv^{\alpha}/dz^{n} = \rho h_{\beta\delta} \frac{\partial h^{\alpha\beta}}{\partial z^{\gamma}} v^{\gamma} v^{\delta}/v^{n} + h_{\beta\gamma} \frac{\partial h^{\alpha\beta}}{\partial z^{n}} v^{\gamma}$$

$$- h^{\alpha\beta}\rho_{\beta}/(\rho^2 v^{n}) - \frac{1}{2} \rho h_{\mu\sigma} h_{\upsilon\tau} h^{\alpha\beta} \frac{\partial h^{\mu\upsilon}}{\partial z^{\beta}} v^{\sigma} v^{\tau}/v^{n}$$

$$dv^{n}/dz^{n} = \rho_{\alpha} v^{\alpha} - \rho\rho_{n} h_{\alpha\beta} v^{\alpha} v^{\beta}/v^{n}$$

$$- \frac{1}{2} \rho^2 h_{\mu\alpha} h_{\upsilon\beta} \frac{\partial h^{\mu\upsilon}}{\partial z^{n}} v^{\alpha} v^{\beta}/v^{n}.$$

These equations are nearly regular, the obstructions being only the v^n factors in the denominators and the third summand in the equation for v^α. However, from (1.14) and the proof of (1.9) we have the a priori bound

$$0 > -c_1 \geq v^n \geq -c_2$$

where c_1, c_2 are independent of the particular geodesic. This resolves the former difficulty; as for the other one write $\rho(z) = a(z')z^n + b$, $b = O((z^n)^2)$. The coefficient function $a(z')$ is naturally associated with the metric g, being simply $\partial\rho/\partial z^n|_{\rho=0}$. Hence $\rho_\beta = O(\rho^2)$ precisely when $a(z')$ is locally constant, and in this case ρ_β/ρ^2 is smooth.

Since we already know that solutions of (1.16) exist down to $z^n = 0$, the standard theory for ordinary differential equations—in particular, smooth dependence on initial conditions—implies that the map

$$(z'(0), v(0)) \longrightarrow (z'_\infty, v_\infty)$$

from the initial values at $t = 0$ to limiting values as $t \longrightarrow \infty$ is a C^∞ diffeomorphism from the energy surface $|v(0)| = 1$ to its image. Of course we need to show that

$$\pi_{\gamma(0)} : v(0) \longrightarrow z'_\infty$$

is a diffeomorphism. It has already been noted that $\pi_{\gamma(0)}$ is known to be a homeomorphism, so it suffices to prove that its differential is nonzero.

To this end a Jacobi field argument is natural. If $w \in T_{\gamma(0)}M$ is orthogonal to $\gamma'(0)$, then essentially by definition

$$d\pi_{\gamma(0)}(w) = \lim_{t\to\infty} J_w(t).$$

Here $J_w(t)$ is the Jacobi field along $\gamma(t)$ such that $J_w(0) = 0$, $J_w'(0) = w$. It satisfies the equation

$$J_w'' + R(J_w,\gamma')\gamma' = 0.$$

By calculations very similar to those in section C, the components of R and of the corresponding curvature operator R_0 for the metric of constant curvature $-a^2$ both blow up as ρ^{-2}. Furthermore, using strongly that $\rho_\alpha = O(\rho^2)$, it is also the case that

$$|R-R_0| = O(1).$$

Hence, by the technique of asymptotic integration we conclude that J_w grows at the same rate as the corresponding constant curvature Jacobi field J_w^0. With respect to the basis $\partial/\partial z^1,\cdots\partial/\partial z^n$, this latter field tends to a finite nonzero limit as $t \longrightarrow \infty$. Thus J_w does also, and the proof is complete.

E. Laplace's Operator

This chapter concludes with a calculation to express the action of the Laplacian for the metric g on differential forms. We use, now and for the rest of this paper, coordinates $(z^1,\cdots,z^n)$ with z^n vanishing simply at ∂M, and $\partial/\partial z^n$ the h-unit normal there.

With the hindsight of experience, the metric on $\Lambda^k M$

$$(1.17) \qquad \langle\ ,\ \rangle_g = \rho^{2k}\langle\ ,\ \rangle_h$$

is best regarded as a nondegenerate metric on singular k-forms. Introduce the space of sections

$$\left\{\frac{dz^1}{\rho},\cdots,\frac{dz^n}{\rho}\right\}$$

over T^*M. In the next chapter we shall define a new vector bundle for which these are nonsingular sections. At any rate, k-fold wedge products of these 1-forms give a basis for $\Lambda^k M$ over any point of the interior. Smooth combinations of these basis forms may be written

$$\tilde{\omega} = \frac{1}{\rho^k}\,\omega$$

where ω is a genuinely smooth form on M. If $\tilde{\eta} = \rho^{-k}\eta$, then

$$\langle\tilde{\omega},\tilde{\eta}\rangle_g = \langle\omega,\eta\rangle_h .$$

Instead of studying the operation $\tilde{\omega} \longrightarrow \Delta_g\tilde{\omega}$ it is more convenient to study the induced operation on ω:

$$\Delta_g(\rho^{-k}\omega) = \rho^{-k}P\omega, \qquad P = \rho^k\Delta_g\rho^{-k}. \tag{1.18}$$

Much of the 'hard' analysis later will focus on determining the mapping properties of a simpler operator which models P.

Now, since there is a factorization

$$P = (\rho^k d\rho^{-k+1})(\rho^{k-1}\delta_g\rho^{-k}) + (\rho^k\delta_g\rho^{-k-1})(\rho^{k+1}d\rho^{-k})$$

it is easier to first calculate the conjugates of d and δ_g. First, then,

$$\begin{aligned} \rho^{j+1}d\rho^{-j}\omega &= \rho^{j+1}(\rho^{-j}d\omega - j\rho^{-j-1}d\rho\wedge\omega) \\ &= \rho d\omega - jd\rho \wedge \omega. \end{aligned} \tag{1.19}$$

As for the adjoint, use duality:

$$\int \langle d\eta, \rho^{-j}\omega\rangle_g dg = \int \langle d\eta, \omega\rangle_h \rho^{j-n} dh$$

$$= \int \langle \eta, \delta_h(\rho^{j-n}\omega)\rangle_h dh$$

$$= \int \langle \eta, \delta_h(\rho^{j-n}\omega)\rangle_g \rho^{2-2j+n} dg$$

$$= \int \langle \eta, \rho^{-j+1}[\rho^{n-j+1}\delta_h \rho^{j-n}\omega]\rangle_g dg$$

for any j-form ω and j-1-form η. In other words

$$\rho^{j-1}\delta_g \rho^{-j} = \rho^{n-j+1}\delta_h \rho^{j-n}.$$

Furthermore, for any metric

$$\delta(f\cdot\alpha) = f\delta\alpha = \iota_{\nabla f}\alpha.$$

Combining these last two formulae yields

$$\rho^{j-1}\delta_g \rho^{-j}\omega = \rho^{n-j+1}(\rho^{j-n}\delta_h\omega - (j-n)\rho^{j-n-1}\iota_{\nabla\rho}\omega)$$

(1.20)

$$= \rho\delta_h\omega + (n-j)\iota_{\nabla\rho}\omega.$$

Here $\nabla\rho$ is the gradient of ρ with respect to h. Finally, (1.19) and (1.20) together, along with a bit of computation, show that

$$P\omega = \rho^2\Delta_h\omega + (2-k)\rho d\rho \wedge \delta_h\omega - k\rho\delta_h(d\rho\wedge\omega)$$

$$(1.21) \qquad + (n-k)\rho L_{\nabla\rho}\omega - 2\rho\iota_{\nabla\rho}d\omega$$

$$+ (n-2k)d\rho \wedge \iota_{\nabla\rho}\omega - k(n-k-1)\iota_{\nabla\rho}(d\rho)\omega$$

$L_{\nabla\rho} = d\iota_{\nabla\rho} + \iota_{\nabla\rho}d$ is the Lie derivative.

Chapter 2. The Calculus of V_0 Pseudodifferential Operators

A. V_0 Vector Fields

The only distinguished submanifold of a manifold with boundary M is its boundary. Related to this is the observation that the ring of differential operators $\mathrm{Diff}(M)$ has two natural subrings of geometric origin: $\mathrm{Diff}_b(M)$ and $\mathrm{Diff}_0(M)$. The former is the space of operators which are sums of products of vector fields, unrestricted in the interior but required to lie tangent to the boundary—the so-called totally characteristic vector fields, the class of which is denoted V_b. The latter space is defined analogously using the V_0-vector fields, the ones vanishing at ∂M. Both V_b and V_0 are Lie algebras under the usual bracket operation for vector fields.

$\mathrm{Diff}_b(M)$ and $\Psi_b(M)$, the related space of totally characteristic pseudodifferential operators, were introduced and studied by R. Melrose in [22]; Melrose-Mendoza [23] contains further developments. It is the purpose of this chapter to define and examine the ring $\Psi_0(M)$ of pseudodifferential operators generalizing $\mathrm{Diff}_0(M)$. Much of the material here derives from information gleaned during conversations with Richard Melrose.

If $z = (y,x) \in \mathbb{R}^{n-1} \times \mathbb{R}^+$ are coordinates of the usual type, with z^n $(= x)$ vanishing on ∂M, then V_0 is generated as a $C^\infty(M)$-module by

$$(2.1) \qquad \left\{z^n \frac{\partial}{\partial z^1}, \dots, z^n \frac{\partial}{\partial z^n}\right\}.$$

Thus a typical element of $\mathrm{Diff}_0^m(M)$ has an expression

$$(2.2) \qquad P = \sum_{|\alpha| \le m} a_\alpha(z)\left(z^n \frac{\partial}{\partial z}\right)^\alpha.$$

The (modified) Laplacian of a conformally compact metric (1.21) lies in $\mathrm{Diff}_0^2(M)$.

Associated to V_0 is the group composed of those diffeomorphisms of M which fix ∂M pointwise. The exponential of a vector field in V_0 belongs to this group. Any diffeomorphism induces a linear action on the tangent space of one of its fixed points. In particular, if $p \in \partial M$ and M_p is the inward pointing half of T_pM, a member of this class induces a linear transformation on T_pM which preserves M_p. Such a map is of the form

$$(2.3) \qquad (x,y) \longrightarrow (sx, y+xu) \qquad s \in \mathbb{R}^+,\ u \in \mathbb{R}^{n-1} \simeq \partial M_p$$

where now (x,y) are linear coordinates on M_p (with the order reversed from the previous paragraph). Let G_p denote this linear group which is the semidirect product of $\mathbb{R}^+$ with $\mathbb{R}^{n-1}$. Its composition and inverse laws are

$$(2.4)\qquad \begin{aligned}(s,u)\cdot(t,v) &= (st, tu+v)\\ (s,u)^{-1} &= (s^{-1}, -s^{-1}u)\end{aligned}$$

(2.3) should be recognized as the left action of G_p on itself. In addition, the vector fields in (2.1) are the infinitesimal generators for this action. For future reference let us record the right action, along with its invariant vector fields

$$(2.5)\qquad \begin{aligned}&(x,y) \longrightarrow (x,y)\cdot(s,u)^{-1} = (s^{-1}x, s^{-1}(y-u))\\ &\left\{-x\,\frac{\partial}{\partial x} - y\cdot\frac{\partial}{\partial y}, \frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^{n-1}}\right\}.\end{aligned}$$

It is interesting and significant to identify G_p with the solvable subgroup $S = AN$ in the Iwasawa decomposition $G = ANK$ of the group of hyperbolic isometries:

$$G = SO_0(1,n), \quad A = \mathbb{R}^+, \quad N = \mathbb{R}^{n-1}, \quad K = SO(n).$$

Furthermore, the S invariant metric on $G/K \simeq M_p$ is hyperbolic. Thus, the introduction of the V_0 vector fields leads rather inexorably to constant negative curvature geometry.

In order to view V_0 somewhat more globally it is convenient to define the V_0 tangent bundle ${}^0TM$ as that bundle, the C^∞ sections of which are precisely the V_0 vector fields

$$V_0 = \Gamma({}^0TM).$$

This way of constructing a bundle will be used several times later on, so we pause to elaborate briefly in this case. The individual fibre ${}^0T_pM$ is obtained by an equivalence relation on V_0:

$$V,\ V' \in V_0; \quad V_p \sim V'_p \Leftrightarrow$$

$$Vf = V'f \quad \forall\ f \in C^\infty(M),\ p \notin \partial M;\ d(V-V')f\big|_p = 0$$

$$\forall\ f \in C^\infty(M),\ p \in \partial M.$$

(2.1) exhibits a spanning set of sections with no relations; these determine a unique C^∞ bundle structure on ${}^0TM$. The dual bundle ${}^0T^*M$ is also quite useful. A basis of sections here, dual to the vector fields of (2.1), is

$$\left\{\frac{dz^1}{z^n},\cdots,\frac{dz^n}{z^n}\right\} = \left\{\frac{dy}{x},\frac{dx}{x}\right\}. \tag{2.6}$$

An operator in $\mathrm{Diff}_0^m(M)$ has a well defined symbol, which is a homogeneous polynomial of degree m on the fibres of ${}^oT^*M$. If P is written as in (2.2), then

$$ {}^o\sigma_m(P)(z,\zeta) = \sum_{|\alpha|=m} a_\alpha(z)\zeta^\alpha \tag{2.7} $$

i.e. each $z^n\,\partial/\partial z^i$ is replaced by ζ^i, and lower order terms are discarded. Invariance follows by continuity from the interior.

The operator P is elliptic in this calculus if

$$ {}^o\sigma_m(P)(z,\zeta) = 0 \Leftrightarrow \zeta = 0. $$

The Laplacian (1.21) is elliptic in this extended sense. Our goal in this paper is to prove Fredholm properties for this particular operator acting on certain weighted L^2 spaces. The general theory applies of course to any elliptic V_0 differential operator.

In the study of $\mathrm{Diff}_0(M)$, as in the usual theory of standard differential operators, it is important to understand simpler models of operators in this class. In the interior of M these are obtained by freezing the coefficients so as to obtain constant coefficient operators on each tangent space, also discarding all but the top order terms, exactly as usual. This last reduction is justifiable since ∂_z^α is 'stronger' when $|\alpha| = m$ than when

$|\alpha| < m$. At a point p on the boundary, P will be modelled more effectively by a certain G_p invariant operator on M_p, which we call the normal operator $N_p(P)$. Its definition is suggested by the fact that near $z^n = 0$

$$1,\ z^n \frac{\partial}{\partial z^i},\ \left[z^n \frac{\partial}{\partial z^i}\right]\left[z^n \frac{\partial}{\partial z^j}\right],\quad \text{and each}\quad \left[z^n \frac{\partial}{\partial z}\right]^\alpha$$

are all equally strong. However, any $(z^n)^k\left[\frac{\partial}{\partial z}\right]^\alpha$ with $k > |\alpha|$ is dominated by one of these operators, hence weaker.

Motivated by these observations we describe a procedure to define $N_p(P)$ when P is differential. (2.43) further on contains a more general definition for pseudodifferential operators. We require first the notion of a normal fibration at $p \in \partial M$. This is a diffeomorphism from a neighbourhood $\mathcal{U}$ of p to a neighbourhood $\mathcal{U}'$ of $0 \in T_pM$ such that

$$f(p) = 0,\quad f(\mathcal{U}) \subset M_p,\quad f(\partial\mathcal{U}) \subset \partial M_p,\quad f_*\big|_0 = \mathrm{Id}.$$

Such a map is readily constructed using for example the geodesic flow of a metric for which ∂M is totally geodesic. Then

$$(2.8)\qquad N_p(P)u = \lim_{r\to 0}\, (R_r)^* f^* P (f^{-1})^* (R_{1/r})^* u$$

where R_r is dilation by the factor r. Letting, for the moment, $\bar{z}$ denote the linear coordinates on M_p induced by z on M, then from (2.8) one may easily check

$$(2.9) \qquad N_p\left(a(z)\left[z^n \frac{\partial}{\partial z}\right]^{\alpha}\right) = a(0)\left[\bar{z}^n \frac{\partial}{\partial \bar{z}}\right]^{\alpha}, \qquad p = \{z = 0\}$$

$$(2.10) \qquad N_p(P \cdot Q) = N_p(P) \cdot N_p(Q), \qquad P, Q \in \mathrm{Diff}_0(M)$$

These formulae show that N_p is a homomorphism from $\mathrm{Diff}_0(M)$ to the G_p (left)-invariant operators on M_p.

B. The Stretched Product Construction

Any attempt to construct a pseudodifferential inverse for an elliptic $P \in \mathrm{Diff}_0(M)$ must reconcile itself with the rather complete degeneracy of P at ∂M. In particular, the Schwartz kernel of such an inverse must possess a singularity somewhat more severe than usual at the submanifold where the diagonal of $M \times M$ intersects the corner $\partial M \times \partial M$. In order the more thoroughly to display this singularity we shall define the appropriate class of kernels on a slightly larger manifold $M \times_0 M$, the V_0 stretched product of M with itself. It is their natural abode; the additional singular behaviour is transferred to the geometry of $M \times_0 M$ and these kernels are as smooth as possible here.

We first examine the product $M \times M$ more closely. It has rather more natural structure than M alone. Thus there are two natural hypersurfaces, the left and right boundaries

$$\partial_\ell(M\times M) = \partial M \times M, \quad \partial_r(M\times M) = M \times \partial M.$$

These are mapped surjectively to M by the right and left projections π_r and π_ℓ, respectively, which send $M \times M$ onto the right and left factors. They intersect in the corner

$$\partial_\ell(M\times M) \cap \partial_r(M\times M) = \partial M \times \partial M.$$

Finally, the fixed point set of the involution I which interchanges the two factors of M is the diagonal $\Delta\iota$. This, in turn, is bounded by the diagonal of the corner $\partial\Delta\iota$.

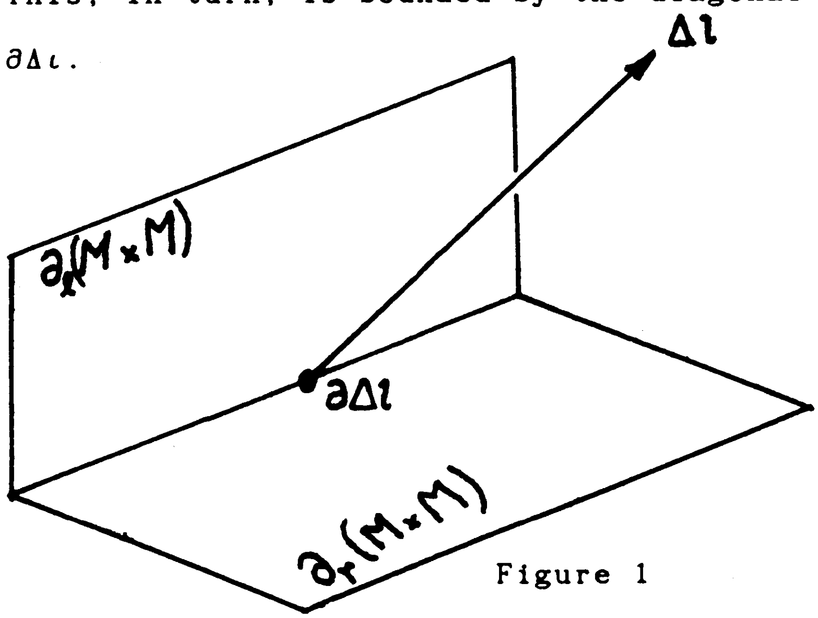

Figure 1

As noted earlier, the kernels of V_0-pseudodifferential operators should exhibit a singularity at $\partial\Delta\iota$ greater than their customary conormal one along the whole diagonal. This new behaviour is best exhibited by passing to the V_0 stretched product $M \times_0 M$ where these kernels have a certain extension property. This manifold is defined by taking the real blow-up of $M \times M$ around $\partial\Delta\iota$, or to put it more familiarly, by introducing polar coordinates around this submanifold. Abstractly $M \times_0 M$ is formed from the disjoint union of $(M\times M)\backslash\partial\Delta\iota$ and $SN_{++}\partial\Delta\iota$, the closed inward pointing sector of the spherical normal bundle to $\partial\Delta\iota$. [25] contains a proof that $M \times_0 M$ has a unique C^∞ structure such that the blow-down map

$$b : M \times_0 M \longrightarrow M \times M \tag{2.11}$$

defined as the identity away from the new face, and as the bundle projection on $SN_{++}\partial\Delta\iota$, is smooth and of rank $n-1$ ($= \dim \partial\Delta\iota$) along this new face. We shall content ourselves by relying on coordinate descriptions.

Using the coordinates (x,y) on the first factor of M in the product, and an identical pair $(\tilde{x},\tilde{y})$ on the second, then

$$\Delta\iota = \{x = \tilde{x},\ y = \tilde{y}\},\quad \partial\Delta\iota = \{x = \tilde{x} = 0,\ y = \tilde{y}\}$$

Let $Y = y - \tilde{y}$, and define the singular coordinate system

$$(R,\omega,\tilde{y})$$

$$(2.12) \qquad R = [x^2 + |Y|^2 + \tilde{x}^2]^{1/2},$$

$$\omega = R^{-1}(x,Y,\tilde{x}) = (\omega_0,\omega',\omega_n)$$

$$R \geq 0, \quad \omega \in S^n_{++} = \{\omega \in \mathbb{R}^{n+1} : |\omega| = 1,$$

$$\omega_0,\ \omega_n \geq 0\}$$

so that

$$(2.13) \qquad \begin{aligned} x = R\omega_0, \quad \tilde{x} &= R\omega_n \\ y = \tilde{y} &+ R\omega'. \end{aligned}$$

This system lifts to be C^∞ on $M \times_0 M$. The new codimension one boundary, where $R = 0$, is called the front face. The original two codimension one boundaries lift to the top and bottom faces:

$$F = \{R = 0\}, \quad T = \{\omega_0 = 0\} = b^{-1}(\{x = 0\})\backslash F$$

(2.14)

$$B = \{\omega_n = 0\} = b^{-1}(\{\tilde{x} = 0\})\backslash F$$

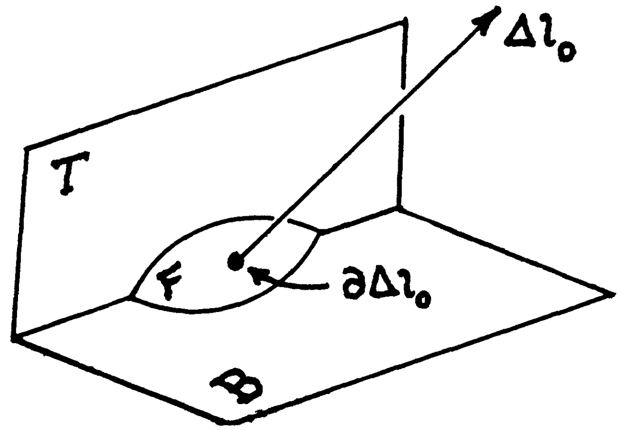

Figure 2

It is often easier to picture $M \times_0 M$ instead as

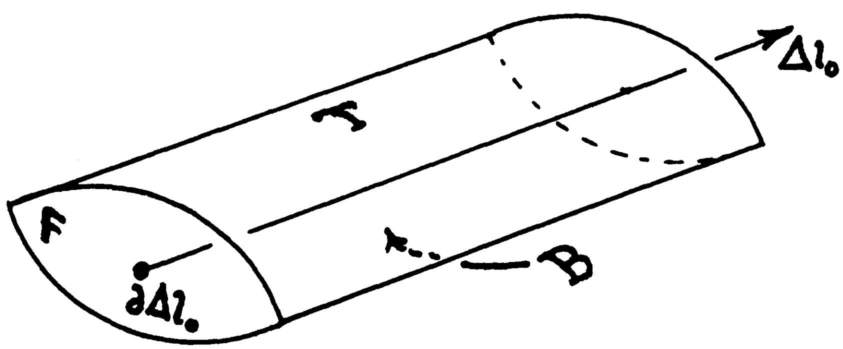

Figure 3

The front face is the location where the analysis of the next chaper takes place. It has more structure than one might first suspect. First and foremost it is a quarter-sphere bundle over $\partial\Delta\iota$ with fibre $F_p \simeq S^n_{++}$ over

(p,p), $p \in \partial M$. This follows either from (2.12) or its intrinsic description as $SN_{++}\partial\Delta\iota$. Each F_p also has a completely natural origin O_p, which is where F_p intersects the lifted diagonal

$$(2.15) \qquad \Delta\iota_0 = \{\omega = (1/\sqrt{2}, 0, 1/\sqrt{2}), \qquad R, \tilde{y} \text{ arbitrary}\}.$$

The bundle of origins $\mathcal{O} = F \cap \Delta\iota_0 = \partial\Delta\iota_0$ is clearly separated from ∂F. In fact $\Delta\iota_0$ intersects $\partial(M\times_0 M)$ transversally, and only at the codimension one boundary. This remark indicates the fundamental gain incurred by passing to the stretched product, as we discuss in the next section.

The fibre F_p admits two separate transitive G_p actions in its interior, where G_p is the group of (2.4). The groups

$$G_p^{\ell} = G_p \times \mathrm{Id}, \quad G_p^r = \mathrm{Id} \times G_p$$

act on the quarter space $M_p \times M_p \subset T_{(p,p)}(M\times M)$, $p \in \partial M$. Since G_p fixes ∂M_p they also fix $T_{(p,p)}\partial\Delta\iota$, hence these actions descend first to the normal bundle

$$N_{++}\partial\Delta\iota\big|_p = T_p(M\times M)_{++}/T_p\partial\Delta\iota$$

thence to its projectivization $SN_{++}\partial\Delta\iota\big|_p = F_p$. Because of the natural origin O_p, F_p may be identified with G_p either by way of the action of G_p^{ℓ} or that of G_p^r. Through this identification these actions on F_p correspond to the left or right action of G_p on itself, and are intertwined by the (lift of the) involution I. Specifically, the map on $M_p \times M_p$

$$(2.16) \quad \phi((x_1,y_1),(x_2,y_2)) = (x_1,y_1)\cdot(x_2,y_2)^{-1} = \left[\frac{x_1}{x_2},\frac{y_1-y_2}{x_2}\right]$$

is invariant both under dilations and translations by elements of $T(\partial\Delta\iota)$. Hence it diffeomorphically identifies the projectivization of (the interior of) $(M_p{\times}M_p)/T_p\partial\Delta\iota$ = F_p with G_p. Then G_p^{ℓ} acts as

$$(t,v)\cdot((x_1,y_1),(x_2,y_2)) = ((tx_1,y_1+x_1v),(x_2,y_2))$$

$$\xrightarrow{\ \phi\ } \left[t\frac{x_1}{x_2},\frac{y_1-y_2}{x_2} + \frac{x_1}{x_2}v\right] = (t,v)\cdot\left[\frac{x_1}{x_2},\frac{y_1-y_2}{x_2}\right]$$

whereas for G_p^r

$$(t,v)\cdot((x_1,y_1),(x_2,y_2)) = ((x_1,y_1),(tx_2,y_2+x_2v))$$

$$\xrightarrow{\ \phi\ } \left[\frac{1}{t}\cdot\frac{x_1}{x_2},\frac{1}{t}\ \frac{y_1-y_2}{x_2} - \frac{v}{t}\right] = \left[\frac{x_1}{x_2},\frac{y_1-y_2}{x_2}\right]\cdot(t,v)^{-1}$$

proving the claim.

Next we describe the infinitesimal G_p^ℓ action, which leads to an important perspective on the nature of F_p.

(2.17) <u>Lemma</u>: The vector fields $x\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial y^i}$ on M lift to the left factor of $M \times M$, thence to $M \times_0 M$. The restrictions of these lifts to F lie tangent to each F_p; there they span the space of G_p^ℓ-invariant vector fields. These restricted lifts are in V_0 with respect to (i.e. vanish on) $F_p \cap T$ and in V_b with respect to (i.e. are tangent to) $F_p \cap B$. Finally, the quarter-sphere F_p may be regarded as a ball B^n blown up around a point on its boundary, with $F_p \cap B$ as the created face. The full space of 'mixed V_0 - V_b' vector fields, generated by the G_p^ℓ-invariant ones described above, is then the lift of $V_0(B^n)$ under this blow-up.

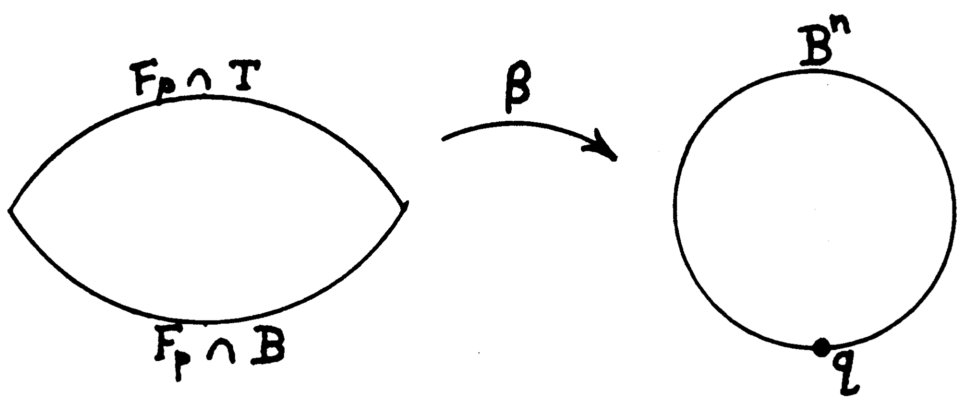

Figure 4

<u>Remark</u>: All assertions concerning tangency properties of various lifts in this statement are special cases of a

general property of the blow-up construction: if a manifold is blown-up around a submanifold, and V is a vector field tangent to this submanifold, then it lifts on the blow-up to a vector field tangent to the new boundary, which is the pre-image of the submanifold.

<u>Proof</u>: Use the coordinates $(x, y, \tilde{x}, \tilde{y})$ on $M \times M$ and $(R, \omega, \tilde{y})$, as in (2.12), on $M \times_0 M$. Then $x\partial_x$, $x\partial_{y_i}$ lift to $M \times M$ to vector fields with the same expression. Next, if b is the map of (2.11), then a brief computation shows that

$$b_*(\omega_0^2 R\partial_R + \omega_0\{\partial_{\omega_0} - \omega_0 \sum_{j=0}^{n} \omega_j \partial_{\omega_j}\}) = x\partial_x$$

(2.18)

$$b_*(\omega_0\omega_i R\partial_R + \omega_0\{\partial_{\omega_i} - \omega_i \sum_{j=0}^{n} \omega_j \partial_{\omega_j}\}) = x\partial_{y_i},$$

$$i = 1, \cdots, n-1.$$

Since b is a diffeomorphism on the interior, and all these vector fields are smooth, the vector fields in parentheses here are the unique smooth lifts of $x\partial_x$, $x\partial_{y_i}$.

Their restrictions to F are

$$X = \omega_0\{\partial_{\omega_0} - \omega_0 \sum_{j=0}^{n} \omega_j \partial_{\omega_j}\}$$

(2.19)

$$Y_i = \omega_0\{\partial_{\omega_i} - \omega_i \sum_{j=0}^{n} \omega_j \partial_{\omega_j}\}, \quad i = 1, \cdots, n-1.$$

These are patently orthogonal to $\sum \omega_j \partial_{\omega_j}$, hence tangent to F. Furthermore, since they contain no $\partial_{\tilde{y}}$ terms, they are also tangent to each F_p. The factor ω_0 ensures their vanishing at $F_p \cap T$ where $\omega_0 = 0$. On the other hand,

$$X \cdot \partial_{\omega_n} = Y_i \cdot \partial_{\omega_n} = 0$$

at $F_p \cap B$, where $\omega_n = 0$ and ∂_{ω_n} is the normal, so they are tangent here, as claimed.

It is by no means obvious that X and Y_i of (2.19) are G_p^{ℓ} invariant. It is convenient therefore to use the projective coordinates of (2.16)

(2.20) $\quad s = x/\tilde{x}, \quad u = \dfrac{y-\tilde{y}}{\tilde{x}} \qquad (x,y),(\tilde{x},\tilde{y}) \in M_p.$

These are nonsingular on the interior of F_p, and even on neighbourhoods of $F_p \cap T$ bounded away from $F_p \cap T \cap B$. Notice that

$$s = \frac{\omega_0}{\omega_n}, \qquad u = \frac{\omega'}{\omega_n}.$$

Another short calculation now shows that

(2.21) $$X = s\partial_s, \quad Y_i = s\partial_{u_i}$$

from which G_p^{ℓ} invariance is obvious.

As for the last assertions of (2.17), first observe that the blow-up of a ball around a point q in its boundary is smoothly equivalent to a quarter-sphere. Denote the blow-down map by

$$\beta : F_p \simeq S^n_{++} \longrightarrow B^n.$$

It carries $F_p \backslash F_p \cap B$ diffeomorphically to $B^n \backslash \{q\}$, and sends $F_p \cap B$ to q. We need to verify that the lifts of $V_0(B^n)$ vector fields vanish at $F_p \cap T$ and lie tangent to $F_p \cap B$. The vanishing is obvious from the bijectivity of β near $F_p \cap T$; the tangency is proved by essentially the same computation as goes into (2.18), which we leave to the reader to check. This completes the proof.

(2.22) <u>Definition</u>: E is the vector bundle over F_p whose full space of C^∞ sections are the vector fields vanishing at $F_p \cap T$ and tangent to $F_p \cap B$ described above. We have shown that

$$\beta^*(\Gamma({}^0T(B^n))) = \Gamma(E).$$

Our description of the basic natural baggage carried by $M \times_0 M$ concludes with a discussion of certain vector bundles we shall be using. These are the V_0 density bundles and the V_0 k-form bundles.

Let us commence with the density bundles, as they are somewhat simpler to understand. Recall the Riemannian density dg of (1.2), which grows as ρ^{-n} near ∂M. Define then $\Gamma_0(M)$ to be the space of all C^∞ densities υ on the interior of M such that $\rho^n \cdot \upsilon$ extends smoothly to the closure. $\Gamma_0(M)$ is the space of sections of a C^∞ line bundle over M, and a typical element may be written in coordinates as

$$\upsilon = h(x,y)\,|x^{-n}dx\ dy| \qquad h \in C^\infty(M).$$

This generalizes naturally to a manifold X with (codimension one) boundary components $X_1, \cdots, X_N$, which have defining functions $\rho_1, \cdots, \rho_N$, respectively. Let

(2.23) $\quad \Gamma_0(X) = \{$smooth densities υ on $\dot{X}$ such that $\rho_1^n \cdots \rho_N^n \upsilon$ extends smoothly to $X\}$.

The connection between these densities and V_0-geometry is illustrated by

(2.24) <u>Lemma</u>: Densities in $\Gamma_0(M\times M)$ lift to densities in $\Gamma_0(M\times_0 M)$ under the usual blow-down map. In fact, the line bundle for which the latter space is the space of sections is the pullback under b of the line bundle corresponding to the former space.

<u>Proof</u>: Away from F and $\partial\Delta\iota$ there is nothing to prove since b is a diffeomorphism there. At F use the coordinates $(R,\omega,\tilde{y})$ of (2.12):

$$\left|\frac{dx\, dy\, d\tilde{x}\, d\tilde{y}}{x^n \tilde{x}^n}\right| = \left|\frac{R^n dR\, d\omega\, d\tilde{y}}{(R\omega_0)^n (R\omega_n)^n}\right| = \left|\frac{dR\, d\omega\, d\tilde{y}}{R^n \omega_0^n \omega_n^n}\right| .$$

The left term spans $\Gamma_0(M\times M)$ since x and $\tilde{x}$ are defining functions for the left and right boundary components, whereas the term on the right generates $\Gamma_0(M\times_0 M)$ as F, T, B are defined by R, ω_0, ω_n, respectively. This proves the lemma.

One may also consider the powers of either of these density bundles; they too correspond under pullback. The half-densities are the ones we use later. Both $\Gamma_0^{1/2}(M\times M)$ and $\Gamma_0^{1/2}(M\times_0 M)$ will commonly be denoted simply $\Gamma_0^{1/2}$; the base space should be clear from the context.

Let us turn now to the k-form bundles; consider first the V_0 cotangent bundle ${}^oT^*M$. By pulling it back to the left factor of $M \times M$, and then to $M \times_0 M$, one obtains a bundle ${}^o\Lambda^1_\ell$ over the stretched product. The subscript ℓ signifies the involvement of the left factor of $M \times M$. By functoriality the pullback of $\Lambda^k({}^oT^*M)$ as above coincides with $\Lambda^k({}^o\Lambda^1_\ell)$. Call this new bundle ${}^o\Lambda^k_\ell$. Using the right factor of the product, ${}^o\Lambda^k_r$ is also defined. Their spaces of sections are denoted ${}^o\Omega^k_\ell$, ${}^o\Omega^k_r$.

The blow-down b carries F_p to the single point $(p,p) \in \partial\Delta\iota$. Hence ${}^o\Lambda^k_\ell$, for example, restricts to a canonically trivial bundle over F_p. To identify this restriction with a more familiar object, compute the lifts of $\frac{dx}{x}$, $\frac{dy_i}{x}$ on M to $M \times_0 M$. Using the coordinate system $(s,u,\tilde{x},\tilde{y})$, with (s,u) as in (2.20), which is nonsingular away from the bottom face B,

$$\frac{dx}{x} = \frac{ds}{s} + \frac{d\tilde{x}}{\tilde{x}}, \qquad \frac{dy_i}{x} = \frac{d\tilde{y}_i}{s \cdot \tilde{x}} + \frac{u_i}{s}\frac{d\tilde{x}}{\tilde{x}} + \frac{du_i}{s} \quad .$$

We define their 'formal pullbacks' to F_p

$$(2.25) \qquad \frac{dx}{x} \longrightarrow \frac{ds}{s}, \quad \frac{dy_i}{x} \longrightarrow \frac{du_i}{s} \quad .$$

(2.26) Lemma: The correspondence (2.25) induces an isomorphism

$$ {}^{0}\Lambda^{k}_{\ell}\Big|_{F_p} \simeq \Lambda^{k}(E^{*}) $$

where E^{*} is the dual of the bundle E described in (2.22).

<u>Proof</u>: $s^{-1}ds$, $s^{-1}du_i$ are dual to $s\partial_s$, $s\partial_{u_i}$, which by (2.17) generate the bundle E. In addition, we showed there that E is a smooth bundle over the closed face F_p, hence E^{*} also extends smoothly to the closure. The rest of the proof is obvious.

<u>Remark</u>: We shall prove a slight extension of this Lemma in (3.3). Also, the rules (2.25) apply to the V_0 density bundles since densities are simply 'absolute values' of n-forms. So analogously

$$ \Gamma_0(M\times_0 M)\Big|_{F_p} \simeq \Gamma_0(F_p). $$

C. V_0 Kernels

The temperate distributions on a manifold with boundary M are those with specific extension properties across this boundary. If $\tilde{M}$ is an open region containing M in its interior, then distributions on $\tilde{M}$ with support in M constitute the space of distributions dual to $C^{\infty}(M)$. On the other hand, restrictions of arbitrary distributions on

$\tilde{M}$ to the interior of M lie in the space dual to $\dot{C}^\infty(M)$—functions vanishing to infinite order on ∂M. The resulting spaces on M, of supported and extendible distributions, respectively, are quite similar but have slightly different extension properties nonetheless.

On a manifold-with-corner, for example $M \times M$, there are various other possibilities. For instance, an ordinary pseudodifferential operator on M has a Schwartz kernel k on $M \times M$ which has a conormal singularity along the diagonal and is C^∞ elsewhere. In fact k then extends to be conormal along the 'longer' diagonal of $\tilde{M} \times \tilde{M}$, and this in some sense fixes its singularity at $\partial\Delta\iota$. There are, however, other quite reasonable extension properties which k could be required to have. To formulate one of these, recall that a manifold with boundary can be doubled across its boundary. In particular, we may form the double $[M \times_0 M]^2$ of the stretched product across its front face.

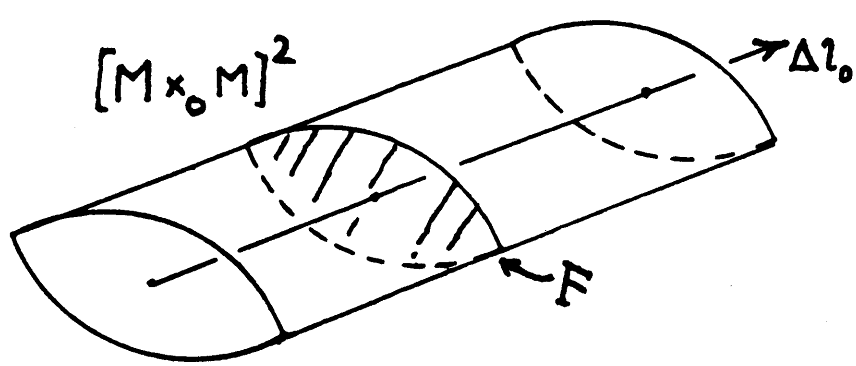

Figure 5

The lifted diagonal joins with its double, resulting in a boundary-less submanifold contained in the interior of $[M\times_0 M]^2$.

(2.27) <u>Definition</u>: $K_0^m(M;\Gamma_0^{1/2})$ is the space of distributional sections of $\Gamma_0^{1/2}(M\times_0 M)$ which extend to $[M\times_0 M]^2$ as distributions conormal of order m (see [19]) along the extended diagonal (tensored with C^∞ sections of a smooth extension of $\Gamma_0^{1/2}$) and vanishing to all orders on $\partial([M\times_0 M]^2)$.

$\Psi_0^m(M;\Gamma_0^{1/2})$, the space of V_0 pseudodifferential operators, is the collection of operators with Schwartz kernels on $M \times M$ lifting to elements of $K_0^m(M;\Gamma_0^{1/2})$.

<u>Remark</u>: If $A \in \Psi_0^m$, then we shall say that its kernel $\kappa(A)$ belongs to K_0^m. Notice also that the pushforward of $\kappa \in K_0^m$ is well defined: the blow-down map b is a diffeomorphism away from F, and the intersection of $\partial\Delta\iota$ with F is transversal so by wave front set considerations this operation makes sense.

Let us examine how $A \in \Psi_0^m(M;\Gamma_0^{1/2})$ acts. Set

$$\mu = \left|\frac{dx\ dy}{x^n}\right|^{1/2}$$

(2.28)

$$\gamma = \left|\frac{ds\ du\ d\tilde{x}\ d\tilde{y}}{s^n\tilde{x}^n}\right|^{1/2} = \left|\frac{dx\ dy\ ds\ du}{s\cdot x^n}\right|^{1/2}$$

where s and u are defined in (2.20). The full system $(s,u,\tilde{x},\tilde{y})$ are regular except along B, and shall frequently be used. Then

$$\kappa(A) = k(s,u,\tilde{x},\tilde{y})\cdot\gamma \tag{2.29}$$

where k is conormal of order m on $\{s = 1,\ u = 0\}$, and decreases rapidly as $s \longrightarrow 0$, $s \longrightarrow \infty$, or $|u| \longrightarrow \infty$. On the half density $f\cdot\mu$, we have

$$(Af)(x,y) = \left\{\iint k(s,u,\tfrac{x}{s},y-\tfrac{x}{s}u)f(\tfrac{x}{s},y-\tfrac{x}{s}u)\tfrac{ds}{s}du\right\}\cdot\mu \tag{2.30}$$

since $\tilde{x} = s^{-1}x$, $\tilde{y} = y - s^{-1}xu$. In particular, the identity is represented by

$$\kappa(Id) = \delta(s-1)\delta(u)\cdot\gamma \in K_0^0(M;\Gamma_0^{1/2}). \tag{2.31}$$

We record another representation of this action, now using the coordinates (x,y,t,v), $t = \tilde{x}/x$, $v = (\tilde{y}-y)/x$. Here the half density γ equals

$$\left|\frac{dx\ dy\ dt\ dv}{x^n t^n}\right|^{1/2}$$

and

$$(2.32) \qquad (Af)(x,y) = \left\{\int k'(x,y,t,v)f(xt,y+xv)\frac{dt}{t^n}dv\right\}\cdot\mu$$

if $\kappa(A) = k'(x,y,t,v)$ in these coordinates. Although (2.32) appears both more natural and neater than (2.30), the first expression is better for our purposes. The reason is that we are frequently interested in the behaviour of Af as $x \longrightarrow 0$, but (t,v) are singular along the top face T, hence less convenient.

It is straightforward to extend these definitions to include operators acting on the k-form bundles of the last section. Thus

$$(2.33) \qquad A \in \Psi_0^m(M;{}^o\Lambda^k \otimes \Gamma_0^{1/2})$$

$$\Longleftrightarrow \kappa(A) \in K_0^m(M;\Gamma_0^{1/2}) \otimes \mathrm{Hom}({}^o\Lambda_r^k,{}^o\Lambda_\ell^k).$$

Unfortunately the class of operators we have defined is not large enough to contain inverses of elliptic V_0 differential operators. For if E is to invert $P \in \mathrm{Diff}_0^m$, it must somehow incorporate information about the asymptotic behaviour of (formal) solutions to $Pu = 0$ near the boundary. The kernels we have already introduced vanish to infinite order at the relevant boundary components T and B.

We remedy this shortcoming as follows. Let T and B denote also the extended top and bottom faces in $[M\times_0 M]^2$, with ρ_T and ρ_B their corresponding defining functions. Suppose V_b is the space of vector fields on this double which are tangent to both boundary components; set

$$\mathcal{A}^{a,b} =$$

$$\{u \in \mathcal{D}'([M\times_0 M]^2) : V_1 \cdots V_j u \in \rho_T^a \rho_B^b (\log \rho_T \log \rho_B)^N L^\infty$$

$$V_i \in V_b, \quad i = 1,\cdots,j, \quad \text{for all } j \text{ and some } N \text{ independent of } j\}.$$

This has an invariantly defined subspace of polyhomogeneous elements

$$\mathcal{A}^{a,b}_{phg} = \{u \in \mathcal{A}^{a,b} : u \sim \sum_{i=0}^{\infty} \sum_{j=0}^{N_i} u_{ij} \rho_T^{a+i} (\log \rho_T)^j \quad \text{near } T,$$

(2.34)

$$u \sim \sum_{i=0}^{\infty} \sum_{j=0}^{N_i'} u_{ij}' \rho_T^{b+i} (\log \rho_B)^j \quad \text{near } B\}.$$

Here u_{ij} and u'_{ij} are C^∞ and depend only on the 'tangential' variables at the appropriate boundary.

(2.35) <u>Definition</u>:

i) $K_0^{-\infty,a,b} = \mathcal{A}_{phg}^{a,b}\Big|_{M\times_0 M} \otimes C^\infty(M\times_0 M;\Gamma_0^{1/2})$.

ii) $K_0^{m,a,b}(M;\Gamma_0^{1/2}) = K_0^m(M;\Gamma_0^{1/2}) + K_0^{-\infty,a,b}(M;\Gamma_0^{1/2})$.

iii) $A \in \Psi_0^{m,a,b} \Leftrightarrow \kappa(A) \in K_0^{m,a,b}$.

The kernels in i) are just sections of $\Gamma_0^{1/2}(M\times_0 M)$ which are C^∞ in the interior and up to the front face, and conormal with classical expansions at T and B.

The extension of these last definitions to operators acting on differential forms is slightly arcane since we need to consider forms with tangential and normal components vanishing with different rates at the boundary. Some additional structure is needed so that these components are well-defined; for the present problem we have the conformal class of the metric h with which to work. Using this, the ordinary form bundle splits at ∂M:

$$\Lambda^k(M)\Big|_{\partial M} = \Lambda_t^k \oplus \Lambda_n^k.$$

Similarly, from the formal pullback (2.25) we have

$$ {}^{o}\Lambda^k(M)\big|_{\partial M} = {}^{o}\Lambda^k_t \oplus {}^{o}\Lambda^k_n. $$

Next, by virtue of the metric h again, we may actually assume that such a splitting is defined in a neighbourhood of the boundary. This is less natural, but the important point is that in any two such splittings the summands agree to first order at the boundary. Another way to phrase this is to say that the conformal class distinguishes the collection of coordinate systems (y,x) for which $x = 0$ on the boundary and $\partial/\partial x$ is orthogonal to ∂M. Henceforth only such coordinates will be used, and we assume that a splitting of the V_0 form bundle is given.

The real purpose of these machinations is so that we may attach an invariant sense to the requirement that the components of $\omega = \omega_t + \omega_n$ vanish at different rates at ∂M:

$$ \omega_t = O(x^a) \qquad \omega_n = O(x^b). $$

If ω_t, ω_n are computed with respect to the coordinates (y,x) and are seen to vanish at these rates, and then recomputed with respect to new coordinates $(\bar{y},\bar{x})$, these rates might be intermingled. If, however, both coordinate systems are of the special type above, then from the fact that

$$\frac{\partial y^i}{\partial \bar{x}} = O(\bar{x}), \qquad \frac{\partial x}{\partial \bar{y}^i} = O(\bar{x})$$

it is easy to see that the new components $\bar{\omega}_t$, $\bar{\omega}_t$ vanish at the rates $\min(a,b+1)$, $\min(a+1,b)$, respectively. Hence, these rates are well defined provided we assume $b-1 \leq a \leq b+1$. Similarly, it is likewise invariant to require that

$$\omega_t \in \mathcal{A}^a_{pgh}, \qquad \omega_n \in \mathcal{A}^b_{phg}$$

if $|a-b| \leq 1$. The spaces here are simply those of (2.34) when only one boundary is present, and of course are extended to contain vector-valued elements.

Now, using all of these technical hypotheses, we assume that both ${}^o\Lambda^k_\ell$ and ${}^o\Lambda^k_r$ split in a neighbourhood of F in $M \times_0 M$ into their tangential and normal subbundles. Thus any $G \in \mathrm{Hom}({}^o\Lambda^k_\ell, {}^o\Lambda^k_r)$ may be associated to a matrix

$$G = \begin{bmatrix} G_{tt} & G_{nt} \\ G_{tn} & G_{nn} \end{bmatrix} \tag{2.36}$$

where

$$G_{tt} : ({}^o\Lambda^k_r)_t \longrightarrow ({}^o\Lambda^k_\ell)_t \qquad G_{nt} : ({}^o\Lambda^k_r)_n \longrightarrow ({}^o\Lambda^k_\ell)_t$$

$$G_{tn} : ({}^{o}\Lambda^{k}_{r})_{t} \longrightarrow ({}^{o}\Lambda^{k}_{\ell})_{n} \qquad G_{nn} : ({}^{o}\Lambda^{k}_{r})_{n} \longrightarrow ({}^{o}\Lambda^{k}_{\ell})_{n}.$$

Thus, for $\omega = \omega_t + \omega_n$, $G\omega = (G\omega)_t + (G\omega)_n$ where

$$(G\omega)_t = G_{tt}\omega_t + G_{nt}\omega_n$$

$$(G\omega)_n = G_{tn}\omega_t + G_{nn}\omega_n.$$

The G_{ij}, $i, j = t, n$, are also called the components of G.

These byzantine preliminaries allow us finally to define kernels generalizing those of (2.35). The idea is simply to let each G_{ij}, when expressed as a matrix, contain entries in $\mathscr{A}^{a,b}_{phg}$. More precisely, suppose σ, τ are two by two real matrices

$$\sigma = \begin{bmatrix} \sigma_{tt} & \sigma_{nt} \\ \sigma_{tn} & \sigma_{nn} \end{bmatrix} \qquad \tau = \begin{bmatrix} \tau_{tt} & \tau_{nt} \\ \tau_{tn} & \tau_{nn} \end{bmatrix} .$$

(2.37) <u>Definition</u>:

i) $K_0^{-\infty,\sigma,\tau}(M;{}^{o}\Lambda^{k}\otimes\Gamma_0^{1/2}) = \{G = (G_{ij}),\ \ i, j = t, n,$

$$G_{ij} \in K_0^{-\infty,\sigma_{ij},\tau_{ij}}(M,\Gamma_0^{1,2}) \otimes \mathrm{Hom}(({}^{o}\Lambda^{k}_{r})_{i},({}^{o}\Lambda^{k}_{\ell})_{j}\}$$

ii) $K_0^{m,\sigma,\tau} = K_0^{m} + K_0^{-\infty,\sigma,\tau}$.

iii) $A \in \Psi_0^{m,\sigma,\tau} \Longleftrightarrow \kappa(A) \in K_0^{m,\sigma,\tau}$.

As before, these spaces are well defined only when

$$|\min(\sigma_{tt},\sigma_{nt}) - \min(\sigma_{tn},\sigma_{nn})| \le 1$$

$$|\min(\tau_{tt},\tau_{nt}) - \min(\tau_{tn},\tau_{nn})| \le 1.$$

To conclude this treatment of V_0 pseudodifferential operators, we discuss their composition properties (especially with V_0 differential operators) and the symbol isomorphism. The following results will only be stated and proved for scalar operators, but obviously still hold for operators on forms, by, for example, referring to a basis.

(2.38) <u>Proposition</u>: $\Psi_0^{*,a,b}$ is a two-sided Diff_0^* module

$$\mathrm{Diff}_0^{\mu} \cdot \Psi_0^{m,a,b} \subset \Psi_0^{m+\mu,a,b}$$

$$\Psi_0^{m,a,b} \cdot \mathrm{Diff}_0^{\mu} \subset \Psi_0^{m+\mu,a,b}.$$

<u>Proof</u>: It suffices first of all only to prove the first inclusion, for the other follows by taking adjoints (note that adjoints of elements in $\Psi_0^{m,a,b}$ lie in $\Psi_0^{m,b,a}$). Furthermore it also suffices to take $\mu = 1$ and then iterate. Consider then the vector fields $x\partial_x$, $x\partial_{y_i}$. By

(2.17) these lift to $M \times_0 M$ to vector fields tangent to all boundaries, whence they preserve the conormal spaces $\mathcal{A}^{a,b}$. Furthermore, using the coordinates (s,u) of (2.20) we showed there that

$$b_*(s\partial_s) = x\partial_x, \quad b_*(s\partial_{u_i}) = x\partial_{y_i}.$$

Thus if $A \in \Psi_0^{m,a,b}$, and $\kappa(A) = k(s,u,\tilde{x},\tilde{y})\left|\dfrac{ds\ du\ d\tilde{x}\ d\tilde{y}}{s^n\tilde{x}^n}\right|^{1/2} = k\cdot\gamma$ then

$$\begin{aligned} \kappa(x\partial_x\cdot A) &= \left[\frac{\partial k}{\partial s} - \frac{n}{2}k\right]\cdot\gamma \\ \kappa(x\partial_{y_i}\cdot A) &= s\frac{\partial k}{\partial u_i}\cdot\gamma. \end{aligned} \tag{2.39}$$

These new kernels each belong to $K_0^{m+1,a,b}$, so we are done.

A symbol map on $\mathrm{Diff}_0(M)$ was introduced in (2.7); there is a natural generalization of this map to $\Psi_0^{*,a,b}$. Recall first that

$$\Psi_0^m \cap \Psi_0^{-\infty,a,b} = \Psi_0^{-\infty}.$$

This implies that if $A = A' + A''$ is a decomposition as in

(2.35) ii), then it makes good sense to let ${}^{0}\sigma_m(A) = {}^{0}\sigma_m(A')$ and only define the symbol on Ψ_0^m. Now, the map here, apart from simplifications in the density factors, is just the usual symbol on the space of distributions conormal along $\Delta\iota_0$:

$$\sigma_m : K_0^m(M;\Gamma_0^{1/2}) \longrightarrow S^m(N^*(\Delta\iota_0);\ \Gamma_0(M) \otimes \Gamma(\text{fibre}))$$

The space on the right contains symbolic densities on the fibres of the conormal bundle of $\Delta\iota_0$; $\Gamma(\text{fibre})$ —densities on the fibres of this bundle—arises through the use of the invariant Fourier transform. Also $\Gamma_0(M)$ results from the restriction of $\Gamma_0^{1/2}(M\times_0 M)$ to the diagonal, for the half densities on each factor combine along $\Delta\iota_0$ to give densities of weight one.

The V_0 symbol map ${}^{0}\sigma_m$ is obtained by first applying the map σ_m above, and then removing the density factors. It suffices to construct a natural density on $N^*(\Delta\iota_0)$ by which to 'divide'. This proceeds in three steps. First note the isomorphism

$$\Gamma_0(M) \otimes \Gamma(\text{fibre}) \simeq \Gamma_0(N^*(\Delta\iota_0)),$$

the latter being the space of densities along $N^*(\Delta\iota_0)$ singular like R^{-n} at F. Next, observe that there is a natural isomorphism

$$\delta : N^*(\Delta\iota_0) \xrightarrow{\sim} {}^0T^*M.$$

This is dual to the isomorphism $N(\Delta\iota_0) \xrightarrow{\sim} {}^0TM$, which is the usual natural isomorphism $N(\Delta\iota) \longrightarrow TM$ away from $\partial\Delta\iota_0$; near this boundary it comes from the identification of $x\partial_x$, $x\partial_{y_i}$, which span ${}^0TM$ near ∂M, with their lifts $s\partial_s$, $s\partial_{u_i}$, which span $N(\Delta\iota_0)$ near $\partial\Delta\iota_0$. Finally, by way of the natural map

$$T^*M \longrightarrow {}^0T^*M$$

(which is not an isomorphism at ∂M) the symplectic density on T^*M corresponds to a density on ${}^0T^*M$. This density is singular at ∂M, yet a nonsingular section of $\Gamma_0({}^0T^*M)$. The resulting naturally defined element of $\Gamma_0(M) \otimes \Gamma(\text{fibre})$ may thusly be removed so as to obtain the V_0 symbol map

$$(2.40) \qquad {}^0\sigma_m : \Psi_0^m \longrightarrow S^m({}^0T^*M).$$

Here we have also used δ to transfer the symbols from $N^*(\Delta\iota_0)$ to ${}^0T^*M$.

(2.41) <u>Theorem</u>: Ψ_0^* is filtered by the symbol maps ${}^0\sigma_m$, $m \in \mathbb{R}$. For any such m we have a short exact sequence

$$0 \longrightarrow \Psi_0^{m-1} \longrightarrow \Psi_0^m \xrightarrow{{}^o\sigma_m} S^m({}^oT^*M)/S^{m-1}({}^oT^*M) \longrightarrow 0$$

and the product formula

$${}^o\sigma_{m_1+m_2}(A_1 \cdot A_2) \equiv {}^o\sigma_{m_1}(A_1)\,{}^o\sigma_{m_2}(A_2).$$

<u>Proof</u>: The exactness of the sequence is a consequence of the same property of the symbol map on distributions conormal to $\Delta\iota_0$. The product formula follows from the usual one on standard pseudodifferential operators, provided we ascertain that $A_1 \cdot A_2$ is in fact a V_0 pseudodifferential operator. For, both the symbol map and the composition rule agree with their ordinary analogues away from ∂M, hence the product rule must extend by continuity, provided the left hand side makes sense.

The first step is to derive an expression for the kernel of a composition. Use the representation (2.30), but write the kernels k_1, k_2 of A_1, A_2 as functions of x, y, s, u (rather than $\tilde{x}, \tilde{y}, s, u$). This is legitimate since we may assume that k_1 and k_2 are supported quite near $\Delta\iota_0$. With

$$s_1 = \frac{x}{\tilde{x}},\ u_1 = \frac{y-\tilde{y}}{\tilde{x}},\ s_2 = \frac{\tilde{x}}{\tilde{x}'},\ u_2 = \frac{\tilde{y}-\tilde{y}'}{\tilde{x}'}$$

and recalling the half densities of (2.28), the composition A_1A_2f becomes, on the level of kernels,

$$f = f(\tilde{x}',\tilde{y}')\cdot\mu \longmapsto \int k_2(\tilde{x},\tilde{y},s_2,u_2)f(\frac{\tilde{x}}{s_2},\tilde{y}-\frac{\tilde{x}}{s_2}u_2)\frac{ds_2}{s_2}du_2 \cdot \mu$$

$$\longmapsto \int k_1(x,y,s_1,u_1)k_2(\frac{x}{s_1},y-\frac{x}{s_1}u_1,s_2,u_2)$$

$$f(\frac{x}{s_1s_2},y-\frac{x}{s_1s_2}(s_2u_1+u_2))\cdot\frac{ds_1\ du_1\ ds_2\ du_2}{s_1s_2} \cdot \mu.$$

Now, introduce

$$\sigma = \frac{x}{\tilde{x}'} = s_1s_2, \qquad \upsilon = \frac{y-\tilde{y}'}{\tilde{x}'} = s_2u_1 + u_2.$$

Hence

$$A_1A_2f = \int k(x,y,\sigma,\upsilon)f(\frac{x}{\sigma},y-x\upsilon/\sigma)\frac{d\sigma}{\sigma}\ d\upsilon \cdot \mu$$

where, by the substitution $s_1 = \sigma/s_2$, $u_1 = (\upsilon-u_2)/s_2$

$$k(x,y,\sigma,\upsilon) =$$

$$\int k_1(x,y,\sigma/s_2,(\upsilon-u_2)/s_2)k_2(\sigma x/s_2,y-x(\upsilon-u_2)/\sigma,s_2,u_2)\frac{ds_2du_2}{s_2^{n-1}} .$$

The k_i are conormal along $\{s_i = 1, u_i = 0\}$, $i = 1,2$, so we must show that k, as given above, is conormal along, and supported near, $\{\sigma = 1, \nu = 0\}$. The support condition is obvious from the similar assumption on k_1, k_2. As for the conormality, consider first the special case $x = y = 0$:

(2.42) <u>Lemma</u>: If k_1, k_2 are distributions on the group G_p of (2.4) which are conormal at, and supported near, the identity $(1,0)$. Then

$$k(\sigma,\nu) = \int k_1((\sigma,\nu)\cdot(s,u)^{-1})k_2(s,u)s^{-n}\, ds\, du$$

enjoys the same properties.

<u>Proof</u>: We reduce this to the corresponding assertion for the group $\mathbb{R}^n$ where this result is well known. The main point is that the space of distributions conormal to some submanifold (e.g. $(1,0)$) is coordinate invariant, and furthermore that, under a submersion, conormal distributions pull back to conormal distributions. Thus, since

$$(\sigma,\nu,\sigma-s,\nu-u) \longmapsto (\sigma/s,(\nu-u)/s)$$

$$\sigma/s = (1 - \frac{\sigma-s}{\sigma})^{-1}, \quad \frac{\nu-u}{s} = \frac{\nu-u}{\sigma}(1 - \frac{\sigma-s}{\sigma})^{-1}$$

is a surjection, there is a distribution k_1' defined by

$$k_1(\sigma/s, \frac{(\upsilon-u)}{s}) = k_1'(\sigma, \upsilon, \sigma-s, \upsilon-u)$$

which is conormal at $\sigma-s = 0$, $\upsilon-u = 0$, and C^∞ elsewhere. The pairing

$$\int k_1'(\sigma, \upsilon, \sigma-s, \upsilon-u) k_2(s,u) s^{-n} \, ds \, du$$

defining k is therefore of the required regularity, which proves the lemma.

To understand the group convolution when the parameters x, y are present, only a slight modification is necessary. Thus, as before, there is a distribution

$$k_2'(x,y,\sigma,\upsilon,s_2,u_2) = k_2(x(1 - \frac{\sigma-s_2}{\sigma})^{-1}, \; y - \frac{x}{\sigma}(\upsilon-u_2), s_2, u_2)$$

also conormal along $s_2 = 1$, $u_2 = 0$, and for which x, y, σ, υ are smooth parameters. As in the lemma, the kernel $k(x,y,\sigma,\upsilon)$ is conormal along $\sigma = 1$, $\upsilon = 0$. The proof of (2.41) is complete.

<u>Remarks</u>: Though we did not verify it, that ${}^{o}\sigma_m(P)$, $P \in \mathrm{Diff}_0^m$, as defined here agrees with the earlier definition

(2.7) is quite easy to check. In addition, the product rule

$$ {}^{0}\sigma_{m+\mu}(P\cdot A) \equiv {}^{0}\sigma_{m}(P)\,{}^{0}\sigma_{\mu}(A), \qquad A \in \Psi_0^{\mu} $$

is simpler to prove, since we may operate directly on the kernel of A with the lifted operators on $M \times_0 M$ covering P.

The residual space for this symbol calculus is $\Psi_0^{-\infty}$; the kernels of the operators in this space are C^∞ on the stretched product $M \times_0 M$, but not on $M \times M$. Thus, although smoothing in the interior of M, they are not compact on any reasonable space. This is discussed at length in the next two sections.

D. Indicial and Normal Operators

The first step in constructing a parametrix for $P \in \mathrm{Diff}_0$ is to iteratively 'remove' the conormal singularities of its kernel along $\Delta\iota_0$. As in the ordinary construction, this is reduced to algebraic manipulation by way of the symbol calculus. But, as noted at the end of the last section, the resulting error after this step is not compact. This necessitates the use of two additional steps, both iterative in nature, and both employing models for the various operators involved. These we describe now.

If E_0 is the operator obtained in the first step:

$$PE_0 = I - Q_0, \quad Q_0 \in \Psi_0^{-\infty}$$

then κ_0, the kernel of Q_0, is C^∞ on $M \times_0 M$ up to all boundaries. In fact, κ_0 vanishes to infinite order at T and B; at F, however, its Taylor series is non-trivial (which causes Q_0 not to be compact). Thus the second step of the construction is the removal of this Taylor series. We seek an E_1 such that

$$PE_1 = Q_0 - Q_1$$

and κ_1, the kernel of Q_1, vanishes to infinite order at F. Unfortunately, in order to accomplish this, we must settle for a correction term E_1 lying in $\Psi_0^{-\infty,a,b}$ for some a, b. (We are restricting discussion to scalar operators for simplicity.) Consequently

$$Q_1 \in R^\infty \Psi_0^{-\infty,a,b}, \quad \kappa_1 \in R^\infty K_0^{-\infty,a,b}$$

where the space of kernels on the right consists of those elements of $K_0^{-\infty,a,b}$ which vanish to infinite order on F. If a and b are sufficiently large then Q_1 is actually compact and the construction is complete. Usually, though, it is necessary to carry through the third step: the removal of the conormal singularity of κ_1 along the top face. Here we seek an $E_2 \in R^{\infty}\Psi_0^{-\infty,a,b}$ such that

$$PE_2 = Q_1 - Q_2, \quad Q_2 \in R^{\infty}\Psi_0^{-\infty,\infty,b}$$

The operator Q_2 is compact on appropriate weighted L^2 spaces and

$$P\cdot(E_0+E_1+E_2) = I - Q_2$$

so that $E = E_0 + E_1 + E_2$ is the desired parametrix.

From this somewhat vague discussion it should be clear that E_1 and E_2 may be 'locally' constructed. Specifically, E_1 should only depend on the operator P frozen at F and the relevant Taylor series of κ_0, whereas E_2 must rely only on the infinitesimal action of P near T and the conormal expansion of κ_1 there.

Consider first the former of these steps. Toward the stated goal, define the normal operator of $A \in \Psi_0^{-\infty,a,b}$

(2.43) $$N_p(A) = \kappa(A)\big|_{F_p}$$

By the remark following (2.26), and recalling that $\kappa(A)$ is actually a section of the half density bundle, $N_p(A)$

is a section of

$$\Gamma_0^{1/2}(F_p) \otimes \mathcal{A}^{a,b}_{phg}(F_p)$$

i.e. smooth in the interior, and with the appropriate conormal singularity at $T \cap F_p$ and $B \cap F_p$. As it stands, this seems to bear no relationship to the previous definition (2.8) for $N_p(P)$, $P \in \mathrm{Diff}_0$. To make this comparison we must interpret $N_p(A)$ as an operator. Remember that F_p is naturally identified (in its interior) with the group G_p in two ways. Using the G_p^ℓ identification, $N_p(A)$ is to be thought of as a left convolution operators on half densities over F_p. Thus, in the coordinates $(s,u,\tilde{x},\tilde{y})$ of (2.20), $p = (0,0)$:

$$N_p(A) = k(s,u,0,0)\cdot\left|\frac{ds\ du\ d\tilde{s}\ d\tilde{u}}{s^n\tilde{s}^n}\right|^{1/2}$$

(2.44)

$$N_p(A)f = \int k((s,u)\cdot(\tilde{s},\tilde{u})^{-1},0,0)f(\tilde{s},\tilde{u})\frac{d\tilde{s}\ d\tilde{u}}{\tilde{s}^n}\cdot\left|\frac{ds\ du}{s^n}\right|^{1/2}$$

where $f = f(s,u)\left|ds\ du/s^n\right|^{1/2}$.

Now, if $P \in \mathrm{Diff}_0^m$ is expressed as in (2.2), then by (2.21) its lift to $M \times_0 M$ is

$$\text{a)} \quad \sum_{|\alpha|\le m} a_\alpha(s\tilde{x},\tilde{y}+\tilde{x}u)(s\partial_u)^{\alpha'}(s\partial_s)^{\alpha_n}, \qquad \alpha = (\alpha',\alpha_n).$$

Hence, applied to the kernel of the identity (2.31),

$$\kappa(P) = \sum_{|\alpha|\le m} a_\alpha(s\tilde{x},\tilde{y}+\tilde{x}u)(s\partial_u)^{\alpha'}(s\partial_s-\tfrac{n}{2})^{\alpha_n}\delta(s-1)\delta(u)\cdot\gamma$$

so that, according to (2.43)

$$\text{b)} \quad N_p(P) =$$

$$\sum_{|\alpha|\le m} a_\alpha(0,0)(s\partial_u)^{\alpha'}(s\partial_s-\tfrac{n}{2})^{\alpha_n}\delta(s-1)\delta(u)\left|\frac{ds\;du\;d\tilde{s}\;d\tilde{u}}{s^n\tilde{s}^n}\right|^{1/2}$$

However, according to (2.8)

$$\text{c)} \quad N_p(P) = \sum_{|\alpha|\le m} a_\alpha(0,0)(s\partial_u)^{\alpha'}(s\partial_s)^{\alpha_n}.$$

Finally then it is not hard to see that the expression c), applied to a half density $f(s,u)|ds\;du/s^n|^{1/2}$, coincides with the result of setting expression b) into (2.44). We have shown that (2.43) is a proper generalization of (2.8).

(2.45) <u>Proposition</u>: There is an exact sequence

$$0 \longrightarrow R\Psi_0^{-\infty,a,b} \longrightarrow \Psi_0^{-\infty,a,b} \xrightarrow{N_p}$$

$$\mathcal{A}^{a,b}_{phg}(F_p) \otimes \Gamma_0^{1/2}(F_p) \longrightarrow 0.$$

If $P \in \mathrm{Diff}_0^m$, $A \in \Psi_0^{-\infty,a,b}$ then

$$N_p(P\cdot A) = N_p(P)\cdot N_p(A).$$

Remark: For $k = 0,1,2,\cdots$, $R^k\Psi_0^{-\infty,a,b}$ is the space of operators whose kernels vanish to order k at F.

Proof: The exactness of the sequence is obvious, save perhaps the surjectivity of N_p. But that too is easy since one may readily construct in local coordinates a kernel with fixed restriction to F_p. As for the product formula, either by applying the operator a) above to $\kappa(A)$ and then setting $\tilde{x} = \tilde{y} = 0$, or by applying the operator c) to $N_p(A)$ in (2.44) one arrives at the same section of $\Gamma_0^{1/2}$, for $\tilde{x}$ and $\tilde{y}$ are merely parameters in a).

The conjugate $R^kN_pR^{-k}$ acts on $R^k\Psi_0^{-\infty,a,b}$, though we shall not use this fact. Nonetheless, as k varies in $\mathbb{Z}^+$, the residual space

$$R^\infty\Psi_0^{-\infty,a,b}$$

is the outer limit of the effectiveness of N_p. Kernels of elements in this space, when pushed down to $M \times M$, lie in

$$\mathscr{A}_0^{a,b}(M\times M) \otimes \Gamma_0^{1/2}(M\times M).$$

The space $\mathscr{A}_0^{a,b}$ consists of those distributions which are conormal (with expansions) along the boundary components $\partial M \times M$, $M \times \partial M$ of $M \times M$, and which vanish to all orders at $\partial\Delta\iota$.

The final step utilizes the indicial operator, which once again models kernels in the last residual space. It is defined by the exact sequence

$$(2.46) \qquad 0 \longrightarrow R^{\infty}\Psi_0^{-\infty,a+1,b} \longrightarrow R^{\infty}\Psi_0^{-\infty,a,b} \xrightarrow{\ I\ } \mathscr{A}_0^{a,b} \otimes \Gamma_0^{1/2}/\mathscr{A}_0^{a+1,b} \otimes \Gamma_0^{1/2} \longrightarrow 0.$$

Thus, for $A \in R^{\infty}\Psi_0^{-\infty,a,b}$, $I(A)$ is simply the top term of the expansion of the kernel of A on $\partial M \times M$. It too may be defined as an operator, but we have no need of this.

On the other hand, $P \in \mathrm{Diff}_0^m(M)$ also has an indicial operator, which we do want to regard as an operator. Note that

$$A \in R^{\infty}\Psi_0^{-\infty,a,b} \Rightarrow P\cdot A \in R^{\infty}\Psi_0^{-\infty,a,b}.$$

$I(P)$ is now defined to be that operator acting 'infinitesimally at ∂M', or equivalently on $N\partial M$, for which

$$I(P\cdot A) = I(P)I(A). \tag{2.47}$$

Strictly speaking there are many operators satisfying (2.47). But since

$$x^{\mu}\partial_x : \mathcal{A}_0^{a,b} \longrightarrow \mathcal{A}_0^{a+1,b} \qquad \mu > 1$$

$$x^{\nu}\partial_{y_i} : \mathcal{A}_0^{a,b} \longrightarrow \mathcal{A}_0^{a+1,b} \qquad \nu \geq 1$$

there is a natural choice for $I(P)$. If

$$P = \sum_{k+|\alpha|\leq m} a_{k,\alpha}(x,y)(x\partial_x)^k(x\partial_y)^\alpha$$

then set

$$I(P) = \sum_{k\leq m} a_{k,0}(0,y)(x\partial_x)^k. \tag{2.48}$$

This is well defined, provided only coordinate systems of the type discussed in section C are used, and obviously satisfies (2.47). The residual space of the filtration induced by I is

$$R^{\infty}\Psi_0^{-\infty,\infty,b}$$

the kernels of which are compact on L^2 (so long as b is large enough), as we prove in the next section.

E. Continuity Properties

We now study the continuity properties of V_0 pseudo-differential operators. Following the usual scheme of such proofs, first the L^2 boundedness of residual operators is demonstrated—this is the most arduous part of the proof—then the symbol calculus immediately implies boundedness for operators of order zero. Finally a few more refined statements, including criteria for compactness, are proved. Other less general results are discussed in the last section of the next chapter. As before, we limit all consideration to scalar operators; the analogues for operators on the form bundles are either obvious or will be commented on when required.

(2.49) Proposition: Any $A \in \Psi_0^{-\infty,a,b}$ induces a bounded map

$$A : x^r L^2_c(x^{-n}dx\, dy) \longmapsto x^{r'} L^2_{loc}(x^{-n}dx\, dy)$$

provided $r' \leq r$, $a - r > \frac{n-1}{2}$, $b + r' > \frac{n-1}{2}$, and $a + b > n-1$.

Proof: The slightly bizarre method we employ is to reduce the statement to the familiar one concerning L^2 boundedness of ordinary pseudodifferential operators of order zero. First note that we may assume that the kernel k of A is supported in $0 \leq x, \tilde{x} \leq 1$, $|y|, |\tilde{y}| \leq 1$. It is of course C^∞ when lifted to $M \times_0 M$, except at T and B where it is conormal. It also suffices to demonstrate boundedness of

$$x^{-r'-(n-1)/2} A \tilde{x}^{r+(n-1)/2} \quad \text{on} \quad L^2_c(x^{-1}dx\, dy).$$

The kernel of this last operator is

$$k_c = x^{r-r'} \left[\frac{x}{\tilde{x}}\right]^{-r-(n-1)/2} k = \tilde{x}^{r-r'} \left[\frac{\tilde{x}}{x}\right]^{r'+(n-1)/2} k.$$

Split k_c into a sum of two kernels, one supported in $x \leq 2\tilde{x}$ and the other in $\tilde{x} \leq 2x$. As $r - r' \geq 0$, let k_1 be the first of these divided by $x^{r-r'}$ and k_2 the second divided by $\tilde{x}^{r-r'}$. We need only demonstrate boundedness of k_1 and k_2 since $k_1 + k_2$ dominates k_c.

Write k_1 as a function of $\tilde{x}$, y, s, u as in (2.30) and k_2 as a function of x, y, t, v as in (2.32). Then by assumption k_1 is supported in $0 \leq \tilde{x} \leq 1$, $|y| \leq 1$,

$0 \le s \le 2$, and k_2 in $0 \le x \le 1$, $|y| \le 1$, $0 \le t \le 2$; both are supported within the domain of regularity of their coordinate systems. Furthermore

(2.50)

a) $k_1 \in C^\infty$ for $s \neq 0$, is conormal at $s = 0$ with an expansion of lowest power $s^{a-r-(n-1)/2}$ and a symbol of order $-a-b$ in u

b) $k_2 \in C^\infty$ for $t \neq 0$, is conormal at $t = 0$ with an expansion of lowest power $t^{b+r'+(n-1)/2}$ and a symbol of order $-a-b$ in u.

Consider first k_1 alone. By the Plancherel formula

$$\int k_1\left[\tilde{x}, y, \frac{x}{\tilde{x}}, \frac{y-\tilde{y}}{\tilde{x}}\right] f(\tilde{x},\tilde{y})\tilde{x}^{-n}\, d\tilde{x}\, d\tilde{y} =$$

$$\int \hat{k}_1\left[\tilde{x}, y, \frac{x}{\tilde{x}}, -\tilde{x}\eta\right] e^{-iy\cdot\eta}\hat{f}(\tilde{x},\eta)\tilde{x}^{-1}\, d\tilde{x}\, d\eta$$

since the Fourier transform of $k_1\left[\text{---}, \frac{y-\tilde{y}}{\tilde{x}}\right]$ with respect to $\tilde{y}$ is $\hat{k}_1(\tilde{x}, y, x/\tilde{x}, -x\eta)e^{-iy\cdot\eta}\tilde{x}^{n-1}$. Next, introduce into this last integral the new variables $e^{-t} = x$, $e^{-\tilde{t}} = \tilde{x}$ to obtain

$$\int \hat{k}_1(e^{-\tilde{t}}, y, e^{-(t-\tilde{t})}, -e^{-\tilde{t}}\eta)e^{-iy\cdot\eta}\hat{f}(e^{-\tilde{t}},\eta)d\tilde{t}\, d\eta$$

where the integrand is supported in $t \geq 0$, $\tilde{t} \geq 0$, $t-\tilde{t} \geq -1$. Notice that by (2.50) a) the first factor increases exponentially unless $a-r > (n-1)/2$, in which case it decreases rapidly. Finally set

$$\kappa_1(\tilde{t},y,\tau,\eta) = \int \hat{k}_1(e^{-\tilde{t}},y,e^{-\sigma},e^{-\tilde{t}}\eta)e^{-iy\cdot\eta}e^{-i\sigma\tau}\, d\sigma$$

so that the former integral becomes

$$\int e^{i(t-\tilde{t})\tau+i(y-\tilde{y})\cdot\eta}\kappa_1(\tilde{t},y,\tau,\eta)f(e^{-\tilde{t}},\tilde{y})d\tilde{t}\, d\tilde{y}\, d\tau\, d\eta.$$

We shall show that κ_1 is a symbol in τ, η of order 0 (and type (1,0)), except for a mild conormal singularity at $e^{-\tilde{t}}\eta = 0$, and so represents an $L^2(dt\, dy)$ bounded operator. Since $L^2(dt\, dy) = L^2(x^{-1}dx\, dy)$, this shows that k_1 has the required boundedness.

To prove the claim we need only examine the effect of differentiating the integral which defines κ_1. First notice that by (2.50) a), the symbolic properties of $k_1(\tilde{x},y,s,u)$ in u imply that its Fourier transform is rapidly decreasing in the dual variable ζ and conormal at $\zeta = 0$. Since $a+b > (n-1)$, $\hat{k}_1$ is bounded there. Hence κ_1 also enjoys these properties. In addition, if $a-r > (n-1)/2$ as assumed, κ_1 is rapidly decreasing in τ. Thus differentiating with respect to either y or τ

preserves this rapid decrease. Differentiating with respect to $\tilde{t}$ either affects the first slot of $\hat{k}_1$ or the fourth. As $\tilde{x}$ is a smooth parameter, the former preserves regularity. In the latter case a factor $e^{-\tilde{t}}\eta$ is produced in front:

$$\partial_{\tilde{t}}\kappa_1 = -\int (e^{-\tilde{t}}\partial_{\tilde{x}}\hat{k}_1 + e^{-\tilde{t}}\eta_j\partial_{\zeta_j}\hat{k}_1)e^{-iy\cdot\eta-i\sigma\tau}\, d\sigma$$

is of the same regularity as κ_1, in particular bounded. Finally,

$$\eta_i\partial_{\eta_j}\kappa_1 = \int -(\eta_i e^{-\tilde{t}}\partial_{\zeta_j}\hat{k}_1 + i\eta_i y_j\hat{k}_1)e^{-iy\cdot\eta-i\sigma\tau}\, d\sigma$$

is also uniformly bounded. By iteration it is readily seen that

$$\left|\partial_{\tilde{t}}^j\partial_y^\alpha\partial_\tau^\ell\partial_\eta^\beta\kappa_1(\tilde{t},y,\tau,\eta)\right| \le C_{j\ell\alpha\beta}(1 + |\tau| + |\eta|)^{-\ell-|\beta|}$$

and, as mentioned before, κ_1 has a bounded conormal singularity at $\eta = 0$, uniform in all other parameters. By the standard theory k_1 induces a bounded transformation on $L^2(x^{-1}\, dx\, dy)$.

The proof for k_2 is quite similar, only here one requires the hypothesis $b + r' > \frac{n-1}{2}$. This completes the proof of (2.49).

Before proving continuity of operators of order zero, let us note that Ψ_0^m is invariant under the operation of taking adjoints, for any m. Thus if $A \in \Psi_0^m$ then the kernel of A^* is

$$\kappa_{A^*}(x,y,\tilde{x},\tilde{y}) \left| \frac{dx\ dy\ d\tilde{x}\ d\tilde{y}}{x^n \tilde{x}^n} \right|^{1/2} =$$

$$\bar{\kappa}_A(\tilde{x},\tilde{y},x,y) \left| \frac{dx\ dy\ d\tilde{x}\ d\tilde{y}}{x^n \tilde{x}^n} \right|^{1/2}$$

which certainly has the same regularity as κ_A on $M \times_0 M$. In addition, it is easy to see that

$${}^o\sigma_m(A^*) = \overline{{}^o\sigma(A)}.$$

(2.51) <u>Theorem</u>: $A \in \Psi_0^{0,a,b}$ is bounded from

$$x^r L^2 \Gamma_0^{1/2} \longrightarrow x^{r'} L^2 \Gamma_0^{1/2}$$

provided $a+b > n-1$, $a-r > \frac{n-1}{2}$, $b + r' > \frac{n-1}{2}$, $r \geq r'$.

<u>Proof</u>: Write $A = A_1 + A_2 \in \Psi_0^0 + \Psi_0^{-\infty,a,b}$. A_2 is bounded between these spaces by (2.49). As for A_1, use the symbol calculus of (2.41) to find $B \in \Psi_0^0$ such that for some sufficiently large $M \in \mathbb{R}$

$$A_1^{*}A_1 + B^{*}B = M\cdot I + R, \qquad R \in \Psi_0^{-\infty}.$$

Then, for $f = f(x,y)|x^{-n}\,dx\,dy|^{1/2}$

$$\int |A_1 f|^2 + \int |Bf|^2 = M\int |f|^2 + \int Rf\cdot f$$

which implies, using the boundedness of R, that

$$\int |A_1 f|^2 \le M' \int |f|^2$$

as desired.

The next step is to consider the action on Sobolev-type spaces. Since the interest here is with parametrices of differential operators, the full theory shall not be developed. For $m = 0,1,2,\cdots$ set

$$(2.52)\qquad H_b^m(M,\Gamma_0^{1/2}) = \{u \in L^2\Gamma_0^{1/2} : (x\partial_x)^i(x\partial_y)^\alpha u \in L^2,$$
$$i + |\alpha| \le m\}$$

(2.53) <u>Corollary</u>: $A \in \Psi_0^{-m,a,b}$ is bounded from

$$x^r L^2\Gamma_0^{1/2} \longrightarrow x^{r'} H_b^m(\Gamma_0^{1/2})$$

for r, r', a, b as in (2.51).

<u>Proof</u>: If k is the kernel of A, then recalling (2.39)

$$(x\partial_x)^i(x\partial_y)^\alpha k \in K_0^{0,a,b} \quad \text{for} \quad i + |\alpha| \le m.$$

The corollary follows.

The space of conormal half densities

$$\mathscr{A}^a(M;\Gamma_0^{1/2}) = \{f \in \Gamma_0^{1/2} : f = f(x,y)\left|\frac{dx\, dy}{x^n}\right|^{1/2},$$

$$(x\partial_x)^i(x\partial_y)^\alpha f(x,y) \in \rho^a(\log \rho)^N L^\infty$$

$$\text{for all } i,\ \alpha, \text{ and some fixed } N\}$$

has, as in (2.34), the subspace of polyhomogeneous elements

(2.54) $$\mathscr{A}^a_{phg}(M;\Gamma_0^{1/2}) =$$

$$\{f \in \mathscr{A}^a : f \sim \sum_{i=0}^{\infty} \sum_{j=0}^{N_i} f_i(y)x^{a+i}(\log x)^j\}.$$

This is the natural range of V_0 pseudodifferential operators applied to smooth functions. Again we only prove a special case:

(2.55) <u>Proposition</u>: For any $b, m \in \mathbb{R}$, $A \in \Psi_0^{m,a,b}$ maps

$$\dot{C}^\infty \longrightarrow \mathcal{A}^a_{phg}$$

(where the space on the left denotes the smooth half densities vanishing to infinite order on ∂M).

<u>Proof</u>: If $k \in K_0^{m,a,b}$ is the kernel of A, then the product

$$k(x,y,\tilde{x},\tilde{y}) \left| \frac{dx\ dy\ d\tilde{x}\ d\tilde{y}}{x^n \tilde{x}^n} \right|^{1/2} f(\tilde{x},\tilde{y}) \left| \frac{d\tilde{x}\ d\tilde{y}}{\tilde{x}^n} \right|^{1/2}$$

$f \in \dot{C}^\infty$, vanishes to infinite order on $\{\tilde{x} = 0\}$ and has an ordinary conormal singularity along the diagonal. Its integral then is that of an ordinary pseudodifferential kernel against f. Furthermore, if

$$k \sim \sum k_i, \quad k_i \in \mathcal{A}^{a+i,b}_{phg}$$

then each

$$\int k_i(x,y,\tilde{x},\tilde{y}) f(\tilde{x},\tilde{y}) \frac{d\tilde{x}\ d\tilde{y}}{\tilde{x}^n} \left| \frac{dx\ dy}{x^n} \right|^{1/2} \in \mathcal{A}^{a+i}$$

which gives the result.

Our final result concerns compactness of operators of negative order with appropriate boundary conditions.

(2.56) <u>Proposition</u>: $A \in R^{\infty}\Psi_0^{-m,a,b}$ is a compact map between

$$x^r L^2 \Gamma_0^{1/2} \longrightarrow x^{r'} L^2 \Gamma_0^{1/2}$$

for $m > 0$, $a+b > n-1$, $a-r > \frac{n-1}{2}$, $b+r' > \frac{n-1}{2}$.

<u>Proof</u>: $x^{-\epsilon}A = B \in \Psi_0^{-m,a-\epsilon,b}$, and if ϵ is sufficiently small, B is bounded between

$$x^r L^2 \longrightarrow x^{r'} H_b^m$$

also. The compactness of $A = x^{\epsilon}B$ is furnished by the uniform smallness of $x^{\epsilon}B$ in a neighbourhood of ∂M and the uniform equicontinuity of functions in H_b^m on $\{(x,y) : x \geq \epsilon_1\}$ for each $\epsilon_1 > 0$, cf. [29].

Chapter 3 Analysis of the Normal Operator

A. Identification of the Model Operators

The novel step in constructing a parametrix for Δ_g, as indicated in section D of the previous chapter, is the use of simpler models for this operator at the boundary, where it degenerates. These models, the normal and indicial operators, are employed in iterative schemes analogous to the initial symbolic step of the construction to obtain a parametrix for which the remainder is compact. It is important then to reach a thorough understanding of their mapping properties; this is the goal of the present chapter. In this section we identify these operators explicitly and analyze the indicial operator. Since it is an ordinary differential operator of Euler type, this is quite easy. The normal operator, however, is a partial differential operator, and the derivation of those of its properties we need later will require more of an effort.

We have given two different, yet equivalent, definitions of the normal operator of $P \in \mathrm{Diff}_0^m(M)$. The first one, (2.8), is an operator on the half tangent space M_p, $p \in \partial M$, while the second, (2.43), is a convolution kernel on the fibre F_p of the front face. This latter interpretation is the one we ultimately use, but the former is easier to manipulate, hence the one of interest in this chapter. Recall now that the action of Δ_g on ${}^0\Lambda^k(M)$ is given by the operator P of (1.21), so that $P = \rho^k \Delta_g \rho^{-k}$. Then $N_p(\Delta_g)$ is an operator on M_p acting on sections of ${}^o\Lambda^k(M_p)$; the induced operator is $N_p(P)$. It turns out that $N_p(\Delta_g)$ is actually the Laplacian of a constant curvature metric, whence it has greater invariance properties than merely the expected G_p-invariance.

(3.1) <u>Proposition</u>: $N_p(\Delta_g)$ is the Laplacian of the constant curvature metric $(d\rho_p)^{-2}h_p$; here $d\rho_p$ is to be thought of as a linear function on M_p. The induced action on ${}^o\Lambda^k(M_p)$ is given by $N_p(P)$.

<u>Proof</u>: Choose coordinates $\bar{z} = (\bar{z}^1, \cdots, \bar{z}^{n-1}, \bar{z}^n) = (\bar{y}, \bar{x})$ on M near p, such that $\bar{y} = (\bar{z}^1, \cdots, \bar{z}^{n-1})$ are geodesic coordinates for h restricted to ∂M, centered at p, and $\bar{z}^n = \bar{x}$ is a defining function for the boundary and $|\partial/\partial\bar{z}^n|_h = 1$. Let $z = (y, x)$ be the corresponding linear coordinates on M_p. Finally, write

$$\rho(\bar{y}, \bar{x}) = a(\bar{y})\bar{x} + b(\bar{y}, \bar{x}), \qquad b = O(\bar{x}^2).$$

Then $|d\rho_p|^2 = a(0)^2$. Now, for $\omega \in \Omega^k(M)$

$$P\omega = \underset{(1)}{\rho^2 \Delta_h \omega} + \underset{(2)}{(2-k)\rho\, d\rho \wedge \delta_h \omega} - \underset{(3)}{k\rho\, \delta_h(d\rho\wedge\omega)}$$

$$+ \underset{(4)}{(n-k)\rho L_{\nabla\rho}\omega} - \underset{(5)}{2\rho\iota_{\nabla\rho}\, d\omega}$$

$$+ \underset{(6)}{(n-2k)d\rho \wedge \iota_{\nabla\rho}\omega} - \underset{(7)}{k(n-k-1)\iota_{\nabla\rho}(d\rho)\omega}$$

is the operator of (1.21), defined by $\Delta_g\left[\frac{\omega}{\rho^k}\right] = \frac{(P\omega)}{\rho^k}$. Suppose $\omega = \omega_I\, d\bar{z}^I$, $I = (i_1,\cdots,i_k)$, $1 \le i_1 <\cdots< i_k \le n$. We shall consider each term (1) - (7) separately, always using the product rule (2.10).

(1) $\rho^2\Delta_h\omega$ has normal operator

$$\left[-a(0)^2 \sum_{i=1}^{n} \frac{\partial^2\omega_I}{(\partial z^i)^2}\right]dx^I$$

(2) $\rho\ d\rho \wedge \delta_h\omega =$

$$(a\bar{z}^n+b)(ad\bar{z}^n+\bar{z}^n da+db)\wedge(-\delta^I_{i_s,I\setminus i_s}(\nabla_{i_s}\omega_I)d\bar{z}^{I\setminus i_s})$$

$$= a^2\bar{z}^n\ d\bar{z}^n \wedge (-1)^s \frac{\partial\omega_I}{\partial z^{i_s}}\ d\bar{z}^{I\setminus i_s} + \cdots$$

which becomes

$$- a^2 z^n \frac{\partial \omega_I}{\partial z^n} \; dz^I \qquad \text{if} \quad i_k = n$$

or

$$\sum_s a^2 (-1)^{k+s-1} z^n \frac{\partial \omega_I}{\partial z^{i_s}} \, dz^{I \setminus i_s} \wedge dz^n \qquad \text{if} \quad i_k < n$$

(3) $$\rho \delta_h (d\rho \wedge \omega) = (a\bar{z}^n + b)\delta_h((ad\bar{z}^n + \cdots) \wedge \omega_I d\bar{z}^I)$$

$$= (a\bar{z}^n + b) \sum (-1)^{k+1} \delta^{I,n}_{i_s, I \setminus i_s, n} \nabla_{i_s} (a\omega_I) d\bar{z}^{I \setminus i_s} \wedge d\bar{z}^n$$

$$+ (a\bar{z}^n + b)(-1)^{k+1} \delta^{I,n}_{n,I} \nabla_n (a\omega_I) d\bar{z}^I + \cdots$$

which reduces to

$$- a^2 z^n \frac{\partial \omega_I}{\partial z^n} \, dz^I + \sum_s (-1)^{k+s} a^2 z^n \frac{\partial \omega_I}{\partial \bar{z}^{i_s}} \, dz^{I \setminus i_s} \wedge dz^n$$

$$\text{if} \quad i_k < n$$

$$0 \qquad\qquad \text{if} \quad i_k = n$$

(4) $\rho L_{\nabla\rho}\omega =$

$$(a\bar{z}^n+b)(\nabla\rho(\omega_I)d\bar{z}^I+\omega_I \frac{\partial\rho^{i_s}}{\partial\bar{z}^j} d\bar{z}^{i_1\cdots i_{s-1}ji_{s+1}\cdots i_k})$$

$$= a^2\bar{z}^n \frac{\partial\omega_I}{\partial\bar{z}^n} d\bar{z}^I + \cdots$$

which remains as $a^2 z^n \dfrac{\partial\omega_I}{\partial z^n} dz^I$ for any I

(5) $\rho\iota_{\nabla\rho}\, d\omega = (a\bar{z}^n+b)\iota_{(\rho^i\partial/\partial\bar{z}^i)}\left[\dfrac{\partial\omega_I}{\partial\bar{z}^j} d\bar{z}^j \wedge d\bar{z}^I\right]$

$$= (a\bar{z}^n+b)\rho^i \frac{\partial\omega_I}{\partial\bar{z}^j}\,\iota_{\partial/\partial\bar{z}^i}(d\bar{z}^j\wedge d\bar{z}^I)$$

$$= a^2 z^n \frac{\partial\omega_I}{\partial\bar{z}^j}\,\iota_{\partial/\partial\bar{z}^n}(d\bar{z}^j\wedge d\bar{z}^I) + \cdots$$

recalling $\rho^i = O(\rho)$ if $i < n$

yielding

$$a^2 z^n \frac{\partial\omega_I}{\partial z^n} dz^I \quad \text{if} \quad i_k < n$$

$$a^2 \sum (-1)^k z^n \frac{\partial\omega_I}{\partial z^j} dz^j \wedge dz^{I\setminus n} \quad \text{if} \quad i_k = n$$

(6) $$d\rho \wedge \iota_{\nabla\rho}\omega = (a d\bar{z}^n + \cdots) \wedge \rho^i \iota_{(\partial/\partial\bar{z}^i)} \omega_I \, d\bar{z}^I$$

$$= a^2 \, d\bar{z}^n \wedge \omega_I \iota_{(\partial/\partial\bar{z}^n)} \, d\bar{z}^I$$

and all that remains is

$$\begin{matrix} a^2\omega_I \, dz^I & i_k = n \\ 0 & i_k < n. \end{matrix}$$

(7) Finally, $d\rho(\nabla\rho)\omega = a^2\omega_I \, d\bar{z}^I + \cdots$

leaves $a^2\omega_I \, dz^I$ for any I.

Heuristically, the procedure throughout is to expand the coefficients of P in a power series, discard any term in which the power of $\bar{z}^n$ exceeds the number of derivatives, and then replace all coordinates $\bar{z}^i$ by their linear analogues z^i.

Collect all terms above with the appropriate coefficients to obtain

$$
(3.2)\qquad N_p(P)(\omega) = \begin{cases} a(0)^2\Big\{\Big[-(z^n)^2 \sum_{i=1}^{n} \dfrac{\partial^2\omega_I}{(\partial z^i)^2} + (n-2)z^n \dfrac{\partial\omega_I}{\partial z^n} - k(n-k-1)\omega_I\Big]dz^I \\ \qquad + 2(-1)^k \sum_s z^n \dfrac{\partial\omega_I}{\partial z^{i_s}} (-1)^{s-1} dz^{I\setminus i_s} \wedge dz^n\Big\} & \text{if } i_k < n \\ a(0)^2\Big\{\Big[-(z^n)^2 \sum_{i=1}^{n} \dfrac{\partial^2\omega_I}{(\partial z^i)^2} + (n-2)z^n \dfrac{\partial\omega_I}{\partial z^n} - (k-1)(n-k)\omega_I\Big]dz^I \\ \qquad + 2(-1)^{k+1} \sum_{j=1}^{n-1} z^n \dfrac{\partial\omega_I}{\partial z^j}\, dz^j \wedge dz^{I\setminus n}\Big\} & \text{if } i_k = n. \end{cases}
$$

Furthermore, the product formula (2.10) shows that

$$N_p(\Delta_g) = (z^n)^{-k} N_p(P)(z^n)^k$$

so that $N_p(\Delta_g)\left[\dfrac{\omega}{(z^n)^k}\right] = \dfrac{N_p(P)\omega}{(z^n)^k}$, i.e. the action induced by $N_p(\Delta_g)$ on ${}^o\Lambda^k(M_p)$ is given by $N_p(P)$, as claimed.

Finally, to see that $N_p(\Delta_g)$ is the Laplacian of the metric $(a(0)z^n)^{-2}\, dz^2$, simply observe that this a special case of all of the calculations above, and preceding (1.21), with $\rho = a(0)z^n$, $h = dz^2$.

The corresponding P is already dilation invariant and so no terms are lost in passing to the limit in (2.8). The assertion now follows, for (3.2) depends only on the dimension n, the degree k, and $|d\rho_p|_n^2$.

In the next section we construct an inverse for the action of $N_p(\Delta_g)$ on ${}^o\Lambda^k(M_p)$, i.e. for $N_p(P)$. However, it is really the normal operator, as defined by (2.43), acting on sections of ${}^o\Lambda_\ell^k \otimes \Gamma_0^{1/2}|_{F_p}$ that we need to invert. It is true, though not immediately obvious, that this bundle and ${}^o\Lambda^k(M_p) \otimes \Gamma_0^{1/2}(M_p)$ are naturally isomorphic. Thus it suffices to study the operator over M_p, then transfer the results back to F_p. Let us now examine the details of this bundle isomorphism. We prove the identification only for the k-form bundle; a similar argument applies equally well to the half-densities. In fact, the density bundles will be systematically neglected through most of this chapter. Their later inclusion is effected by 'conjugating the whole discussion by $x^{n/2}$.'

(3.3) <u>Lemma</u>: Both ${}^o\Lambda^k(M_p)$ and ${}^o\Lambda^k_\ell|_{F_p}$ are trivial, though not naturally so. There is a canonical equivalence between these two bundles over the interiors of their bases. Furthermore, each is also naturally equivalent to ${}^o\Lambda^k(B^n) = \Lambda^k({}^oT^*B^n)$, again only over the interiors of their bases.

<u>Proof</u>: ${}^o\Lambda^k(M_p)$ lifts to the left factor of $M_p \times M_p$, thence to $M_p \times_0 M_p$; the resulting bundle we denote ${}^o\Lambda^k_\ell(M_p \times_0 M_p)$. It then pulls back to the fibre F_0 over $(0,0) \in \partial M_p \times \partial M_p$. Examination of the rules (2.25) for formal pullback show that this restriction is isomorphic to ${}^o\Lambda^k(M_p)$. Now, the linear model M_p is equivalent to first order with M at p. This means that the stretched products $M_p \times_0 M_p$ and $M \times_0 M$ are also first order equivalent near F_0 and F_p, respectively. Hence ${}^o\Lambda^k(M_p)(M_p \times_0 M_p)$ and ${}^o\Lambda^k_\ell(M \times_0 M)$ agree to order zero at these fibres, so their restrictions to F_0 and F_p coincide in a natural manner. Finally, Lemma (2.19) asserts the equivalence of any of these bundles with ${}^o\Lambda^k(B^n)$, at least over the interiors of the bases. Notice that the isomorphism ${}^o\Lambda^k(M_p) \simeq {}^o\Lambda^k(B^n)$ is <u>not</u> induced by the usual conformal map $\mathbb{R}^n_+ \longrightarrow B^n$.

As we have indicated earlier, the indicial operator of P is an ordinary differential operator of Euler type; it is actually an uncoupled system. Using the coordinates of (3.1) with p corresponding to 0, from (3.2) we derive

$$(3.4)\qquad I_p(P)(\omega_I\, dz^I) = \begin{cases} a(0)^2\left[-(z^n)^2 \dfrac{\partial^2\omega_I}{(\partial z^n)^2} + (n-2)\dfrac{\partial\omega_I}{\partial z^n} - k(n-k-1)\omega_I\right]dz^I & \text{if } i_k < n \\[2ex] a(0)^2\left[-(z^n)^2 \dfrac{\partial^2\omega_I}{(\partial z^n)^2} + (n-2)\dfrac{\partial\omega_I}{\partial z^n} - (k-1)(n-k)\omega_I\right]dz^I & \text{if } i_k = n. \end{cases}$$

Reverting now to the notation $z = (y,x)$, the solutions of

$$I_p(P)(\omega) = 0$$

are of the form $\omega = x^{s_1}\, dy^I$, $\omega = x^{s_2}\, dy^J \wedge dx$, $|I| = k$, $|J| = k-1$. The numbers s_1 and s_2 are called the indicial roots. A short calculation shows that

$$(3.5)\qquad \begin{aligned} s_1 &= k \quad \text{or} \quad n-k-1 \quad \text{unless} \quad k = \tfrac{n-1}{2} \\ s_2 &= k-1 \quad \text{or} \quad n-k \quad \text{unless} \quad k = \tfrac{n+1}{2}. \end{aligned}$$

The solutions in the exceptional cases $k = (n\pm1)/2$ involve additional logarithmic factors, but we are not concerned with these here.

The basic reason the parametrix construction breaks down in these exceptional cases is that only for these values of k do the indicial roots not vanish at different rates at $x = 0$. For the other values of k, only one of the solutions corresponding to each pair of indicial roots lies in the L^2 space which is natural for the problem. Thus:

$$\text{(3.6)}\qquad \begin{aligned} &\text{Both } x^{n-k},\ x^{n-k-1} \in L^2_{loc}(x^{-n}\,dx) \quad \text{iff} \quad k < \tfrac{n-1}{2} \\ &\text{Both } x^{k},\ x^{k-1} \in L^2_{loc}(x^{-n}\,dx) \qquad \text{iff} \quad k > \tfrac{n+1}{2}\ . \end{aligned}$$

Slightly restated, if ω solves $I_p(P)\omega = 0$ and $\omega \in L^2_{loc}(x^{-n}\,dx)$, then (with summation intended)

$$\text{(3.7)}\qquad \begin{aligned} &\omega = c_I x^{n-k-1} dy^I + c_J x^{n-k} dy^J \wedge dx \quad \text{if} \quad k < \tfrac{n-1}{2} \\ &\omega = c_I x^{k} dy^I + c_J x^{k-1} dy^J \wedge dx \qquad \text{if} \quad k > \tfrac{n+1}{2}\ . \end{aligned}$$

We shall prove in the next chapter that if $\omega \in L^2(dg)$, $P\omega = 0$, then it is polyhomogeneous conormal at ∂M, and (3.7) is the leading term in its asymptotic expansion there.

We should remark, as a general note, that the anaylsis of this particular V_0 elliptic problem is greatly simplified by the fact that the indicial roots (3.5) are independent of the base point p. In the general problem they may well vary, and complications arise from repeated roots, etc.

B. The Inverse of the Hyperbolic Laplacian

Let N_p denote the operator of (3.2), neglecting the $a(0)^2$ factor. From Lemma (3.3), N_p may be interpreted as the operator induced by the constant curvature Laplacian either on ${}^o\Lambda^k(\mathbb{R}^n_+)$ or ${}^o\Lambda^k(B^n)$. It is important that we have both of these models available. The inverse for N_p is constructed using the Fourier transform on the y variables in the upper half space. This is quite natural, for these hypersurfaces are horospheres, hence have natural Euclidean structures. Then, however, the additional rotational symmetry of the ball is useful in understanding the behaviour of this inverse near $\infty \in \mathbb{R}^n_+$.

So, N_p has a partial symbol given by

$$N_p(x\partial_x, x\partial_y) \int e^{iy\eta}\hat{\omega}(\eta)d\eta = \int e^{iy\cdot\eta}\cdot N_p(x\partial_x, ix\eta)\hat{\omega}(\eta)d\eta$$

where $\eta = \eta_1 dy^1 + \cdots + \eta_{n-1}dy^{n-1}$ is dual to y, and

$$\hat{\omega}(\eta) = \int e^{-iy\cdot\eta}\omega(y)dy$$

is the usual Fourier transform on the components of ω. It is easy to see from (3.2) that

$$N_P(x\partial_x, ix\eta)\hat{\omega}_I dy^I = \left[-x^2 \frac{\partial^2\hat{\omega}_I}{\partial x^2} + (n-2)x\frac{\partial\hat{\omega}_I}{\partial x} + (x^2|\eta|^2 - k(n-k-1))\hat{\omega}_I\right]dy^I + 2(-1)^k\iota(ix\tilde{\eta})\hat{\omega}_I dy^I \wedge dx$$

(3.8)

$$N_P(x\partial_x, ix\eta)\hat{\omega}_J dy^J \wedge dx = \left[-x^2 \frac{\partial^2\hat{\omega}_J}{\partial x^2} + (n-2)x\frac{\partial\hat{\omega}_J}{\partial x} + (x^2|\eta|^2 - (k-1)(n-k))\hat{\omega}_J\right]dy^J \wedge dx + 2(-1)^{k+1} ix\eta \wedge \hat{\omega}_J dy^J.$$

The $\tilde{\eta}$ in the first of these formulae is the h-dual of η:

$$\tilde{\eta} = \eta_1 \frac{\partial}{\partial y^1} + \cdots + \eta_{n-1} \frac{\partial}{\partial y^{n-1}}$$

and is to be contracted with $dy^I \wedge dx$.

The operator of (3.8), which we also call N_P, is only mildly coupled. This allows all solutions to $N_P\hat{\omega} = 0$

to be found, whence the inverse may be constructed and analyzed. Thus, regarding η now as a fixed parameter, the y coordinates may be rotated so that (3.8) assumes a particularly simple form. In fact, rotate y so that $\eta = |\eta|\, dy^1$. Then

$$\eta \wedge dy^J \neq 0 \Leftrightarrow j_1 > 1$$

$$\iota(\tilde{\eta}) dy^I \wedge dx \neq 0 \Leftrightarrow i_1 = 1.$$

There are three cases to consider:

(3.9) i) $\hat{\omega} = \hat{\omega}_I\, dy^I, \quad i_1 > 1.$

$$\Rightarrow N_P\hat{\omega} = \left[-x^2 \frac{\partial^2 \hat{\omega}_I}{\partial x^2} + (n-2)x\frac{\partial \hat{\omega}_I}{\partial x} + (x^2|\eta|^2 - k(n-k-1))\hat{\omega}_I\right] dy^I$$

ii) $\hat{\omega} = \hat{\omega}_J\, dy^J \wedge dx, \quad j_1 = 1.$

$$\Rightarrow N_P\hat{\omega} = \left[-x^2 \frac{\partial^2 \hat{\omega}_J}{\partial x^2} + (n-2)x\frac{\partial \hat{\omega}_J}{\partial x} + (x^2|\eta|^2 - (k-1)(n-k))\hat{\omega}_J\right] dy^J \wedge dx$$

iii) $\hat{\omega} = \hat{\omega}_I\, dy^I + \hat{\omega}_J\, dy^J \wedge dx,$

$$I = (i_1, \cdots, i_k) = (1, j_1, \cdots, j_{k-1}) = (1, J)$$

$$\Rightarrow N_p\hat{\omega} = \left[-x^2 \frac{\partial^2\hat{\omega}_I}{\partial x^2} + (n-2)x\frac{\partial\hat{\omega}_I}{\partial x} + (x^2|\eta|^2-k(n-k-1))\hat{\omega}_I + 2(-1)^{k+1}ix|\eta|\hat{\omega}_J\right]dy^I$$

$$+ \left[-x^2 \frac{\partial^2\hat{\omega}_J}{\partial x^2} + (n-2)x\frac{\partial\hat{\omega}_J}{\partial x} + (x^2|\eta|^2-(k-1)(n-k))\hat{\omega}_J + 2(-1)^k ix|\eta|\hat{\omega}_I\right]dy^J \wedge dx.$$

The first two of these operators are quite easy to invert. The third one is slightly less so, since not all of its solutions are known explicitly. Nonetheless, it may be regarded as a compact perturbation of the diagonal system formed by i) and ii).

The program now is as follows. Ultimately we seek to prove that the partial differential operator N_p of (3.2), acting on

$$L^2(\mathbb{R}^+\times\mathbb{R}^{n-1}, x^{-n}\,dx\,dy) \simeq L^2(\mathbb{R}^+, x^{-n}\,dx; L^2(\mathbb{R}^{n-1}, dy))$$

has a bounded inverse. Yet more refined regularity properties of this inverse are proved in the next section. Upon conjugation by the Fourier transform it suffices to show that the ordinary differential operator N_p of (3.8), acting on

$$L^2(\mathbb{R}^+, x^{-n}\, dx; L^2(\mathbb{R}^{n-1}, d\eta))$$

also has a bounded inverse.

(3.10) <u>Proposition</u>: There exists a unique kernel $G(x,\tilde{x},\eta)$ which is bounded on

$$L^2(\mathbb{R}^+, x^{-n}\, dx; L^2(\mathbb{R}^{n-1}, d\eta))$$

and such that

$$N_p(x\partial_x, ix\eta)G(x,\tilde{x},\eta) = x^n\delta(x-\tilde{x}).$$

(The right side of this equation is the Schwartz kernel of the identity on the space above.)

The proof is quite long and involves several lemmas along the way. Consider first the two operators (3.9) i), ii). Set

$$P_1 = -x^2\frac{d^2}{dx^2} + (n-2)x\frac{d}{dx} + (x^2|\eta|^2 - k(n-k-1))$$

$$P_2 = -x^2\frac{d^2}{dx^2} + (n-2)x\frac{d}{dx} + (x^2|\eta|^2 - (k-1)(n-k)).$$

These are Fuchsian operators, the indicial roots of which, as expected, are k, $n-k-1$ and $k-1$, $n-k$. In fact, for $k < \frac{n-1}{2}$ (the case $k > \frac{n+1}{2}$ is treated similarly, and shall be mentioned only rarely) we find, cf. [19],

(3.11)

$$P_1u_1 = 0 \Rightarrow u_1(x) = c_1x^{\frac{n-1}{2}} I_{\nu_1}(x|\eta|) + c_2x^{\frac{n-1}{2}} K_{\nu_1}(x|\eta|),$$

$$\nu_1 = \frac{n-2k-1}{2}$$

$$P_2u_2 = 0 \Rightarrow u_2(x) = c_1x^{\frac{n-1}{2}} I_{\nu_2}(x|\eta|) + c_2x^{\frac{n-1}{2}} K_{\nu_2}(x|\eta|),$$

$$\nu_2 = \frac{n-2k+1}{2} .$$

The I_ν amd K_ν are Bessel functions—specifically, the modified Bessel function of the first kind and Macdonald's function, respectively, of order $\nu > 0$. Their asymptotics are well-known [19]

(3.12)

$$I_\nu(x) \sim x^\nu/2^\nu\Gamma(1+\nu), \quad K_\nu(x) \sim 2^{\nu-1}\Gamma(\nu)/x^\nu \qquad x \longrightarrow 0^+$$

$$I_\nu(x) \sim e^x/\sqrt{2\pi x}, \quad K_\nu(x) \sim e^{-x}\sqrt{\pi/2x} \qquad x \longrightarrow \infty.$$

Thus, near $x = 0$,

$$x^{\frac{n-1}{2}} I_{\nu_1}(x|\eta|) \sim x^{n-k-1}|\eta|^{\nu_1},$$

$$x^{\frac{n-1}{2}} K_{\nu_1}(x|\eta|) \sim x^k |\eta|^{-\nu_1}$$

(3.13)

$$x^{\frac{n-1}{2}} I_{\nu_2}(x|\eta|) \sim x^{n-k} |\eta|^{\nu_2},$$

$$x^{\frac{n-1}{2}} K_{\nu_2}(x|\eta|) \sim x^{k-1} |\eta|^{-\nu_2}$$

(up to constant factors). Recalling (3.7), $u_i \in L^2_{loc}(x^{-n}\,dx)$ only if $c_2 = 0$, $i = 1,2$. On the other hand, these solutions are exponentially increasing at infinity. The other pair, with $c_1 = 0$, decrease suitably at infinity, but do not vanish to sufficiently high order at $x = 0$. (For the exceptional degrees $k = (n\pm 1)/2$, the I_ν and K_ν differ only by a logarithmic factor at $x = 0$.)

Now we construct kernels inverting P_1 and P_2. In fact, these two operators are treated so similarly that only the work for P_1 is displayed. With

$$u(x) = x^{\frac{n-1}{2}} I_{\nu_1}(x|\eta|), \quad v(x) = x^{\frac{n-1}{2}} K_{\nu_1}(|\eta|)$$

we know the kernel is of the form

$$G_1(x,\tilde{x},\eta) = \alpha(\tilde{x})u(x)H(\tilde{x}-x) + \beta(\tilde{x})v(x)H(x-\tilde{x})$$

and satisfies

$$P_1G_1(x,\tilde{x},\eta) = x^n\delta(x-\tilde{x}),$$

which is the kernel of the identity on $L^2(x^{-n}\,dx)$. This form is chosen so that G_1 is a 'free' solution for $x \neq \tilde{x}$, satisfying the boundary conditions (with respect to x) of sufficiently rapid decay at $x = 0, \infty$. A bit of calculation, with $\alpha(\tilde{x}) = \gamma(\tilde{x})v(\tilde{x})$, $\beta(\tilde{x}) = \gamma(\tilde{x})u(\tilde{x})$, shows

$$P_1G_1 = x^n\delta(x-\tilde{x})[-2\gamma(x)x^{2-n}(u(x)v'(x)-u'(x)v(x)].$$

Now, as $P_1 = -x^n \frac{d}{dx}(x^{2-n} \frac{d}{dx}) + (x^2|\eta|^2-k(n-k-1))$, the expression

$$x^{2-n}(u(x)v'(x)-u'(x)v(x))$$

is simply the Sturm-Liouville Wronskian of P_1, hence independent of x. By examining, for example, its power series as $x \longrightarrow 0$, it is also independent of η. Choosing $\gamma(x)$ to be the appropriate constant

$$(3.14)\qquad G_1(x,\tilde{x},\eta) = c_{k,n} x^{\frac{n-1}{2}} \tilde{x}^{\frac{n-1}{2}} [I_{\upsilon_1}(x|\eta|)K_{\upsilon_1}(\tilde{x}|\eta|)H(\tilde{x}-x)$$

$$+ I_{\upsilon_1}(\tilde{x}|\eta|)K_{\upsilon_1}(x|\eta|)H(x-\tilde{x})].$$

Analogously, the kernel G_2 for P_2 is obtained by replacing υ_1 with υ_2.

It is convenient to introduce the space

$$(3.15)\qquad \mathcal{H}_r^{(p)} = \hat{H}^p([0,2];x^{-n}\,dx) + x^r H^p([1,\infty);dx)$$

where $p \in \mathbb{Z}^+$, $r \in \mathbb{R}$, H^p is the usual Sobolev space of order p and

$$\hat{H}^p = \{f : x^i \frac{d^j f}{dx^j} \in L^2([0,2];x^{-n}\,dx) \quad \text{for} \quad p \geq i \geq j \geq 0\}$$

so as to formulate the

(3.16) <u>Proposition</u>: For $k < \frac{n-1}{2}$ or $k > \frac{n+1}{2}$

$$P_i : \mathcal{H}_r^{(2)} \longrightarrow \mathcal{H}_{r+2}^{(0)} \qquad i = 1,2$$

is an isomorphism, with inverse represented by the kernel $G_i(x,\tilde{x},\eta)$.

<u>Proof</u>: Obviously P_1 and P_2 map $\mathcal{H}_r^{(2)}$ into $\mathcal{H}_{r+2}^{(0)}$, so it suffices to show that

$$G_i \; : \; \mathcal{H}^{(0)}_{r+2} \longrightarrow \mathcal{H}^{(2)}_{r} \qquad i = 1,2$$

is bounded. By construction it must then actually invert P_i. Uniqueness is also immediate since $P_i u = 0$ has no solutions in $\mathcal{H}^{(2)}_r$ for any r.

The basic tool to prove boundedness of a kernel is

<u>Schur's</u> <u>Test</u>:

The positive kernel $G(x,\tilde{x})$ induces a bounded map on $L^2(d\mu)$ if there exist positive measurable functions p and q such that

$$\int G(x,\tilde{x})q(\tilde{x})d\mu(\tilde{x}) \le \alpha p(x); \quad \int G(x,\tilde{x})p(x)d\mu(x) \le Bq(\tilde{x})$$

Now, given the decomposition of $\mathcal{H}^{(0)}_{r+2}$ as a sum in (3.14) it suffices to prove boundedness on each summand. Thus if $f \in \mathcal{H}^{(0)}_{r+2}$, write $f = f_1 + f_2$, $f_1 \in L^2([0,1];x^{-n}\,dx)$, $f_2 \in x^{r+2}L^2([1,\infty);dx)$. Accordingly

$$\|f\|^2_{r,(0)} = \int_0^1 |f_1(x)|^2 x^{-n}\,dx + \int_1^\infty (x^{-r-2}f_2(x))^2\,dx.$$

<u>Case I</u>: $f \in L^2([0,1],x^{-n}\,dx) = x^{n/2}L^2([0,1],dx)$.

It is convenient to include the various weight factors in the kernel. Thus let

$$J(x,\tilde{x}) = x^{-n/2}G_1(x,\tilde{x})\tilde{x}^{-n/2}.$$

Then $G_1 : L^2([0,1],x^{-n}\,dx) \longrightarrow \hat{H}^2([0,1],x^{-n}\,dx)$ is bounded if and only if

$$L^2(dx) \ni f \longmapsto x^{n/2}f \longmapsto \int G_1(x,\tilde{x})\tilde{x}^{n/2}f(\tilde{x})\tilde{x}^{-n}\,d\tilde{x}$$

$$\longmapsto \int x^{-n/2}G_1(x,\tilde{x})\tilde{x}^{-n/2}f(\tilde{x})d\tilde{x}$$

$$= \int J(x,\tilde{x})f(\tilde{x})d\tilde{x} \in \hat{H}^2([0,1];dx)$$

is also bounded. Furthermore, it even suffices to prove boundedness of J on $L^2(dx)$ since xdJ/dx, x^2d^2J/dx^2 are similar to J in their asymptotics as $x \longrightarrow 0$ and $x \longrightarrow \infty$. (These derivatives contain terms with $\delta(x-\tilde{x})$ as a factor, but never $\delta'(x-\tilde{x})$, multiplied by a bounded coefficient; these are obviously bounded on L^2.)

Now, since x and $\tilde{x}$ lie in $[0,1]$ we may replace J by

$$J_a(x,\tilde{x}) = x^{-n/2}[x^{n-k-1}\tilde{x}^kH(\tilde{x}-x) + x^k\tilde{x}^{n-k-1}H(x-\tilde{x})]\tilde{x}^{-n/2}$$

$$= x^{\frac{n}{2}-k-1}\tilde{x}^{k-\frac{n}{2}}H(\tilde{x}-x) + x^{k-\frac{n}{2}}\tilde{x}^{\frac{n}{2}-k-1}H(x-\tilde{x}).$$

For $k < \frac{n}{2} - 1$, apply Schur's test with $p = q = 1$:

$$\int_0^1 J_a(x,\tilde{x})d\tilde{x} = \int_0^x x^{k-\frac{n}{2}}\tilde{x}^{\frac{n}{2}-k-1}\,d\tilde{x} + \int_x^1 x^{\frac{n}{2}-k-1}\tilde{x}^{k-\frac{n}{2}}\,d\tilde{x}$$

$$= x^{k-\frac{n}{2}} \cdot \frac{\tilde{x}^{\frac{n}{2}-k}}{(\frac{n}{2}-k)}\Bigg|_0^x + x^{\frac{n}{2}-k-1}\,\frac{\tilde{x}^{k-\frac{n}{2}}}{(k-\frac{n}{2}+1)}\Bigg|_x^1$$

$$= x^{k-\frac{n}{2}}(x^{\frac{n}{2}-k}/(\tfrac{n}{2}-k)) + x^{\frac{n}{2}-k-1}(1 - x^{k-\frac{n}{2}+1})/(k-\tfrac{n}{2}+1) \leq C$$

since neither $k-\frac{n}{2}$ nor $\frac{n}{2}-k-1$ equal -1, and $\frac{n}{2}-k-1 > 0$.

Symmetrically $\int_0^1 J(x,\tilde{x})dx \leq C$.

For $k = \frac{n}{2} - 1$ (the only excluded case), use $p = q = (-\log x)/\sqrt{x}$.

$$\int_0^1 J_a(x,\tilde{x})(-\log \tilde{x})/\tilde{x}^{1/2}\,d\tilde{x}$$

$$= \int_0^x x^{-1}\,\frac{-\log \tilde{x}}{\tilde{x}^{1/2}}\,d\tilde{x} + \int_x^1 \frac{-\log \tilde{x}}{\tilde{x}^{3/2}}\,d\tilde{x}$$

$$= x^{-1}(-2\tilde{x}^{1/2} \log \tilde{x} + 4\tilde{x}^{1/2})\Big|_0^x + (2\tilde{x}^{-1/2} \log \tilde{x} + 4\tilde{x}^{-1/2})\Big|_x^1$$

$\leq C(-\log x)/\sqrt{x}$ and likewise for

$$\int_0^1 J_a(x,\tilde{x})(-\log x)/\sqrt{x}\, dx.$$

On the other hand, as $x \longmapsto \infty$ the only nonvanishing term in

$$\int G_1(x,\tilde{x})f(\tilde{x})\tilde{x}^{-n}\, d\tilde{x}$$

is, by (3.14)

$$\int_0^x v(x)u(\tilde{x})f(\tilde{x})\tilde{x}^{-n}\, d\tilde{x} = \int_0^1 v(x)u(\tilde{x})f(\tilde{x})\tilde{x}^{-n}\, d\tilde{x}$$

since $\operatorname{supp} f \subset [0,1]$. This decreases exponentially provided the integral makes sense. But $u \sim x^{n-k-1}$, so that

$$\int_0^1 \tilde{x}^{n-k-1} f(\tilde{x})\tilde{x}^{-n}\, d\tilde{x}$$

$$\leq \left[\int_0^1 f^2\tilde{x}^{-n}\, d\tilde{x}\right]^{1/2}\left[\int_0^1 \tilde{x}^{n-2k-2}\, d\tilde{x}\right]^{1/2} < \infty.$$

Case II: $f \in x^{r+2}L^2([1,\infty);dx)$.

Again we combine weight factors into the kernel. Thus we need to prove boundedness of the composition

$$L^2[1,\infty);dx) \ni f \longmapsto x^{r+2}f \longmapsto \int G_1(x,\tilde{x})\tilde{x}^{r+2-n}f(\tilde{x})d\tilde{x}$$

$$\longmapsto \int x^{-r}G_1(x,\tilde{x})\tilde{x}^{r+2-n}f(\tilde{x})d\tilde{x}$$

$$\equiv \int J(x,\tilde{x})f(\tilde{x})d\tilde{x} \in H^2([1,\infty],dx)$$

where $J(x,\tilde{x}) = x^{-r}G_1(x,\tilde{x})\tilde{x}^{r+2-n}$. As before it suffices to prove that J is bounded on L^2, for its derivatives are handled similarly. We require the

(3.17) Lemma: If $\mu \in \mathbb{R}$ then as $x \longrightarrow \infty$,

$$\left(\int_1^x \tilde{x}^{\mu}e^{\tilde{x}}\, d\tilde{x}\right)x^{-\mu}e^{-x} \longrightarrow 1$$

$$\left(\int_x^{\infty} \tilde{x}^{\mu}e^{-\tilde{x}}\, d\tilde{x}\right)x^{-\mu}e^{x} \longrightarrow 1.$$

Proof: Apply L'Hôpital's rule.

Now by (3.12) we may replace J by

$$J_a(x,\tilde{x}) = x^{\frac{n}{2}-r-1}\tilde{x}^{r+1-\frac{n}{2}}[e^{x-\tilde{x}}H(\tilde{x}-x) + e^{\tilde{x}-x}H(x-\tilde{x})].$$

Apply Schur's test with $p = q = 1$:

$$\int_1^\infty J_a(x,\tilde{x})d\tilde{x} = \int_1^x x^{\frac{n}{2}-r-1} e^{-x} \tilde{x}^{r+1-\frac{n}{2}} e^{\tilde{x}} d\tilde{x}$$

$$+ \int_x^\infty x^{\frac{n}{2}-r-1} e^{x} \tilde{x}^{r+1-\frac{n}{2}} e^{-\tilde{x}} d\tilde{x} \le C \quad \text{by (3.17).}$$

By symmetry, $\displaystyle\int_1^\infty J_a(x,\tilde{x})dx \le C.$

Finally, if $\operatorname{supp} f \subset [1,\infty]$, then as $x \longrightarrow 0$

$$\int G_1(x,\tilde{x})f(\tilde{x})\tilde{x}^{-n}\, d\tilde{x} = \int_x^\infty u(x)v(\tilde{x})f(\tilde{x})\tilde{x}^{-n}\, d\tilde{x}$$

$$= u(x)\int_1^\infty v(\tilde{x})f(\tilde{x})\tilde{x}^{-n}\, d\tilde{x} \in \hat{H}^2([0,1];x^{-n}\, dx)$$

since the integral exists and is independent of x, and $u(x)$ lies in this space.

This concludes the proof of Proposition (3.16).

We must now construct an inverse for the operator of (3.9) iii). Set

$$L_0 = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 2(-1)^{k+1} ix|\eta| \\ 2(-1)^k ix|\eta| & 0 \end{bmatrix}.$$

Then that operator is $L_1 = L_0 + A$. More generally, let

$$L_t = L_0 + tA, \quad 0 \le t \le 1.$$

By (3.16), for any r

$$L_0 : \mathcal{H}_r^{(2)} \longrightarrow \mathcal{H}_{r+2}^{(0)}$$

is invertible. (Strictly speaking, these spaces are defined to contain only scalar functions. Here of course we extend the definition to allow vector functions by requiring each component to lie in the space.) Clearly each L_t maps $\mathcal{H}_r^{(2)}$ into $\mathcal{H}_{r+2}^{(0)}$.

(3.18) <u>Lemma</u>: $A : \mathcal{H}_r^{(2)} \longrightarrow \mathcal{H}_{r+2}^{(0)}$ is compact.

<u>Proof</u>: Certainly $A : x^r H^2([1,\infty),dx) \longrightarrow x^{r+1}H^2([1,\infty),dx)$ is bounded. The inclusion

$$x^{r+1}H^2 \longrightarrow x^{r+2}L^2$$

is compact by the L^2 version of the Arzela-Ascoli Theorem. On the other hand, between

$$\hat{H}^2([0,2],x^{-n}\,dx) \longrightarrow L^2([0,2],x^{-n}\,dx)$$

A is the strong limit of the same multiplication operator acting between

$$\hat{H}^2([\epsilon,2],x^{-n}\,dx) \simeq \eta^2([\epsilon,2],x^{-n}\,dx) \longrightarrow L^2([\epsilon,2],x^{-n}\,dx)$$

which, for each $\epsilon > 0$, is compact as before. Such strong limits are compact provided a 'uniform smallness' condition at the boundary holds, cf. [29]. Here this states that if f varies over a bounded set in $\hat{H}^2([0,2],x^{-n}\,dx)$, the functions Af have uniformly small norm in $L^2([0,\epsilon],x^{-n}\,dx)$. This follows from the factor of x in A. The proof is complete.

By the Lemma,

$$L_t : \mathscr{H}^{(2)}_r \longrightarrow \mathscr{H}^{(0)}_{r+2}$$

is an analytic family of Fredholm operators. Thus, for any t:

(3.19) $$\text{index } L_t : \mathcal{H}_r^{(2)} \longrightarrow \mathcal{H}_{r+2}^{(0)}$$

$$= \text{index } L_0 : \mathcal{H}_r^{(2)} \longrightarrow \mathcal{H}_{r+2}^{(0)} = 0.$$

To prove that L_1 is an isomorphism we need only show

(3.20) <u>Proposition</u>: $L_1 u = 0$, $u \in \mathcal{H}_r^{(2)}$ for some $r \Rightarrow$ $u = 0$.

(3.21) <u>Corollary</u>: $L_1 : \mathcal{H}_r^{(2)} \longrightarrow \mathcal{H}_{r+2}^{(0)}$ is invertible for each r.

Note that from (3.16), a solution to $L_1 u = 0$ lying in some $\mathcal{H}_r^{(2)}$ must actually lie in every $\mathcal{H}_r^{(2)}$, $-\infty < r < \infty$. For

$$u \in \mathcal{H}_r^{(2)}, \quad (L_0+A)u = 0 \Rightarrow L_0 u = -Au \in \mathcal{H}_{r+1}^{(0)} \Rightarrow u \in \mathcal{H}_{r-1}^{(2)}$$

and the assertion follows inductively.

In essence, (3.20) is nothing but the statement that $\mathbb{H}^n$ has no L^2 harmonic forms outside the middle degrees. However, rather than developing this connection directly, which requires a fair bit of effort, it seems better to give a proof more nearly consonant with our overall techniques. There is actually an easy proof that L_1 is strictly positive based on the inequality

$$- x^2 \frac{d^2}{dx^2} + (n-2)x\frac{d}{dx} \geq \frac{(n-1)^2}{4}$$

—which is equivalent to the well-known fact that the spectrum of the hyperbolic Laplacian on functions is bounded below by $(n-1)^2/4$. Unfortunately this fails in the two degrees $k = \frac{n}{2} \pm 1$.

The method which covers all cases involves an integration by parts to replace the second order system $N_P\hat{\omega} = 0$ by the larger first order system—which may be solved—

$$N_d\hat{\omega} = 0, \quad N_\delta\hat{\omega} = 0.$$

Obviously this is just the normal operator analogue of the fact that L^2 harmonic forms on $\mathbb{H}^n$ are both closed and coclosed.

Recall first that the conjugated hyperbolic Laplacian may be factored

$$N_P = x^k \Delta_H x^{-k}$$

$$= (x^k\ d\ x^{-k+1})(x^{k-1}\ \delta_H\ x^{-k}) + (x^k\ \delta_H\ x^{-k-1})(x^{k+1}\ d\ x^{-k}).$$

Here δ_H is the coboundary operator on $\mathbb{H}^n$. Furthermore, by the product formula (2.10),

$$N_P = N(x^k\, d\, x^{-k+1})N(x^{k-1}\, \delta_H\, x^{-k})$$

$$+ N(x^k\, \delta_H\, x^{-k-1})N(x^{k+1}\, d\, x^{-k})$$

(neglecting basepoints). Now, from (1.19) and (1.20) a calculation similar to that in Proposition (3.1) shows that the (conjugate by the Fourier transform of) the normal operators are

$$N_d\hat{\omega} = (-1)^k(x\frac{\partial\hat{\omega}_I}{\partial x} - k\hat{\omega}_I)dy^I \wedge dx + ix\eta \wedge (\hat{\omega}_I dy^I + \hat{\omega}_J dy^J \wedge dx)$$

(3.22)

$$N_\delta\hat{\omega} = (-1)^k(x\frac{\partial\hat{\omega}_J}{\partial x} - (n-k)\hat{\omega}_J)dy^J - ix\iota(\tilde{\eta})(\hat{\omega}_I dy^I + \hat{\omega}_J dy^J \wedge dx).$$

Using again the reduction $|\eta|dy^1 = \eta$, this system splits into three cases, just as in (3.9). Only the last of these possibilities is of interest here, and arises when $I = (1,J)$: then, for $\hat{\omega} = \hat{\omega}_I dy^I + \hat{\omega}_J dy^J \wedge dx$

$$L_d\hat{\omega} = \left[(-1)^k(x\frac{d\hat{\omega}_I}{dx} - k\hat{\omega}_I) + ix|\eta|\hat{\omega}_J\right]dy^I \wedge dx$$

(3.23)

$$L_\delta\hat{\omega} = \left[(-1)^k(x\frac{d\hat{\omega}_J}{dx} - (n-k)\hat{\omega}_J) - ix|\eta|\hat{\omega}_I\right]dy^I,$$

and the relationship

$$L_1 = L_d L_\delta + L_\delta L_d$$

is still valid. (The degree k in L_d and L_δ has been suppressed, so in this last equation it is different in the two L_d's, and also in the two L_δ's.)

(3.24) <u>Lemma</u>: Let $\hat{\omega} = f_I dy^I + g_J dy^J \wedge dx$ and $\hat{\mu} = \alpha_K dy^K + \beta_L dy^L \wedge dx$ be k-form and (k-1)-form valued functions, respectively, on $\mathbb{R}^+$. Then using the pointwise Euclidean scalar product on forms we have

$$\int_0^\infty \langle N_d\hat{\omega},\hat{\mu}\rangle x^{-n}\, dx = \int_0^\infty \langle\hat{\omega}, N_\delta\hat{\mu}\rangle x^{-n}\, dx$$

if and only if for each I: $\displaystyle\int_0^\infty \frac{d}{dx}(x^{1-n} f_I \bar{\beta}_I)dx = 0.$

<u>Proof</u>: $\displaystyle\int_0^\infty \langle N_d\hat{\omega},\hat{\mu}\rangle x^{-n}\, dx$

$$= \int_0^\infty \langle (-1)^k (x\frac{df_I}{dx} - kf_I)dy^I \wedge dx + ix\eta \wedge \hat{\omega},\hat{\mu}\rangle x^{-n}\, dx$$

$$= \int_0^\infty \left[(-1)^k (x\frac{df_I}{dx} - kf_I)\bar{\beta}_I + \langle\hat{\omega}, -ix\iota(\tilde{\eta})\hat{\mu}\rangle\right] x^{-n}\, dx$$

$$= \int_0^\infty (-1)^k(-f_I \frac{d}{dx}(x^{1-n}\bar{\beta}_I)-k\bar{\beta}_I x^{-n}) - \langle\hat{\omega}, ix\iota(\tilde{\eta})\hat{\mu}\rangle x^{-n}\ dx$$

$$+ \int_0^\infty (-1)^k \frac{d}{dx}(x^{1-n}f_I\bar{\beta}_I)dx$$

$$= \int_0^\infty \langle\hat{\omega}, N_\delta\hat{\mu}\rangle x^{-n}\ dx + \int_0^\infty (-1)^k \frac{d}{dx}(x^{1-n}f_I\bar{\beta}_I)dx.$$

Hence N_d and N_δ are adjoints iff this boundary term vanishes.

<u>Proof</u> of Proposition (3.20): suppose $\hat{\omega} = f\ dy^I + g\ dy^J \wedge dx$ solves

$$L_1\hat{\omega} = 0$$

(here I and J are fixed, $I = (1,J)$), and $\hat{\omega} \in \mathcal{H}_r^{(2)}$ for some r. The comments following (3.21) indicate that $\hat{\omega} \in L^2(x^{-n}\ dx)$ and decreases rapidly at infinity. Then

$$0 = \int_0^\infty \langle L_1\hat{\omega}, \hat{\omega}\rangle x^{-n}\ dx = \int_0^\infty \langle L_d L_\delta\hat{\omega} + L_\delta L_d\hat{\omega}, \hat{\omega}\rangle x^{-n}\ dx$$

$$= \int_0^\infty \left(|L_\delta \hat{\omega}|^2 + |L_d \hat{\omega}|^2\right) x^{-n}\, dx$$

provided

$$\text{a)} \quad \int_0^\infty \frac{d}{dx}\left[x^{1-n}\left(x\frac{dg}{dx} - (n-k)g + (-1)^{k+1} ix|\eta|f\right)\bar{g}\right] dx = 0$$

and

$$\text{b)} \quad \int_0^\infty \frac{d}{dx}\left[x^{1-n}\left(x\frac{df}{dx} - kf + (-1)^{k} ix|\eta|g\right)\bar{f}\right] dx = 0.$$

Now L_1 is a Fuchsian operator, the indicial roots of which coincide with those of the indicial operator of P. From (3.6) and a bit of calculation, if $k < \frac{n-1}{2}$, f and g have convergent expansions

$$\text{(3.25)} \quad \begin{aligned} f &= c_0 x^{n-k-1} + \cdots + \log x \cdot (c_0' x^{n-k+1} + \cdots) \\ g &= d_0 x^{n-k} + \cdots + \log x \cdot (d_0' x^{n-k} + \cdots). \end{aligned}$$

The logarithmic factors arise since the indicial roots differ by a positive integer. That these factors actually occur follows from an explicit computation of the first few terms of the Frobenius series. For later reference we record that when $k > \frac{n+1}{2}$

$$(3.26)\qquad \begin{aligned} f &= c_0 x^k + \cdots + \log x \cdot (c_0' x^k + \cdots) \\ g &= d_0 x^{k-1} + \cdots + \log x \cdot (d_0' x^{k+1} + \cdots) \end{aligned}$$

To prove that the boundary contributions vanish, use these expansions near $x = 0$ and the rapid decrease as $x \longrightarrow \infty$. Then, for example in a), when $k < \frac{n-1}{2}$, the integrand is

$$\frac{d}{dx}(a_0 x^{n-2k+1} + \cdots)$$

while in b) it is

$$\frac{d}{dx}(b_0 x^{n-2k-1} + \cdots)$$

By the second fundamental theorem of calculus the boundary terms vanish as desired; similar reasoning holds when $k > \frac{n+1}{2}$.

We have so far shown that an L^2 solution of $N_p \hat{\omega} = 0$ also satisfies

$$L_d \hat{\omega} = 0, \qquad L_\delta \hat{\omega} = 0.$$

In terms of the coordinate components f and g these equations are

$$x\frac{df}{dx} - kf + (-1)^k ix|\eta|g = 0$$

$$x\frac{dg}{dx} - (n-k)g + (-1)^{k+1} ix|\eta|f = 0.$$

These equations may actually be uncoupled and solved. Thus, with a bit of work one sees that

$$x^2\frac{d^2f}{dx^2} - nx\frac{df}{dx} + (k(n-k+1)-x^2|\eta|^2)f = 0$$

$$x^2\frac{d^2g}{dx^2} - nx\frac{dg}{dx} + ((k+1)(n-k)-x^2|\eta|^2)g = 0.$$

Once again f and g are Bessel functions, [19]:

$$f = c_1 x^{\frac{n+1}{2}} I_{\nu_2}(x|\eta|) + c_2 x^{\frac{n+1}{2}} K_{\nu_2}(x|\eta|) \quad \nu_2 = \left|\frac{n-2k+1}{2}\right|$$

(3.27)

$$g = d_1 x^{\frac{n+1}{2}} I_{\nu_1}(x|\eta|) + d_2 x^{\frac{n+1}{2}} K_{\nu_1}(x|\eta|) \quad \nu_1 = \left|\frac{n-2k-1}{2}\right|.$$

In order that f, g lie in $L^2(x^{-n}\,dx)$ near zero, c_2 and d_2 must vanish, as in (3.13). But the solutions thus obtained increase exponentially, which is a contradiction. This proves Proposition (3.20).

Notice that the solutions f and g with $c_2 = d_2 = 0$ above are the factors of the logarithmic terms in (3.25) and (3.26). Finally we may complete the

Proof of (3.10): By virtue of (3.16) and (3.21) there is a kernel $G(x,\tilde{x},\eta)$ for which

$$N_p(x\partial_x, ix\eta)G(x,\tilde{x},\eta) = x^n\delta(x-\tilde{x})$$

and such that, for each η, the corresponding operator

$$G : L^2(x^{-n}\, dx) \longrightarrow L^2(x^{-n}\, dx)$$

is bounded. As of yet we may not assert this uniformly in η, which is what we require. Of course this uniformity is obvious for the 'uncoupled part' of G (3.14); for the 'coupled part' we know little save that it is an analytic solution of N_p, of the form (3.25) for $x < \tilde{x}$ and (3.26) for $x > \tilde{x}$. For this extra information we use the dilation invariance of N_p.

Thus let $G(x,\tilde{x},\tilde{\eta})$ be the solution above when $|\hat{\eta}| = 1$. It is obviously smooth in $\hat{\eta}$. The solution G to

$$N_p(x\partial_x, ix|\eta|\hat{\eta})G = |\eta|^{n-1}x^n\delta(x-\tilde{x})$$

obtained by replacing x, $\tilde{x}$ by $x|\eta|$, $\tilde{x}|\eta|$ must be

$$(3.28) \qquad G(x,\tilde{x},\eta) = |\eta|^{1-n} G(x|\eta|,\tilde{x}|\eta|,\eta/|\eta|)$$

where $\hat{\eta} = \eta/|\eta|$. The known part of the kernel (3.14) does indeed satisfy this.

The L^2 boundedness of G as in (3.10) is now quite easy. Define the transformation

$$D_a : f(x) \longrightarrow f_a(x) = a^{\frac{1-n}{2}} f(ax), \qquad a \in \mathbb{R}^+.$$

This correspondence is readily seen to be an $L^2(x^{-n}\,dx)$ isometry. Set f_a into the inequality

$$(3.29) \quad \int \left| \int G(x,\tilde{x},\hat{\eta}) f(\tilde{x}) \tilde{x}^{-n}\, d\tilde{x} \right|^2 x^{-n}\, dx \le C \int |f|^2 x^{-n}\, dx$$

where C may be assumed independent of $\tilde{\eta} \in S^{n-2}$. The new right hand side, with f_a replacing f, is independent of a. On the other side, a change of variables $\tilde{x} \longmapsto a\tilde{x}$ results in a new inner integral on the left side

$$\int G(x,\tilde{x},\hat{\eta}) f_a(\tilde{x}) \tilde{x}^{-n}\, d\tilde{x} = \int a^{\frac{n-1}{2}} G(x,a^{-1}\tilde{x},\hat{\eta}) f(\tilde{x}) \tilde{x}^{-n}\, d\tilde{x}.$$

Applying the dilation $D_{a^{-1}}$ transforms this to

$$\int a^{n-1} G(a^{-1}x,a^{-1}\tilde{x},\hat{\eta}) f(\tilde{x}) \tilde{x}^{-n}\, d\tilde{x}$$

which retains the same $L^2(x^{-n}\,dx)$ norm. Thus, with $a = |\eta|^{-1}$, the definition (3.28) implies

$$(3.30) \quad \int \left| \int G(x,\tilde{x},\eta) f(\tilde{x}) \tilde{x}^{-n}\, d\tilde{x} \right|^2 x^{-n}\, dx \le C \int |f(x)|^2 x^{-n}\, dx$$

where C is independent of $\eta \in \mathbb{R}^{n-1}\setminus\{0\}$. If f is now allowed to depend on η also, then by integrating both sides of (3.30) with respect to η we obtain the required estimate of (3.10).

The kernel inverting the partial differential operator N_P is

$$(3.31) \qquad G(x,y,\tilde{x},\tilde{y}) = c_n \int e^{i(y-\tilde{y})\cdot\eta} G(x,\tilde{x},\eta)\, d\eta.$$

It is bounded on $L^2(x^{-n}\,dx\,dy)$, hence is the unique Green function for the constant curvature Laplacian. It also has some isometry invariance, namely that of translation and dilation:

$$G(x,y+v,\tilde{x},\tilde{y}+v) = G(x,y,\tilde{x},\tilde{y})$$

$$G(az,a\tilde{z}) = G(z,\tilde{z}), \quad z = (x,y), \quad \tilde{z} = (\tilde{x},\tilde{y})$$

the latter following from a change of variables in (3.31).

C. Structure of the Kernel

We now discuss the structure of the kernel G of (3.31), which is of the type discussed in the last chapter.

(3.32) <u>Theorem</u>: $G(z,\tilde{z})\cdot\gamma \in K_0^{-2,\sigma,\tau}(\mathbb{R}^n_+ \times_0 \mathbb{R}^n_+;\Gamma_0^{1/2} \otimes {}^0\Lambda^k)$ where

$$\sigma = \begin{bmatrix} n-k-1 & n-k+1 \\ n-k & n-k \end{bmatrix}, \quad \tau = \sigma^t, \quad \text{for} \quad k < \frac{n-1}{2},$$

$$\sigma = \begin{bmatrix} k & k \\ k-1 & k+1 \end{bmatrix}, \quad \tau = \sigma^t, \quad \text{for} \quad k > \frac{n-1}{2}, \quad \text{and} \quad \gamma$$

is the half-density of (2.28).

As always, the proof is rather involved. First we prove uniform symbol estimates in the regions $x \le \tilde{x}$, $\tilde{x} \le x$. This implies the requisite regularity in the interior and, together with properties of the indicial operator, yields the correct estimates at the boundary.

We commence by examining the 'uncoupled' part of the kernel:

(3.33) <u>Lemma</u>: The kernels G_1 and G_2 of (3.14), which furnish inverses for the operators P_1 and P_2, satisfy the estimates, for $|\eta| \ge 1$,

$$|\partial_x^i \partial_{\tilde{x}}^j \partial_\eta^\alpha G_i(x,\tilde{x},\eta)| \leq C_{\epsilon,i,j,\alpha}(1+|\eta|)^{-1-|\alpha|+i+j} e^{-|x-\tilde{x}||\eta|}$$

uniformly in $\epsilon \leq x \leq \tilde{x} \leq \epsilon^{-1}$, $\epsilon \leq \tilde{x} \leq x \leq \epsilon^{-1}$ for any $\epsilon > 0$, $i = 1,2$. In particular, neglecting the $(x,\tilde{x})$ dependence, they are symbols of order -1, uniformly in these regions of the $(x,\tilde{x})$ plane. Finally, each G_i has a classical expansion as $\eta \longrightarrow 0$ of lowest homogeneity $|\eta|^0$.

Proof: Consider only G_1 in the region $x \leq \tilde{x}$; the other cases are quite similar. Then

$$G_1(x,\tilde{x},\eta) = (x\tilde{x})^{\frac{n-1}{2}} I_\nu(x|\eta|)K_\nu(\tilde{x}|\eta|)$$

where $\nu = (n-2k-1)/2$ for $k < (n-1)/2$, which we also assume. Both I_ν and K_ν have asymptotic expansions for large values of their arguments, the first terms of which are given in (3.12). All derivatives of I_ν, K_ν also have expansions which must equal the ones obtained by differentiating those for I_ν, K_ν. All these expansions are of step size one in descending powers of $x|\eta|$ or $\tilde{x}|\eta|$, hence it suffices to argue only for the first terms. Thus replace G_1 by

$$(x\tilde{x})^{\frac{n-2}{2}} e^{-(\tilde{x}-x)|\eta|} |\eta|^{-1}$$

as $|\eta| \longrightarrow \infty$, $\epsilon \leq x \leq \tilde{x}$. The estimates of the Lemma are quite easy to verify; the only point to be noted is that

$$\partial_\eta^\alpha e^{-(\tilde{x}-x)|\eta|} = \sum_{|\beta|\leq|\alpha|} p_\beta(\eta, x-\tilde{x}) e^{-(\tilde{x}-x)|\eta|}$$

where each $p_\beta = (\tilde{x}-x)^{|\beta|} \cdot$ {a term homogeneous in η of degree $|\beta| - |\alpha|$}. Since

$$|\eta|^{|\beta|-|\alpha|}(\tilde{x}-x)^{|\beta|}e^{-(\tilde{x}-x)|\eta|} \leq C_\beta(1+|\eta|)^{-|\alpha|}$$

this too has the correct growth.

As for the behaviour as $\eta \longrightarrow 0$, both I_ν and K_ν have convergent Frobenius series (the one for K_ν containing logarithms) so that from (3.13)

$$G_1(x, \tilde{x}, \eta) = c_0 x^{n-k-1}\tilde{x}^k + |\eta|(c_1 x^{n-k}\tilde{x}^k + c_1' x^{n-k-1}\tilde{x}^{k+1}) + \cdots$$

(with an additional $|\eta| \log |\eta|$ term when $k = \frac{n}{2} - 1$). This proves all claims.

To extend this to the 'coupled' part of the kernel G_c, which inverts L_1, we must understand its structure better. This necessitates a closer examination of the solutions to $L_1 u = 0$. Altogether, this nullspace is four dimensional and two independent elements are identified in

(3.27). In one of these, both components are exponentially increasing—call this solution v_1; the other solution u_1 has both components exponentially decreasing. Let us record that when $|\eta| = 1$

$$(3.34)\quad \begin{aligned} u_1 &= \begin{bmatrix} a_0 x^{n-k+1} + \cdots \\ b_0 x^{n-k} + \cdots \end{bmatrix}, \quad u_1 \sim \begin{bmatrix} a_0' x^{n/2} e^{x} + \cdots \\ b_0' x^{n/2} e^{x} + \cdots \end{bmatrix} \quad x \longrightarrow \infty \\ v_1 &= \begin{bmatrix} c_0 x^{k} + \cdots \\ d_0 x^{k+1} + \cdots \end{bmatrix}, \quad v_1 \sim \begin{bmatrix} c_0' x^{n/2} e^{-x} + \cdots \\ d_0' x^{n/2} e^{-x} + \cdots \end{bmatrix} \quad x \longrightarrow \infty \end{aligned}$$

Other solutions are not known explicitly, but fortunately their asymptotics may be derived as follows. A Frobenius series calculation posits two additional solutions of the form (also when $|\eta| = 1$)

$$(3.35)\quad \begin{aligned} u_1 &= \begin{bmatrix} \bar{a}_0 \quad x^{n-k-1} + \cdots \\ \bar{b}_0 \quad x^{n-k} + \cdots \end{bmatrix} + Au_1 \log x \\ v_2 &= \begin{bmatrix} \bar{c}_0 \quad x^{k} + \cdots \\ \bar{d}_0 \quad x^{k-1} + \cdots \end{bmatrix} + Bv_1 \log x \end{aligned}$$

In (3.34), (3.35) none of the coefficients a_0, a_0', $\bar{a}_0$, A, b_0, b_0', $\bar{b}_0$, B, c_0, c_0', $\bar{c}_0$, d_0, d_0', $\bar{d}_0$ vanish.

On the other hand, the singularity of L_1 at $x = \infty$ is not regular. One may find formal solutions which are (possibly divergent) series in descending powers of x multiplied by an exponential. Thus a straightforward computation produces series

$$x^{\frac{n}{2}-1} e^{x} \begin{bmatrix} \alpha_1 x + \alpha_0 + \alpha_{-1} x^{-1} + \cdots \\ \beta_1 x + \beta_0 + \cdots \end{bmatrix},$$

$$x^{\frac{n}{2}-1} e^{x} \begin{bmatrix} \alpha_{-1} x^{-1} + \alpha_{-2}\, x^{-2} + \cdots \\ \beta_{-1} x^{-1} + \cdots \end{bmatrix}$$

$$x^{\frac{n}{2}-1} e^{x} \begin{bmatrix} \alpha_1 x + \cdots \\ \beta_1 x + \cdots \end{bmatrix}, \quad x^{\frac{n}{2}-1} e^{x} \begin{bmatrix} \alpha_{-1} x^{-1} + \cdots \\ \beta_{-1} x^{-1} + \cdots \end{bmatrix}.$$

The α_i, β_i are different of course in each of these solutions, and none of the leading coefficients vanish. Once these formal solutions have been found, the theory developed in Chapter 5 of [8] (for which the analyticity of the coefficients of L_1 is essential) implies the existence of actual solutions obeying each of these asymptotics as $x \longrightarrow \infty$. Indeed both u_1 and v_1 of (3.34) fit this pattern—their series being the ones commencing in x^1. Thus for some choice of $\bar{a}_i$, $\bar{b}_i$, $\bar{c}_i$, $\bar{d}_i$ in (3.35)

$$u_2 \sim \begin{bmatrix} a_0' x^{\frac{n}{2}-2} e^{x} + \cdots \\ b_0' x^{\frac{n}{2}-2} e^{x} + \cdots \end{bmatrix}, \quad a_0', \; b_0' \neq 0$$

(3.36)

$$v_2 \sim \begin{bmatrix} c_0' x^{\frac{n}{2}-2} e^{-x} + \cdots \\ d_0' x^{\frac{n}{2}-2} e^{-x} + \cdots \end{bmatrix}, \quad c_0', \; d_0' \neq 0$$

The kernel $G_c(x,\tilde{x},\hat{\eta})$, $|\hat{\eta}| = 1$, has a form similar to (3.14):

$$G_c(x,\tilde{x},\hat{\eta}) = U(x)A(\tilde{x})H(\tilde{x}-x) + V(x)B(\tilde{x})H(x-\tilde{x})$$

where U, V are two by two matrices, the columns of which are linear combinations of u_1, u_2, respectively v_1, v_2, and A, B are matrices chosen so that

$$L_1 G_c = x^n \delta(x-\tilde{x}) \cdot I.$$

But L_1 is self-adjoint with respect to $x^{-n}\,dx$ (both formally and by (3.24) as an operator), so that G_c is too:

$$G_c(x,\tilde{x}) = G_c^*(\tilde{x},x)$$

Hence $A(\tilde{x}) = V(\tilde{x})^*$, $B(\tilde{x}) = U(\tilde{x})^*$ and

(3.37) $$G_c = U(x)V(\tilde{x})^*H(\tilde{x}-x) + V(x)U(\tilde{x})H(x-\tilde{x}).$$

The only obstruction to completing our task is the explicit identification of the columns of U, V in terms of the u_i, v_i. However, the condition that $L_1G_c = x^n\delta$ entails that

$$U'(x)V(x)^* - V'(x)U(x)^* = -\tfrac{1}{2}x^{n-2}I.$$

Of necessity then, from (3.34) and (3.36)

(3.38) $$U(x) = \begin{bmatrix} au_2 & bu_1 \end{bmatrix}, \quad V(x) = \begin{bmatrix} cv_1 & dv_2 \end{bmatrix}$$

for some a, b, c, d, none of which may vanish.

Since

$$G_c(x,\tilde{x},\eta) = |\eta|^{1-n}G_c(x|\eta|,\tilde{x}|\eta|,\eta/|\eta|)$$

a verbatim repeat of the proof of (3.33) shows that

(3.39) <u>Lemma</u>: G_c satisfies the estimates, for $|\eta| \geq 1$,

$$|\partial_x^i\partial_{\tilde{x}}^j\partial_\eta^\alpha G_c(x,\tilde{x},\eta)X \leq C_{\epsilon,i,j,\alpha}(1+|\eta|)^{-1-|\alpha|+i+j}e^{-|x-\tilde{x}||\eta|}$$

uniformly in $\epsilon \leq x \leq \tilde{x} \leq \epsilon^{-1}$, $\epsilon \leq \tilde{x} \leq x \leq \epsilon^{-1}$. Furthermore, each term of G_c has a classical expansion (in fact a convergent series) as $\eta \longrightarrow 0$, uniform in $x, \tilde{x} \leq C$ and of lowest homogeneity $|\eta|^0$.

We now have sufficient information to study the kernel $G(z,\tilde{z})$ of (3.31).

(3.40) <u>Lemma</u>: $G(z,\tilde{z})$ is C^∞ when $z \neq \tilde{z}$ and $x, \tilde{x} > 0$. The singularity along the diagonal is classical conormal, with an expansion involving (quasi-) homogeneous terms of increasing degree in $z - \tilde{z}$; the first such term, up to a constant factor, is $|z-\tilde{z}|^{2-n}$.

<u>Proof</u>: By virtue of the uniform symbol estimates of the two previous lemmas, the usual arguments imply that $G(z,\tilde{z})$ is C^∞ in the stated region. The singularity along the diagonal is the normal one for the inverse of an elliptic differential operator. From our vantage point, though, the symbolic behaviour of $G(x,\tilde{x},\eta)$ away from $\eta = 0$ shows that when $x = \tilde{x}$ $G(z,\tilde{z})$ has an appropriate expansion in powers of $y - \tilde{y}$, the most singular of which has degree of homogeneity $-(n-1) - (-1) = 2-n$. That the singularity behaves thusly in all directions along $\{z = \tilde{z}\}$ follows by invoking the rotation invariance of G. Alternately one might consider the Fourier transform of $G(x,x+t,\eta)\rho(t)$ with respect to t (where ρ is a cutoff function identically one near $t = 0$ and compactly supported in,

say, $(-\frac{1}{2}x, \frac{1}{2}x))$. It is straightforward to show that this is symbolic in (τ, η) of degree -2 (τ being dual to t) and polyhomogeneous.

(3.41) Proposition: $G(z, \tilde{z})$ is conormal on $\{x = 0, \tilde{x} > 0\}$, $\{\tilde{x} = 0, x > 0\}$ and $\{x = \tilde{x} = 0, y \neq \tilde{y}\}$. Its components each have expansions on either of these codimension one faces, with leading terms determined by the two-by-two matrices σ for $\{x = 0\}$ and τ for $\{\tilde{x} = 0\}$ of (3.32).

Proof: We prove all assertions only for the inverse transform of the kernel G_1 of (3.14), and assuming $k < \frac{n-1}{2}$. The methods apply equally well to the full kernel, though the notation becomes more intricate.

Consider first the region near $x = 0$, $\tilde{x} \geq \epsilon$. From (3.33) the integral

$$(x\partial_x)^{\ell}\partial_{\tilde{x}}^{j}\partial_y^{\alpha}\partial_{\tilde{y}}^{\beta} \int e^{i(y-\tilde{y})\cdot\eta} G_1(x, \tilde{x}, \eta)\, d\eta$$

$$= \int e^{i(y-\tilde{y})\cdot\eta}(x\partial_x)^{\ell}\partial_{\tilde{x}}^{j}(i\eta)^{\alpha}(-i\eta)^{\beta}(x\tilde{x})^{\frac{n-1}{2}} I_{\nu}(|\eta|)K_{\nu}(\tilde{x}|\eta|)d\eta$$

converges absolutely for any ℓ, j, α, β, and uniformly in $x < \frac{1}{2}\epsilon$, $\epsilon \leq \tilde{x} \leq \epsilon^{-1}$, $|y|$, $|\tilde{y}| \leq C$; indeed it decreases exponentially there. Furthermore, it obviously has the same regularity as $x \longrightarrow 0$ independently of these various indices. This yields the conormality along $\{x = 0, \tilde{x} > 0\}$ since ϵ was arbitrary. Similar arguments apply to $\{\tilde{x} = 0, x > 0\}$.

For the corner, where both $x, \tilde{x} \sim 0$, it is convenient to observe that we may assume $|y-\tilde{y}| > \epsilon$, since the dilation invariance $G(az, a\tilde{z}) = G(z, \tilde{z})$ allows for all computations to be performed well away from $\{x = \tilde{x} = y - \tilde{y} = 0\}$. Now, assuming only that $x \leq \tilde{x}$, we may multiply both sides of the equation above by $(y-\tilde{y})^{\gamma}$ and replace $\partial_{\tilde{x}}$ by $\tilde{x}\partial_{\tilde{x}}$ to get

$$\int e^{i(y-\tilde{y})\cdot\eta}(-i\partial_{\eta})^{\gamma}$$

$$\left[(x\partial_x)^{\ell}(\tilde{x}\partial_{\tilde{x}})^{j}(i\eta)^{\alpha}(-i\eta)^{\beta}(x\tilde{x})^{\frac{n-1}{2}} I_{\nu}(x|\eta|)K_{\nu}(\tilde{x}(|\eta|)\right] d\eta$$

as the new right hand side. (The integration by parts in η is permissible since the whole expression is to be considered as an oscillatory integral.) Now, by choosing γ so that $-1 + \ell + j + |\alpha| + |\beta| - |\gamma| < -n$, the integral over $|\eta| \geq 1$ is convergent, whereas the integral over $|\eta| \leq 1$ can only produce a C^{∞} contribution. Furthermore, the regularity of this expression is stable as

$x, \tilde{x} \longrightarrow 0$. Since $y - \tilde{y} \neq 0$, the conormality along the corner is now established.

As for the various expansions, notice that when $x \longrightarrow 0$, $\tilde{x} \geq \epsilon$, the first factor in

$$\int I_\nu(x|\eta|)K_\nu(\tilde{x}|\eta|)(x\tilde{x})^{\frac{n-1}{2}} e^{i(y-\tilde{y})\cdot\eta}\, d\eta$$

may be replaced by its Frobenius series. Again, provided $x < \frac{1}{2}\epsilon$, each term of this (convergent) sum in the integrand decreases exponentially with η, hence we have a series

$$\sum_{j=0}^{\infty} x^{-n-1+j} c_j(\tilde{x}, y-\tilde{y})$$

furnishing the asymptotic behaviour as $x \longrightarrow 0$. To obtain this expansion uniformly down to $\tilde{x} = 0$, but still assuming $x \leq \tilde{x}$ of course, we may instead replace $I_\nu(x|\eta|)$ by the first N terms of its Frobenius series plus an exponentially increasing remainder. Each term in the resulting integrand exhibits symbolic behaviour, and the first N summands still decrease exponentially in η. These provide the first N terms of the asymptotic expansion; the remainder term is indeed lower order as may be seen by multiplying it by $(y-\tilde{y})^\gamma$ for $|\gamma|$ sufficiently large and integrating by parts so that the resulting integral converges.

Applying the arguments of the last paragraph to the full kernel yields the correct expansions in all components, for these all are derived from (3.34), (3.35).

<u>Proof</u> of Theorem (3.32): The lift of $G(z,\tilde{z})$ to $\mathbb{R}^n_+ \times_0 \mathbb{R}^n_+$ has the correct behaviour along $\Delta\iota_0$, T, and B, as demonstrated in (3.40) and (3.41). Indeed the only point not yet discussed is the behaviour near the front face F. However this is trivial since the dilation invariance of G is equivalent to the independence of the lifted kernel on R. Hence not only is G smooth on F, except at $\partial\Delta\iota_0$, but the expansions at $\Delta\iota_0$, T, B all continue uniformly down to F.

Let us now untangle the various definitions and identify both the operator induced by the Laplacian on ${}^o\Lambda^k \otimes \Gamma_0^{1/2}$ and its inverse. First recall that the operator P, for which G is an inverse, satisfies

$$\Delta(x^{-k}\omega) = x^{-k}(P\omega) \iff P = x^k \Delta x^{-k}$$

G then is the kernel induced by the standard Green function $G_\Lambda \in \Gamma\,\mathrm{Hom}(\Lambda^k,\Lambda^k)$. In fact, since Λ^k and ${}^o\Lambda^k$ are canonically identified over the interior we may let G_Λ act on ${}^o\Lambda^k$:

$$\frac{\omega(x,y)}{x^k} \longrightarrow \int G_\Lambda(x,y,\tilde{x},\tilde{y})\tilde{x}^{-k}\omega(\tilde{x},\tilde{y})\tilde{x}^{-n}\,d\tilde{x}\,d\tilde{y}$$

$$= x^{-k}\int G(x,y,\tilde{x},\tilde{y})\omega(\tilde{x},\tilde{y})\tilde{x}^{-n}\,d\tilde{x}\,d\tilde{y}$$

i.e.

$$G_\Lambda(x,y,\tilde{x},\tilde{y}) = x^{-k}G(x,y,\tilde{x},\tilde{y})\tilde{x}^k$$

(3.42)

$$\Delta G_\Lambda = x^n\delta(x-\tilde{x})\delta(y-\tilde{y})\cdot I.$$

To incorporate half-densities, use μ and γ of (2.28) and define $\tilde{P}$ by

$$\Delta(x^{-k}\omega\cdot\mu) = x^{-k}(\tilde{P}\omega)\cdot\mu \iff \tilde{P} = x^{k+n/2}\Delta x^{-k-n/2} = x^{n/2}Px^{-n/2}.$$

It is of course the operator induced by Δ on ${}^o\Omega^k \otimes \Gamma_0^{1/2}$. The corresponding kernel $\tilde{G}_\Lambda \in \Gamma\ \mathrm{Hom}({}^o\Lambda^k, {}^o\Lambda^k)$ must satisfy

$$\Delta\cdot(\tilde{G}_\Lambda\cdot\gamma) = (x^{n/2}\Delta x^{-n/2}\tilde{G}_\Lambda)\cdot\gamma = x^n\delta(x-\tilde{x})\delta(y-\tilde{y})\cdot I$$

—which is the kernel of the identity with respect to $\Gamma_0^{1/2}$. From (3.42) one may check that

$$\text{(3.43)}\qquad \tilde{G}_\Lambda = x^{n/2}G_\Lambda\tilde{x}^{-n/2} = x^{-k+n/2}G\tilde{x}^{k-n/2}.$$

The kernel G_Λ transfers to B^n and is invariant under all hyperbolic isometries there. The preceding discussion may be carried through on the ball, the only difference being that x must be replaced by $1 - |w|^2$ (w is the Euclidean coordinate on B^n). The appropriate kernel inverting the Laplacian acting on ${}^o\Lambda^k(B^n) \otimes \Gamma_0^{1/2}(B^n)$ is

$$(3.44) \qquad \tilde{G}_\Lambda = (1-|w|^2)^{n/2} G_\Lambda(w,\tilde{w})(1-|\tilde{w}|^2)^{-n/2}.$$

Suppose now that σ is any two-by-two matrix—the components of which regulate decay rates of the tangential and normal components of a section of $\mathrm{Hom}({}^o\Lambda^k, {}^o\Lambda^k)$ —and define the new matrix

$$\sigma + q = (\sigma_{ij}+q), \quad i, j = t, n, \quad q \in \mathbb{R}.$$

In summary, we have proved

(3.45) <u>Corollary</u>:

$$\tilde{G}_\Lambda(w,\tilde{w}) \cdot \gamma \in K_0^{-2,\sigma-k+n/2,\tau+k-n/2}(B^n; {}^o\Lambda^k \otimes \Gamma_0^{1/2})$$

σ, τ as in (3.32). Furthermore

$$\Delta(\tilde{G}_\Lambda \cdot \gamma) = (1-|w|^2)^n \delta(w-\tilde{w}) \cdot \gamma \cdot I.$$

D. Refined Mapping Properties

In this section we further refine the mapping properties of the Laplacian and its inverse. Two results are proved; the first, a straightforward extension of Proposition (2.55), is valid more generally, but its statement and proof are simpler in the present circumstances. Let us define a form valued version of the space (2.54).

$$\omega \in \mathcal{A}^{\alpha}_{phg}(M, {}^{o}\Lambda^{k}), \qquad \alpha = (\alpha_t, \alpha_n)$$

$$(3.46) \qquad \Leftrightarrow \omega = \omega_t + \omega_n \in {}^{o}\Omega^{k}_{t} \oplus {}^{o}\Omega^{k}_{n} \quad \text{near} \quad \partial M$$

$$\omega_i \in \mathcal{A}^{\alpha_i}_{phg}(M) \otimes {}^{o}\Omega^{k}_{i}, \qquad i = t, n.$$

The decomposition ${}^{o}\Lambda^{k} = {}^{o}\Lambda^{k}_{t} \oplus {}^{o}\Lambda^{k}_{n}$ near ∂M is defined following the discussion in Chapter 2, Section C, and the spaces above are invariant under coordinate changes only when $|\alpha_t - \alpha_n| \leq 1$.

(3.47) <u>Proposition</u>: Let $f \in \mathcal{A}^{\alpha+q}_{phg}(B^n, {}^{o}\Lambda^{k})$, where $\alpha = (n-k-1, n-k)$ if $k < (n-1)/2$, $\alpha = (k, k-1)$ if $k > (n+1)/2$ and $\alpha + q = (\alpha_i + q)$, $q \in \mathbb{Z}^{+}$, in either case. Then there is a unique $\omega \in \mathcal{A}^{\alpha}_{phg}(B^n, {}^{o}\Lambda^{k})$ such that

$$\Delta\omega = f.$$

<u>Proof</u>: Let ρ be a defining function for ∂B^n, $\rho \geq 0$, and θ the variable in $\partial B^n = S^{n-1}$. Now, the tangential and normal components of f have expansions

$$f_i \sim \sum_{j=0}^{\infty} f_{ij} \qquad i = t,\ n$$

$$f_{ij} = \sum_{\ell=0}^{N_{ij}} C_{ij\ell}(\theta)\rho^{\alpha_i+q+j}(\log \rho)^{\ell}$$

near the boundary. Inasmuch as

$$\Delta - I_\Delta : \mathscr{A}^{\alpha} \longrightarrow \mathscr{A}^{\alpha+1}$$

for any α, we proceed by first choosing ω_{io}, $i = t,n$ so that

$$I_\Delta \omega_{io} = f_{io}$$

$$\omega_{io} = \sum_{\ell=0}^{N_{io}} c'_{io\ell}(\theta)\rho^{\alpha_i+q}(\log \rho)^{\ell}.$$

Then, assuming ω_{im} have been chosen, $m < j$, pick ω_{ij} so that

$$I_\Delta \omega_{ij} = f'_{ij}$$

$$\omega_{ij} = \sum_{\ell=0}^{N_{ij}} c'_{ij\ell}(\theta)\rho^{\alpha_i+q+j}(\log \rho)^\ell .$$

The f'_{ij} are quasihomogeneous terms of the correct degree which depend only on f_{ij} and $(\Delta - I_\Delta)\omega_{tm}$, $(\Delta - I_\Delta)\omega_{nm}$, $m < j$. All these equations are solvable by finite series methods. Note only that uniqueness holds at each step since each exponent $\alpha_i + q + j$ is greater than the largest of the indicial roots of I_P —the operator induced by I_Δ on ${}^o\Lambda^k$. For the same reason the highest power of the logarithm at each stage is never increased.

If $\omega_0 \sim \sum \omega_{ij}$ then $\Delta\omega_0 - f \simeq g \in \dot{C}^\infty(B^n) \otimes {}^o\Lambda^k$. (2.55) and (3.45) together guarantee the existence of a ω_1 such that

$$\Delta\omega_1 = g.$$

Then $\Delta(\omega_0 - \omega_1) = f$, as desired.

The other solvability result is of the same type as (3.47), but with somewhat more singular right hand sides. Recall from (2.17) that it is useful to think of the quarter-sphere S^n_{++} as the ball B^n blown up around a point w_0 in its boundary. Analogous to the spaces (3.46) are

$$(3.48)\qquad \mathscr{A}^{\alpha,\beta}_{phg}(S^n_{++}, {}^{o}\Lambda^k) \longleftrightarrow \mathscr{A}^{\alpha,\beta}_{phg}(B^n, {}^{o}\Lambda^k)$$

$\alpha = (\alpha_t, \alpha_n)$, $\beta = (\beta_t, \beta_n)$. An element is a section of ${}^{o}\Lambda^k$ which is conormal, with classical expansion, at the top and bottom edges of the quarter-sphere. The two-vectors α, β determine the most singular terms in these expansions; as in (3.46) we require $|\alpha_t - \alpha_n|$, $|\beta_t - \beta_n| = 0$ or 1. The analogous spaces over B^n are those with expansions at $\partial B^n - \{w_0\}$ and at $\{w_0\}$. In particular, using the defining function $r(w) = |w - w_0|$, the tangential component of an element has an expansion

$$(3.49)\qquad f_t \sim \sum_{i=0}^{\infty} \sum_{\ell=0}^{N_i} \phi_{i\ell} r^{\beta_t + i} (\log r)^{\ell} \qquad r \longrightarrow 0$$

and similarly for the normal component. The $\phi_{i\ell}$ here are functions of $\theta \in S^{n-1}_{+}$, the sphere in the half tangent space to B^n at w_0. We may now state the

(3.50) <u>Theorem</u>: For any $f \in \mathscr{A}^{\alpha+q,\beta}_{phg}(B^n, {}^{o}\Lambda^k)$, $q \in \mathbb{Z}^+$, there is always a solution

$$\omega \in \mathscr{A}^{\alpha,\beta}_{phg}(B^n, {}^{o}\Lambda^k)$$

to $\Delta\omega = f$, so long as the entries of β are integers and α is the 2-vector of (3.47).

The typically arduous proof involves removing the additional singularity at w_0 so as to reduce the problem to (3.47). This step is essentially local, so we replace B^n by $\mathbb{R}^n_+$ with w_0 corresponding to 0. In addition, replace Λ by N_p and regard ω and f as sections of Λ^k rather than ${}^o\Lambda^k$. The principle tool is the Mellin transform, taken in the radial variable r, where $z \in \mathbb{R}^n_+$ has polar coordinates $r \in \mathbb{R}^+$, $\theta \in S^{n-1}_+$. This is the transform

$$\begin{aligned} u(r,\theta) \longrightarrow u_M(\zeta,\theta) &= \int r^{-\zeta-1} u(r,\theta)\, dr \\ u(r,\theta) &= (2\pi)^{-1} \int r^{\zeta} u_M(\zeta,\theta)\, d\zeta . \end{aligned} \tag{3.51}$$

In general, u_M is only defined (and analytic) in a half-space $\operatorname{Re} \zeta < a$, and so the integral defining the inverse transform is taken over a line $\operatorname{Re} \zeta = c$, $c < a$ fixed. (For all facts quoted here regarding the transform, see [23].) The constant a which limits the domain of u_M is closely related to the decay of u as $r \longrightarrow 0$. If $u = 0$ for large values of r and vanishes at a definite rate at $r = 0$, then u_M is rapidly decreasing as $|\operatorname{Im} \zeta| \longrightarrow \infty$, $\operatorname{Re} \zeta$ fixed.

The most important property of u_M for us is that if $u(r,\theta)$ has an expansion as in (3.49), and vanishes for r large, then u_M continues meromorphically to the whole ζ-plane, with poles only at $\zeta \in \beta_0 + \mathbb{Z}^+$ (if β_0 is the most singular exponent in the expansion). Furthermore, the order of the pole at $\beta_0 + i$ is $N_i + 1$. Conversely, if $v(\zeta,\theta)$ is meromorphic in ζ with real poles and decreases rapidly on each line $\operatorname{Re}\zeta$ = constant, then

$$u(r,\theta) = \int_{\operatorname{Re}\,\zeta=c} r^{\zeta} v(\zeta,\theta)\, d\zeta$$

has an expansion as in (3.49) with the exponent s in each term $r^s(\log r)^{\ell}$ the location of a pole of v. Only those s occur for which $\operatorname{Re} s > c$, and the order of the pole at $\zeta = s$ is one greater than the highest power of the logarithm multiplying r^s.

Define the operator L_{ξ} by

$$(N_p u)_M = L_{\zeta} u_M(\zeta,\theta).$$

Clearly

$$L_{\zeta} = |z|^{-\zeta} N_p |z|^{\zeta}.$$

Since N_p is invariant under the homothety $z \longrightarrow az$, this conjugate is also. Therefore it is differential only in the θ variable; in ζ it is purely algebraic. Furthermore, as an operator on $S_+^{n-1} = \{(\theta_1, \cdots, \theta_n) : |\theta| = 1, \theta_n \geq 0\}$ it is in Diff_0^2 —with respect to the boundary $\theta_n = 0$ of course. A short calculation shows that its indicial operator is one we know:

$$I(L_\zeta) = I_P. \tag{3.52}$$

The present goal is to invert L_ζ, so as to solve the equation $L_\zeta \omega_M = f_M$. Naturally, L_ζ is not invertible for every value of ζ. Its inverse G_ζ is meromorphic, and its poles contribute to the expansion of ω_M. Let $\rho(r) \in C_0^\infty(\mathbb{R})$ equal one for $|r| \leq 1/2$, and define, at least formally, the inverse for L_ζ

$$G_\zeta f = \lim_{t\to 0} M_t^*\{r^{-\zeta} G(r^\zeta \rho(r) f(\theta))\}$$

$f \in \dot{C}^\infty(S_+^{n-1}) \otimes {}^o\Lambda^k$, G the inverse of N_P, and M_t the dilation by t. Substituting the integral for G in this formula we get

$$G_\zeta f(\theta) = \int_{S_+^{n-1}} H_\zeta(\theta, \tilde{\theta}) f(\tilde{\theta}) (\tilde{\theta}_n)^{-(n-1)}\, d\tilde{\theta} \tag{3.53}$$

where

$$(3.54) \qquad H_\zeta(\theta,\tilde\theta) = \int_0^\infty G(\theta,\tau\tilde\theta)\tau^\zeta \frac{d\tau}{\tau\tilde\theta_n} .$$

For any value ζ at which this last integral converges, G_ζ then serves as an actual inverse for L_ζ. In fact, but for an insignificant difference, G_ζ is actually a V_0-pseudodifferential operator. Let us now examine the values of ζ where (3.54) converges. Fix $\theta \neq \tilde\theta$ and notice that by the dilation invariance of G, $G(\theta,\tau\tilde\theta) = G(\tau^{-1}\theta,\tilde\theta)$. The components of G decays at various rates as $\tau \longrightarrow 0$ and $\tau \longrightarrow \infty$. However, the extreme case is G_{tt}; all other components exhibit 'more favourable' rates of decay. Suppose $k < \frac{n-1}{2}$; then

$$G_{tt}(\theta,\tau\tilde\theta) \sim \tau^{n-k-1}, \qquad \tau \longrightarrow 0$$

$$G_{tt}(\tau^{-1}\theta,\tilde\theta) \sim \tau^{-(n-k-1)}, \qquad \tau \longrightarrow \infty.$$

Hence (3.54) converges when $-(n-k-1) < \mathrm{Re}\,\zeta < n-k-1$.

(3.55) <u>Lemma</u>: H_ζ extends meromorphically to the whole ζ-plane with poles at $\pm\zeta \in n-k-1 + \mathbb{Z}^+$ if $k < \frac{n-1}{2}$ and $\pm\zeta \in k-1 + \mathbb{Z}^+$ if $k > \frac{n-1}{2}$. These poles are of at most order two; the residues at $\zeta = N$ of H_ζ and $(\zeta-N)H_\zeta$ are kernels of finite rank.

<u>Proof</u>: Choose a partition of unity $1 = \phi_1(\tau) + \phi_2(\tau) + \phi_3(\tau)$, where $\phi_3(t) = \phi_1(\tau^{-1})$, ϕ_1 is supported in $\tau \leq 1/2$, and ϕ_2 in $(1/4, 4)$. Write $H_\zeta = H_\zeta^1 + H_\zeta^2 + H_\zeta^3$ by letting H_ζ^i be defined by the integral (3.54) with $\phi_i(\tau)$ inserted into the integrand.

H_ζ^2 is entire in ζ for $\theta \neq \tilde{\theta}$, which singularity we discuss later. By a change of variables

$$H_\zeta^1(\theta, \tilde{\theta}) = \int \phi_1(\tau) G(\theta, \tau\tilde{\theta}) \tau^\zeta \frac{d\tau}{\tau\tilde{\theta}_n}$$

$$H_\zeta^3(\theta, \tilde{\theta}) = \int \phi_1(\tau) G(\tau\theta, \tilde{\theta}) \tau^{-\zeta} \frac{d\tau}{\tau\tilde{\theta}_n} .$$

Replace τ^λ by τ_+^λ, $\lambda = \pm \zeta$, in each of these expressions. The family τ_+^λ of homogeneous distributions is well known to extend meromorphically to $\mathbb{C}$ with simple poles at $\lambda \in -\mathbb{Z}^+$; the residue at $\lambda = -N$ is $(-1)^{N-1} \cdot \delta^{(N-1)}/(N-1)!$.

Now, replace $G(\theta, \tau\tilde{\theta})$ and $G(\tau\theta, \tilde{\theta})$ by their expansions as $\tau \longrightarrow 0$. These expansions involve terms $\tau^m G_{mo}$, $\tau^m \log \tau \, G_{m1}$, where the $G_{m\ell}$ are kernels in $(\theta, \tilde{\theta})$ which are polynomial in θ or $\tilde{\theta}$ for $G(\tau\theta, \tilde{\theta})$ or $G(\theta, \tau\tilde{\theta})$, respectively. The expansions commence with τ^{n-k-1}, $k < \frac{n-1}{2}$, or τ^{k-1}, $k > \frac{n+1}{2}$. Inasmuch as $\tau_+^\lambda \tau^m = \tau_+^{\lambda+m}$ for <u>any</u> λ, m, and $\tau_+^\lambda \log \tau = d_\lambda(\tau_+^\lambda)$, the integrals of each term of the expansions should be thought of as a pairing of

$\tau_+^{\zeta+m}$ or $\tau_+^{\zeta+m} \log \tau$ with $\phi_1(\tau)$; the coefficient here is a kernel of finite rank since it is polynomial in θ or $\tilde{\theta}$. Finally, since $\phi_1^{(i)}(0) = 0$, $i > 0$, only one or two terms will actually be singular. Specifically, at $\zeta = -(n-k-1+i)$, for example, H_ζ^1 will have a pole resulting only from the two terms $\tau^{n-k+i}G_{io}(\theta,\tilde{\theta})$, $\tau^{n-k-1+i} \log \tau$ $G_{i1}(\theta,\tilde{\theta})$ in the expansion for $G(\theta,\tau\tilde{\theta})$, and H_ζ^3 is regular. Similarly, the pole at $\zeta = n-k-1+i$ occurs only in H_ζ^3, and not in H_ζ^1. The proof is complete.

Since $L_\zeta G_\zeta = (\theta_n)^{n-1}\delta(\theta-\tilde{\theta})$ in the region where (3.54) converges, it must continue to hold in the extended domain, away from the poles, by the uniqueness of analytic continuation. The kernel H_ζ essentially lies in $K_0^{-2,\sigma,\tau-1}$ over $S_+^{n-1} \times_0 S_+^{n-1}$, as we shall now demonstrate. First of all, H_ζ is the pushforward of G (with a few extra τ factors) under the submersion $(\theta,\tilde{\theta},\tau) \longrightarrow (\theta,\tilde{\theta})$. The restriction of this map to the submanifold $\{\theta = \tilde{\theta}\}$ is still a submersion onto its image, and $G(\theta,\tau\tilde{\theta})$ is conormal along $\{\theta = \tilde{\theta},\ \tau = 1\}$, so H_ζ is conormal along $\{\theta = \tilde{\theta}\}$. Next, H_ζ has the proper singularities along the top and bottom faces, as may be ascertained by setting the expansion for G into (3.54). Finally, the one way in which H_ζ differs from the V_0 kernels we have already studied is that it has a logarithmic singularity at the front face. The easiest way to check this is to observe that the kernel G_Λ of (3.42) is actually a function of the hyperbolic distance between z and $\tilde{z}$. This simplifies the calculations necessary to display this singular behaviour, which are then straightforward and much the same as those in [20], hence will be omitted.

Despite this one difference, H_ζ still enjoys all the continuity properties discussed in Chapter 2. From this, and using (3.52), the proof of (3.47) may be repeated to show that

$$L_\zeta \omega = f \in \mathscr{A}^{\alpha+q}_{phg}(S^{n-1}_+, \Lambda^k)$$

has a unique solution $\omega \in \mathscr{A}^{\alpha}_{phg}(S^{n-1}_+, \Lambda^k)$ depending meromorphically on ζ. Its poles are precisely those of H_ζ.

<u>Proof</u> of (3.50): Take $f \in \mathscr{A}^{\alpha+q,\beta}_{phg}(B^n, {}^o\Lambda^k)$ and transfer it to the upper half-space, with $w_0 = 0$. Also, multiply it by x^k so as to get an ordinary k-form. Now, express its components in polar coordinates, and take their Mellin transform with respect to r. The new form, $f_M(\zeta, \theta)$, is meromorphic; its poles are at $\zeta \in \min(\beta_t, \beta_n) + \mathbb{Z}^+$, and it is still in $\mathscr{A}^{\gamma+q}_{phg}$ in θ. Let $\omega_M(\zeta, \theta) = G_\zeta f_M(\zeta, \theta)$. It too is meromorphic in ζ, and lies in $\mathscr{A}^{\alpha}_{phg}$ in θ. By virtue of the hypothesis that the β_i are integers, ω_M

has poles at $\pm\zeta \in n-k-1+\mathbb{Z}^{+}$ (or $k-1+\mathbb{Z}^{+}$) and $\min(\beta_t,\beta_n)+\mathbb{Z}^{+}$.

Define

$$\omega_1(r,\theta) = (2\pi)^{-1}\int r^{\zeta}\omega_M(\zeta,\theta)\,d\zeta$$

where the integral is over $\operatorname{Re}\zeta = \min(\beta_t,\beta_n) - 1/2$. Then

$$N_P\omega_1 - f = g$$

vanishes to infinite order at $r = 0$, and upon being transferred back to B^n, lies in $\mathcal{A}^{\alpha+q}_{phg}(B^n,\Lambda^k)$. Now apply (3.47) to find ω_2 such that

$$N_P\omega_2 = g.$$

$\omega_1 - \omega_2$ is the desired solution: $N_P(\omega_1-\omega_2) = f$.

Chapter 4: Hodge Cohomology

A. The Parametrix Construction

We now have all the analytic tools at our disposal necessary to construct a suitable parametrix for the Laplacian on k-forms for a conformally compact metric (1.2). As described in section D of Chapter 2, the construction proceeds in three stages. Each of these involve inverting model operators for Δ_g —the symbol, the normal operator, and the indicial operator at the diagonal, the front face, and the top face of $M \times_0 M$, respectively.

The first step uses the symbol calculus of Theorem (2.41), but is otherwise identical to the usual microlocal procedure. Thus we seek an operator

$$E_0 \in \Psi_0^{-2}(M; {}^0\Lambda^k \otimes \Gamma_0^{1/2})$$

the kernel of which is supported quite near $\Delta\iota_0$ in $M \times_0 M$, and of course such that

$$\Delta_g E_0 = I - Q_0, \quad Q_0 \in \Psi_0^{-\infty}. \tag{4.1}$$

First choose $E_{0,0} \in \Psi_0^{-2}$ satisfying ${}^0\sigma_2(\Delta_g){}^0\sigma_{-2}(E_{0,0}) = 1$, which is possible by the V_0 ellipticity of Δ_g.

Then

$$\Delta_g E_{0,0} - I = -Q_{0,1} \in \Psi_0^{-1}.$$

Similarly, by induction, we may find $E_{0,j} \in \Psi_0^{-2-j}$ such that

$$\Delta_g E_{0,j} - Q_{0,j} = -Q_{0,j+1} \in \Psi_0^{-j-1}$$

$j = 1,2,\cdots$. Obviously it may be assumed that the kernel of each $E_{0,j}$, and thus $Q_{0,j}$, is supported near $\Delta\iota_0$. Now take

$$E_0 \sim \sum E_{0,j}$$

and the first step is complete.

Next we seek an operator E_1 for which

$$\Delta_g E_1 = Q_0 - Q_1 \tag{4.2}$$

where the kernel of Q_1 vanishes to infinite order on the front face F. Although $\kappa(Q_0)$ is supported well away from all other boundaries, it is not possible to choose E_1 and Q_1 with this property. In fact, we will find that

$$E_1 \in \Psi_0^{-\infty,\sigma',\tau'}, \qquad Q_1 \in R^\infty\Psi_0^{-\infty,\sigma',\tau'} \tag{4.3}$$

where $\sigma' = \sigma-k+n/2$, $\tau' = \tau+k-n/2$, σ and τ the matrices of (3.32). We note that it is precisely at this step where the proof breaks down in the middle degrees $k = n/2$, $(n\pm1)/2$.

As in the first step, E_1 is obtained by an iterative process. The first term $E_{1,0}$ is chosen to satisfy

$$N_p(\Delta_g)N_p(E_{1,0}) = N_p(Q_0)$$

for each $p \in \partial M$. In fact, by (3.47) there is a function

$$N_p(E_{1,0}) \in \mathcal{A}_{phg}^{\sigma',\tau'}(F_p) \otimes \Gamma_0^{1/2}$$

solving this equation, and then, by the exactness of the sequence in (2.45) there is an $E_{1,0}$ with this function as its normal operator. Thus

$$\Delta_g E_{1,0} - Q_0 = -Q'_{1,1} \in R\Psi_0^{-\infty,\sigma',\tau'}.$$

It is crucial to observe now that $Q'_{1,1}$ is actually slightly better:

$$Q'_{1,1} \in R\Psi_0^{-\infty,\sigma'+1,\tau'}$$

$$\Rightarrow Q'_{1,1} = xQ''_{1,1}, \quad Q''_{1,1} \in \Psi_0^{-\infty,\sigma',\tau'}.$$

The reason for this gain is that the normal operator of Δ_g annihilates the top order term of the asymptotic expansion for $E_{1,0}$ at the top face. Indeed, this is so because $N_p(E_{1,0})$ is given by applying the Green function for this normal operator to $N_p(Q_0)$ (which is supported in the interior), and now (3.47) implies that $E_{1,0}$ has the correct top order term.

To get further terms in the series for E_1, let us assume that, instead of a Taylor series in R, we let

$$E_1 \sim \sum \tilde{x}^j E_{1,j}$$

so that

$$\Delta_g E_1 \sim \sum \tilde{x}^j \Delta_g E_{1,j}.$$

Define $Q_{1,j} = \frac{x}{\tilde{x}} Q''_{1,j}$; then the inductive step is to solve

$$\Delta_g E_{1,j} - Q_{1,j} = -xQ''_{1,j+1}$$

where

$$Q_{1,j} \in \Psi_0^{-\infty,\sigma'+1,\tau'-j} \Leftrightarrow Q''_{1,j} \in \Psi_0^{-\infty,\sigma',\tau'-j+1}.$$

This has already been accomplished for $j = 0$. For greater values of j, we solve this equation by first solving

$$N_p(\Delta_g)N_p(E_{1,j}) = N_p(Q_{1,j}) \in \mathcal{A}^{\sigma'+1,\tau'-j}(F_p; {}^{o}\Lambda^k \otimes \Gamma_0^{1/2}).$$

By Theorem (3.50) an appropriate solution may indeed be found; its extension into the interior of $M \times_0 M$ then satisfies

$$\Delta_g E_{1,j} - Q_{1,j} = -RQ'_{1,j+1}$$

$$Q'_{1,j+1} \in \Psi_0^{-\infty,\sigma',\tau'-j}.$$

However, by reasoning as above—and now the point to note is that by hypothesis the top order term in the expansion for $Q_{1,j}$ is of type $\sigma' + 1$ and so doesn't interfere with the argument—we actually have

$$\Delta_g E_{1,j} - Q_{1,j} = -xQ''_{1,j+1}$$

$$Q''_{1,j+1} \in \Psi_0^{-\infty,\sigma',\tau'-j}$$

as desired. Finally, since $E_{1,j} \in \Psi_0^{-\infty,\sigma',\tau'-j}$

$$\tilde{x}^j E_{1,j} \in \Psi_0^{-\infty,\sigma',\tau'}$$

and

$$E_1 \sim \sum \tilde{x}^j E_{1,j} \in \Psi_0^{-\infty,\sigma',\tau'}.$$

This E_1 solves (4.2).

The final step of this construction is to find $E_2 \in R^\infty\Psi^{-\infty,\sigma',\tau'}$ such that

$$\Delta_g E_2 - Q_1 = -Q_2 \in R^\infty\Psi_0^{-\infty,\infty,\tau'}. \tag{4.3}$$

This is quite easy: one merely solves for each term in the expansion at T using the indicial operator, cf. the proof of (3.47). This completes the construction.

Set

$$E = E_1 + E_2 + E_3.$$

(4.4) <u>Theorem</u>: $E \in \Psi_0^{-2,\sigma',\tau'}(M;{}^o\Lambda^k\otimes\Gamma_0^{1/2})$ is a parametrix for Δ_g:

$$\Delta_g E = I - Q,\ Q \in R^\infty\Psi_0^{-\infty,\infty,\tau'}(M;{}^o\Lambda^k\otimes\Gamma_0^{1/2}).$$

(4.5) <u>Corollary</u>: $Q : L^2({}^o\Lambda^k\otimes\Gamma_0^{1/2}) \longrightarrow L^2({}^o\Lambda^k\otimes\Gamma_0^{1/2})$ is compact, and thus

$$\Delta_g : L^2({}^o\Lambda^k \otimes \Gamma_0^{1/2}) \longrightarrow L^2({}^o\Lambda^k \otimes \Gamma_0^{1/2}) \quad \text{is Fredholm.}$$

B. **Proof of the Main Theorem**

It follows from Corollary (4.5) that the harmonic space

$$\mathcal{H}^k = \{\omega \in L^2\Omega^k(dg) : \Delta_g\omega = 0\}$$

is finite dimensional. It remains for us to identify it in terms of the topological cohomology of M. First observe that the adjoint of the remainder

$$Q^* \in R^\infty\Psi_0^{-\infty,\sigma',\infty}(M; {}^o\Lambda^k \otimes \Gamma_0^{1/2}).$$

To double check the order of $\kappa(Q^*)$ on the top face, notice that

$$E^*\Delta_g = I - Q^*$$

implies

$$\text{(4.6)} \qquad \omega \in \mathcal{H}^k \Rightarrow Q^*(\omega\cdot\mu) = \omega\cdot\mu.$$

Thus the order σ' is the only one possible which ensures that

$$I(\Delta_g)\omega \equiv 0$$

modulo terms of one order higher.

Remove the density factor and regard ω as a section of the ordinary form bundle. From (2.55) and (4.6)

$$\omega \in \mathcal{H},\ \left|k-\frac{n}{2}\right| > \frac{1}{2} \Rightarrow \omega \in \mathcal{A}^{\alpha}_{phg}(M,\Lambda^k)$$

(4.7)

$$\alpha = \begin{cases} (n-2k-1, n-2k) & k < \frac{n-1}{2} \\ (0,-1) & k > \frac{n+1}{2} \end{cases}.$$

These explicit asymptotics of L^2 harmonic forms, together with various consequences of the existence of the parametrix, allow us to identify $\mathcal{H}^k$ with the deRham cohomology spaces of M.

Let us first briefly review the deRham theory on a compact manifold with boundary. There are, of course, two flavours of cohomology to consider, the relative and the absolute, and these are dual. Inasmuch as M and its interior share the same topological cohomology, it is reasonable to let the absolute cohomology be defined by

$$H^k(M) = \{\omega \in \Omega^k(\mathring{M}) : d\omega = 0\}/\{d\eta : \eta \in \Omega^{k-1}(\mathring{M})\}.$$

In this equality we must specify what sort of regularity— and boundary values—these forms should possess. In fact, it suffices either to let ω and η be C^∞ in $\mathring{M}$ or just distributional (currents), cf. [9], with no boundary constraints. However, when proving the Hodge theorem for a nondegenerate metric it is more convenient to use smooth forms on M which satisfy 'absolute' boundary conditions, cf. [26], which we shall not define here since we do not need them. On the other hand, the relative cohomology not surprisingly always requires constraints at the boundary. Hence, it is well known that we may compute $H^k(M,\partial M)$ using the complex of forms satisfying relative boundary conditions

$$H^k(M,\partial M) = \frac{\{\omega \in C^\infty\Omega^k(M) \,:\, i^*\omega = 0,\ d\omega = 0\}}{\{d\eta \,:\, \eta \in C^\infty\Omega^{k-1}(M),\ i^*\omega = 0\}} .$$

Here $i : \partial M \longrightarrow M$ is the inclusion. We could equally well use forms supported in the interior of M, as shown by an argument using the chain homotopy operator in the proof of our next theorem.

We need to define the relative groups using somewhat less regular forms. There are two spaces of distributions on M, briefly mentioned already in Chapter 2, but see [22] for more details. Let $\dot{C}^\infty$ denote the smooth functions vanishing to infinite order at ∂M and let

$$C^{-\infty}(M) = (\dot{C}^{\infty}(M))', \quad \dot{C}^{-\infty}(M) = (C^{\infty}(M))'.$$

These are the 'extendible' and 'supported' distributions, respectively. If $\tilde{M}$ is an open extension of M, then the former space contains the restrictions to $\mathring{M}$ of arbitrary distributions on $\tilde{M}$, while elements of the latter are distributions on $\tilde{M}$ actually supported in M. The sequence

$$0 \longrightarrow C^{-\infty}(M, \partial M) \longrightarrow \dot{C}^{-\infty}(M) \longrightarrow C^{-\infty}(M) \longrightarrow 0$$

is exact, the initial space being of distributions on M supported in ∂M. One of the few differences between these spaces arises by comparing how differential operators act on them. As may be checked from duality, an extendible distribution is to be differentiated in $\mathring{M}$, the result extending to the boundary by continuity; a supported distribution is differentiated as a distribution on $\tilde{M}$, so that boundary layers may well occur.

Having stated these facts, we now come to the point of the discussion. It is the case (which unfortunately is not to be found in the literature) that the absolute and relative cohomologies of M may be computed from the complexes of forms with values in $C^{-\infty}$ and $\dot{C}^{-\infty}$, respectively. We mention all this to justify a minor technicality concerning regularity in the proof of our

(4.8) Main Theorem: There are natural isomorphisms

$$\mathcal{H}^k \ni \omega \longmapsto [\omega] \in H^k(M, \partial M) \qquad k < \frac{n-1}{2}$$

$$\mathcal{H}^k \ni \omega \longmapsto [\omega] \in H^k(M) \qquad k > \frac{n+1}{2}.$$

Proof: L^2 harmonic forms are closed and coclosed since the metric g is complete. Hence when $k > \frac{n+1}{2}$, $\omega \in \mathcal{H}^k$ represents an absolute cohomology class. If $k < \frac{n-1}{2}$, then from (4.7) both components of ω vanish at ∂M. We may then consider it as lying in $\dot{C}^{-\infty}(M)$, and it is still closed in the sense of supported distributions. Thus ω represents a relative class. It will suffice to show that below the middle degree the map is injective, while above the middle degree it is surjective.

Suppose then that for $k < \frac{n-1}{2}$, $\omega \in \mathcal{H}^k$, we have

$$\omega = d\eta, \quad i^*\eta = 0.$$

Choose finitely many coordinate patches near ∂M such that $h_{\alpha n} = 0$, $h_{nn} = 1$, and the coordinate z^n is defined globally near the boundary. Then, since $\delta_g\omega = 0$, we compute

$$\|\omega\|^2 = \langle d\eta, \omega\rangle - \langle \eta, \delta_g \omega\rangle = \int_M d(\eta \wedge *_g \omega)$$

$$= \lim_{\epsilon\to 0} \int_{\{z^n=\epsilon\}} \eta \wedge *_h \omega \ \rho^{2k-n}.$$

By assumption on the coordinates, only η_t and $\omega_n = (*_h\omega)_t$ enter into this last integral. We know that

$$\omega_n = O(\rho^{n-2k} \log \rho)$$

but it is less clear at what rate η_t vanishes.

We study this issue by using the chain homotopy operator of [26]. Thus if

$$\omega = \omega_I \ dy^I + \omega_J \ dy^J \wedge dx \in \Omega^k$$

and $\omega_I(y,0) = 0$, then we define $R\omega = (R\omega)_J \ dy^J$ by

$$(R\omega)_J = (-1)^{k-1} \int_0^1 \omega_J(y,tx)x \ dt.$$

This is actually coordinate independent, and satisfies

$$dR\omega + Rd\omega = \omega$$

so long as $i^*\omega = 0$. Note also that $i^*R\omega = 0$.

If ω is closed then $\omega = dR\omega$ in the neighbourhood of the boundary where $R\omega$ is defined. If ψ is a function supported in this neighbourhood which is identically one near ∂M, then

$$\omega - d(\psi R\omega)$$

represents the same relative cohomology class and vanishes near the boundary. We are assuming that this class is trivial, so there exists a form β for which

$$i^*\beta = 0, \quad \omega - d(\psi R\omega) = d\beta$$

which is C^∞ up to ∂M. This shows that we may take

$$\eta = \beta + \psi R\omega$$

and so

$$\eta_t = O(\rho \log \rho).$$

The limit above must now vanish, hence $\omega = 0$ and the map into relative cohomology is injective.

On the other hand, when $k > \frac{n+1}{2}$, take a smooth representative α for an arbitrary absolute cohomology class. Obviously α is square-integrable with respect to

ρ^{2k-n} since $|\alpha|_h^2$ is bounded. We know that Δ_g has closed range with a finite dimensional complement, so we may write

$$\alpha = \Delta_g\beta + \omega.$$

for some $\beta \in L^2\Omega^k$ and $\omega \in \mathcal{H}^k$. Obviously β is C^∞ in the interior of M, and since $d\alpha = 0$ it follows that

$$\alpha = d\delta_g\beta + \omega.$$

Hence $[\alpha] = [\omega]$ and the theorem is proved.

As the last part of this proof shows, we have actually proved something about the L^2 cohomology spaces discussed in the introduction.

(4.9) Proposition: For $k < \frac{n-1}{2}$ or $k > \frac{n+1}{2}$ we have

$$L^2H^k \simeq \mathcal{H}^k$$

and so, in these degrees, L^2H^k is finite dimensional.

Proof: We have established a strong Hodge decomposition for these values of k:

$$L^2\Omega^k(M,dg) \qquad \alpha = d\delta_g\beta + \delta_g d\beta + \omega$$

$\omega \in \mathcal{H}^k$. α is closed iff the second summand vanishes, and so $[\alpha] = [\omega]$ in L^2H^k.

The existence of a parametrix away from the middle degrees implies that the spectrum of Δ_g on $L^2\Omega^k$ is bounded below by a positive constant for these values of k. However, when $k = (n\pm1)/2$ —so that the dimension n is odd—then judging from the special case $\mathbb{H}^n$, cf. [11], the continuous spectrum should in general extend down to zero, and Δ_g would not have closed range. About the final case, when n is even and $k = n/2$, we can reach a definite conclusion.

(4.10) <u>Proposition</u>: $\mathcal{H}^{n/2}$ is of infinite dimension, but on the orthocomplement $(\mathcal{H}^{n/2})^\perp$ the spectrum of Δ_g has a positive lower bound. Hence the range of the Laplacian on $L^2\Omega^{n/2}$ is closed, a strong Hodge decomposition holds, and $L^2H^{n/2} \simeq \mathcal{H}^{n/2}$.

<u>Proof</u>: Since $L^2\Omega^{n/2}(M,dg) = L^2\Omega^{n/2}(M,dh)$ and $\delta_g = \rho^2\delta_h$ on $\Omega^{n/2}$ (the space $\mathcal{H}^{n/2}$ depends only on the conformal class of the metric), it is necessary only to solve the elliptic boundary problem

$$\Delta_h \omega = 0, \quad i^* \omega = \mu \in C^\infty \Omega^{n/2}(\partial M), \quad i^* \delta_h \omega = 0$$

for the nondegenerate metric h; each solution is C^∞, hence in $L^2(dg)$, and an infinite dimensional subspace of $\mathcal{H}^{n/2}$ is thereby obtained.

The argument for the next part follows [14] closely. Choose $\epsilon > 0$ so that $[0,\epsilon]$ does not intersect the spectrum of $\Delta_g^{n/2\pm1}$ (the superscript indicating the degree of form on which Δ_g is supposed to act). Now suppose

$$\omega = \chi(\Delta_g^{n/2})\alpha \neq 0.$$

By the assumption on ϵ,

$$d\omega = d\chi(\Delta_g^{n/2})\alpha = \chi(\Delta_g^{n/2+1})d\alpha = 0$$

$$\delta_g \omega = \delta_g \chi(\Delta_g^{n/2})\alpha = \chi(\Delta_g^{n/2-1})\delta_g \alpha = 0$$

and so $\omega \in \mathcal{H}^{n/2}$. This proves that

$$\text{spec}(\Delta_g^{n/2}) \cap [0,\epsilon] = \{0\}$$

as desired.

The last assertions are now immediate, and we are done.

It seems quite likely that a fairly complete spectral picture of the Hodge Laplacian for a conformally compact metric in any degree could be obtained from refinements of the analysis in this dissertation. Perhaps this issue will be addressed later if it seems warranted.

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