THEORY OF TWO POINT CORRELATION FUNCTION IN A VLASOV PLASMA

by

Teymour Boutros-Ghali

B.A., Cambridge University

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M.A., Cambridge University

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Signature of Author ______________________

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Certified by ______________________

Thomas H. Dupree, Thesis Supervisor

Accepted by ______________________

A. H. Henry, Chairman, Departmental Committee on Graduate Students

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Abstract

A self-consistent theory of phase space granulations, called "clumps", has been derived. These fluctuations are produced when regions of different phase space density are mixed by the fluctuating electric fields. The source and turbulent scattering operator for these fluctuations is obtained through a renormalization of the one and two point equations for a Vlasov plasma. We treat throughout the case of electrostatic turbulence.

Our equations are similar to the "clump" model of Dupree[2] and the direct interaction formulation of Orzag and Kraichnan[10], and Dubois and Espedal[11].They differ from Ref. [2] in that self-consistency is included in the formulation. Many aspects, however, of the underlying "clump" model remain the same. The equations in Ref. [11] are similar in that they contain many, but not all, of the terms (necessary for conservation laws) which are generated through our approach. If we neglect the "clump" contribution then the equations reduce to the "coherent approximation" described by Krommes and Kleva[13]. Our solution method is based on the concept of two disparate time scales which allow us to treat the equal time two point equation as an initial condition for its two time counterpart. The picture of a "test" clump emerges quite naturally within such a framework. The source term for the clump correlation function is identified and certain intrinsic properties investigated. The anaylysis of the coefficients in the renormalization is examined with reference to conservation laws such as energy and momentum. The self-sustaining criterion for such fluctuations is addressed in a two species plasma and a novel state of turbulence is observed where a non-linear instability is generated *before* the boundaries of linear instability.

Thesis Supervisor:Thomas H. Dupree

Title: Professor of Physics and Nuclear Engineering

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This document was typed (by me!) using TEX and the EMACS editor on the ITS system at LCS.

To my Mother and Father

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Chapter 1

Introduction

It has been theoretically predicted that non wave-like fluctuations, called clumps[1-5], are an integral element in Vlasov turbulence. These particle-like modes can be viewed as phase space granulations arising from the incompressible nature of the flow. Since the Vlasov equation conserves phase space density along particle orbits, regions of different density cannot interpenetrate. The imperfect mixing leads to a graininess of the distribution function with a resulting potential spectrum. This spectrum can in turn rearrange the density gradients and in the process regenerate the turbulence. Qualitatively one can argue that if the phase space volume of a clump is sufficiently small, then the particles within the clump will be scattered turbulently as a group. This group will persist for a characteristic time period (the clump lifetime[2]) before the orbits of the individual members diverge. Thus one can view a clump as a macroparticle whose effective charge decreases with time. If the spectrum is to be self sustaining then this decay has to be balanced by an energy source. The problem can therefore be analyzed in two steps; the first seeks the characteristic lifetime of the fluctuations while the second investigates their source. In a manner analogous to discrete particle calculations, the relevant quantity in the theory is the self correlation function.

In this work we address two distinct but nonetheless closely related problems. The first deals with the self-consistent renormalization of the Vlasov equation, while the second treats the clump problem within such a framework. In a turbulent plasma a "test" particle immersed in the system will induce fluctuations not only from the average background, but also form the existing fluctuations. This process occurs through non linear coupling and proceeds indefinitely as induced fluctuations couple back. A number of self-consistent renormalizations have been proposed in which the effect of clumps has been neglected or not explicitly dealt with[10]. In particular Orzag and Kraichnan[10], Dubois[12] and Espedal[11], and Krommes[13–14] have applied various versions of the direct interaction approximation[8–9] to the Vlasov problem. Similarly Rudakov and Tsytovich[16], developing the work of Kadomstev[15], have obtained analogous equations. Our approach develops the work of Dupree and Tetreault[6] to include self-consistency and a contribution from a "discrete" quantity such as clumps. This important contribution leads to a set of equations[7] whose physical content and properties are quite different from the "standard" weak turbulence renormalizations. If, however, we neglect the clump contribution the equations can be shown to reduce to those in the cited references.

Mathematically we can trace the origin of these fluctuations in the following way. Let us assume that the fluctuating part of the distribution function obeys an equation of the form

$$\left(\frac{\partial}{\partial t} + v_1\frac{\partial}{\partial x_1} + C_{11}\right)\delta f(1) = -\frac{q}{m}\delta E(1)\frac{\partial}{\partial v_1}\langle f(1)\rangle \tag{1.1}$$

Here δE is the fluctuating electric field, $\langle f\rangle$ the average distribution and C_{11} is a schematic representation of a "collision" operator arising from collective interactions. This collision integral represents a selective summing of a certain subset of non-linear terms and physically accounts for the perturbation of δf away from its ballistic orbit plus other non-linear effects. In the absence of such a renormalization conventional perturbation analysis gives rise to a resonance denominator $(\omega - kv)$, where ω and k describe a wave $\exp i(kx-\omega t)$ and v is the particle velocity. This resonance, which is fundamental to the damping and growing of waves also leads to time secularities in the individual terms of the perturbation analysis. One of the goals of the renormalization is to eliminate these secularities which are due to the vanishing of the lowest order operator

$$\frac{\partial}{\partial t} + v_1\frac{\partial}{\partial x_1} \tag{1.2}$$

at a wave particle resonance. The earliest treatement[17] of such an operator (C_{11}) resulted in diffusion of δf in velocity space. This followed quite naturally from Quasi-Linear [18] theory where the average distribution also obeyed a diffusion equation.

While such an approach resolves the singular behaviour of (1.2) other *secular* contributions arise. In particular the strong mode coupling and harmonic distortion at a wave particle resonance is not properly described. If we consider the distribution function as a superposition of velocity streams then each stream will be resonant with a wave going at the same speed. This interaction cannot be described by a conventional perturbation scheme since the stream quickly develops a number of higher harmonics with complicated spatial dependance. These then get propagated ballistically at the stream speed. An analysis of such a problem could in theory be carried out in a one point frame. The perturbations of the distribution function, however, are extremely complicated and of a random nature in such an interaction. It is therefore more appropriate to investigate this contribution through a statistical framework which deals with the correlation of two points at close separation. In other words we need to develop a theory for the ensemble averaged two point correlation function $\langle \delta f(1) \delta f(2) \rangle$.

One can easily obtain an equation (incorrectly as we shall see) for the correlation function $\langle \delta f(1) \delta f(2) \rangle$ by multiplying (1.1) by $\delta f(2)$ and *vice versa* for the equation governing $\delta f(2)$. Ensemble averaging we get

$$\left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + C_{11} + C_{22}\right)\langle \delta f(1) \delta f(2) \rangle = \\ -\frac{q}{m}\langle \delta E(1) \delta f(2) \rangle \frac{\partial}{\partial v_1}\langle f(1) \rangle - \frac{q}{m}\langle \delta E(2) \delta f(1) \rangle \frac{\partial}{\partial v_2}\langle f(2) \rangle \tag{1.3}$$

The lowest order operator is

$$\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} \tag{1.4}$$

Time asymptotically and in a spatially homogeneous system it reduces to

$$(v_1 - v_2)\frac{\partial}{\partial x_-} \tag{1.5}$$

($x_- = x_1 - x_2$). In this case the divergence occurs because two points coming arbitrarily close to each other will experience the same forces and follow the same orbit. As such one would expect the renormalization to account for the interaction of two points which are very close to each other. If we take (1.3) as our renormalization we find that the LHS operator states that two points will always move independantly of each other *whatever* their spatial and velocity separation. On physical grounds this cannot be correct and we would expect some terms which specifically correlate the interaction between points 1 and 2. Let us call these C_{12} and C_{21} so that (1.3) becomes

$$\left(\frac{\partial}{\partial t} + v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2} + C_{11} + C_{22} + C_{12} + C_{21}\right)\langle\delta f(1)\delta f(2)\rangle = \\ -\frac{q}{m}\langle\delta E(1)\delta f(2)\rangle\frac{\partial}{\partial v_1}\langle f(1)\rangle - \frac{q}{m}\langle\delta E(2)\delta f(1)\rangle\frac{\partial}{\partial v_2}\langle f(2)\rangle \tag{1.6}$$

We derive an equation of this form in Chapter 3. Let us rewrite (1.6) as

$$\left(\frac{\partial}{\partial t} + T_{12}\right)\langle\delta f(1)\delta f(2)\rangle = S \tag{1.7}$$

where S represents the RHS of (1.6) and T_{12} represents the renormalization plus the convective terms. We show that in the relative coordinate system x_-, v_- ($v_\pm = v_1 \pm v_2, x_\pm = x_1 \pm x_2$), $T_{12} \to 0$ as $x_-, v_- \to 0$ while S does not. Consequently $\langle\delta f(1)\delta f(2)\rangle$ is a very peaked function of $\{x_-, v_-\}$. The difference between (1.6) and (1.3), which represents the "clump", occurs in a very locallized region of velocity space where the C_{ij} terms dominate $v_-\partial/\partial x_-$. It is clear that (1.3), and by default (1.1), does not contain this information. Thus we must conclude that there exists a set of terms in the one point formulation which are not resummed by conventional renormalizations. Indeed we show in Chapter 3 that the clump contribution can also be viewed as a *secular* contribution arising from a set of "incoherent" terms which are nominally of second order in the perturbation analysis. If f^c ("coherent") is the solution to (1.1), the total solution must contain an added contribution $\tilde{f}$ ("incoherent") which generates the cross operators (C_{ij}).

While different regimes of turbulence have been characterized in the literature[19], we will be primarily concerned with the so called weak turbulence limit. By which we mean that the spectrum auto-correlation time (τ_c) is much less than the trapping time (τ_{tr}). τ_c and τ_{tr} are characterized by $\simeq (k\Delta v_{ph})^{-1}$ and $\simeq (kv_{tr})^{-1}$ where $v_{tr}^2 \simeq D\tau_{tr}$. v_{tr} is the trapping width in velocity space, Δv_{ph} is the spread in phase velocity of the fluctuations, k is the average wavenumber while D is the diffusion coefficient of Quasi-Linear theory. q and m are the particle charge and mass. These two time scales are closely related to another physical concept: if the "clump" is treated as a macro-particle of typical width v_{tr} then τ_{tr} is the slow or "long" time scale associated with the decay of clump structure. τ_c, on the other hand, represents the fast or "short" time scale which is associated with the ballistic motion of the centre of mass of the clump. These time scales have to be disparate for the concept of a clump as a *test* particle to be meaningful. If the condition $\tau_c \ll \tau_{tr}$ is satisfied then the decay of the clump will occur on a much slower time scale than the decay of the (two time) autocorerelation function. It is then approriate and expedient to handle the problem in a manner similar to the test particle model of Rosenbluth and Rostoker[20]. In Fourier space, if the clump generates a spectrum $\langle\tilde{\phi}^2\rangle_{k\omega}$ then the total shielded potential is given by

$$\langle\phi^2\rangle_{k\omega} = \frac{\langle\tilde{\phi}^2\rangle_{k\omega}}{|\epsilon_{k\omega}|^2} \tag{1.8}$$

where $\epsilon_{k\omega}$ is the non-linear dielectric which we derive in Chapter 2. The symbol $\langle AB\rangle_{k\omega}$ is a Fourier transform on the relative coordinate x_1-x_2 and t_1-t_2 (where we have assumed temporal and spatial homogeneity).

We start in Chapter 2 with the derivation of a renormalized, self consistent, one point equation for an infinite spatial and temporally homogeneous electrostatic plasma. We introduce the incoherent contribution $\tilde{f}$ as an initial condition. The properties of the resulting equations are analyzed in the framework of conservation laws such as energy and momentum. In the long wavelength limit the "collision" operator reduces to a perturbed Fokker-Planck operator which conserves energy and momentum. An unperturbed version of this collision operator leads to a Lenard-Balescu like equation for the average distribution[1]

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial v}F - \frac{\partial}{\partial v}D\frac{\partial}{\partial v}\right)\langle f\rangle = 0 \tag{1.10}$$

Here the drag (F) is due to the reaction of the shielding cloud on the "discrete" clump while the diffusion (D) results from the shielded clump spectrum. Chapter 3 continues in the same vien with a derivation of the two point equation. We make use of the two time scaling ($\tau_{tr} \gg \tau_c$) to *decouple* the two time, and equal time two point equations. The result is a Markovian theory in which we use the equal time equation as an initial value for the two time equation. The analysis is carried to nominally second order in the electric field strength. The important property of phase space conservation ($T_{12} \to 0$ as $x_-, v_- \to 0$) is retained in the final equation: this result is independant of the Markovian assumption.

We can compare, schematically, the equations we derive to previous formulations in the following way. Dupree's original theory[1,2] and subsequent papers[21–23] considered solutions of the basic equation

$$\left(\frac{\partial}{\partial t} + T^0_{12}\right)\langle\delta f\delta f\rangle = S^0 \tag{1.11}$$

The zero superscripts refer to stochastic acceleration variables. For example T_0 might include diffusion in the relative coordinate system while S^0 would be the $\langle \delta E \delta f \rangle \partial / \partial \langle f \rangle$ term evaluated through the approximation $f = f^c$ only. The self-consistent approach which treats $\tilde{f}$ *on par* with f^c changes (1.11) to

$$\left(\frac{\partial}{\partial t} + T^0_{12} + T^s_{12}\right)\langle \delta f \delta f \rangle = S^0 + S^s \tag{1.12}$$

T^s_{12} is a contribution to the T^0_{12} operator arising from the perturbation of the medium through the coupling of δf to the background fluctuations. A systematic analysis of these contributions is carried out in the long wavelength limit where numerous cancellations between these terms and the T^0_{12} operator are demonstrated on the basis of momentum and energy conservation.

The analysis of the source $S = S^0 + S^s$ is investigated in Chapter 4. A useful identification is made between the source and the relaxaxtion of the average distribution. This allows us to show that for a one species, one dimensional plasma the source term (which now resembles a Lenard-Balescu operator) is approximately zero. This result is directly related to the fact that in a one dimensional problem electron-electron (or ion-ion) collisions cannot relax the average distribution because of momentum constraints. Important cases exist where it is non zero. For example, in a two species plasma or for a spectrum containing normal modes of the system. The latter ensures that the one dimensional collision operator is none zero, and in this case the procedure can be viewed as a correction to Quasi-Linear theory.

To complete the analysis we require an equation for the *two time* correlation function since spectral functions such as $\langle \phi^2 \rangle_{k\omega}$ require a knowledge of $\langle \phi(t_1)\phi(t_2)\rangle_k$. This last quantity appears in the evaluation of the C_{ij} operators. Our basic equation is obtained quite simply by taking (1.1) and multiplying by $\delta f(t_2)$ to obtain

$$\left(\frac{\partial}{\partial t_1} + v_1 \frac{\partial}{\partial x_1} + C_{11}\right)\langle \delta f(t_1)\delta f(t_2)\rangle = -\frac{q}{m}\langle \delta E(t_1)\delta f(t_2)\rangle \frac{\partial}{\partial v_1}\langle f \rangle \tag{1.13}$$

This equation is valid for $t_1 \geq t_2 \geq 0$ and is solved with the solution to (1.12) as an initial condition. (1.13) and (1.12) underline our approach and solution technique. We have neglected the cross operators in (1.13) but not in (1.12). Physically this approximation is related to the idea that the clumping phenomena is intrinsically an *equal time* mechanism. It is only when two particles see the same electric field at the same point in space and time that a strong correlation will exist between them. Furthermore this effect is a *secular* contribution arising from the steady state (or time asymptotic) solution of (1.12).

Thus in principle we could solve (1.13) with the cross terms but we would need to look at the solution as $t_1, t_2 \to \infty$ with $t_1 - t_2 \ll \omega_p^{-1}$. Instead we treat the initial value problem which considers the equal time and two time equations as independant entities. In such an approach the equal time equation generates the incoherent response which then gets propagated through what, we will show, is essentially a ballistic operator to obtain its fast spectral dependance.

In Chapter 5 we consider the formal solutions to the set of equations (1.12) and (1.13). We can anticipate some of the results in the following intuitive way. The distribution f is conserved along a particles orbit. Thus the value of f at two neighbouring points may be quite different since these points might originally have been widely separated. Let $g_0(v, v_0, t)$ be the Green's function which solves the equation governing the average distribution function

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial v}F - \frac{\partial}{\partial v}D\frac{\partial}{\partial v}\right)g_0(v, v_0, t) = \delta(v - v_0) \tag{1.14}$$

and consider a small volume of phase space x_-, v_- located at x, v. We define τ_{cl} as the characteristic e-folding time of the solution to (1.12) (i.e $\tau_{cl} \simeq T_{12}^{-1}$). Physically, if we follow the orbits of two points located about x, v back in time, all the particles within x_-, v_- will move together for a time $\tau_{cl}(x_-, v_-)$ (at which point they will be at coordinate v_0). Further back in time the orbits will have diverged and the particles will move independantly. Thus the density in the volume x_-, v_- at time t and position v, is approximately equal to the density of the average distribution at an earlier time $t - \tau_{cl}$ and position v_0. The coordinate v_0 is distributed according to the Green's function g_0 thus we can write for the fluctuations at v, t

$$\langle \delta f \delta f \rangle = \int dv_0 g_0(v, v_0, \tau_{cl}) f_0^2(v_0, t - \tau_{cl}) - f_0^2(v, t) \tag{1.15}$$

If $(D\tau_{cl})^{1/2} \ll \Delta v_{ph}$, v_{th}, where v_{th} is the "thermal" or characteristic velocity associated with the average distribution we can expand g_0 to obtain the operator relation

$$\int dv_0 g_0(v, v_0, \tau_{cl}) \simeq 1 + \tau_{cl}\left[\frac{\partial}{\partial v}D\frac{\partial}{\partial v} - \frac{\partial}{\partial v}F\right] \tag{1.16}$$

The clump contribution is obtained by subtracting the solution to (1.3) from (1.15). If the characteristic e-folding time of (1.3) is τ_{tr} then the same arguments lead to

$$\langle \tilde{f}\tilde{f} \rangle \simeq [\tau_{cl} - \tau_{tr}]\left[\frac{\partial}{\partial v}D\frac{\partial}{\partial v} - \frac{\partial}{\partial v}F\right]f_0^2 \tag{1.17}$$

We can write (1.17) as $[\tau_{cl}-\tau_{tr}][S^0 + S^s]$ where S^0 is the diffusive part of the source and S^s is the friction term. We obtain an expression similar to (1.17) in Chapter 5. We must remember that (1.17) is an equal time result and to obtain spectral functions we need the two time version of $\langle \tilde{f}\tilde{f}|0\rangle$. We show that $\langle \tilde{f}\tilde{f}|t\rangle$ is obtained by propagating (1.17) through $g_k(v, v_0, t)$ which is a spatially *inhomogeneous* generalization of the $g_0(v, v_0, t)$ operator. In the long wave-length limit g_k is a ballistic operator renormalized by terms which are equivalent to a simple iterative solution of a Fokker-Planck equation.

This system of equations is extremely complicated and at all stages we attempt to present models which explain the underlying physics. To this aim the picture of clumps being generated by the mixing of the average gradients is extremely useful. While the existence of such a mechanism can easily be justified on physical grounds some confusion has arisen on the magnitude, hence importance, of such an effect. In particular we examine the conclusion reached by Dubois *et al.*[11] who in their treatment of a version of renormalized equations for the Klimontovich system conjectured that these fluctuations were down an order of ϕ^2 compared to the coherent response. The nominal ordering of the expansion is fully investigated and we show how to recover the correct ordering and source term in the limit of small x_-, v_-.

We focus on some models of the self energy interaction in Chapter 6. In this context self energy is seen as the ability of a clump to act on itself because of its finite size. In other words the electric fields generated by the structure acts on different points within the structure altering its lifetime. The discussion is more qualitative than quantitative due to the complexity of the effects taking place.

In Chapter 7 we give these results a more tangible perspective. The analysis of an electron ion plasma, in which a current exists due to an electron drift, is carried out in a simplified version of the "test" clump picture. Model equations are used to describe the formation of ion and electron clumps. These interact through a two species source term which couples ion and electron density gradients. We treat an equation of the form

$$\left(\frac{\partial}{\partial t} + T_{12}^0\right)\langle \delta f \delta f \rangle = S^0 + S^s \tag{1.18}$$

The question of regeneration is addressed through a self consistent numerical calculation. Results of the simulation in the pre ion-acoustic regime ($T_e/T_i \simeq .1, 10$) indicate that the clump spectrum regenerates at electron drift velocities which are appreciably below ($\simeq$ 50%) those needed for the onset of linear instability. An approximate analytical result is in agreement with these predictions.

In conclusion we summarize the salient features in this work. A self-consistent renormalization of the one point and two point equations in a Vlasov plasma is performed through a procedure analogous to the direct interaction approximation. The singular element arising from phase space conservation is treated within the framework of the renormalization. Our equations are similar to those in Ref. [2] and [11]. They differ from Ref. [2] in that self-consistency is included in the formulation. Many aspects, however, of the underlying "clump" model remain the same. The equations in Ref. [11] are similar in that they contain many, but not all, of the terms (necessary for conservation laws) which are generated through our approach. If we neglect the "clump" contribution then the equations reduce to the "coherent approximation" described by Krommes and Kleva[13]. Our solution method is based on the concept of two disparate time scales which allow us to treat the equal time two point equation as an initial condition for its two time counterpart. The picture of a "test" clump emerges quite naturally within such a framework. The source term for the clump correlation function is identified and certain intrinsic properties investigated. We examine and analyze the properties of the coefficients in the renormalization through conservation laws such as energy and momentum. The self-sustaining criterion for such fluctuations is addressed in a two species plasma and a novel state of turbulence is observed where a non-linear instability is generated *before* the boundaries of linear instability.

Chapter 2

One Point Equation

The subject of plasma turbulence and the renormalization of the governing equations, has received a fair amount of attention in recent years. We present a renormalization procedure which takes its roots in two distinct methodologies. The first generalizes the work of Dupree and Tetreault[6] to take into account self-consistent contributions to the renormalization. This yields the same results as some of the recent renormalizations[11–14] based on the direct interaction approximation[8–10]. The methods are presumably equivalent, however we believe our approach presents a considerable simplification and allows a clearer insight into the iterative scheme. The second relies on an a priori realization that the standard one point ("coherent") renormalized theories fail to include an "incoherent" ($\tilde{f}$) contribution which we will show is of the same order of magnitude. As a result these renormalizations are incomplete inspite of the more "educated" way of computing the iterative process. This effect was first pointed out by Dupree in his "clump" theory[1–5]. Though having been shown to be a simpler, and in some instances deficient version of the more rigorous renormalizations, the theory nonetheless lays the groundwork for what follows.

In this chapter we will derive the set of equations governing the evolution of the one point equation, to nominally second order in the fluctuation field strength. We interpret the various terms, and show how they are necessary for momentum and energy conservation. To make contact with previous theories we illustrate their use for a collisional plasma, where the singular behaviour of the fluctuations is assumed to originate from particle discretness effects only.

2.1. One point Renormalization

Our starting point is the time honoured Vlasov equation coupled with Poisson's equation:

$$\begin{aligned}(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}+\frac{q}{m}E(x,t)\frac{\partial}{\partial v})f(x,v,t)=0\\ \frac{\partial}{\partial x}E(x,t)=-4\pi q\int dv[f(x,v,t)-n_0\delta(v)]\end{aligned} \tag{2.1}$$

$f(x,v,t)$ is the distribution function, q, m, x, and v are the charge, mass, position, and velocity. n_0 is the density of the uniform background of particles with charge q.

If one considers the Vlasov equation as describing a fictious plasma in which the discreteness parameters (n^{-1}, q, m) approach zero in such a way that mn, qn, and nkT remain constant, then it is clear that this system exhibits an infinite number of degrees of freedom. We therefore seek to deal with statistical averages of the distribution function, covariance and higher order correlations. We will use $\langle\ldots\rangle$ to represent this average, which is interpreted as an ensemble average over a large number of realizations.

We write the fields as the sum of a mean plus a fluctuation:

$$\begin{aligned}f(x,v,t)=&\langle f(x,v,t)\rangle+\delta f(x,v,t)\\ E(x,t)=&\langle E(x,t)\rangle+\delta E(x,t)\end{aligned} \tag{2.2}$$

where $\langle\delta f\rangle=\langle\delta E\rangle=0$. Furthermore we will assume spatial homogeneity so that the ensemble average becomes synonymous to a spatial average. In that case $\langle E\rangle=0$ (due to charge neutrality), and $\langle f(x,v,t)\rangle=f_0(v,t)$.

On the basis of the arguments in the previous section we can also write the fluctuating part of distribution function as

$$\delta f=f^c+\tilde{f}=f^c+f^d+f^v+f^m$$

f^c, is the phase coherent response to the applied electric field, and $\tilde{f}$ describes incoherent fluctuations which can be due to a variety of physical processes. In particular f^d is the discrete particle noise * , f^v, its Vlasov equivalent which, as will be shown is a direct consequence of phase space conservation.

*Strictly speaking the inclusion of f^d would mean we are dealing with the Klimontovich distribution. Operationally this makes no difference. We include the term for completness, though through out most of this work we will assume collisionless turbulence so that f^d is zero.

f^m represents all other effects, which can include mode coupling to modes of different phase than the applied electric .

To simplify the analysis we will consider a one dimensional plasma of length L and proceed to the infinite case once we obtain the renormalized equations. (The multidimensional case, with weak inhomogeneities is a straightforward extension.) We expand the fluctuating part of the field and distribution function in a Fourier series

$$\delta f(v,x,t) = \sum_k f_k(v,t) \exp ikx \qquad (k = \frac{n\pi}{2L})$$
$$\delta E(x,t) = \sum_k E_k(t) \exp ikx \tag{2.3}$$

Eq. (2.1) becomes

$$\frac{\partial f_k(t)}{\partial t} + ikvf_k(t) + \frac{q}{m}E_k(t)\frac{\partial f_0}{\partial v} + \frac{q}{m}\frac{\partial}{\partial v}\sum_{k'} E_{k'}(t) f_{k-k'}(t) = 0$$
$$ikE_k(t) = 4\pi ne \int dv f_k(v,t) \tag{2.4}$$
$$E_k(t) = -ik\phi_k(t)$$

Conventional perturbation analysis assumes that there exists some ordering parameter λ ($\ll 1$) which allows the solution to be written as a power series in λ:

$$f_k(t) = \lambda f_k^{(1)}(t) + \lambda^2 f_k^{(2)}(t) + \cdots$$
$$E_k(t) = \lambda E_k^{(1)}(t) + \lambda^2 E_k^{(2)}(t) + \cdots \tag{2.5}$$

The coefficients of the series represent succesive improvements to the previous order solution. In such an approach the non-linear term does not appear in the first order solution being nominally of second order. It is well known that expansions in terms of the resulting "free" or ballistic propagator $(\partial_t + ikv)$ exhibit un-acceptable time secularities. One can anticipate such a behaviour on physical grounds since the ballistic motion does not take into account energy transfer (a "second order" process) in and out of a given mode. But a turbulent state is most commonly envisioned as one in which a large number of modes are excited with the interchange of energy between them being integral to the evolution of the system. The neglect of these contributions is cumulative so that the level of fluctuations can be quite weak while still translating into a sizeable secular contribution. The goal of the renormalization is to extract a "collision" operator out of the non-linear term and incorporate that in the "linearized" result as a remedy. Of the infinite set of non-linear terms we will only retain those which have the same phase ("phase coherent") as the driven mode f_k. Let us call the coherent portion of the non-linear term, $\int dt' C_k(t-t') f_k(t')$, where C_k contains the amplitude of the fluctuations but no phase information. We rewrite the Vlasov equation as

$$\frac{\partial f_k(t)}{\partial t} + ikvf_k(t) + \frac{q}{m}E_k(t)\frac{\partial f_0}{\partial v} + \int_0^t dt' C_k(t-t') f_k(t') = \\ -\lambda\left(\frac{q}{m}\frac{\partial}{\partial v}\sum_{k'} E_{k'}(t) f_{k-k'}(t) - \int_0^t dt' C_k(t-t') f_k(t')\right) \tag{2.6}$$

where the difference between the non-linear term and C_k is assumed to be an order smaller than the rest of the equation. We now reinstate the perturbation expansion and associate λ with the electric field amplitude.

Equating order by order we get

$$\frac{\partial f_k^{(1)}(t)}{\partial t} + ikvf_k^{(1)}(t) + \frac{q}{m}E_k^{(1)}(t)\frac{\partial f_0}{\partial v} + \int_0^t dt' C_k(t-t') f_k^{(1)}(t') = 0 \tag{2.7}$$

$$\frac{\partial f_{k-k'}^{(2)}(t)}{\partial t} + ikvf_{k-k'}^{(2)}(t) + \frac{q}{m}E_{k-k'}^{(2)}(t)\frac{\partial f_0}{\partial v} + \int_0^t dt' C_{k-k'}(t-t') f_k^{(2)}(t') = \\ -\frac{q}{m}E_k^{(1)}(t)\frac{\partial}{\partial v}f_{k'}^{(1)*}(t) - \frac{q}{m}E_{k'}^{(1)*}(t)\frac{\partial}{\partial v}f_k^{(1)}(t) \tag{2.8}$$

and

$$\int_0^t dt' C_k(t-t') f_k^{(1)}(t') = \left(\frac{q}{m}\frac{\partial}{\partial v}\sum_{k'}(E_{k'}^{(1)} f_{k-k'}^{(2)} + E_{k-k'}^{(2)} f_{k'}^{(1)} + \cdots) + (E_{k'}^{(1)} f_{k-k'}^{(1)})\right)_{Phase\ Coherent} \tag{2.9}$$

where we have included terms up to second order. It is important to note that the terms in the first set of brackets will give the correct phase dependance for C_k. The other term, however, cannot contribute phase coherently since the fluctuation $f_{k-k'}^{(1)}$ cannot be decomposed into ones driven by k and k'. This is illustrated by the RHS of (2.8) where, of the infinite set of non-linear terms, we have only retained the subset which when iterated in (2.9) will give terms proportional to $f_k(t)$ or $E_k(t)$.

We anticipate this last observation by writing

$$\int_0^t dt' C_k(t-t') f_k(t') = \int_0^t dt' C_k^f(t-t') f_k(t') + \frac{q}{m}\int_0^t dt' E_k(t') \frac{\partial}{\partial v} C_k^{\phi}(t-t') \tag{2.10}$$

Eq. (2.7) is solved by *defininig* the "coherent" (f_k^c) and "incoherent" ($\tilde{f}_k$) responses through the following partition

$$\frac{\partial f_k^{c(1)}(t)}{\partial t} + ikv f_k^{c(1)}(t) + \int_0^t dt' C_k^f(t-t') f_k^{c(1)}(t') = -\frac{q}{m} E_k^{(1)}(t) \frac{\partial f_0}{\partial v} - \frac{q}{m}\int_0^t dt' E_k^{(1)}(t') \frac{\partial}{\partial v} C_k^{\phi}(t-t') \tag{2.11}$$

and

$$\frac{\partial \tilde{f}_k^{(1)}(t)}{\partial t} + ikv \tilde{f}_k^{(1)}(t) + \int_0^t dt' C_k^f(t-t') \tilde{f}_k^{(1)}(t') = 0 \tag{2.12}$$

with initial conditions $f_k^{c(1)}(t=0) = f_k^{c(1)}(0)$ and $\tilde{f}_k^{(1)}(t=0) = \tilde{f}_k^{(1)}(0)$. Note that (2.11) and (2.12) add up to the original equation (2.7). This division tracks linear response theory. f^c is associated with the induced fields which shield perturbations in the plasma. In this case, however, the ballistic operator is renormalized through what we will show is a Fokker-Planck operator and the average distribution through C^{ϕ}. We can neglect the initial condition $f_k^c(0)$ by setting the lower limit of the dt' integral in (2.11) to $-\infty$. This presumes that the coherent initial condition decays very quickly $O(\omega_p^{-1})$. We will show, however, that the ballistic contribution arising from the popagation of the initial condition $\tilde{f}_k(0)$ decays on a much longer time scale so that such a stratagem is not particulary useful. Instead we define a backward equation for $f_k(-t)$ where we use $C_k(t-t') = C_k(|t-t'|)\mathrm{sgn}(t-t')$. $\tilde{f}_k(0)$ *cannot* be obtained in an iterative way. In fact the exact structure of $\tilde{f}_k(0)$ is far too complicated and in practice we will only need the correlation function $\langle \tilde{f}(0)\tilde{f}(0)\rangle_k$. This quantity, which is extremely localized in velocity, can be obtained from a solution of the *equal time* two point equation for small separation.

If we define the Green's function $g_k(t)$ through

$$\begin{aligned} &\frac{\partial g_k(t)}{\partial t} + ikv g_k(t) + \int_0^t dt' C_k^f(t-t') g_{k'}(t') = 0 \quad t > 0 \\ &g_k(t=0^+) = 1 \\ &g_k(t) = 0 \qquad t < 0 \end{aligned} \tag{2.13}$$

and the relevant transforms as

$$g_{k\omega}\int_0^\infty dt g_k(t)\exp i\omega t$$

$$f_k(t)=\sum_\omega f_{k\omega}\exp i\omega t \tag{2.14}$$

we can synthesise the one point results in the following form

$$f^{(1)}_{k\omega}=\tilde{f}^{(1)}_{k\omega}+f^{c(1)}_{k\omega}=\tilde{f}^{(1)}_{k\omega}+\frac{q}{m}g_{k\omega}ik\phi^{(1)}_{k\omega}\frac{\partial\overline{F}_{k\omega}}{\partial v} \tag{2.15a}$$

where

$$-i(\omega-kv+iC^f_{k\omega})g_{k\omega}=1,\qquad C^f_{k\omega}=\int_0^\infty dtC^f_k(t)\exp i\omega t \tag{2.15b}$$

and

$$\overline{F}_{k\omega}=f_0+C^\phi_{k\omega},\qquad C^\phi_{k\omega}=\int_0^\infty dtC^\phi_k(t)\exp i\omega t \tag{2.15c}$$

are respectively, the renormalized Green function and "equivalent " background distribution function [20]. If we use Poisson's equation we get

$$\phi^{(1)}_{k\omega}=\frac{4\pi ne}{|k|^2}\int dvf^{(1)}_{k\omega} \tag{2.15d}$$

$$\tilde{\phi}^{(1)}_{k\omega}=\frac{4\pi ne}{|k|^2}\int dv\tilde{f}^{(1)}_{k\omega} \tag{2.15e}$$

which yields

$$\phi^{(1)}_{k\omega}=\frac{\tilde{\phi}^{(1)}_{k\omega}}{\epsilon_{k\omega}} \tag{2.15f}$$

$\epsilon_{k\omega}$ is the non-linear dielectric given by

$$\epsilon_{k\omega}=1-i\frac{\omega_p^2}{|k|^2}\int dvg_{k\omega}k\frac{\partial}{\partial v}\overline{F}_{k\omega} \tag{2.16}$$

Given the set (2.15) the next step in the calculation is to obtain explicit expressions for the coefficients in the collision operators $C^f_{k\omega}$ and $C^\phi_{k\omega}$. This requires the quantities $f^{(2)}_{k-k'}(t)$ and $E^{(2)}_{k-k'}(t)$, which are functions of $\phi_{k'}^{(1)}$ and $f_{k'}^{(1)}$. Because of the phase coherent approximation the modes at k' will only appear as products of the form $\sum_{k'}\langle\phi_{k'}^{(1)*}(t')f_{k'}^{(1)}(t)\rangle$ and $\sum_{k'}\langle\phi_{k'}^{(1)*}(t')\phi_{k'}^{(1)}(t)\rangle$. Assuming time stationarity we can write, for example,

$$\sum_{k'}\phi_{k'*}^{(1)}(t)f_{k'}^{(1)}(t') = \sum_{k',\omega'}\phi_{k'\omega'}^{(1)*}f_{k'\omega'}^{(1)}\exp -i\omega'(t-t') \equiv \int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}\langle\phi f\rangle_{k'\omega'}\exp -i\omega'(t-t') \tag{2.17}$$

where

$$\langle\phi f\rangle_{k'\omega'} = \int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}dxe^{i(\omega't-k'x)}\langle\phi(v_1,x+x',t+t')f(v_2,x',t')\rangle \tag{2.18}$$

We can use the set (2.15) to obtain

$$\begin{aligned}\sum_{k',\omega'}\langle\phi_{k'\omega'}^*f_{k'\omega'}\rangle &= \frac{q}{m}\sum_{k',\omega'}\frac{\langle\tilde{\phi}_{k'\omega'}^*\tilde{\phi}_{k'\omega'}\rangle}{|\epsilon_{k'\omega'}|^2}ig_{k'\omega'}k'\frac{\partial}{\partial v}\overline{F}_{k'\omega'} + \sum_{k',\omega'}\frac{\langle\tilde{\phi}_{k'\omega'}^*\tilde{f}_{k\omega}\rangle}{|\epsilon_{k'\omega'}|^2}\epsilon_{k'\omega'}\\ &= \int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}\left(\frac{\langle\tilde{\phi}^2\rangle_{k'\omega'}}{|\epsilon_{k'\omega'}|^2}ig_{k'\omega'}k'\frac{\partial}{\partial v}\overline{F}_{k'\omega'} + \frac{\langle\tilde{\phi}\tilde{f}\rangle_{k'\omega'}}{|\epsilon_{k'\omega'}|^2}\epsilon_{k'\omega'}\right)\end{aligned} \tag{2.19}$$

where we have expressed all quantities in terms of velocity moments of the incoherent correlation function $\langle\tilde{f}\tilde{f}\rangle_{k'\omega'}$. This quantity can be obtained quite simply by noting that

$$\int\frac{d\omega'}{2\pi}\sum_{k'}\langle\tilde{f}_{k'}(1)\tilde{f}_{k'}^*(2)\rangle_{\omega'} = \int\frac{d\omega'}{2\pi}\int\frac{dk'}{2\pi}(\langle\tilde{f}^+(1,\omega')\tilde{f}(2,0)\rangle_{k'} + \langle\tilde{f}(1,0)\tilde{f}^-(2,\omega')\rangle_{k'}) \tag{2.20}$$

where

$$\tilde{f}^{\pm}(\omega') = \int_0^{\infty}dtf(t)\exp \pm i\omega't \tag{2.21}$$

Using Eq. (2.12) we immediately get

$$\sum_{k',\omega'}\langle\tilde{f}_{k'\omega'}(1)\tilde{f}_{k'\omega'}^*(2)\rangle = \int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}[g_{k'\omega'}(1) + g_{k'\omega'}^*(2)]\langle\tilde{f}(0)\tilde{f}(0)\rangle_{k'} \tag{2.22}$$

Note that if the turbulence is weak this reduces to the familiar result of ballistic propagation. If the fluctuations are localized in velocity such that $\langle\tilde{f}\tilde{f}\rangle \approx \delta(v_1 - v_2)$, (2.22) reduces to $2Re\, g_{k'\omega'}\langle\tilde{f}\tilde{f}\rangle_{k'}$.

(2.19) and (2.22) are obtained in a slightly different way in Chapter 5. There, in a two point formulation, we treat the initial condition from both $f^c(0)$ and $\tilde{f}(0)$ to obtain the same expressions.

Proceeding with this scheme we partition the expression for $f^{(2)}_{k-k'}$ into $f^{c(2)}_{k-k'}(t)$ and $\tilde{f}^{(2)}_{k-k'}(t)$ through

$$f^{c(2)}_{k-k'}(t) = \int_0^t dt' g_{k-k'}(t-t') \times i\frac{q}{m}\left(i(k-k')\frac{\partial}{\partial v}(f_0 + C^{\phi}_{k-k'}(t'))\phi^{(2)}_{k-k'}(t') - k'\frac{\partial f^{(1)}_{k'}(t')}{\partial v}\phi^{(1)*}_{k'}(t') + k\frac{\partial f^{c(1)*}_{k'}(t')}{\partial v}\phi^{(1)}_{k}(t')\right) \tag{2.23}$$

and

$$\tilde{f}^{(2)}_{k-k'}(t) = \int_0^t dt' g_{k-k'}(t-t') i\frac{q}{m}k\frac{\partial \tilde{f}^{(1)*}_{k'}(t')}{\partial v}\phi^{(1)}_{k}(t') \tag{2.24}$$

we have assumed that the only initial condition is $\tilde{f}^{(1)}_{k}(0)$. Note that $\tilde{f}^{(2)}_{k-k'}$ is a purely *perturbative* quantity. It represents the modification, on the ballistic time scale, of the *non perturbative* quantity $\tilde{f}^{(1)}_{k'}$ through the action of the electric field $\phi^{(1)}_{k}$.

Using (2.15),(2.18) and (2.19) coupled with Poisson's equation in (2.9) we get, after transforming, the following set of coefficients:

$$C_{k\omega}f_{k\omega} = C^{f}_{k\omega}f_{k\omega} - \frac{q}{m}ik\frac{\partial}{\partial v}C^{\phi}_{k\omega}\phi_{k\omega} \tag{2.25}$$

where $C^{f}_{k\omega}f_{k\omega}$ is defined through:

$$C^{f}_{k\omega}f_{k\omega} \equiv -\frac{\partial}{\partial v}\left(D_{k\omega}\frac{\partial}{\partial v} - F_{k\omega}\right)f_{k\omega} - \frac{\partial}{\partial v}\left((d^f + d^t)\frac{\partial}{\partial v} - \frac{\partial}{\partial v}(\mathcal{F}^f + \mathcal{F}^t)\right)\bar{f} \tag{2.26}$$

The various symbols in equation (2.26) are given by

$$D_{k\omega} = \frac{q^2}{m^2}\sum_{k',\omega'} g_{k-k',\omega-\omega'} k'k' \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} \tag{2.27}$$

$$F_{k\omega} = -\frac{q}{m}\sum_{k',\omega'} i\Big(k'\tilde{f}^{(2)}_{k-k',\omega-\omega'}\frac{\tilde{\phi}^{(1)}_{k'\omega'}}{\epsilon_{k'\omega'}} + (k-k')\tilde{f}^{(1)}_{k'\omega'}\frac{\tilde{\phi}^{(2)}_{k-k',\omega-\omega'}}{\epsilon_{k-k',\omega-\omega'}}\Big)\frac{1}{f_{k\omega}} \tag{2.28}$$

$$d^t\frac{\partial \bar{f}}{\partial v} = -\frac{q^2}{m^2}\sum_{k',\omega'} ik'k-k'\frac{\tilde{\phi}^{(1)}_{k'\omega'}}{\epsilon_{k'\omega'}}\frac{\tilde{\phi}^{(2)}_{k-k',\omega-\omega'}}{\epsilon_{k-k',\omega-\omega'}}$$

$$\times\Big(\frac{\partial \overline{F}_{k-k',\omega-\omega'}}{\partial v} g_{k-k',\omega-\omega'} + \frac{\partial \overline{F}_{k'\omega'}}{\partial v} g_{k'\omega'}\Big) \tag{2.29}$$

In the above equations $\tilde{f}^{(2)}_{k-k',\omega-\omega'}$ and $\tilde{\phi}^{(2)}_{k-k',\omega-\omega'}$ are *defined* through

$$\tilde{f}^{(2)}_{k-k',\omega-\omega'} = \frac{q}{m} g_{k-k',\omega-\omega'} ik\phi^{(1)}_{k\omega} + \frac{\partial \tilde{f}^{(1)*}_{k'\omega'}}{\partial v}$$

$$\tilde{\phi}^{(2)}_{k-k',\omega-\omega'} = \frac{4\pi ne}{|k-k'|^2}\int dv \tilde{f}^{(2)}_{k-k',\omega-\omega'} \tag{2.30}$$

The remaining terms satisfy

$$d^f\frac{\bar{\mathcal{E}}f}{\partial v} = \frac{q^2}{m^2}\omega_p^2\sum_{k',\omega'} ik'\frac{k-k'}{|k-k'|^2}\frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2}\int dv' \frac{g_{k-k',\omega-\omega'}}{\epsilon_{k-k',\omega-\omega'}} k'\frac{\partial f_{k\omega}}{\partial v'}$$

$$\times\Big(\frac{\partial \overline{F}_{k-k',\omega-\omega'}}{\partial v} g_{k-k',\omega-\omega'} + \frac{\partial \overline{F}_{k'\omega'}}{\partial v} g_{k'\omega'}\Big) \tag{2.31}$$

$$(\mathcal{F}^t + \mathcal{F}^f)\bar{f} = -\frac{q}{m}\omega_p^2\sum_{k',\omega'} k'\frac{k-k'}{|k-k'|^2}\frac{\langle\tilde{\phi}^*_{k'\omega'}\tilde{f}_{k'\omega'}\rangle}{\epsilon^*_{k'\omega'}\epsilon_{k-k',\omega-\omega'}}\int dv' g_{k-k',\omega-\omega'}\frac{\partial f_{k\omega}}{\partial v'} \tag{2.32}$$

The $C^{\phi}_{k\omega}$ operator is defined by:

$$C^{\phi}_{k\omega}\phi_{k\omega} \equiv (\beta_{k\omega} + \gamma_{k\omega} + \delta_{k\omega})\phi_{k\omega} \tag{2.33}$$

where

$$\beta_{k\omega} = \frac{q^2}{m^2} \sum_{k',\omega'} k'k' \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} g_{k-k',\omega-\omega'} \frac{\partial}{\partial v} g^*_{k'\omega'} \frac{\partial}{\partial v} \overline{F}^*_{k'\omega'} \tag{2.34}$$

$$\gamma_{k\omega} = \frac{q^2}{m^2} \sum_{k',\omega'} k'k' \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} \omega_p^2 \frac{i(k-k')}{|k-k'|^2} \int dv' \frac{g_{k-k',\omega-\omega'}}{\epsilon_{k-k',\omega-\omega'}} \frac{\partial}{\partial v'} g^*_{k'\omega'} \frac{\partial}{\partial v'} \overline{F}^*_{k'\omega'}$$

$$\times \left(\frac{\partial \overline{F}_{k-k',\omega-\omega'}}{\partial v} g_{k-k',\omega-\omega'} + \frac{\partial \overline{F}_{k'\omega'}}{\partial v} g_{k'\omega'} \right) \tag{2.35}$$

$$\delta_{k\omega} = \frac{q}{m} \omega_p^2 \sum_{k',\omega'} k' \frac{k-k'}{|k-k'|^2} \frac{\langle \tilde{\phi}^*_{k'\omega'} \tilde{f}_{k'\omega'} \rangle}{\epsilon^*_{k'\omega'} \epsilon_{k-k',\omega-\omega'}} \int dv' g_{k-k',\omega-\omega'} \frac{\partial}{\partial v'} g^*_{k'\omega'} \frac{\partial}{\partial v'} \overline{F}^*_{k'\omega'} \tag{2.36}$$

In appendix A we illustrate through a specific example the method by which these results are obtained. It is also helpful to use the first two columns of Fig. (3.2) which indicate diagramatically the steps in the iterative process. Time stationarity and an implicit assumption of steady state are used throughout the formulation.

The operator $C^f_{k\omega}$ has been written in the suggestive form of (2.26) to emphasize the physical origins of the individual terms. We start with an analysis of that operator. One must add, however, that the interpretations that follow are approximate. The equations as they stand are extremely complicated integro-differential equations and only under certain restrictive conditions (§2.4), can they be unfolded into the more familiar Fokker-Planck coefficients.

Not withstanding these difficulties we can interpret $D_{k\omega}$ as a generalized non-Markovian diffusion coefficient. In the long wave-length limit ($k \to 0$) and near a wave-particle resonance when $g_{k'\omega'}$ acts as a function rather than an operartor, the term describes the diffusion of the particles away from their ballistic orbits. This effect has often been interpreted as a broadening[17] of the resonance function which causes a non-linear damping of the fluctuations on the trapping time scale. (This is the time for the position of the particles to become randomized with respect to the phase of the fluctuations:$\tau_{tr} \simeq (k^2D/3)^{-1/3}$). In the same limit F can be interpreted as the drag or friction coefficient, which is the reaction of the plasma shielding cloud on the test fluctuation. With the polarization drag due to the velocity dependance of $D\,(\partial D/\partial v)$ it introduces a frequency shift in the resonance function. This effect has also been used to model the resistivity of a collisionless plasma[1].

The remaining terms arise from the self-consistent nature of the calculation. Loosely they can also be interpreted as drag and diffusion coefficients. The essential difference is that instead of being driven by the average distribution and acting on the fluctuations, for example

$$\frac{\partial}{\partial v} F f_{k\omega} \simeq \frac{\partial}{\partial v}(\frac{\partial}{\partial v}\langle f\rangle) f_{k\omega}, \quad \text{and} \quad \frac{\partial}{\partial v} D \frac{\partial}{\partial v} f_{k\omega} \simeq \frac{\partial}{\partial v}(\langle f\rangle)\frac{\partial}{\partial v} f_{k\omega},$$

they are driven *by the fluctuations* and act back on the average distribution *. This effect could be anticipated from a perturbation scheme. Consider $\partial/\partial t\, f = Cf$; C is a generalized "collision" operator. If we ensemble average this expression, and expand according to (2.3) we would get

$$\frac{\partial}{\partial t}\delta f - \langle C\rangle \delta f = \delta C\langle f\rangle$$

The right hand side is representative of the terms in question. Of more importance these terms are essential for energy and momentum conservation. Of course we can only talk of these conservation properties in the $k = 0$ limit since they are pertinent to the system as whole. For finite k, energy and momentum can "leak" out of any phase space element into neighbouring ones. However this still allows us to retain the picture of, say, fluctuations being diffused (D) and to conserve momentum these same fluctuations act back on whatever is diffusing them (through $\mathcal{F}$). The difference between the "t" and "f" superscripts is discussed in §2.4 where the $k = 0$ case is treated. The $D_{k\omega}$ and d^f terms have appeared in several theories[11–13] and are the "diffusion" and "polarization" terms of Krommes and Kleva[13].

The $C^{\phi}_{k\omega}$ operator unfortunately eludes such a straighforward interpretation. Various terms within that operator have appeared in previous theories. For example the beta term is the velocity equivalent of the drift wave "β" term which appears in [4]. There it was identified as a mode coupling contribution necessary for energy conservation. Krommes and Kleva[13] have obtained the $\beta_{k\omega}$ and $\gamma_{k\omega}$ terms in what they refer to as a "coherent" approximation of the D. I. A. . They interpret these as a ponderomotive renormalizations of the background distribution, while Dubois[12] refers to the same elements as sources of "quasi-particles". We have not been able to find a simple physical interpretation of these terms. The $\delta_{k\omega}$ contribution arises from the inclusion of the incoherent fluctuation $\tilde{f}_{k\omega}$. This term is included in the $C^{\phi}_{k\omega}$ operator rather than the $C^{f}_{k\omega}$ one, because in one dimension and in the long wavelength limit it cancels against the $\beta_{k\omega}$ and $\gamma_{k\omega}$ terms. This cancellation is shown to be a consequence of energy and momentum conservation. We elaborate on this point in §2.4.

* The analogy used is strictly valid only for discrete particle where $\langle\phi\rangle^2 \simeq \langle f\rangle$, but it allows one to describe the essential features of the physics in familiar terms.

The set of equations is closed with a knowledge of the incoherent self correlation function $\langle\tilde{f}(0)\tilde{f}(0)\rangle_k$. But even given this quantity the equations are rather complex and further approximations are necessary to make them tractable.

2.2. The Equation for the Average Distribution Function

We can make contact with the previous discussion by deriving the collision operator for $\langle f\rangle = f_0$. In the notation of §2.1 this could be written as $\langle C\rangle\langle f\rangle$. From the equation for the average distribution

$$\frac{\partial}{\partial t}f_0 = -\frac{q}{m}\frac{\partial}{\partial v}\sum_{k',\omega'} ik'\langle\phi^*_{k'\omega'}f_{k'\omega'}\rangle \tag{2.37}$$

we can use (2.19) to recast (2.37) as

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial v}J(v)$$

where

$$J(v) = \omega_p^2\frac{q}{m}\int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}\frac{k'k'}{|k'|^2}\frac{1}{|\epsilon_{k'\omega'}|^2}\int dv'(\langle\tilde{\phi}\tilde{f}(v)\rangle_{k'\omega'}Re[g_{k'\omega'}(v')\frac{\partial\overline{F}_{k'\omega'}}{\partial v'}]$$
$$- \langle\tilde{\phi}\tilde{f}(v')\rangle_{k'\omega'}Re[g_{k'\omega'}(v)\frac{\partial\overline{F}_{k'\omega'}}{\partial v}]) \tag{2.38}$$

It is fairly easy to see that if we used the discrete particle spectrum, $\langle\tilde{f}(1)\tilde{f}(2)\rangle_{k'} = n^{-1}\delta(v_1 - v_2)\langle f\rangle$, (2.38) would reduce to the Lenard-Balescu collision intergral. In this case, however, the source is the "discrete" clump. The first term is the dynamical friction due to the shielding cloud acting on the discrete fluctuation. The second is the diffusion of Quasi-Linear theory. We note that one of the effects of the renormalization is to introduce additional friction and diffusion coefficients in the equation for the the average distribution function. For example the friction term instead of being driven by the gradient of f_0 only, contains contributions from the gradient of $C^{\phi}_{k\omega}$. By the same token, the diffusive process rearranges $\overline{F}_{k\omega}$ rather than f_0.

If we define

$$N \equiv n\int dv, \qquad M \equiv nm\int dvv, \qquad E \equiv \frac{1}{2}nm\int dvv^2$$

as respectively the number, momentum and energy operators it is straightforward to show that

$$\frac{\partial}{\partial t}Nf_0 = \frac{\partial}{\partial t}Mf_0 = \frac{\partial}{\partial t}Ef_0 = 0$$

The first two properties are self evident from the structure of equation (2.38).

Let us prove the third by operating E on the RHS of (2.38). After an integration by parts and adding and subtracting ω', we are left with

$$\omega_p^2 nq \int \frac{dk'}{2\pi} \int \frac{d\omega'}{2\pi} \int dv \int dv' \frac{k'}{|k'|^2} \frac{1}{|\epsilon_{k'\omega'}|^2} (\omega' - k'v + \omega')$$

$$\times \left(2Re g_{k'\omega'}(v) \langle \tilde{\phi} \tilde{f}(v) \rangle_{k'} Re[g_{k'\omega'}(v') \frac{\partial F_{k'\omega'}}{\partial v'}] - 2Re g_{k'\omega'}(v') \langle \tilde{\phi} \tilde{f}(v') \rangle_{k'} Re[g_{k'\omega'}(v) \frac{\partial F_{k'\omega'}}{\partial v}] \right)$$

From the equation of continuity $\int dv(\omega' - k'v) Re g_{k'\omega'} \langle \tilde{f}\tilde{f} \rangle_{k'} = 0$. We are thus left with the ω' factor out of the expression $(\omega' - k'v + \omega')$. The remaining terms cancel since the v and v' integrals can be performed to yield (with opposite sign) the expression

$$\int \frac{dk'}{2\pi} \int \frac{d\omega'}{2\pi} k'\omega' \langle \tilde{\phi}^2 \rangle_{k'\omega'} \frac{Im\, \epsilon_{k'\omega'}}{|\epsilon_{k'\omega'}|}$$

2.3. Potential Energy Conservation

In the previous sections we implicitly assumed that the plasma responded adiabatically to the perturbation. By this we mean that the characteristic relaxation time of f_0, $\tau_r \simeq \lambda_{mfp}/v_{th}$, was much longer than the correlation time of the plasma excitation, τ_c. As a result we were able to neglect the time dependance of $f_0(v,t)$ when solving for $f_{k\omega}$. This assumption is violated for large wave-length, and Quasi-Linear theory attempts a correction by taking τ_r^{-1} as the Landau damping rate and incorporating it through a WKB ansatz into the Kinetic equation. The neglect of the temporal variation of f_0 does not allow for changes in the potential energy of the fluctuations: the kinetic energy can change since we are shuffling particles over phase space through the velocity operator. To have a theory which conserves *total energy* we need to include the time dependance of the average distribution when computing the plasma response[27,28]. This task can be facilitated by assuming that this time dependance is weak compared to the real frequency response of the plasma. We will interpret these discrepant time scales as the ballistic motion of the "centre of mass" of the fluctuations and the slow decay of their structure.

We can incorporate these ideas by defining a two time Laplace transform[24] such that

$$\int_0^\infty dte^{i\omega t}\frac{\partial g_k(t)}{\partial t} \simeq -1 - i\omega g'_{k\omega}(t) + \frac{\partial}{\partial t}g'_{k\omega}(t) \tag{2.39}$$

the explicit "t" dependance is now understood to operate on slowly varying quantities only.

The one point Green function now satisfies

$$\left(\frac{\partial}{\partial t} - i(\omega - kv + iC^f_{k\omega})\right)g'_{k\omega}(t) = 1 \tag{2.40}$$

We can expand $g'_{k\omega}$ in a Taylor series about $g_{k\omega}$; $g_{k\omega}$ satisfies (2.15b). Keeping only the first order correction the fluctuations are determined from

$$\begin{aligned} f_{k\omega} &= \tilde{f}_{k\omega} + \frac{q}{m}ig_{k\omega}k\left(1 - g_{k\omega}\frac{\partial}{\partial t}\right)\frac{\partial}{\partial v}\overline{F}_{k\omega}\phi_{k\omega} \\ \epsilon_{k\omega}\phi_{k\omega} &= \tilde{\phi}_{k\omega} - i\frac{\partial\epsilon_{k\omega}}{\partial\omega}\frac{\partial\phi_{k\omega}}{\partial t} - i\phi_{k\omega}\frac{\partial^2\epsilon_{k\omega}}{\partial\omega\partial t} \end{aligned} \tag{2.41}$$

In the weak turbulence limit of Quasi-Linear theory we would get the wave Kinetic equation as

$$\frac{\partial}{\partial t}|\phi_{k\omega}|^2 = 2Re\left(\epsilon_{k\omega} - i\frac{\partial^2\epsilon_{k\omega}}{\partial\omega\partial t}\epsilon_{k\omega}\right)\left(i\frac{\partial\epsilon_{k\omega}}{\partial\omega}\right)^{-1}|\phi_{k\omega}|^2 \tag{2.42}$$

with damping coefficient given by $\simeq -\epsilon^i_{k\omega}/(\partial/\partial\omega\,\epsilon^r_{k\omega})$.

When we have a source, charge continuity demands that

$$\tilde{J} + \frac{\partial\mathfrak{D}}{\partial t} = 0, \qquad \frac{\partial\mathfrak{D}}{\partial x} = \tilde{\rho} \tag{2.43}$$

$\tilde{J}$ is the source current, $\tilde{\rho}$ the source charge, and $\mathfrak{D}$ the electric displacement vector. Fourier transforming this expression on the same lines as above we get an expression for the current

$$\tilde{J}_{k\omega} = i\omega\epsilon_{k\omega}E_{k\omega} - \frac{\partial E_{k\omega}}{\partial t}\frac{\partial}{\partial\omega}(\omega\epsilon_{k\omega}) \tag{2.44}$$

Now if we use (2.41) in (2.33) rather than (2.15) we obtain a Kinetic equation which explicitly retains the variation of the electric field. With the help of (2.44) we can then show that for an arbitrary source term (which obeys the equation of continuity)

$$\frac{\partial}{\partial t}Ef_0 = -\sum_{k',\omega'}k'k'\frac{|\phi_{k'\omega'}|^2}{8\pi} \tag{2.45}$$

2.4. Interpretation of the Equations

One can gain considerable insight by looking at the long wavelength limit of the above set of equations. For simplicity, and to make contact with previous theories$rf25$, we will consider the discrete particle case where the self correlation is given by

$$\langle \tilde{f}(1)\tilde{f}(2)\rangle_{k\omega} = \frac{2\pi}{n}(g_{k\omega}(1) + g^*_{k\omega}(2))\delta(v_1 - v_2)\langle f\rangle \tag{2.46}$$

We will assume $\overline{F}_{k\omega} \simeq f_0$ and take $g_{k\omega} \simeq 1/(\omega - kv + i\epsilon)$. These assumptions do not make any of the underlying physics less ge.ieral.

If we take the $k = 0$ limit with $lim_{k,\omega\to 0} f_{k\omega} = f_0^1$, and

$$\langle \tilde{f}^{(1)}\tilde{f}^{(2)*}\rangle_{k'\omega'} + \langle \tilde{f}^{(2)}\tilde{f}^{(1)*}\rangle_{k'\omega'} = \frac{2\pi}{n}(g_{k'\omega'}(1) + g^*_{k'\omega'}(2))f_0^1 \tag{2.47}$$

(relation (2.47) is demonstrated in Appendix B) equation (2.26) for $C^f_{k\omega}$ reduces to

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial v}F - \frac{\partial}{\partial v}D\frac{\partial}{\partial v}\right)f_0^1 = \left(-\frac{\partial}{\partial v}(\mathcal{F}^t + \mathcal{F}^f) + \frac{\partial}{\partial v}(d^t + d^f)\frac{\partial}{\partial v}\right)f_0^0 \tag{2.48}$$

The expression for $\mathcal{F}$ (equation (2.32) has been expanded into two terms by noting that

$$Re(i\frac{\chi^*_{k\omega}}{\epsilon^*_{k\omega}\epsilon^*_{k\omega}}) = \frac{Im\chi_{k\omega}}{|\epsilon_{k\omega}|^2} - [\frac{\chi_{k\omega}}{\epsilon_{k\omega}} + \frac{\chi^*_{k\omega}}{\epsilon^*_{k\omega}}]\frac{Im\epsilon_{k\omega}}{|\epsilon_{k\omega}|^2} \tag{2.49}$$

χ is defined by (2.52).

We can reproduce this equation from a simple linearization of the Lenard-Balescu equation. Let us write the average distribution function as a series expansion

$$f_0 = f_0^0 + f_0^1 + \ldots \tag{2.50}$$

f_0^0 represents the background distribution, while f_0^1 represents the $0'th$ fourier component of the fluctuations. Typically, f_0^1 could be a disturbance due to a very long wave-length fluctuation (such as an eigen-mode of the dispersion relation). The presence of this small amplitude disturbance implies that the dielectric ϵ will have a perturbative component due to f_0^1. That is we can write

$$\epsilon_{k\omega} = \epsilon^0_{k\omega} + \chi^1_{k\omega} + \ldots = 1 + \chi^0_{k\omega} + \chi^1_{k\omega} + \ldots \tag{2.51}$$

where χ is the standard succeptibility, defined by

$$\chi^1_{k\omega} = \frac{\omega_p^2}{|k|^2} \int dv \, \frac{k\partial/\partial v_1 f_0^1}{\omega - kv + i\delta} \tag{2.52}$$

The fluctuating fields can be expanded in a similar fashion. In this case the "1" superscript might represent the result of ballistic motion,while the "2" superscript the distortion to these orbits due to the presence of f_0^1. We note that in the spirit of a test particle picture we would expect second order perturbed quantities to be made up of two distinct physical processes: the first would affect the "test" particle while the second would affect the "field" particle. Schematically if $\phi \simeq \phi^{test} e^{-r/\lambda}$ then the perturbation will affect *both* ϕ^{test} and the coherent (shielding) response.

If we linearize

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial v} F f + \frac{\partial}{\partial v} D \frac{\partial}{\partial v} f \tag{2.53a}$$

$$Ff = \frac{q}{m} \sum_{k',\omega'} k' \frac{\epsilon^i_{k'\omega'}}{|\epsilon_{k'\omega'}|^2} \langle \tilde{\phi}^*_{k'\omega'} \tilde{f}_{k'\omega'} \rangle \qquad D = \sum_{k',\omega'} \frac{q^2}{m^2} \pi \delta(\omega' - k'v) \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} \tag{2.53b}$$

according to the above prescription we will immediately recover (2.48) with the same *coefficients* as obtained through the renormalization. The details of the calculation are presented in Appendix B.

The physical interpretation of the terms is now simple: D and F are the standard diffusion and friction coefficients in the absence of the perturbation. d is the modification to the diffusion coefficient due to the perturbation. As previously indicated it consists of two terms, one decribing the rearrangement of the test particles (perturbation of the orbits:d^t), the other describing the distortion of the shielding cloud (d^f). $\mathcal{F}$ is the modification to the drag coeifficient, and likewise has two components.

This distinction is important in terms of energy and momentun conservation. It is straight forward to show by taking the v^2, and v integral of equation (2.48) that the conservation properties are achieved through the following cancellation of individual terms:

$$\begin{aligned} (N;M;E)\left(\frac{\partial}{\partial v} F f_0^1 - \frac{\partial}{\partial v} d^t \frac{\partial}{\partial v} f_0^0\right) &= 0 \\ (N;M;E)\left(\frac{\partial}{\partial v} D \frac{\partial}{\partial v} f_0^1 - \frac{\partial}{\partial v} \mathcal{F}^t f_0^0\right) &= 0 \\ (N;M;E)\left(\frac{\partial}{\partial v} \mathcal{F}^f f_0^0 - \frac{\partial}{\partial v} d^f \frac{\partial}{\partial v} f_0^0\right) &= 0 \end{aligned} \tag{2.54}$$

Field and test perturbations balance independantly.

In one dimension, when we treat the resonance functions as approximately $(\omega - kv)^{-1}$, the collision operator exhibits one further property; namely

$$\lim_{k,\omega \to 0} \quad C_{k\omega} f_{k\omega} \to 0 \tag{2.55}$$

The various terms in the $C^f_{k\omega}$ operator cancel in the same pairs as in Eq. (2.54), while the $\beta_{k\omega}$, $\gamma_{k\omega}$ and $\delta_{k\omega}$ in the $C^\phi_{k\omega}$ operator can also be shown to pair, and cancel. This cancellation is easily reconcilled on physical grounds. Collision like processes cannot change the average distribution in one dimension since momentum constraints insure that an encounter between two particles moving at v and v' will result in the same velocity partitioning after the collision. In higher dimensions or for different mass encounters this is not the case. Equally, keeping the broadened resonance functions etc. leads to a non-zero operator since this is equivalent to taking *three body* encounters into account. Finally we point out that a plasma has the added capability, in the presence of a wave, of transmitting momentum through non-resonant interactions. This would also invalidate the previous considerations; the effect, however, is not included in our collision operator since we do not consider the zeroes of the dielectric.

We are left with a clear physical picture of the operator $C^f_{k\omega}$: it describes the divergence of "test" particles away from their ballistic orbit due to their interaction with the electric fields of "field" particles. However because it is a self-consistent calculation these same particles, to conserve momentum and energy, act back on the plasma. Clearly when $k \neq 0$ there are more complicated effects taking place. In particular the probing nature of the k wave vector through the $k - k'$ convolution is not self evident. However we still believe these interpretations are helpful in understanding the fundamental actions of the operator.

Chapter 3

Two Point Equation

In this Chapter we derive equations for the equal time and two time two point equations. The validity of the expansion parameter (λ) used in the one point equation is examined. We show that on the short time scale ($t/\tau_c \simeq 1$) this expansion is meaningfull while on the long one ($t/\tau_{tr} \simeq 1$) some of the terms left out in Chapter 2 become of the same order as the "collision" integral C_k. These terms were excluded because they could not be expressed as a phaseless factor operating on f_k or E_k. The main feature of the two point renormalization is to capture any contribution from these elements. We go to a two point formulation because it is only by squaring such terms that their phases can be made to cancel. The intrinsic difference between the *equal time* and *two time* two point equation emerges quite simply from the analysis. In particular the singular behaviour of the equal time equation is shown to be a direct consequence of phase space conservation. The relationship of the iterative scheme to the more conventional expansions, such as the BBGKY hierarchy, is demonstrated.

3.1. Phase Space Conservation

Let us start by considering the exact two point equation (with spatial homogeneity)

$$
\begin{aligned}
(\frac{\partial}{\partial t} + v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2})\langle\delta f(1)\delta f(2)\rangle = &-\frac{q}{m}\langle\delta E(1)\delta f(2)\rangle\frac{\partial}{\partial v_1}\langle f\rangle \\
&-\frac{q}{m}\frac{\partial}{\partial v_1}\langle\delta E(1)\delta f(2)\delta f(1)\rangle + (1 \leftrightarrow 2)
\end{aligned}
\tag{3.1}
$$

A standard weak turbulence expansion would assume that $|\delta E|^2 \ll mv_{th}^2$, and use δE as an expansion parameter. The linearized solution, which neglects the "third" order terms has a non-integrable singularity as v_1 approaches v_2. (For a stationary state the left hand side goes to zero while the right hand side does not.) A renormalization takes this into account and incorporates some of the "third" order term into the left hand operator so that its inversion will not be singular. We contend that this process is still inadequate.

Suppose that through some clever scheme we manage to incorporate the *exact* third order result and proceed to solve the equation. Since the singular behaviour arises for small separation we can change to "+, −" coordinates, and neglect the "+" contribution. Let

$$\begin{aligned} x_{\pm} &= x_1 \pm x_2 \\ v_{\pm} &= v_1 \pm v_2 \\ t_{\pm} &= t_1 \pm t_2 \end{aligned} \tag{3.2}$$

Equation (3.1) can now be written as

$$\left(\frac{\partial}{\partial t_+}\langle + v_-\frac{\partial}{\partial x_-}\langle + \frac{q}{m}\frac{\partial}{\partial v_-}\langle\delta E(1) - \frac{q}{m}\frac{\partial}{\partial v_-}\langle\delta E(2)\right)\delta f\delta f|v_-, x_-, t_+\rangle = S \tag{3.3}$$

with

$$S = -\frac{q}{m}\langle\delta E(1)\delta f(2)\rangle\frac{\partial}{\partial v_1}\langle f\rangle - \langle\delta E(2)\delta f(1)\rangle\frac{\partial}{\partial v_2}\langle f\rangle \tag{3.4}$$

But when $v_-, x_- \rightarrow 0$ we are once again left with the singular behaviour since the non-linear terms exactly cancel, while the right hand side which is independant of the relative coordinate does not. (In Chapter 4 we identify the latter as a source term, S, for the fluctuations.)

It is not hard to trace the origin of this behaviour. The Vlasov equation preserves phase space density along particle orbits and the singular behaviour is just an alternative way of formulating that same statement. Consider the exact distribution $f(x, v, t)$; the conservation property can be stated as

$$\frac{d}{dt}f(x(t), v(t), t) = 0 \tag{3.5}$$

where the differential is now taken along the particles orbit. Multiplying the above equation by f, ensemble averaging and integrating over the velocity coordinate we get

$$\int dv(\frac{\partial}{\partial t}\langle\delta f^2\rangle + \frac{\partial}{\partial t}\langle f\rangle^2) = 0 \tag{3.6}$$

Using

$$\frac{\partial}{\partial t}\langle f\rangle = -\frac{q}{m}\frac{\partial}{\partial v}\langle \delta E \delta f\rangle \tag{3.7}$$

and integrating the second term by parts we get

$$\int dv\left(\frac{\partial}{\partial t}\langle \delta f^2\rangle + 2\frac{q}{m}\langle \delta E\delta f\rangle\frac{\partial}{\partial v}\langle f\rangle\right) = 0 \tag{3.8}$$

This is equation (3.3) in the limit of $v_-, x_- \to 0$. The integral over velocity averages the "+" coordinate, which was previously neglected. The important point to note is that no perturbative scheme to *any* order will get rid of the singular effect. It is entrenched as a basic property of the equation. We can even go further and state that any approximate set of equations which does not conserve this property is incapable of describing small scale fluctuations in a plasma.

We will show that the two point renormalization preserves the singular nature of the original equation (3.3). On the other hand the two point equation which defines the "coherent" response does not, and is therefore inadequate for the description of small scale fluctuations. Schematically, the difference appears in the following way. Eq. (3.3) can be written as

$$\left(\frac{\partial}{\partial t} + T_{12}\right)\langle \delta f\delta f\rangle = S \qquad T_{12}\to 0 \quad x_-, v_- \to 0 \tag{3.9}$$

The precise details of the LHS operator are not important at this point. The fundamental property we wish to focus on is the vanishing of T_{12} for small separation. In the simplest case T_{12} might represent the two point turbulent diffusion operator of Ref. [4]:

$$v_-\frac{\partial}{\partial x_-} + \frac{\partial}{\partial v_1}D_{11}\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_1}D_{12}\frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_2}D_{21}\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}D_{22}\frac{\partial}{\partial v_2} \tag{3.10}$$

In the relative coordinate system the diffusion coefficients cancel as x_- and v_- approach zero.

$\langle \delta f^c \delta f^c\rangle$ satisfies a similar equation

$$\left(\frac{\partial}{\partial t} + T_1 + T_2\right)\langle \delta f^c\delta f^c\rangle = S \qquad T_1 + T_2 \neq 0 \quad x_-, v_- \to 0 \tag{3.11}$$

where at the level of simplification of (3.10), $T_1 + T_2$ would be the T_{12} operator without the bivariate diffusion elements D_{12} and D_{21}. Equation (3.11) predicts no singular behaviour since the LHS does not dissapear for small separation. As will become apparent (3.11) is correct in the long wave-length limit where it describes the collective behaviour of the plasma. (By which we mean the eigenmodes of the dispersion relation.)

Notice that we have partitioned the two point equation into two distinct elements, Eqs. (3.3) and (3.4). We see from (3.6) and (3.8) that the first preserves the square of the fluctuating part of the distribution while the second is related to the conservationof $\langle f \rangle^2$. Physically this is a natural division and allows us to identify S, as defined through (3.4), as the source of fluctuations. The LHS of (3.3) is some none-linear operator which acts on the fluctuations through the self consistent interactions of the turbulent electric fields set up by the fluctuations. This operator might destroy (through turbulent diffusion, ballistic motion, etc.) the spectrum or enhance it through some kind of non-linear instability. The RHS of (3.3), on the other hand, does not act on the fluctuations directly, but through the indirect mechanism of changing the average distribution. When the gradients of the average distribution are modified a mixing process occurs as elements of phase space rearrange to generate the new average distribution. The rearrangement creates new fluctuations and a steady state can be envisioned as a result of the competition between creation (RHS) and destruction (LHS) of the fluctuations.

At this point we briefly look at the question of "ordering". Let us take as our ordering parameter δE. Furthermore let us assume that $\langle \delta f \delta f \rangle$ is $\simeq \langle \delta f^c \delta f^c \rangle$. That is the singular behaviour is negligible, and to "lowest" order the plasma can be described by $\langle \delta f^c \delta f^c \rangle$. The governing equation is (3.6). S is second order while for x_-, v_- large, T_1 and T_2 are nominally of first order (proportional to kv_-). Thus $\langle \delta f \delta f \rangle$ is second order. However as v_- and x_- tend to zero the dominant contribution to the T operators comes from the diffusion and drag coefficients (the $C^f_{k\omega}$ operator). We can get a rough estimate of its dependance on the expansion parameter by setting $C^f \simeq 1/\tau_{tr} \simeq (Dk^2)^{1/3} \simeq \delta E^{1/2}$, from which $\langle \delta f \delta f \rangle$ is proportional to $\delta E^{3/2}$.

Clearly we are witnessing the breakdown of the expansion parameter. We started an expansion in integer powers of δE and end up with a result which could not under any circumstances be obtained from such an expasion. Thus through a process of contradiction we are once again forced to conclude that in this regime $\langle \delta f^c \delta f^c \rangle$ is incapable of describing the total plasma response. In fact we can estimate the ratio of the two terms as

$$\langle \delta f \delta f \rangle / \langle \delta f^c \delta f^c \rangle \simeq T_{12}/(T_1 + T_2) \simeq (\Delta v_{ph}/v_{tr})^2 \gg 1$$

(v_{tr} is a correlation length in v_-, which is interpreted as a trapping length in velocity space, and is typically much less than the thermal spread.) We return to these ideas in Chapter 4 where the formal solution of the two point equation is addressed.

3.2. The Singular Behaviour: An Alternative Perspective

We have already seen that the singular behaviour in Eq. (3.1) arises because the lowest order operartor vanishes as $v_- \to 0$. Our arguments focused primarily on the structure of the equation in *velocity* space. We now propose to make similar arguments but in the *time* domain. We postulate the existence of two disparate time scales. These have been identified in Chapter 2 as the period associated with the ballistic motion of the centre of mass of the clump ($(k_0 v_+)^{-1} \simeq \lambda_d/\Delta v_{ph}$) and the characteristic decay of such structures ($(k_0 v_-)^{-1} \simeq \lambda_d/v_{tr}$). Here k_0 is the average wavenumber of the spectrum which is set approximately equal to the inverse of the debye length (λ_d^{-1}). Notice that this time scaling is directly related to the two velocity scales and that the spatial scale does not influence the temporal ones. At first glance one might expect a disparity in the spatial scale also. Further analysis will show, however, that this is not the case. The spatial scale in the problem is determined by the RHS of (3.3) only: the correlation length of the two point function which characteristically is the debye length.

We now consider Eq.(2.6) and Eq.(2.12). Suppose λ is set equal to 1 in Eq.(2.6). The equation for $\tilde{f}_k(t)$ can then be written as

$$\begin{aligned}
&\frac{\partial \tilde{f}_k^{(1)}(t)}{\partial t} + ikv\tilde{f}_k^{(1)}(t) + \int_0^t dt' C_k^f(t-t')\tilde{f}_k^{(1)}(t') = \mathcal{S}_k(t) \\
&\mathcal{S}_k(t) = -\frac{q}{m}\frac{\partial}{\partial v}\sum_{k'} E_{k'}^{(1)}(t) f_{k-k'}^{(1)}(t) \\
&\tilde{f}_k^{(1)}(t{=}0) = \tilde{f}_k^{(1)}(0)
\end{aligned} \tag{3.12}$$

with solution

$$\tilde{f}_k(t) = g_k(t)\tilde{f}_k(0) + \int_0^t dt' g_k(t-t')\mathcal{S}_k(t') \tag{3.13}$$

where $g_k(t)$ is the Green's function which satisfies (2.13). $\mathcal{S}_k(t)$ was treated as a second order term in Chapter 1 (being proportional to $E_{k'}^{(1)} f_{k-k'}^{(1)}$. To obtain the correlation function we can square this expression and ensemble average to get

$$\langle\tilde{f}_k(1,t_1)\tilde{f}^*_k(2,t_2)\rangle \simeq g_k(1,t_1)g^*_k(2,t_2)\langle\tilde{f}_k(1,0)\tilde{f}^*_k(2,0)\rangle + \int_0^{t_1} ds g_k(1,s)\int_0^{t_2} du g^*_k(2,u)\langle\mathcal{F}_k(1,t_1{-}s)\mathcal{F}^*_k(2,t_2{-}u)\rangle \tag{3.14}$$

We have assumed *a priori* that cross terms are not important in the formulation. The rationalle for such a step will become clear as we proceed.

For stationary turbulence we can write

$$\begin{aligned}\langle\mathcal{F}_k(1,t_1-s)\mathcal{F}^*_k(2,t_2-u)\rangle &\equiv\langle\mathcal{F}_k(1)\mathcal{F}^*_k(2)|t_1{-}t_2{-}(s{-}u)\rangle\\ \langle\tilde{f}_k(1,t_1)\tilde{f}^*_k(2,t_2)\rangle &\equiv\langle\tilde{f}_k(1)\tilde{f}^*_k(2)|t_1{-}t_2\rangle\\ \langle\tilde{f}_k(1,0)\tilde{f}^*_k(2,0)\rangle &\equiv\langle\tilde{f}_k(1)\tilde{f}^*_k(2)|t_1{=}t_2\rangle\end{aligned} \tag{3.15}$$

The first of these expressions is peaked about $t_1{-}t_2{-}(s{-}u) = 0$ with width proportional to $k_0\Delta v_{ph}{}^{-1}$. This is a statement that the characteristic *time* scale associated with the pair correlation function is approximately the inverse plasma frequency ω_p^{-1}. We will show in (§4.3) and (§3.3) that $\langle\mathcal{F}_k(1)\mathcal{F}^*_k(2)\rangle$ can be expressed as two terms:

$$\langle\mathcal{F}_k(1)\mathcal{F}^*_k(2)\rangle = \langle\tilde{\mathcal{F}}_k(1)\tilde{\mathcal{F}}^*_k(2)\rangle + \langle\overline{\mathcal{F}}_k(1)\overline{\mathcal{F}}^*_k(2)\rangle$$

$\langle\tilde{\mathcal{F}}_k(1)\tilde{\mathcal{F}}^*_k(2)\rangle$ and $\langle\overline{\mathcal{F}}_k(1)\overline{\mathcal{F}}^*_k(2)\rangle$ are defined through (5.33). It turns out that when $\{x_1, v_1\}$ approach $\{x_2, v_2\}$ the action of the $\langle\tilde{\mathcal{F}}_k(1)\tilde{\mathcal{F}}^*_k(2)\rangle$ contribution is to cancel the renormalization in the g_k operators. This result is proved in §5.3. At this stage we only remark that this is nothing but the cancellation of the non-linear terms referred to in §3.1.

Thus, for small separations, we take $g_k(1,t) \simeq \exp ikv_1t$ and use $\tau = (t_1{-}t_2{-}(s{-}u))$ to recast (3.14) into

$$\langle\tilde{f}_k(1)\tilde{f}^*_k(2)|t_1{-}t_2\rangle \simeq \exp ik\{(v_1{-}v_2)t_1{+}v_2(t_1{-}t_2)\}\langle\tilde{f}_k(1)\tilde{f}^*_k(2)|t_1{=}t_2\rangle + \int_0^{t_1} ds \exp ik(v_1{-}v_2)s\int_{\tau_1}^{\tau_2} d\tau \exp ikv_2\{(t_1{-}t_2){-}\tau\}\langle\overline{\mathcal{F}}_k(1)\overline{\mathcal{F}}^*_k(2)|\tau\rangle \tag{3.16}$$

where $\tau_1 = \{t_1{-}t_2{-}s\}$ and $\tau_2 = \{t_1{-}t_2{-}(s{-}t_2)\}$.

We will discuss the properties of (3.16) in terms of t_+ $(t_1{+}t_2)$ and t_- $(t_1\text{-}t_2)$ coordinates. Several points emerge quite simply from this expression.

(i) If $t_- \simeq 0$ and $t_+ \gg \omega_p^{-1}$ we can set $\tau_1 = -\infty$ and $\tau_2 = \infty$ since $\langle \bar{\mathcal{F}}(1)\bar{\mathcal{F}}(2)|\tau\rangle$ is peaked about $\tau \simeq 0$ with width $O(\omega_p^{-1})$. Under these conditions the integral over ds becomes independant of the integral over $d\tau$ and for $v_- \simeq 0$ the ds integration is proportional to t_+.

(ii) For $t_+ \ll \tau_{tr}$ the ballistic contribution dominates in (3.16). The second term is small (being proportionalto t_+), since the secular contribution has not had time to grow. Note that the ballistic contribution is sharply peaked about t_- because of the $\exp ikv_+t_-$ term. This is straightforward to see if, for the purposes of this discussion, we assume that the distribution for both velocity scales is Gaussian (with width v_{th} and v_{tr}). When integrating over the v_+ scale (to obtain the potential spectrum) the result will decay as $\simeq \exp-(kv_{th}t_-)^2 \simeq \exp-(\omega_p t_-)^2$. Thus relevant quantities such as the spectrum decay in a couple of plasma times in the t_- coordinate. In the t_+ coordinate, however, the spectrtum decays on the trapping scale since the integral over v_- is proportional to $\simeq \exp-(kv_-t_+)^2 \simeq \exp-(t_+/\tau_{tr})^2$.

(iii) For $t_+ \simeq \tau_{tr}$ the ballistic portion will have decayed. The second term will likewise be small *unless* $t_- \simeq 0$ and $v_- \simeq 0$. If t_1 and t_2 are large ($> \tau_{tr}$) but $t_1 - t_2 \simeq \omega_p^{-1}$ the second term develops a secular contribution which as t_+ tends to infinity generates a term $\simeq \delta(v_-)$. This result is *not* a failure of the renormalization but a direct consequence of phase space conservation. In the t_- coordinate the arguments used in (ii) can be applied to the $\exp ikv_2t_-$ factor to show that the secular contribution is peaked in t_- with width ω_p^{-1}.

(iv) It becomes clear that the cross terms which were dropped in (3.14) contribute in the nebulous regime of $\omega_p^{-1} \ll t_+ \ll \tau_{tr}$. But since we will only be using equations which require information from the two outer limits of the inequality we can neglect these terms.

We can now analyze the ordering parameter λ. For $t_+\omega_p \simeq 1$, $\lambda \ll 1$ since the contribution from $\langle \bar{\mathcal{F}}(1)\bar{\mathcal{F}}(2)\rangle$ has not had time to grow, being proportional to t_+. For $t_+/\tau_{tr} \simeq 1$ and periods greater than τ_{tr}, $\lambda \simeq 1$. By that stage the initial condition has decayed and the solution is described through the $\langle \bar{\mathcal{F}}\bar{\mathcal{F}}\rangle$ term. In a steady state one can envision the solution to $\langle \tilde{f}\tilde{f}\rangle$ through Figs. (3.1). The secular and ballistic contribution always add up to the same *total* solution. This means that on the ω_p^{-1} time scale we can use the ballistic representation of the solution while for much longer periods we have to revert to the formulation which contains the secular result. This is in fact the procedure we adopt. It is important to realize that when calculating the coefficients in the renormalization it is perfectly legitimate to use the ballistic representation since the coefficients depend on factors proportional to $\langle \tilde{f}(t_1)\tilde{f}(t_2)\rangle$ (and its

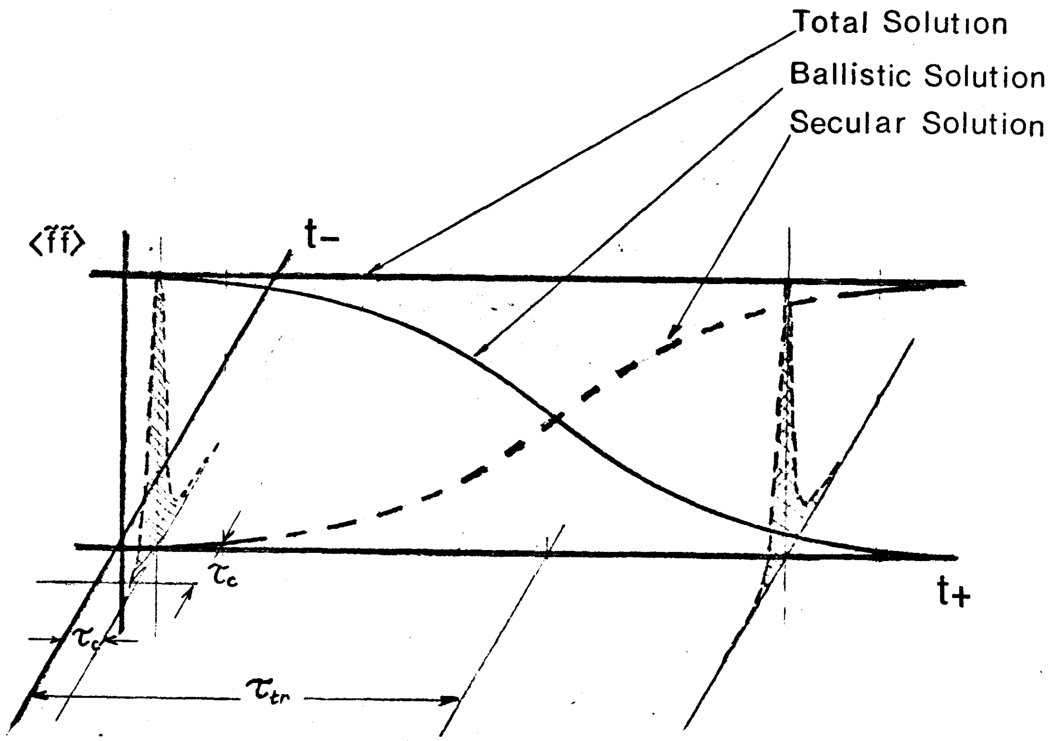

Fig. 3.1 Time History of $\langle \tilde{f}(t_1)\tilde{f}(t_2)\rangle$ Solution

various velocity moments): these do not require a knowledge of the correlation function for periods larger than a couple of plasma frequencies.

We conclude by adding that the ordering associated with the $\langle \tilde{\mathcal{F}}(1)\tilde{\mathcal{F}}(2)\rangle$ term is even more subtle than indicated by the previous discussion. We will show (Chapter 5) that this term, which was nominally of "fourth-order" on the ballistic time scale, actually becomes "second-order" on the equal or clump time scale. This effect coupled with the secular contribution generates the "clump" spectrum.

3.3. Two Point Renormalization: Two Time

The two point renormalization is performed by taking the one point equation of Chapter 2 for $f_k(t_1)$, multiplying by $f_k^*(t_2)$ and ensemble averaging the result. In this case we will retain the non-linear terms proportional to λ in (2.6) as our ultimate goal is to obtain an equal time equation for $\langle f_k(1)f_k(2)\rangle$. We know from the discussion in the previous section that in this regime these terms are part of the mechanism which generates the clump spectrum.

We write the equation for $\langle f_k(t_1)f_k^*(t_2)\rangle$ as:

$$\frac{\partial}{\partial t_1}\langle f_k(t_1)f_k^*(t_2)\rangle + ikv_1\langle f_k(t_1)f_k^*(t_2)\rangle + \frac{q}{m}\langle E_k(t_1)f_k^*(t_2)\rangle\frac{\partial f_0}{\partial v_1} = \\ -\int_0^{t_1} dt' C_{11}(k, t_1-t')\langle f_k(t')f_k^*(t_2)\rangle - \int_0^{t_2} dt' C_{12}(k, t_1-t', t'-t_2)\langle f_k(t')f_k^*(t_2)\rangle \tag{3.17}$$

where

$$\int_0^{t_2} dt' C_{12}(k, t_1-t', t'-t_2)\langle f_k^{(1)}(t')f_k^{(1)*}(t_2)\rangle \equiv \left(\frac{q}{m}\frac{\partial}{\partial v_1}\sum_{k'}\langle E_{k'}^{(1)}(t_1)f_{k-k'}^{(1)}(t_1)f_k^{(2)*}(t_2)\rangle\right)_{Phase\,Coherent} \tag{3.18}$$

Let us note, *en passant*, the following points: (1) $C_{11}(k, t-t')$ is the C_k operator of Chapter 2. (2) By assuming that the collision integrals can be expressed as the difference of time coordinates we have already made a statement of time sationarity. In general $C_{11}(t-t')$ would be expressed as $C_{11}(t, t')$. (3) In (3.18) the first term which will contribute to the $E_{k'}^{(1)}f_{k-k'}^{(1)}$ product is $f_k^{(2)*}$. Here there is an implicit assumption that the phases of the terms with "$^{(1)}$" superscripts are randomly distributed so that the $\langle E^{(1)}f^{(1)}f^{(1)}\rangle$ term does not contribute.

We select, as before, only those terms out of $f_k^{(2)*}(t_2)$ whose phases will cancel the phases of $E_{k'}^{(1)}$ and $f_{k-k'}^{(1)}$. That is $f_k^{(2)*}(t_2)$ is given by:

$$f_k^{c(2)*}(t_2) = -\int_0^{t_2} dt' g_k(t_2-t') \\ \times\frac{q}{m}\left(k\frac{\partial}{\partial v}(f_0 + C_k^{\phi}(t'))\phi_k^{(2)*}(t') + (k-k')\frac{\partial}{\partial v_2}f_{k'}^{c(1)*}(t')\phi_{k-k'}^{(1)*}(t') + k'\frac{\partial}{\partial v_2}f_{k-k'}^{(1)*}(t')\phi_{k'}^{(1)*}(t')\right) \tag{3.19}$$

and

$$\tilde{f}_k^{(2)*}(t_2) = -\int_0^{t_2} dt' g_k(t_2-t') i\frac{q}{m}(k-k')\frac{\partial}{\partial v_2}\tilde{f}_{k'}^{(1)*}(t')\phi_{k-k'}^{(1)*}(t') \tag{3.20}$$

(3.19) and (3.20) are just the one point results of Chapter 2 for the mode k (Eqs. (2.18) and (2.19)). When this result is substituted in (3.18) coupled with Poisson's equation we obtain the *two time*, two point, equation. The domain of validity of the equation is $t_1 > t_2 \geq 0$.

Unless $t_1 \simeq t_2$ the C_{12} operator does not contribute to the equation ($\lambda \ll 1$). In the Markovian limit one can show that the cross operators are a function of exp $ik(x_- - v_+ t_-)$. For large t_- ($t_-\omega_p > 1$) these terms can be neglected. This is just a statement that the action of the fields at point 2 and time t_2 will not appreciably influence the motion of point 1 at time t_1 since these points will be separated by a large distance $v_+(t_1 - t_2)$. Thus for the analysis of *localized fluctuations* we will drop the cross terms from the two time formulation.The two time equation therfore reduces to a product of one point equations. This has to be solved with initial condition $\langle f(t_1)f(t_2)|t_1 = t_2\rangle$.

3.4. Two Point Renormalization: Equal Time

To obtain the equal time equation we take the one point equation for $f(t_2)$, perform the same excercise as in §3.3, and add the result to (3.17). We use

$$\frac{\partial}{\partial t}\langle f(1,t)f(2,t)\rangle \equiv \langle f(1,t)\frac{\partial}{\partial t}f(2,t)\rangle + \langle f(2,t)\frac{\partial}{\partial t}f(1,t)\rangle$$

to express the two time derivatives as a single operator. The next step is to take the limit $t_1, t_2 \to \infty$ for the arguments in the time integrals of the collision operators. This is consistent with our two time scaling procedure in which we assume that $\partial/\partial t_+ \ll \partial/\partial t_-$. It is also motivated by the discussion in §3.2 were we saw that we had to approach the asymptotic limit in t_+ to obtain the secular contribution. We will assume that the resulting equations are still valid for weak departures from steady state and stationarity.

We expand field and distribution function in a Fourier series such that

$$\delta f(x,t) = \sum_{k\omega} f_{k\omega} \exp i(kx - \omega t)$$

$$\delta E(x,t) = \sum_{k\omega} E_{k\omega} \exp i(kx - \omega t) \tag{3.21}$$

Using (3.17) and taking the limit $t_1 = t_2 = \infty$ we get the following equal time equation:

$$\left(\frac{\partial}{\partial t} + ikv_1\right)\langle f_k(1)f_k^*(2)|t\rangle + \sum_{\omega}(C_{11}^f + C_{12}^f)\langle f_{k\omega}(1)f_{k\omega}^*(2)|t\rangle + (1 \leftrightarrow 2) = S_k$$

$$+\frac{q}{m}ik\frac{\partial}{\partial v_1}\sum_{\omega} C_{11}^{\phi}\langle \phi_{k\omega}f_{k\omega}^*(2)|t\rangle - \frac{q}{m}i(k-k')\frac{\partial}{\partial v_1}\sum_{\omega} C_{12}^{\phi}\langle f_{k\omega}(1)\phi_{k\omega}^*|t\rangle + (1 \leftrightarrow 2) \tag{3.22}$$

Here C_{11} is the $C_{k\omega}$ operator of Chapter 2, and S_k is the Fourier version of (3.4). The intrinsic non-Markovian nature of the equation is apparent in (3.22): we have not, as yet, managed to decouple the slow and fast time scales. At the end of this section we present an approximation which allows such a simplification.

Eq. (3.17) is very similar to the product of two one point equations except for the bivariate operators which originate from the iteration of the incoherent terms. These are defined by

$$C_{12}^{f}\langle f_{k\omega}(1)f_{k\omega}^{*}(2)\rangle \equiv \left(\frac{\partial}{\partial v_1}D_{12}*\frac{\partial}{\partial v_2}-\frac{\partial}{\partial v_1}F_{12}*\right)\langle f_{k-k',\omega-\omega'}(1)f_{k-k',\omega-\omega'}^{*}\rangle + \frac{\partial}{\partial v_1}\left(\langle(d_{12}^{t}*+d_{12}^{f}*)f_{k-k',\omega-\omega'}(1)\rangle\right)\frac{\partial\bar{f}}{\partial v_2} \tag{3.23}$$

The "$*$" represents a convolution of the $\{k',\omega'\}$ sum with the correlation function at $\{k-k',\omega-\omega'\}$. That is

$$D_{12}*\langle f_{k-k',\omega-\omega'}(1)f_{k-k',\omega-\omega'}^{*}(2)\rangle = -\frac{q^2}{m^2}\sum_{k',\omega'}g_{k\omega}^{*}(2)k'k'\frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2}\langle f_{k-k',\omega-\omega'}(1)f_{k-k',\omega-\omega'}^{*}(2)\rangle \tag{3.24}$$

$$F_{12}*\langle f_{k-k',\omega-\omega'}(1)f_{k-k',\omega-\omega'}^{*}(2)\rangle = \frac{q}{m}\sum_{k',\omega'}ik'\frac{\tilde{\phi}_{k'\omega'}}{\epsilon_{k'\omega'}}\frac{\tilde{f}_{k\omega}^{(2)*}(2)}{f_{k-k',\omega-\omega'}^{*}(2)}\langle f_{k-k',\omega-\omega'}(1)f_{k-k',\omega-\omega'}^{*}(2)\rangle \tag{3.25}$$

$$\langle d_{12}^{t}*f_{k-k',\omega-\omega'}(1)\rangle\frac{\partial\bar{f}}{\partial v_2} = -\frac{q^2}{m^2}\sum_{k',\omega'}ik'k\frac{\tilde{\phi}_{k'\omega'}}{\epsilon_{k'\omega'}}\frac{\tilde{\phi}_{k\omega}^{(2)*}}{\epsilon_{k\omega}^{*}}g_{k\omega}^{*}(2)f_{k-k',\omega-\omega'}(1)\frac{\partial\bar{F}_{k\omega}^{*}}{\partial v_2} \tag{3.26}$$

$\tilde{\phi}_{k\omega}^{(2)*}$ and $\tilde{f}_{k\omega}^{(2)*}(2)$ are given by

$$\begin{aligned}\tilde{f}_{k\omega}^{(2)*} &= -\frac{q}{m}g_{k\omega}^{*}(2)i(k-k')\phi_{k-k',\omega-\omega'}^{*}\frac{\partial\tilde{f}_{k'\omega'}^{(1)*}}{\partial v_2}\\ \tilde{\phi}_{k\omega}^{(2)*} &= \frac{4\pi ne}{|k|^2}\int dv_3\tilde{f}_{k\omega}^{(2)*}(3)\end{aligned} \tag{3.27}$$

Similarly $d_{12}^{f}*$ satisfies

$$\langle d^{f}_{12} * f_{k-k',\omega-\omega'}(1)\rangle \frac{\partial \bar{f}}{\partial v_2} = \frac{q^2}{m^2}\omega_p^2 \sum_{k',\omega'} ik' \frac{k}{|k|^2} \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} g^*_{k\omega}(2) \frac{\partial \overline{F}^*_{k\omega}}{\partial v_2} \times \int dv_3 \frac{g^*_{k\omega}(3)}{\epsilon^*_{k\omega}} k' \frac{\partial}{\partial v_3} \langle f_{k-k',\omega-\omega'}(1) f^*_{k-k',\omega-\omega'}(3)\rangle \tag{3.28}$$

The C^{ϕ}_{12} operator is defined through

$$C^{\phi}_{12} \langle \phi^*_{k\omega} f_{k\omega}(1)\rangle \equiv (\beta_{12} * + \gamma_{12} *) \langle f_{k-k',\omega-\omega'}(1) \phi^*_{k-k',\omega-\omega'}\rangle \tag{3.29}$$

where

$$\beta_{12}* = \frac{q^2}{m^2} \sum_{k',\omega'} k'k' \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} \frac{\partial}{\partial v_2} g^*_{k\omega}(2) \frac{\partial}{\partial v_2} \overline{F}^*_{k\omega} \tag{3.30}$$

$$\gamma_{12}* = -\frac{q^2}{m^2} \sum_{k',\omega'} k'k' \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} \omega_p^2 \frac{ik}{|k|^2} g^*_{k\omega}(2) \frac{\partial}{\partial v_2} \overline{F}^*_{k\omega} \int dv_3 \frac{g^*_{k\omega}(3)}{\epsilon^*_{k\omega}} \frac{\partial}{\partial v_3} g^*_{k\omega}(3) \frac{\partial}{\partial v_3} \overline{F}^*_{k\omega} \tag{3.31}$$

A this stage one can query a seeming asymmetry in the renormalized eqaution. Why do certain terms which appear in the C_{11} have their counterpart in the C_{12} operator while others do not ? For example D_{11} has its equivalent in D_{12}, while $\mathfrak{F}_{11}$ does not. It is difficult to give a rigorous explanation to this observation from the final renormalized result. On the other hand if we go back to the original expansion it is fairly easy to see how this comes about.

Consider the limit $\{k \to 0\}$ of the Fourier version of (3.1). S disappears since there are no average fields and we get

$$\lim_{k\to 0} \frac{\partial}{\partial t} \langle f_k(1) f^*_k(2)|t\rangle = -\frac{q}{m}\frac{\partial}{\partial v_1} \sum_{k'} \Big(E^{(1)}_{k'} f^{(2)*}_{k'}(1) f^1_0(2) + E^{(2)*}_{k'} f^{(1)}_{k'}(1) f^1_0(2) + E^{(1)*}_{k'} f^{(1)}_{k'}(1) f^2_0(2) \Big) + (1 \leftrightarrow 2) \tag{3.32}$$

where we have kept the same non-linear terms which we used to evaluate the collision operators (see (2.9) and (3.18)). We identify

$$\frac{\partial f_0^1(1)}{\partial t} = -\frac{q}{m}\frac{\partial}{\partial v_1}\sum_{k',\omega'}(E_{k'}^{(1)}f_{k'}^{(2)*}(1) + E_{k'}^{(2)*}f_{k'}^{(1)}(1)) \tag{3.33}$$

and

$$\frac{\partial f_0^0(1)}{\partial t} = -\frac{q}{m}\frac{\partial}{\partial v_1}\sum_{k',\omega'}E_{k'}^{(1)*}f_{k'}^{(1)}(1) \tag{3.34}$$

Thus the exact equation in the limit of small k becomes

$$\lim_{k\to 0} \quad \frac{\partial}{\partial t}\langle f_k(1)f_k^*(2)|t\rangle = f_0^1(1)\frac{\partial f_0^1(2)}{\partial t} + \frac{\partial f_0^1(1)}{\partial t}f_0^1(2) + f_0^2(1)\frac{\partial f_0^0(2)}{\partial t} + \frac{\partial f_0^0(1)}{\partial t}f_0^2(2) \tag{3.35}$$

The first two terms are nothing but the limit of the C_{11}^f and C_{22}^f operators as $\{k \to 0\}$. They are, as we have seen, the *perturbed* Lenard-Balescu operators. The third and fourth term are the elements which for finite k yield the cross operators since they are associated with $f_k^{(2)}$. But we see that their limit is the *unperturbed* collision operator. Thus we would expect those terms to reduce to a Fokker-Planck drag and diffusion. This result does not follow directly from our equations because we have approximated $f_k^{(2)}$ by the subset of terms (3.19) and (3.20). On the other hand it explains qualitatively why the C_{12} operator does not contain all the "companion" elements of the C_{11} operator.

On a more quantitative basis it is helpful to see the origin of the various terms in the iterative process as they relate to Eqs. (2.9) and (3.18). We will use the following notation to differentiate terms which have two components. For example the terms in Eqs. (2.14), (2.15), (2.17), and (2.21) all consist of two parts. These will be written $F_{11}(1) + F_{11}(2)$, $d_{11}^t(1) + d_{11}^t(2)$, etc., where "(1)" refers to the first term in the parentheses and "(2)" to the second.

Then the iteration of $f_{k-k'}^{(2)}$ and $\phi_{k-k'}^{(2)}$ in the first term of (2.9) yields

$$D_{11},\ \beta_{11},\ F_{11}(1),\ d_{11}^t(1),\ d_{11}^f(1),\ \text{and } \gamma_{11}(1)$$

The iteration of $\phi_{k-k',\omega-\omega'}^{(2)}$ in the second term of (2.9) yields

$$\mathcal{F},\ \delta_{11},\ F_{11}(2),\ d_{11}^t(2),\ \text{and } \gamma_{11}(2)$$

Finally

$$D_{12},\ F_{12},\ d^t_{12},\ d^f_{12},\ \gamma_{12},\ \text{and } \beta_{12}$$

come from the iteration of $f^{(2)}_{k\omega}$ and $\phi^{(2)}_{k\omega}$ in (3.18). The steps of the iteration are illustrated diagramatically in Fig. 3.2.

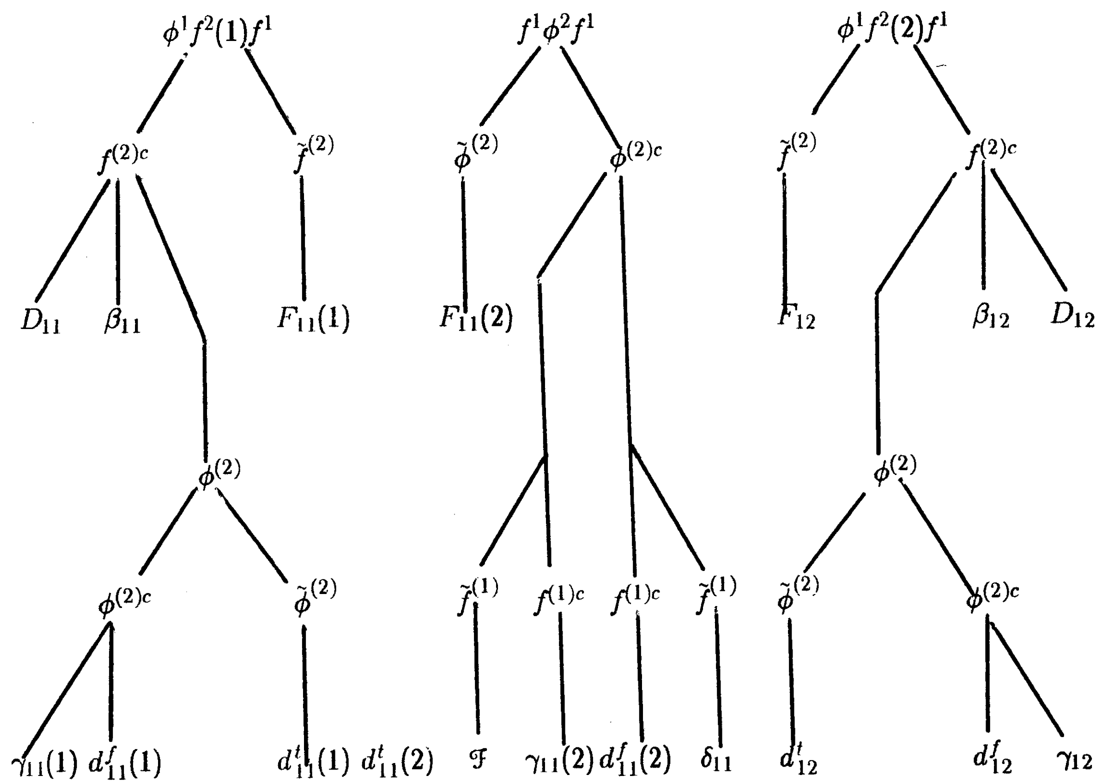

Fig. 3.2 Iteration of Two Point Equation

We must remember that the Fourier series are defined over a finite interval of time T and length L. Thus equation (3.17) is only defined over that domain. We now wish to consider an infinite system and pass to the Fourier integral limit. For the k transform this can be easily accomplished by multiplying (3.17) by the length L of the system and taking the limit $L \to \infty$. In a spatially homogeneous sytem we have (A.7)

$$\lim_{L\to\infty} \quad L\langle f_k f_{-k}\rangle \to \langle ff\rangle_k \tag{3.36}$$

Note that terms which have a convolution between the $\{k'\}$ sum and the correlation function at $\{k-k'\}$ are transformed through

$$\lim_{L\to\infty} \qquad L\sum_{k'}|\phi_{k'}|^2\langle f_{k-k'}f_{k'-k}\rangle \to \int\frac{dk'}{2\pi}\langle\phi^2\rangle_{k'}\langle ff\rangle_{k-k'} \tag{3.37}$$

The T limit is slightly more tricky. As a first approximation we assume that the turbulence is stationary. (i.e. $\partial/\partial t = 0$ in (3.17).) In general this is not the case since we are going to allow the system to evolve on the "slow" relaxation time scale. However we can write the temporal solution as a superposition of a stationary state and a weak directional (function of t_+) state. This is tantamount to using the multiple time scaling of §2.3, and allows us to pass to infinite T by using (3.30) and (3.31) with k, k', and L replaced by ω, ω', and T.

To decouple the equal time and two time equations we will make a Markovian approximnation. This is of course consistent with our assumption $\tau_c \ll \tau_{tr}$. We thus assume that for small separations $\langle f(1)f(2)\rangle_{k\omega}$ and $\langle f(1)f(2)\rangle_{k-k',\omega-\omega'}$ are strongly peaked about $(\omega-kv_+)$ and $(\omega-\omega'-(k-k')v_+)$ respectively. Coefficients in the renormalization are transformed through the following example

$$\begin{aligned}
&\frac{\partial}{\partial v_1}\int d\omega D_{12} * \frac{\partial}{\partial v_2}\langle f(1)f(2)\rangle_{k-k',\omega-\omega'} = \\
&\quad -\frac{q^2}{m^2}\frac{\partial}{\partial v_1}\int d\omega\int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}g^*_{k\omega}(2)k'k'\frac{\langle\tilde{\phi}^2\rangle_{k'\omega'}}{|\epsilon_{k'\omega'}|^2}\frac{\partial}{\partial v_2}\langle f(1)f(2)\rangle_{k-k',\omega-\omega'} \to \\
&\quad -\frac{q^2}{m^2}\frac{\partial}{\partial v_1}\int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}g^*_{k'\omega'}(2)k'k'\frac{\langle\tilde{\phi}^2\rangle_{k'\omega'}}{|\epsilon_{k'\omega'}|^2}\frac{\partial}{\partial v_2}\langle f(1)f(2)\rangle_{k-k'}
\end{aligned} \tag{3.38}$$

Notice that in the last expression $g^*_{k\omega}(2) \to g^*_{k'\omega'}(2)$ and that $\int d\omega\langle f(1)f(2)\rangle_{k-k',\omega-\omega'} \to \langle f(1)f(2)\rangle_{k-k'}$. Transformations of the type described by (3.38) allow us to recast the equal time equation into

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + ikv_1 + C^f_{11}(k) + C^f_{12}(k) + (1\leftrightarrow 2)\right)\langle f(1)f(2)\rangle_k = S_k \\
&\quad +\frac{q}{m}ik\frac{\partial}{\partial v_1}C^\phi_{11}(k)\langle\phi f(2)\rangle_k - \frac{q}{m}i(k-k')\frac{\partial}{\partial v_1}C^\phi_{12}(k)\langle f(1)\phi\rangle_k + (1\leftrightarrow 2)
\end{aligned} \tag{3.39}$$

We have explicitly indicated that the C operators are a function of k only.

3.5. Properties of the Two Point Equation

The first property we would like to investigate is that of phase space conservation. If we sum equation (3.22) over k and take the $v_1 \to v_2$ limit (neglecting "+" dependances) one can trivially show that

$$
\begin{aligned}
&\sum_{k\omega}[D_{11}+D_{12}+D_{21}+D_{22}]=0\,, \qquad \sum_{k\omega}[F_{11}(1)+F_{12}+F_{21}+F_{22}(1)]=0\\
&\sum_{k\omega}[\beta_{11}+\beta_{12}+\beta_{21}+\beta_{22}]=0\,, \qquad \sum_{k\omega}[d^t_{11}(1)+d^t_{12}+d^t_{21}+d^t_{22}(1)]=0\\
&\sum_{k\omega}\left[d^f_{11}(1)+d^f_{12}+d^f_{21}+d^f_{22}(1)\right]=0\,, \qquad \sum_{k\omega}[\gamma_{11}(1)+\gamma_{12}+\gamma_{21}+\gamma_{22}(1)]=0\\
&\sum_{k\omega}[F_{11}(2)+F_{22}(2)]=0\,, \qquad \sum_{k\omega}[\mathcal{F}_{11}+\mathcal{F}_{22}]=0\\
&\sum_{k\omega}[d^t_{11}(2)+d^t_{22}(2)]=0\,, \qquad \sum_{k\omega}\left[d^f_{11}(2)+d^f_{22}(2)\right]=0\\
&\sum_{k\omega}[\delta_{11}+\delta_{22}]=0\,, \qquad \sum_{k\omega}[\gamma_{11}(2)+\gamma_{22}(2)]
\end{aligned}
\tag{3.40}
$$

Note that the summation over $\{k, \omega\}$ is equivalent to taking the limit x_-, $t_- \to 0$. In affecting the cancellations of (3.40) the following trends appear. If a term contains two velocity derivatives in the minus coordinate the "11" term will cancel with its "21" counterpart and vice versa. If the term contains only one v_- derivative, "11" will cancel with "22" and "21" (if any) will cancel with "12". Referring to Fig. (3.2), we note that the renormalization originates from three groups. The second group (which comes from allowing the perturbed electric field, ϕ^2, to act back on the fluctuations) produces the elements which do not have a bivariate counterpart. These, as we see in (3.40), cancel "11" with "22".

We give an example to illustrate (3.40). Consider the d^t_{ij} terms. If we take the equation for d^t_{21} (which is identical to Eq. (3.26) with $(1 \leftrightarrow 2)$*) and sum it over $\{k, \omega\}$ we can change $\{k', \omega'\}$ to $\{-k', -\omega'\}$. At the same time we set $\{k + k', \omega + \omega'\}$ equal to $\{k, \omega\}$ by changing the order of summation. The resulting expression is identical to the one for $d^t_{11}(1)$. Going to the relative coordinate system in velocity we use

*remember that $(1 \leftrightarrow 2)$ implies $\{k\omega \to -k, -\omega\}$, and $\{k'\omega' \to -k', -\omega'\}$ in addition to $\{v_1 \leftrightarrow v_2\}$.

$$\frac{\partial}{\partial v_1} \rightarrow \frac{\partial}{\partial v_+} + \frac{\partial}{\partial v_-}$$
$$\frac{\partial}{\partial v_2} \rightarrow \frac{\partial}{\partial v_+} - \frac{\partial}{\partial v_-} \tag{3.41}$$

At this stage it is important to realize that $\overline{F}_{k-k',\omega-\omega'}$ consists of a part which has no v_- dependance ($\langle f \rangle$) and the $C^{\phi}_{k-k',\omega-\omega'}$ operator which exhibits, indirectly a v_- dependance. If we take $C^{\phi}_{k-k',\omega-\omega'}(1)$ we know that this contains expressions of the form $(\omega - \omega' - (k - k')v_1)$. For small separation $\omega = kv_2$ and we immediately see the appearance of the v_- dependance. Using (3.36), and neglecting the gradients on v_+, changes the sign of one of the expressions so that they cancel. The other terms follow suit in much the same fashion.

The second property we which to examine is the behaviour of the bivariate terms for large separation in phase space. We use the the Markovian approximation. Consider, for example, the diffusion coefficients D_{ij}. If $v_- \simeq 0$ we can inverse Fourier transform these terms to get

$$\frac{\partial}{\partial v_-} D_-(x_-) \frac{\partial}{\partial v_-} \langle \delta f \delta f | x_-, v_-, t \rangle; \qquad D_-(x_-) = D_{11} + D_{22} - D_{12}(x_-) - D_{21}(x_-) \tag{3.42}$$

where for example

$$D_{12}(x_-) = \frac{q^2}{m^2} \int \frac{dk'}{2\pi} \int \frac{d\omega'}{2\pi} \frac{ik'k'\langle\phi\rangle^2_{k'\omega'}}{(\omega' - k'v_2 - iC'f)} \exp{(ik'x_-)} \tag{3.43}$$

D_- is a diffusion coefficient in the relative coordinate. From (3.38) the properties of D_- become

$$D_- \rightarrow 0, \qquad x_- \rightarrow 0;$$
$$D_- \rightarrow 2D_{11}, \qquad x_- \rightarrow \infty \tag{3.44}$$

One can understand (3.44) on the following physical grounds. Two particles which are close together in phase-space experience roughly the same forces and therfore move together even though their average coordinates x_+ and v_+ may change significantly. On the other hand if $|k_0 x_-| \gg 1$ (where k_0 is a measure of the spectrum width), then D_{12} and D_{21} are small and the particles diffuse independantly. In general one expects all the bivariate operators to exhibit a strong dependance on x_- and possibly v_-. The latter appears as a Doppler shift in the Green function. The x_- dependance appears through the convolution of the $\{k'\}$ sum with the correlation function at $\{k-k'\}$ since coefficients which have an x_- dependance will transform through

$$\int dx_- e^{ikx_-} A_{12}(x_-)\langle \delta f \delta f | x_- \rangle \equiv \sum_{k',\omega'} A_{12}(k')\langle ff | k_- k' \rangle \tag{3.45}$$

The cross operators describe the correlated motion between points 1 and 2. This may take the form of a drag, diffusion or other non-linear process. On the other hand it is physically clear that this correlated motion will dissapear for sufficiently large spatial distances (the "sufficiently" is determined by the spectrum width).

3.6. Equivalence to the BBGKY Hierarchy

An exact omparison with the BBGKY hierarchy is not possible since our procedure renormalizes the propagators and distribution function. However if we neglect these renormalization effects and consider the iterative process only, it is fairly easy to see that the "phase-coherent" approximation which leads to (3.39) is similar to a truncation of the Mayer cluster expansion at the four point irreducible function. That is $G_4(1,2,3,4)$ in

$$\begin{aligned}\langle \delta f(1)\delta f(2)\delta f(3)\delta f(4)\rangle = G_2(1,2)G_2(3,4) + G_2(1,3)G_2(2,4) + G_2(1,4)G_2(2,3)\\ + G_4(1,2,3,4);\end{aligned} \tag{3.46}$$

$$G_2(1,2) = \langle \delta f(1)\delta f(2)\rangle$$

is set equal to zero.

We define the propagator of the linearized Vlasov equation through[19]

$$\left(\frac{\partial}{\partial t} + L(1,k)\right)P(1,k,t) = 0\,; \qquad L(1,k) = ikv_1 - i\frac{\omega_p^2}{|k|^2}k\frac{\partial}{\partial v_1}f_0\int dv_1\,. \tag{3.47}$$

together with the initial condition

$$P(1,k,t=0) = 1$$

The application of $P(t)$ to an arbitrary time independant function $g(0)$ produces a time dependant function $g(t)$ whose elements satisfy the linearized Vlasov-Poisson equations and whose initial condition is $g(0)$.

We can write the second equation of the hierarchy[26] (with discretness parameter set to zero) as

$$\left(\frac{\partial}{\partial t} + L(1,k) + L(2,-k)\right)\langle f(1)f(2)|k\rangle =$$

$$\frac{q}{m}\frac{\partial}{\partial v_1}\int_0^t d\tau \int dv_3 \int \frac{dk'}{2\pi} P(1,k{-}k',\tau)P(2,-k,\tau)P(3,k',\tau)$$
$$\times i\frac{\omega_p^2}{k'^2}k'\left(\langle f(3)E|k'\rangle \frac{\partial}{\partial v_1}\langle f(1)f(2)|k\rangle + \langle Ef(2)|k\rangle \frac{\partial}{\partial v_1}\langle f(3)f(1)|k'\rangle\right) \tag{3.48}$$
$$\vdots$$
$$+ (1 \leftrightarrow 2)$$

The P operators propagate points $\{1,2,3\}$ (from $t - \tau$ to t) through the electric field structure of the plasma. We will simplify the analysis by assuming homogeneity in velocity space so that they become ballistic operators. Concentrating on the temporal propagation we get for the first term

$$\int_0^\infty d\tau P(1, k - k', \tau)\langle f(3)E|k', \tau\rangle\langle f(1)f(2)|k, -\tau\rangle \tag{3.49}$$

$P(2)$ and $P(3)$ have inserted a (fast) τ dependance into $\langle \delta f \delta f\rangle$ (making them two time correlation functions). We have assumed that the one time correlation functions are slowly varying functions of t compared to the P operators, and have taken the time asymptotic solution.

Using

$$\langle f(3)E|k', \tau\rangle = \sum_{\omega'}\langle f(3)E|k', \omega'\rangle e^{i\omega'\tau} \tag{3.50}$$

and

$$\langle f(1)f(2)|k, -\tau\rangle = \langle f(1)f(2)|k\rangle e^{-ikv_2\tau} \tag{3.51}$$

(3.45) becomes

$$\sum_{\omega'}\int_0^\infty d\tau e^{i(\omega' - k'v_1 + kv_-)\tau}\langle f(3)E|k'\rangle\langle f(1)f(2)|k\rangle \tag{3.52}$$

Evaluating the τ integral we get the un-renormalized green function, and the first term in (3.48) becomes the D_{11} diffusion coefficient. A similar calculation for all the other terms in (3.48) would yield the $D_{ij}, \beta_{ij}, \mathcal{F}$, etc. terms. Note that we would have to use Eq. (2.15a) to recover terms which come from the incoherent spectrum. For example the second term in (3.48) gives β_{11} *and* $F_{11}(1)$ when one uses $f_{k'\omega'} = f^c_{k'\omega'} + \tilde{f}_{k'\omega'}$. The d_{ij} terms, however, would not appear. This is just a result of using the ballistic approximation for P. In this context we can therefore interpret the d_{ij} coefficients as terms accounting for the shielding effects in the P propagator.

We thus see that an un-renormalized version of our procedure is akin to a solution of a BBGKY/-Vlasov cumulant hierarchy, which includes all terms up to the irreducible four point correlation function. The renormalization resummes "higher" order terms which make (3.39) a much more robust equation than its counterpart (3.48). The inclusion of the (Vlasov) incoherent fluctuation in the iterative process takes into account physics which is outside the scope of any of these expansions.

Chapter 4

Source Term

This chapter is concerned with the source of small scale fluctuations in a Vlasov plasma. In Chapter 2 we derived a two point equation and labelled certain elements as source terms. The rationalle for such a nomenclature is justified and the required properties of such a term are discussed. In particular the importance of momentum and energy conservation is shown, and the implications of these constraints to a one and two species source term are analyzed.

4.1. Mixing Length Theory

The mixing length theories of fluid turbulence, originally formulated by Prandtl, postulate a mechanism for turbulence based on the movement of small discrete clumps or particles of fluid.

We imagine the fluid to be made of a large number of these elements each carrying a transferable property such as mass, momentum, and energy. It is further assumed that these particles of fluid are displaced some distance "l" before any of the transferable properties are changed by the new environment. In other words within this mixing length "l" the particles of fluid retain their own identity and properties until they suddenly mix with the new surroundings at $x = x_0 + l$ (x_0 is the starting point at which the fluid element was identified). Thus for a period $\tau = l/v_+$ (v_+ is the velocity of the centre of mass) the fluid particle may change its structure subject to the strict conditions of conserving momentum, energy and mass *locally*. Obviously it would be more plausible to suppose that the mixing proceeds by continuous movement of the fluid. On the other hand this mechanism is amenable to a simple treatment and, in the case of fluid turbulence, is ultimately justified in that it leads to results which are compatible with experimental observation.

On the basis of the these arguments the fluctuation in the velocity (δv), and the correlation function ($\langle\delta v \delta v\rangle$) can be expressed in terms of the average gradients and a mixing length by

$$\delta v \approx l\frac{\partial}{\partial x}\langle v\rangle ; \qquad \langle\delta v \delta v\rangle \approx \langle l^2\rangle\Big(\frac{\partial}{\partial x}\langle v\rangle\Big)^2 \tag{4.1}$$

This means that the fluctuations in velocity depend upon the changes in the mean velocity at two points a distance l apart. While the analogy is not necessarily exact we can analyze the singular behaviour of the Vlasov equation in the light of such a model.

The Vlasov distribution $f(x, v, t)$ describes the density of an incompressible self interacting fluid which flows in x, v phase space. The incompressible nature of the flow implies that two neighbouring points in phase space can have quite different densities since they may have come from points, which at an earlier time, were widely separated (Figure 4.1). To obtain an expression for the magnitude of these fluctuations one requires a solution of (3.39). However, we can obtain a qualitative solution in the following way. We recall equation (3.9)

$$\Big(\frac{\partial}{\partial t} + T_{12}\Big)\langle\delta f \delta f\rangle = S \qquad T_{12} \to 0 \quad x_-, v_- \to 0$$

The operator T_{12} is a function of v_-, while the source is a function of v_+ only. We have thus separated the average properties from the fluctuating ones, since S describes the changes on the thermal scale

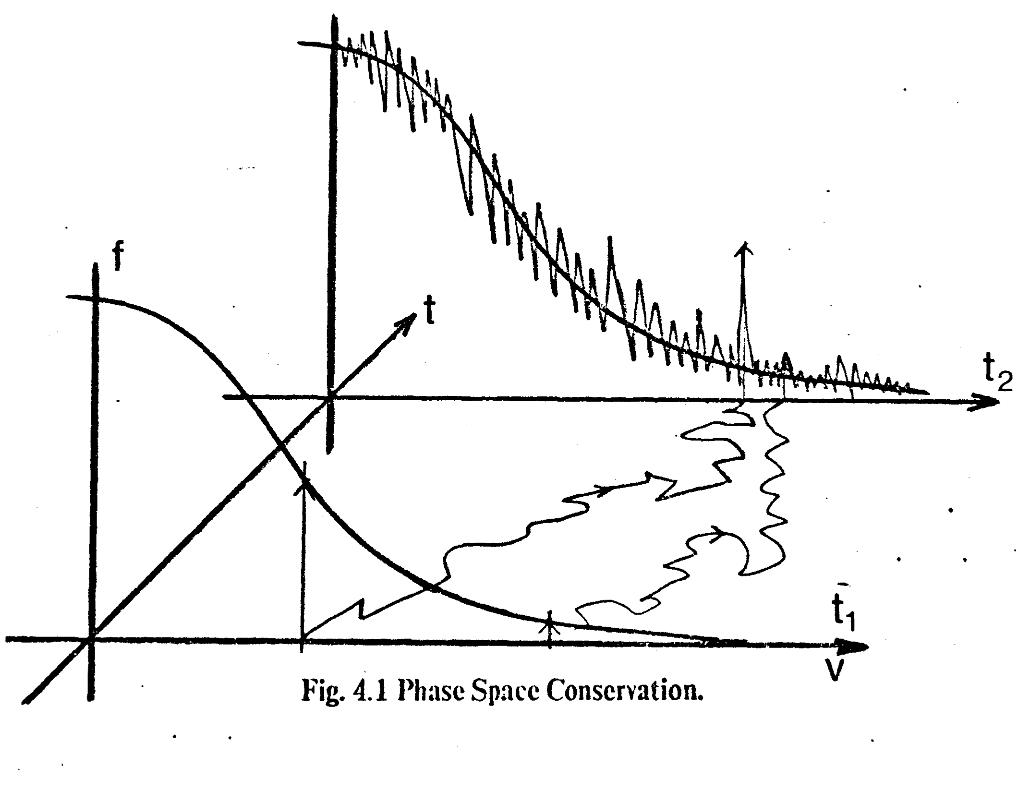

Fig. 4.1 Phase Space Conservation.

while T_{12} describes those on a trapping scale. The solution of this equation can be written as

$$\int d\tau e^{-T_{12}\tau} S(v_+, t-\tau) \equiv \tau_{cl}(x_-, v_-) S \tag{4.2}$$

Previous approximations[4–7] used

$$S(x_-) = (D_{12}(x_-) + D_{21}(x_-)) \frac{\partial^2}{\partial v_1 \partial v_2} \langle f(1) \rangle \langle f(2) \rangle \tag{4.3}$$

This can be obtained by substituting $f = f^c$ *only*, in the Fourier version of (3.4). The steps leading to (4.2) are discussed in (§4.3).

Equation (4.2) becomes

$$\langle f(1) f(2) \rangle = \tau_{cl}(x_-, v_-)(D_{12} + D_{21}) \frac{\partial^2}{\partial v_1 \partial v_2} \langle f(1) \rangle \langle f(2) \rangle \tag{4.4}$$

We can understand (4.4) in the following way. $\tau_{cl}(D_{12} + D_{21})$ is a "mixing-length" in velocity space, and contains all the fine grain information. This consists of a "clumping" time (τ_{cl}) which represents the time for which a fluid particle retains its identity. Clearly this is strongly dependant on the separation of the points in phase space. When 1 approaches 2, $T_{12} \to 0$ so that τ_{cl} becomes infinite. This is consistent with the notion of phase space conservation since the fluid particle becomes a point which conserves f along its orbit indefinitely. The diffusion coefficients contain the spectrum of the fields which determine the rate at which the mixing occurs, and the level of turbulence.

We can thus view the incoherent fluctuations of Chapters 1 and 2 in the framework of a mixing length theory where the mixing occurs in velocity space off the average velocity gradients, and the "length" originates in the incompressible nature of the flow.

4.2. General Properties of a Source Term

We wish to analyze some generic properties of the source as defined in the previous section. Equation (2.4) which is rewritten below

$$\int dv \left(\frac{\partial}{\partial t} \langle \delta f^2 \rangle + \frac{\partial}{\partial t} \langle f \rangle^2\right) \equiv \int dv \left(\frac{\partial}{\partial t} \langle \delta f^2 \rangle + 2 \frac{q}{m} \langle \delta E \delta f \rangle \frac{\partial}{\partial v} \langle f \rangle\right) \tag{4.5}$$

immediately shows that the source term

$$S = \frac{q}{m} \langle \delta E(1) \delta f(2) \rangle \frac{\partial}{\partial v_1} \langle f \rangle + \frac{q}{m} \langle \delta E(2) \delta f(1) \rangle \frac{\partial}{\partial v_2} \langle f \rangle \tag{4.6}$$

in the limit of small separation, is related to the rate of change of the average distribution function.The underlying mechanism is one of an increase in the level of fluctuations at the expense of the average distribution (and vice versa). For example if the average distribution is unstable it can relax by changing its shape to a more stable configuration. This new configuration is produced through a mixing of fluids of different density. In the process granulations are generated since these different densities cannot interpenetrate.

We reiterate that the source term is treated as an independant entity because it does not act on the fluctuations directly, but through the indirect mechanism of changing the average distribution. For example we can easily envision a region of phase space where the average distribution is flat (say at the top of the Maxwellian). Given a set of fluctuations at that point, their evolution would be governed by

$$(\frac{\partial}{\partial t} + T_{12})\langle \delta f \delta f \rangle \simeq 0$$

T_{12} would be identified as the reciprocal of the e-folding time of the fluctuations. An alternative way of looking at the problem is to realize that the source term is also the mechanism by which the plasma shields fluctuations. When $\langle f \rangle$ is flat the debye length becomes infinite and no shielding occurs. This is equivalent to saying that the plasma does not redistribute itself (and by default the average distribution) to minimize the charge modulations caused by the fluctuations.

We can examine, more closely, the partitioning implied by (4.5) in the case of a one dimensional plasma where normal mode interactions are neglected. The latter is an important restriction because it leads to

$$\frac{\partial}{\partial t}\overline{\langle \delta f \rangle^2} \simeq 0 \tag{4.7}$$

(The "___ " represents the average over velocity space.) This comes about since an unrenormalized collision operator of the Lenard-Balescu type goes to zero in one dimension, so that (4.5) reduces to (4.7). We already identified this property in Chapter 2 as the result of momentum constraints. However we can make an even stronger statement than (4.7).

Let $\langle f \rangle = \langle f(v,0) \rangle + \Delta \langle f(v,t) \rangle$. $\langle f(v,0) \rangle$ is the initial value of the average distribution function while $\Delta \langle f(v,t) \rangle$ is the change in $\langle f \rangle$. Equation (4.5) can be rewritten as

$$\int dv (\frac{\partial}{\partial t}(\langle \delta f^2 \rangle + \Delta \langle f \rangle^2) + \langle f(v,0) \rangle \frac{\partial}{\partial t} \Delta \langle f \rangle) = 0 \tag{4.8}$$

Since the fluid particles can only transfer momentum locally we can approximate the initial distribution by a Taylor series centered about some average coordinate

$$f(v,0) \simeq a + bv \tag{4.9}$$

and equation (4.8) becomes

$$\int dv\left(\frac{\partial}{\partial t}(\langle\delta f^2\rangle + \Delta\langle f\rangle^2)\right) = 0 \tag{4.10}$$

We have used

$$a\frac{\partial}{\partial t}\overline{\Delta\langle f\rangle} = 0\,; \qquad b\frac{\partial}{\partial t}\overline{v\Delta\langle f\rangle} = 0 \tag{4.11}$$

which represent number and momentum conservation, to obtain (4.10).

If we integrate (4.10) over time, we get

$$\langle\overline{\delta f^2}\rangle = \langle\overline{\delta f^2(0)}\rangle - \overline{\Delta\langle f\rangle}^2 \tag{4.12}$$

$\delta f(0)$ represents the initial level of fluctuations. This last equation shows that not only does the level of fluctuations stay constant (4.7), but in one dimension it will decrease since the last term is positive definite.

The same arguments can be used to show that if there is an energy source the fluctuations can increase. Consider for example a two species problem in an ion-acoustic regime. The energy source is the drifting electron maxwellian. In that case

$$b\frac{\partial}{\partial t}\overline{v\Delta\langle f_{ion}\rangle} = -b'\frac{\partial}{\partial t}\overline{v\Delta\langle f_{elec}\rangle}$$

since momentum can now be exchanged between the electrons and ions. This implies that

$$\langle\overline{\delta f_{ion}^2}\rangle = \langle\overline{\delta f_{ion}^2(0)}\rangle + b'\overline{v\Delta\langle f_{elec}\rangle} - \overline{\Delta\langle f_{ion}\rangle}^2 \tag{4.13}$$

If b' is positive the gradients can be used to generate a turbulent state where the fluid particles are the dominant contribution. Of course this state may also contain eigenmodes of the plasma and one has yet to demonstrate that these are any less efficient transport agents.

It is worthwhile emphasizing that our discussions rely on two important assumptions. The first is the "localness" of the interaction. The second is the conservation properties of the source term. The former need not be true if the turbulent spectrum contains waves, since these can transport momentum through non-resonant interactions and expansion (4.9) would not be valid. The latter are investigated in detail in the next section.

4.3. One Species Source Term

We can obtain an expression for the source term by using (2.19) in the Fourier version of (4.6). The result is

$$S_{k'\omega'} = \left(\frac{\varsigma^2}{m^2}k'k'\frac{\langle\tilde{\phi}^*_{k'\omega'}\tilde{\phi}_{k'\omega'}\rangle}{|\epsilon_{k'\omega'}|^2}g^*_{k'\omega'}\frac{\partial}{\partial v_2}\overline{F}_{k'\omega'} + ik'\frac{\langle\tilde{\phi}_{k'\omega'}\tilde{f}^*_{k'\omega'}\rangle}{|\epsilon_{k'\omega'}|^2}\epsilon^*_{k'\omega'}\right)\frac{\partial}{\partial v_1}\langle f\rangle \tag{4.14}$$
$$+ (1 \leftrightarrow 2)$$

we pass to the Fourier integral limit and write

$$S(k,\omega) = \int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}\delta(\omega-\omega')\delta(k-k')S(k',\omega') \tag{4.15}$$

where $S(k',\omega')$ is identical to (4.14) except that the spectrums are expressed in terms of the integral transforms, i.e. $\langle f_{k'\omega'}f_{k'\omega'}\rangle \to \langle ff\rangle_{k'\omega'}$.

If we take $\overline{F}_{k'\omega'} \simeq \langle f\rangle$, we can write (4.15) in the more symmetric form of

$$S(k) = (D^0_{12}(k) + D^0_{21}(k))\frac{\partial^2}{\partial v_1\partial v_2}\langle f(1)\rangle\langle f(2)\rangle - (F^0_{12}(k)\frac{\partial}{\partial v_1} + F^0_{21}(k)\frac{\partial}{\partial v_2})\langle f(1)\rangle\langle f(2)\rangle \tag{4.16}$$

The zero superscripts mean that the terms are the Markovian version of the cross operators. For example D_{12} can, in that limit, be written as (2.38)

$$D_{12}(x_-) = -\frac{q^2}{m^2}\int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}\frac{ik'k'\langle\phi^2\rangle_{k'\omega'}}{(\omega'-k'v_2-iC^f)}e^{ik'x_-} \equiv \int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}H_{k'\omega'}e^{ik'x_-}$$

The Fourier transform of this expression is given by

$$D^0_{12}(k) = \int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}\delta(k-k')H_{k'\omega'} \tag{4.17}$$

Similarly the Markovian limit of the cross drag terms can be defined through

$$F^0_{12}(x_-) = \frac{q}{m}\int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}ik'\frac{\langle\tilde{\phi}\tilde{f}\rangle_{k'\omega'}}{|\epsilon^*_{k'\omega'}|^2}\frac{\epsilon_{k'\omega'}}{\langle f\rangle}e^{ik'x_-} \equiv \int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}R_{k'\omega'}e^{ik'x_-}$$

so that

$$F^0_{12}(k) = \int \frac{dk'}{2\pi} \int \frac{d\omega'}{2\pi} \delta(k-k') R_{k'\omega'} \tag{4.18}$$

Using (4.17) and (4.18) in (4.14) we obtain (4.16).

Note that we have integrated all expressions over ω to reduce the results to their one time version. If we had kept the two time dependance this would have appeared as an addition "$i\omega t_-$" to the factor $\exp(ikx_-)$. This reflects the idea that the clumping mechanism is intrinsically an *equal time* phenomenan of the two point equation. For $|t_-\omega_p| \geq 1$ the cross operators become small and the whole mechanism for phase space granulation disintegrates.

We see that the single time version of the source term S contains additional terms (in comparison to (4.3)) which originate from the inclusion of the incoherent fluctuation. These terms are important, since in one dimension and in the limit of weak turbulence (so that $Re\, g_{k\omega} \to \delta(\omega - kv)$) they cancel the source as defined through (4.3). This cancellation occurs for small separation (we need F_{12} and $D_{12} \to F_{11}$ and D_{11}) and is directly related to the cancellation of these same terms in a Lenard-Balescu collision integral. This is striaght-forward to see by considering the expression for the current driving the average distribution, Eq (2.38). Our source, in the limit of small separation, is this same expression without the integral over dk' and multiplied by $\partial/\partial v \langle f \rangle$. If we take $\overline{F} \simeq \langle f \rangle$ and $Re\, g_{k\omega} \simeq \delta(\omega - kv)$ then (2.38) is identically zero for *every* mode k' (i.e. we do not need to sum over all modes) and the source likewise dissapears. We have already discussed the momentum considerations which lead to such a result. Moreover this is in agreement with (4.7), where the exact equations predicted that such a term should dissapear, since there is no relaxation of the background distribution. We must emphasize the underlying assumption of stationarity and the neglect of any unstable wave-like modes since these can lead to a non-zero relaxation of $\langle f \rangle$ even in one dimension.

It is interesting to view the inclusion of these self consistent contributions (F_{12}) from another viewpoint. The inclusion of f^c *and* $\tilde{f}$ in the equation for the average distribution insures that momentum and energy are conserved. If we iterated $f = f^c$ only these propertise could not be proved*. With this in mind, let us examine once again (4.5). The right hand side can be written as

$$\int dv \left(-2\frac{q}{m} \langle \delta E \delta f \rangle' \frac{\partial}{\partial v} \langle f \rangle + 2\frac{q}{m} \langle \delta E \delta f \rangle'' \frac{\partial}{\partial v} \langle f \rangle \right) \tag{4.19}$$

*Unless we treated the case of Quasi-Linear theory which relies on a further property, namely $\epsilon_{k\omega} = 0$, to prove the conservation laws. Thus the above statement should be interpreted as "... given that we do not use $\epsilon_{k\omega} = 0$ then ... etc."

Here the "′" indicates that the term comes from whatever approximation we use to evaluate S, while "″" indicates the same but arising from whatever approximation we use to evaluate the relaxation of $\langle f\rangle$. Obviously in the exact equations these are identical. However if we iterate $f = f^c$ in the first and $f = f^c + \tilde{f}$ in the second then this inconsistency translates into a violation of phase space conservation.

We can extend this argument to *infer* energy and momemtum conservation in the source term. This concept is more a mathematical statement that in the *exact* equations the following relations hold

$$I_m = mn_0 \sum_k \int dv_1 \int dv_2 \frac{(v_1 + v_2)}{2} S(k) \equiv mn_0 \int dv_1 v_1 \frac{\partial \langle f\rangle}{\partial t} \tag{4.20}$$

$$I_e = \frac{1}{2} mn_0 \sum_k \int dv_1 \int dv_2\, v_1 v_2 S(k) \equiv \frac{1}{2} mn_0 \int dv_1 v_1^2 \frac{\partial \langle f\rangle}{\partial t} \tag{4.21}$$

In essence these are nothing but self consistency conditions dressed in the guise of conservation laws. Clearly they are not satisfied unless we iterate $f = f^c + \tilde{f}$ in *both* the expression for S and $\partial/\partial t\langle f\rangle$. It is only in that sense that we talk of "conservation" properties of the source term. On the other hand on a more intuitive plane these results are fairly plausible. Suppose the source term is seen as a mechanism by which these "chunks" of plasma are moved through phase space. The equation of continuity, for example, would demand that for any chunk moved from x_1 to x_2, a similar one should move from x_2 to x_1. Furthermore this process can only occur if the total energy and momentum transported are conserved. In the case of the purely diffusive source one can see that this is not the case. So long as there exist some average gradients, this term will shuffle them around to produce fluctuations. Nowhere is there evidence of the reaction of the plasma to this rearrangement. The F_{ij} terms in the source provide this response.

An added refinement can be obtained if instead of (2.19) we use equations (2.41) in (4.6) (that is take into account the two-time scaling procedure of §2.3). The source term becomes slightly more complicated since it explicitly contains terms describing the slowly varying potentials. This implies that even without a renormalization the source will be non-zero in one dimension, being proportional to the rate of change of the potential energy.

4.4. Two Species Source Term

We consider a two component plasma made of electrons and ions. The ions are no longer stationary, and participate in the mixing process. We want to obtain an expression for the source term in the equation for the "i'th" species. We will use these expressions in Chapter 7 when we consider the problem of ion acoustic turbulence. At this stage, however, they are interesting because they demonstrate how the two species term remains finite. Once again one will clearly see the influence of momentum constraints on the problem.

Let ϕ^e and ϕ^i be the electron and ion potentials. The total, self consistent, plasma potential is ϕ $(= \phi^e + \phi^i)$. Through a simple extension of the procedure in Chapter 2 the fields of the dressed ions and electrons can be calculated as

$$
\begin{aligned}
\phi^e_{k\omega} &= \tilde{\phi}^e_{k\omega} - \phi_{k\omega}\chi^e_{k\omega} \\
\phi^i_{k\omega} &= \tilde{\phi}^i_{k\omega} - \phi_{k\omega}\chi^i_{k\omega}
\end{aligned}
\tag{4.22}
$$

$\phi^i_{k\omega}$ and $\phi^e_{k\omega}$ are the incoherent ion and electron fluctuations, while χ is the standard susceptability defined through 2.52.

We redefine a dielectric

$$
\epsilon_{k\omega} = 1 + \chi^e_{k\omega} + \chi^i_{k\omega} \tag{4.23}
$$

through which we can solve (4.22) to obtain

$$
\phi_{k\omega} = \frac{\tilde{\phi}_{k\omega}}{\epsilon_{k\omega}}; \qquad \tilde{\phi}_{k\omega} = \tilde{\phi}^i_{k\omega} + \tilde{\phi}^e_{k\omega} \tag{4.24}
$$

and

$$
\begin{aligned}
\phi^e_{k\omega} &= (1 - \frac{\chi^e_{k\omega}}{\epsilon_{k\omega}})\tilde{\phi}^e_{k\omega} - \frac{\chi^e_{k\omega}}{\epsilon_{k\omega}}\tilde{\phi}^i_{k\omega} \\
\phi^i_{k\omega} &= (1 - \frac{\chi^i_{k\omega}}{\epsilon_{k\omega}})\tilde{\phi}^i_{k\omega} - \frac{\chi^i_{k\omega}}{\epsilon_{k\omega}}\tilde{\phi}^e_{k\omega}
\end{aligned}
\tag{4.25}
$$

If we neglect any correlations between incoherent fluctuations of different species, and follow the procedure of §4.3, we get for the ion source term

$$
S^i(k) = \sum_{j=i,e} (D^{ij}_{12}(k) + D^{ij}_{21}(k))\frac{\partial^2}{\partial v_1 \partial v_2}\langle f^i(1)\rangle\langle f^j(2)\rangle - (F^{ij}_{12}(k)\frac{\partial}{\partial v_1} + F^{ij}_{21}(k)\frac{\partial}{\partial v_2})\langle f^i(1)\rangle\langle f^j(2)\rangle \tag{4.26}
$$

Four new terms appear which consist of diffusion and drag on the ion distribution driven by the gradients of the electron distribution. For example

$$D^{ie}_{12}(k)\frac{\partial^2}{\partial v_1\partial v_2}\langle f^i(1)\rangle\langle f^i(2)\rangle = \int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}\frac{\langle\tilde{\phi}^e\rangle^2_{k'\omega'}}{\langle\epsilon_{k'\omega'}\rangle^2}g^*_{k'\omega'}\delta(k-k')\frac{\partial^2}{\partial v_1\partial v_2}\langle f^i(1)\rangle\langle f^i(2)\rangle \tag{4.27}$$

represents the generation of fluctuations from the mixing of the gradients of the ion distribution, the mixing process being generated by the turbulent spectrum of electron fluctuations. Similarly

$$F^{ie}_{12}(k)\frac{\partial}{\partial v_1}\langle f^i(1)\rangle\langle f^i(2)\rangle\frac{q}{m}\int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}ik'\frac{\langle\tilde{\phi}^i\tilde{f}^i\rangle_{k'\omega'}}{|\epsilon^*_{k'\omega'}|^2}\frac{\epsilon_{k'\omega'}}{\langle f(2)^i\rangle}\frac{\partial}{\partial v_1}\langle f^i(1)\rangle\langle f^i(2)\rangle \tag{4.28}$$

describes the mixing of the ion distribution through the dynamical drag driven by the gradients of the average electron distribution.

For the one dimensional problem, we note that D^{ii} and F^{ii} approximately cancel so that the ion source reduces to

$$S^i(k) \simeq \left[D^{ie}_{12}(k) + D^{ie}_{21}(k)\right]\frac{\partial^2}{\partial v_1\partial v_2}\langle f^i(1)\rangle\langle f^i(2)\rangle - \left[F^{ie}_{12}\frac{\partial}{\partial v_2} + F^{ie}_{21}\frac{\partial}{\partial v_1}\right]\langle f^i(1)\rangle\langle f^i(2)\rangle \tag{4.29}$$

This is an important result since for low frequency turbulence the two terms can reinforce rather than subtract. For example if the average electron distribution has a bulk drift there will be regions where the gradient of the ion distribution has opposite sign to that of the electron. Thus the two terms in (4.29) will add since F^{ie}_{12} is proportional to $\partial/\partial\langle f^e\rangle$. This is in contrast to the one species case where the terms in the source cancelled for small separation. Once again we can reconcile this behaviour in terms of momentum conservation arguments. For the two species problem the ion distribution can relax independantly of ion-ion "collisions" (in fact these terms, F_{ii} and D_{ii}, cancel) since it can redistribute its average density in velocity space by exchanging momentum with the electron distribution.

In conclusion we must add that this general procedure is only justified when we are analyzing the two point equation for small scales. It is in that regime that we can make a precise distinction between the left and right hand side of (3.3) and (3.4). One represents the relaxation on local trapping scales while the other those on thermal scales. For large velocity separations the T_{12} operator (which reduces to $T_1 + T_2$) describes phenomena which occur on the same scale as S. In that regime it becomes more expedient to treat $S \simeq \langle\delta E\delta f\rangle$ as part of a homogeneous integro-differential equation rather than treating it as a given quantity which drives the fluctuations. This point is treated more fully in the next chapter.

Chapter 5

Solutions

We have emphasized in the past Chapters the problems of disparate scales in an iterative solution of the Vlasov equation. The regimes of small ($\simeq v_{tr}$) and large velocity ($\simeq v_{th}$) separation were identified as the two fundamental scales. Closely related to these characteristic velocities are the two *time* scales associated with the equal time equation ($\simeq \tau_{tr}$) and the two time equation ($\simeq \omega_p^{-1}$). These equations are mathematically and physically quite different. One is a boundary value problem while the other is an initial value problem. In fact we will use the solution of the one time equation as an initial value for the two time equation.

In this Chapter we outline the steps which connect these solutions, and obtain formal expressions for the correlation functions in the two regimes. We investigae the breakdown of the expansion parameter and relate the results to the discussion in (§2.2). Our approach is compared to that of Dubois and Espedal[11], who explicitly obtain an equation governing the incoherent fluctuations. On the basis of the latter they conclude that the these fluctuations are down by an order of $|\phi^2|$ compared to the coherent (or wave) response. We believe that for small separation their conclusion is incorrect, and we indicate how to retrieve the singular behaviour within their framework.

5.1. One Time Equation

Of the number of equations we have developed, the one time (or equal time) two point equation describes the more involved interactions in a plasma. It is only for $t_1 = t_2$ that the cross operators become important in the evolution of two neighbouring points. The enhancement of the correlation between such points is in part due to these terms, and allows the existence of a "clumping" mechansim.

The one time equation dwells on the creation and destruction of these "fluid" elements. For example if the phase space volume of such elements is sufficiently small then all the particles within that fluctuation will move together since they feel approximately the same forces. The period for which the fluctuations exist, as independant discrete elements, is determined through the "time constant" (T_{12}^{-1}) of the governing equation. The source term regenerates the fluctuations through the mixing of fluids of different density. This picture and the action of the T_{12} operator is complicated by the fact that the fluctuations do not originate from point (delta function) structures. In classical mechanics discrete particles cannot act on themselves. In this case, however, since the elements have a finite physical extent self interactions can occur which may enhance the lifetime of the structures. This section analyzes a method by which one can "solve" for the singular portion ($\tilde{G}$) of the correlation function. Such a quantity describes the structure (in an ensemble averaged sense) of these fluid elements or "macro particles". Our governing equation is Eq. (3.39).

We define (symbolically) the following operators

$$E_1 = E_1^0(k) + C_1^\phi(k) \tag{5.1}$$

E_1 is a renormalized Coulomb operator. That is $E_1^0 G_k \equiv \langle\delta E\delta f(2)\rangle_k \partial/\partial v_1 \langle f(1)\rangle$ and $C_1^\phi G_k \equiv \langle\delta E\delta f(2)\rangle_k \partial/\partial v_1 C_{11}^\phi(k)$: note that $[E_1^0 + E_2^0]G_k \equiv S_k$. We also write

$$\begin{aligned} T_1 =& ikv_1 + C_{11}^f(k) \\ \Delta T_{12}* =& C_{12}^f(k)* + C_{12}^\phi(k)* \end{aligned} \tag{5.2}$$

(The "$*$" is a reminder that the "12" terms are in fact convolutions of a k' sum with functions at k–k'.) Eq. (3.39) can be written in terms of these operators as

$$(\frac{\partial}{\partial t} + T_1 + T_2)G_k(1,2,t) = -\Delta[T_{12}* + T_{21}*]G_k(1,2,t) + [E_1 + E_2]G_k(1,2,t) \tag{5.3}$$

In a manner analogous to the test particle picture we will assume that $G_k(1,2,t)$ consists of two parts

$$G_k(1,2) = \overline{G}_k(1,2) + \tilde{G}_k(1,2) \tag{5.4}$$

$\tilde{G}(1,2)$ represents that part of the correlation function which describes the singular behaviour for small separation (in the case of discrete particles this would be the self correlation (1.35), where the delta functions describe the point structure of the particle) while $\overline{G}_k(1,2,t)$ will be associated with the shielding properties of the plasma.

We *define* the equation for $\overline{G}$ through

$$(\frac{\partial}{\partial t} + T_1 + T_2)\overline{G}_k(1,2,t) = [E_1 + E_2][\overline{G}_k(1,2,t) + \tilde{G}_k(1,2,t)] \tag{5.5}$$

This immediately defines $\tilde{G}_k(1,2)$ since $\tilde{G}_k(1,2) = G_k(1,2) - \overline{G}_k(1,2)$. We recognize that $(E_1 + E_2)\tilde{G}_k(1,2)$ acts as a source in that equation. This format is very reminiscent of the second equation in the BBGKY hierarchy with discretness effects *included.* In fact in the absence of any renormalization we would identify $(E_1 + E_2)\tilde{G}_k(1,2)$ as the exact discrete particle source. This is straightforward to see if we use $\tilde{G}_k(1,2) = n^{-1}\delta(v_1 - v_2)\langle f(1)\rangle$. In that case we also know that $\overline{G}_k(1,2)$ will describe the shielding of the discrete particles by the collective interactions of the plasma. Indeed, it is this analogy which motivated this particular choice in the first place. Time asymptotically, one can solve (5.3) and (5.5) to get

$$G_k(1,2) = \frac{T_1 + T_2}{T_1 + T_2 - E_1 - E_2}\tilde{G}_k(1,2) \tag{5.6}$$

The next step is to obtain the solution for small velocity separation ($v_- v_{tr}^{-1} \leq 1$). $\tilde{G}_k$ is defined as the difference between the exact solution G_k and $\overline{G}_k(1,2)$. From (5.3) we have

$$(\frac{\partial}{\partial t} + T_{12}(k)*)G_k(1,2) = S_k \tag{5.7}$$

In this formulation T_{12} $(= T_1 + T_2 + \Delta(T_{12} * + T_{21} *))$ contains all the v_- dependance while S_k is assumed *given* and the solution (4.19) is used to explicitly evaluate that term (Chapter 4). This is in contrast to the way we treated that term when evaluating $\overline{G}_k(1,2)$. There we took S_k $(= (E_0^1 + E_0^2)G_k(1,2))$ to be part of a homogeneous equation for $\overline{G}_k(1,2)$. $\tilde{G}_k(1,2,t)$ is then obtained from

$$\tilde{G}_k(1,2,t) = \int dt' g_{12}(k,t,t')S_k(t') - \overline{G}_k(1,2,t) \tag{5.8}$$

where $g_{12}(k,t,t')$ is the Greens function which solves (5.7), with the RHS set equal to $\delta(t)$. The incoherent self correlation can be expressed in terms of the T and E operators as

$$\tilde{G}_k(1,2,t) = \left(1 - \frac{E_1 + E_2}{T_1 + T_2}\right) \int dt' g_{12} S_k \tag{5.9}$$

For small separation, equation (5.9) can be written in the physically appealing form

$$\tilde{G}_k(1,2,t) = (\langle \tau_{cl} \rangle - \tau_{tr}) S_k \tag{5.10}$$

(5.10) derives from noting that as $x_-, v_- \rightarrow 0$, $E_1 + E_2 \rightarrow E_1^0 + E_2^0$ and $(T_1 + T_2)^{-1} \simeq \tau_{tr}$. Thus the clump portion of the correlation function is the difference between the total solution $(\langle \tau_{cl} \rangle S)$ and the shielding solution $(\tau_{tr} S)$. $\langle \tau_{cl} \rangle$ is some e-folding time characteristic of the solution[2] to (5.7). For example in the case where (in real space) we approximate T_{12} by

$$T_{12}(x_-, v_-) \simeq v_- \frac{\partial}{\partial x_-} - \frac{\partial}{\partial v_-} D_- \frac{\partial}{\partial v_-} \tag{5.11}$$

with

$$\begin{aligned} D_- =& D_{11} + D_{22} - D_{12} - D_{21} \\ \simeq& \frac{q^2}{m^2} \int \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} k^2 \langle \phi^2 \rangle_{k\omega} 2 Re g_{k\omega}(v_+)(1 - \cos kx_-) \end{aligned} \tag{5.12}$$

one can obtain the expression

$$\begin{aligned} \langle \tau_{cl}(x_-, v_-) \rangle =& \tau_0 \ln \frac{3}{k_0^2 [x_-^2 - 2x_- v_- \tau_0 + 2v_-^2 \tau_0^2]} \quad \arg \ln > 1 \\ =& 0 \qquad \text{otherwise} \end{aligned} \tag{5.13}$$

$$\tau_0 = (4k_0^2 D)^{-\frac{1}{3}} = (12)^{-\frac{1}{3}} \tau_{tr} \ .$$

by calculating the length of time during which particles that are initially separated by x_-, v_-, will move together before they separate by k_0^{-1}. This can be achieved by computing the moments

$$\langle x_-^n(t) v_-^m(t) \rangle \equiv \int dx_- \int dv_- x_-^n v_-^m g_{12} \tag{5.14}$$

setting $k_0^2 \langle x_-^2(\tau_{cl}) \rangle \simeq 1$, and solving the resulting equation.

In general the following observations can be made from the simple form (5.10). First as x_-, v_- approach zero $\tau_{cl} \gg \tau_{tr}$ since the first is singular while the second is not. Thus $\tilde{G}_k$ approaches $\langle \tau_{cl} \rangle S_k$ which is equal to the total response G_k. Second for large separation $\tau_{cl} \simeq \tau_{tr}$ so that $\tilde{G}_k \rightarrow 0$.

There are several points in this procedure which deserve some comment. Equation (5.9) is an integral equation and as such is not really a solution. On the other hand it has the distinct advantage of putting the v_- dependance of $\tilde{G}$ into the T_{12} operator. In that sense it is a step forward since one can analyze the T_{12} operator independantly of the v_- dependance of S (which has had the v_- dependance integrated out). The second point reiterates the previous discussion on the treatment of the "RHS" of the equation in the two regimes. For large velocity separation we solved $\overline{G}$ with S as an unknown. For small separation we treated that term as given. This procedure reflects the notion of disparate velocity scales. In the first case the T operator has an ikv_- dependance which for v_- large is of the same order as $\partial\langle f\rangle/\partial v$. Thus E_1 cannot be treated independantly of the T operator. When v_- is small, however, we can separate these quantities and look upon S as a distinct and independant quantity in the two point equation.

5.2. Two Time Equation

We now wish to show that this particular choice of $\overline{G}$ and $\tilde{G}$ leads to the shielded test particle picture where $\tilde{G}$ obeys a ballistic equation of motion (with Fokker-Planck renormalization). The assumption of time stationarity allows us to take Eq. (3.17) of Chapter 3 and set $t_2 = 0$. Neglecting the cross terms we have

$$\frac{\partial}{\partial t_1}\langle f(t_1)f(0)\rangle_k + ikv_1\langle f(t_1)f(0)\rangle_k + \int_0^{t_1} dt' C^f(t_1-t')\langle f(t')f(0)\rangle_k = \\ -\frac{q}{m}\int_0^{t_1} dt'\langle E(t')f(0)\rangle_k \frac{\partial}{\partial v_1}\overline{F}_k(t_1-t') \qquad (5.15)$$

The term containig $\partial/\partial v\langle f\rangle$ has been included with C_k^φ to produce $\overline{F}_k$ (see (2.15c)). This equation has to be solved with $\langle f(t_1)f(0)\rangle_k = \langle f(0)f(0)\rangle_k = G_k(1,2,t)$ as an initial condition. (5.10) propagates point 1 keeping point 2 fixed and is therfore valid for $t_1 \geq t_2 = 0$. For $t_2 \geq t_1 = 0$ the operator is changed form coordinate 1 to 2.

The solution to (5.10) is given by

$$\langle f(1)f(2)\rangle_{k\omega} = \{\overline{P}(1,k,\omega) + \overline{P}^*(2,k,\omega)\}\langle f(1)f(2)|t_1 = t_2\rangle_k \qquad (5.16)$$

where the $\overline{P}$ propagator is identical to P in (3.47) except for the inclusion of the renormalization from the collision operator C. That is $\overline{P}(1,k,\omega)$ is the solution to

$$-i\left(\omega - kv_1 + iC^f_{11}(k) - \frac{\omega_p^2}{|k|^2}k\frac{\partial \overline{F}_k}{\partial v_1}\int dv_1\right)\overline{P}(1,k,\omega) = 1 \tag{5.17a}$$

and can be written as

$$\overline{P}_{k\omega} = g_{k\omega}(1)\left\{1 - \frac{\omega_p^2}{|k|^2}k\frac{\partial \overline{F}_k}{\partial v_1}\frac{1}{\epsilon_{k\omega}}\int dv_1 g_{k\omega}(1)\right\} \tag{5.17b}$$

To obtain (5.11) we have used

$$\begin{aligned}\langle f(1)f(2)\rangle_{k\omega} &\equiv \int_{-\infty}^{\infty} dx\_ \int_{-\infty}^{\infty} dt\_ e^{ikx\_} e^{-i\omega t\_} \langle f(v_1,x_1,t_1)f(v_2,x_2,t_2)\rangle \\ &= \int_0^{\infty} dt\_ e^{-i\omega t\_} P(1,k,t_1-t_2)\langle f(1)f(2)|t_1 = t_2\rangle_k \qquad t_1 > t_2 \\ &+ \int_{-\infty}^{0} dt\_ e^{-i\omega t\_} P(2,k,t_2-t_1)\langle f(1)f(2)|t_1 = t_2\rangle_k \qquad t_2 > t_1\end{aligned} \tag{5.18}$$

From Eq. (2.12) we know that on the fast time scale $\langle \tilde{f}(1,t_1)\tilde{f}(2,0)\rangle$ satisfies

$$\frac{\partial}{\partial t_1}\langle \tilde{f}(1,t_1)\tilde{f}(2,0)\rangle + ikv_1\langle \tilde{f}(1,t_1)\tilde{f}(2,0)\rangle + \int_0^{t_1} dt' C^f_{11}(k,t_1-t')\langle \tilde{f}(1,t')\tilde{f}(2,0)\rangle = 0 \tag{5.19}$$

with solution

$$\langle \tilde{f}(1)\tilde{f}(2)\rangle_{k\omega} = \tilde{G}_{k\omega}(1,2) = [g_{k\omega}(1) + g^*_{k\omega}(2)]\tilde{G}_k(1,2) \tag{5.20}$$

Applying the P propagators we find that, *given* (5.20), $\overline{G}_{k\omega}(1,2)$ satisfies

$$\begin{aligned}\overline{G}_{k\omega}(1,2) = &\left(g_{k\omega}(1)\left\{1 - \frac{\omega_p^2}{|k|^2}k\frac{\partial \overline{F}_k}{\partial v_1}\frac{1}{\epsilon_{k\omega}}\int dv_1 g_{k\omega}(1)\right\} + (1 \leftrightarrow 2)\right)G_k(1,2) \\ &- [g_{k\omega}(1) + g^*_{k\omega}(2)]\tilde{G}_k(1,2)\end{aligned} \tag{5.21}$$

We substitute the expression for $G_k(1,2)$ in (5.21) to get

$$\overline{G}_{k\omega} = \left(\frac{i}{(\omega + iT_1 - iE_1)} - \frac{i}{(\omega - iT_2 + iE_2)}\right)\frac{T_1 + T_2}{T_1 + T_2 - E_1 - E_2}\tilde{G}_k(1,2) - \tilde{G}_{k\omega}(1,2) \tag{5.22}$$

we have expressed the P operators as $i/(\omega + i[T - E])$. To be consistent with our earlier assumptions on the nature of the time integral in the collision operator we have to set $D_{k\omega}$ etc. equal to $D(k)$ in the P operators. (5.22) can then be simplified to

$$\begin{aligned}\overline{G}_{k\omega}(1,2) &= \frac{T_1 + T_2}{[\omega + iT_1 - iE_1][\omega - iT_2 + iE_2]}\tilde{G}_k(1,2) - \tilde{G}_{k\omega}(1,2) \\ &= -\tilde{G}_{k\omega}(1,2) + \left(1 + i\frac{\omega_p^2}{|k|^2} g_{k\omega}(1)\frac{k}{\epsilon_{k\omega}}\frac{\partial \overline{F}_k}{\partial v_1}\int dv_1\right) \\ &\qquad \times \left(1 - i\frac{\omega_p^2}{|k|^2} g^*_{k\omega}(2)\frac{k}{\epsilon^*_{k\omega}}\frac{\partial \overline{F}^*_k}{\partial v_2}\int dv_2\right)\frac{i[\omega - iT_2] - i[\omega + iT_1]}{[\omega + iT_1][\omega - iT_2]}\tilde{G}_k(1,2)\end{aligned} \tag{5.23}$$

which is identical to $\langle f^c(1)f^{c*}(2)\rangle_{k\omega} + \langle f^c(1)f^*(2)\rangle_{k\omega} + \langle f(1)f^{c*}(2)\rangle_{k\omega}$. This is of course the result we set out to prove. As previously advertised, we can also identify $\overline{G}_{k\omega}(1,2)$ as the shielding response to the incoherent spectrum $\int dv_1 \int dv_2 \tilde{G}_{k\omega}(1,2)$, since (5.23) can be integrated over v_1 and v_2 to yield

$$\langle \phi^2 \rangle_{k\omega} = \frac{\langle \tilde{\phi}^2 \rangle_{k\omega}}{|\epsilon_{k\omega}|^2} \tag{5.24}$$

If we neglect the incoherent fluctuation then the solution becomes $\langle f^c f^c \rangle$. The equations revert to the more common weak turbulence expansions (including renormalizations). These solutions are ultimately concerned with wave, and mode coupling type of interactions, since the driving mechanism is the zeroes of the dielectric function.

5.3. Dubois and Espedal Solution

In their paper on the D. I. A. as applied to plasma turbulence , Dubois and Espedal[11] derived a set of renormalized two point equations. One of their conclusions was that to nominally second order in the electric field strength no "...additional noise terms arise which can be interpreted as due to phase space clumps." This statement is given more physical substance by pointing out that the D. I. A. cannot account for correlated objects such as clumps since it is known to be an exact solution to the random coupling model of Kraichnan[8,9]. In such a model localized effects would tend to be smoothed over and lost.

On a more mathematical basis Dubois and Espedal base their conclusion on the structure of the equation for the unscreened correlation function. This equation is one of the novel aspects of their work and allows a direct calculation of the self correlation function $\langle \tilde{f}\tilde{f} \rangle$. The authors estimate this component of the fluctuations and propose that it is down by a factor of $|\phi|^2$ compared to $\langle f^c f^c \rangle$. The result is arrived at by observing that no source term of the form described in Chapter 4 appears in their equation. In fact the only "source" term is proportional to $|\phi|^4$.

While the technique used to obtain their two point equation is certainly elegant and rigorous, we believe that it tends to obscure some of the underlying physics. In this section we follow the derivation and show how the singular behaviour is contained in the equation for $\tilde{G}(1,2)$ in a somewhat convoluted fashion.

The following method (or a close variant) is adopted by the authors. They define a slightly different $\overline{G}$ and $\tilde{G}$ which we will denote by primes so that $G = \overline{G}' + \tilde{G}'$. The equations satisfied by these quantities are

$$(\frac{\partial}{\partial t} + T_1 + T_2)\overline{G}'_k(1,2,t) = [E_1 + E_2][\overline{G}'_k(1,2,t) + \tilde{G}'_k(1,2,t)] - [R_{12} * + R_{21} *]G_k \quad (5.26a)$$

$$(\frac{\partial}{\partial t} + T_1 + T_2)\tilde{G}'_k(1,2,t) = -\Delta[T_{12} * + T_{21} *][\overline{G}'_k(1,2,t) + \tilde{G}'_k(1,2,t] + [R_{12} * + R_{21} *]G_k \quad (5.26b)$$

$$R_{12} * = d^t_{12} * + d^f_{12} * + \gamma_{12} * \quad (5.26c)$$

We can use (5.26a) in (5.26b) to obtain the equation for $\tilde{G}'$

$$\begin{aligned}(\frac{\partial}{\partial t} + T_1 + \Delta T_{12} * - R_{12} * + (1 \leftrightarrow 2))\tilde{G}'_k(1,2,t) = \\ -[\Delta T_{12} * - R_{12} * + (1 \leftrightarrow 2)](\frac{E_1 - R_{12} * + (1 \leftrightarrow 2)}{T_1 - E_1 + R_{12} * + (\leftrightarrow 2)})\tilde{G}_k(1,2,t)\end{aligned} \quad (5.27)$$

Equation (5.27) can after some algebra be reduced to the format (Appendix C) in Ref.[11]. It is worth mentioning that in Ref. [11] the one point equation is obtained by iterating $f = f^c$ rather than $f = f^c + \tilde{f}$. In other words the renormalized collision operators do not contain terms such as $F, \mathfrak{F}$ etc.. We do not elaborate on this discrepancy at this point since it does not affect our fundamental concern: the existence of a $\tilde{G}$ driven by the source described in Chapter 4. It is fairly simple to show that the singular behaviour will occur whether you iterate $f = f^c + \tilde{f}$ or just $f = f^c$.

We want to investigate this equation in the limit of small separation. Going to "–" coordinates and taking the limit $x_-, v_- \to 0$ (the former is achieved by integrating over k) we have

$$\begin{aligned}\int dk(T_1 + T_2 + \Delta[T_{12}* + T_{21}*]) &\to \int dk[E_1 + E_2 - E_1^0 - E_2^0] \\ \int dk\Delta[T_{12}* + T_{21}*] &\to \int dk[-T_1 - T_2 + E_1 + E_2 - E_1^0 - E_2^0] \\ \int dk[R_{12}* + R_{21}*] &\to \int dk - [R_1 + R_2]\end{aligned} \tag{5.28}$$

R_1 and R_2 are the one point versions of (5.26c). Eq. (5.28) follows quite simply from the properties of (3.40). We use this result in (5.27) and get

$$(\frac{\partial}{\partial t} + \int dk[E_1 - E_1^0 + R_1 + (1 \leftrightarrow 2)])\tilde{G}'_k = \int dk[T_1 - E_1 + E_0^1 - R_1 + (1 \leftrightarrow 2)]\overline{G}'_k \tag{5.29}$$

but from the definition of $\overline{G}'_k$

$$\int dk[T_1 - E_1 + E_1^0 - R_1 + (1 \leftrightarrow 2)]\overline{G}'_k = \int dk[E_1 + R_1 + (1 \leftrightarrow 2)]\tilde{G}' + \int dk[E_1^0 + E_2^0]\overline{G}'_k \tag{5.30}$$

so that for small separation eqaution (5.27) reduces to (remember that $\int dkG_k \equiv G(x_- = 0)$)

$$\lim_{x_-, v_- \to 0} \frac{\partial}{\partial t}\tilde{G}(x_-, v_-)' = [E_1^0 + E_2^0]\overline{G}'(x_-, v_-) + [E_1^0 + E_2^0]\tilde{G}'(x_-, v_-) = S(x_-, v_-) \tag{5.31}$$

The ordering of the source term is back to $|\phi|^2$. It is interesting to see how this ordering changes (and becomes meaningless) for different velocity scale lengths. Consider the operator on the RHS of (5.27). For large v_-, ikv_- dominates the terms in the denominator, and $V\tilde{G}$, where V is given by

$$V = [E_1 - R_{12} * + (1 \leftrightarrow 2)]/[T_1 - E_1 + R_{12} * + (1 \leftrightarrow 2)]$$

is of order $|\phi|^2$. $\Delta[T_{12} + R_{21}]$ is also of order $|\phi|^2$ so that for large separation the term become a "source" of $O(|\phi|^4)$. For small separation, however, V becomes of O(1)($\simeq (E_1 + R_1)/(E_1 + R_1)$) and one recovers the correct source term.

With reference to the discussion in §2.2, $\overline{\mathcal{F}}$ and $\tilde{\mathcal{F}}$ are given by

$$\lim_{t_1,t_2\to\infty} \quad \tilde{\mathcal{F}}(t_1)\int_0^{t_2} dt' g_k(t_2-t')\tilde{\mathcal{F}}(t') = -\,[\Delta T_{12} - R_{12}]\tilde{G}_k(1,2)$$
$$\lim_{t_1,t_2\to\infty} \quad \overline{\mathcal{F}}(t_1)\int_0^{t_2} dt' g_k(t_2-t')\overline{\mathcal{F}}(t') = -\,[\Delta T_{12} - R_{12}]\overline{G}_k(1,2) \tag{5.33}$$

We can see from (5.28) that part of the action of the $\tilde{\mathcal{F}}$ contribution is to cancel the T_1 and T_2 renormalization. At the same time the $\overline{\mathcal{F}}$ part is seen, from (5.28) and (5.30), to change its ordering from $|\phi|^4$ to $|\phi|^2$.

This change in the ordering is rather difficult to see when the equation is written out explicitly. An example might serve to clarify the point. We simplify the analysis by assuming that the only terms which contribute to the renormalization are the Markovian diffusion coefficients. The equation for the total correlation function $G_k(1,2)$ is

$$\left(\frac{\partial}{\partial t} + ikv_- - \frac{\partial}{\partial v_1}D_{11}\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2}D_{22}\frac{\partial}{\partial v_2}\right)G_k(1,2,t) =$$
$$\left(\frac{\partial}{\partial v_1}D_{12} * \frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_2}D_{21} * \frac{\partial}{\partial v_1}\right)G_k(1,2,t)$$
$$+ \left(ik\frac{\partial}{\partial v_1}\langle f\rangle\frac{\omega_p^2}{|k|^2}\int dv_1 - ik\frac{\partial}{\partial v_2}\langle f\rangle\frac{\omega_p^2}{|k|^2}\int dv_2\right)G_k(1,2,t) \tag{5.32}$$

Using (5.26) the equations for $\overline{G}$ and $\tilde{G}$ are

$$\left(\frac{\partial}{\partial t} + ikv_- - \frac{\partial}{\partial v_1}D_{11}\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2}D_{22}\frac{\partial}{\partial v_2}\right)\overline{G}_k(1,2,t) =$$
$$\left(ik\frac{\partial}{\partial v_1}\langle f\rangle\frac{\omega_p^2}{|k|^2}\int dv_1 + (1\leftrightarrow 2)\right)[\overline{G}_k(1,2,t) + \tilde{G}_k(1,2,t)] \tag{5.34a}$$

and

$$(\frac{\partial}{\partial t} + ikv_- + \frac{\partial}{\partial v_-} D_- * \frac{\partial}{\partial v_-})\tilde{G}_k(1,2,t) =$$

$$-(\frac{\partial}{\partial v_1} D_{12} * \frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_2} D_{21} * \frac{\partial}{\partial v_1}) \int d\omega (\frac{\partial}{\partial v_1} \langle f \rangle g_{k\omega} \frac{ik}{\epsilon_{k\omega}} \frac{\omega_p^2}{|k|^2} \int dv_1 + (1 \leftrightarrow 2))\tilde{G}_{k\omega}(1,2,t)$$

$$-(\frac{\partial}{\partial v_1} D_{12} * \frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_2} D_{21} * \frac{\partial}{\partial v_1}) \int d\omega \frac{q^2}{m^2} k^2 g_{k\omega} g_{k\omega}^* \frac{\langle \phi^2 \rangle_{k\omega}}{|\epsilon_{k\omega}|^2} \frac{\partial^2}{\partial v_1 \partial v_2} \langle f(1) \rangle \langle f(2) \rangle \tag{5.34b}$$

A cursory examination of equations (5.34a) and (5.34b) leads to an intriguing conclusion: the information on the singular behaviour seems to have been lost. There is no doubt that equation (5.33) is singular for small separation since the diffusion coefficients cancel while the Coulomb operator does not. The equation for $\overline{G}$ does not contain that information since the operator on the LHS remains finite for small separations. The equation for $\tilde{G}$ does not seem to contain that information either; D_- goes to zero but we are still left with the operator on the second line which is finite as $x_-, v_- \to 0$. Furthermore even if that operator dissapeared the "source" term (presumably the third line in (5.34b) since it looks like a mixing of the average gradients by the turbulent electric fields) is now down by an order of ϕ^2 (since D_{12} is proportional to ϕ^2).

This seeming discrepancy or loss of information can easily be reconciled. If we consider "–" coordinates only, and take the limit $1 \to 2$ then

$$\frac{\partial}{\partial v_1} D_{12} \frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_2} D_{21} \frac{\partial}{\partial v_1} \to -\frac{\partial}{\partial v_1} D_{11} \frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2} D_{22} \frac{\partial}{\partial v_2}$$

Moreover equation (5.33) (for $\partial/\partial t = 0$) gives

$$(\frac{\partial}{\partial v_1} D_{11} \frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2} D_{22} \frac{\partial}{\partial v_2})\overline{G}_k(1,2) = [E_0^1 + E_0^2] G_k(1,2)$$

Thus (as $1 \to 2$) lines 2 *and* 3 of (5.34b) approach, in a somewhat convoluted manner, the source term of Chapter 3.

5.4. One Point Review

We have presented an approximate technique for solving the two time and equal time two point equatons. This approach has relied on the presence of two time scales which allow us to decouple these equations and treat them independantly. Starting from the two point equation for G_k the partition of G_k, defined through (5.5), into $\overline{G}_k$ and $\tilde{G}_k$ has led quite naturally to a "test-clump" picture. It is interesting to see how this partition is related to the one point equations. Contrary to one's first inclination, our $\overline{G}$ and $\tilde{G}$ are *not* consistent with

$$\left(\frac{\partial}{\partial t} + T_1\right) f^c \tilde{f} = E \tilde{f} \frac{\partial}{\partial v} \overline{F} \tag{5.35a}$$

$$\left(\frac{\partial}{\partial t} + T_1\right) \tilde{f} \tilde{f} = \mathcal{I} \tilde{f} \tag{5.35b}$$

but rather with

$$\left(\frac{\partial}{\partial t} + T_1\right) f^c \tilde{f} = E \tilde{f} \frac{\partial}{\partial v} \overline{F} - \mathcal{I} f^c \tag{5.36a}$$

$$\left(\frac{\partial}{\partial t} + T_1\right) \tilde{f} \tilde{f} = \mathcal{I} + \mathcal{I} f^c \tag{5.36b}$$

On the fast or ballistic time scale *both* these set of equations reduce to (2.11) and (2.12) namely

$$\left(\frac{\partial}{\partial t} + T_1\right) f^c \simeq E \frac{\partial}{\partial v} \overline{F} \tag{5.37a}$$

$$\left(\frac{\partial}{\partial t} + T_1\right) \tilde{f} \simeq 0 \tag{5.37b}$$

From which we recover the one point shielding results of Chapter 2.

We have shown that the set (5.36) will yield the shielded clump picture when the "slow" (equal time) version (5.36) is used as an initial condition for the "fast" (two time) version (5.37). The following question arises: what is the effect of using the same procedure with, instead of (5.36), the set (5.35). In fact one might worry that an inconsistency is generated since we will be propagating a *different* incoherent response ($\tilde{G}'$) through (5.37b). This paradox is easily resolved by noting that the initial condition "$\overline{G}$" will also be different ($\overline{G}'$). In fact it is simple to show that this different initial condition produces the missing part (as it clearly should) of the incoherent response. In other words the total potential will be given by

$$\langle \phi^2 \rangle_{k\omega} = \frac{\langle \tilde{\phi}'^2 \rangle_{k\omega}}{|\epsilon_{k\omega}|^2} + R_{k\omega} \tag{5.38}$$

$R_{k\omega}$ is the remainder generated by the different $\overline{G}'$ condition and $\tilde{\phi}'$ is the potential generated by the $\tilde{f}$ defined through (5.35b).This can also be written as

$$\langle\phi^2\rangle_{k\omega} = \frac{\langle\tilde{\phi}^2\rangle_{k\omega}}{|\epsilon_{k\omega}|^2} \tag{5.39}$$

where now $\tilde{\phi}$ is the potential generated by the $\tilde{f}$ defined through (5.36b). Thus both partitions yield the same *total* potential on the other hand (5.36) is emminently more useful since it leads to the concept of a shielded "macro-particle" which can be treated in much the same way as a shielded "test-particle".

In conclusion we add that if by the D. I. A. we understand a scheme which iterates the coherent response only then this procedure will break down. For small separation the incoherent response is certainly of the same "order", if not larger than f^c. This is in agreement with the physical models behind the coherent and incoherent response. The former represents a weak coupling, sufficient to describe shielding and other non local phenomena. The latter is concerned with the much more violent interactions at wave particle resonances: there results a strong distortion and modulation of resonant velocity streams of the distribution function. This is a strong coupling problem where the stream develops a complicated or "incoherent" phase dependance due to the highly non-linear interaction at the resonance.

Chapter 6

Fluctuation Self-Interaction

We have established in the past Chapters the existence of non wave-like fluctuations which we believe are integral to the relaxation processes in a collisionless plasma. These appear as complicated spatial granulations of the phase space density $f(x, v, t)$, and are a result of the imcompressible nature of the Vlasov equation. In this context $f(x, v, t)$ can be thought of as the density of an incompressible, self interacting, fluid which flows in the two dimensional phase space $\{x, v\}$. (As opposed to the hydrodynamic density $\rho(x, t)$ which describes a fluid in the three dimensional space $\{x\}$.) A turbulent state will mix (rearrange) the particle distribution and produce fluctuations in f about its average value $\langle f \rangle$. The resulting phase space distribution will have local excesses δf_+ of charge[2], and local depletions δf_- ("holes"). This distinction is important since the physical behaviour of these entities is somewhat different. δf_+ being an aglomeration of like charges is self repulsive. Holes can be viewed as gravitational bodies[29] which are self binding.

This Chapter investigates, qualitatively, the self interaction of such fluctuations. Our discussion will treat the case $\epsilon_{k\omega} > 0$, for which the velocity of the fluctuations is less than the thermal velocity The self energy of clumps was considered negligible in Ref.[2]. Later work[30], recognizing that any mixing process would also generate bound states such as holes[29] came to an opposite conclusion. Thermodynamically[31], holes can play an important role in a turbulent situation because they represent the most probable state for local equilibria[17]. The most persuasive arguments for the existence of such entities comes from computer simulations. Berk *et al.*[14] have studied the interaction of these modes and seen the persistence of such structures in the evolution of a two-stream instability. More recently[32] numerical simulations treating strongly turbulent states have identified holes as a major component in the relaxation process.

From simple energy considerations it is possible to see that a hole is self binding. Consider a hole with a certain spatial and velocity extent: ballistic streaming will quickly cause the structure to shear apart. In the process the potential energy of the hole will decrease. To conserve energy the hole must increase its kinetic energy. The only way this can be done is by pushing particles out in velocity which in turn *contracts* the hole. These arguments gain a different perspective when one considers the hole as the dual of a gravitational mass[33]. The picture becomes reminiscent of a cluster of gravitating bodies. Thin streams of more energetic fluid are ejected from the main body and rotate clockwise about its centre. This rotation occurs because of the gravitational attraction between the main body and the spiralling arms. The elements travelling with a positive velocity on the right hand side of the phase plane are attracted to those on the left. This reduces their velocity and they start moving downwards. If the orbits are "trapped", they will actually reverse their direction of motion and end up moving to the left. The same set of events causes elements on the left to move upward and to the right. The precise details of the motion depend on the initial energy and equilibrium states of such structures.

It is important to realize that these interactions are "self-energy" ones in the truest sense of the expression. The structures have a certain velocity and spatial extent. This allows the fluctuations to act on themselves and co-ordinate the energy and momentum exchange *within* the structure. It is this feature which is the essential element in the "self-energy" relaxation. Furthermore, in the case of a hole, this mechanism *enhances* the lifetime of the structure.

We wish to investigate in what way it might be possible to incorporate such effects within a Kinetic Theory. It is important, however, to realize that the covariance $\langle \delta f(1) \delta f(2) \rangle$ *cannot* distinguish between δf_+ and δf_-. For small separation it becomes a variance (self correlation) which is indifferent to the sign of δf.

6.1. Stochastic Acceleration Problem

We want to consider the action of the operator

$$\frac{\partial}{\partial t} + v_- \frac{\partial}{\partial x_-} + \frac{\partial}{\partial v_-} D_- \frac{\partial}{\partial v_-} \tag{6.1}$$

in the context of the previous discussion. This operator is the basis of the clump lifetime calculation in Ref.[2]. It can be obtained as a reduced version of the T_{12} operator in the following way: (i) neglect all contributions from the iteration of ϕ^2. (ii) In the resulting equation neglect β_- and consider the Markovian limit.

The first assumption derives from treating the problem as a stochastic acceleration. In other words the fields are prescribed externally and f is not related to ϕ through Poisson's equation. This means that the problem is not self consistent and the medium cannot act back on itself through the intermediary of the electric fields. These fields, however, can randomly diffuse the "test" particles off their ballistic orbits.

The stochastic acceleration approach is clearly a somewhat reduced and incomplete description of the problem. Furthermore from Chapter 4 we know that it generates the incorrect source term. For the purposes of this discussion, however, we will neglect the source term and consider the action of this operator on a set of initial fluctuations comprised of holes and and positive fluctuations. The important point is that (6.1) retains one of the essential features of the exact equations; the T_{12} operator dissapears for small separation. This leads to an enhanced correlation between two points and produces the logarithmic clump lifetime of Chapter 5.

It is interesting to see what is the effect of this operator on a localized structure such as a hole. The ballistic operator will shear the structure apart in real space while the diffusion will tear it in velocity. Two neighbouring points will "co-exist" for a time τ_{cl} before they eventually diverge. The theme is one of indiscriminate destruction. That it can not reproduce qualitative results such as those previously described is not surprising since we have thrown out *any* possibility of such an interaction by specifying a lack of correlation between field and fluctuation. More important, however, is the consideration that if holes are the major protagonists in the turbulent spectrum then such an approximation will seriously *under estimate* the lifetime of these structures.

6.2. Gravitational Instability

We have seen that the stochastic acceleration operator does not have any means of preventing decay; in fact it accelerates it. The question therefore arises as to what kind of effect one should look for which might slow down this decay. To allow us to develop some qualitative ideas on the subject we begin by treating the dual of the hole: the gravitating mass.

A first step in the formation of gravitating bodies, such as stars, is the attraction of a large mass of gas to form a single condensate that is gravitationally bound. An idealized model which provides insight into this mechanism is the Jeans gravitational instability. In its simplest form a medium in equilibrium is subjected to a small perturbation. Under certain conditions the gravitational forces will cause an exponential growth of this perturbation. The resulting regions of increasing density represent the initial formation of the body.

We assume a plane wave solution of the form

$$\tilde{\rho} = a e^{i(kx+\omega t)}. \tag{6.2}$$

$\tilde{\rho}$ is the perturbative piece of the density. On linearizing the fluid momentum equation, coupled with Poisson's equation[27] one obtains the dispersion relation

$$\omega^2 = k^2 \bar{v}^2 - 4\pi G \rho_0 \tag{6.3}$$

$\bar{v}$ is the mean spread in velocity of the body. G is the gravitational constant and ρ_0 is the unperturbed density. Equation (6.3) predicts that for wavelengths greater than $\lambda_j = (\pi \bar{v}^2 / G\rho_0)^{1/2}$ the system will be gravitationally unstable. The disturbance will grow exponentially causing a condensation of matter. This critical length is known as the "Jeans length" and signals the onset of the gravitational collapse.

It is fairly easy to show that this instability is the dual of the more familiar two stream instability[33]. Consider the step distribution of Figure 6.1 (a). For velocities greater than $|\Delta v|$ the distribution is

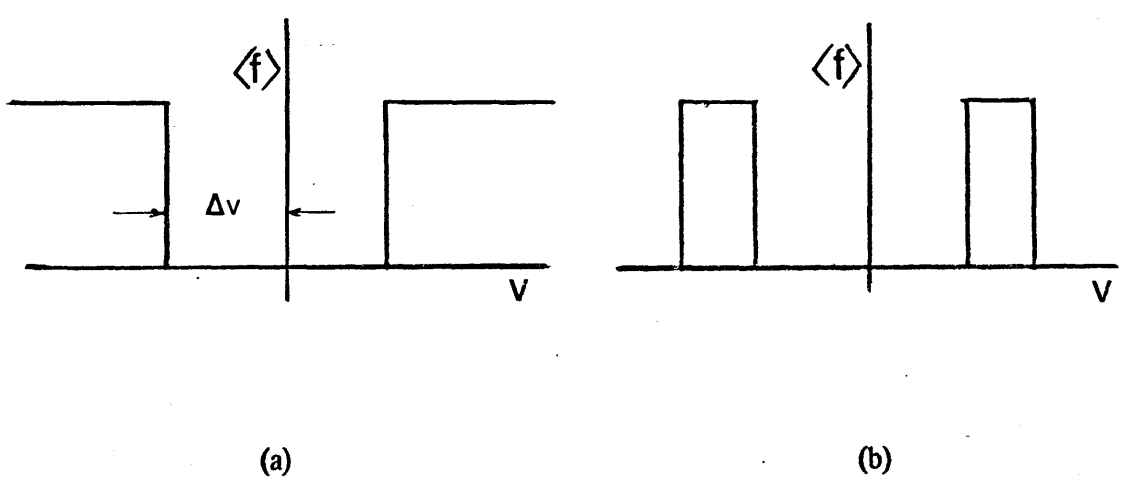

Fig. 6.1 Distribution Function for Two Stream Instability

flat and has value 1. For $|v| \leq |\Delta v|$ the distribution has value zero. The dynamics of the instability are primarily determined by the boundary at the step. They are relatively insensitive to the tails of the distribution which participate in a shielding role. Moreover, since $\partial\langle f\rangle/\partial v$ is zero in that region, the only phenomena which survives is undamped oscillations or an instability. As far as the instability is concerned, the distributions in figures 6.1 (a) and 6.1 (b) are identical. The second, is the one more commonly associated with the two stream instability.

To obtain the dispersion relation we use

$$\frac{\partial}{\partial v}\langle f\rangle = \frac{1}{2\Delta v}[\delta(v-\Delta v)-\delta(v+\Delta v)] \tag{6.4}$$

in

$$\epsilon_{k\omega} = 0 = 1 + \frac{\omega_p^2}{|k|^2}\int dv \frac{ik\partial\langle f\rangle/\partial v}{(\omega - kv + i\delta)} \tag{6.5}$$

The eigenvalue equation becomes

$$\omega^2 = k^2\Delta v^2 - \omega_p^2 \tag{6.6}$$

In this case the Jeans length is given by $\lambda_j' = (\pi m\Delta v^2/ne^2)^{1/2}$. This is identical to λ_j if we identify Gm^2 as the equivalent "gravitational charge".

The distribution used in (6.5) is a spatially averaged function. We can extend these arguments to an $\langle f\rangle \simeq f_k$ so long as thw wavelength of the instability is less than k^{-1}. In that case the dielectric does not distinguish between a small $\{k\}$ fluctuation and the average distribution. One might try to construct an analytic theory in which an equivalent, local f, plays the role of $\langle f\rangle$ as the source of a trapping mechanism. For example ballistic motion causes holes to elongate in real space. The longer they become the more prone they are to gravitational collapse. This is clear from (6.6) which shows that they are susceptible to long wavelength perturbations. These perturbations generate an instability which tends to hold the structures together.

This simple model is open to criticism, and the more pertinent question are: Is this information on the behaviour of single elements transferable to a Kinetic Theory which involves $\langle\delta f\delta f\rangle$ only? And if so, how can we use this type of model to describe a statistical ensemble of such structures?

The most striking objection to such a theory is the undue emphasis it places on the negative element of the fluctuations. Charge conservation demands that for every hole there be an equivalent charge excess. It is not clear *a priori* why a statistical quantity such as the variance should single out holes.

A model, which is consistent with numerical simulations[32], sees the emphasis on holes displayed in the following way. Regions of positive δf could blow themselves apart onto large surface areas of phase space. Holes would be dispersed within this sea. To conserve charge density the "depth" of the holes would be much greater than the "height" of the charge excess. This would mean that the dynamics of the system would be entirely dominated by hole-hole interactions and the correlation function would be primarily meausuring hole material.

We have conducted some preliminary work along such lines. Some of the terms proportional to $\langle \phi f \rangle_k$ in the two point equation can be expressed as an equivalent background distribution which looks like a negatively peaked "$\langle f \rangle$". This could generate an instability in the *relative* coordinate system $\{x_-, v_-\}$. Whether this instability specifically enhances the lifetime of *individual* fluctuations or just generates new ones is lost in the ensemble averaging process. The result appears as a longer "clumping" time. We have not carried detailed calculations due to the complexity of the terms present. The interpretation of part of the C^{ϕ} terms through this or a similar model remains an intriguing possibility.

Chapter 7

Role of Clumps in Ion-Acoustic Turbulence

Most models of ion-acoustic turbulence rely on the interaction of ion-sound *waves* with each other, and particles in the average distribution. The dissipative effects due to the presence of turbulence are often characterized by an instability induced through particle streaming. The simplest case of such an instability, in the absence of magnetic fields, is the ion-acoustic or two stream instability. This has been investigated at some length[34] within the framework of weak turbulence theory. In particular non-linear effects such as resonance broadening have been invoked as additional ingredients in the dissipation and stabilization of the growing modes[35–36]. In this Chapter we consider a simplified version of our set of equations for a two species plasma. The role of electron and ion clumps is investigated as a possible constituent in the dynamic processes.

As demonstrated in Chapter 4, the source term for the correlation function is finite for a two species plasma. This is a result of the added degree of freedom due to the presence of the second species. Energy and momentum conservation do not impose the strict constraints present in the one dimensional, one species problem. If there exits an external source, such as a current within the plasma, fluctuations are generated which relax the average distribution. In the process the mixing off the gradients of $\langle f \rangle$ regenerate the fluctuations. A self sustaining state exists when the decay of these fluctuations due to the turbulent fields is balanced by the creation of new ones. Only when this condition is met can the effect of clumps be considered an important element in the description of turbulence. We thus which to address the intrinsic question of "regeneration" within an *a priori* self consistent formulation.

We proceed by solving numerically a set of idealized equations for ion and electron clumps. The background distributions are assumed to be Maxwellians; the ions being stationary and the electrons drifting with a velocity v_d. At each time step the source and diffusion coefficients are calculated self consistently from the solution. A wide range of parameter space (T_e/T_i, the electron to ion temperature ratio, and v_d the electron drift velocity) is investigated. We find that the fluctuations regenerate at drift velocities which are appreciably below the linear instability level.

7.1. The Basic Equation

The solution of the exact equations for a two species plasma presents a somewhat formidable task even for numerical analysis. Our aim is less ambitious in that we will deal with what we believe is the minimum amount of information necessary for a relevant description of the problem. The two species equations are easily derived using the method of §4.4 coupled with the renormalization technique of Chapters 2 and 3.

We assume that ions and electrons obey the following *model* equation (α, refers to the species).

$$\left(\frac{\partial}{\partial t} + v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2} - \sum_{i,j=1,2} \frac{\partial}{\partial v_i} D^{\alpha}_{ij} \frac{\partial}{\partial v_j}\right)\langle \delta f^{\alpha} \delta f^{\alpha}\rangle = S^{\alpha} \tag{7.1}$$

This model equation is a considerably reduced version of the formulation in Chapter 3 and it is worth pausing to analyze the terms which have been discarded.

We start with the perturbed Fokker Planck operator C^f. This consists, in addition to the diffusion D, of a drag F, and perturbed diffusion and friction coefficients $d^f + d^t$ and $\mathfrak{F}^f + \mathfrak{F}^t$. The first point to note is that $d^f + d^t$ and $\mathfrak{F}^f + \mathfrak{F}^t$ conserve (in the long wavelength limit) momentum and energy against F and D. d and $\mathfrak{F}$ are back reactions of the plasma to the disturbance caused by D and F. It is therefore plausible to assume that they would *reduce* the effect of the test particle quantities (D and F), rather than enhance them. Clearly the interaction is quite complicated for finite k, and mode coupling between different wavelength might result in a different picture. While more investigation is required to determine the exact magnitude of these terms, we believe that the the exclusion of the perturbed quantities represents a *lower* limit on the clump lifetime.

On a more concrete level, we notice that in the relative coordinate, v_-, the F and $d^f + d^t$ terms contain a single velocity derivative compared to the diffusion operator (D). For small v_- we conclude that the latter destroys far more effectively (by a factor of v_a/v_{tr}) the correlabtion function.

The second class of terms belong to the C^{ϕ} operator. We have attempted in Chapter 6 to give a qualitative feel for the possible effects of such terms. While some of them can enhance the clump lifetime, it is not clear what the net effect of these terms is. In the interest of simplicity we neglect them in this model.

The final modification concerns the source term. It is helpful to visualize the self consistent aspect of the calculation in two distinct steps. The first adds the $F\partial/\partial_v\langle f\rangle$ term to the source. This as previously mentioned, reflects directly the notion of momentum and energy conservation for the *average* distribution function. The second appears in the renormalization of the green functions etc., within these expressions. We neglect the latter contributions. Physically and mathematically, it is clear that the presence of $F\partial/\partial_v\langle f\rangle$ is the most important difference to the source, compared to previous calculations.

Bearing these limitations in mind we can, accordind to Chapter 5, write $\langle\delta f^a\delta f^a\rangle \equiv G^a$ as the sum of two parts $\tilde{G} + \overline{G}$. In the relative coordinate system G satisfies

$$\left(\frac{\partial}{\partial t} + v_-\frac{\partial}{\partial x_-} - \frac{\partial}{\partial v_-}D^a_-\frac{\partial}{\partial v_-}\right)G^a = S^a \tag{7.2}$$

where the source term is defined through

$$S^a = -2(q_a/m_a)\langle E\delta f^a\rangle\frac{\partial}{\partial v_+}\langle f^a\rangle \tag{7.3}$$

and the diffusion coefficient through

$$D^a_- = (q_a^2/m_a^2)\sum_{k',\omega'} k'^2|\phi_{k'\omega'}|^2 2Re g_{k'\omega'}(v_+)(1 - cos kx_-) \tag{7.3}$$

$\overline{G}$ satisfies a similar equation with D_- replaced by D_+,

$$\left(\frac{\partial}{\partial t} + v_-\frac{\partial}{\partial x_-} - \frac{\partial}{\partial v_-}D^a_+\frac{\partial}{\partial v_-}\right)\overline{G}^a = S^a \tag{7.4}$$

D^a_+ is the spatially homogeneous diffusion coefficient given by

$$D^a_+ = (q_a^2/m_a^2)\sum_{k',\omega'} k'^2|\phi_{k'\omega'}|^2 2Re g_{k'\omega'}(v_+) \tag{7.5}$$

The incoherent spectrum is related to the total potential through

$$|\phi_{k'\omega'}|^2 = \frac{\left|\tilde{\phi}^e_{k'\omega'}\right|^2 + \left|\tilde{\phi}^i_{k'\omega'}\right|^2}{|\epsilon_{k'\omega'}|^2} \tag{7.6}$$

We simplify matters by taking $\epsilon_{k'\omega'}$ to be the lowest order, unrenormalized dielectric,

$$\epsilon_{k'\omega'} = 1 - \frac{1}{k'^2\lambda^2_{d_e}}\left[Z'(\frac{\omega'-v_d}{k'v_e}) + \frac{T_e}{T_i}Z'(\frac{\omega'}{k'v_i})\right] \tag{7.7}$$

In the above expressions λ_{d_a} is the debye length for the "a" species. T_e, T_i, v_e, v_i, are respectively electron and ion temperature and thermal velocities. Z' is the derivative of the plasma dispersion function

$$Z(z) = i\pi^{\frac{1}{2}}exp(-z^2)(1 - Erf(-iz)) \tag{7.12}$$

where $Erf(z)$ is the error function.

From the these expressions it is clear that to obtain the diffusion coefficients we require the incoherent or "clump" contribution. This can easily be obtained from the solution of (7.2), and (7.4) since according to (5.10) $\langle\delta f\delta f\rangle$ is the difference between these expressions.

We are now in a position to reduce the source and diffusion coefficients to the following form:

$$\begin{aligned} D^a_+ &= 2\pi\omega^4_{p_a}\lambda^3_{d_a}v_+^{-1}\sum_{k'}\frac{\Delta^e_{k'} + \Delta^i_{k'}}{|k\lambda_{d_a}|^3}\frac{v_+}{\left|\epsilon_{k',k'v_+}\right|^2} \equiv 2\pi\omega^4_{p_a}\lambda^3_{d_a}v_+^{-1}\left[d^e_0 + d^i_0\right] \\ d^e_0 &= \sum_{k'}\frac{\Delta^e_{k'}}{|k\lambda_{d_a}|^3}\frac{v_+}{\left|\epsilon_{k',k'v_+}\right|^2} \end{aligned} \tag{7.13}$$

and

$$\begin{aligned} D^a_- &= 2\pi\omega^4_{p_a}\lambda^3_{d_a}v_+^{-1}\sum_{k'}\frac{(\Delta^e_{k'} + \Delta^i_{k'})}{|k\lambda_{d_a}|^3}\frac{v_+}{\left|\epsilon_{k',k'v_+}\right|^2}(1 - \cos kx_-) \equiv 2\pi\omega^4_{p_a}\lambda^3_{d_a}v_+^{-1}\left[d^e_0 + d^i_0 - d^e_x - d^i_x\right] \\ d^e_x &= \sum_{k'}\frac{\Delta^e_{k'}}{|k\lambda_{d_a}|^3}\frac{v_+}{\left|\epsilon_{k',k'v_+}\right|^2}\cos k'x_- \end{aligned} \tag{7.14}$$

d^a is a dimensionless quantity, while Δ_k is the fourier transform of the clump charge density in the v_- coordinate. That is

$$\Delta_k^a \equiv \int dx e^{ikx_-} \int dv_- \langle \delta \tilde{f}^a \delta \tilde{f}^a | v_-, x_- \rangle \tag{7.15}$$

The source terms from chapter 4 can be simplified to

$$\begin{aligned} S^e &= - S_0 \, Im \, Z'((v_+ - v_d)/v_e)/\pi^2 \lambda_{d_e}^2 \\ S^i &= S_0 \, Im \, Z'(v_+/v_i)/\pi^2 \lambda_{d_i}^2 \end{aligned} \tag{7.16}$$

where

$$S_0 = \frac{\lambda_{d_e}}{v_+} \left[\frac{T_e}{T_i} Im \, Z'(v_+/v_i) d_x^e - Im \, Z'((v_+ - v_d)/v_e) d_x^i \right] \tag{7.17}$$

We have used

$$Im \, Z'(v_+/v_a) = \pi \omega_{p_a}^2 \lambda_{d_a}^2 \frac{\partial \langle f \rangle}{\partial v_+} \tag{7.18}$$

and taken $Re g_{k\omega} \approx \pi \delta(\omega - kv)$ to reformulate some of the expressions.

7.3. Method of Solution

The method of solution is quite straightforward. We integrate numerically four partial differential equations (two for each species), to obtain the total G, and $\overline{G}$ response. (Remember that $\overline{G}$ represents *three* terms, $f^c \tilde{f} + \tilde{f} f^c + f^c f^c$). The difference between these solutions represents $\tilde{G}$. The result is integrated over v_- to obtain the charge density which is fourier transformed through a highly optimized FFT algorithm. The resulting quantity is used to evaluate the diffusion coefficients and the source term from equations (7.13) through (7.18). (Note that the integrations involved are nothing but inverse fourier transforms.) These new coefficients are used to advance the equations in the next time step. Several schemes were investigated for the finite difference equations. These are outlined in Appendix C.

Some further restrictions are imposed in that we neglect the time variation of the average distributio in the source term. This means that we do not allow the average distribution to relax during our simulation. A more complete description would take this into account. However since we are primarily interested in the onset of instability (surplus regeneration) rather than the saturation mechanism *per se*, this assumption is unimportant for the purposes of this model. Within this framework v_+ becomes a constant which is treated on par with the temperature and drift velocity as an external parameter for each run.

Computational difficulties arise when one tries to integrate two interelated equations whose functions evolve on extremely disparate time scales such as electron and ion plasma frequencies. For the purposes of this calculation we have restricted ourselves to mass ratios of the order of $m_i/m_e \approx 4 \rightarrow 40$, the computational costs involved for larger mass ratios being prohibitive.

For the two stream or ion acoustic instability one can generate a linear instability boundary in the drfit velocity and temperature ratio domain[37]. By this we mean that for a given temperature ratio there exists a threshold drift velocity beyond which an eigenmode of the plasma dielectric will go linearly unstable. The most familiar case is probably in the $T_e/T_i \gg 1$ regime where the ion landau damping becomes small enough to allow unstable waves to be generated off the positive gradients of the electron distribution. The data available in the literature deals with real mass ratios. Thus for purposes of comparison we generate the same stability boundary for the artificial mass ratios. The method used is the standard technique of solving the simultaneous set of equations consisting of the functions

$$H_1(v_+) = \frac{\partial}{\partial v_+}\Big(\langle f^e\rangle + (m_e/m_i)\langle f^i\rangle\Big) = 0 \qquad \frac{\partial}{\partial v_+}H_1(v_+) > 0 \tag{7.19}$$

coupled with

$$H_2(v_+) = Re\Big((T_e/T_i)Z'(v/v_i) + Z'((v - v_d)/v_e)\Big) = 0 \tag{7.20}$$

The last equation assumes that the $k = 0$ mode is the first to go unstable. In the temperature regimes which we investigate ($T_e/T_i \approx .1 \rightarrow 10.$) this is the case.

As a practical point we use the diffusion coefficient as the yardstick for the measurement of decay or growth. If that quantity remains constant or increases we consider the state to be self sustaining or unstable.

7.4. Results of Numerical Simulations

Fig. (7.1) plots the critical drift velocity as a function of T_e/T_i for the onset of electrostatic instability in an electron ion plasma. The curves are for three mass ratios, $m_i/m_e = 1840, 160$ and 4. Fig. (7.2) shows an amplified version of the case $m_i/m_e = 4$. On the same figure the region for clump regeneration is plotted. The shaded area indicates the region of non linear instability generated by the numerical solution of the differential equations. For $T_e/T_i = 1$ we see that $v_{nl} \approx .55 v_l$. (v_{nl} is the critical drift for non-linear instability while v_l is the analogous quantity for the linear case.)

Fig. (7.3) illustrates the x_- dependance of the diffusion coefficient D_- for two diffrent cases. In the first the incoherent fluctuations are decaying while in the second they are growing. To facilitate the

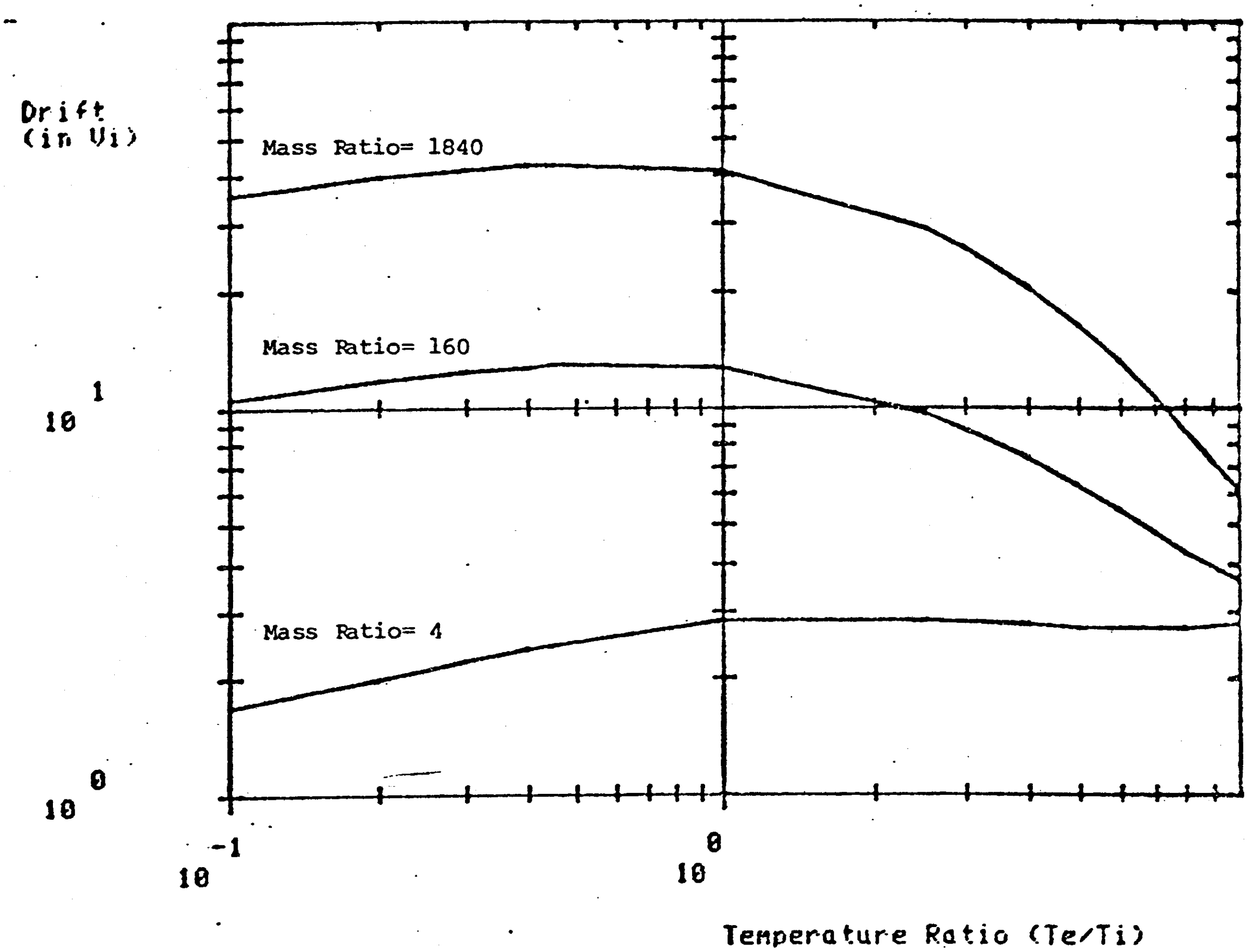

Fig. 7.1 Plot of Critical Drift Velocity vs. Temperature Ratio
for Onset of Electrostatic Instability

discussion let us assume the following generic form for these curves

$$D_{-} \approx D_0(1 - e^{-k_0|x_-|})$$

We have paramaterized D_{-} through its size D_0 and a characteristic width k_0^{-1}. The most striking feature which differentiates the curves is their "k_0". For the case of decay this is roughly a debye length while in the unstable regime this is closer to 3 or 4 debye lengths. At the same time the magnitude of D_0 in the unstable case is approximately ten times that in the decaying regime.

With this information, the mechanism for the instability can be characterized in the following way. As the parameter set v_+, v_d and T_e/T_i approach values such that ϵ becomes small the width of D_{-} in x space increases. Mathematically this is fairly easy to see since $\epsilon \to 0$ makes $D(k)$ peaked about $k = 0$. On transforming back to x space this produces a broader function. In other words the typical clump size increases. Furthermore $\overline{G}$ *decreases* since D_0 (which is equivalent to D_+) increases. The same holds for the source term which becomes larger and broader in phase space. The net result is an increase in the clump charge density with an equivalent increase in the magnitude of the potential spectrum.

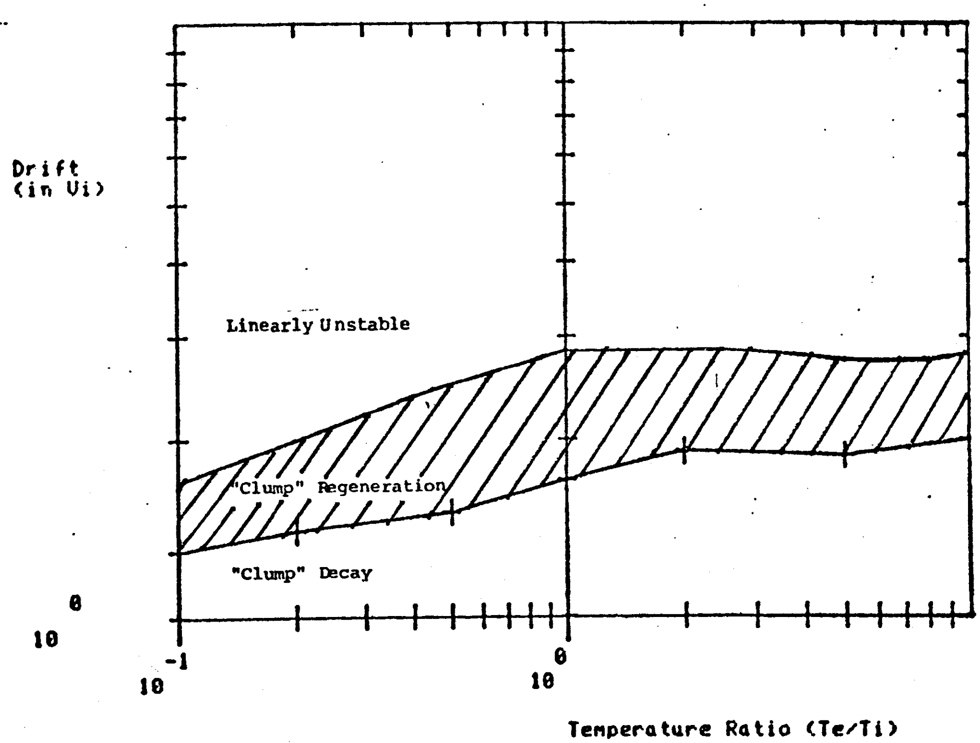

Fig. 7.2 Clump Regeneration Regime for $m_i/m_e = 4$

Thus when one passes a threshold value, determined by the proximity of the parameter space to an undamped eigenvalue of the dielectric, the system will always regenerate provided that S remains positive finite. In previous one dimensional calculations one of the problems encountered was the dissapearance of S as $\epsilon \to 0$. For a one species problem an undamped mode has $Re\,\epsilon \approx Im\,\epsilon \approx 0$ and S

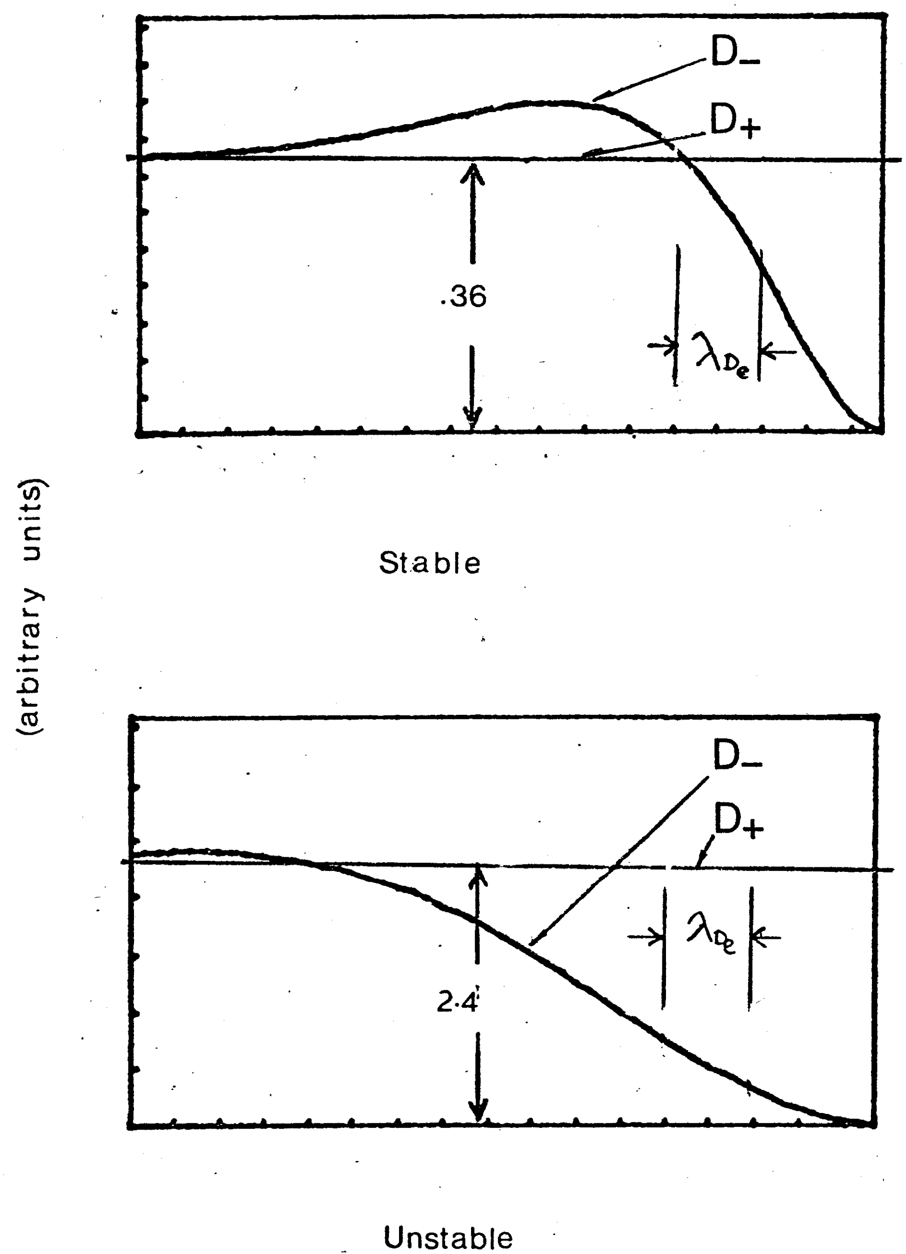

Fig. 7.3 D_- as a Function of x_- for Unstable and Stable Regimes

being proportional to $Im\,\epsilon$ becomes small in that same limit. For the two species formulation this is not the case since $\epsilon \rightarrow 0$ can be satisfied for $Im\,\chi^e + Im\,\chi^i \approx 0$ rather than requiring the individual terms to dissapear. These arguments are made more quantitative in §7.5 where we derive an approximate analytic solution which illustrates the trends we have described.

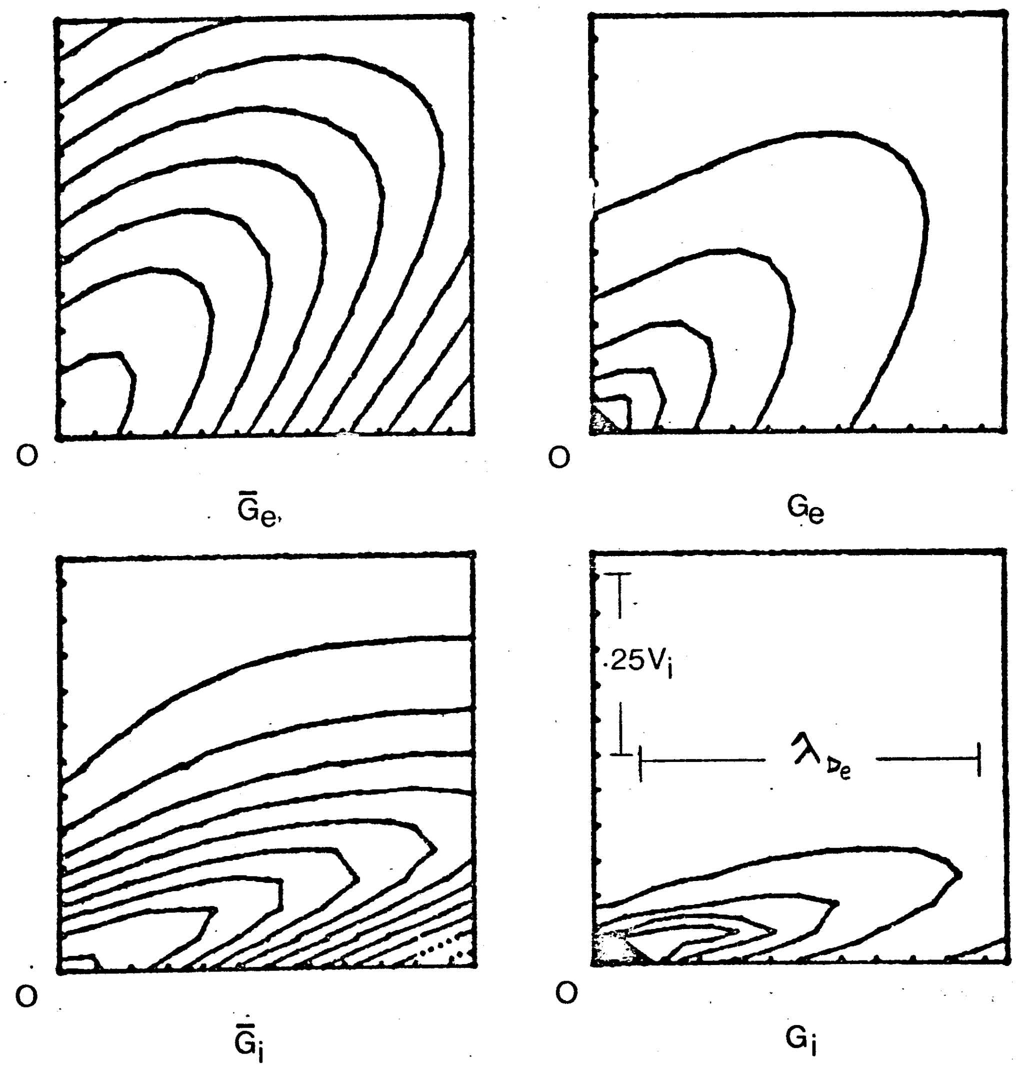

Fig. 7.3 Contour Maps of the Electron and Ion Correlation Functions

Figure 7.3 illustrates the contour lines for the ion and electron correlation functions. Figures 7.4 and 7.5 show cross sections in x and v of these functions. The different tilt of the contour lines (between the electron and ion picture) arises because $D^e \gg D^i$.

Two further aspects of the simulation are of interest. The first is the characteristic growth rate of the

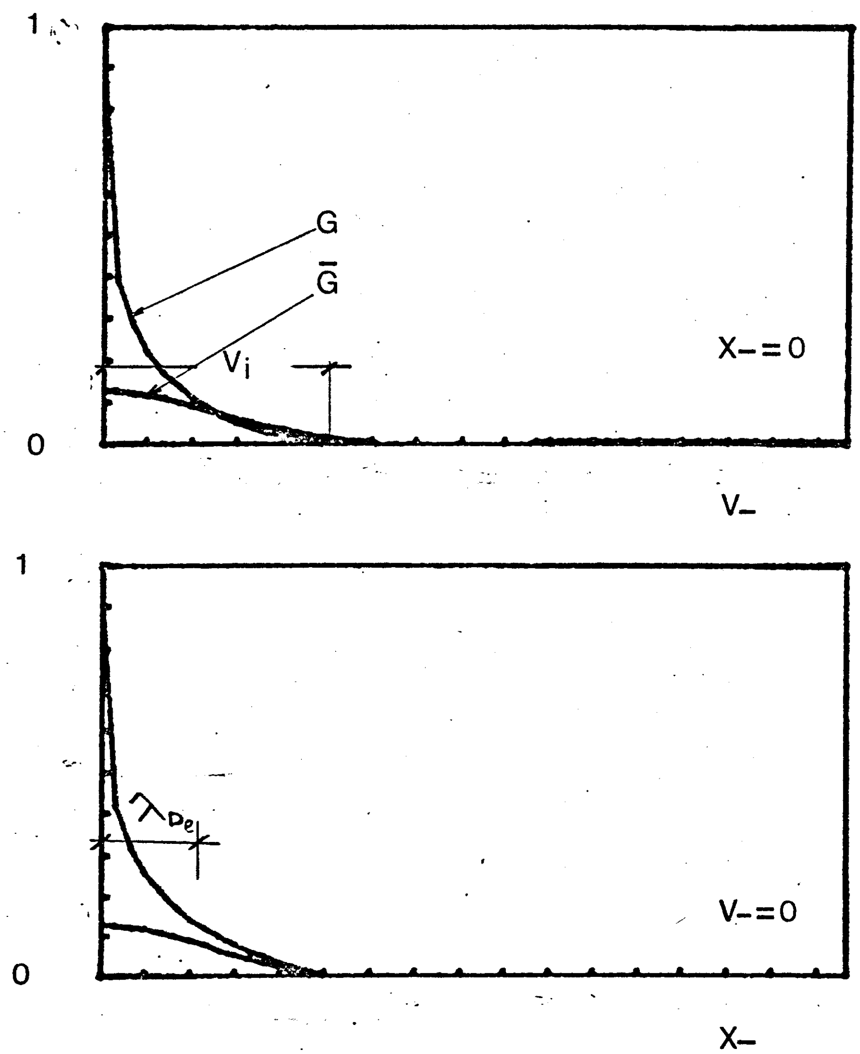

Fig. 7.4 Cross Section for Electron Distribution

instability and the second is the linewidth of the fluctuations.

To answer the second question, investigations of the solution well inside the regeneration regime indicate that the phase velocity width $\Delta v_{ph} = (\omega/k)_{max} - (\omega/k)_{min}$ is much larger than the linear instability result. For example, fluctuations whose growth rate is maximized for phase velocities around

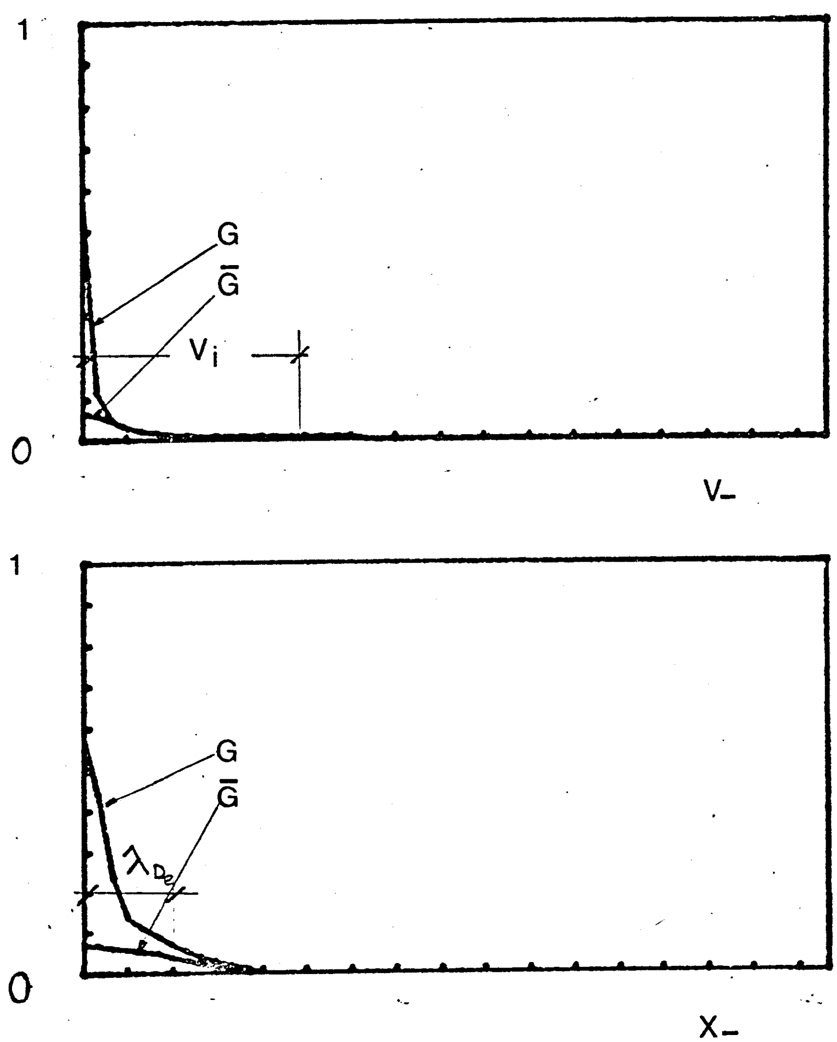

Fig 7.5 Cross Section for Ion Distribution

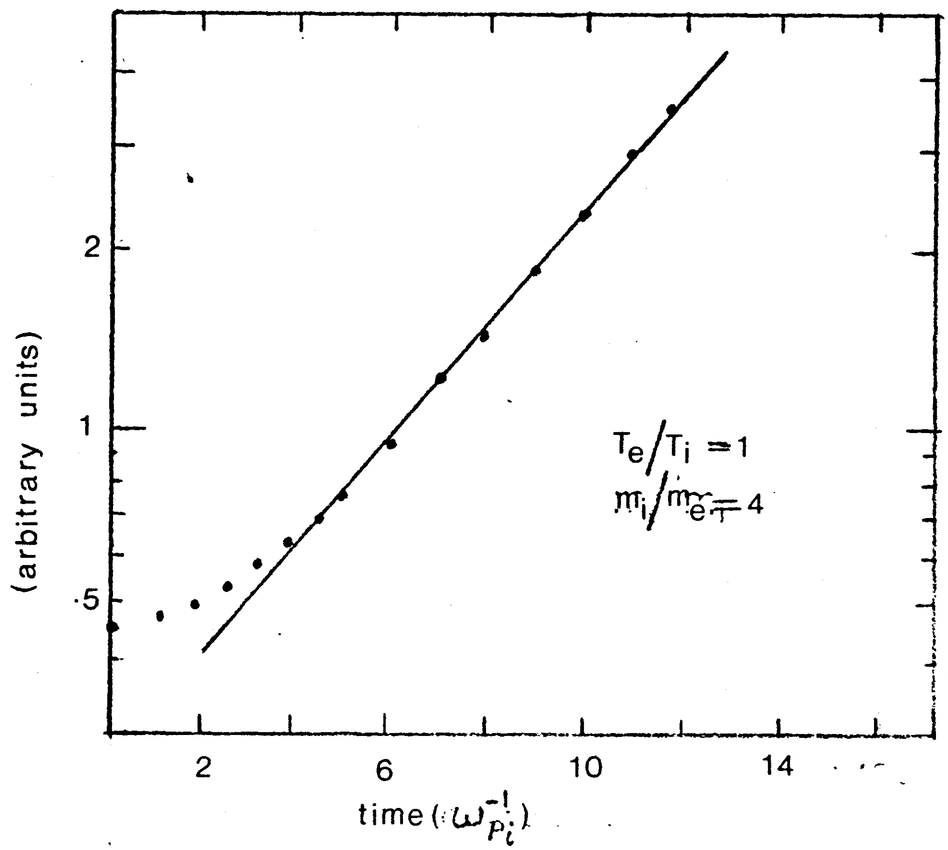

Fig. 7.6 Evolution of Diffusion Coefficient for Unstable Mode

v_i will extend in velocity up to $\pm 35\%$ of that value. That is the spectrum will include phase velocities from $.65v_i$ to $1.35v_i$. This is an interesting result when coupled with a growth rate which is of an exponential nature. A typical curve demonstrating the growth of the diffusion coefficient is shown in Fig. 7.6.

7.5. Approximate Analytic Solution

One can obtain an approximate self consistency condition in the following way. According to (5.10) we can write the solution to the differential equations as (α is once again a species superscript)

$$\langle \tilde{f}^{\alpha}\tilde{f}^{\alpha} \rangle = (\tau_{cl}^{\alpha} - \tau_{tr}^{\alpha})S^{\alpha} \tag{7.21}$$

The clump lifetime can be approximated by (5.13),

$$\tau_{cl}(k, v_-) \simeq \delta(v_-)\int \tau_{cl}(k, v_-) = \delta(v_-)\frac{2\pi}{|k|^2}\Big(1 - J_0(6^{1/2}k/k_0)\Big) = \delta(v_-)A(k) \tag{7.22}$$

while the trapping time, as a function of k and v_-, can be modelled by

$$\tau_{tr} = \frac{\tau_t}{4 + (kv_-\tau_t)^2}; \qquad \tau_t = (k_0^2 D/3)^{1/3} \tag{7.23}$$

We have approximated the resonance broadening by a constant factor τ_t. k_0 is an average wave number characterizing the spectrum which we define through

$$|k_0^\alpha|^2 \simeq (D_+^\alpha)^{-1}\left[\frac{\partial}{\partial v}\frac{\partial D_-^\alpha}{\partial v}\right]_{x_-=0} \tag{7.24}$$

The source term is given by

$$\begin{aligned} S^\alpha \simeq & 2D^{\alpha\beta}\left[\frac{\partial}{\partial v}\langle f^\alpha\rangle\right]^2 - 2F^{\beta\alpha}\left[\langle f^\alpha\rangle\frac{\partial}{\partial v}\langle f^\alpha\rangle\right] \\ \simeq & 2D^{\alpha\beta}\left[\frac{\partial}{\partial v}\langle f^\alpha\rangle\right]^2 - (m_\beta/m_\alpha)2D^{\beta\alpha}\left[\frac{\partial}{\partial v}\langle f^\alpha\rangle\frac{\partial}{\partial v}\langle f^\beta\rangle\right] \end{aligned} \tag{7.25}$$

We have used in (7.25)

$$2F^{\beta\alpha}\left[\langle f^\alpha\rangle\frac{\partial}{\partial v}\langle f^\alpha\rangle\right] \simeq (m_\beta/m_\alpha)2D^{\beta\alpha}\left[\frac{\partial}{\partial v}\langle f^\alpha\rangle\frac{\partial}{\partial v}\langle f^\beta\rangle\right]$$

This last equality is motivated by energy and momentum considerations since theses terms are identically equal when one takes their v and v^2 moments. Physically this is just a statement that if, say, electrons are diffused by ion fluctuations then the displaced electrons, to conserve momentum, will act back through a balancing dynamical drag.

The potential fluctuations are related to the correlation function

$$|\tilde{\phi}^\alpha_{k\omega}|^2 = \left(\frac{4\pi nq}{k^2}\right)^2 \int dv_1 \int dv_2 2\pi\delta(\omega - kv_1)\langle\tilde{f}^\alpha_k(1)\tilde{f}^\alpha_k(2)\rangle \tag{7.26}$$

If we substitute Eq. (7.21) through (7.23) and (7.25) in (7.26), using

$$\int dv_- \frac{2\tau_t}{4 + (kv_-\tau_t)^2} = \frac{\pi}{|k|} \tag{7.27}$$

we get after some algebra

$$|\tilde{\phi}^{\alpha}_{ku}|^2 = \frac{A(k)}{\pi^2} \frac{|\epsilon_{ku}|^2}{|\epsilon_{ku}|^2 - 2Im\,\chi^{\alpha}_{ku} Im\,\chi^{\beta}_{ku}}$$

$$\times\Big[B^{\alpha}(u)(Im\,\chi^{\alpha}_{ku})^2 - Im\,\chi^{\alpha}_{ku} Im\,\chi^{\beta}_{ku}\Big]\int dk'|k'||\tilde{\phi}^{\alpha}_{k'u}|^2|\epsilon_{ku}|^{-2} \tag{7.28}$$

where

$$B^{\alpha}(u) = \frac{I_{\beta\beta}(u)}{1 + I_{\alpha\beta}(u)} \tag{7.29}$$

with

$$I_{\beta\beta}(u) = \int dk|k|\frac{A(k)}{\pi^2}\frac{(Im\,\chi^{\beta}_{ku})^2}{|\epsilon_{ku}|^2 - 2Im\,\chi^{\alpha}_{ku} Im\,\chi^{\beta}_{ku}}$$

$$I_{\alpha\beta}(u) = \int dk|k|\frac{A(k)}{\pi^2}\frac{Im\,\chi^{\beta}_{ku} Im\,\chi^{\alpha}_{ku}}{|\epsilon_{ku}|^2 - 2Im\,\chi^{\alpha}_{ku} Im\,\chi^{\beta}_{ku}} \tag{7.30}$$

The solution to Eq. (7.28) is

$$|\tilde{\phi}^{\alpha}_{ku}|^2 = |\epsilon_{ku}|^2 N(u) R(k, u) \tag{7.31}$$

where

$$R^{\alpha}(k, u) = \frac{A(k)}{\pi^2}\frac{\Big(B^{\alpha}(u)(Im\,\chi^{\alpha}_{ku})^2 - Im\,\chi^{\alpha}_{ku} Im\,\chi^{\beta}_{ku}\Big)}{|\epsilon_{ku}|^2 - 2Im\,\chi^{\alpha}_{ku} Im\,\chi^{\beta}_{ku}} \tag{7.32}$$

$N(u)$ is arbitrary if

$$\int dk|k|R^{\alpha}(k, u) = 1 \tag{7.33}$$

and $N(u) = 0$ otherwise. Equation (7.29) is the first equation which must be satisfied for a steady, self-sustaining, state to exit. The second condition, which determines k_0, is given by (7.24). If we use (7.31) this can be recast as

$$\int dk|k|k^2\Big[R^{e}(k, u) + R^{i}(k, u)\Big] = 2k_0^2 \tag{7.34}$$

The simultaneous solution of Equations (7.29) and (7.30) determine the parameter range for which a self consistent steady state can exist.

The above equations were solved numerically for the real mass ratio, and for mass ratios in the range of the computer simulation. The results are shown in Fig. (7.7). Given the approximate nature

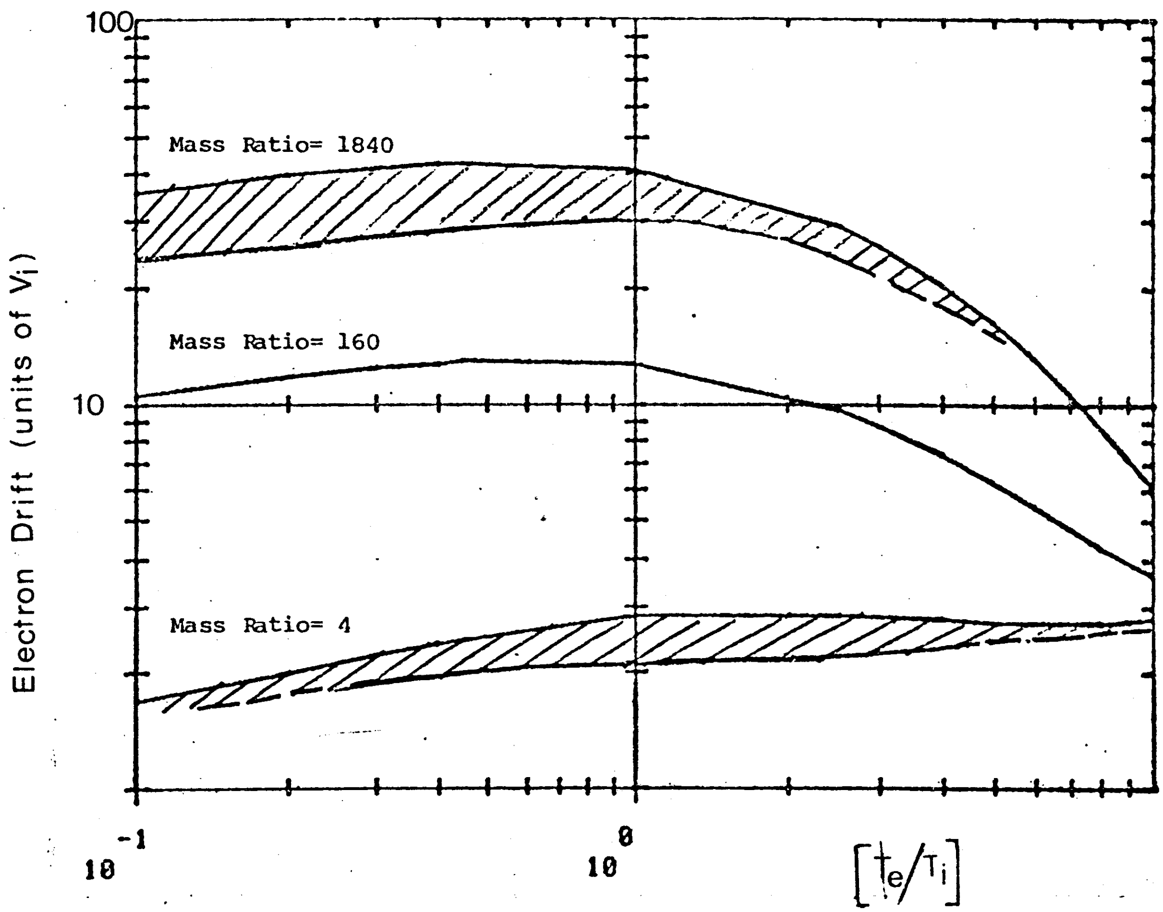

Fig. 7.7 Clump Regeneration from Analytic Results

of the analytic expressions for τ_{cl} and τ_{tr} the results are surprisingly close. The computer experiment predicts a reduction of approximately 50% in the drift velocity required for instability, while the analytic result predicts a more modest 25%. This discrepancy is easily attributed to the logarithmic expression for the clump lifetime which is a serious approximation to the exact result for $x_- \geq k_0^{-1}$ and $v_- \geq v_{tr}$. Since a major contribution to the clump charge density comes from its finite extent in phase space the regeneration condition is quite sensitive to the approximation in (7.22).

Nonetheless this approximate result illustrates effectively the essential aspects of the calculation. The first shows how the approach to an eigenvalue of the dielectric will always insure regeneration since the numerator in (7.32) will be small and will allow (7.33) to be satisfied. (The integral, far away from a solution of $\epsilon_{ku} = 0$ is less than 1.) The second element is the reduction of this effect due to the subtraction of $\overline{G}$. If one did not subtract that contribution then the equations would only differ in that denominators with the expression $(|\epsilon_{ku}|^2 - 2Im\,\chi^{\alpha}_{ku} Im\,\chi^{\beta}_{ku})$ would reduce to $|\epsilon_{ku}|^2$.

Appendix A

Transforms

A.1 Fourier Transforms

The finite transform and its inverse are defined, over a spatial length $2L$ or temporal length $2T$, through

$$f(x,t)=\sum_{k\omega} f_{k\omega} \exp i(kx-\omega t) \qquad k=\frac{n\pi}{2L}, \quad \omega=\frac{m\pi}{2T}$$
$$f_{k\omega}=\int_{-T}^{T}\frac{dt}{2T}\int_{-L}^{L}\frac{dx}{2L} f(x,t)\exp -i(kx-\omega t) \tag{A.1}$$

The integral transforms are the limit of (A.1) as L and T become infinite. For example the spatial transform is given by

$$f(x)=\int_{-\infty}^{\infty}\frac{dk}{2\pi} f(k)\exp(ikx)$$
$$f(k)=\int_{-\infty}^{\infty} dx f(x)\exp(-ikx) \tag{A.2}$$

A.2 Laplace Transforms

Laplace and inverse Laplace transforms are defined by

$$f^{\pm}(\omega)=\int_{0}^{\infty} dt f(t)\exp(\mp i\omega t)$$
$$f(t)=\int_{-\infty+i\sigma}^{\infty+i\sigma} d\omega f^{\pm}(\omega)\exp(\mp i\omega t) \tag{A.2}$$

A.3 Averages

In the two point formulation one often deals with products of δf. For a spatially and temporally homogeneous system it follows that

$$\langle f(k,\omega)f(k',\omega')\rangle \equiv (2\pi)^2\langle ff\rangle_{k\omega}\delta(k+k')\delta(\omega+\omega') \qquad (A.4)$$

where

$$\langle f(1)f(2)\rangle_{k\omega} = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dx e^{i(\omega t - kx)}\langle f(v_1, x+x', t+t')f(v_2, x', t')\rangle \qquad (A.5)$$

One can go from the discrete limit of the transform to the continuous limit through the following transformation

$$\lim_{L\to\infty} \qquad L\langle f_k(1)f_k^*(2)\rangle \to \langle f(1)f(2)\rangle_k \qquad (A.6)$$

(A.6) can be demonstrated in the following way. From (A.4) we have

$$\langle ff\rangle_k = \lim_{L\to\infty} \int dx e^{-ikx}\int_{-L}^{L}\frac{dx'}{2L}\sum_{k'}\sum_{k''} f_{k'}f_{k''}e^{ik''x'}e^{ik'(x+x')} \qquad (A.7)$$

where we have interpreted the ensemble average as a spatial average. Using

$$\lim_{L\to\infty} \int_{-L}^{L}\frac{dx'}{2L}e^{i(k'+k'')x'} = \delta_{k',k''} \qquad (A.8)$$

where $\delta_{a,b}$ is the Kroenecker delta function we get

$$\langle ff\rangle_k = \sum_{k'}\sum_{k''} f_{k'}f_{k''}\delta_{k',k''}\delta(k-k'') \qquad (A.9)$$

But $\delta(k-k'') = \lim_{L\to\infty} L\delta_{k,k''}$ from which (A.6) immediately follows. Similarly, using (A.1) and (A.2), one can show that

$$\sum_{k',\omega'}\langle f_{k'\omega'}(1)f_{k'\omega'}^*(2)\rangle = \int\frac{dk}{2\pi}\int\frac{d\omega}{2\pi}\langle f(1)f(2)\rangle_{k'\omega'} \qquad (A.10)$$

A.4 Renormalization of One Point Equation

From Chapter 1 the collision operator is defined through

$$\int_0^t dt' C_k(t-t') f_k(t') \equiv -\left(\frac{q}{m}\frac{\partial}{\partial v}\sum_{k'} ik'\phi_{k'}^{(1)}(t) f_{k-k'}^{(2)}(t) + i(k-k')\phi_{k-k'}^{(2)}(t) f_{k'}^{(1)}(t)\right)_{Phase\ Coherent} \tag{A.11}$$

We recall the greens function $g_k(t)$ through

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + ikv\right) g_k(t) + \int_0^t dt' C_k^f(t-t') g_{k'}(t') = 0 \quad t > 0 \\ &g_k(t = 0^+) = 1 \\ &g_k(t) = 0 \qquad t < 0 \end{aligned} \tag{A.12}$$

Using (A.12) the iteration for $f_{k-k'}^{(2)}(t)$ is given by

$$\begin{aligned} f_{k-k'}^{(2)}(t) = &\int_0^t dt' g_{k-k'}(t-t') \\ &\times i\frac{q}{m}\left((k-k')\frac{\partial}{\partial v}(f_0 + C_{k-k'}^{\phi}(t'))\phi_{k-k'}^{(2)}(t') - k'\frac{\partial f_k^{(1)}(t')}{\partial v}\phi_{k'}^{(1)*}(t') + k\frac{\partial f_{k'}^{(1)*}(t')}{\partial v}\phi_k^{(1)}(t')\right) \end{aligned} \tag{A.13}$$

Consider, for example, the diffusion term $D_{k\omega}$. This is obtained by iterating the second term of (A.13) into the first term of (A.11). Performing the substitution we obtain:

$$\frac{q^2}{m^2}\frac{\partial}{\partial v}\sum_{k'}\int_0^t dt' g_{k-k'}(t-t')\langle\phi_{k'}^*(t')\phi_{k'}(t)\rangle\frac{\partial}{\partial v}f_k(t') \tag{A.14}$$

We assume stationary turbulence so that $\langle\phi(t)\phi(t')\rangle \rightarrow \langle|\phi_{k'}|^2|t-t'\rangle$. Fourier transforming (A.14) and interchanging the order of integration we get:

$$\frac{q^2}{m^2}\frac{\partial}{\partial v}\int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}\int_0^\infty dt'\int_{t'}^\infty dt e^{i\omega t} g_{k-k'}(t-t')\langle\phi^2\rangle_{k'\omega'} e^{-i\omega(t-t')}\frac{\partial}{\partial v}f_k(t') \tag{A.15}$$

We have made use of (A.5) to express $\langle|\phi_{k'}|^2|t-t'\rangle$ in its Fourier representation and used (A.10) to go to the integral limit of the k' sum. Defining

$$g_{k\omega} = \int_0^\infty dt g_k(t) \exp(i\omega t) \tag{A.16}$$

(A.15) immediately reduces to

$$\frac{q^2}{m^2}\frac{\partial}{\partial v}\int \frac{dk'}{2\pi}\int \frac{d\omega'}{2\pi} g_{k-k',\omega-\omega'} \langle \phi^2 \rangle_{k'\omega'} \frac{\partial}{\partial v} f_{k\omega} \tag{A.17}$$

The next step is to express $\langle \phi^2 \rangle_{k'\omega'}$ as the shielded spectrum $\langle \tilde{\phi}^2 \rangle_{k'\omega'} / |\epsilon_{k'\omega'}|^2$. This can easily be deduced form the Fourier transform of equations (1.15). The same technique can be applied to the remaining terms in the iteration. It is important to stress that the format of results depends on the assumption of time sationarity. For a non-stationary system the coefficients would have to be left in the format of (A.14).

Appendix B

Perturbation of the Lenard-Balescu Equation

In §1.4 we interpreted some of the terms in the collision operator C^f by considering an expansion of the discrete particle Lenard-Balescu collision integral

$$\left.\frac{\partial f}{\partial t}\right]_{LB} = -\frac{\partial}{\partial v}Ff + \frac{\partial}{\partial v}D\frac{\partial}{\partial v}f \tag{B.1}$$

F and D are the drag and diffusion coefficients given by

$$Ff = \frac{q}{m}\sum_{k',\omega'} k'\frac{\epsilon^i_{k'\omega'}}{|\epsilon_{k'\omega'}|^2}\langle\tilde{\phi}^*_{k'\omega'}\tilde{f}_{k'\omega'}(1)\rangle \qquad D = \sum_{k',\omega'}\frac{q^2}{m^2}\pi\delta(\omega' - k'v)\frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} \tag{B.2}$$

The spectrum of fluctuations is given by

$$\langle\tilde{f}(1)\tilde{f}(2)\rangle_{k'\omega'} = \frac{2\pi}{n}\delta(\omega' - k'v)\delta(v_1 - v_2)\langle f\rangle \tag{B.3}$$

where n is the average density of particles. Using (B.3) and (B.2) in (B.1) we can write the collision integral as

$$\left.\frac{\partial f(1)}{\partial t}\right]_{LB} = \pi\frac{\omega_p^4}{n}\sum_{k'}\frac{k'k'}{|k'|^4}\frac{\partial}{\partial v_1}\int dv_3\frac{\delta(k'v_1 - k'v_3)}{|\epsilon_{k',k'v_1}|^2}\left[\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_3}\right]f(1)f(3) \tag{B.4}$$

We consider the response to a *wave*, of a plasma described by the equation

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}+\frac{q}{m}E\frac{\partial}{\partial v}\right)f=\left.\frac{\partial f}{\partial t}\right]_{LB} \tag{B.5}$$

We will treat the Fokker-Planck coefficients as a perturbation acting on the correlated motion described by the Vlasov operator. Thus the wave field is present both as the smooth macrofield E in the Vlasov operator and in its effects on the drift and diffusion coefficients. Following the procedure in §2.4 we linearize f and $\epsilon_{k'\omega'}$ according to (2.50) and (2.51). This yields

$$-i(\omega-kv)f_{k\omega}-i\frac{q}{m}k\frac{\partial f_0^0}{\partial v}\phi_{k\omega}=\left.\frac{\partial f_{k\omega}}{\partial t}\right]_{LB} \tag{B.6}$$

If we write the perturbed collision operator as $-Cf_{k\omega}$ we have

$$f_{k\omega}=-\frac{q}{m}k\frac{\partial f_0^0}{\partial v}\phi_{k\omega}(\omega-kv+iC)^{-1} \tag{B.7}$$

and

$$f_0^1=\lim_{k,\omega\to 0}f_{k\omega} \tag{B.8}$$

The perturbed collision operator becomes

$$\begin{aligned}\left.\frac{\partial f_0^1(1)}{\partial t}\right]_{LB}=&\pi\frac{\omega_p^4}{n}\sum_{k'}\frac{k'k'}{|k'|^4}\frac{\partial}{\partial v_1}\int dv_3\frac{\delta(k'v_1-k'v_3)}{|\epsilon^0_{k',k'v_1}|^2}\left[\frac{\partial}{\partial v_1}-\frac{\partial}{\partial v_3}\right]\left[f_0^1(1)f_0^0(3)+f_0^0(1)f_0^1(3)\right]\\&-\pi\frac{\omega_p^4}{n}\sum_{k'}\frac{k'k'}{|k'|^4}\frac{\partial}{\partial v_1}\int dv_3\frac{\delta(k'v_1-k'v_3)}{|\epsilon^0_{k',k'v_1}|^2}\left[\frac{\chi^1_{k'\omega'}}{\epsilon^0_{k'\omega'}}+\frac{\chi^{1*}_{k'\omega'}}{\epsilon^{0*}_{k'\omega'}}\right]\left[\frac{\partial}{\partial v_1}-\frac{\partial}{\partial v_3}\right]f_0^0(1)f_0^0(3)\end{aligned} \tag{B.9}$$

which can be rewritten as

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial v}F-\frac{\partial}{\partial v}D\frac{\partial}{\partial v}\right)f_0^1=\left(-\frac{\partial}{\partial v}(\mathcal{F}^t+\mathcal{F}^f)+\frac{\partial}{\partial v}(d^t+d^f)\frac{\partial}{\partial v}\right)f_0^0 \tag{B.10}$$

where

$$D = \pi \frac{\omega_p^4}{n} \sum_{k'} \frac{k'k'}{|k'|^4} \int dv_3 \frac{\delta(k'v_1 - k'v_3)}{|\epsilon^0_{k',k'v_1}|^2} f_0^0(3)$$

$$F = \pi \frac{\omega_p^4}{n} \sum_{k'} \frac{k'k'}{|k'|^4} \int dv_3 \frac{\delta(k'v_1 - k'v_3)}{|\epsilon^0_{k',k'v_1}|^2} \frac{\partial}{\partial v_3} f_0^0(3) \tag{B.11}$$

are the diffusion and drag coefficients in the absence of the wave field $E_{k\omega}$

$$\mathcal{F}^t = \pi \frac{\omega_p^4}{n} \sum_{k'} \frac{k'k'}{|k'|^4} \int dv_3 \frac{\delta(k'v_1 - k'v_3)}{|\epsilon^0_{k',k'v_1}|^2} \frac{\partial}{\partial v_3} f_0^1(3)$$

$$\mathcal{F}^f = -\pi \frac{\omega_p^4}{n} \sum_{k'} \frac{k'k'}{|k'|^4} \int dv_3 \frac{\delta(k'v_1 - k'v_3)}{|\epsilon^0_{k',k'v_1}|^2} \left[\frac{\chi^1_{k'\omega'}}{\epsilon^0_{k'\omega'}} + \frac{\chi^{1*}_{k'\omega'}}{\epsilon^{0*}_{k'\omega'}} \right] \frac{\partial}{\partial v_3} f_0^0(3) \tag{B.12}$$

and

$$d^t = \pi \frac{\omega_p^4}{n} \sum_{k'} \frac{k'k'}{|k'|^4} \int dv_3 \frac{\delta(k'v_1 - k'v_3)}{|\epsilon^0_{k',k'v_1}|^2} f_0^1(3)$$

$$d^f = -\pi \frac{\omega_p^4}{n} \sum_{k'} \frac{k'k'}{|k'|^4} \int dv_3 \frac{\delta(k'v_1 - k'v_3)}{|\epsilon^0_{k',k'v_1}|^2} \left[\frac{\chi^1_{k'\omega'}}{\epsilon^0_{k'\omega'}} + \frac{\chi^{1*}_{k'\omega'}}{\epsilon^{0*}_{k'\omega'}} \right] f_0^0(3) \tag{B.13}$$

are the modifications to the Fokker-Planck coefficients due to the wave induced distortion of the distribution function.

We wish to compare the above to the $k, \omega \to 0$ limit of the $C^f_{k\omega}$ operator

$$-C^f_{k\omega} f_{k\omega} \equiv \frac{\partial}{\partial v}\left(D_{k\omega}\frac{\partial}{\partial v} - F_{k\omega}\right) f_{k\omega} + \frac{\partial}{\partial v}\left((d^f + d^t)\frac{\partial}{\partial v} - (\mathcal{F}^t + \mathcal{F}^f)\right)\bar{f} \tag{B.14}$$

For the spectrum given by (B.3) it is clear that setting $\{k, \omega \to 0\}$ in (2.27) will reduce the diffuion $D_{k\omega}$ to the "zero" order Lenard-Balescu coefficient. Furthermore in the long wavelength limit, and time asymptotically we have

$$\lim_{k,\omega\to 0} \quad (\mathcal{F}^t + \mathcal{F}^f) f_{k\omega} = -\frac{q}{m}\omega_p^2 \sum_{k',\omega'} \frac{k'k'}{|k'|^2} \frac{\langle \tilde{\phi}^*_{k'\omega'} \tilde{f}_{k'\omega'}(1)\rangle}{\epsilon^*_{k'\omega'}\epsilon^*_{k'\omega'}} \int dv_3 \frac{i}{\omega' - k'v_3 - i\delta} \frac{\partial}{\partial v_3} f_0^1(3)$$

$$= -\frac{q}{m}\sum_{k',\omega'} k' \langle \tilde{\phi}^*_{k'\omega'} \tilde{f}_{k'\omega'}(1)\rangle i \frac{\chi^*_{k'\omega'}}{\epsilon^*_{k'\omega'}\epsilon^*_{k'\omega'}} \tag{B.15}$$

If we use equation (2.49)

$$Re\left[i\frac{\chi^*_{k'\omega'}}{\epsilon^*_{k'\omega'}\epsilon^*_{k'\omega'}}\right] = \frac{Im\chi_{k'\omega'}}{|\epsilon_{k'\omega'}|^2} - \left[\frac{\chi_{k'\omega'}}{\epsilon_{k'\omega'}} + \frac{\chi^*_{k'\omega'}}{\epsilon^*_{k'\omega'}}\right]\frac{Im\epsilon_{k'\omega'}}{|\epsilon_{k'\omega'}|^2} \tag{B.16}$$

in (B.15) and perform the ω' integral we recover the coefficients obtained from the linearization (B.12).

At the end of this appendix we show that

$$\left[\langle \tilde{\phi}^{(1)}\tilde{f}^{(2)}\rangle_{k'\omega'} + \langle \tilde{f}^{(1)}\tilde{\phi}^{(2)}\rangle_{k'\omega'}\right] = \frac{(4\pi ne)}{|k'|^2}\frac{2\pi}{n}\delta(\omega' - k'v) f_0^1$$

$$\left[\langle \tilde{\phi}^{(1)}\tilde{\phi}^{(2)}\rangle_{k'\omega'} + \langle \tilde{\phi}^{(1)}\tilde{\phi}^{(2)}\rangle_{k'\omega'}\right] = \frac{(4\pi ne)^2}{|k'|^4}\int dv \frac{2\pi}{n}\delta(\omega' - k'v) f_0^1 \tag{B.17}$$

where the expressions in (B.17) are the limit $k, \omega \to 0$, of

$$\tilde{f}^{(2)}_{k-k',\omega-\omega'}\tilde{\phi}^{(1)}_{k'\omega'} + \tilde{f}^{(1)*}_{k'\omega'}\phi^{(2)}_{k+k',\omega+\omega'}$$

when one uses (A.5) to go from the discrete to the continuous limit of the Fourier transform.

We also have

$$\lim_{k,\omega\to 0} \quad d^t\frac{\partial \bar{f}}{\partial v_1} = -\frac{q^2}{m^2}\sum_{k',\omega'} k'k' \frac{\left[\tilde{\phi}^{(1)}_{k'\omega'}\tilde{\phi}^{(2)*}_{k'\omega'} + \tilde{\phi}^{(1)*}_{k'\omega'}\tilde{\phi}^{(2)}_{k'\omega'}\right]}{[\omega' - k'v_1 - i\delta]} \frac{i}{|\epsilon^0_{k'\omega'}|^2}\frac{\partial}{\partial v_1} f_0^0(1)$$

$$\lim_{k,\omega\to 0} \quad d^f\frac{\partial \bar{f}}{\partial v_1} = -\frac{q^2}{m^2}\sum_{k',\omega'} k'k' \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon^0_{k'\omega'}|^2} \frac{-i}{[\omega' - k'v_1 - i\delta]}\left[\frac{\chi_{k'\omega'}}{\epsilon_{k'\omega'}} + \frac{\chi^*_{k'\omega'}}{\epsilon^*_{k'\omega'}}\right]\frac{\partial}{\partial v_1} f_0^0(1) \tag{B.18}$$

and

$$\lim_{k,\omega\to 0} \quad F_{k\omega}f_{k\omega} = \frac{q}{m}k'\sum_{k',\omega'} i\left[\tilde{f}^{(2)*}_{k'\omega'}\tilde{\phi}^{(1)}_{k'\omega'} + \tilde{f}^{(1)*}_{k'\omega'}\tilde{\phi}^{(2)}_{k'\omega'}\right]\frac{\epsilon^*_{k'\omega'}}{|\epsilon_{k'\omega'}|^2} \tag{B.19}$$

Using (B.17) in (B.18) and (B.19), and retaining the real part of these terms it is easy to see that they reduce to their counterparts (B.11) and (B.13).

To show (B.17) we take (2.30) multiply respective terms by $\phi_{k'\omega'}$ and $f^*_{k'\omega'}$ to get

$$\begin{aligned}\tilde{f}^{(2)}_{k-k',\omega-\omega'}\tilde{\phi}^{(1)}_{k'\omega'} + \tilde{f}^{(1)*}_{k'\omega'}\phi^{(2)}_{k+k',\omega+\omega'} &= \frac{q}{m}g_{k-k',\omega-\omega'}(1)ik\phi_{k\omega}\frac{\partial}{\partial v_1}\langle\tilde{f}^{(1)*}_{k'\omega'}(1)\tilde{\phi}^{(1)}_{k'\omega'}\rangle \\ &+ i\frac{\omega_p^2}{|k+k'|^2}\int dv_3 g_{k+k',\omega+\omega'}(3)\frac{\partial}{\partial v_3}\langle\tilde{f}^{(1)*}_{k'\omega'}(1)\tilde{f}^{(1)}_{k'\omega'}(3)k\phi_{k\omega}\rangle\end{aligned} \tag{B.20}$$

Since these expression are going to be summed over k' and ω' we can set $\langle\tilde{f}^*_{k'\omega'}\tilde{f}_{k'\omega'}\rangle$ equal to $\langle ff\rangle_{k'\omega'}$ by changing the summation to an integration. Using (B.3) we get for the right hand side of (B.20)

$$\begin{aligned}&\frac{(4\pi ne)}{|k'|^2}\frac{2\pi}{n}\delta(\omega'-k'v)g_{k-k',\omega-\omega'}\frac{q}{m}ik\phi_{k\omega}\frac{\partial f_0^0}{\partial v} \\ &+\frac{(4\pi ne)}{|k'|^2}\frac{2\pi}{n}\frac{q}{m}k\phi_{k\omega}f_0^0\left[\frac{(k+k')\delta(\omega'-k'v)}{[\omega+\omega'-(k+k')v+i\delta]^2} - \frac{\partial/\partial v\delta(\omega'-k'v)}{[\omega-\omega'-(k-k')v+i\delta]}\right]\end{aligned} \tag{B.21}$$

The term on the first line is the desired answer since we can use (B.7) and (B.8) to reduce it to (B.17). We thus want to show that the remaining terms are zero in the limit of $k,\omega\to 0$. Using

$$\delta(x) = \lim_{\delta\to 0}\frac{\delta}{(x^2+\delta^2)} \tag{B.22}$$

the term in square brackets can be written as

$$\lim_{\Delta,\delta\to 0} \quad k'\frac{\delta}{(\Delta'^2+\delta^2)}\left[\frac{(\Delta+\Delta'-i\delta)^2}{[(\Delta+\Delta')^2+\delta^2]^2} - \frac{2\Delta'[\Delta-\Delta'-i\delta]}{[\Delta'^2+\delta^2][(\Delta-\Delta')^2+\delta^2]}\right] \tag{B.23}$$

where $\Delta = \omega - kv$ and $\Delta' = \omega' - k'v$. As Δ approaches zero the imaginary parts exactly cancel and we are left with an expression that is entirely real. Furthermore since any k' dependance in Δ' will get eliminated by the ω' inetgral the only k' element to survive is the one outside the square brackets in (B.23). For F and d^t, the final result has to be real. Since $|\epsilon_{k'\omega'}|^2$ is an even function of k' we are left with k' integrals which integrate odd functions. In the case of $F_{k\omega}$ this is $\approx \int dk'k'^2 Im\, \epsilon_{k',k'v}$ while for the diffusion, d^t, it is $\approx \int dk'k'^3$. Both integrals are identically equal to zero. Thus the only term to survive in the long wavelength limit is the first one in equation (B.20).

Appendix C

Program Listing

This appendix contains the listing of the program which solves the ion and electron clump equations. The version in this program uses a Levellier weighting method for the hyperbolic equation and an explicit method for the parabolic equation. Other version were tested which used an implicit method for the parabolic equations. The results were essentially the same, but the amount of memory neccesary for the second technique made the explicit method with a small time step more advantageous. If we denote the dependant variable as $u^t_{v,x}$ the finite difference scheme becomes[38,39]

$$\begin{aligned} u^{t+1/2}_{v,x} &= (1 - 2r^t_x)u^t_v + r^t_x(u^t_{v+1,x} + u^t_{v-1,x}) + s^t_x \Delta t/2 \\ u^{t+1}_{v,x} &= u^{t+1/2}_{v,x} - a^{t+1/2}_v\left(u^{t+1/2}_{v,x} - u^{t+1/2}_{v,x-1}\right) + s^{t+1/2}_x \Delta t/2 \end{aligned} \tag{C.1}$$

where

$$\begin{aligned} r^t_x &= \Delta t D(x,t)/((\Delta v)^2 2) \\ a^{t+1/2}_v &= v(\Delta t/2\Delta x) \end{aligned} \tag{C.2}$$

The von Neumann stability condition requires that $a^t_v \leq 1$ for the hyperbolic equation. The parabolic equation requires $r^t_x \leq .5$ for stability. To insure that the stability conditions are not violated the program continually revises the time step $\Delta t/2$ so that r and a are bounded in the region specified by equation (D.2). In practice we used a smaller region $a^t_v \leq .2$ and $r^t_x \leq .125$. To check the results we used different time splitting in (D.1): for example $t + 1/3$, $t + 1/4$ etc.. We also investigated a Crank-Nicholson scheme coupled with a Levellier method. The results were similar to within $\simeq 2\%$.

The program which implements these equations with the equations for the source and diffusion follows:

```
*select box=m16,account=430mas
*file name=pdiff
c...............................................................
c
c
c     program name: PDIFF IE        9/7/80    T.B-G
c
c     VERSION: 5  UPDATE: 25/7/80   (..vectorized.)
c     VERSION: 4  UPDATE: 24/7/80   (..no vertical trans.)
c     VERSION: 3  UPDATE: 12/7/80   (..with vertical trans.)
c
c     This program solves the CLUMP equation for Electrons
c     and Ions.
c
c     SYMBOLS used:
c
c...............................................................
      integer tim,ee
      dimension uec1(101,101),uec2(101,101),uet1(101,101),uet2(101,101)
      dimension uic1(101,101),uic2(101,101),uit1(101,101),uit2(101,101)
      dimension dstar(101),dstar0(101)
      dimension sex(101),six(101),hv(101)
      dimension xx(101),yy(101)
c
      common/b/nvm1,nvc,nvm51,nvm31,nvm15,v0
      common/c/mxm1,mxm2,mxc,mxcp1,mxcm1,x0
      common/e/mxcp14,mxcm25,mxcp25,mxcm50
      common/eps/num,ntime,dcx,aa,bb,drift,rl,rl3,rm,rt,vs0,beta
c
      namelist/out/namout

c
      data namout/'xout'/
      data nv/101/,mx/101/
c
      call link('unit6=tty//')
      write(6,1)
1     format(2x,'input: name of output file.',/)
      read(6,out)
      call create(20,namout,1,-1)
c
      t1=second(zz)
       call timedate(tim,dat,mach)
c
c     Initialize Graphics
c
      call keep80(1,3)
      call fr80id(tim,1,1)
      call plts
      call dders(-1)

c
c     Initialize Data
c
      call number(nv,mx)
      call init(uet1,uec1,uit1,uic1,
     1          dstar,dstar0,
     2          sex,six,hv,
     3          diffe,diffi,adv,nv,mx)
      call scale(xx,yy,nv,mx)
      call graph(uec1,uet1,uic1,uit1,
     1           dstar,dstar0,sex,six,
     2           xx,yy,nv,mx)
      write(20,5)rm,rt,drift
5     format(3x,'evolution of diffusion coefficient for',//,
     1       3x,'  rm=',f5.2,' rt=',f5.2,' drift=',f5.2,//)
c
c     Solve
c
      do 10 it=1,ntime
c
```

```
        call solver(EE,uec1,uec2,dstar0,sex,hv,diffe,adv,nv,mx)
        call solver(EE,uet1,uet2,dstar,sex,hv,diffe,adv,nv,mx)
        call solver(II,uic1,uic2,dstar0,six,hv,diffi,adv,nv,mx)
        call solver(II,uit1,uit2,dstar,six,hv,diffi,adv,nv,mx)
        call diff(uet2,uec2,uit2,uic2,
     1           dstar,dstar0,
     2           sex,six,
     3           diffe,diffi,adv,nv,mx)
        if(mod(it,num).eq.0)call graph(uec2,uet2,uic2,uit2,
     1                                         dstar,dstar0,sex,six,
     2                                           xx,yy,nv,mx)
10    continue
c
c     all done
c
      call frame
      t1=second(zz)-t1
      write (6,40)t1
40    format(16H whew!! all done, f10.4)
      call plote
      call donepl
      call exit
      end
c
c..............................................................
c
      subroutine solver(AA,u1,u2,d,s,h,r,a,nv,mx)
c
c     Solves parabolic and hyperbolic equation for species
c     AA.
c
      dimension u1(nv,mx),u2(nv,mx)
      dimension d(mx),s(mx),h(nv)
c
       call solvep(u1,u2,d,s,h,r,nv,mx)
       call solveh(u1,u2,s,h,a,nv,mx)
       call reset(u1,u2,nv,mx)
c
      return
      end
c
c..............................................................
c
      subroutine solvep(u1,u2,d,s,h,r,nv,mx)
c
c     This subroutine solves the parabolic
c     part of the differential equation.
c
      dimension u1(nv,mx),u2(nv,mx)
      dimension d(mx),s(mx),h(nv)
c
c     These vectors were added for vectorization
c     and are not integral to the calculation
c
      dimension r0(101),r1(101),r2(101)
c
      common/b/nvm1,nvc,nvm51,nvm31,nvm15,v0
      common/c/mxm1,mxm2,mxc,mxcp1,mxcm1,x0
c
c     vectorize
c
      do 5 ix=1,mx
      r0(ix)=r*d(ix)
      r1(ix)=2.*r0(ix)
5     r2(ix)=1.-r1(ix)
c
c     Boundaries
c
      do 20 ix=1,mx
      u2(1,ix)=r2(ix)*u1(1,ix)+r1(ix)*u1(2,ix)
       u2(nv,ix)=u2(nv-2,mx-ix+1)
c
c     Main
c
      do 10 iv=2,nvm1
10    u2(iv,ix)=u1(iv,ix)*r2(ix)+(u1(iv-1,ix)
```

```
     1          +u1(iv+1,ix))*r0(ix)+s(ix)*h(iv)
20    continue
c
      return
      end
c.................................................................
c
      subroutine solveh(u1,u2,s,h,a,nv,mx)
c
c     This subroutine solves the hyperbolic
c     part of the equation
c
      dimension u1(nv,mx),u2(nv,mx),s(mx),h(nv)
c
      dimension v1(101)
c
      common/b/nvm1,nvc,nvm51,nvm31,nvm15,v0
      common/c/mxm1,mxm2,mxc,mxcp1,mxcm1,x0
c
      call reset(u1,u2,nv,mx)
c
c     Positive Velocity
c
      do 5 iv=1,nvm1
      v=v0-iv
5     v1(iv)=a*v
c
      do 20 iv=1,nvm1
      do 10 ix=2,mx
10    u2(iv,ix)=u1(iv,ix)-v1(iv)*(u1(iv,ix)-u1(iv,ix-1))+s(ix)*h(iv)
20    continue
c
c     Negative Velocity
c
      do 30 ix=1,mxm1
30    u2(nv,ix)=u2(nv-2,mx-ix+1)
c
c     sets boundary conditions [illegible]hyp problem
c     by linear fit
c
      do 40 iv=1,nvm1
40    u2(iv,1)=2.*u2(iv,2)-u2(iv,3)
      do 50 iv=1,nvm1
50    u2(iv,1)=cvmgp(u2(iv,1),0.,u2(iv,1))
c
      u2(nv,mx)=u2(nv,1)
c
      return
      end

c
c.................................................................
c
      subroutine reset(u1,u2,nv,mx)
c
c     set old to new
c
      dimension u1(nv,mx),u2(nv,mx)
c
      do 10 iv=1,nv
      do 10 ix=1,mx
10    u1(iv,ix)=u2(iv,ix)
c
      return
      end
c.................................................................
c
      subroutine diff(uet2,uec2,uit2,uic2,
     1                dstar,dstar0,
     2                sex,six,
     3                diffe,diffi,adv,nv,mx)
c
c      This Subroutine Finds the INCOHERENT fluctuations by
c     subtracting coherent form total ff. The charge ditribution
c     is computed in the vector Z(1025) which is fourier transformed.
c     diffusion coefficients are claculated for all spatial positions
c     through another Fourier transform.
```

```
c
      dimension uet2(nv,mx),uec2(nv,mx),uit2(nv,mx),uic2(nv,mx)
      dimension dex(101),dex0(101),dix(101),dix0(101)
      dimension sex(mx),six(mx)
c
      dimension dstar(mx),dstar0(mx)
      dimension chrex(30),chrix(30)
      common //z(1025),y(1025)
c
      common/b/nvm1,nvc,nvm51,nvm31,nvm15,v0
      common/c/mxm1,mxm2,mxc,mxcp1,mxcm1,x0
      common/eps/num,ntime,dcx,aa,bb,drift,rl,rl3,rm,rt,vs0,beta
      common/scle/xtime,nk,n2n,xnt,nxi,nvi
      common/source/scxe,scxi,scc1,scc2
c
      data seight/2.82843/,stwo/1.414214/,ntime/0/
c
c     calculate ion & electron charge density
c
      do 5 ix=1,30
      chrex(ix)=0.
      chrix(ix)=0.
5     continue

c
      do 10 ix=1,30
      do 10 iv=nvm31,nvm1
      chrex(ix)=uet2(iv,mxc-ix+1)-uec2(iv,mxc-ix+1)+
     1 chrex(ix)+uet2(iv-1,mxc+ix-1)-uec2(iv-1,mxc+ix-1)
      chrix(ix)=uit2(iv,mxc-ix+1)-uic2(iv,mxc-ix+1)+
     1 chrix(ix)+uit2(iv-1,mxc+ix-1)-uic2(iv-1,mxc+ix-1)
10    continue
c
c     fourier transforms
c
c     electrons
c
      call fft(chrex)
      do 20 ix=1,mxc
      dex0(mxc-ix+1)=y(1)
20    dex(mxc-ix+1)=y(1)-y(ix)
      do 30 ix=mxc,mx
      dex0(ix)=y(1)
30    dex(ix)=dex(mx-ix+1)
c
c     ions
c
      call fft(chrix)
      do 40 ix=1,mxc
      dix0(mxc-ix+1)=y(1)
40    dix(mxc-ix+1)=y(1)-y(ix)
      do 50 ix=mxc,mx

      dix0(ix)=y(1)
50    dix(ix)=dix(mx-ix+1)
c
c     time: find if parameters of equation are
c           stable, and set xnt equal to greatest
c           fraction of time step. i.e. smallest
c           time step.
c
      dmax=0.
      do 55 ix=1,mx
      dstar0(ix)=dex0(ix)+dix0(ix)
55    dstar(ix)=dex(ix)+dix(ix)
      do 56 ix=1,mxc
56    dmax=amax1(dmax,dstar(ix))
c
      xntd=rl3*rm*rm*nvi*nvi*nvi*dmax
      xnta=stwo*nvm1*nxi/(.9*nvi*rl)
c
      xnt=xntd
      adv=stwo*nxi/(xnt*nvi*rl)
      diffi=rl3*nvi*nvi*nvi/(xnt*seight)
      diffe=rm*rm*diffi
c
      if(xnta.lt.xntd)goto 60
```

```
c
      xnt=xnta
      adv=stwo*nxi/(xnt*nvi*rl)
      diffi=rl3*nvi*nvi*nvi/(xnt*seight)
      diffe=rm*rm*diffi
c
60    continue
      xtime=xtime+1./xnt
      ntime=ntime+1
      if(mod(ntime,100).eq.0)write(20,100)xtime,dmax
100   format(3x,f8.5,3x,f6.3)
c
c     source terms
c
      do 70 ix=1,mx
      comm=sccl*(dex0(ix)-dex(ix))+scc2*(dix0(ix)-dix(ix))
      s00=comm/xnt
      six(ix)=scxi*s00
70    sex(ix)=scxe*s00
c

      return
      end
c..................................................................
c
      subroutine fft(ch)
c
c     Transforms potential Spectrum and calculates
c     diffusion coefficient using predetermined
c     expression for dielectric function.
c
      dimension ch(30)
c
      common //z(1025),y(1025)
c
      common/eps/num,ntime,dcx,aa,bb,drift,rl,rl3,rm,rt,vs0,beta
      common/scle/xtime,nk,n2n,xnt,nxi,nvi
c
      do 10 i=1,1025
10    z(i)=0.
c
      do 20 i=1,30
20    z(i)=ch(i)
c
      call setf79(2,n2n)
      call four79(2,n2n)
      do 30 ik=1,nk
      xk=ik-1.
      xk2=xk*xk
30    z(ik)=(dcx*xk*y(ik))/(xk2*xk2+aa*xk2+bb)
      call four79(2,n2n)
c
      return
      end
c..................................................................
c
      subroutine number(nv,mx)
c
c     works out all the constants in the program
c
      common/b/nvm1,nvc,nvm51,nvm31,nvm15,v0
      common/c/mxm1,mxm2,mxc,mxcp1,mxcm1,x0
      common/e/mxcp14,mxcm25,mxcp25,mxcm50
c
      nvm1=nv-1
      nvm15=nv-15
      nvm31=nv-31
      nvm51=nv-51
      v0=float(nvm1)
c
      mxm1=mx-1
      mxm2=mx-2
      mxc=mx/2+1
      mxcp1=mxc+1
      mxcm1=mxc-1
      x0=float(mx+1)/2.
```

```
c
      mxcp14=mxc+14
      mxcm25=mxc-25
      mxcp25=mxc+25
      mxcm50=mxc-50
c
      return
      end
c.................................................................
c
      subroutine init(uet1,uec1,uit1,uic1,
     1                dstar,dstar0,
     2                sex,six,hv,
     3                diffe,diffi,adv,nv,mx)
c
c     Sets the initial level of fluctuations and works out
c     certain parameters for the run.
c
      complex ze,zi,zre,zri,dzre,dzri
c
      dimension uec1(nv,mx),uet1(nv,mx),uic1(nv,mx),uit1(nv,mx)
      dimension dstar(mx),dstar0(mx)
      dimension sex(mx),six(mx),hv(nv)
      dimension n2(9)
c
      common/b/nvm1,nvc,nvm51,nvm31,nvm15,v0
      common/c/mxm1,mxm2,mxc,mxcp1,mxcm1,x0
      common/eps/num,ntime,dcx,aa,bb,drift,rl,rl3,rm,rt,vs0,beta
      common/scle/xtime,nk,n2n,xnt,nxi,nvi
      common/source/scxe,scxi,scc1,scc2
c
      namelist/int/num,ntime,n2n,nxi,nvi,rm,rt,drift
c
      data n2/3,5,9,17,33.65,129,257,513/
      data turb/.01/
c
c     basic constants
c
      write(6,1)
1     format(2x,'input: num,ntime,',/,
     1          '       n2n,nxi,nvi,',/,
     2          '       rm,rt,drift',/)
      read(6,int)
c
      xtime=0.
c
      rv=sqrt(rm*rt)
      rl=sqrt(rt)
      rl3=rl**3
      rv2m1=rv*rv-1.
      if(vs0.eq.0)vs0=rv*sqrt(drift*drift+rv2m1*alog(rv))/rv2m1-
     1                drift/rv2m1

       vs02=vs0*vs0
       beta=(drift/vs0-1.)
       xe=(vs0-drift)/rv
       xi=vs0
       ze=cmplx(xe,0.)
       zi=cmplx(xi,0.)
c
c     dielectric from z function
c
      call zeta(ze,zre,dzre,ddzre,dddzre)
      call zeta(zi,zri,dzri,ddzri,dddzri)
c
      rze=real(dzre)
      aze=aimag(dzre)
      rzi=real(dzri)
      azi=aimag(dzri)
c
      a0=-2.*(rze+rt*rzi)
      b0=(a0/2.)**2+(aze+rt*azi)**2
c
      pi=3.1415927
      sq2=sqrt(2.)
      nk=n2(n2n)
```

```
      l=nk-1
      const=float(l)/(pi*nxi)
      const2=const*const
      aa=a0*const2
      bb=b0*const2*const2
      cx=2.*sq2*l*l/(pi*pi*nxi**3)
c
      scxe=-sq2*aze/(nvi*rl*pi*pi)
      scxi=sq2*azi*rl/(nvi*pi*pi)
      scc1=rt*azi
      scc2=-aze

c
c     write parameters for this run
c
      call frame
      call setch(5.,40.,1,0,2,0)
      write(100,100)
100   format(10x,'  Two species Electron/Ion Clump Problem  ',///)
      write(100,200)rm,.t,vs0,drift,beta,n2n,nxi,nvi
200   format(5x,'parameters for this run',///,
     1       5x,'ion to electron mass ratio = ',f6.3,//,
     2       5x,'electron to ion temp. ratio= ',f6.3,//,
     3       5x,'vs      (v+ coordinate)     = ',f6.3,//,
     4       5x,'electron drift velocity    = ',f6.3,//,
     5       5x,' (units of ion thermal vel.) ',///,
     6       5x,'beta (magnification factor)= ',f6.3,//,
     6       5x,'power of  f.f.t.           = ',i4,//,
     7       5x,'x axis (n debye lenghts)   = ',i4,//,
     8       5x,'v axis (n ion thermal v.)  = ',i4)
c
c     draw maxwellians
c
      call frame
      call setch(5.,40.,1,0,2,0)
      write(100,300)
300   format(10x,' Configuration of Average Distributions',//)
c
      do 400 iv=1,31
      v=float(iv-16.)
      vi=v/4.
      vi2=vi*vi
      ve=(v-4.*drift)/(4.*rv)
      ve2=ve*ve
      dstar0(iv)=vi
      sex(iv)=exp(-ve2)/rv
400   six(iv)=exp(-vi2)

c
      call maps(dstar0(1),dstar0(31),0.,1.2,.2,.9,.2,.75)
      call trace(dstar0(1),six,31)
      call trace(dstar0(1),sex,31)
c
c     set ff
c
      do 10 iv=1,nv
      do 10 ix=1,mx
      uec1(iv,ix)=0.
      uic1(iv,ix)=0.
      uet1(iv,ix)=0.
      uit1(iv,ix)=0.
10    continue
c
      do 20 iv=1,nv
      do 20 ix=1,mx
      x=(ix-mxc)/2.
      v=(iv-nvm1)/2.
      x2=x*x
      v2=v*v
      if(v2.gt.25.)v2=25.
      if(x2.gt.25.)x2=25.
c
      uet1(iv,ix)=uet1(iv,ix)+turb*exp(-(x2+v2))
      uit1(iv,ix)=uet1(iv,ix)
c
20    continue
c
```

```
      vc=float(nv/2.)
      do 30 iv=1,nv
      vp=iv-vc
30    hv(iv)=cvmgp(1.,0.,vp)
c
      call diff(uet1,uec1,uit1,uic1,
     1         dstar,dstar0,
     2         sex,six,
     3         diffe,diffi,adv,nv,mx)
c
      return
      end
c.............................................................
c
      subroutine scale(xx,yy,nv,mx)
c
      dimension xx(mx),yy(nv)
      common/scle/xtime,nk,n2n,xnt,nxi,nvi
c
      do 10 ix=1,mx
10    xx(ix)=float(ix-1)/nxi
c
      do 20 iv=1,nv
20    yy(iv)=float(iv-1)/nvi
c
      return
      end
c.............................................................
c
      subroutine graph(uec,uet,uic,uit,
     1                 dex,dix,sex,six,
     2                    xx,yy,nv,mx)
c
c     does all the plotting routines
c
      dimension uec(nv,mx),uet(nv,mx)
      dimension uic(nv,mx),uit(nv,mx)
      dimension dex(mx),dix(mx)
      dimension sex(mx),six(mx)
      dimension cs(2),xx(mx),yy(nv)
      dimension screxc(15,15),scrixc(15,15)
      dimension scrext(15,15),scrixt(15,15)
      dimension xmi(4),xma(4),ymi(4),yma(4)
      dimension x1(51),y1(51)
c
      common/b/nvm1,nvc,nvm51,nvm31,nvm15,v0
      common/c/mxm1,mxm2,mxc,mxcp1,mxcm1,x0
      common/e/mxcp14,mxcm25,mxcp25,mxcm50
      common/scle/xtime,nk,n2n,xnt,nxi,nvi
c
      data xmi/.11328,.61328,.11328,.61328/
      data xma/.5,1.,.5,1./
      data ymi/.61328,.61328,.11328,.11328/
      data yma/1.,1.,.5,.5/
c
c     find minimum and maximum values
c
      call minmax(uec,rmaxec,rminec,nv,mx)
      call minmax(uet,rmaxet,rminet,nv,mx)
      call minmax(uic,rmaxic,rminic,nv,mx)
      call minmax(uit,rmaxit,rminit,nv,mx)
c
      do 20 iv=nvm15,nvm1
      do 20 ix=mxc,mxcp14
      screxc(ix-mxc+1,iv-nvm15+1)=uec(nvm15+nvm1-iv,ix)
      scrext(ix-mxc+1,iv-nvm15+1)=uet(nvm15+nvm1-iv,ix)
      scrixc(ix-mxc+1,iv-nvm15+1)=uic(nvm15+nvm1-iv,ix)
20    scrixt(ix-mxc+1,iv-nvm15+1)=uit(nvm15+nvm1-iv,ix)
c
c    write data
c
      call frame
      call setch(5.,40.,1,0,2,0)
      write(100,100)xtime,rmaxet,rmaxec,rmaxit,rmaxic,
```

```
     1              dex(1),sex(mxc),six(mxc)
100   format(10x,'value of diffusion and source at t=',f8.4,///,
     1        5x,'elec. max (total)    =',f10.6,//,
     2        5x,'elec. max (coherent) =',f10.6,//,
     3        5x,'ion   max (total)    =',f10.6,//,
     4        5x,'ion   max (coherent) =',f10.6,///,
     5        5x,'diffusion (sum)      =',f10.6,//,
     5        5x,'elec. source         =',f10.6,//,
     6        5x,'ion   source         =',f10.6,//)
c
      call frame
      call ddérs(-1)
      ncont=12
      cs(1)=0.
c
      df=(rmaxec-rminec)/ncont
      cs(2)=df
      call maps(xx(1),xx(15),yy(1),yy(15),xmi(1),xma(1),ymi(1),yma(1))
      call rcontr(k1,cs,k2,screxc,15,xx,1,15,1,yy,1,15,1)
      df=(rmaxet-rminet)/ncont
      cs(2)=df
      call maps(xx(1),xx(15),yy(1),yy(15),xmi(2),xma(2),ymi(2),yma(2))
      call rcontr(k1,cs,k2,scrext,15,xx,1,15,1,yy,1,15,1)
      df=(rmaxic-rminic)/ncont
      cs(2)=df
      call maps(xx(1),xx(15),yy(1),yy(15),xmi(3),xma(3),ymi(3),yma(3))
      call rcontr(k1,cs,k2,scrixc,15,xx,1,15,1,yy,1,15,1)
      df=(rmaxit-rminit)/ncont
      cs(2)=df
      call maps(xx(1),xx(15),yy(1),yy(15),xmi(4),xma(4),ymi(4),yma(4))
      call rcontr(k1,cs,k2,scrixt,15,xx,1,15,1,yy,1,15,1)
c
      write(6,990)
990   format('contour all done')
c
c     cross section for large region
c
      call frame
c
      do 40 iv=nvm51,nvm1
40    y1(iv-nvm51+1)=uet(nvm51+nvm1-iv,mxc)
      call maps(yy(1),yy(51),0.,.15,.2,.9,ymi(1),yma(1))
      call trace(yy(1),y1,51)
      do 45 iv=nvm51,nvm1
45    y1(iv-nvm51+1)=uec(nvm51+nvm1-iv,mxc)
      call trace(yy(1),y1,51)
c
      do 50 ix=mxcm50,mxc
50    y1(ix-mxcm50+1)=uet(nvm1,mxcm50+mxc-ix)
      call maps(xx(1),xx(51),0.,.15,.2,.9,ymi(3),yma(3))
      call trace(xx(1),y1,51)
      do 55 ix=mxcm50,mxc
55    y1(ix-mxcm50+1)=uec(nvm1,mxcm50+mxc-ix)
      call trace(xx(1),y1,51)
c
      call frame
c
      do 60 iv=nvm51,nvm1
60    y1(iv-nvm51+1)=uit(nvm51+nvm1-iv,mxc)
      call maps(yy(1),yy(51),0.,.15,.2,.9,ymi(1),yma(1))
      call trace(yy(1),y1,51)
      do 65 iv=nvm51,nvm1
65    y1(iv-nvm51+1)=uic(nvm51+nvm1-iv,mxc)
      call trace(yy(1),y1,51)
c
      do 70 ix=mxcm50,mxc
70    y1(ix-mxcm50+1)=uit(nvm1,mxcm50+mxc-ix)
      call maps(xx(1),xx(51),0.,.15,.2,.9,ymi(3),yma(3))
      call trace(xx(1),y1,51)
      do 75 ix=mxcm50,mxc
75    y1(ix-mxcm50+1)=uic(nvm1,mxcm50+mxc-ix)
      call trace(xx(1),y1,51)
c
c     diffusion and source
```

```
c
      call frame
c
      ymax=1.5*dex(1)
      do 80 ix=mxcm50,mxc
80    y1(ix-mxcm50+1)=dex(ix)
      call maps(xx(1),xx(51),0.,ymax,.2,.9,ymi(1),yma(1))
      call trace(xx(1),y1,51)
c
      ymin=-.5*sex(mxc)
      do 90 ix=mxcm50,mxc
90    y1(ix-mxcm50+1)=sex(ix)
      call maps(xx(1),xx(51),ymin,sex(mxc),.2,.9,ymi(3),yma(3))
      call trace(xx(1),y1,51)
c
      return
      end
c...............................................................
c
      subroutine minmax(u,rmax,rmin,nv,mx)
c
c     Find maximum and minum values of ff
c
      dimension u(nv,mx)
c
      common/b/nvm1,nvc,nvm51,nvm31,nvm15,v0
      common/c/mxm1,mxm2,mxc,mxcp1,mxcm1,x0
      common/e/mxcp14,mxcm25,mxcp25,mxcm50
c
      rmin=10.
       rmax=0.
c
      do 10 iv=nvm15,nv
      do 10 ix=mxcm25,mxcp25
      rmin=amin1(rmin,u(iv,ix))
10    rmax=amax1(rmax,u(iv,ix))
c
      return
      end
c...............................................................
c
      subroutine zeta(z,zetaoz,dzetaz,ddzeta,dddzet)
c
c        purpose
c             to compute the plasma dispersion function and its first
c             three derivatives for a complex argument z.
c
c        usage
c             call zeta(z,zetaoz,dzetaz,ddzeta,dddzet) .
c
c        description of parameters
c             z = given argument (complex)
c             zetaoz = plasma dispersion function (complex)
c             dzetaz = first derivative of plasma dispersion function
c               (complex)
c             ddzeta = second derivative of plasma dispersion function
c               (complex)
c             dddzet = third derivative of plasma dispersion function
c             (complex)
c
c        description of program
c             when abs(z) .gt. 4, the plasma dispersion function and
c             its derivatives are evaluated by computing the third
c             derivative from its asymptotic expansion.  then the
c             function and its first 2 derivatives are computed using
c             relations derived from the differential equation.
c             when abs(z) .le. 4 and abs(imag(z)) .gt. 1, the function
c             is evaluated using the continued fraction method.
c             when abs(z) .le. 4 and abs(imag(z)) .le. 1, the function
c             is computed in double-preciaion using its power series
c             expansion.  in the continued fraction and power series
c             methods the derivatives are computed using the
c             differential equation and formulas derived from it.
c
```

```
c        references
c             zeta was written by prof. james d. callen for his ph.d.
c             thesis, "absolute and convective microinstabilities of a
c             magnetized plasma," department of nuclear engineering,
c             m.i.t., 1968.
c             fried, b.d. and conte, samuel d., "the plasma dispersion
c             function," academic press, 1961.
c
      complex z,zetaoz,dzetaz,term,fmult,terme,an1,bn1,zsquar,hold,
     1temp1,temp2,ddzeta,dddzet,cmplx,cexp,conjg
      double precision realmu,imagmu,realsu,imagsu,realte,imagte,realse,
     1imagse
      data expmax/174./
c  expmax = the maximum no. to which e may be raised on the machine.
      data error/1.e-06/
      zsquar=z*z
      x=real(z)
      y=aimag(z)
      fn=real(zsquar)
      if(y .gt. 0.) go to 99
      if(abs(fn).lt.expmax.and.abs(aimag(zsquar)).lt.5.e+04)go to 98
      if(fn.gt.0.)go to 97
    1 format(76h argument z of subroutine zeta has too large a negative
     1imaginary part, z= ,1pe14.7,3h + ,1pe14.7,2h i)
      write(6,1) z
   97 hold=(0.,0.)
      go to 99
   98 hold=(0.,1.772454)*cexp(-zsquar)
   99 if(x*x+y*y.gt.16.)go to 200
      if(abs(y).ge.1.)go to 300
c
c  power series method - double precision
c
      realte=-2.*x
      imagte=-2.*y
      realmu=.5*(imagte*imagte-realte*realte)
      imagmu=-imagte*realte
      realsu=realte
      imagsu=imagte
      if(x.eq.0..and.y.eq.0.)go to 103
      fn=3.
  100 realse=realte

      imagse=imagte
      realte=(realse*realmu-imagse*imagmu)/fn
      imagte=(realse*imagmu+imagse*realmu)/fn
      realse=realsu
      imagse=imagsu
      realsu=realsu+realte
      imagsu=imagsu+imagte
      fn=fn+2.
      if(sngl(realse)-sngl(realsu).ne.0..or.sngl(imagse)-sngl(imagsu).ne
     1.0.)go to 100
  103 x=realsu
      fn=imagsu
      if(y.gt.0.)hold=(0.,1.772454)*cexp(-zsquar)
      zetaoz=cmplx(x,fn)+hold
      go to 401
c
c  asymptotic series method - compute third derivative
c
  200 fn=5.
      dddzet=6.
      term=dddzet
      fmult=.5/zsquar
  201 terme=term
      term=term*fmult*fn*(fn-1.)/(fn-3.)
      zetaoz=term/terme
      if(abs(real(zetaoz))+abs(aimag(zetaoz)).gt.1.)go to 250
      zetaoz=dddzet
      dddzet=dddzet+term
      fn=fn+2.
      if(real(zetaoz).ne.real(dddzet).or.aimag(zetaoz).ne.aimag(dddzet))
```

```
     1go to 201
  250 dddzet=dddzet/(zsquar*zsquar)
      if(y.gt.0.)go to 260
      fn=1.
      if(y.lt.0.)fn=2.
      dddzet=dddzet-4.*fn*hold*z*(2.*zsquar-3.)
  260 ddzeta=-(4.+(zsquar-.5)*dddzet)/(z*(2.*zsquar-3.))
      dzetaz=(2.-z*ddzeta)/(2.*zsquar-1.)
      zetaoz=-(1.+.5*dzetaz)/z
      return
c
c  continued fraction method
c
c   (terme=a(n-1), term=a(n), dzetaz=b(n-1), fmult=b(n))
  300 if(y.lt.0.)z=conjg(z)
      terme=(1.,0.)
      term=(0.,0.)
      dzetaz=term
      fmult=terme
      n=0
      an1=z
      bn1=-z*z+0.5
  301 temp1=bn1*term+an1*terme
      temp2=bn1*fmult+an1*dzetaz
       zetaoz=temp1/temp2
      dzetaz=(zetaoz-term/fmult)/zetaoz
      if(abs(real(dzetaz)).lt.error.and.abs(aimag(dzetaz)).lt.error) go
     1to 302
      bn1=bn1+2.
      n=n+1
      an1=-.5*float(n*(2*n-1))
      terme=term
      dzetaz=fmult
      term=temp1
      fmult=temp2
      if(n.lt.30)go to 301
  302 if(y.ge.0.) go to 401
      zetaoz=conjg(zetaoz)+2.*hold
      z=conjg(z)
  401 dzetaz=-2.*(1.+z*zetaoz)
      ddzeta=-2.*(zetaoz+z*dzetaz)
      dddzet=-2.*(2.*dzetaz+z*ddzeta)
      return
      end

c..................................................................
*cft i=pdiff,b=bdif,l=ldif
*ldr i=bdif,lib=(ftlib,disspla,tv80lib,fortlib),x=xdif
```

The following program is used to solve for the analytic regeneration condition of §6.5.

```
*select box=m16,account=430mas
*file name=st
c...............................................................
c
c
c      program name: st f10     1/8/80    t.b-g
c
c      solves for
c      analytical regeneration conditions, equations
c      (7.33) and (7.34).
c
c      SYMBOLS USED:
c
c         rt= temperature ratio (Te/Ti)
c         rm= mass ratio (Mi/Me)
c         drift= electron drift velocity
c         vs0= v(+) coordinate
c         xk0= average wavenumber of fluctuations
c
c         xj0= 0 order Bessel function
c         zre= derivative of z function (electrons)
c         zri= derivative of z function (ions)
c
c
c...............................................................
      external f
      dimension x(2),w(8),par(4)
      data eps/.0001/,nsig/5/,itmax/1000/
      data x/1.,1./,w/8*0./
c
      namelist/temp/rt,rm,drift
      namelist/guess/x
c
      call dropfile("+xst")
      call link("unit6=tty,unit20=(xout,hc,create)//")
c
      write(6,1)
1     format(3x,"input: rt,rm,drift",/)
      read(6,temp)
      write(6,2)
2     format(3x,"input: initial guess, x(1),x(2)",/)
      read(6,guess)
c
      par(1)=rt
      par(2)=rm
      par(3)=drift
      par(4)=sqrt(rm*rt)
c
      call zsystm(f,eps,nsig,2,x,itmax,w,par,ier)
c
      write(6,10)(x(i),i=1,2),itmax
10    format(3x,2(f8.5,2x),"number of iterations:",i6)
c
c     ALL DONE
c
       call timeused(icpu,io,isys)
        write (6,40)icpu
40       format(16H whew.. all done,i8)
          call exit
      end
c...............................................................
c
      function f(x,k,par)
c
      complex ze,zi,zre,zri,dzre,dzri
      dimension x(2),par(4)
      data pi2/8.825/
c
      external func,funck
c
      rt=par(1)
      rm=par(2)
      drift=par(3)
      rv=par(4)
      vs0=x(1)
```

```
      xk0=x(2)
c
      xe=(vs0-drift)/rv
      xi=vs0
      ze=cmplx(xe,0.)
      zi=cmplx(xi,0.)
c
c     dielectric from z function
c
      call zeta(ze,zre,dzre,ddzre,dddzre)
      call zeta(zi,zri,dzri,ddzri,dddzri)
c
      rze=real(dzre)
      aze=aimag(dzre)
      rzi=real(dzri)
      azi=aimag(dzri)
c
        a0=-2.*(rze+rt*rzi)
        b0=(a0/2.)**2+(aze+rt*azi)**2
c
c    do integrals
c
      a1=rt*rt*azi*azi
      a2=rt*azi*aze
      b0=b0-2.*a2
      call gauss(0.,7.,xi1,func,a0,b0,a1,xk0)
      call gauss(0.,7.,xi2,func,a0,b0,a2,xk0)
      alpha=xi1/(pi2+xi2)
      a3=aze*(alpha*aze-rt*azi)/pi2
      call gauss(0.,7.,xi3,func,a0,b0,a3,xk0)
c
      f=xi3-1.
      if(k.eq.1)return
c
c
      a1=aze*aze
      call gauss(0.,7.,xi1,func,a0,b0,a1,xk0)
      beta=xi1/(pi2+xi2)
      a3e=a3
      a3i=rt*azi*(rt*beta*azi-aze)/pi2
      call gauss(0.,10.,xke3,funck,a0,b0,a3e,xk0)
      call gauss(0.,10.,xki3,funck,a0,b0,a3i,xk0)
c
      f=(xke3+xki3)/2.-xk0*xk0
      return
c
      end
c...................................................................
c
        function func(x,a0,b0,x2,xk0)
c
        data pi/3.1415927/
c
        xp=2.45*x/xk0
        func=4.*pi*x2*(1.-xj0(xp))/((x**4+a0*x**2+b0)*x)
        return
        end
c.......................................................................
c
        function funck(x,a0,b0,x2,xk0)
c
        data pi/3.1415927/
c
        xp=2.45*x/xk0
        funck=4.*pi*x2*x*(1.-xj0(xp))/(x**4+a0*x**2+b0)
        return
        end
c.......................................................................
c
        function xj0(x)
c
c       Calculates J0(x) (the zero order BESSEL
c       function)
c
        if(x.gt.3.)goto 10
c
        x=x/3.
```

```
      x2=x*x
      xj0=1.+x2*(-2.2499+x2*(1.26562+x2*(-.316386+x2*(.04444+x2*
     1          (-.00394+x2*.00021)))))
      return
c
10    xold=x
      x=3./x
      f0=.797885+x*(-.00000077+x*(-.00552+x*(-.00009512+x*(.001372+x*
     1          (-.00072805+x*.00014476)))))
      theta0=xold-.78539+x*(-.041664+x*(-.00003954+x*(.0026257+x*
     1          (-.000541+x*(-.000293+x*.00013558)))))
      xj0=f0*cos(theta0)/sqrt(xold)
      return
c
      end
c
c..........................................................................
c
c Program Name:gauss    8/7/80    T.Boutros-Ghali
c
c     16 POINT GAUSS QUADRATURE.
c
      subroutine  gauss(xlo,xhi,xint,func,a0,b0,x2,xk0)
c
c     xint    = value of integral
c     xlo     = lower limit of integration
c     xhi     = upper limit
c     func    = function to be integrated
c     a0,b0,
c     x2,xk0  = parameters for func
c
      dimension x0(16),w0(16)

c
      data x0/-.9894009349,-.9445750230,-.8656312023,-.7554044083,
     1        -.6178762444,-.4580167776,-.2816035507,-.0950125098,
     1        0.6178762444,0.4580167776,0.2816035507,0.0950125098,
     1        0.9894009349,0.9445750230,0.8656312023,0.7554044083/
c
      data w0/0.0271524594,0.0622535239,0.0951585116,0.1246289712,
     1        0.1495959888,0.1691565193,0.1826034150,0.1894506104,
     1        0.1495959888,0.1691565193,0.1826034150,0.1894506104,
     1        0.0271524594,0.0622535239,0.0951585116,0.1246289712/
c
      a=(xhi-xlo)/2.
      b=(xhi+xlo)/2.
c
      xint=0.
      do 10 i=1,16
      xx=a*x0(i)+b
10    xint=xint+a*w0(i)*func(xx,a0,b0,x2,xk0)
c
      return
      end
c.......................................................................
```

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