On Dual-Weighted Residual Error Estimates for p-Dependent Discretizations

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Abstract

This report analyzes the behavior of three variants of the dual-weighted residual (DWR) error estimates applied to the p-dependent discretization that results from the BR2 discretization of a second-order PDE. Three error estimates are assessed using two metrics: local effectivities and global effectivity. *A priori* error analysis is carried out to study the convergence behavior of the local and global effectivities of the three estimates. Numerical results verify the *a priori* error analysis.

1 p-Dependence of DG Discretizations

Let $u \in V$, where V is some appropriate function space, be the weak solution to a general second-order PDE described by the semilinear form $R(\cdot,\cdot) : V \times V \to \mathbb{R}$. That is, u satisfies

$$R(u, v) = 0, \quad \forall v \in V.$$

The space $V_{h,p}$ is a finite-dimensional space of piecewise polynomial functions of degree at most p on a triangulation $\mathcal{T}_h$ of domain $\Omega \subset \mathbb{R}^n$, i.e.

$$V_{h,p} \equiv \{ v_{h,p} \in L^2(\Omega) |\ v_{h,p}|_K \in P^p(K), \forall K \in \mathcal{T}_h \},$$

where $P^p(K)$ denotes the space of p-th degree polynomial on element K. A finite element approximation to the problem, $u_{h,p} \in V_{h,p}$, is induced by the semilinear form $R_{h,p}(\cdot,\cdot) : V_{h,p} \times V_{h,p} \to \mathbb{R}$ and satisfies

$$R_{h,p}(u_{h,p}, v_{h,p}) = 0, \quad \forall v_{h,p} \in V_{h,p}.$$

Definition 1.1 (p-Dependence)**.** *Let $q < p$. A semilinear form $R_{h,p}(\cdot,\cdot) : V_{h,p} \times V_{h,p} \to \mathbb{R}$ is said to be p-independent if*

$$R_{h,p}(w_{h,q}, v_{h,q}) = R_{h,q}(w_{h,q}, v_{h,q}), \quad \forall w_{h,q}, v_{h,q} \in V_{h,q} \subset V_{h,p}.$$

If a semilinear form is not p-independent, then it is said to be p-dependent.

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We now show that the semilinear form arising from the second discretization of Bassi and Rebay (BR2)[1] of a second-order PDE is p-dependent. For simplicity, let us consider the Poisson equation with homogeneous Dirichlet boundary conditions on domain Ω,

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The appropriate function space for the problem is $V = H_0^1(\Omega)$. The semilinear form is given by

$$R(w, v) = \ell(w) - a(w, v), \tag{1.1}$$

where the source functional $\ell \in V'$ and the bilinear form $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ are given by

$$\ell(w) = \int_\Omega f v dx \quad \text{and} \quad a(w, v) = \int_\Omega \nabla v \cdot \nabla w dx.$$

The BR2 discretization of the Poisson equation is given by the semilinear form

$$R_{h,p}(w_{h,p}, v_{h,p}) = \ell_{h,p}(w_{h,p}) - a_{h,p}(w_{h,p}, v_{h,p}), \tag{1.2}$$

where

$$\begin{aligned} \ell_{h,p}(v_{h,p}) &= \ell(v_{h,p}) \\ a_{h,p}(w_{h,p}, v_{h,p}) &= \sum_{K \in \mathcal{T}_h} \int_K \nabla v_{h,p} \cdot \nabla w_{h,p} dx - \sum_{F \in \mathcal{F}_h} \int_F \{\nabla v_{h,p}\} \cdot [\![w_{h,p}]\!] + [\![v_{h,p}]\!] \cdot \{\nabla w_{h,p}\} ds \\ &\quad + \sum_{F \in \mathcal{F}_h} a_{h,p}^{F,\mathrm{BR2}}(w_{h,p}, v_{h,p}) \end{aligned}$$

where $\mathcal{F}_h$ denotes the set of faces of the triangulation. On the interior faces, the jump operator, $[\![\cdot]\!]$, for a scalar quantity x is defined by

$$[\![x]\!] = x^- \hat{n}^- + x^+ \hat{n}^+.$$

and the average operator, $\{\cdot\}$, for a vector quantity y is defined by

$$\{y\} = \frac{1}{2}(y^- + y^+).$$

Due to the homogeneous Dirichlet boundary condition, the operators on the boundary faces are given by (see e.g. [2] for general case)

$$[\![x]\!] = x\hat{n} \quad \text{and} \quad \{y\} = y.$$

The BR2 face penalty term for the face $F \in \mathcal{F}_h$ is given by

$$a_{h,p}^{F,\mathrm{BR2}}(w_{h,p}, v_{h,p}) = -\int_F \beta [\![v_{h,p}]\!] \cdot r_{h,p}^F([\![w_{h,p}]\!]) ds,$$

where the lifting operator, $r^F_{h,p}(\llbracket w_{h,p}\rrbracket) \in [V^F_{h,p}]^d$, satisfies

$$\sum_{K\in K_F} \int_K g_{h,p} \cdot r^F_{h,p}(\llbracket w_{h,p}\rrbracket)dx = -\int_F \{g_{h,p}\} \cdot \llbracket w_{h,p}\rrbracket ds, \quad \forall g_{h,p} \in [V^F_{h,p}]^d,$$

where $V^F_{h,p} \equiv \{v_{h,p} \in L^2(K_F)|\ v_{h,p}|_K \in P^p(K), K \in K_F\}$ with K_F denoting the set of elements neighboring face K. The stability parameter, β, must be set to a number greater than the number of faces for coercivity [3].

Theorem 1.1. *The BR2 lifting operator, $r^F_{h,p}(\cdot)$, is p-dependent in the sense that*

$$r^F_{h,q}(\llbracket w_{h,q}\rrbracket) \neq r^F_{h,p}(\llbracket w_{h,q}\rrbracket)$$

for some $w_{h,q} \in V_{h,q}$ with $q < p$.

Proof. By definition, the lifting operator, $r^F_{h,p}(\llbracket w_{h,q}\rrbracket)$, satisfies

$$\sum_{K\in K_F} \int_K g_{h,p} \cdot r^F_{h,p}(\llbracket w_{h,q}\rrbracket)dx = -\int_F \{g_{h,p}\} \cdot \llbracket w_{h,q}\rrbracket ds, \quad \forall g_{h,p} \in [V^F_{h,p}]^d.$$

Because $V^F_{h,p}$ is finite dimensional, there exist basis functions that span $V^F_{h,p}$. In particular, let us denote the basis functions that span the restriction of $V^F_{h,p}$ to K, one of the elements in K_F, by $\{\phi_m\}$. The dimension of $V_{h,p}|_K$ is $\mathcal{N}(p)$, where $\mathcal{N}(p)$ is the dimension of the p-th degree polynomial space. For example, for triangular elements, $\mathcal{N}(p) = (p+1)(p+2)/2$. We will chose ϕ_m to be a hierarchical orthogonal basis with respect to K, i.e.,

$$\phi_m \in P^r(K), \quad \forall m \leq \mathcal{N}(r)$$
$$\int_K \phi_n \phi_m dx = \begin{cases} c_n, & n = m \\ 0, & n \neq m. \end{cases}$$

The i-th spatial component of the lifting operator restricted to element K, $r^{F,i}_{h,p}(\llbracket w_{h,q}\rrbracket)|_K$, can be represented as

$$r^{F,i}_{h,p}(\llbracket w_{h,q}\rrbracket)|_K = \sum_{n=1}^{\mathcal{N}(p)} B^i_n \phi_n$$

where $B^i \in \mathbb{R}^{\mathcal{N}(p)}$. The coefficients, B^i, of the lifting operator restricted to K must satisfy the system of algebraic equations

$$\sum_{n=1}^{\mathcal{N}(p)} \left[\int_K \phi_m \phi_n dx\right] B^i_n = -\alpha \int_F \phi_m n_i \cdot \llbracket w_{h,q}\rrbracket ds, \quad \forall m = 1, \dots, \mathcal{N}(p),$$

where $\alpha = 1/2$ on the interior face and $\alpha = 1$ on the boundary face. Due to the orthogonality of the basis functions, we arrive at an explicit expression for the coefficients,

$$B^i_n = -\frac{\alpha}{c_n} \int_F \phi_n \hat{n}_i \cdot \llbracket w_{h,q}\rrbracket ds, \quad n = 1, \dots, \mathcal{N}(p).$$

The face integral term does not vanish in general. In particular,

$$B_n^i = -\frac{\alpha}{c_n} \int_F \phi_n \hat{n}_i \cdot [\![w_{h,q}]\!] ds \neq 0, \quad n = \mathcal{N}(q) + 1, \ldots, \mathcal{N}(p),$$

for some $[\![w_{h,q}]\!] \in P^q(F)$. Having finite coefficients for $n > \mathcal{N}(q)$, the lifting operator $r_{h,p}^{F,i}([\![w_{h,q}]\!])|_K$ is not in the space $P^q(K)$. In contrast, $r_{h,q}^{F,i}([\![w_{h,q}]\!])|_K \in P^q(K)$ by construction. Thus, $r_{h,q}^{F,i}([\![w_{h,q}]\!]) \neq r_{h,p}^{F,i}([\![w_{h,q}]\!])$ and the lifting operator is p-dependent. □

As the lifting operator is p-dependent, the semilinear form arising from the BR2 discretization of a second-order PDE is p-dependent.

Remark 1.1. *The interior penalty (IP) DG discretization is also p-dependent. The bilinear form for the IP method is obtained by replacing the BR2 face penalty term, $a_{h,p}^{F,BR2}(\cdot,\cdot) : V_{h,p} \times V_{h,p} \to \mathbb{R}$, with the IP face penalty term,*

$$a_{h,p}^{F,IP}(w_{h,p}, v_{h,p}) = C^{IP} \int_F \frac{p^2}{h} [\![v_{h,p}]\!] \cdot [\![w_{h,p}]\!] ds,$$

which is p-dependent due to the explicit presence of the p^2 term.

2 The Dual-Weighted Residual Error Estimation

In this section, we review the dual-weighted residual (DWR) error estimate of Becker and Rannacher [4, 5] applied to the DG methods.

2.1 Problem Setup

For simplicity, we consider the Poisson equation with homogeneous Dirichlet boundary conditions, as in Section 1, with a linear output functional of the form

$$J(w) = J_{h,p}(w) = -\ell^O(w) = -\int_\Omega g w dx,$$

for some $g \in L^2(\Omega)$. Our objective is to quantify

$$\mathcal{E} \equiv J_{h,p}(u_{h,p}) - J(u),$$

where $u \in V$ and $u_{h,p} \in V_{h,p}$ satisfy the residual expressions Eq. (1.1) and (1.2), respectively. In the DWR framework, the output error is quantified in terms of the adjoint solution, ψ. For the Poisson problem of interest, the strong form of the dual problem is given by

$$\begin{aligned} -\Delta\psi &= g \quad \text{in } \Omega \\ \psi &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Equivalently, the weak form of the dual problem is: Find $\psi \in V = H_0^1(\Omega)$ such that

$$R^\psi(v, \psi) = \ell^O(v) - a(v, \psi) = 0, \quad \forall v \in V.$$

Similarly, the finite element approximation to the dual problem is: Find $\psi_{h,p} \in V_{h,p}$ such that

$$R_{h,p}^\psi(v_{h,p}, \psi_{h,p}) = \ell^O(v_{h,p}) - a_{h,p}(v_{h,p}, \psi_{h,p}) = 0, \quad \forall v_{h,p} \in V_{h,p}.$$

2.2 Local and Global Consistency Results

Let us develop properties of the discrete primal and dual residual that facilitate the development of error estimates for the DG method.

Lemma 2.1 (Extended Local Consistency)**.** *The semilinear form possesses local consistency in the following sense: Given the true solution, $u \in V = H^1(\Omega)$, the residual satisfies*

$$R_{h,p}(u, v|_K) = 0, \qquad \forall v \in H^1(\Omega),$$

where $v|_K \in L^2(\Omega)$ is understood as the restriction of v to K with zero extension in $\Omega \backslash K$. Similarly, given the true adjoint, $\psi \in V$, the adjoint residual satisfies

$$R^{\psi}_{h,p}(v|_K, \psi) = 0, \qquad \forall v \in H^1(\Omega).$$

These results are referred to as the extended local primal and dual consistency, respectively, because it encompasses the traditional statement of local consistency for $v|_K \in V_{h,p}(K) \subset H^1(K)$.

Proof. Since $u \in H^1(\Omega)$, all terms related to jumps in u in the primal residual vanish. The remaining expression is

$$\begin{aligned}
R_{h,p}(u, v|_K) &= \ell(v|_K) - a_{h,p}(u, v|_K) \\
&= \sum_{K' \in \mathcal{T}_h} \int_{K'} f v|_K dx - \sum_{K' \in \mathcal{T}_h} \int_{K'} \nabla v|_K \cdot \nabla u dx + \sum_{F \in \mathcal{F}_h} \int_F [\![v|_K]\!] \cdot \{\nabla u\} ds \\
&= \int_K f v dx - \int_K \nabla v \cdot \nabla u dx + \int_{\partial K} v \hat{n} \cdot \nabla u ds \\
&= \int_K v(f + \Delta u) dx = 0, \quad \forall v \in V = H^1(\Omega).
\end{aligned}$$

Similarly, since $\psi \in H^1(\Omega)$, all terms related to jumps in ψ in the dual residual vanish. The remaining expression is

$$\begin{aligned}
R^{\psi}_{h,p}(v|_K, \psi) &= \ell^O(v|_K) - a_{h,p}(v|_K, \psi) \\
&= \sum_{K' \in \mathcal{T}_h} \int_{K'} g v|_K dx - \sum_{K' \in \mathcal{T}_h} \int_{K'} \nabla \psi \cdot \nabla v|_K dx + \sum_{F \in \mathcal{F}_h} \int_F \cdot \{\nabla \psi\} [\![v|_K]\!] ds \\
&= \int_K g v dx - \int_K \nabla \psi \cdot \nabla v dx + \int_{\partial K} \nabla \psi \cdot \hat{n} v ds \\
&= \int_K v(g + \Delta \psi) dx = 0, \quad \forall v \in V = H^1(\Omega).
\end{aligned}$$

□

Lemma 2.2 (Extended Global Consistency)**.** *Given the true primal solution, $u \in V = H^1(\Omega)$, the discrete primal residual is globally consistent in the sense that*

$$R_{h,p_2}(u, v) = 0, \quad \forall v \in V_{h,p_1} \oplus V, \forall p_1, p_2 \in \mathbb{N}$$

Similarly, given the true dual solution, $\psi \in V = H^1(\Omega)$, the discrete dual residual is globally consistent in the sense that

$$R^{\psi}_{h,p_2}(v, \psi) = 0, \quad \forall v \in V_{h,p_1} \oplus V, \forall p_1, p_2 \in \mathbb{N}$$

Proof. First, we note that

$$\begin{aligned} V_{h,p_1} \oplus V &= (\oplus_K V_{h,p_1}(K)) \oplus V \subset (\oplus_K V_{h,p_1}(K)) \oplus \left(\oplus_K H^1(K)\right) \\ &= \oplus_K \left(V_{h,p_1}(K) \oplus H^1(K)\right) = \oplus_K H^1(K). \end{aligned}$$

The proof then follows from the extended local consistency. Since $v = \sum_{K \in \mathcal{T}_h} v|_K$, we have

$$R_{h,p_2}(u, v) = \sum_{K \in \mathcal{T}_h} R_{h,p_2}(u, v|_K) = 0, \qquad \forall v \in \oplus_K H^1(K) \supset (V_{h,p_1} \oplus V),$$

where the second equality follows from the extended local consistency, i.e., $R_{h,p_2}(u, v|_K) = 0$, $\forall v \in H^1(K)$. The proof for the global dual consistency is identical. □

2.3 DWR Error Estimates

Theorem 2.1 (Functional Error Representation Formula)**.** *The error in the finite element approximation of the output, $J_{h,p}(u_{h,p})$, is represented in terms of the adjoint solution, $\psi \in V$, by*

$$\mathcal{E} \equiv J_{h,p}(u_{h,p}) - J(u) = R_{h,p}(u_{h,p}, \psi - \psi_{h,p}).$$

Proof. Using the definition of the adjoint, we obtain the error representation formula

$$\begin{aligned} \mathcal{E} \equiv J_{h,p}(u_{h,p}) - J(u) &= \ell^O(u - u_{h,p}) \\ &= a_{h,p}(u - u_{h,p}, \psi) && \text{(extended global dual consistency)} \\ &= a_{h,p}(u - u_{h,p}, \psi - \psi_{h,p}) && \text{(Galerkin orthogonality)} \\ &= \ell(\psi - \psi_{h,p}) - a_{h,p}(u_{h,p}, \psi - \psi_{h,p}) && \text{(extended global primal consistency)} \\ &= R_{h,p}(u_{h,p}, \psi - \psi_{h,p}). \end{aligned}$$

Note that $\psi_{h,p}$ could be replaced by any $v_{h,p} \in V_{h,p}$ since $R_{h,p}(u_{h,p}, v_{h,p}) = 0$, $\forall v_{h,p} \in V_{h,p}$. □

Definition 2.1 (Local Functional Error Representation Formula)**.** *The functional output error, $\mathcal{E}$, is localized to element K according to*

$$\eta_K \equiv R_{h,p}(u_{h,p}, (\psi - \psi_{h,p})|_K).$$

Let us state a few important properties of the local error η_K. First, the output error is the sum of the local errors, i.e.

$$\mathcal{E} = \sum_{K \in \mathcal{T}_h} \eta_K.$$

Second, the local error representation requires that a local residual, which vanishes with mesh refinement, results from the elemental restriction of test functions. While DG discretizations have this property, continuous Galerkin discretizations do not. For continuous Galerkin discretizations, the global error representation formula must be integrated by parts to yields an expression with the strong form of residual, which vanishes with mesh refinement.

In practice, the true adjoint, $\psi \in V$, is not computable. Thus, we replace the adjoint with the surrogate solution obtained on a enriched space, i.e., $\psi_{h,p'} \in V_{h,p'}$ such that

$$R^{\psi}_{h,p'}(v_{h,p'}, \psi_{h,p'}) = 0, \quad \forall v_{h,p'} \in V_{h,p'},$$

for some $p' = p + p_{\text{inc}} > p$, where p_{inc} is the increase in the polynomial degree in the enrichment process.

We now introduce three different forms of the error estimates.

Definition 2.2 (Error Estimate 1)**.** *The error estimate 1 is given by*

$$\begin{aligned}\mathcal{E}^{(1)} &\equiv R_{h,p'}(u_{h,p}, \psi_{h,p'} - \psi_{h,p})\\ \eta_K^{(1)} &\equiv R_{h,p'}(u_{h,p}, (\psi_{h,p'} - \psi_{h,p})|_K).\end{aligned}$$

The error estimate 1 arises naturally if the discrete formulation of the adjoint is used (see, e.g., [6, 7, 8]).

Definition 2.3 (Error Estimate 2)**.** *The error estimate 2 is given by*

$$\begin{aligned}\mathcal{E}^{(2)} &\equiv R_{h,p'}(u_{h,p}, \psi_{h,p'})\\ \eta_K^{(2)} &\equiv R_{h,p'}(u_{h,p}, \psi_{h,p'}|_K).\end{aligned}$$

Error estimate 2 eliminates the need to compute $\psi_{h,p}$ by appealing to the local Galerkin orthogonality of DG discretizations, and this is one of the error estimates advocated in [9]. However, with the form presented, the local Galerkin orthogonality does not hold due to the p-dependence of the semilinear form. In particular, while

$$R_{h,p}(u_{h,p}, v_{h,p}) = 0, \quad \forall v_{h,p} \in V_{h,p},$$

the same does not hold if the p about which the residual is evaluated is replaced by $p' \neq p$, i.e.,

$$R_{h,p'}(u_{h,p}, v_{h,p}) \neq 0, \quad \text{for some } v_{h,p} \in V_{h,p}.$$

This implies that

$$\begin{aligned}\mathcal{E}^{(2)} &\equiv R_{h,p'}(u_{h,p}, \psi_{h,p'}) \neq R_{h,p'}(u_{h,p}, \psi_{h,p'} - \psi_{h,p}) \equiv \mathcal{E}^{(1)}\\ \eta_K^{(2)} &\equiv R_{h,p'}(u_{h,p}, \psi_{h,p'}|_K) \neq R_{h,p'}(u_{h,p}, (\psi_{h,p'} - \psi_{h,p})|_K) \equiv \eta_K^{(1)},\end{aligned}$$

and the error estimate 2 is different from error estimate 1.

Definition 2.4 (Error Estimate 3)**.** *The error estimate 3 is given by*

$$\begin{aligned}\mathcal{E}^{(3)} &\equiv R_{h,p}(u_{h,p}, \psi_{h,p'})\\ \eta_K^{(3)} &\equiv R_{h,p}(u_{h,p}, \psi_{h,p'}|_K).\end{aligned}$$

The error estimate 3 is obtained by simply replacing ψ in the error representation formula by $\psi_{h,p'}$. Note that because the residual is evaluated about p, the Galerkin orthogonality holds, and we have

$$\begin{aligned}\mathcal{E}^{(3)} &\equiv R_{h,p}(u_{h,p}, \psi_{h,p'}) = R_{h,p}(u_{h,p}, \psi_{h,p'} - v_{h,p}) \quad \forall v_{h,p} \in V_{h,p}\\ \eta_K^{(3)} &\equiv R_{h,p}(u_{h,p}, \psi_{h,p'}|_K) = R_{h,p}(u_{h,p}, (\psi_{h,p'} - v_{h,p})|_K) \quad \forall v_{h,p} \in V_{h,p}.\end{aligned}$$

2.4 Assessment of the Error Estimates

For each of the error estimates considered, we will develop a bound for the absolute error in the global error estimate

$$|\mathcal{E} - \mathcal{E}^{(i)}|$$

and the absolute error in the local error estimate

$$|\eta_K - \eta_K^{(i)}|.$$

In practice, however, we are more interested in the quality of the error estimates with respect to the true error. In particular, we want to ensure that the error in the error estimate is a small fraction of the true error; otherwise the estimates would be useless. The relative error in the global error estimate i is given by

$$\theta_{\text{global}}^{(i)} \equiv \frac{|\mathcal{E} - \mathcal{E}^{(i)}|}{|\mathcal{E}|}.$$

The relative error is related to the error effectivity I^{eff} defined in, for example, [4, 5] by

$$\frac{|\mathcal{E} - \mathcal{E}^{(i)}|}{|\mathcal{E}|} = \left| \frac{\mathcal{E} - \mathcal{E}^{(i)}}{\mathcal{E}} \right| = \left| 1 - \frac{\mathcal{E}^{(i)}}{\mathcal{E}} \right| = \left| 1 - I^{\text{eff}} \right|.$$

That is, the relative error measures the deviation of the error effectivity from unity. Ideally, the effectivity of the error estimate should improve with mesh refinement such that $I^{\text{eff}} \to 1$ as $h \to 0$. Equivalently, the relative error should ideally vanish as $h \to 0$.

Similarly, the relative error in the local error estimate i is given by

$$\theta_{\text{local},K}^{(i)} \equiv \frac{|\eta_K - \eta_K^{(i)}|}{|\eta_K|} = \left| 1 - \frac{\eta_K^{(i)}}{\eta_K} \right|.$$

Again, the relative error in the local error estimate measures the deviation of the local error effectivity from unity.

3 *A Priori* Error Analysis

In this section, we perform *a priori* analysis of the three error estimates to establish the bound on the output estimation errors. In particular, we are interested in the convergence of the estimates with grid refinement.

Throughout this section, we will use the notation $A \lesssim B$ to imply that $A \leq cB$ for some $c < \infty$ independent of h, in order to avoid proliferation of constants. Similarly, $A \gtrsim B$ implies that $A \geq cB$ for some $c > 0$ independent of h. Moreover, $A \approx B$ implies that $A \lesssim B$ and $B \lesssim A$.

3.1 Assumptions

We assume that the DG-FEM approximation to both the primal and the dual problems are optimal in the L^2 sense, i.e.,

$$\|u - u_{h,p}\|_{L^2(K)} \lesssim \|u - \Pi_{h,p} u\|_{L^2(K)}, \quad \forall K \in \mathcal{T}_h$$
$$\|\psi - \psi_{h,p}\|_{L^2(K)} \lesssim \|\psi - \Pi_{h,p} \psi\|_{L^2(K)} \quad \forall K \in \mathcal{T}_h,$$

where $\Pi_{h,p} : V \to V_{h,p}$ is the L^2 projection operator such that $\Pi_{h,p} v \in V_{h,p}$ satisfies

$$\|v - \Pi_{h,p} v\|_{L^2(\Omega)} = \inf_{w_{h,p} \in V_{h,p}} \|v - w_{h,p}\|_{L^2(\Omega)}.$$

Furthermore, we will assume u and ψ are analytic for convenience. Under the analyticity assumption, the scaling argument results in the following interpolation results:

$$\|v - \Pi_{h,p} v\|_{H^m(K)} \lesssim h^{p+1-m} \|v\|_{H^{p+1}(K)}$$
$$\|v - \Pi_{h,p} v\|_{H^m(F)} \lesssim h^{p+1/2-m} \|v\|_{H^{p+1}(K)}.$$

3.2 Useful Relationships

This section introduces lemmas that facilitate the development of the error bounds for the output error estimates.

Lemma 3.1 (Local Residual-Error Mapping). *For all $p_1, p_2, p_3 \in \mathbb{N}$, the local dual-weighted residual can be represented as*

$$R_{h,p_3}(w_{h,p_1}, (\psi - v_{h,p_2})|_K) = a_{h,p_3}(u - w_{h,p_1}, (\psi - v_{h,p_2})|_K), \quad \forall w_{h,p_1} \in V_{h,p_1}, v_{h,p_2} \in V_{h,p_2}.$$

where u and ψ are the solutions to the primal and dual problems respectively.

Proof. The proof relies on the definition of the primal residual, the extended local consistency (Lemma 2.1), and the linearity of the bilinear form, i.e.,

$$\begin{aligned} R_{h,p_3}(w_{h,p_1}, (\psi - v_{h,p_2})|_K) &\equiv \ell_{h,p_3}((\psi - v_{h,p_2})|_K) - a_{h,p_3}(w_{h,p_1}, (\psi - v_{h,p_2})|_K) \\ &= a_{h,p_3}(u, (\psi - v_{h,p_2})|_K) - a_{h,p_3}(w_{h,p_1}, (\psi - v_{h,p_2})|_K) \\ &= a_{h,p_3}(u - w_{h,p_1}, (\psi - v_{h,p_2})|_K), \quad \forall w_{h,p_1} \in V_{h,p_1}, v_{h,p_2} \in V_{h,p_2}. \end{aligned}$$

□

Lemma 3.2 (Global Residual-Error Mapping). *For all $p_1, p_2, p_3 \in \mathbb{N}$, the global dual-weighted residual can be represented as*

$$\begin{aligned} R_{h,p_3}(w_{h,p_1}, \psi - v_{h,p_2}) = a_{h,p_3}(u - w_{h,p_1}, \psi - v_{h,p_2}) &= R^{\psi}_{h,p_3}(u - w_{h,p_1}, v_{h,p_2}) \\ &\forall w_{h,p_1} \in V_{h,p_1}, v_{h,p_2} \in V_{h,p_2}. \end{aligned}$$

where u and ψ are the solutions to the primal and dual problems respectively.

Proof. The first equality follows from the local residual-error mapping, i.e.,

$$R_{h,p_3}(w_{h,p_1}, \psi - v_{h,p_2}) = \sum_{K \in \mathcal{T}_h} R_{h,p_3}(w_{h,p_1}, (\psi - v_{h,p_2})|_K) = \sum_{K \in \mathcal{T}_h} a_{h,p_3}(u - w_{h,p_1}, (\psi - v_{h,p_2})|_K)$$
$$= a_{h,p_3}(u - w_{h,p_1}, \psi - v_{h,p_2}), \quad \forall w_{h,p_1} \in V_{h,p_1}, v_{h,p_2} \in V_{h,p_2}.$$

The second equality results from the definition of the adjoint residual and the extended global consistency, i.e.,

$$\begin{aligned} R^{\psi}_{h,p_3}(u - w_{h,p_1}, v_{h,p_2}) &\equiv \ell^O_{h,p_3}(u - w_{h,p_1}) - a_{h,p_3}(u - w_{h,p_1}, v_{h,p_2}) \\ &= a_{h,p_3}(u - w_{h,p_1}, \psi) - a_{h,p_3}(u - w_{h,p_1}, v_{h,p_2}) \\ &= a_{h,p_3}(u - w_{h,p_1}, \psi - v_{h,p_2}), \quad \forall w_{h,p_1} \in V_{h,p_1}, v_{h,p_2} \in V_{h,p_2}. \end{aligned}$$

□

Lemma 3.3 (*h*-Scaling of the Lifting Operator)**.** *The BR2 lifting operator is bounded by the face jump according to*

$$\|r^F_{h,p}([\![v]\!])\|_{L^2(K_F)} \lesssim h^{-1/2} \|[\![v]\!]\|_{L^2(F)}.$$

Proof. The lemma is stated in, for example, [10]. Here, we present the proof for completeness. The inequality follows from setting the test function equal to $r^F_{h,p}([\![v]\!])$ in the definition of the lifting operator, applying the Schwarz inequality, and invoking the trace scaling argument, i.e.,

$$\begin{aligned} \|r^F_{h,p}([\![v]\!])\|^2_{L^2(K_F)} &= \int_{K_F} r^F_{h,p}([\![v]\!]) \cdot r^F_{h,p}([\![v]\!]) dx \\ &= \int_F \{r^F_{h,p}([\![v]\!])\} \cdot [\![v]\!] ds && \text{(definition of lifting operator)} \\ &\le \|r^F_{h,p}([\![v]\!])\|_{L^2(F)} \|[\![v]\!]\|_{L^2(F)} && \text{(Schwarz)} \\ &\lesssim h^{-1/2} \|r^F_{h,p}([\![v]\!])\|_{L^2(K_F)} \|[\![v]\!]\|_{L^2(F)}. && \text{(trace scaling)} \end{aligned}$$

Division of the both sides by $\|r^F_{h,p}([\![v]\!])\|_{L^2(K_F)}$ yields the desired result,

$$\|r^F_{h,p}([\![v]\!])\|_{L^2(K_F)} \lesssim h^{-1/2} \|[\![v]\!]\|_{L^2(F)}.$$

□

Remark 3.1. *The face jump is also bounded by the lifting operator as* $h^{-1/2}\|[\![v]\!]\|_{L^2(F)} \lesssim \|r^F_{h,p}([\![v]\!])\|_{L^2(K_F)}$*. The proof is provided in [10].*

Lemma 3.4 (Local Bilinear Form Error Bound)**.** *Under the optimality assumption, the following error bound holds on element K for all $p_1, p_2, p_3 \in \mathbb{N}$:*

$$|a_{h,p_3}(u - u_{h,p_1}, (\psi - \psi_{h,p_2})|_K)| \lesssim h^{p_1+p_2} \|u\|_{H^{p_1+1}(\hat{K})} \|\psi\|_{H^{p_2+1}(K)},$$

where u and ψ are the true solution to the primal and dual problems, respectively, and u_{h,p_1} and ψ_{h,p_2} are the DG-FEM approximation to the primal and dual problems, respectively, and $\hat{K}$ is the set of elements sharing common face with K.

Proof. Substitution of the expression for the bilinear form yields

$$
\begin{aligned}
&a_{h,p_3}(u-u_{h,p_1},(\psi-\psi_{h,p_2})|_K) \\
&= \underbrace{\sum_{K'}\int_{K'}\nabla(\psi-\psi_{h,p_2})|_K\cdot\nabla(u-u_{h,p_1})dx}_{\text{(I)}} - \underbrace{\sum_F\int_F\{\nabla(\psi-\psi_{h,p_2})|_K\}\cdot[\![u-u_{h,p_1}]\!]ds}_{\text{(II)}} \\
&\quad - \underbrace{\sum_F\int_F[\![(\psi-\psi_{h,p_2})|_K]\!]\cdot\{\nabla(u-u_{h,p_1})\}ds}_{\text{(III)}} - \underbrace{\sum_F\int_F\beta[\![(\psi-\psi_{h,p_2})|_K]\!]\cdot r^F_{h,p_3}([\![u-u_{h,p_1}]\!])ds}_{\text{(IV)}}
\end{aligned}
$$

Now we bound each one of the braced terms. The interior term becomes

$$
\begin{aligned}
|\text{(I)}| &= \left|\sum_{K'}\int_{K'}\nabla(\psi-\psi_{h,p_2})|_K\cdot\nabla(u-u_{h,p_1})dx\right| = \left|\int_K\nabla(\psi-\psi_{h,p_2})\cdot\nabla(u-u_{h,p_1})dx\right| \\
&\le \|\psi-\psi_{h,p_2}\|_{H^1(K)}\|u-u_{h,p_1}\|_{H^1(K)} \lesssim h^{p_1+p_2}\|\psi\|_{H^{p_2+1}(K)}\|u\|_{H^{p_1+1}(K)}
\end{aligned}
$$

The first face term is bounded by

$$
\begin{aligned}
|\text{(II)}| &= \left|\sum_F\int_F\{\nabla(\psi-\psi_{h,p_2})|_K\}\cdot[\![u-u_{h,p_1}]\!]ds\right| = \left|\int_{\partial K}\alpha\nabla(\psi-\psi_{h,p_2})\cdot[\![u-u_{h,p_1}]\!]ds\right| \\
&\le \|\alpha\nabla(\psi-\psi_{h,p_2})\|_{L^2(\partial K)}\|[\![u-u_{h,p_1}]\!]\|_{L^2(\partial K)} \\
&\lesssim h^{p_2-1/2}\|\psi\|_{H^{p_2+1}(K)}h^{p_1+1/2}\|u\|_{H^{p_1+1}(\hat{K})} = h^{p_1+p_2}\|\psi\|_{H^{p_2+1}(K)}\|u\|_{H^{p_1+1}(\hat{K})},
\end{aligned}
$$

where $\alpha = 1$ if F is a boundary face, and $\alpha = 1/2$ if F is an interior face. The second face term is bounded in a similar manner as the first term, resulting in

$$
\begin{aligned}
|\text{(III)}| &= \left|\sum_F\int_F[\![(\psi-\psi_{h,p_2})|_K]\!]\cdot\{\nabla(u-u_{h,p_1})\}ds\right| = \left|\int_{\partial K}(\psi-\psi_{h,p_2})\hat{n}\cdot\{\nabla(u-u_{h,p_1})\}ds\right| \\
&\lesssim h^{p_1+p_2}\|\psi\|_{H^{p_2+1}(K)}\|u\|_{H^{p_1+1}(\hat{K})}
\end{aligned}
$$

Finally, we bound the term involving the lifting operator as

$$
\begin{aligned}
|\text{(IV)}| &= \left|\sum_{F\in\mathcal{F}}\int_F\beta[\![(\psi-\psi_{h,p_2})|_K]\!]\cdot r^F_{h,p_3}([\![u-u_{h,p_1}]\!])ds\right| \\
&= \left|\sum_{F\in\partial K}\int_F\beta(\psi-\psi_{h,p_2})\hat{n}\cdot r^F_{h,p_3}([\![u-u_{h,p_1}]\!])\right| && \text{(finite support of } (\psi-\psi_{h,p_2})|_K) \\
&\le \sum_{F\in\partial K}\beta\|\psi-\psi_{h,p_2}\|_{L^2(F)}\|r^F_{h,p_3}([\![u-u_{h,p_1}]\!])\|_{L^2(F)} && \text{(Schwarz inequality)} \\
&\lesssim \sum_{F\in\partial K}\|\psi-\psi_{h,p_2}\|_{L^2(F)}h^{-1/2}\|r^F_{h,p_3}([\![u-u_{h,p_1}]\!])\|_{L^2(K_F)} && \text{(trace scaling)} \\
&\lesssim \sum_{F\in\partial K}\|\psi-\psi_{h,p_2}\|_{L^2(F)}h^{-1}\|[\![u-u_{h,p_1}]\!]\|_{L^2(F)} && \text{(Lemma 3.3)} \\
&\lesssim h^{p_1+p_2}\|\psi\|_{H^{p_2+1}(K)}\|u\|_{H^{p_1+1}(\hat{K})} && (L^2 \text{ optimality assumption})
\end{aligned}
$$

Combining the bounds for (I), (II), (III), and (IV), we obtain the desired result:

$$|\eta_K| \leq |(\mathrm{I})| + |(\mathrm{II})| + |(\mathrm{III})| + |(\mathrm{IV})| \lesssim h^{p_1+p_2} \|\psi\|_{H^{p_2+1}(K)} \|u\|_{H^{p_1+1}(\hat{K})}.$$

□

Lemma 3.5 (Global Bilinear Form Error Bound)**.** *Under the optimality assumption, the following error bound holds for all $p_1, p_2, p_3 \in \mathbb{N}$:*

$$|a_{h,p_3}(u - u_{h,p_1}, \psi - \psi_{h,p_2})| \lesssim h^{p_1+p_2} \|u\|_{H^{p_1+1}(\Omega)} \|\psi\|_{H^{p_2+1}(\Omega)}$$

where u_{h,p_1} and ψ_{h,p_2} are the DG-FEM approximation to the primal and dual problems, respectively.

Proof. The global error bound is a direct consequence of the local error bound, i.e.,

$$\begin{aligned}
|a_{h,p_3}(u - u_{h,p_1}, \psi - \psi_{h,p_2})| &= \left| \sum_{K \in \mathcal{T}_h} a_{h,p_3}(u - u_{h,p_1}, (\psi - \psi_{h,p_2})|_K) \right| \\
&\lesssim \sum_{K \in \mathcal{T}_h} h^{p_1+p_2} \|u\|_{H^{p_1+1}(\hat{K})} \|\psi\|_{H^{p_2+1}(K)} \\
&\leq h^{p_1+p_2} \left(\sum_{K \in \mathcal{T}_h} \|u\|_{H^{p_1+1}(\hat{K})} \right) \left(\sum_{K \in \mathcal{T}_h} \|\psi\|_{H^{p_2+1}(K)} \right) \\
&\lesssim h^{p_1+p_2} \|u\|_{H^{p_1+1}(\Omega)} \|\psi\|_{H^{p_2+1}(\Omega)}
\end{aligned}$$

□

3.3 *A Priori* Error Analysis of the True Output Error

In this section, we analyze the convergence behavior of the true output error.

Theorem 3.1 (Convergence of True Error)**.** *Let $u_{h,p} \in V_{h,p}$ be the DG-FEM solution to the Poisson equation. The local and global error are bounded by*

$$\begin{aligned}
|\eta_K| &\lesssim h^{2p} \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^{p+1}(K)} \\
|\mathcal{E}| &\lesssim h^{2p} \|u\|_{H^{p+1}(\Omega)} \|\psi\|_{H^{p+1}(\Omega)},
\end{aligned}$$

respectively, where $\hat{K}$ is the set of elements sharing a common face with K.

Proof. We prove the local convergence bound by invoking the local residual-error mapping, Lemma 3.1, for $w_{h,p_1} = u_{h,p}$ and $v_{h,p_2} = \psi_{h,p}$ and by applying the local bilinear form error bound, Lemma 3.4, for $p_1 = p_2 = p_3 = p$, i.e.,

$$|\eta_K| \equiv |R_{h,p}(u_{h,p}, (\psi - \psi_{h,p})|_K)| = |a_{h,p}(u - u_{h,p}, (\psi - \psi_{h,p})|_K)| \lesssim h^{2p} \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^{p+1}(K)}.$$

Similarly, we obtain the global convergence bound by applying the global residual-error mapping, Lemma 3.2, for $w_{h,p_1} = u_{h,p}$ and $v_{h,p_2} = \psi_{h,p}$ and the global bilinear form error bound, Lemma 3.5, for $p_1 = p_2 = p_3 = p$, i.e.,

$$|\mathcal{E}| \equiv |R_{h,p}(u_{h,p}, \psi - \psi_{h,p})| = |a_{h,p}(u - u_{h,p}, \psi - \psi_{h,p})| \lesssim h^{2p} \|u\|_{H^{p+1}(\Omega)} \|\psi\|_{H^{p+1}(\Omega)}.$$

Thus, both the global and local errors superconverge at the rate of h^{2p}. □

3.4 *A Priori* Error Analysis of Output Error Estimate 3

In this section, we analyze the convergence behavior of the output error estimate 3.

Theorem 3.2 (Convergence of Local Error Estimate 3)**.** *The error in the local error estimate 3 is bounded by*

$$|\eta_K - \eta_K^{(3)}| \lesssim h^{p+p'} \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^{p'+1}(K)}.$$

Proof. By linearity of the semilinear form with respect to the second argument, we have

$$\eta_K - \eta_K^{(3)} = R_{h,p}(u_{h,p}, \psi|_K) - R_{h,p}(u_{h,p}, \psi_{h,p'}|_K) = R_{h,p}(u_{h,p}, (\psi - \psi_{h,p'})|_K)$$

From here on, the proof is similar to that of the convergence of the true error. By invoking the local residual-error mapping, Lemma 3.1, for $w_{h,p_1} = u_{h,p}$ and $v_{h,p_2} = \psi_{h,p'}$ and by applying the local bilinear form error bound, Lemma 3.4, for $p_1 = p_3 = p$ and $p_2 = p'$, we obtain

$$|\eta_K - \eta_K^{(3)}| = |a_{h,p}(u - u_{h,p}, (\psi - \psi_{h,p'})|_K)| \lesssim h^{p+p'} \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^{p'+1}(K)}.$$

□

Corollary 3.1. *Assuming the true local error converges as* $\eta_K \approx h^{2p} \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^{p+1}(K)}$*, the effectivity of the local error estimate 3 converges to unity as*

$$\theta_{\text{local},K}^{(3)} = \left| 1 - \frac{\eta_K^{(3)}}{\eta_K} \right| = \frac{|\eta_K^{(3)} - \eta_K|}{|\eta_K|} \lesssim \frac{C h^{p+p'} \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^{p'+1}(K)}}{h^{2p} \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^{p+1}(K)}} = h^{p'-p} = h^{p_{\text{inc}}}$$

where p_{inc} *is the increase in the polynomial degree for the truth surrogate adjoint solve.*

Theorem 3.3 (Convergence of Global Error Estimate 3)**.** *The error in the global error estimate 3 is bounded by*

$$|\mathcal{E}^{(3)} - \mathcal{E}| \lesssim h^{p+p'} \|u\|_{H^{p+1}(\Omega)} \|\psi\|_{H^{p'+1}(\Omega)}.$$

Proof. The convergence of the global error estimate 3 follows from that of the local counterpart, i.e.,

$$\begin{aligned} |\mathcal{E} - \mathcal{E}^{(3)}| = \left| \sum_{K \in \mathcal{T}_h} (\eta_K - \eta_K^{(3)}) \right| &\lesssim \sum_{K \in \mathcal{T}_h} h^{p+p'} \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^{p'+1}(K)} \\ &\lesssim h^{p+p'} \|u\|_{H^{p+1}(\Omega)} \|\psi\|_{H^{p'+1}(\Omega)}. \end{aligned}$$

□

Corollary 3.2. *If* $\mathcal{E} \approx h^{2p} \|u\|_{H^{p+1}(\Omega)} \|\psi\|_{H^{p'+1}(\Omega)}$*, then the effectivity of the global error estimate 3 converges to unity as*

$$\theta_{\text{global}}^{(3)} = \left| 1 - \frac{\mathcal{E}^{(3)}}{\mathcal{E}} \right| = \frac{|\mathcal{E}^{(3)} - \mathcal{E}|}{|\mathcal{E}|} \lesssim \frac{h^{p+p'} \|u\|_{H^{p+1}(\Omega)} \|\psi\|_{H^{p'+1}(\Omega)}}{h^{2p} \|u\|_{H^{p+1}(\Omega)} \|\psi\|_{H^{p'+1}(\Omega)}} \lesssim h^{p'-p} = h^{p_{\text{inc}}}.$$

3.5 *A Priori* Error Analysis of Output Error Estimate 1

In this section, we analyze the convergence behavior of the output error estimate 1.

Theorem 3.4 (Convergence of Local Error Estimate 1)**.** *The error in the local error estimate 1 is bounded by*

$$|\eta_K - \eta_K^{(1)}| \lesssim h^{2p} \|\psi\|_{H^{p+1}(K)} \|u\|_{H^{p+1}(\hat{K})}$$

Proof. Expanding the difference in the local error using the error representation formula,

$$\begin{aligned}
\eta_K - \eta_K^{(1)} &= R_{h,p}(u_{h,p}, (\psi - \psi_{h,p})|_K) - R_{h,p'}(u_{h,p}, (\psi_{h,p'} - \psi_{h,p})|_K) \\
&= \underbrace{R_{h,p}(u_{h,p}, (\psi - \psi_{h,p})|_K) - R_{h,p'}(u_{h,p}, (\psi - \psi_{h,p})|_K)}_{\text{(I)}} + \underbrace{R_{h,p'}(u_{h,p}, (\psi - \psi_{h,p'})|_K)}_{\text{(II)}}
\end{aligned}$$

Term (II) can be bounded following a similar argument as that used to bound $\eta_K - \eta_K^{(3)}$. By invoking the local residual-error mapping, Lemma 3.1, for $w_{h,p_1} = u_{h,p}$ and $v_{h,p_2} = \psi_{h,p'}$ and by applying the local bilinear form error bound, Lemma 3.4, for $p_1 = p$ and $p_2 = p_3 = p'$, we obtain

$$\begin{aligned}
|\text{(II)}| &= |R_{h,p'}(u_{h,p}, (\psi - \psi_{h,p'})|_K)| = |a_{h,p'}(u - u_{h,p}, (\psi - \psi_{h,p'})|_K)| \\
&\lesssim h^{p+p'} \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^{p'+1}(K)}.
\end{aligned}$$

The only difference in the terms constituting (I) stems from the difference in the lifting spaces. Thus, term (I) can be expressed as

$$\begin{aligned}
|\text{(I)}| &= \left| - \sum_{F \in \partial K} \left(\int_F \beta [\![(\psi - \psi_{h,p})|_K]\!] \cdot r^F_{h,p'}([\![u - u_{h,p}]\!]) ds - \int_F \beta [\![(\psi - \psi_{h,p})|_K]\!] \cdot r^F_{h,p}([\![u - u_{h,p}]\!]) ds \right) \right| \\
&= \left| - \sum_{F \in \partial K} \int_F \beta [\![(\psi - \psi_{h,p})|_K]\!] \cdot \left(r^F_{h,p'}([\![u - u_{h,p}]\!]) - r^F_{h,p}([\![u - u_{h,p}]\!]) \right) ds \right| \\
&\leq \sum_{F \in \partial K} \beta \|\psi - \psi_{h,p}\|_{L^2(F)} \|r^F_{h,p'}([\![u - u_{h,p}]\!]) - r^F_{h,p}([\![u - u_{h,p}]\!])\|_{L^2(F)} \\
&\lesssim \sum_{F \in \partial K} \|\psi - \psi_{h,p}\|_{L^2(F)} h^{-1/2} \|r^F_{h,p'}([\![u - u_{h,p}]\!]) - r^F_{h,p}([\![u - u_{h,p}]\!])\|_{L^2(K_F)} \\
&\lesssim \sum_{F \in \partial K} \|\psi - \psi_{h,p}\|_{L^2(F)} h^{-1} \|[\![u - u_{h,p}]\!]\|_{L^2(F)} \\
&\lesssim h^{2p} \|\psi\|_{H^{p+1}(K)} \|u\|_{H^{p+1}(\hat{K})}
\end{aligned}$$

Combining the bounds for (I) and (II), we obtain

$$\begin{aligned}
|\eta_K^{(1)} - \eta_K| &\leq |\text{(I)}| + |\text{(II)}| \lesssim h^{2p} \|\psi\|_{H^{p+1}(K)} \|u\|_{H^{p+1}(\hat{K})} + h^{p+p'} \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^{p'+1}(K)} \\
&\lesssim h^{2p} \|\psi\|_{H^{p+1}(K)} \|u\|_{H^{p+1}(\hat{K})}
\end{aligned}$$

□

Corollary 3.3. *If $\eta_K \approx h^{2p}\|u\|_{H^{p+1}(\hat{K})}\|\psi\|_{H^{p+1}(K)}$, then the local effectivity does not converge to unity as the mesh is refined, i.e.,*

$$\theta^{(1)}_{\text{local},K} = \left|1 - \frac{\eta^{(1)}_K}{\eta_K}\right| = \frac{|\eta_K - \eta^{(1)}_K|}{|\eta_K|} \lesssim \frac{h^{2p}\|\psi\|_{H^{p+1}(K)}\|u\|_{H^{p+1}(\hat{K})}}{h^{2p}\|\psi\|_{H^{p+1}(K)}\|u\|_{H^{p+1}(\hat{K})}} \lesssim 1.$$

Theorem 3.5 (Convergence of Global Error Estimate 1)**.** *The error in the global error estimate 1 is bounded by*

$$|\mathcal{E} - \mathcal{E}^{(1)}| \lesssim h^{2p}\|\psi\|_{H^{p+1}(\Omega)}\|u\|_{H^{p+1}(\Omega)}.$$

Proof. The convergence of the global error estimate 1 follows from that of the local counterpart, i.e.,

$$\begin{aligned}|\mathcal{E} - \mathcal{E}^{(1)}| = \left|\sum_{K\in\mathcal{T}_h}(\eta_K - \eta^{(1)}_K)\right| &\lesssim \sum_{K\in\mathcal{T}_h} h^{2p}\|u\|_{H^{p+1}(\hat{K})}\|\psi\|_{H^{p+1}(K)} \\ &\lesssim h^{2p}\|u\|_{H^{p+1}(\Omega)}\|\psi\|_{H^{p+1}(\Omega)}.\end{aligned}$$

□

Corollary 3.4. *If $\mathcal{E} \approx h^{2p}\|\psi\|_{H^{p+1}(\Omega)}\|u\|_{H^{p+1}(\Omega)}$, then the global effectivity does not converge to unity as the mesh is refined, i.e.,*

$$\theta^{(1)}_{\text{global}} = \left|1 - \frac{\mathcal{E}^{(1)}}{\mathcal{E}}\right| = \frac{|\mathcal{E} - \mathcal{E}^{(1)}|}{|\mathcal{E}|} \lesssim \frac{h^{2p}\|\psi\|_{H^{p+1}(\Omega_h)}\|u\|_{H^{p+1}(\Omega_h)}}{h^{2p}\|\psi\|_{H^{p+1}(\Omega_h)}\|u\|_{H^{p+1}(\Omega_h)}} \lesssim 1.$$

3.6 *A Priori* Error Analysis of Output Error Estimate 2

In this section, we analyze the convergence behavior of the output error estimate 2.

Theorem 3.6 (Convergence of Local Error Estimate 2)**.** *The error in the local error estimate 2 is bounded by*

$$|\eta_K - \eta^{(2)}_K| \lesssim h^p\|u\|_{H^{p+1}(\hat{K})}\|\psi\|_{H^1(K)}.$$

Proof. We will first bound the local error estimate, $\eta^{(2)}_K$. By the definition of the primal residual and the linearity of the bilinear form,

$$\eta^{(2)}_K = R_{h,p'}(u_{h,p}, \psi_{h,p'}|_K) = \ell(\psi_{h,p'}|_K) - a_{h,p'}(u_{h,p}, \psi_{h,p'}|_K) = a_{h,p'}(u - u_{h,p}, \psi_{h,p'}|_K).$$

As $a_{h,p'}(u - u_{h,p}, v_{h,p}) \neq 0$ in general for $p' > p$, we cannot subtract $\psi_{h,p}|_K$ from the second argument. The substitution of the BR2 bilinear form to the expression for $\eta^{(2)}_K$ yields

$$\begin{aligned}\eta^{(2)}_K = &\underbrace{\int_K \nabla(u - u_{h,p})\cdot\nabla(\psi_{h,p'})dx}_{\text{(I)}} - \underbrace{\int_{\partial K}\{\nabla\psi_{h,p'}|_K\}\cdot[\![u - u_{h,p}]\!]ds}_{\text{(II)}} \\ &- \underbrace{\int_{\partial K}[\![\psi_{h,p'}|_K]\!]\cdot\{\nabla(u - u_{h,p})\}ds}_{\text{(III)}} - \underbrace{\sum_{F\in\partial K}\int_F \beta[\![\psi_{h,p'}|_K]\!]\cdot r^F_{h,p}([\![u - u_{h,p}]\!])ds}_{\text{(IV)}}\end{aligned}$$

The interior term is bounded by

$$|(\mathrm{I})| \leq \|u - u_{h,p}\|_{H^1(K)} \|\psi_{h,p'}\|_{H^1(K)} \lesssim h^p \|u\|_{H^{p+1}(K)} \|\psi_{h,p'}\|_{H^1(K)}.$$

The first face term is bounded by

$$\begin{aligned}|(\mathrm{II})| &\leq \|\{\nabla\psi_{h,p'}|_K\}\|_{L^2(\partial K)} \|[\![u - u_{h,p}]\!]\|_{L^2(\partial K)} \lesssim h^{-1/2}\|\nabla\psi_{h,p'}\|_{L^2(K)} h^{p+1/2}\|u\|_{H^{p+1}(\hat{K})} \\ &\lesssim h^P \|\psi_{h,p'}\|_{H^1(K)} \|u\|_{H^{p+1}(\hat{K})}.\end{aligned}$$

The second face term is bounded by

$$\begin{aligned}|(\mathrm{III})| &\leq \|[\![\psi_{h,p'}|_K]\!]\|_{L^2(\partial K)} \|\{\nabla(u - u_{h,p})\}\|_{L^2(\partial K)} \lesssim h^{-1/2}\|\psi_{h,p'}\|_{L^2(K)} h^{p+1/2}\|u\|_{H^{p+1}(K)} \\ &\lesssim h^p \|\psi_{h,p'}\|_{L^2(K)} \|u\|_{H^{p+1}(K)}.\end{aligned}$$

The term involving the lifting operator is bounded by

$$\begin{aligned}|(\mathrm{IV})| &= \sum_{F\in\partial K} \int_F \beta[\![\psi_{h,p'}|_K]\!] \cdot r^F_{h,p}([\![u - u_{h,p}]\!]) ds \leq \sum_{F\in\partial K} \beta\|\psi_{h,p'}\|_{L^2(F)} \|r^F_{h,p}([\![u - u_{h,p}]\!])\|_{L^2(F)} \\ &\lesssim \sum_{F\in\partial K} \|\psi_{h,p'}\|_{L^2(F)} h^{-1/2} \|r^F_{h,p}([\![u - u_{h,p}]\!])\|_{L^2(K_F)} \lesssim \sum_{F\in\partial K} \|\psi_{h,p'}\|_{L^2(F)} h^{-1} \|[\![u - u_{h,p}]\!]\|_{L^2(F)} \\ &\lesssim h^p \|\psi_{h,p'}\|_{L^2(K)} \|u\|_{H^{p+1}(\hat{K})}.\end{aligned}$$

Combining the bounds for (I), (II), (III), and (IV), we obtain

$$|\eta^{(2)}_K| \leq |(\mathrm{I})| + |(\mathrm{II})| + |(\mathrm{III})| + |(\mathrm{IV})| \lesssim h^p \|u\|_{H^{p+1}(\hat{K})} \|\psi_{h,p'}\|_{H^1(K)}.$$

We further note that

$$\begin{aligned}\|\psi_{h,p'}\|_{H^1(K)} &= \|\psi_{h,p'} - \psi + \psi\|_{H^1(K)} \leq \|\psi_{h,p'} - \psi\|_{H^1(K)} + \|\psi\|_{H^1(K)} \\ &\lesssim h^{p'} \|\psi\|_{H^{p'+1}(K)} + \|\psi\|_{H^1(K)} \lesssim \|\psi\|_{H^1(K)}\end{aligned}$$

for h sufficiently small. Thus, we obtain the bound for $\eta^{(2)}_K$ in terms of u and ψ, i.e.,

$$|\eta^{(2)}_K| \lesssim h^p \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^1(K)}$$

An immediate consequence of this result is that

$$\begin{aligned}|\eta^{(2)}_K - \eta_K| &\lesssim |h^p \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^1(K)} - h^{2p} \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^{p+1}(K)}| \\ &\lesssim h^p \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^1(K)}.\end{aligned}$$

□

Corollary 3.5. *If $\eta_K \approx h^{2p} \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^{p+1}(K)}$, then the effectivity of the local error estimate 3 diverges in the sense that*

$$\theta^{(2)}_{\mathrm{local},K} = \left|1 - \frac{\eta^{(2)}_K}{\eta_K}\right| = \frac{|\eta_K - \eta^{(2)}_K|}{|\eta_K|} \lesssim \frac{h^p \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^1(K)}}{h^{2p} \|u\|_{H^{p+1}(\hat{K})} \|\psi\|_{H^{p+1}(K)}} \lesssim h^{-p},$$

i.e., the local error estimator degrades (relative to the true local error) as the mesh is refined.

Theorem 3.7 (Convergence of Global Error Estimate 2)**.** *The error in the global error estimate 2 is bounded by*

$$|\mathcal{E} - \mathcal{E}^{(2)}| \lesssim h^{2p'} \|u\|_{H^{p+1}(\Omega)} \|\psi\|_{H^{p'+1}(\Omega)}$$

Proof. Unlike the analysis for the global error estimate 1 and 3, simply summing the local error estimator bounds results in a loose bound. Thus, we will pursue a different approach to obtain a tighter bound. We first note that

$$R_{h,p}(u_{h,p}, v) = R_{h,p'}(u_{h,p}, v), \qquad \forall v \in H^1(\Omega),\ \forall p, p' \in \mathbb{N},$$

as the lifting operator is always multiplied by the jump in the second argument and $[\![v]\!] = 0$, $\forall v \in H^1(\Omega)$. In particular, we can rewrite the true error representation as

$$\mathcal{E} = R_{h,p}(u_{h,p}, \psi) = R_{h,p'}(u_{h,p}, \psi)$$

The error in the global error estimate becomes

$$\begin{aligned}
\mathcal{E} - \mathcal{E}^{(2)} &= R_{h,p'}(u_{h,p}, \psi) - R_{h,p'}(u_{h,p}, \psi_{h,p'}) \\
&= R_{h,p'}(u_{h,p}, \psi - \psi_{h,p'}) && \text{(linearity)} \\
&= R^{\psi}_{h,p'}(u - u_{h,p}, \psi_{h,p'}) && \text{(Lemma 3.2 for } w_{h,p} = u_{h,p},\ v_{h,p} = \psi_{h,p'}) \\
&= \inf_{v_{h,p'} \in V_{h,p'}} R^{\psi}_{h,p'}(u - u_{h,p} - v_{h,p'}, \psi_{h,p'}) && \text{(dual Galerkin orthogonality)} \\
&= \inf_{v_{h,p'} \in V_{h,p'}} a_{h,p'}(u - v_{h,p'}, \psi - \psi_{h,p'}) && \text{(Lemma 3.2 for } w_{h,p} = v_{h,p'},\ v_{h,p} = \psi_{h,p'})
\end{aligned}$$

By applying the global bilinear form error bound, Lemma 3.5, for $p_1 = p_2 = p_3 = p'$, we obtain

$$|\mathcal{E}^{(2)} - \mathcal{E}| = |a_{h,p'}(u - v_{h,p'}, \psi - \psi_{h,p'})| \lesssim h^{2p'} \|u\|_{H^{p'+1}(\Omega_h)} \|\psi\|_{H^{p'+1}(\Omega_h)}$$

□

Corollary 3.6. *If $\mathcal{E} \approx h^{2p} \|\psi\|_{H^{p+1}(\Omega)} \|u\|_{H^{p+1}(\Omega)}$, then the effectivity of the global error estimate 2 converges to unity as*

$$\theta^{(2)}_{\text{global}} = \left|1 - \frac{\mathcal{E}^{(2)}}{\mathcal{E}}\right| = \frac{|\mathcal{E} - \mathcal{E}^{(2)}|}{|\mathcal{E}|} \lesssim \frac{h^{2p'} \|u\|_{H^{p+1}(\Omega_h)} \|\psi\|_{H^{p'+1}(\Omega_h)}}{h^{2p} \|\psi\|_{H^{p'+1}(\Omega)} \|u\|_{H^{p+1}(\Omega)}} \lesssim h^{2(p'-p)} = h^{2p_{\text{inc}}}$$

3.7 Summary of *A Priori* Error Analysis

Table 1 summarizes the result of the *a priori* error analysis. The table shows that neither the local nor global effectivity of the estimate 1 approaches unity as $h \to 0$. The estimate 2 results in a superconvergent global estimate; however, the local error effectivity diverges with mesh refinement, and thus the estimator is not suited for driving adaptation. The estimate 3 is the only estimate whose effectivity converges to unity both locally and globally as $h \to 0$.

(a) local estimates

	$\eta_K^{(i)}$	$\lvert\eta_K^{(i)} - \eta_K\rvert$	$\theta_{\text{local},K} = \lvert 1 - \eta_K^{(i)}/\eta_K\rvert$
1	$R_{h,p'}(u_{h,p}, (\psi_{h,p'} - \psi_{h,p})\vert_K)$	$h^{2p}\lVert u\rVert_{H^{p+1}(\hat{K})}\lVert\psi\rVert_{H^{p+1}(K)}$	h^0
2	$R_{h,p'}(u_{h,p}, \psi_{h,p'}\vert_K)$	$h^{p}\lVert u\rVert_{H^{p+1}(\hat{K})}\lVert\psi\rVert_{H^{1}(K)}$	h^{-p}
3	$R_{h,p}(u_{h,p}, \psi_{h,p'}\vert_K)$	$h^{p+p'}\lVert u\rVert_{H^{p+1}(\hat{K})}\lVert\psi\rVert_{H^{p'+1}(K)}$	$h^{p_{\text{inc}}}$
true	$R_{h,p}(u_{h,p}, \psi\vert_K)$	-	-

(b) global estimates

	$\mathcal{E}^{(i)}$	$\lvert\mathcal{E}^{(i)} - \mathcal{E}\rvert$	$\theta_{\text{global}} = \lvert 1 - \mathcal{E}^{(i)}/\mathcal{E}\rvert$
1	$R_{h,p'}(u_{h,p}, \psi_{h,p'} - \psi_{h,p})$	$h^{2p}\lVert u\rVert_{H^{p+1}(\Omega)}\lVert\psi\rVert_{H^{p+1}(\Omega)}$	h^0
2	$R_{h,p'}(u_{h,p}, \psi_{h,p'})$	$h^{2p'}\lVert u\rVert_{H^{p'+1}(\Omega)}\lVert\psi\rVert_{H^{p'+1}(\Omega)}$	$h^{2p_{\text{inc}}}$
3	$R_{h,p}(u_{h,p}, \psi_{h,p'})$	$h^{p+p'}\lVert u\rVert_{H^{p+1}(\Omega)}\lVert\psi\rVert_{H^{p'+1}(\Omega)}$	$h^{p_{\text{inc}}}$
true	$J_{h,p}(u_{h,p}) - J(u) = R_{h,p}(u_{h,p}, \psi)$	-	-

Table 1: Summary of the local and global error estimate convergence.

4 Numerical Results

This section provides numerical verification of the *a priori* error analysis results presented in Section 3. In particular, we apply the three error estimates to a one dimensional Poisson problem given by

$$\begin{aligned} -\frac{d^2u}{dx^2} &= \exp(x)(1+x), \quad \text{on } (0,1), \\ u(0) &= u(1) = 0, \end{aligned}$$

and the functional output of interest,

$$J(u) = \int_0^1 \sin(\pi x) u(x) dx.$$

Note that the analytical solution to the primal and dual problems are given by

$$u = (\exp(x) - 1)(1 - x) \quad \text{and} \quad \psi = \sin(\pi x),$$

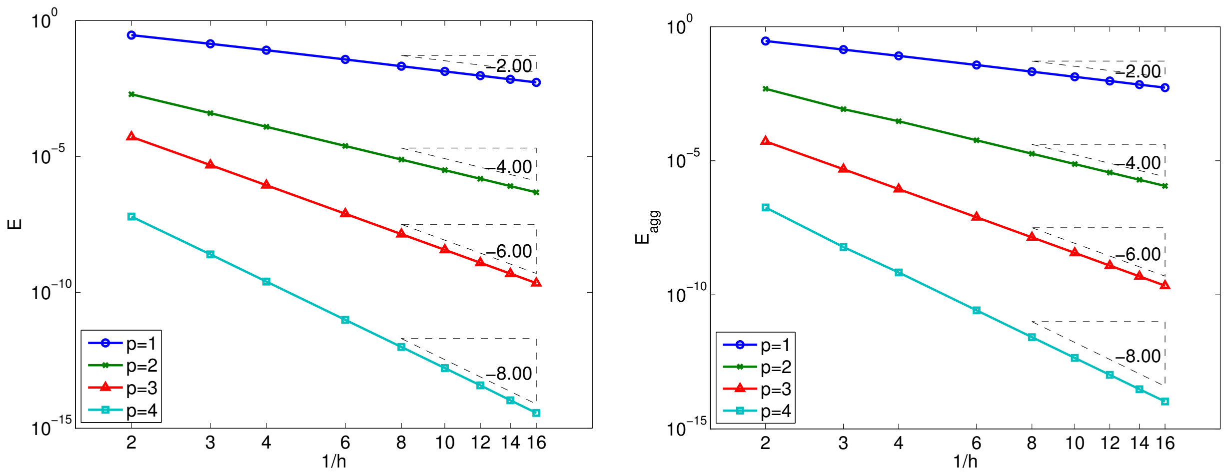
both of which are in C^∞ and have finite and non-vanishing measures in $H^m(\Omega)$, $\forall m \in \mathbb{N}$.

We will use two different metrics to assess the performance of the error estimates. The first measure is the relative error in the global estimate as defined earlier, i.e.

$$\theta^{(i)}_{\text{global}} \equiv \frac{|\mathcal{E} - \mathcal{E}^{(i)}|}{|\mathcal{E}|} = \left| 1 - \frac{\mathcal{E}^{(i)}}{\mathcal{E}} \right|,$$

where $\mathcal{E}$ is the true error and $\mathcal{E}^{(i)}$ is the error estimate provided by the estimator i. Recall that the relative error is equivalent to the deviation of the error effectivity from unity. The second measure is the agglomerated local effectivity, which is a single measure intended to capture the effectivity of the local, element-wise error estimates. The agglomerated local effectivity is defined by

$$\theta^{(i)}_{\text{local}} \equiv \left| 1 - \frac{\mathcal{E}^{(i)}_{\text{agg}}}{\mathcal{E}_{\text{agg}}} \right| = \frac{|\mathcal{E}_{\text{agg}} - \mathcal{E}^{(i)}_{\text{agg}}|}{|\mathcal{E}_{\text{agg}}|}$$



(a) true error, $\mathcal{E}$ (b) true agglomerated local error, $\mathcal{E}_{\text{agg}} \equiv \sum_K |\eta_K|$

Figure 1: The convergence of the true output error.

where

$$\mathcal{E}_{\text{agg}} \equiv \sum_{K \in \mathcal{T}_h} |\eta_K| \quad \text{and} \quad \mathcal{E}_{\text{agg}}^{(i)} \equiv \sum_{K \in \mathcal{T}_h} |\eta_K^{(i)}|.$$

Note that this is different from the relative local error $\theta_{\text{local},K}^{(i)}$ associated with each element K, but it is an agglomerated measure of the quality of the local estimates.

4.1 True Output Error

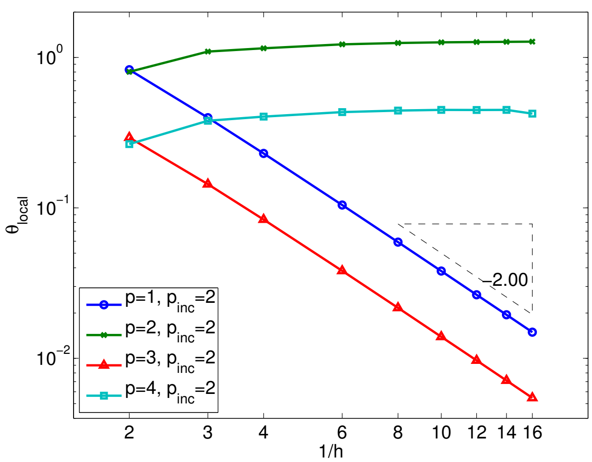
We first analyze the behavior of the true error, measured in the standard sense and in the agglomerated local sense. Figure 1 shows the convergence results for $p = 1, 2, 3, 4$. Since both the primal and dual solutions are infinitely smooth, Theorem 3.1 predicts the superconvergence of both the local and global errors at the rate of h^{2p}. The numerical result confirms the analysis. Since the solutions have well-behaved higher order derivatives, the convergence with grid refinement is very smooth. We note that $p = 4$ solution achieves machine precision accuracy using just 16 elements; while this is an encouraging result, it makes the assessment of the error estimates more difficult, as the results are affected by the finite precision arithmetics. Thus, $p = 3$ and $p = 4$ results are sometimes truncated or omitted, if the results have been deemed polluted by rounding errors.

4.2 Output Error Estimate 1

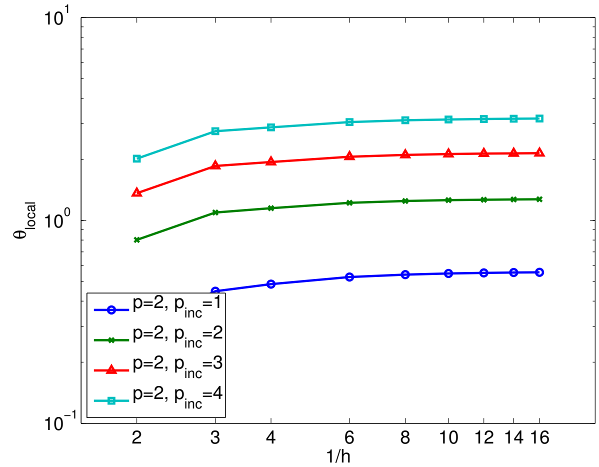
By the *a priori* error analysis, Theorem 3.4 and 3.5, we expect

$$\theta_{\text{local}}^{(1)} \equiv \left| 1 - \frac{\mathcal{E}_{\text{agg}}^{(1)}}{\mathcal{E}_{\text{agg}}} \right| \lesssim h^0 \quad \text{and} \quad \theta_{\text{global}}^{(1)} \equiv \left| 1 - \frac{\mathcal{E}^{(1)}}{\mathcal{E}} \right| \lesssim h^0,$$

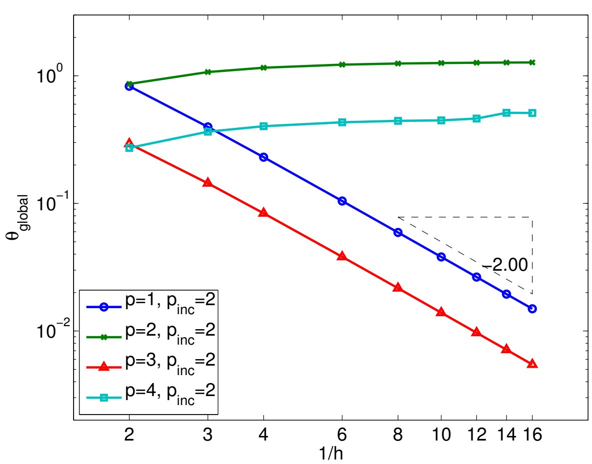
i.e., neither the local nor the global effectivity converge to unity with grid refinement. Figure 2 shows the convergence of the local and global effectivity of error estimate 1. The result must be



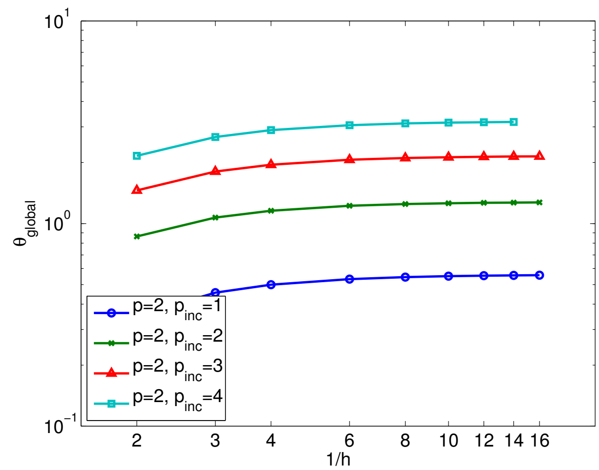
(a) local error effectivity, varying p



(b) local error effectivity, varying $p_{\rm inc}$



(c) global error effectivity, varying p



(d) global error effectivity, varying $p_{\rm inc}$

Figure 2: The local and global effectivity of the error estimate 1.

interpreted carefully, as the *a priori* error analysis results are upper bound and the cancellation can give a false sense of convergence. For example, Figure 2(a) and 2(c) show that the local and global effectivities converge to unity for odd p but not for even p. The cause of this odd-even behavior is unclear, but similar results have been observed in [11, 12]. In these cases, we should always compare the worst convergence rate with the *a priori* analysis, i.e. the even results for this case. The numerical experiment confirms that the effectivity of the error estimate 1 does not converge to unity in either the local or the global sense.

4.3 Output Error Estimate 2

By the *a priori* error analysis, Theorem 3.6 and 3.7, we expect

$$\theta^{(2)}_{\text{local}} \equiv \left|1 - \frac{\mathcal{E}^{(2)}_{\text{agg}}}{\mathcal{E}_{\text{agg}}}\right| \lesssim h^{-p} \quad \text{and} \quad \theta^{(2)}_{\text{global}} \equiv \left|1 - \frac{\mathcal{E}^{(2)}}{\mathcal{E}}\right| \lesssim h^{2p_{\text{inc}}},$$

i.e., the local error effectivity diverges at the rate of h^{-p}, but the global error effectivity superconverges at the rate of $h^{2p_{\text{inc}}}$. The divergence of the local effectivity is captured in Figure 3(a) and 3(b). In particular, the local effectivity diverges at the rate of h^{-2} and h^{-4} for $p = 2$ and 4, respectively. The local effectivity is not a function of p_{inc} as $p_{\text{inc}} = 1, 2, 3, 4$ all diverges at the rate of h^{-2} for $p = 2$. On the other hand, Figure 3(c) and 3(d) show that the global effectivity exhibit superconvergence. In particular, the global effectivity convergence rate is a function of p_{inc} showing the convergence rates of h^2, h^4, and h^6 for $p_{\text{inc}} = 1$, 2, and 3, respectively. The global effectivity convergence rate is not a function of p, as $p_{\text{inc}} = 2$ results in the convergence rate of h^4 for all $p = 1, 2, 3$. These results are consistent with the *a priori* analysis.

4.4 Output Error Estimate 3

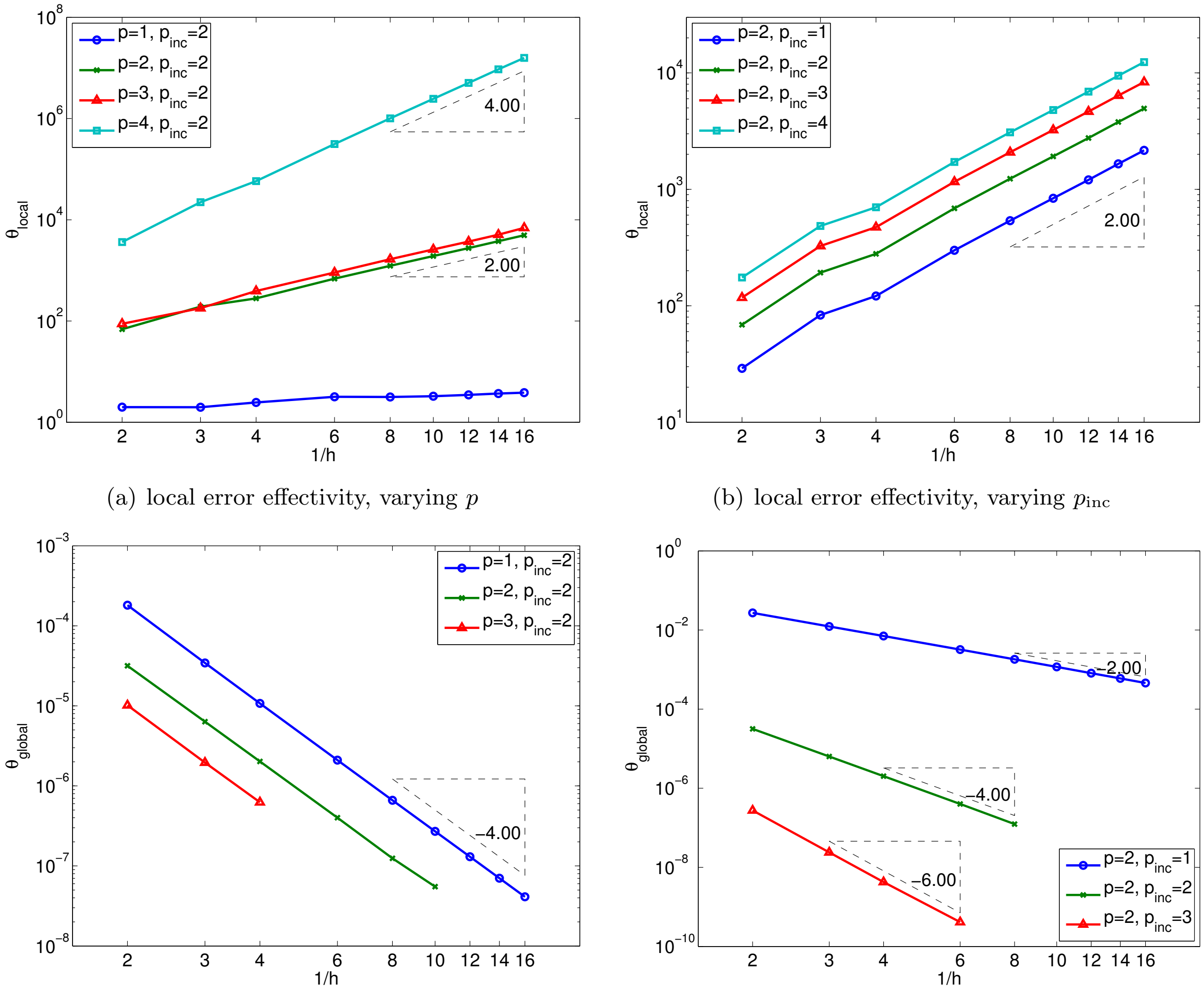
By the *a priori* error analysis, Theorem 3.2 and 3.3, we expect

$$\theta_{\text{local}} \equiv \left|1 - \frac{\mathcal{E}^{(3)}_{\text{agg}}}{\mathcal{E}_{\text{agg}}}\right| \lesssim h^{p_{\text{inc}}} \quad \text{and} \quad \theta_{\text{global}} \equiv \left|1 - \frac{\mathcal{E}^{(3)}}{\mathcal{E}}\right| \lesssim h^{p_{\text{inc}}},$$

i.e., both the local and global error effectivities converge at the rate of $h^{p_{\text{inc}}}$. Figure 4(a) shows that $p_{\text{inc}} = 2$ results in the local effectivity convergence of h^2 for $p = 2, 4$. Figure 4(b) shows that the convergence rate improves to h^4 for $p_{\text{inc}} = 4$. The same behavior is shown for the global effectivity in Figure 4(c) and 4(d), converging at the rate of at least $h^{p_{\text{inc}}}$.

5 Conclusion

This report analyzed the behavior of three variants of the DWR error estimates applied to a p-dependent discretization. We showed that the BR2 discretization of second-order PDEs results in a p-dependent discretization due to the presence of the p-dependent lifting operator. Then, we analyzed three commonly used variants of DWR error estimates. The *a priori* error analysis showed that the effectivity of error estimate 1—which naturally results from the discrete interpretation of the adjoint—converges in neither the local nor global sense. Error estimate 2 exhibited superconvergent global effectivity; however, its local effectivity diverges, making it unsuited for grid



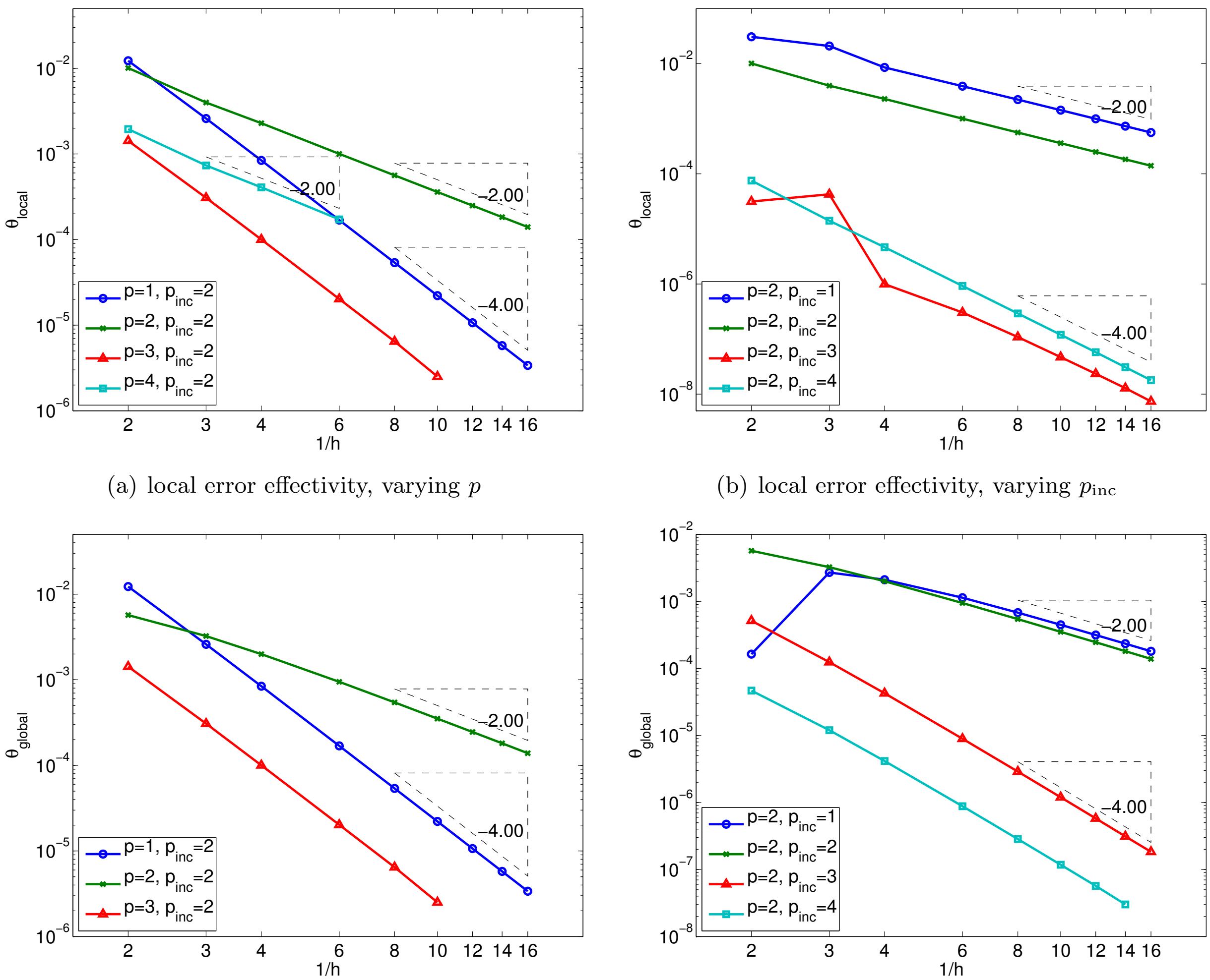
(a) local error effectivity, varying p

(b) local error effectivity, varying p_{inc}

(c) global error effectivity, varying p

(d) global error effectivity, varying p_{inc}

Figure 3: The local and global effectivity of the error estimate 2.



(a) local error effectivity, varying p

(b) local error effectivity, varying p_{inc}

(c) global error effectivity, varying p

(d) global error effectivity, varying p_{inc}

Figure 4: The local and global effectivity of the error estimate 3.

adaptation. The effectivity of error estimate 3 converges both in the local and global sense, making it an attractive choice for both error estimation and adaptation. A simple one-dimensional Poisson problem numerically verified the *a priori* error analysis.

References

[1] F. Bassi, S. Rebay, GMRES discontinuous Galerkin solution of the compressible Navier-Stokes equations, in: K. Cockburn, Shu (Eds.), Discontinuous Galerkin Methods: Theory, Computation and Applications, Springer, Berlin, 2000, pp. 197–208.

[2] F. Bassi, A. Crivellini, S. Rebay, M. Savini, Discontinuous Galerkin solution of the Reynolds averaged Navier-Stokes and k-ω turbulence model equations, Comput. & Fluids 34 (2005) 507–540.

[3] D. N. Arnold, F. Brezzi, B. Cockburn, L. D. Marini, Unified analysis of discontinuous Galerkin methods for elliptical problems, SIAM J. Numer. Anal. 39 (5) (2002) 1749–1779.

[4] R. Becker, R. Rannacher, A feed-back approach to error control in finite element methods: Basic analysis and examples, East-West J. Numer. Math. 4 (1996) 237–264.

[5] R. Becker, R. Rannacher, An optimal control approach to a posteriori error estimation in finite element methods, in: A. Iserles (Ed.), Acta Numerica, Cambridge University Press, 2001.

[6] D. A. Venditti, Grid adaptation for functional outputs of compressible flow simulations, Ph.D. thesis, Massachusetts Institute of Technology, Cambridge, Massachusetts (2002).

[7] K. J. Fidkowski, D. L. Darmofal, A triangular cut-cell adaptive method for higher-order discretizations of the compressible Navier-Stokes equations, J. Comput. Phys. 225 (2007) 1653–1672.

[8] N. K. Burgess, D. J. Mavriplis, An *hp*-adaptive discontinuous Galerkin solver for aerodynamic flows on mixed-element meshes, AIAA 2011-490 (2011).

[9] K. Fidkowski, D. Darmofal, Review of output-based error estimation and mesh adaptation in computational fluid dynamics 49 (4) (2011) 673–694.

[10] F. Brezzi, M. Manzini, D. Marini, P. Pietra, A. Russo, Discontinuous finite elements for diffusion problems, in: Francesco Brioschi (1824-1897) convegno di studi matematici, October 22-23, 1997, Ist. Lomb. Acc. Sc. Lett., Incontro di studio N. 16, 1999, pp. 197–217.

[11] K. Harriman, P. Houston, B. Senior, E. Süli, hp-version discontinuous Galerkin methods with interior penalty for partial differential equations with nonnegative characteristic form., Tech. Rep. Technical Report NA 02/21, Oxford University Computing Lab Numerical Analysis Group (2002).

[12] T. A. Oliver, A higher-order, adaptive, discontinuous Galerkin finite element method for the Reynolds-averaged Navier-Stokes equations, PhD thesis, Massachusetts Institute of Technology, Department of Aeronautics and Astronautics (Jun. 2008).