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ROBUSTNESS OF INFINITE DIMENSIONAL SYSTEMS

By

STUART TOWNLEY

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To Sharon

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Summary

The results contained within this thesis concern an abstract framework for a robustness analysis of exponential stability of infinite dimensional systems. The abstract analysis relies on the strong relationship between exponential stability and L^2-stability which exists for many classes of linear systems.

In Chapter 1 a "stability radius", for systems governed by semi-groups, is developed, for a class of "structured" perturbations of its generator. The abstract theory is illustrated by examples of perturbations of the boundary data for homogeneous boundary value problems and also perturbations arising due to neglected delay terms in differential delay equations.

In Chapter 2 a related problem of a non standard linear quadratic problem is studied, which leads to a stability analysis for certain non-linear systems.

In Chapter 3 an abstract L^2-stability theory is developed and then applied to integrodifferential equations and time-varying systems, to investigate the robustness of exponential stability of such systems.

PRINCIPAL NOTATION

$L(B_1,B_2)$	- Banach space of bounded linear maps $f:B_1 \rightarrow B_2$, B_1 and B_2 Banach spaces, with the induced operator norm.
B^*	- Dual of a Banach space, B .
T^*	- Adjoint of a linear operator, T on B .
$D(T:B_1 \rightarrow B_2)$	- $\{b \in B_1 \mid T(b) \in B_2\}$
$D_B(T)$	- $D(T:B \rightarrow B)$
$<\cdot\ \cdot>_{B,B^*}$	- The duality pairing $<b,b^*>_{B,B^*} = b^*(b)$ for $b \in B$, $b^* \in B^*$.
$L^2(t_0,T;B)$	- Banach space of functions $f:[t_0,T] \rightarrow B$, strongly measurable and square integrable in the sense of Bochner, with $\Vert f(\cdot)\Vert_{L^2(t_0,T;B)} = (\int_{t_0}^{T} \Vert f(s)\Vert_B^2 ds)^{\frac{1}{2}}$
L_B^2	- $L^2(0,\infty;B)$
$\hat{f}$	- Fourier-Plancherel transform of a function $f(\cdot) \in L^2(0,\infty;B)$.
$\mathcal{B}^{\infty}(t_0,\infty;L(B_1,B_2))$	- Space of functions $f:[t_0,\infty) \rightarrow L(B_1,B_2)$ such that $f(\cdot)$ is strongly measurable and $\operatorname{ess\,sup}_{t \geq t_0} \Vert f(\cdot)\Vert < \infty$.
$\mathcal{B}_-^1(t_0,\infty;L(B_1,B_2))$	- Set of functions $f:[t_0,\infty) \rightarrow L(B_1,B_2)$ such that $f(\cdot)$ is strongly measurable and $e^{\beta t}f(t)$ is integrable and $\int_{t_0}^{\infty} e^{\beta t}\Vert f(t)\Vert \leq \infty$, $\beta > 0$.

$L^2_{B,R}$ - Set of functions $f:[0,\infty) \to B$, $f(\cdot) \in L^2(0,T;B)$ for all $T > 0$ and $\int_0^\infty e^{-2Rt}||f(t)||_B^2 dt < \infty$.

$\sigma(T)$ - Spectrum of an operator $T:D_B(T) \to B$.

$\underline{\sigma}(M)$ - Lowest singular value of a matrix $M \in K^{m \times n}$.

ρ_T - The map restricting a function on $[0,\infty)$ to the domain $[0,T]$.

σ_T - The map extending a function defined on $[0,T]$ to $[0,\infty)$.

$J(x_0,v)$ - Quadratic cost functional (in this instance of indefinite sign).

$\bar{x},\bar{\lambda}$ - Complex conjugate of $x \in B$, B a complex Banach space and of a scalar $\lambda \in \mathbb{C}$ (pages 52-55, only).

0. Introduction

In the last five to ten years a considerable amount of research in the systems theory area has been concentrated on the robustness issues of control systems design. This research has been mostly centred around the H^{∞} approach to the robustness of linear systems, see references [1]-[5], [46]. Of the many interesting features of this input-output description of robustness is the knowledge, a priori, of the maximum achievable robustness of a given feedback system. However, in addition to the computational problems involved in solving the H^{∞} optimisation problem, there are other more fundamental problems concerning the choice of the H^{∞} norm. This choice of norm restricts the analysis to those perturbations conserving the number of right half planes poles of the nominal transfer function. More recently, the state space approach to robustness has grown as an alternative to the H^{∞} approach. However, unlike the H^{∞} approach, no unified state space approach exists. Of the existing state space approaches there are those of Patel and Toda [13], Petersen and Hollot [10] and Hinrichsen and Pritchard [7,8]. The former two use a Liapunov based approach to stability. The approach of Hinrichsen and Pritchard uses spectrum analysis, Liapunov techniques and also (of greater importance in the sequel) the equivalence of exponential stability and L^2-stability for linear differential equations. This equivalence, between L^2-stability and exponential stability for linear finite dimensional systems, and its generalisations to various classes of infinite dimensional systems, provides the basic tool for robustness analysis in the sequel.

In Francis [3] the input output stability of a linear system is shown to be robust, with respect to both the H^∞ norm and the L^1-norm of the convolution kernel, but not with respect to its L^p-norm, $p > 1$. It is the purpose of the work in the sequel to establish a framework within which the exponential stability of infinite dimensional systems is robust to a class of highly structured, unbounded perturbations. The class of such highly structured/unbounded perturbations is shown to include examples of perturbations of the boundary data for homogeneous boundary value problems and also perturbations due to neglected delay terms in differential-delay equations. Such perturbations might be considered typical for any given infinite dimensional (control) system.

In Chapter 1 the nominal system is given as an exponentially stable, strongly continuous semigroup $(S(t))_{t\geq 0}$ on a Banach Space X , such that $x(t) = S(t)x_0$ is the mild solution of the abstract differential equation

$$(N) \quad \dot{x} = Ax \qquad x(0) = x_0 \; .$$

The property of exponential stability of the nominal system is shown to be robust with respect to unbounded/structured perturbation BDC, where $B:U \to \bar{X} \supseteq X$, $C:\underline{X} \subseteq X \to Y$ are fixed, and $D:Y \to U$ is arbitrary, in the sense that $x(\cdot)$, the continuous solution of

$$(P) \quad x(t) = S(t)x_0 + \int_0^t S(t-s)BDCx(s)ds \qquad x_0 \in X \; ,$$

is exponentially decreasing for all D , $||D|| < \alpha$, $\alpha > 0$. The maximum allowable α is characterised as a stability radius, for system (P) and its characterisation constitutes the main part of Chapter 1. This stability radius characterisation is applied to some typical (but simple) examples of infinite dimensional systems.

In Chapter 2 the connection between the stability radius of Chapter 1 and a certain non standard linear quadratic problem is developed. The results of this chapter are generalisations of the results of Hinrichsen and Pritchard [8] and the analysis proceeds via the abstract framework of Salamon and Pritchard [22]. As well as the theoretical importance, of this non-standard linear quadratic problem, is the use of a solution of the corresponding algebraic Riccati equation as a Liapunov functional for a class of non-linear systems. The stability analysis for the nonlinear system again relies on L^2-stability of a perturbed system.

In Chapter 3, the robustness analysis for those systems described in Chapter 1 is extended considerably to encompass systems governed by time varying and integrodifferential equations. Again the notion of L^2-stability plays an important role and this type of stability is analysed in an abstract framework in section 1 of Chapter 3. In section 2, the abstract results are applied to systems of integrodifferential equations. The main result of this section is an exact characterisation of the robustness of integrodifferential equations to different types of perturbations. In section 3 the main results are that, whilst uniform asymptotic stability of evolution operators is robust to a class of time-varying, structured, unbounded, perturbations, the exact robustness is not given by the abstract analysis of section 1.

For any Hilbert space H , $\langle\cdot,\cdot\rangle_H$ denotes the inner product on H . For any Banach space B , $||\cdot||_B$ denotes the norm on B . When the norm of a map is required its norm is denoted by $||\cdot||$ unless otherwise stated. If A is a closed, densely defined operator on a Banach space B then A^* denotes the adjoint of A . If B is a Banach space, $L^2(t_0,T;B)$ denotes the space of strongly measurable functions $[t_0,T] \to B$ which are square integrable in the sense of Bochner. When $t_0 = 0$ and $T = \infty$ this space is denoted by L^2_B .

Since throughout the sequel continued reference is made to nominal and perturbed systems, for a family of maps $\theta(\gamma)$, parameterised by $\gamma \in \Gamma$ (some parameter set, usually time), refers to the nominal system, $\theta^P(\gamma)$ refers to the perturbed system, whilst $\theta_S(\gamma)$ refers to some subsidiary family of maps (for example feedback semigroups).

CHAPTER 1.

A Stability radius for systems defined by strongly continuous semigroups of linear operators.

§0. Introduction

Perturbation theory for linear operators defined on real or complex Banach spaces has received considerable interest in the last thirty years (Kato [15]). The following two famous theorems concern the exponential growth of a semigroup of linear operators and bounded perturbation of its generator, and might be considered as a first step towards a robustness theory for infinite dimensional linear systems.

Theorem (0.1)

Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup of linear operators on a real or complex Banach space X, then there exists $\omega_0 \in \mathbb{R}$ such that for all $\omega \geq \omega_0$, there exists $M(\omega) \geq 0$ such that

$$||S(t)|| \leq M(\omega)e^{\omega t} \quad \text{for all} \quad t \geq 0 .$$

Definition (0.2)

If $\omega_0 < 0$ then $(S(t))_{t\geq 0}$ is said to be exponentially stable.

Theorem (0.3)

Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup of linear operators on a Banach space X, with generator $A:D_X(A) \to X$, and $D \in L(X)$, then $A+D$ generates $(S^D(t))_{t\geq 0}$ defined by

$$(1.0.1) \qquad S^D(t)x_0 = S(t)x_0 + \int_0^t S(t-s)DS^D(s)x_0 ds \ , \quad x_0 \in X \ .$$

Moreover, if

$$||S(t)|| \le Me^{\omega t} \quad \text{for all} \quad t \ge 0 \quad \text{then}$$

$$||S^D(t)|| \le Me^{(\omega + M||D||)t} \quad \text{for all} \quad t \ge 0 \ .$$

Corollary (0.4)

Let $S(t)$ be an exponentially stable semigroup with generator A and $D \in L(X)$.

If $||D|| < \frac{-\omega}{M}$, where $\omega \in [\omega_0,0)$ is guaranteed by proposition (0.1), then $A+D$ generates an exponentially stable semigroup. □

Remark (0.5)

Corollary (0.3) says that the exponential stability of a semigroup is "robust" to bounded perturbation of its generator.

Remark (0.6)

Optimisation of the bound $\frac{-\omega}{M(\omega)}$ with respect to $M(\omega)$, $\omega \in [\omega_0,0)$ would result in the following conjecture:

Let $A:D(A) \to X$ be the generator of a strongly continuous semigroup of linear operators on X , a Banach space over $K = \mathbb{R}$, $\mathbb{C}$, with $\omega_0 < 0$, then

$$r_K(A) := \sup_d \{ ||D|| < d, \ D \in L(X) \text{ implies } (A+D) \text{ generates a strongly continuous, exponentially stable semigroup } (S^D(t))_{t\ge 0} \text{ on } X \}$$

$$= \sup\{\tfrac{-\omega}{M(\omega)} \mid \omega \in [\omega_0,0)\} \ .$$

Unfortunately even if such a result were true, its implications for systems governed by partial differential or differential-delay equations is limited in that, firstly

a) calculation of $\sup_{\omega_0 \le \omega \le 0} \{\frac{-\omega}{M(\omega)}\}$ would in general be very difficult

and secondly, and more importantly

b) the class of <u>bounded</u> perturbations is <u>very</u> limited.

<u>Example</u> 0.8

Consider the following simple differential-delay equation:

$$(1.0.2) \begin{cases} \dot{z}(t) = A_0 z(t) + A_1 z(t-1), t>0, & A_0, A_1 \in K^{n\times n}, \ K = \mathbb{R}, \mathbb{C} \\ z(\tau) = z_\tau & \tau \in [-1,0) . \\ z(0) = z_0 \end{cases}$$

It can be shown that if $\text{Det } (\lambda I - A_0 - e^{-\lambda} A_1) \neq 0$ for all $\lambda \in \bar{\mathbb{C}}_+$, then there exists an exponentially stable semigroup $S(t) \in \mathcal{L}(X)$, $X = K^n \times L^2(-1,0;K^n)$ with generator A defined by

$$(1.0.3) \qquad (Ah)(\theta) = \begin{cases} A_0 h(0) + A_1 h(-1) & \theta = 0 \\ \frac{dh}{d\theta}(\theta) & \theta \in [-1,0) \end{cases}$$

and $\qquad D(A) = H^1(-1,0;K^n)$.

Consider now the perturbed system

$$(1.0.4) \quad \begin{cases} \dot{z}(t) = A_0 z(t) + Pz(t-\alpha) + A_1 z(t-1) \quad , \quad t > 0 \\ z(\tau) = z_\tau \quad \tau \in [-1,0) \\ z(0) = z_0 \end{cases}$$

with $P \in K^{n\times n}$, $0 \le \alpha \le 1$. This results in a perturbation of the abstract operator A in (1.0.3) of the form $A + BDC$, $B = [{I \atop 0}]$, $D = P$ and $C = [0, E_{-\alpha}]$, $E_{-\alpha}h = h(-\alpha)$. Notice that although $B \in L(K^n, K^n \times L^2(-1,0;K^n))$ and $D \in L(K^n)$, C is unbounded as a map $K^n \times L^2(-1,0;K^n) \to K^n$, however $C \in L(K^n \times H^{\frac{1}{2}+\varepsilon}(-1,0;K^n),K^n)$. (For a definition of the space $H^s(-1,0;K^n)$ see Kappel and Kunisch [36], and also Lions and Magenes [57]).

<u>Example</u> (0.9)

Consider the following parabolic system

$$(1.0.5) \quad \begin{cases} z_t = z_{\xi\xi} \\ \\ z_\xi(0,t) = 0 = z(1,t) \quad z(\xi,0) = z(\xi) \ . \end{cases}$$

It can be shown that there exists an exponentially stable semigroup $S(t) \in L(X)$, where $X = L^2(0,1)$, with generator

$$(1.0.6) \qquad Ah = \frac{d^2h}{d\xi^2} \ , \quad D(A) = \{h \in H^2 \mid h_\xi(0) = 0 = h(1)\} \ .$$

Suppose the boundary data is subject to perturbation (for example, in the nominal system the interaction of the distributed system with its environment is neglected and this results in perturbation of the boundary data) resulting in

$$(1.0.7)\quad \begin{cases} z_t = z_{\xi\xi} \\ z_\xi(0,t) = dz(0,t)\ , \quad z(1,t) = 0\ ,\ z(\xi,0) = z(\xi)\ . \end{cases}$$

The perturbed system can be written

$$(1.0.8)\quad \begin{cases} z_t = z_{\xi\xi} + d\delta_0(\xi)\ z(0)\ . \\ z_\xi(0,t) = 0 = z(1,t)\ ,\ z(\xi,0) = z(\xi)\ . \end{cases}$$

See Curtain Pritchard [21]. With respect to this abstract semigroup formulation the perturbation of the boundary data results in a perturbed operator $A + BDC$, $Bu = \delta_0 u$, $u \in K$, $\delta_0(\cdot)$ the Dirac-delta distribution at 0 , $D = d \in K$ and $Ch = h(0)$. Notice again that BDC is unbounded as a map $L^2(0,1) \to L^2(0,1)$ but that with respect to this decomposition BDC ,

$$B \in L(K,H^{-\frac{1}{2}-\varepsilon}(0,1)) \text{ and } C \in L(H^{\frac{1}{2}+\varepsilon}(0,1),K)\ ,\ \varepsilon > 0\ .$$

It is the purpose of this chapter to establish a framework for the study of such abstract differential equationswith regards to both well-posedness and stability of the perturbed systems. In fact these two examples typify the class of systems to be studied in that the perturbation term is

- unbounded
- highly structured.

In Hinrichsen and Pritchard [7] [8] the question of robustness of exponential stability is considered. In [7] the concepts of real and complex "stability radii" are introduced for unstructured perturbations of a stable matrix $A \in K^{n\times n}$. In [8] the concept of a structured stability radius is introduced, for the case $K = \mathbb{C}$. These results are summarised in section 1. In section 2 the framework for the study of the well-posedness of the perturbed system is developed. In section 3 it is shown, in the case that X is complex, how these well posedness considerations lead to different stability radii, depending upon the degree of unboundedness in the perturbations. The main result of this section is the exact characterisation of these stability radii. In section 4, these stability radii are applied to various simple (but typical) examples to illustrate both the characterisation of robustness and the various conditions its well posedness requires.

§1. Stability radii for finite dimensional systems.

The question of a state space approach to the robustness of systems is becoming more important as both an alternative and an ally to the well established input-output or H^∞ approach to robustness of time invariant linear systems. Amongst the state space approaches are the works of Patel and Toda [13] and also Petersen and Hollot [10]. The approach in the sequel is based upon that of Hinrichsen and Pritchard [7], [8] and it is their results that are summarised below. Throughout this section $A \in K^{n\times n}$, $\sigma(A) \subseteq \mathbb{C}_-$, and the usual inner product, Euclidean norm and induced Euclidean norm are used.

Definition 1.1

(1.1.1) $$\mathcal{U}_n(K) = \{M \in K^{n\times n} \mid \sigma(M) \cap \bar{\mathbb{C}}_+ \neq \emptyset\} \quad .$$

- $\mathcal{U}_n(K)$ is the set of $n\times n$ non-asymptotically stable matrices over K .

Let $B \in K^{n\times m}$, $C \in K^{p\times n}$.

Definition 1.2

(1.1.2) $$r_K(A;B,C) := \inf_{D\in K^{m\times p}} \{||D|| \mid (A + BDC) \in \mathcal{U}_n(K)\} \quad .$$

Theorem (1.3) (Hinrichsen and Pritchard, [8])

$$r_{\mathbb{C}}(A;B,C) = \infty \quad \text{if} \quad G(i\omega) \equiv 0$$

$$\left.\begin{array}{lll} (1.1.3) & = & \inf\limits_{\omega} \dfrac{1}{||G(i\omega)||} \\ & & \\ (1.1.4) & = & \dfrac{1}{||L||} \end{array}\right\} \quad \text{if} \quad G(i\omega) \not\equiv 0$$

where $G(i\omega) = C(i\omega-A)^{-1}B$ and $L \in \mathcal{L}(L^2_{K^m}, L^2_{K^p})$ is defined by

$$(Lu)(t) = C\int_0^t e^{A(t-s)}Bu(s)ds \;, \quad u(\cdot) \in L^2_{K^m} \quad .$$

Theorem (1.4) (Hinrichsen and Pritchard, [7])

(1.1.5) $$r_{\mathbb{R}}(A;I,I) = \min_{\omega\in\mathbb{R}} \; \inf_{\substack{x,y\in\mathbb{R}^n \\ ||x||^2+||y||^2=1}} \mu(x,y,-Ax-\omega y,\omega x-Ay)$$

where the inf is taken over linear independent x,y and $\mu^2(x,y,u,v)$ is the maximum root of the equation

$$\det \begin{bmatrix} ||u||^2-\lambda||x||^2 & <u,v> - \lambda<x,y> \\ <u,v>-\lambda <x,y> & ||v||^2 - \lambda ||y||^2 \end{bmatrix} = 0 .$$

Remark (1.5)

In obtaining characterisation (1.1.3) for $r_{\mathbb{C}}(A;B,C)$ and also characterisation (1.1.5) for $r_{\mathbb{R}}(A;I,I)$ emphasis is placed upon a direct analysis of the spectrum, $\sigma(A+BDC)$, of the perturbed operator $(A+BDC)$. Of greater interest, in the case of infinite dimensional systems, is the second characterisation concerning the equivalence of $\sigma(A+BDC) \subseteq \mathbb{C}_-$ and the exponential stability of $e^{(A+BDC)t}$, where $x(t) = e^{(A+BDC)t}x_0$ is the solution of

$$(1.1.6) \qquad x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}BDCx(s)ds , \qquad x_0 \in \mathbb{C}^n$$

the mild (integral) version of

$$(1.1.7) \qquad \dot{x}(t) = (A + BDC)x(t) \qquad x(0) = x_0 .$$

Also important in this characterisation is the auxiliary equation

$$(1.1.8) \qquad y(t) = Ce^{At}x_0 + C\int_0^t S(t-s)BDy(s)ds , \qquad x_0 \in \mathbb{C}^n$$

obtained when isolating the arbitrary $D \in \mathbb{C}^{p\times m}$ in (1.1.6).

Due to the distinct possibility of unboundedness in the perturbation term the semigroup analogue of (1.1.6) will be taken as the perturbed system and only within a more restrictive framework will this system be related to the abstract analogue of (1.1.7). As well as this relationship between the mild solution of the abstract differential equation and the solution of the differential equation itself, the notion of a weak solution of an associated differential equation will be discussed when the abstract equation (D) models a class of partial differential equations.

This section is completed with the following result, concerning the equivalence of L^2-stability and exponential stability, for systems defined by semigroups of linear operators. (For more general results, concerning evolution operators and also semigroups of non-linear operators see Datko [18] and Ichikawa [19] respectively.)

Lemma (1.6) (Datko [17]) (See also Pritchard and Zabczyk [50]).

Assume $S(t) \in \mathcal{L}(X)$ is a strongly continuous semigroup on a real or complex Banach space X . Then $S(t)$ is exponentially stable if and only if

$$(1.1.9) \qquad \int_0^\infty ||S(t)x||_X^2 dt < \infty \quad \text{for all} \quad x \in X . \qquad \square$$

Remark (1.7)

This result allows a norm based analysis of the exponential stability of perturbed semigroups.

§2. An Abstract Framework and the well posedness of the perturbed system.

Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup of linear operators on a Banach space X over $K = \mathbb{R}$ or $\mathbb{C}$. (In section 3 the main results are restricted to the case $K = \mathbb{C}$.) Let $A : D_X(A) \to X$ be the generator of $(S(t))_{t\geq 0}$. Examples (0.8) and (0.9) are accommodated within the abstract differential equation

$$(D) \quad \dot{x}(t) = (A + BDC)x(t) , \quad x(0) = x_0 ,$$

a (formal) perturbation of

$$(N) \quad \dot{x}(t) = Ax(t) , \quad x(0) = x_0 ,$$

where $B \in L(U,\bar{X})$ $C \in L(\underline{X},Y)$ are fixed structure operators and $D \in L(Y,U)$ is arbitrary and measures the size of the perturbation BDC. $\underline{X}$, $\bar{X}$, U and Y are auxiliary Banach spaces over K, arising from the particular system under consideration, satisfying

(A1) $\underline{X} \subseteq X \subseteq \bar{X}$, where the inclusions are continuous and dense.

Due to this unboundedness in the perturbation term BDC, the mild form

$$(P) \quad x(t) = S(t)x_0 + \int_0^t S(t-s)BDCx(s)ds , \quad x_0 \in X ,$$

is taken as the perturbed system. At this stage equation (P) could be taken as defining a dynamical system on the Banach space X, with $D \equiv 0$ defining the mild, nominal (unperturbed) system $x_0 \to S(t)x_0$. Of importance in the sequel is the auxiliary equation

(Y) $y(t) = CS(t)x_0 + C\int_0^t S(t-s)BDy(s)ds \ , \quad x_0 \in X \ .$

In order to make sense of the perturbed system (P) and the auxiliary equation (Y) , conditions must be imposed on the various operators A , B and C and the underlying spaces $\underline{X}$, $X \bar{X}$.

(A2) $S(t)$ extends (restricts) to a strongly continuous semigroup $\bar{S}(t)(\underline{S}(t))$ on $\bar{X}$ $(\underline{X})$ with the domain of A on $\bar{X}$, $D_{\bar{X}}(A)$, satisfying

$D_{\bar{X}}(A) \subseteq \underline{X}$, with continuous dense injection, where $D_{\bar{X}}(A)$

is endowed with the graph norm

(A3) there exists $M,\alpha > 0$ such that $||T(t)|| \leq Me^{-\alpha t}$ for all $t \geq 0$ where $T(t) = \underline{S}(t)$, $S(t)$, $\bar{S}(t)$ on $\underline{X}$, X , $\bar{X}$ respectively.

(A4) $\int_0^t \bar{S}(t-s)Bu(s)ds \in \underline{X}$ for almost all $t \geq 0$, $u(\cdot) \in L_U^2$ and there exists $k > 0$ such that

$$||\int_0^{\cdot} \bar{S}(\cdot-s)Bu(s)ds||_{L^2_{\underline{X}}} \leq k||u(\cdot)||_{L^2_U} \quad \text{for all} \quad u(\cdot) \in L_U^2 \ .$$

(A5) For all $0 \leq t_0 \leq T$, there exists $b(t_0,T) \geq 0$ such that for all $u(\cdot) \in L^2(t_0,T;U)$

$$\int_{t_0}^T \bar{S}(T-s)Bu(s)ds \in X \qquad \text{and}$$

$$||\int_{t_0}^T \bar{S}(T-s)Bu(s)ds||_X \leq b(t_0,T)||u(\cdot)||_{L^2(t_0,T;U)} \ ,$$

(A6) i) there exists $c_1(t_0,T) \geq 0$ such that

$$\|CS(\cdot)x_0\|_{L^2(t_0,T;Y)} \leq c_1(t_0,T)\|x_0\|_X \quad \text{for all } x_0 \in \underline{X} .$$

ii) there exists $c_2(t_0,T) \geq 0$ such that

$$\|Cx(\cdot)\|_{L^2(t_0,T;Y)} \leq c_2(t_0,T)\|x(\cdot)\|_{L^2(t_0,T;X)}$$

for all $x(\cdot) \in L^2(t_0,T;\underline{X})$.

(N.B.) $L^2(t_0,T;\underline{X}) \subseteq L^2(t_0,T;X)$ with continuous dense injection.

(A6)(iii) there exists $c_3(t_0,T) \geq 0$ such that

$$\|CS(\cdot)x_0\|_{L^2(t_0,T;Y)} \leq c_3(t_0,T)\|x_0\|_{\bar{X}} \quad \text{for all } x_0 \in \underline{X} .$$

Remarks (2.1)

a) The integrations of (A4) and (A5) are taken in $\bar{X}$ and it is the combined smoothing action of $\bar{S}(\cdot)$ and this integration which makes the assumptions reasonable (see examples (4.1)-(4.5)). The reason for three variations of assumption (A6) is due to the properties of the particular perturbed system under consideration. If for example, the dynamical system (P) is being analysed only to consider an abstract framework for the study, of stability, of known unique solutions to equation (P), then only assumption (A6)(i) is imposed. If a unique semigroup $S^D(t)$ is sought such that $x(t) = S^D(t)x_0$, $x_0 \in D_{\bar{X}}(A)$ solves equation (D) in $\bar{X}$ then (A6)(iii) is imposed. If, however, the perturbed system is taken only as the dynamical system (P) then in order to obtain uniqueness of solutions assumption (A6)(ii) is imposed. (It is shown in the sequel that this "0^{th} order" assumption (A6)(ii) is rather restrictive and it can be replaced by a much less restrictive "r^{th} order" assumption.)

Lemma (2.2)

i) Assume (A6)(ii) holds then necessarily (A6)(i) holds.

ii) Assume (A6)(iii) and (A1) holds then necessarily (A6)(i) holds.

iii) (A3) implies that $b(t_0,T)$, $c_1(t_0,T)$ and $c_3(t_0,T)$ can be taken as independent of t_0 and T .

Proof. i) is trivial.

ii) Let $c > 0$ be such that $||x||_{\bar{X}} \leq c||x||_X$ for all $x \in X$. If there exist $c_3(t_0,T) \geq 0$ such that

$$||CS(\cdot)x||_{L^2(t_0,T;Y)} \leq c_3(t_0,T)||x||_{\bar{X}} \text{ for all } x \in \underline{X}$$

then

$$||CS(\cdot)x||_{L^2(t_0,T;Y)} \leq c.c_3(t_0,T)||x||_X \text{ for all } x \in \underline{X} ,$$

and (A6)(i) follows.

iii) Let $0 \leq t_0 \leq T \leq t$, $x \in \underline{X}$, then

$$\int_0^t ||CS(s)x||_Y^2 ds = \int_0^{t_0} ||CS(s)x||_Y^2 ds + \sum_{r=0}^{n-1} \int_{t_0+r(T-t_0)}^{t_0+(r+1)(T-t_0)} ||CS(s)x||_Y^2 ds$$

$$+ \int_{t_0+n(T-t_0)}^{t} ||CS(s)x||_Y^2 ds, \quad n(T-t_0) \le t-t_0 \le (n+1)(T-t_0)$$

$$\le c(0,t_0)^2 ||x||_X^2 + \sum_{r=0}^{n-1} c(t_0,T)^2 ||S(T-t_0)||^{2r} ||x||_X^2$$

$$+ c(0,t-(t_0+n(T-t_0)))^2 ||S(t_0+n(T-t_0)||^2 ||x||_X^2 \ .$$

If $(T-t_0)$ is such that $Me^{-\alpha(T-t_0)} < 1$ then $\int_0^t ||CS(s)x||_Y^2 ds \le K||x||_X^2$ independent of t. A similar argument holds for assumption (A6)(iii). Now assumption (A5) is the dual of (A6)(i) (Salamon [51]) and therefore the independence of $b(t_0,T)$ in t_0, T follows immediately. Denote these constants by c_1, c_3 and b. □

Remark (2.3)

Assumption (A3) and (A6)(i) ((A6)(iii)), together with the conclusions of lemma (2.2)(iii), imply that the bounded map $x \to CS(\cdot)x$, $\underline{X} \to L_Y^2$, extends, uniquely, to a bounded map $X \to L_Y^2$, $(\bar{X} \to L_Y^2)$. For each $x \in X (x \in \bar{X})$ the unique function in L_Y^2 guaranteed by these assumptions, is denoted by $CS(\cdot)x$ throughout this chapter.

Definition (2.4)

(i) The perturbed system (P) is said to be well-posed in the sense of (A6)(i) if there exists a unique strongly continuous semigroup $S^D(t) \in L(X)$ such that

a) for all $x_0 \in \underline{X}$, $S^D(\cdot)x_0 \in L^2(0,T;\underline{X})$ for all $T > 0$

and b) for all $x_0 \in X$, $x(t) = S^D(t)x_0$ solves equation (P).

(ii) The perturbed system (P) is said to be well-posed as a dynamical system on X if there exists a strongly continuous semigroup, $S^D(t) \in L(X)$, such that for all $x_0 \in X$, $x(t) = S^D(t)x_0$ is the unique, continuous solution of

$$x(t) = S(t)x_0 + \int_0^t \bar{S}(t-s)BDCx(s)ds ,$$

where $Cx(\cdot)$ is interpreted in $L^2(0,T;Y)$ for each $T \geq 0$.

(iii) The perturbed system (P) is said to be well-posed, if there exists a unique strongly continuous semigroup $\bar{S}^D(t) \in L(\bar{X})$, such that

a) $\bar{S}^D(t)$ restricts to a strongly continuous semigroup on X.

b) $\bar{S}^D(t)$ is strongly continuous on $D_{\bar{X}}(A) = D_{\bar{X}}(A+BDC)$

and c) for all $x_0 \in D_{\bar{X}}(A)$, $\bar{S}^D(t)x_0$ is continuously differentiable

and

$$\dot{\bar{S}}^D(t)x_0 = (A + BDC)\bar{S}^D(t)x_0 .$$

Notation

Denote by L, the linear map $u(\cdot) \to (Lu)(\cdot)$

$$(Lu)(t) = C\int_0^t \bar{S}(t-s)Bu(s)ds . \quad (1.2.1)$$

Lemma (2.5)

Assume $C \in L(\underline{X},Y)$ and (A4) holds then

$$L \in L(L^2_U, L^2_Y) .$$

Theorem (2.6)

If (A1)-(A5)(A6)(i) hold and $||D|| < \frac{1}{||L||}$ then the perturbed system is well posed in the sense of (A6)(i).

Proof.

Consider first the auxiliary equation (Y) in L^2_Y

$$\text{(Y)} \qquad y(\cdot) = CS(\cdot)x_0 + (LDy)(\cdot) .$$

The right-hand side is well defined for each $x_0 \in X$. Moreover, since $||LD|| < 1$, this equation has a unique solution for $y(\cdot) \in L^2_Y$.

Now define $x(t,x_0)$ by

$$\text{(1.2.2)} \qquad x(t,x_0) = S(t)x_0 + \int_0^t S(t-s)B\, Dy(s)ds , \quad x_0 \in X .$$

By (A3) and (A4), $x(\cdot,x_0) \in L^2_X$ and $Cx(\cdot,x_0) \in L^2_Y$. Hence $y(\cdot) = Cx(\cdot,x_0)$ and therefore $x(\cdot,x_0)$ solves (P) in L^2_X.

The continuity of $x(\cdot,x_0)$, $x_0 \in X$ and the semigroup property for $S^D(t) \in L(X)$ defined by $S^D(t)x_0 = x(t,x_0)$ are proved in the appendix. If $x_0 \in \underline{X}$ then $S^D(\cdot)x_0 \in L^2_{\underline{X}}$ and therefore $S^D(t)$ satisfies all the conditions of definition (2.4)(i). Suppose $T(t) \in L(X)$ is a second strongly continuous semigroup satisfying definition (2.4)(i). Let $x \in \underline{X}$, then for all $T > 0$,

$$C(T(\cdot)x - S^D(\cdot)x) = LDC(T(\cdot)x - S^D(\cdot)x)$$

holds in $L^2(0,T;Y)$. Hence since, $||LD|| < 1$,

$$T(t)x = S^D(t)x \quad \text{for almost all} \quad t \geq 0$$

and therefore

$$T(t)x = S^D(t)x \quad \text{in} \quad X \quad \text{for all} \quad t \geq 0 \text{ .}$$

Hence by assumption (A2) and the strong continuity of $T(t)$ and $S^D(t)$ in X , $S^D(t) = T(t)$. □

This theorem is essentially an existence theorem for the perturbed semigroup. It is, however, unique amongst those semigroups mapping $\underline{X} \to L^2(0,T;\underline{X})$, $T > 0$. Its existence provides a means of analysing already well posed systems. This type of well-posedness is considered again in a more abstract setting in Chapter 3.

Corollary (2.7)

Assume (A1)-(A5)(A6)(ii) hold and $||D|| < \dfrac{1}{||L||}$ then equation (P) is well posed as a dynamical system on X .

Proof.

Existence of a strongly continuous semigroup $S^D(t) \in L(X)$, satisfying the conditions of definition (2.4)(ii), is guaranteed by Theorem (2.6). Suppose that $T(t) \in L(X)$ is a second, strongly continuous semigroup satisfying the conditions of definition (2.4)(ii). Then

$$\bar{C}(T(\cdot)x - S^D(\cdot)x) = LD\bar{C}(T(\cdot)x - S^D(\cdot)x)(\cdot), \quad \text{for all} \quad x \in X$$

on $L^2(0,T;Y)$, $\bar{C}$ denotes the extension of $x(\cdot) \to Cx(\cdot)$ $L^2(0,T;\underline{X}) \to L^2(0,T;Y)$ to a bounded map $L^2(0,T;X) \to L^2(0,T;Y)$. Therefore $\bar{C}(T(\cdot)x) = \bar{C}(S^D(\cdot)x) = CS^D(\cdot)x$ in $L^2(0,T;Y)$ and therefore $T(t)x = S^D(t)x$ for all $t \geq 0$, $x \in X$. □

Remark (2.8)

Unfortunately, in applications, assumption (A6)(ii) is quite restrictive since it involves no smoothing action of the semigroup. For example if $Y = U = X$ and there exists $\alpha > 0$, $\beta \geq 0$ such that

$$||S(t)x||_{\underline{X}} \leq \frac{Me^{-\alpha t}}{t^\beta}||x||_X \quad \text{for all} \quad t > 0 \quad ,$$

(A6)(i) requires that $\beta < \frac{1}{2}$ whereas (A6)(ii) requires

$$||Cx(\cdot)||_{L^2(0,T;Y)} \leq K_T||x(\cdot)||_{L^2(0,T;X)} \quad \text{for which}$$

the inherent smoothing action plays no role. An alternative, involving the smoothing action of the semigroup is to suppose that for each $D \in L(Y,U)$, $T \geq 0$ there exists $K_T(D) \geq 0$ such that

$$\text{(A7)} \quad \int_0^T ||C\int_0^t S(t-s)BDCx(s)ds||_Y^2 dt \leq k_T^2(D)||x(\cdot)||^2_{L^2(0,T;X)}$$

for all $x(\cdot) \in L^2(0,T;\underline{X})$ and the left hand side of (A7) is interpreted via assumption (A4) and lemma (2.5). To see (at least intuitively) how this helps, assume $A \le 0$ and the unboundedness of B and C are $(-A)^{\alpha}$, $(-A)^{\beta}$ respectively. Single application of the operator L results in a smoothing of $(A^{-1+\alpha+\beta})$ and therefore in order that (A7) holds, $\alpha+2\beta < 1$. More generally assume, instead of (A7), that the following r^{th} order condition holds, that is there exists $r \in \mathbb{N}_0$, such that for all $D \in L(Y,U)$ and $T \ge 0$, there exists $K_T(D) \ge 0$ such that for all $x(\cdot) \in L^2(0,T;\underline{X})$

$$\text{(A8)}\quad \int_0^t ||C\int_0^{t_r} S(t_r-t_{r-1})BD\ldots C\int_0^{t_1} S(t_1-s)BDCx(s)ds\ldots dt_{r-1}||_Y^2 dt_r$$

$$= ||(LD)^r Cx(\cdot)||^2_{L^2(0,T;Y)} \le K_T^2(D)||x(\cdot)||^2_{L^2(0,T;X)} \quad .$$

If (A8) holds for some $r \in \mathbb{N}_0$ and $\tilde{x}(\cdot,x_0)$ is a second solution of (P) then

$$\text{(1.2.3)}\qquad (LD)^r C\tilde{x}(\cdot,x_0) = (LD)^r CS(\cdot)x_0 + (LD)^{r+1} C\tilde{x}(\cdot,x_0) \ .$$

But $||LD|| < 1$ and $(LD)^r C\tilde{x}(\cdot,x_0) \in L^2(0,T;Y)$ and therefore

$$(LD)^r(C\tilde{x}(\cdot,x_0)) = (LD)^r(CS^D(\cdot)x_0) \quad \text{in} \quad L^2(0,T;Y) \ .$$

Applying $(LD)^k C$ to

$$\tilde{x}(\cdot,x_0) = S(\cdot)x_0 + \int_0^{\cdot} S(\cdot-s)BDC\tilde{x}(s,x_0)ds$$

for each $k = r-1,\ldots, 0$ results in $(LD)^k(C\tilde{x}(\cdot,x_0)) \in L^2(0,T;Y)$

and therefore

$$\tilde{x}(\cdot,x_0) = S^D(\cdot)x_0 \quad \text{in} \quad L^2(0,T;X)$$

for each $T \geq 0$ and hence by continuity

$$\tilde{x}(t,x_0) = S^D(t)x_0 \quad \text{for all} \quad t \geq 0 \ .$$

N.B. Of course, by $(LD)^k(Cx(\cdot,x_0))$ is understood the unique extension of this function guaranteed by (A8), but since at each stage $k = r-1,\dots, 0$ this extension is shown to be $(LD)^k(CS(\cdot,x_0))$, $\tilde{x}(\cdot,x_0)$ satisfies $\tilde{x}(\cdot,x_0) \in D(C)$, when C is considered as an unbounded operator $L^2(0,T;X) \to L^2(0,T;Y)$.

This additional assumption (A8) is considered in a detailed case study in section 4, when the degrees of unboundedness allowed by each assumption is determined. This case study illustrates that (A8) is restrictive only in the extreme case of $\alpha = \beta = \frac{1}{2}$, the maximum degree of unboundedness allowed by (A4)(A5) and (A6)(i).

Corollary (2.9)

Assume (A1)-(A5)(A6)(iii) hold and $||D|| < \frac{1}{||L||}$ then equation (P) is well-posed (in the sense of definition (2.4)(iii)).

Proof.

If (A6)(iii) holds and $||D|| < \frac{1}{||L||}$ then equation (Y) in theorem (2.6) has a well defined solution for $y(\cdot) \in L_Y^2$, for all $x_0 \in \bar{X}$.

The strong continuity and semigroup property for $\bar{S}^D(t) \in L(\bar{X})$ defined by $\bar{S}^D(t)x_0 = \bar{x}(t,x_0)$ where

$$\bar{x}(t,x_0) = \bar{S}(t)x_0 + \int_0^t S(t-s)BDy(s)ds \ , \quad x_0 \in \bar{X}$$

follow just as in theorem (2.6). In order to establish the differentiability of $\bar{S}^D(t)x_0$, let $x_0 \in D_{\bar{X}}(A)$. Then $(A + BDC)x_0 \in \bar{X}$ by assumption (A2). Just as in the proof theorem (2.6) there exists a unique solution of

$$(1.2.4) \qquad z(t) = C\bar{S}(t)(A+BDC)x_0 + C\int_0^t \bar{S}(t-s)BDz(s)ds$$

for $z(\cdot) \in L_Y^2$. In fact $\int_0^t z(s)ds = C\bar{S}^D(t)x_0 - Cx_0$ and therefore $C\bar{S}^D(\cdot)x_0$ is differentiable and $(\frac{d}{dt}(C\bar{S}^D(t)x_0))(\cdot) \in L_Y^2$. The details for this are contained in the appendix. Define $w(\cdot)$ by

$$(1.2.5) \qquad w(t) = \bar{S}(t)(A+BDC)x_0 + \int_0^t \bar{S}(t-s)BD\dot{y}(s)ds \ .$$

As above, it is easy to show that $\int_0^t w(s)ds = \bar{S}^D(t)x_0 - x_0$. Therefore $\bar{S}^D(t) \in L(X) \cap L(\bar{X})$ is strongly, continuously differentiable and

$$(1.2.6) \qquad \dot{\bar{S}}^D(t)x_0 = \bar{S}^D(t)(A+BDC)x_0 \ .$$

Also, since for $x_0 \in D_{\bar{X}}(A)$, $y(t) = C\bar{S}^D(t)x_0$ satisfies $y(\cdot)$, $\dot{y}(\cdot) \in L_Y^2$, it follows that $\bar{S}^D(t)x_0 \in D_{\bar{X}}(A)$, for all $x_0 \in D_{\bar{X}}(A)$ and therefore

$$(1.2.7) \qquad \dot{\bar{S}}^D(t)x_0 = (A+BDC)\bar{S}^D(t)x_0 \ , \quad \text{for all } x_0 \in D_{\bar{X}}(A) \ .$$

Suppose $\bar{T}(t) \in L(\bar{X}) \cap L(X)$ is a second strongly continuous semigroup satisfying definition (2.4)(iii). Let $x_0 \in D_{\bar{X}}(A)$, then

$$\bar{T}(t)x_0 - \bar{S}^D(t)x_0 = \int_0^t \frac{d}{ds} (\bar{S}^D(t-s)\bar{T}(s)x_0)ds$$
$$= 0 .$$

Therefore by (A1)(A2) and the strong continuity of $\bar{T}(t)$ and $\bar{S}^D(t)$ on $\bar{X}$, uniqueness of $\bar{S}^D(t)$ is proved. □

Remark (2.10)

Corollary (2.9) is the best result since it is possible that $\text{Im } B \cap X = \{0\}$, for example inboundary perturbations. The result relates the generator A^D of the semigroup $(\bar{S}^D(t))_{t\geq 0}$ and the perturbed operator $(A+BDC)$, by the following formula,

$$A^D\Big|_{D_{\bar{X}}(A)} = (A + BDC)\Big|_{D_{\bar{X}}(A)} .$$

As with assumption (A6)(ii), in applications (A6)(iii) severely limits the degree of unboundedness allowable within the given framework. For example if $Y = U = X$ and there exists $\beta,\gamma \geq 0$ and $\alpha > 0$ such that

$$||S(t)x||_{\underline{X}} \leq \frac{Me^{-\alpha t}}{t^{\beta}} ||x||_X$$

$$||\bar{S}(t)x||_X \leq \frac{Me^{-\alpha t}}{t^{\gamma}} ||x||_{\bar{X}} \quad , \quad \text{for all} \quad t > 0 ,$$

then (A6)(iii) requires that $\beta+\gamma < \frac{1}{2}$.

Remark (2.10)

In the case that either $\underline{X} = X$ or else $\bar{X} = X$ then corollaries (2.7) (2.9) hold respectively. However in many applications this is not the case and the system is well-posed only in the sense of definition (2.4)(i). When this is the case, frequently, some other associated machinery guarantees that the solution of the mild equation (P) is still unique in some other sense. For example, when (N) models a parabolic or hyperbolic equation on some suitable function space, with homogeneous boundary values, any solution of the mild equation (P), with suitable B,C and D operators satisfying a Green's formula, can be shown to be a "weak solution" of a mixed boundary value problem (see Curtain and Pritchard [21]) for which $D = 0$ corresponds to the nominal boundary value problem. Consider again example (0.9) of section 1 with the semigroup $S(t) \in L(X)$ given by

$$S(t)x = \Sigma\, e^{\lambda_n t} x_n \phi_n \ , \quad x = \Sigma x_n \phi_n \ , \quad \Sigma x_n^2 < \infty$$

$$\lambda_n = -\left(\frac{2n+1}{2}\right)^2 , \phi_n(\xi) = \sqrt{2} \cos \left(\frac{2n+1}{2}\right) \pi \xi$$

$$X = L^2(0,1) \quad Ah = \frac{\partial^2 h}{\partial \xi^2} \ , \quad D(A) = \{h \in H^2 | h_\xi(0) = h(1)\} \ ,$$

corresponding to the homogeneous boundary value problem

$$(1.2.8) \quad \begin{cases} x_t(\xi,t) = x_{\xi\xi}(\xi,t) \quad x(\xi,0) = x(\xi) \\ x_\xi(0,t) = x(1,t) = 0 \quad . \end{cases}$$

If $x(\cdot,x_0)$ is a continuous function satisfying

$$x(t) = S(t)x_0 + \int_0^t S(t-s)BDCx(s)ds$$

with $D = d \in \mathbb{C}$, $Ch(\cdot) = h(0)$ and $Bu = \delta_0(\cdot)u$, where $\delta_0(\cdot)$ is the Dirac-delta distribution at 0 then $x(\cdot,x_0)$ can be shown to be a "weak solution" of the mixed boundary value problem

$$(1.2.9) \quad \begin{cases} x_t(\xi,t) = x_{\xi\xi}(\xi,t) \quad x(\xi,0) = x(\xi) \\ x_\xi(0,t) = dx(0,t) \ , \quad x(1,t) = 0 \ . \end{cases}$$

Thus when $x(\cdot,x_0)$ is exponentially decaying the mixed boundary value problem has a generalised solution which is exponentially stable. For a more detailed study of this correspondence between abstract differential equations (D) and weak solutions of boundary value problems when A is a partial differential operator, see Curtain and Pritchard [21] and Salamon [24]. See also Pritchard and Salamon [23] for the connection between the mild solution (P) and the abstract differential equation (D) for the case of differential delay equations.

As yet, well-posedness of the perturbed system, defined by equation (P), has been considered only for those D lying in a suitable subset of $L(Y,U)$. The reason for this becomes clear in section 3 and results from the requirement that the solutions be exponentially decaying. In order to obtain well-posedness results for arbitrary $D \in L(Y,U)$ further assumptions must be imposed upon the nominal system and the structure operators B and C. One possibility is to assume that for all $D \in L(Y,U)$ there exists $T \geq 0$ such that $H_D(T) \in L(L^2(0,T;Y))$, defined by

$$(1.2.10) \quad H_D(T) = \rho_T(I - LD)\sigma_T \ ,$$

is boundedly invertible, where ρ_T and σ_T are defined by

$$\rho_T : L^2_H \to L^2(0,T;H) \quad \rho_T : h(\cdot) \to h(\cdot)\big|_{[0,T]} \quad \text{and}$$

$$\sigma_T : L^2(0,T;H) \to L^2_H \quad \sigma_T : h(\cdot) \to h_T(\cdot) \;, \quad \rho_T h_T = h \text{ and } h_T(t) = 0 \quad t > T \;.$$

(For a detailed study of the well-posedness of equation (P) in this setting see Salamon [24] and Chapman et al [52].) An alternative to this method, more closely related to the requirement of exponential stability of solutions, is to proceed as follows:-

Let $R \geq 0$ and suppose instead of (A4) that

$$\text{(A9)} \quad \left|\left|\int_0^{\cdot} e^{-R(\cdot-s)}\bar{S}(\cdot-s)Bu(s)ds\right|\right|_{L^2_{\underline{X}}} \leq k_R||u(\cdot)||_{L^2_U}$$

for all $u(.) \in L^2_U$, where $k_R \to 0$ as $R \to \infty$.

In section 4 the results of this section and section 3 are applied to semigroups satisfying the following properties

$$||S(t)x||_{\underline{X}} \leq \frac{Me^{-\alpha t}}{t^{\beta}}||x||_X \qquad t > 0 \;, \quad \beta \geq 0 \quad \alpha > 0$$

and

$$||\bar{S}(t)x||_X \leq \frac{Me^{-\alpha t}}{t^{\gamma}}||x||_{\bar{X}} \qquad t > 0 \;, \quad \gamma \geq 0 \;.$$

If these assumptions hold then property (A9) reduces to

$$\int_0^{\infty} \frac{Me^{-(R+\alpha)t}}{t^{\beta+\gamma}} dt \to 0 \quad \text{as} \quad R \to \infty \;.$$

To show this let $\varepsilon > 0$. Choose $(R+\alpha)\hat{t}$ such that $Me^{-(R+\alpha)\hat{t}} = \delta$ $\hat{t} < 1$ and δ sufficiently small. Then

$$\int_0^\infty \frac{Me^{-(R+\alpha)t}}{t^{\beta+\gamma}}\,dt = \int_0^{\hat{t}} \frac{Me^{-(R+\alpha)t}}{t^{\beta+\gamma}}\,dt + \int_{\hat{t}}^1 \frac{Me^{-(R+\alpha)t}}{t^{\beta+\gamma}}\,dt$$

$$+ \int_1^\infty \frac{Me^{-(R+\alpha)t}}{t^{\beta+\gamma}}\,dt$$

$$\leq M\int_0^{\hat{t}} \frac{1}{t^{\beta+\gamma}}\,dt + \delta\int_{\hat{t}}^1 \frac{1}{t^{\beta+\gamma}}\,dt$$

$$+ 1\int_1^\infty Me^{-(R+\alpha)t}\,dt$$

$$\leq \frac{M\hat{t}^{(1-\beta-\gamma)}}{(1-\beta-\gamma)} + \eta(1 - \hat{t}^{(1-\beta-\gamma)})$$

$$+ \frac{M}{(R+\alpha)} \;,\quad \gamma+\beta < 1 \;,\quad \eta = \frac{\delta}{(1-\beta-\gamma)} \;.$$

Choose R large enough such that $\eta < \frac{\varepsilon}{3}$, $\frac{\hat{t}^{(1-\beta-\gamma)}}{(1-\beta-\gamma)} < \min(1, \frac{\varepsilon}{3})$ and $\frac{M}{(R+\alpha)} < \frac{\varepsilon}{3}$ results in

$$k_R \to 0 \quad \text{as} \quad R \to \infty \;.$$

Therefore, for such systems condition (A9) is automatic and is a direct consequence of the separating of the exponential decay of the semigroup and its inherent smoothing action on the spaces $\underline{X}$, X and $\bar{X}$.

Notation.

Denote by $L_R \in \mathcal{L}(L^2_U, L^2_Y)$ the map

$$u(\cdot) \to C\int_0^{\cdot} e^{-R(\cdot-s)}S(\cdot-s)Bu(s)ds \ , \quad R > 0$$

guaranteed by (A9).

Theorem (2.11)

Assume conditions (A1)-(A3)(A5)(A6)(i) and (A9) hold. If $D \in \mathcal{L}(Y,U)$ then the system is well posed in the sense of definition (2.4)(i).

Proof.

Denote by $L^2_{H,R}$ the set of functions

$$L^2_{H,R} = \{f(\cdot) \in L^2(0,T;H) \text{ for all } T > 0 \text{ , such that } \int_0^\infty e^{-2Rt}||f(t)||^2_H dt < \infty\} \ .$$

Choose R sufficiently large so that $||L_R D|| < 1$ and consider the following fixed point problem on L^2_Y

$$(1.2.11) \quad y_R(\cdot) = y^R_0(\cdot) + (L_R D y_R)(\cdot)$$

where $y^R_0(t) = Ce^{-R(t)}S(t)x_0$. (N.B. $y^R_0(\cdot) \in L^2_Y$ follows from the fact that (A6)(i) is preserved under bounded perturbation and $S_R(t) = e^{-Rt}S(t)$ is exponentially stable on all three spaces $\underline{X}$, X , $\bar{X}$.)

Equation (1.2.11) has a unique solution for $y_R(\cdot) \in L_Y^2$. As in the proof of theorem (2.6) define $x(t,x_0)$ by

$$(1.2.12) \quad x(t,x_0) = S(t)x_0 + \int_0^t \bar{S}(t-s)BDe^{Rs}y_R(s)ds \ .$$

Then it is easy to show that $x(\cdot,x_0) \in L_{X,R}^2$ and $Cx(\cdot,x_0) \in L_{Y,R}^2$. Therefore $x(t,x_0)$ satisfies (P) in $L_{H,R}^2$. Continuity and the semi-group property for $(S^D(t))_{t\geq 0}$ defined by $S^D(t)x_0 = x(t,x_0)$ are immediate. Also $x(\cdot,x_0) \in L^2(0,T;\underline{X})$ for all $x \in \underline{X}$ and is unique with this property. Consequently the system is well posed for all $D \in \mathcal{L}(Y,U)$. □

Remark (2.12)

It turns out that well-posedness of the perturbed system for arbitrary $D \in \mathcal{L}(Y,U)$ is unnecessary from the robustness viewpoint, although theorem (2.11) gives a complete picture for the results of this section.

Remark (2.13)

Again as for the case when $||D|| < \frac{1}{||L||}$ the corollaries (2.7) and (2.9) hold true for all $D \in \mathcal{L}(Y,U)$ under the necessary additional assumptions. These are omitted here since the important aspect is the existence theorems for arbitrary $D \in \mathcal{L}(Y,U)$.

§3. A robustness measure.

The well-posedness results of §2 now allow the following definition of various stability radii (depending upon the degree of unboundedness) for perturbed systems defined by equation (P).

Definition (3.1)

For $j = 1,2$ or 3 define

(1.3.1) $r^j_K(A;B,C) = \sup_d\{||D|| < d$ implies the system (P) is well-posed in the sense of definition (2.4)(j) and the guaranteed semigroup is exponentially stable$\}$.

Remarks (3.2)

a) The possibility that $r^j_K(A;B,C) = \infty$ is not discounted.

b) In Chapter 2, $r^3_{\mathbb{C}}(A;B,C)$ is related to a certain non-standard linear quadratic problem.

c) In Chapter 3 a robustness measure or stability radius analogous to this one is developed for certain other infinite-dimensional, linear systems.

d) Theorem (2.6) (and corollaries (2.7) and (2.9)) involved the bound $||D|| < ||L||^{-1}$. In this section $r^j_{\mathbb{C}}(A;B,C)$ is shown to be equal to $||L||^{-1}$ for each $j = 1,2,3$.

Proposition (3.3)

If conditions (A1)-(A6)(j) hold then

$$r^j_K(A;B,C) \geq \frac{1}{||L||} \quad .$$

Note: This establishes that exponential stability of the perturbed semigroup is robust.

Proof.

If $||D|| < \frac{1}{||L||}$ then $y(\cdot)$, the unique solution of (Y) satisfies

$$||y(\cdot)||_{L^2_Y} \leq ||(I - LD)^{-1}|| \; ||CS(\cdot)x_0||_{L^2_Y}$$

$$\leq k||x_0||_X \quad \text{for some} \quad k \geq 0 \ .$$

Hence

$$\int_0^\infty ||S^D(t)x_0||^2_X dt \leq 2\int_0^\infty ||S(t)x_0||^2_X dt + 2\int_0^\infty ||\int_0^t \bar{S}(t-p)BDy(p)dp||^2_X dt$$

$$\leq \bar{k}^2||x_0||^2_X \quad \text{for some} \quad \bar{k} \geq 0 \ .$$

Therefore exponential stability of $(S^D(t))_{t\geq 0}$ follows from lemma (1.6). □

Remark (3.4)

In fact for $j = 3$ $(\bar{S}^D(t))_{t\geq 0}$ is exponentially stable also as a semigroup on $\bar{X}$.

Lemma (3.5)

If (A1)-(A3) hold, then

$$(1.3.2) \qquad (i\omega I-A)^{-1}x = \int_0^\infty e^{-i\omega t}\bar{S}(t)x dt \in \underline{X} \text{ for all } x \in \bar{X}$$

$$\text{and} \quad (i\omega-A)^{-1} \in L(\bar{X},\underline{X}) \quad .$$

Proposition (3.6)

Assume (A1)-(A4) hold and U and Y are Hilbert spaces. If $\sup_\omega ||G(i\omega)|| < \infty$, where $G(i\omega) = C(i\omega-A)^{-1}B$, then

$$(1.3.3) \qquad ||L|| = \sup_\omega ||G(i\omega)|| \ .$$

Proof.

See Appendix.

Theorem (3.7)

Suppose (A1)-(A6)(j) hold for j = 1,2 or 3, $0 < \sup_\omega ||G(i\omega)|| < \infty$ and U and Y are Hilbert spaces then

$$(1.3.4) \qquad r^j_{\mathbb{C}}(A;B,C) = \frac{1}{\sup_\omega ||G(i\omega)||} \ , \quad j = 1,2 \text{ or } 3 \ .$$

Proof.

From propositions (3.3) and (3.6) it follows that

$$(1.3.5) \qquad r^j_K(A;B,C) \geq \frac{1}{\sup_\omega ||G(i\omega)||} \ .$$

In order to prove the exactness of the estimate (1.3.5) in the case $K = \mathbb{C}$ it is demonstrated that for all $\varepsilon > 0$ there exists $D \in L(Y,U)$, $x \in X$ with $||D|| < \frac{1}{\sup_\omega ||G(i\omega)||} + \varepsilon$, such that the corresponding solution $x^D(\cdot,x)$ of (P) is not exponentially decreasing.

Let $\omega_0 \in \mathbb{R}$ and $\delta > 0$ be such that

$$||G(i\omega_0)|| \geq \sup_{\omega}||G(i\omega)|| - \delta/2 > 0 .$$

For this ω_0, choose $u \in U$, $||u|| = 1$ such that

$$||G(i\omega_0)u||_Y \geq ||G(i\omega_0)|| - \delta/2 .$$

Set $G(i\omega_0)u = ||G(i\omega_0)u||_Y v$, $||v||_Y = 1$ and define $D^u \in L(Y,U)$ by

$$D^u y = \frac{<y,v>_Y u}{||G(i\omega_0)u||_Y} \quad , \text{ for all } y \in Y .$$

Then

$$\frac{1}{\sup_{\omega}||G(i\omega)||-\delta} \geq \frac{1}{||G(i\omega_0)u||_Y} = ||D^u|| > \frac{1}{\sup_{\omega}||G(i\omega)||}$$

and

$$C(i\omega_0 I-A)^{-1}BD^u v = \frac{C(i\omega_0 I-A)^{-1}Bu}{||C(i\omega_0 I-A)^{-1}Bu||_Y} = v .$$

Thus $\qquad x = (i\omega_0 I-A)^{-1}BD^u v \in D_{\bar{X}}(A)$ satisfies

$$(i\omega_0 I-A)x = BD^u Cx .$$

Consequently for this D^u and $x \in X$, $x(t) = e^{i\omega_0 t}x$ satisfies the differential equation (D) as in corollary (2.9). Also $x(t)$ is continuous with values in $D_{\bar{X}}(A)$ and therefore

$$\begin{aligned} x(t) - \bar{S}(t)x &= \int_0^t \frac{d}{ds}(\bar{S}(t-s)x(s))ds \\ &= \int_0^t \bar{S}(t-s)(i\omega-A)x(s)ds \\ &= \int_0^t \bar{S}(t-s)BDCx(s)ds \quad . \end{aligned}$$

Hence $x(t) = e^{i\omega_0 t}x$ is a solution of (P) for $D = D^u$. Now given $\varepsilon > 0$ choose $\delta > 0$, sufficiently small such that

$$||D|| < \frac{1}{\sup_\omega ||G(i\omega)||} + \varepsilon \qquad \text{and hence}$$

$$r^j_{\mathbb{C}}(A;B,C) = \frac{1}{\sup_\omega ||G(i\omega)||} \quad , \quad j = 1,2,3 \ . \qquad \square$$

Remark (3.8)

In fact contained within the proof of this theorem is the following result concerning the assumption $\sup_\omega ||G(i\omega)|| < \infty$.

Corollary (3.9)

Assume conditions (A1)-(A3) hold. If (A4) holds then $\sup_\omega ||G(i\omega)|| < \infty$.

Proof.

Suppose $\sup_\omega ||G(i\omega)|| = \infty$ then given any $R > 0$ there exists $\omega_0 \in \mathbb{R}$, $u \in U$, $||u|| = 1$ such that

$$||G(i\omega_0)u||_Y \geq R \ . -$$

For this ω_0 and u let D^u and x be constructed as in theorem (3.5). Then $Cx \neq 0$

$$||D^u|| \le \frac{1}{R} \quad \text{and} \quad y(t) = e^{i\omega_0 t} Cx$$

is a solution of equation (Y), but $y(\cdot) \notin L^2_Y$. However for R sufficiently large $||LD|| < 1$ and therefore equation (Y) has a unique solution $y(\cdot) \in L^2_Y$. This leads to a contradiction and therefore $\sup\limits_{\omega} ||G(i\omega)|| < \infty$. □

The results of this section allow analysis of the robustness of various infinite dimensional systems. In general, however, calculation of this robustness margin or stability radius is very difficult (even for the case X finite dimensional). In the final section of this chapter, calculation of this stability radius is carried out for some simple examples.

§4. Applications.

Example (4.1)

Consider again example (0.8) of section 1, where $X = \mathbb{C}^n \times L^2(-1,0;\mathbb{C}^n)$ and A is given by

$$(Ah)(\theta) = \begin{cases} A_0 h(0) + A_1 h(-1) & \theta = 0 \\ \frac{dh}{d\theta} & \theta \in [-1,0) \end{cases}$$

$$D(A) = H^1(-1,0;\mathbb{C}^n)$$

This example illustrates perfectly all the various difficulties involving the structure and unboundedness of the perturbation term and the various assumptions required in obtaining the results of sections 2 and 3. Here $B = [{I \atop 0}]$ and $C = [0,E_{-\alpha}]$, where $E_{-\alpha}h = h(-\alpha)$, result in $\bar{X} = X$ but $\underline{X} = \mathbb{C}^n \times H^{\frac{1}{2}+\varepsilon}(-1,0;\mathbb{C}^n)\frac{1}{2} > \varepsilon > 0$

If all the roots of $\mathrm{Det}(\lambda I-A_0-e^{-\lambda}A_1) = 0$ lie in the right half plane then it is easy to verify conditions (A1)-(A6)(i) and (iii). For example, (A6)(i) requires

$$||CS(\cdot)x||_{L^2_Y} \leq c||x||_X \qquad \text{for all}$$

$$x = (z_0,z_\tau) \in \mathbb{C}^n \times H^{\frac{1}{2}+\varepsilon}(-1,0;\mathbb{C}^n) \quad .$$

Now $$||CS(\cdot)x||^2_{L^2_Y} = \int_0^\infty ||z(t-\alpha)||^2_{\mathbb{C}^n}dt$$

where $z(\cdot)$ is the solution of

$$\dot{z}(t) = A_0z(t) + A_1z(t-1) \quad z(\tau) = z_\tau \quad \tau \in [-1,0] \quad .$$

Taking Fourier-Plancherel transforms yields

$$\int_0^\infty ||z(t-\alpha)||^2_{\mathbb{C}^n}dt \leq \frac{1}{2\pi}\int_{-\infty}^\infty ||\hat{z}(i\omega)||^2_{\mathbb{C}^n}d\omega + \int_{-1}^0 ||z(t)||^2_{\mathbb{C}^n}dt$$

where $\hat{z}(i\omega) = (i\omega-A_0-A_1e^{-i\omega})^{-1}(z_0) + A_1\int_{-1}^0 e^{-i\omega(\tau+1)}z(\tau)d\tau)$

and therefore using the condition on $\mathrm{Det}(\lambda I-A_0-e^{-\lambda}A_1)$

$$||CS(\cdot)x||_{L^2_Y} \leq c||x||_X \quad \text{for all} \quad x \in \underline{X} \quad .$$

Clearly abstraction of the perturbed equation

$$\dot{z}(t) = A_0 z(t) + Pz(t-\alpha) + A_1 z(t-1) \quad t > 0$$

results in a unique semigroup on X. (See Curtain and Pritchard [21]). This semigroup can be obtained via the construction of section 2.

Let $u \in \mathbb{C}^n$ then

$$(i\omega I-A)^{-1}\binom{I}{0}u = \binom{h}{h(\theta)}$$

if

$$u = i\omega h(0) - A_0 h(0) - A_1 h(-1)$$

$$0 = i\omega h(\theta) - \frac{dh}{d\theta}(\theta) .$$

Hence

$$h(\theta) = e^{i\omega\theta}h(0) = e^{i\omega\theta}[i\omega I-A_0-A_1 e^{-i\omega}]^{-1}u .$$

Therefore

$$C(i\omega I-A)^{-1}Bu = e^{-i\omega\alpha}[i\omega I-A_0-A_1 e^{-i\omega}]^{-1}u$$

and

$$r_{\mathbb{C}}^j(A;B,C) = \inf_{\omega} \underline{\sigma}[i\omega I-A_0-A_1 e^{-i\omega}] \quad , \quad j = 1 \text{ and } 3 .$$

Therefore the system

$$\dot{z}(t) = A_0 z(t) + Pz(t-\alpha) + A_1 z(t-1), \; t > 0$$

is exponentially stable providing that $||P|| < r_{\mathbb{C}}(A;B,C)$. Note, however, the surprising result that the bound on P is independent of the delay term α ! The reason for this is that complex perturbations are allowed in the framework. If the perturbation P is restricted to $\mathbb{R}^{n\times n}$ then clearly the bound would depend on α. This complex stability radius does give a (lower) bound on the size of allowable $P \in \mathbb{R}^{n\times n}$ when the delay term α is unknown.

Example (4.2)

Consider the nominal, second order partial differential equation

$$(1.4.1)\quad \begin{cases} z_{tt} + \nu z_t - z_{\xi\xi} = 0 \ , \quad \nu > 0 \\ \\ z(0,t) = z(1,t) = 0 \quad z(\xi,0) = z(\xi) \end{cases} .$$

If $X = H_0^1(0,1) \times L^2(0,1)$, $Ax = \begin{bmatrix} 0 & I \\ \Delta & -\nu \end{bmatrix}$, $\Delta z = z_{\xi\xi}$, $D(A) = H_0^1(0,1) \cap H^2(0,1) \times H_0^1(0,1)$, then this partial differential equation may be reformulated as an abstract differential equation

$$\dot{x}(t) = Ax(t) \ , \quad x(0) = x_0 \ .$$

Suppose that robustness of this system to perturbations in the damping term of the form $(Dz_t(\cdot))(\xi)$, is required. That is, a perturbed system

$$(1.4.2)\quad \begin{cases} z_{tt} + \nu z_t - (Dz_t(\cdot))(\xi) - z_{\xi\xi} = 0 \ , \quad \nu > 0 \\ \\ z(0,t) = z(1,t) = 0 \quad z(\xi,0) = z(\xi) \end{cases} .$$

Such a perturbation of (1.4.1) results in an abstract structured perturbation BDC with $B = \binom{0}{I}$, $C = (0,I)$, $D \in \mathcal{L}(L^2(0,1))$ and $\underline{X} = \bar{X} = X$, $U = Y = L^2(0,1)$.

Since $\nu > 0$ and B and C are bounded it is easy to check that assumptions (A1)-(A6) are satisfied. Let $\phi_n(\xi) = \sqrt{2} \sin n\pi\xi$ $n = 1,2,\dots$ and

$$x = \begin{bmatrix} \Sigma x_n^1 \phi_n \\ \Sigma x_n^2 \phi_n \end{bmatrix},$$

then an equivalent norm on $H_0^1(0,1) \times L^2(0,1)$ is given by

$$||x||_X^2 = \Sigma n^2\pi^2(x_n^1)^2 + (x_n^2)^2 .$$

Let $u \in L^2(0,1)$ with $u = \Sigma u_n\phi_n$, $\Sigma\ |u_n|^2 < \infty$.
Then

$$(i\omega-A)^{-1}\binom{0}{I}\Sigma u_n\phi_n = \begin{bmatrix} \Sigma h_n^1\phi_n \\ \Sigma h_n^2\phi_n \end{bmatrix} \qquad \text{if}$$

$$0 = \Sigma(i\omega h_n^1 - h_n^2)\phi_n$$

$$\Sigma u_n\phi_n = \Sigma\,(+\ n^2\pi^2 h_n^1 + (i\omega+\nu)h_n^2)\phi_n \quad .$$

So $\quad h_n^2 = i\omega h_n^1, u_n = (+\ n^2\pi^2 - \omega^2 + i\omega\nu)h_n^1$, $n = 1,2 \dots$.

Thus

$$C(i\omega I-A)^{-1}Bu = \Sigma h_n^2\phi_n = \Sigma \left(\frac{i\omega}{+n^2\pi^2 - \omega^2 + i\omega\nu}\right) u_n\phi_n \quad .$$

This yields

$$r_{\mathbb{C}}^{j}(A;B,C) = \inf_{\omega}\ \inf_{n}\left(\frac{(n^2\pi^2-\omega^2)^2 + \omega^2\nu^2}{\omega^2} \right)^{\frac{1}{2}}$$

$= \nu \ , \quad j = 1,2,3 \ .$

It must be emphasised, however, that this is the exact bound for all $D \in L(L^2(0,1))$ and not just the linear operator defined via scalar multiplication.

It is also of interest to compute $r_{\mathbb{C}}(A;I,I)$ for which one obtains $r_{\mathbb{C}}(A;I,I) \sim \nu/2$ as $\nu \to 0$ and $r_{\mathbb{C}}(A;I,I) \sim \pi^2/\nu$ as $\nu \to \infty$. These results are again of some surprise since as $\nu \to \infty$ the decay rate of the semigroup tends to infinity. The reason for this result is that the unstructured case means that both 0 and I are perturbed in $\begin{bmatrix} 0 & I \\ \Delta & -\nu \end{bmatrix}$.

Example (4.3) - Case study I.

Let $(S(t))_{t\geq 0}$ be the semigroup of example (0.9) defined on $X = L^2(0,1)$ with generator A , $Ah = h_{\xi\xi}$, $D(A) = \{h \in H^2 | h_\xi(0) = h(1) = 0\}$. Of importance in performing the calculations are the eigenvalues $\lambda_n = -(\frac{2n+1}{2})^2\pi^2$ and eigenvectors $\phi_n(\xi) = \sqrt{2}\cos(\frac{2n+1}{2})\pi\xi$, $n = 0,1,\ldots.$ of A .

Case (i) Let $B = C = I_{L^2(0,1)}$ and $D \in L(L^2(0,1))$ then the abstract operator $A + D$ corresponds to the partial differential equation

$$(1.4.3) \quad \begin{cases} z_t = z_{\xi\xi} + (Dz)(\cdot) \\ \\ z_\xi(0,t) = z(1,t) = 0 \end{cases} .$$

Clearly, since B and C are bounded, assumptions (A1)-(A6) hold.

Let $x = \Sigma x_n \phi_n$, $\Sigma x_n^2 < \infty$ then

$$(i\omega - A)^{-1} x = \Sigma (i\omega - \lambda_n)^{-1} x_n \phi_n \; .$$

Therefore

$$\begin{aligned} r_{\mathbb{C}}^{j}(A,I,I) &= \frac{1}{\sup\limits_{\omega} \|(i\omega - A)^{-1}\|} \\ &= |\lambda_1| \\ &= \pi^2/4 \; , \quad j = 1,2,3 \; . \end{aligned}$$

Again this is the expected result for the case scalar multiplication operators but it must be emphasised that it is also the case for all $D \in \mathcal{L}(L^2(0,1))$.

<u>Case (ii)</u> Let $C \in \mathcal{L}(L^2(0,1),\mathbb{C})$ be given by

$$Cx = \langle x,c \rangle_{L^2(0,1)}$$

for some $c \in L^2(0,1)$ and B be given by $Bu = \delta_0 u$ where $u \in \mathbb{C}$ and δ_0 is the Dirac-delta function at 0 . Then $B \in \mathcal{L}(\mathbb{C}, H^{-\frac{1}{2}-\varepsilon})$ $\frac{1}{2} > \varepsilon > 0$ where $H^s = \{x = \Sigma x_n \phi_n$ such that $\Sigma |\lambda_n|^s x_n^2\}$ with norm $\|x\|_{H^s}^2 = \Sigma |\lambda_n|^s x_n^2$. Note $H^0 = L^2(0,1)$. The abstract structure operators result in a perturbed system

$$(1.4.4) \qquad \begin{cases} z_t = z_{\xi\xi} \\ z_\xi(0,t) = d\int_0^1 c(\sigma) z(\sigma,t) d\sigma \; , \quad c(\cdot) \in L^2(0,1) \\ z(1,t) = 0 \; . \end{cases}$$

Since C is bounded it is easy to check assumptions (A1)-(A3) and (A6)(i)(ii). Now

$$||\int_0^{\cdot}\bar{S}(\cdot-s)Bu(s)ds||^2_{L^2_X} \leq 2\ \Sigma \frac{1}{|\lambda_n|^2}||u(\cdot)||^2_{L^2(0,\infty)}$$

from which (A4) follows. (A5) requires that

$$||\int_0^t \sqrt{2}\ \Sigma e^{\lambda_n s}\phi_n u(s)ds||^2_{L^2(0,1)} \leq k^2(t)||u(\cdot)||^2_{L^2(0,t)} \quad .$$

But the left hand side is equal to

$$\Sigma \frac{2}{|\lambda_n|^\alpha} \ (\int_0^t e^{\lambda_n s}|\lambda_n|^{\alpha/2}u(s)ds)^2 \ , \quad \alpha \geq 0 \ .$$

Now $\sup_n e^{\lambda_n s}|\lambda_n|^{\alpha/2} \leq e^{-\alpha/2}(\frac{\alpha}{2s})^{\alpha/2}$. Therefore

$$||\int_0^t \sqrt{2}\ \Sigma e^{\lambda_n s}\phi_n u(s)ds||^2_{L^2(0,1)}$$

$$\leq \ e^{-\alpha}(\frac{\alpha}{2})^\alpha(\int_0^t s^{-\alpha}ds)(\int_0^t|u(s)|^2ds)\ \Sigma \frac{1}{|\lambda_n|^\alpha} \quad .$$

Any $\alpha \in (\frac{1}{2}\ , 1)$ yields (A5).

Now for all $x \in L^2(0,1)$,

$$||CS(\cdot)x||^2_{L^2(0,\infty)} = \int_0^\infty \Sigma\ e^{2\lambda_n t}<x,\phi_n>^2<\phi_n,c>^2dt$$

$$= \int_0^\infty \Sigma\ e^{2\lambda_n t}\lambda_n^{\frac{1}{2}+\varepsilon} \frac{<x,\phi_n>^2}{|\lambda_n|^{\frac{1}{2}+\varepsilon}} \quad <\phi_n,c>^2dt$$

$$\leq k^2||x||^2_{H^{-\frac{1}{2}-\varepsilon}} \quad , \quad k > 0 \ .$$

Therefore the system is well posed in the sense of definition (2.4). In order to obtain a specific formula for the associated stability radii choose $c(\xi) = \sqrt{2} \cos \pi/2\xi$. Then it is easy to show that

$$C(i\omega-A)^{-1}Bu = \sqrt{2}\ (i\omega-\lambda_1)^{-1}u\ , \quad \text{for all} \quad u \in \mathbb{C}\ ,$$

and for $j = 1,2,3$

$$r^j_{\mathbb{C}}(A;B,C) = \frac{\pi^2}{4\sqrt{2}} \quad .$$

Therefore the system

$$\begin{cases} z_t = z_{\xi\xi} \\ z_\xi(0,t) = \sqrt{2}\ d\int_0^1 \cos \pi\ \xi/2\ z(\xi,t)d\xi\ , \quad z(1,t) = 0 \end{cases}$$

is exponentially stable if $|d| < \pi^2/4\sqrt{2}$.

Case (iii). Let $Bu = \delta_0 u$, as in case (ii) but now let $Cz = z(\xi_1)$, $0 \le \xi_1 \le 1$. With $\bar{X} = H^{-\frac{1}{2}-\varepsilon}$ as in case (ii) and $\underline{X} = H^{\frac{1}{2}+\varepsilon}$, $0 < \varepsilon < \frac{1}{2}$ $B \in L(\mathbb{C},H^{-\frac{1}{2}-\varepsilon})$ $C \in L(H^{\frac{1}{2}+\varepsilon},\mathbb{C}\ )$ and $D = d \in \mathbb{C}$. The abstract perturbation operator $A + BDC$ corresponds to the mixed boundary value problem

$$(1.4.5) \quad \begin{cases} z_t = z_{\xi\xi} \\ \\ z_\xi(0,t) = dz(\xi_1,t), \quad z(1,t) = 0 \end{cases} \quad .$$

In fact for the spaces $\underline{X}$ and $\bar{X}$ the semigroup, $(S(t))_{t\geq 0}$, satisfies the following properties:- there exists $\alpha > 0$ such that

$$(1.4.6) \qquad ||S(t)x||_{\underline{X}} \leq \frac{Me^{-\alpha t}}{t^{\beta}} ||x||_X \quad , \quad t > 0$$

and

$$(1.4.7) \qquad ||\bar{S}(t)x||_X \leq \frac{Me^{-\alpha t}}{t^{\gamma}} ||x||_{\bar{X}} \quad , \quad t > 0$$

and $\gamma = \beta = 1/4 + \varepsilon/2$. To see (1.4.7) let $x = \Sigma\, x_n\phi_n$ then

$$\begin{aligned} ||S(t)x||_X^2 &= \Sigma |x_n|^2 e^{2\lambda_n t} \\ &= \Sigma \frac{|x_n|^2}{|\lambda_n|^{\mu}} |\lambda_n|^{\mu} e^{2\lambda_n t} \quad , \quad \mu = \tfrac{1}{2} + \varepsilon \\ &\leq \sup_n \left\{ e^{2\lambda_n t} |\lambda_n|^{\mu} \right\} \Sigma \frac{|x_n|^2}{|\lambda_n|^{\mu}} \\ &\leq e^{-\mu} \left(\frac{\mu}{2t}\right)^{\mu} \Sigma \frac{x_n^2}{|\lambda_n|^{\mu}} \\ &= e^{-(\frac{1}{2}+\varepsilon)} \left(\frac{\frac{1}{2}+\varepsilon}{2t}\right)^{\frac{1}{2}+\varepsilon} ||x||^2_{H^{-\frac{1}{2}-\varepsilon}} \quad , \end{aligned}$$

and (1.4.7) follows using the exponential stability of $S(t)$.

It is easy to show that for systems satisfying (1.4.6) and (1.4.7) assumptions (A4)-(A6)(i) require

(A4) $\quad \beta + \gamma < 1$

(A5) $\quad \beta < \frac{1}{2}$

(A6)(i) $\quad \gamma < \frac{1}{2}$

and (A6)(iii) $\beta + \gamma < \frac{1}{2}$. Therefore assumptions (A1)-(A5)(A6)(i) hold for $0 < \varepsilon < \frac{1}{2}$. Also

$$C(i\omega-A)^{-1}B = \sqrt{2}\,\Sigma(i\omega-\lambda_n)^{-1}\phi_n(\xi_1)$$

and therefore

$$r_{\mathbb{C}}^1(A,B,C) = \frac{1}{\sqrt{2}\,\Sigma\,\frac{|\phi_n(\xi_1)|}{|\lambda_n|}} \quad .$$

In particular if $\xi_1 = 0$, $r_{\mathbb{C}}^1(A;B,C) = 1$. So the perturbed system (1.4.5)

$$\begin{cases} z_t = z_{\xi\xi} \\ z_\xi(0,t) = dz(0,t) \ , \quad z(1,t) = 0 \end{cases}$$

will be exponentially stable providing $|d| < 1$.

Case (iii) provides a classic example when the study of the mild solution of (P) provides the unique solution of an associated problem, in this case the weak solution of a parabolic equation with mixed boundary values.

This case study illustrates well the interplay between the smoothing action of the semigroup and the structure and unboundedness of the perturbed operator. However all the systems studied in example (4.1)-(4.3) are known a priori to be well posed in some sense. In order to illustrate how the theorems of section 2 and 3 can be used to study both the problems of well-posedness and stability of the perturbed system and also how (A8) relates to the uniqueness problem for solutions of the perturbed system (P) consider a second case study.

Example (4.4) Case study II

Let $\{\phi_n\}_{n=1}^{\infty}$ be a complete orthonormal basis for a real Hilbert space X , and $\{\lambda_n\}_{n=1}^{\infty}$ a set of real numbers with $\ldots\lambda_n<\ldots<\lambda_1 < 0$. Then

$$(1.4.8) \qquad S(t)x = \Sigma\ e^{\lambda_n t}<x,\phi_n>\phi_n \qquad \text{for} \quad x \in X$$

defines a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X with generator

$$(1.4.9) \begin{cases} Ax = \Sigma \quad \lambda_n<x,\phi_n>\phi_n \\ \\ D(A) = \{x \in X : \Sigma\ \lambda_n^2<x,\phi_n>^2 < \infty\} \end{cases} .$$

It is easy to see that $r_{\mathbb{C}}(A;I,I) = |\lambda_1|$.

Consider now the following (unbounded) structure operators:

$$(1.4.10) \qquad Bu = \Sigma\ b_n<u,\phi_n>\phi_n \qquad b_n \neq 0 \qquad n = 1,2 \ldots$$

$$(1.4.11) \qquad Cx = \Sigma\ c_n<x,\phi_n>\phi_n \qquad c_n \neq 0 \qquad n = 1,2 \ldots \quad .$$

Then if

$$\bar{X} = \{x = \Sigma x_n\phi_n \ , \ \Sigma\beta_n^2x_n^2 < \infty \quad \beta_n \neq 0\} \ , \ ||x||_{\bar{X}}^2 = \Sigma\beta_n^2x_n^2$$

$$\underline{X} = \{\underline{x} = \Sigma \underline{x}_n\phi_n \ , \ \Sigma\gamma_n^2\underline{x}_n^2 < \infty \quad \gamma_n \neq 0\} \ , \ ||\underline{x}||_{\underline{X}}^2 = \Sigma\gamma_n^2\underline{x}_n ,$$

$B \in L(X,\bar{X})$, $C \in L(\underline{X},X)$, providing that

$$\beta_n^2 b_n^2 \le K \quad \text{for large } n$$

$$c_n^2 \le K\gamma_n^2 \quad \text{for large } n$$

for some generic constant K .

It is easy to show that assumptions (A1)-(A6)(i)(iii) are valid if

(A1) $\gamma_n^2 \ge K \ge \beta_n^2$ for large n ,

(A2) $\gamma_n^2 \le K\beta_n^2|\lambda_n|^2$ for large n ,

(A4) $b_n^2\gamma_n^2 \le K|\lambda_n|^2$ for large n ,

(A5) $b_n^2 \le K|\lambda_n|$ for large n ,

(A6)(i) $c_n^2 \le K|\lambda_n|$ for large n ,

(A6)(iii) $c_n^2 \le K\beta_n^2|\lambda_n|$ for large n .

Consider now assumption (A8) and for simplicity assume that $D = dI_X$ where I_X is the identity map on X . Let $x(\cdot) = \Sigma x_n(\cdot)\phi_n \in L^2(0,T;\underline{X})$ then

$$(LD)(Cx(\cdot))(t) = d\int_0^t \sum_{n=1}^{\infty} c_n^2 b_n \, e^{-\lambda_n(t-s)} x_n(s)\phi_n ds$$

and (A7) follows providing that $c_n^2|b_n| \le k|\lambda_n|$, for n large. In general (A8) follows for some $r \in \mathbb{N}_0$ if $|c_n|^{r+1}|b_n|^r \le k|\lambda_n|^r$, for n large. Therefore if $|c_n b_n| < k|\lambda_n|$ then $|c_n|^{r+1}|b_n|^r \le k|\lambda_n|^r$ for

r sufficiently large. Thus if $|b_n c_n| \leq k|\lambda_n|$

$$r^1_{\mathbb{C}}(A;B,C) = \inf_n \frac{|\lambda_n|}{|b_n c_n|} \quad \text{and if } |b_n c_n| < k|\lambda_n|$$

then the system is well posed as a dynamical system in the sense of remark (2.8).

In fact if $c_n^2 < k|\lambda_n|$, $b_n^2 < k|\lambda_n|$ and $\beta_n = \frac{1}{|b_n|}$, $\gamma_n = |c_n|$ then the system (P) is well posed as a dynamical system on X .

The conditions $c_n^2 \leq k|\lambda_n|$, $b_n^2 < k|\lambda_n|$ imply that B and C are of the order $(-A)^{\frac{1}{2}-\varepsilon}$ for $\varepsilon > 0$. However the condition $c_n^2 \leq k\ \beta_n^2|\lambda_n|$, required for assumption (A6)(iii) restricts the unboundedness to be that of $(-A)^{\frac{1}{2}}$.

Note that in the calculation of $r^1_{\mathbb{C}}(A;B,C)$ the $\sup_\omega ||G(i\omega)||$ occurs at $\omega = 0$ and therefore this stability radius for complex perturbations is also the stability radius for real perturbations.

Example (4.5) Case study III.

Let $\{\lambda_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$ be a set of complex numbers with $\lambda_n = \lambda_n^R + i\lambda_n^I$ and $\lambda_n^R \leq -\nu$, $\nu > 0$ and $n = 1,2,\ldots$. Assume $x_n = x_n^R + ix_n^I$ is such that $\{x_n^R, x_n^I\}$ is linear independent set, spanning a vector space X . For $x = \Sigma\alpha_n x_n^R + \beta_n x_n^I$ define

$$||x||^2_{X_\delta} = \Sigma(\alpha_n^2 + \beta_n^2)\delta_n^2, \ \delta_n \neq 0$$

and denote by X_δ the Banach space obtained by taking the closure of X

with respect to this norm. For $x = \Sigma\alpha_n x_n^R + \beta_n x_n^I$ set

$$(1.4.12) \quad S(t)x = \Sigma e^{\lambda_n t}(\cos\lambda_n^I t[\alpha_n x_n^R + \beta_n x_n^I] + \sin\lambda_n^I t[\beta_n x_n^R - \alpha_n x_n^I]) \ .$$

Clearly $(S(t))_{t\geq 0}$ defines a strongly continuous semigroup on X_1 with $\delta_n = 1$ for all n and

$$||S(t)|| \leq e^{-\nu t} \ , \quad t \geq 0 \ .$$

If A is the generator of $S(t)$, then

$$(1.4.13) \quad \begin{aligned} Ax &= \Sigma\lambda_n^R(\alpha_n x_n^R + \beta_n x_n^I) + \lambda_n^I(\beta_n x_n^R - \alpha_n x_n^I) \\ D(A) &= \{x \in X_1 , \ x = \Sigma\alpha_n x_n^R + \beta_n x_n^I \ , \ \Sigma|\lambda_n|^2(\alpha_n^2 + \beta_n^2) < \infty\} \ . \end{aligned}$$

Also

$$A[x_n^R x_n^I] = [x_n^R x_n^I]\begin{bmatrix} \lambda_n^R & \lambda_n^I \\ -\lambda_n^I & \lambda_n^R \end{bmatrix} \quad \text{and} \quad x_n \ , \ \bar{x}_n$$

and $\lambda_n, \bar{\lambda}_n$, $n = 1,2,\ldots$ are the eigenvectors and eigenvalues of A. It is easy to show that for $x = \Sigma\alpha_n x_n^R + \beta_n x_n^I$

$$(i\omega I - A)^{-1}x = \Sigma \frac{((i\omega - \lambda_n^R)\alpha_n + \lambda_n^I\beta_n)x_n^R + ((i\omega - \lambda_n^R)\beta_n - \alpha_n\lambda_n^I)x_n^I}{((i\omega - \lambda_n^R)^2 + (\lambda_n^I)^2)} \ .$$

Hence

$$||(i\omega I - A)^{-1}||^2 = \sup_n \frac{\omega^2 + |\lambda_n|^2}{((\omega^2 + |\lambda_n|^2)^2 - 4(\lambda_n^I)^2\omega^2)}$$

and therefore

$$r_{\mathbb{C}}^{j}(A;I,I) = \inf_{\omega}\inf_{n}\left(\frac{(\omega^2 + |\lambda_n|^2)^2 - 4(\lambda_n^I)^2\omega^2}{\omega^2 + |\lambda_n|^2}\right)^{\frac{1}{2}} , \quad j = 1,2,3 .$$

One can show, viewing this minimisation problem as an infinite family of 2×2 problems, that $r_{\mathbb{C}}^{j}(A;I,I)$ is the minimum of

$$\inf_{n\in\Lambda}\{|\lambda_n|\} \quad \text{and} \quad \inf_{n\in\mathbb{N}\setminus\Lambda}\{(4|\lambda_n^I|(|\lambda_n|-|\lambda_n^I|))^{\frac{1}{2}}\}$$

where $\quad \Lambda = \{n \in \mathbb{N} \mid |\lambda_n|^2 \geq 4\lambda_n^{I^2}\}$.

This gives a relationship between the eigenvalues λ_n of A and the robustness of the nominal system with respect to bounded perturbation.

For example $\lambda_n^R = -n^2\pi^2$, $\lambda_n^I = 0$ $r_{\mathbb{C}}(A;I,I) = \pi^2$. (However, this construction is artificial since it concerns complex perturbations of a real operator A whose spectrum is complex.) To examine the effect of unbounded perturbations suppose $C = I$ and for $x = \Sigma(\alpha_n x_n^R + \beta_n x_n^I)$

$$\text{(U)} \quad Bx = \Sigma\, \gamma_n(\alpha_n x_n^R + \beta_n x_n^I) \quad \gamma_n \neq 0 \ , \ B \in \mathcal{L}(X_1, X_\delta),\ \gamma_n^2\delta_n^2 \leq k \ .$$

It is easy to show that

$$r_{\mathbb{C}}^{j}(A;B,I) = \inf_{\omega}\inf_{n}\left(\frac{(|\lambda_n|^2 + \omega^2)^2 - 4(\lambda_n^I)^2\omega^2}{\gamma_n^2(\omega^2 + |\lambda_n|^2)}\right)^{\frac{1}{2}}, \quad j = 1,2$$

which results in $r_{\mathbb{C}}^{j}(A;B,I)$ being the minimum of

$$\inf_{n\in\Lambda}\{\frac{|\lambda_n|}{|\lambda_n|}\} \quad \text{and} \quad \inf_{n\in\mathbb{N}\setminus\Lambda}\{\frac{(4|\lambda_n^I|(|\lambda_n| - |\lambda_n^I|))^{\frac{1}{2}}}{|\gamma_n|}\} .$$

If $|\lambda_n|^2 \geq 4(\lambda_n^I)^2$ $n \in \mathbb{N}$ then the system withstands unboundedness so that $|\gamma_n| \leq k|\lambda_n^R|$. If however $|\lambda_n|^2 \leq 4(\lambda_n^I)^2$ one can show that

$$\sqrt{\frac{4}{3}}\ \frac{|\lambda_n^R|}{|\gamma_n|} < r_{\mathbb{C}}(A;B,I) < \sqrt{2}\ \frac{|\lambda_n^R|}{|\gamma_n|}$$

and therefore $|\gamma_n| \sim |\lambda_n^R|$. This illustrates how the distribution of the spectrum restricts the allowable unboundedness of the perturbation. This framework allows analysis of the robustness of some stabilisation procedures for hyperbolic systems. It is well known that a certain degree of unboundedness in the feedback is required to stabilise such systems.

Example (4.6)

Consider

$$(1.4.14)\ \begin{cases} z_{tt} = z_{\xi\xi} \\ z_\xi(0,t) = fz_t(0,t)\ ,\quad z(1,t) = 0 \quad f > 1\ . \end{cases}$$

It is easy to show that the eigenvalues and eigenvectors of the corresponding first order (in time) problem

$$(1.4.15)\ \begin{cases} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_t = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ z_{1_\xi}(0,t) = fz_2(0,t)\ ,\quad z_1(1,t) = 0 \end{cases}$$

are respectively $\lambda_n = -\nu+n\pi i, \bar{\lambda}_n, -\nu = \frac{1}{2}\log\frac{f-1}{f+1}$ and

$$x_n = \begin{pmatrix} (\frac{1+f}{1-f} e^{-\nu\xi} + e^{\nu\xi})(\cos n\pi\xi \ + \ i\sin n\pi\xi) \\ \\ \lambda_n(\frac{1+f}{1-f} e^{-\nu\xi} + e^{\nu\xi})(\cos n\pi\xi \ + \ i\sin n\pi\xi) \end{pmatrix} \text{ and } \bar{x}_n$$

for $n = 0,1,2 \ldots$. Using the analysis of case study III it is easy to show that $r_{\mathbb{C}}(A;I,I) = \nu$. This agrees with the observation by Chen [39] that positive boundary damping will dominate over negative distributed damping. However the system does not withstand any unbounded perturbation of type (U) , since $\gamma_n \leq k|\lambda_n^R|$ for large n . Unfortunately representation, via (U), of particular B operators, respecting the second order structure is very difficult, and therefore it is not possible to say whether the feedback system (1.4.14) is robust to perturbations of, say, the boundary data. The analysis does, however, indicate that care must be taken in the design of feedback controllers for undamped, hyperbolic systems, for example large flexible space structures. (See Chen [38], [39] for a study of the boundary stabilisation of wave equations and Datko et al [37] for a study of the effect of time delays in such stabilisation procedures.)

§5. Conclusions.

The results of sections 2and 3 allow the analysis of the exact robustness of infinite dimensional systems to unbounded/structured perturbations. The robustness characterisation turns out to be the same, independent of the degree of unboundedness in the perturbation term, and only the interpretation of this measure changes. The interpretation depends upon whether the abstract equation (P) a) represents some system known a priori to be well posed, b) yields solutions of the abstract differential equation (D) or else c) has meaning only as a dynamical system on X . In all of these cases the major problem is to prove that solutions constructed via the auxiliary equation (Y) are unique. This leads to various types of assumption. If the system is studied only as a dynamical system on X then the " zero order" assumption (A6)(ii) is imposed. This assumption, however, turns out to be quite restrictive. The "r^{th} order" assumption (A8) recovers the high degree of unboundedness allowed within the framework and this is illustrated by the case study II in section 4.

The reason why the robustness measures are characterised by the same formula is because, in all three cases, the unique solution is constructed via the same auxiliary (Y) equation. The stability radius introduced in this chapter could be termed a stability radius for the solution of the (Y) equation on L^2_Y . This concept is considered in detail in Chapter 3 where an abstract version of equation (Y) is used to study more general systems governed by evolution or resolvent operators.

All the analysis in this chapter with the exception of example (4.4) is carried out for the case $K = \mathbb{C}$. This limitation is best illustrated by example (4.1) and indicates a need for a real stability radius for infinite dimensional systems. Another area for possible future research, arising also in the study of the robustness of a simple delay equation, is as follows. Let the nominal system be given as in example (4.1) by

$$\dot{z}(t) = A_0 z(t) + A_1 z(t-1) , \quad t > 0$$

and consider instead the perturbed system

$$\dot{z}(t) = A_0 z(t) + \sum_{i=1}^{\ell} P_i z(t-\alpha_i) + A_1 z(t-1), \quad t > 0$$

where $0 \leq \alpha_1 \leq \ldots \leq \alpha_\ell \leq 1$. This results in the perturbed structured operator $\sum_{i=1}^{\ell} B_i D_i C_i$ with

$$B_i = [{I \atop 0}] , \quad D_i = P_i \in \mathbb{C}^{n \times n} \text{ and } C_i = [0 \quad E_{-\alpha_i}] \quad .$$

The problem would then be to determine a measure of robustness to such "multi perturbations" and then to obtain a characterisation of this measure.

Probably a more obvious area of further research is to construct feedback controllers which maximise this stability radius, hence obtaining the maximum robustness for the system. This would be particularly interesting in applications to systems of large flexible structures. In Szumko [53] this problem is studied for the case X is finite dimensional using alternative characterisations of the complex stability radius, via a certain non-standard algebraic Riccati equation. This Riccati equation and the associated linear quadratic problem is studied in the next chapter.

CHAPTER 2. The stability radius and the algebraic Riccati equation.

§0. Introduction.

In this chapter the connection between the stability radius for semigroups of operators and a certain non-standard, linear quadratic problem is developed. This connection is treated in the finite dimensional case by Hinrichsen and Pritchard in [8]. In order to generalise their theory to an infinite dimensional setting with unbounded structure operators a framework conceptually similar to the one for $r_{\mathbb{C}}^3$ of Chapter 1 is used. At a technical level the assumptions placed upon the system are more restrictive requiring the dual of A6(iii) (as is expected for the control problem) in order to obtain a strict algebraic Riccati equation (see Salamon [24] for a weaker treatment of the standard linear quadratic problem). This conceptual similarity in the frameworks required for the study of these two connected, but quite different, problems is, after some consideration, of little surprise since both problems deal with the L^2-boundedness of certain functions. This relationship is due, essentially, to the equivalence of L^p-stability (L^2-stability is used in Chapter 1 so that the norm induced on D as a map from L_Y^2 to L_U^2 is the same as that on $L(Y,U)$) and exponential stability for semigroups of operators. For a detailed account of this equivalence and generalisations to evolution operators and non-linear semigroups see Datko [17], [18] and Ichikawa [19] (also Curtain-Pritchard [21], Pritchard-Zabczyk [50]).

In section 1 it is shown how study of the structured stability radius of Chapter 1 leads, quite naturally, onto a non-standard, infinite time, linear quadratic problem. In section 2 the strengthening of the assumptions of Chapter 1 and the consequences pertaining to the current problem are detailed. The framework used is that of Pritchard and Salamon [22] in their detailed study of the standard linear quadratic problem with unbounded sensing and control. Also in section 2, the question of well-posedness of the linear quadratic problem on the finite time interval is approached. In section 3 the infinite time problem is studied, by a limiting procedure on the finite time problem. This leads to a non-standard algebraic Riccati equation. Finally in section 4 a solution of the algebraic Riccati equation is used as a Liapunov functional, to estimate regions of attraction of the origin for an unusual class of Lipschitz bounded perturbations of the nominal system. This Liapunov functional will not be a Liapunov functional in the classical sense of La Salles principle. Instead the Liapunov functional is used only to obtain a region of initial states where the evolution of the system is in L^2. This together with the mild equation results in asymptotic stability of the origin for the perturbed systems. This is unusual in that the normal approach is to obtain relatively compact orbits which would be difficult to guarantee in this infinite dimensional setting.

§1. The stability radius and a non-standard linear quadratic problem.

Let A be the generator of an exponentially stable semigroup of operators, $S(t) \in L(X)$, on a complex Hilbert space X. Assume

$B \in L(U,\bar{X})$ and $C \in L(\underline{X},Y)$ where U,Y, $\underline{X}$ and $\bar{X}$ are complex Hilbert spaces such that A1 of Chapter 1 holds. Recall that, under suitable assumptions upon A, B and C , $r^3_{\mathbb{C}}(A;B,C)$ is characterised as $\frac{1}{||L||}$ where $L:L^2_U \to L^2_Y$; $u(\cdot) \to C \int_0^{\cdot} \bar{S}(\cdot-s)Bu(s)ds$.

Consider now the following optimisation problem

$$\inf_{v \in L^2_U} J(x_0,v) \quad \text{where}$$

$$J(x_0,v) = \int_0^\infty ||v(s)||^2_U - r^2||y(s)||^2_Y ds$$

and $$y(t) = CS(t)x_0 + C\int_0^t \bar{S}(t-s)Bv(s)ds , \quad x_0 \in \bar{X} .$$

This first elementary proposition relates the functional $J(0,v)$ and the quantity $||L|| = \sup_\omega ||G(i\omega)||$ of Chapter 1.

Proposition (1.1)

Assume (A1)-(A4) of Chapter 1 hold then

$$r^2 \leq r^2_{\mathbb{C}} \text{ if and only if } \quad (i) \qquad J(0,v) \geq 0$$

$$\text{if and only if } \quad (ii) \quad I - r^2G^*(i\omega)G(i\omega) \geq 0$$

for all $\omega \in \mathbb{R}$, where $G(i\omega) = C(i\omega-A)^{-1}B$ and $r_{\mathbb{C}} = \inf_\omega \frac{1}{||G(i\omega)||}$.

Proof.

For any $v(\cdot) \in L^2_U$, $y(\cdot) \in L^2_Y$, where

$$y(t) = C\int_0^t \bar{S}(t-s)Bv(s)ds = (Lv)(t) .$$

Therefore $J\,(0,v) = ||v||^2_{L^2_U} - r^2||y(\cdot)||_{L^2_Y}$.

By proposition (3.6), Chapter 1 $||Lv||^2_Y \le \frac{1}{r^2_{\mathbb{C}}}\,||v||^2_U$. Therefore $r^2 < r^2_{\mathbb{C}}$ implies $J(0,v) \ge 0$ for all $v \in L^2_U$. Conversely if $r \ge r_{\mathbb{C}}$ then $J(0,v) < 0$ since $||L|| = \frac{1}{r_{\mathbb{C}}}$. For the second inequality, use is made of Plancherel's theorem, Yosida [54], Kappel-Kunisch [36], and the boundedness of $G(i\omega)$, to deduce that

$$J(0,v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} <\hat{v}(i\omega),\hat{v}(i\omega)>_U - r^2<G(i\omega)\hat{v}(i\omega),\ G(i\omega)\hat{v}(i\omega)>_Y\ d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} <\hat{v}(i\omega)(I - r^2G^*(i\omega)G(i\omega))\ ,\ \hat{v}(i\omega)>_U\ ,\ d\omega$$

where $\hat{v}(i\omega) = \int_0^{\infty} e^{-i\omega t}v(t)dt\ ,\ v(\cdot) \in L^2_U$.

Remark (1.2)

If all the assumptions A1-A6 (j) are imposed then proposition 1 gives an equivalence between the bound $r^j_{\mathbb{C}}$, upon perturbations retaining exponential stability and the non-negativity of $J(0,v)$ for $v(\cdot) \in L^2_U$.

Remark (1.3)

The following proposition concerns $J(x_0,v)$ and replaces the assumption in Pritchard and Salamon [22] of there existing a control function with finite cost.

Proposition (1.4)

Assume (A1)-(A4)(A6)(iii) (of Chapter 1) hold. If $r < r_{\mathbb{C}}$

then $\inf_{v \in L^2_U} J(x_0,v) > -\infty$ for all $x_0 \in \bar{X}$.

Proof.

Recall from Chapter 1 that $CS(\cdot)x_0$ extends by (A6)(iii) to a unique function in L^2_Y for all $x_0 \in \bar{X}$. By (A1)(A3)(A4) and (A6)(iii) $y(\cdot) = CS(\cdot)x_0 + (Lv)(\cdot)$ is well defined for any $x_0 \in \bar{X}$ and $v(\cdot) \in L^2_U$. Therefore

$$J(x_0,v) = J(0,v) - r^2\int_0^\infty (||CS(s)x_0||^2_Y - 2r^2 \mathrm{Re} <CS(s)x_0,(Lv)(s)>_Y)ds$$

$$\geq -r^2(1 + \frac{1}{\alpha})\int_0^\infty ||CS(s)x_0||^2_Y ds + [(r^2_{\mathbb{C}} - r^2) - \alpha]\int_0^\infty ||(Lv)(s)||^2_Y ds$$

for any $\alpha > 0$. Hence for α sufficiently small

$$J(x_0,v) \geq -r^2(1 + \frac{1}{\alpha}) \int_0^\infty ||CS(s)x_0||^2_Y ds$$

$$> -\infty \text{ by assumption (A6)(iii).} \qquad \square$$

This optimisation problem has received considerable interest Brockett [47], Willems [55] in the finite dimensional case, due to the geometry of the solution of the corresponding Riccati equation. The connection between this non standard optimisation problem and the stability radius of Chapter 1 motivates its study in this infinite dimensional framework. In fact study of such non-standard Riccati equations is relevant to the problem of robustness improvement via state space methods (see Szumko [53]).

§2. The abstract framework and the finite time problem.

In section 1 the relationship between the stability radius of Chapter 1 and a certain non-standard, infinite time linear quadratic problem was introduced. Consider now the corresponding finite time problem on $[t_0,T]$ fixed.

$$(2.2.1) \qquad \inf_{v \in L^2(t_0,T;U)} J^T_{t_0}(x_0,v) \ , \quad J^T_{t_0}(x_0,v) = \int_{t_0}^{T} ||v(s)||^2_U \ -r^2||Cx(s)||^2_Y ds$$

$$(2.2.2) \qquad x(t) = S(t)x_0 + \int_{t_0}^{t} \bar{S}(t-s)Bv(s)ds \quad , \quad t_0 \le t \le T \ , \ x_0 \in \bar{X} \ .$$

As in Chapter 1, in order to make sense of (2.2.2) when B and C are unbounded operators, conditions (A1)-(A6)(iii) are assumed to hold. In fact in the sequel the following additional assumption is required.

(A5ii) there exists $b(t_0,T) \ge 0$ such that for all $v(\cdot) \in L^2(t_0,T;U)$

$$\int_{t_0}^{T} \bar{S}(T-s)Bv(s)ds \in \underline{X} \quad \text{and}$$

$$||\int_{t_0}^{T} \bar{S}(T-s)Bv(s)ds||_{\underline{X}} \le b(t_0,T)||v(\cdot)||_{L^2(t_0,T;U)} \ .$$

Note. As in lemma (2.2) Chapter 1, $b(t_0,T)$ can be assumed to be independent of t_0,T .

Remark (2.1)

If, under the identification of X with X^* , $\underline{X}^* = \bar{X}$ then (A5)(ii) is the dual of (A6)(iii). (Salamon [51]).

Remark (2.2)

$\bar{X}^* \subseteq X \subseteq \underline{X}^*$ with continuous dense embeddings and $S^*(t)$ is a strongly continuous semigroup on $\bar{X}^*$, X , $\underline{X}^*$. See Pritchard and Salamon [22].

Remark (2.3)

Of importance in the sequel are the dual statements of (A6)(iii) and (A5)(ii).

(A6)(iii)* for all $y(\cdot) \in L^2(t_0,T;Y)$

$$||\int_{t_0}^{T} S^*(s-t_0)C^*y(s)ds||_{\bar{X}^*} \leq c||y(\cdot)||_{L^2(t_0,T;Y)}$$

and

(A5ii)* for all $x \in \bar{X}^*$ $\quad ||B^*S^*(T-\cdot)x||_{L^2(t_0,T;U)} \leq b||x||_{\underline{X}^*}$.

(See Salamon [55]).

It is necessary, in the sequel, to consider the system (2.2.2) with $v(t) = F(t)x(t)$, where $F(t) \in L(\bar{X},U)$ is strongly continuous. The following theorem summarises the results of Pritchard and Salamon [22] concerning the mild evolution operator corresponding to this feedback operator.

Theorem (2.4)

Assume (A1)-(A5)(ii)(A6)(iii) hold and let $F(t) \in L(\bar{X},U)$ be a strongly continuous family of bounded linear operators on $[t_0,T]$, then there exists a mild evolution operator $\phi_F(t,s) \in L(\bar{X})$ defined for $t_0 \leq s \leq t \leq T$ and $x_0 \in \bar{X}$ by

$$(2.2.3) \qquad \phi_F(t,s)x_0 = \bar{S}(t-s)x_0 + \int_s^t \bar{S}(t-\sigma)BF(\sigma)\phi_F(\sigma,s)x_0 d\sigma$$

satisfying

$$\text{i) } (2.2.4) \quad \phi_F(t,s)x_0 - x_0 = \int_s^t \phi_F(t,\sigma)[A + BF(\sigma)]x_0 d\sigma \ , \quad t_0 \leq s \leq t \leq T$$

for all $x \in D_{\bar{X}}(A)$ or alternatively $\phi_F(t,s)$ is strongly continuously differentiable in s on $[t_0,t]$ and

$$(2.2.5) \qquad \frac{\partial\phi_F(t,s)}{\partial s} x_0 = -\phi_F(t,s)[A + BF(s)]x_0$$

$$\text{(ii) } (2.2.6) \quad \phi_F(t,s)x_0 = \bar{S}(t-s)x_0 + \int_s^t \phi_F(t,\sigma)BF(\sigma)\bar{S}(\sigma-s)x_0 d\sigma$$

for all $t_0 \leq s \leq t \leq T$ and $x \in \bar{X}$.

Moreover if $w(\cdot) \in L^2(t_0,T;U)$ then the solution of (2.2.2) with $v(t) = F(t)x(t)+w(t)$ is given by

$$(2.2.7) \qquad x(t) = \phi_F(t,t_0)x_0 + \int_{t_0}^t \phi_F(t,\sigma)Bw(\sigma)d\sigma \ , \quad t_0 \leq t \leq T \ .$$

(iii) $\phi_F(t,s)$ is also a strongly continuous evolution operator on $\underline{X}$ and X and there exists $c'(t_0,T),b'(t_0,T) \geq 0$ such that

$$(A6)(iii) \quad ||C\phi_F(\cdot,s)||_{L^2(s,T;Y)} \leq c'(t_0,T)||x||_{\bar{X}} \ ,$$

for all $x \in \underline{X}$ and $s \in [t_0,T]$.

$$(A5)'(ii) \quad \left\|\int_{t_0}^{t} \phi_F(t,\sigma)Bu(\sigma)d\sigma\right\|_{\underline{X}} \le b'(t_0,T)\|u(.)\|_{L^2(t_0,T;U)}$$

for all $u(\cdot) \in L^2(t_0,t;U)$ and $t \in [t_0,T]$ and the dual statements

$$(A6)(iii)^{*}{}' \quad \left\|\int_{s}^{T} \phi_F^*(\tau,s)C^*y(\tau)d\tau\right\|_{\bar{X}^*} \le c'(t_0,T)\|y(\cdot)\|_{L^2(s,T;Y)}$$

for all $y(\cdot) \in L^2(t_0,T;Y)$ and $s \in [t_0,T]$.

$$(A5)^{*}{}'(ii) \quad \|B^*\phi_F^*(t,\cdot)x\|_{L^2(t_0,t;U)} \le b'(t_0,T)\|x\|_{\underline{X}^*}$$

for all $x \in \bar{X}^*$ and $t \in [t_0,T]$.

Remark (2.5)

For the details concerning parts (i) and (ii) see Curtain and Pritchard [21], Pritchard and Salamon [22].

Proposition (2.6)

Let $F(t) \in L(\bar{X},U)$ be strongly continuous for $t_0 \le t \le T$ and assume (A1)-(A4)(A5)(ii)(A6)(iii) hold, then $P_F(t) \in L(\bar{X},\bar{X}^*)$, defined for all $x \in \bar{X}$ and $t_0 \le t \le T$ by

$$(2.2.8) \qquad P_F(t)x = \int_{t}^{T} \phi_F^*(\tau,t)[F^*(\tau)F(\tau) - r^2C^*C]\phi_F(\tau,t)x d\tau \quad ,$$

is strongly continuous. Moreover, for $v(t) = F(t)x(t)$, $x_0 \in \bar{X}$,

$$(2.2.9) \qquad J_{t_0}^{T}(x_0,v) = \langle x_0 \; , \; P_F(t_0)x_0\rangle_{\bar{X},\bar{X}^*} \quad . \qquad \square$$

Remark (2.7)

As in the standard linear quadratic problem a formula like (2.2.9) determines the optimal cost.

The following two lemmas are important in the sequel when comparing the cost of two controls. Their proofs follow immediately from lemmas (2.4) and (2.5) Pritchard and Salamon [22] since these are independent of the non-negativity of the cost functional. They are included here for completeness.

Lemma (2.8) (Pritchard and Salamon [22] lemma (2.4))

Assume conditions (A1)(A2)(A4)(A5)(ii)(A6)(iii) hold, $F(t) \in \mathcal{L}(\bar{X},U)$ is strongly continuous on $[t_0,T]$, $v(\cdot) \in L^2(t_0,T;U)$ and $y(\cdot) \in L^2(t_0,T;Y)$ then

$$(2.2.10) \quad \int_{t_0}^{T} \int_{s}^{T} <C\phi_F(t,s)Bv(s),y(t)>_Y dt\, ds$$

$$= \int_{t_0}^{T} <C\int_{t_0}^{t} \phi_F(t,s)Bv(s)ds,y(t)>_Y dt ,$$

where $\phi_F(t,s) \in \mathcal{L}(\bar{X}) \cap \mathcal{L}(\underline{X})$ is the mild evolution operator guaranteed by theorem (2.4). □

Lemma (2.9) (Pritchard and Salamon [22] lemma (2.5))

Assume conditions (A1)(A2)(A4)(A5)(ii)(A6)(iii) hold, $F(t) \in \mathcal{L}(\bar{X},U)$ is strongly continuous on $[t_0,T]$ and $v(\cdot) \in L^2(t_0,T;U)$ then

(2.2.11) $$J^T_{t_0}(x_0,v) - \langle x_0,P_F(t_0)x_0\rangle_{\bar{X},\bar{X}^*}$$

$$= \int_{t_0}^{T} ||B^*P_F(t)x(t) + v(t)||_U^2 dt$$

$$- \int_{t_0}^{T} ||B^*P_F(t)x(t) + F(t)x(t)||_U^2 dt$$

for all $x_0 \in \bar{X}$ where $P_F(t)$ is defined by (2.2.8) and $x(\cdot)$ is defined by (2.2.2). □

Theorem (2.10)

Assume conditions (A1)-(A4)(A5)(ii)(A6)(iii) hold, $r < r^3_{\mathbb{C}}(A;B,C)$ then there exists a unique strongly continuous, self adjoint, non-positive operator $P(t) \in L(\bar{X},\bar{X}^*), t_0 \le t \le T$, solving the integral Riccati equation

(2.2.12) $$P(t)x = \int_t^T \phi^*(s,t)[P(s)BB^*P(s) - r^2C^*C]\phi(s,t)x ds$$

for all $x \in \underline{X}$ and $t_0 \le t \le T$, where $\phi(s,t)$ is the mild evolution operator guaranteed by theorem (2.4) for $F(t) = -B^*P(t) \in L(\bar{X},U)$. Moreover

(2.2.13) $$v_0(t) = -B^*P(t)x(t)$$

achieves the optimal cost and the optimal cost is

(2.2.14) $$J^T_{t_0}(x_0,v_0) = \langle x_0,P(t_0)x_0\rangle_{\bar{X},\bar{X}^*} .$$

Proof.

Firstly, by proposition (1.4)

$$\inf_{v\in L^2(t_0,T;U)} J^T_{t_0}(x_0,v) \ge \inf_{v\in L^2_U} J(x_0,v) > -\infty .$$

Consider now the following iterative procedure (compare with Curtain and Pritchard [21], Pritchard and Salamon [22]).

$$(2.2.15)\begin{cases} P_{k+1}(t)x = \int_t^T \phi_k^*(s,t)[P_k(s)BB^*P_k(s)-r^2C^*C]\phi_k(s,t)x\,ds \\ \\ P_0(t) \quad \equiv \ 0 \end{cases}$$

where $\phi_k(s,t) \in L(\bar{X}) \cap L(\underline{X})$ is guaranteed by theorem (2.4) for $F = -B^*P_k(t)$, and $P_{k+1}(t) \in L(\bar{X},\bar{X}^*)$ is guaranteed by proposition (2.6) for $F = -B^*P_k(t)$ and $t_0 \leq t \leq s \leq T$. Setting $u(t) = -B^*P_k(t)x(t)$ and $F(t) = -B^*P_{k-1}(t)$ in lemma (2.9) yields

$$\langle x_0,P_{k+1}(t)x_0\rangle_{\bar{X},\bar{X}^*} - \langle x_0,P_k(t)x_0\rangle_{\bar{X},\bar{X}^*}$$
$$= -\int_t^T ||B^*P_k(t)x(t) - B^*P_{k-1}(t)x(t)||_U^2 dt ,$$

for all $x_0 \in \bar{X}$, $t \in [t_0,T]$ and $k = 1,2,\ldots.$. Therefore $\langle x_0,P_k(t)x_0\rangle_{\bar{X},\bar{X}^*}$ is a monotonically decreasing, $k = 0,1,\ldots,$ non-positive and

$$\begin{aligned} \langle x_0,P_{k+1}(t)x_0\rangle_{\bar{X},\bar{X}^*} &= J_t^T(x_0,v_k) \\ &\geq \inf_v J_t^T(x_0,v) \\ &> -\kappa^2||x_0||_{\bar{X}}^2 \quad \text{where} \quad v_k(t) = -B^*P_k(t)x(t) . \end{aligned}$$

Therefore $P_k(t)$ converges strongly to a non-positive, self adjoint operator $P(t) \in L(\bar{X},\bar{X}^*)$ for all $t \in [t_0,T]$. (Kato [15], p.452).

Also $$|<x_0,P_k(t)x_0>_{\bar{X},\bar{X}^*}| \leq r^2(1+\frac{1}{\alpha})\int_0^\infty ||CS(t)x_0||_Y^2 dt$$

$$\leq r^2(1+\frac{1}{\alpha})\ c^2||x_0||_{\bar{X}}^2$$

for α sufficiently small.

Therefore $P_k(t)x$ is a uniformly bounded sequence of continuous functions in $\bar{X}^*$. Hence $P_k(\cdot)x$ is strongly measurable and bounded in $\bar{X}^*$. Hence $\phi(s,t) \in L(\bar{X})$ defined by (2.2.3) for $F(t) = -B^*P(t)$ is a strongly continuous evolution operator, for $t_0 \leq t \leq s \leq T$.

Consider

$$\phi(s,t)x - \phi_k(s,t)x$$

$$= -\int_t^s \bar{S}(s-\sigma)BB^*P(\sigma)\phi(\sigma,t)xd\sigma + \int_t^s \bar{S}(s-\sigma)BB^*P_k(\sigma)\phi_k(\sigma,t)xd\sigma$$

$$= \int_t^s \bar{S}(s-\sigma)BB^*[P_k(\sigma)-P(\sigma)]\phi(\sigma,t)xd\sigma$$

$$- \int_t^s \bar{S}(s-\sigma)BB^*P_k(\sigma)[\phi(\sigma,t)-\phi_k(\sigma,t)]xd\sigma \quad .$$

Applying Gronwalls lemma to this identity together with the fact that $||\int_t^s \bar{S}(s-\sigma)BB^*[P_k(\sigma)-P(\sigma)]\phi(\sigma,t)xd\sigma||_{\bar{X}} \to 0$ as $k \to \infty$, by the Lebesgue dominated convergence theorem, yields the uniform convergence of $\phi_k(s,t)x$ to $\phi(s,t)x$. Consequently,

$$||\phi_k(s,t)x||_{\bar{X}} \leq ||\phi(s,t)x||_{\bar{X}} + \varepsilon_k \quad \varepsilon_k \to 0$$

independently of $t_0 \leq t \leq s \leq T$, $x \in \bar{X}$, and therefore $||\phi_k(s,t)||_{L(\bar{X})}$ is uniformly bounded for all $k \in \mathbb{N}$ and $t \leq s \leq T$.

Also for all $x \in \underline{X}$

$$\phi_k(s,t)x = S(s-t)x - \int_t^s S(s-\sigma)BB^* P_k(\sigma)\phi_k(\sigma,t)x d\sigma \quad ,$$

and therefore

$$||\phi_k(s,t)x||_{\underline{X}} \le ||S(s-t)x||_{\underline{X}} + b(t_0,T)||B^* P_k(\cdot)\phi_k(\cdot,t)x||_{L^2(t,s;U)}$$

from which it follows that $||\phi_K(s,t)||_{L(\underline{X})}$ is uniformly bounded for all $k \in \mathbb{N}$ and $t \le s \le T$. This allows application of the Lebesgue dominated convergence theorem in (2.2.15) to obtain

$$P(t)x = \int_t^T \phi^*(s,t)[P(s)BB^* P\ (s) - r^2 C^* C]\phi(\sigma,t)x d\sigma$$

for all $x \in \underline{X}$.

Strong continuity of $P(t)x \in L(\bar{X},\bar{X}^*)$ follows from the inequality

$$||P(t)x - P(\tau)x||_{\bar{X}^*} \le ||P(t)x - P(t)\hat{x}||_{\bar{X}^*} + ||P(\tau)\hat{x} - P(\tau)x||_{\bar{X}^*}$$
$$+ ||P(t)\hat{x} - P(\tau)\hat{x}||_{\bar{X}^*}$$

for $\hat{x} \in \underline{X}$ such that $||x-\hat{x}||_{\bar{X}}$ is sufficiently small and the strong continuity of $\phi(s,t)$ on $\bar{X}$ and $\underline{X}$.

In order to establish the uniqueness of $P(t)$ suppose that $Q(t) \in L(\bar{X},\bar{X}^*)$ is any strongly continuous, non-positive solution of (2.2.12) and let $v(\cdot) \in L^2(t_0,T;U)$ and $x_0 \in \bar{X}$ be given. Define

$w(t) = v(t) + B^*Q(t)x(t)$ for $t_0 \le t \le T$ where $x(t)$ is the solution of (2.2.2) corresponding to $v(t)$. It is easy to show that for this control

$$J_{t_0}^T(x_0,v) = \langle x_0,Q(t_0)x_0\rangle_{\bar{X},\bar{X}^*} + \int_{t_0}^T ||w(t)||_U^2 dt$$

therefore in particular for $v(t) = - B^*P(t)x(t)$. Hence

$$\langle x_0,Q(t_0)x_0\rangle_{\bar{X},\bar{X}^*} = \langle x_0,P(t_0)x_0\rangle_{\bar{X},\bar{X}^*}$$

for all $x_0 \in \bar{X}$. This shows that $P(t) \in L(\bar{X},\bar{X}^*)$ is unique and additionally that $v(t) = - B^*P(t)x(t)$ is optimal. Non-positivity of $P(t) \in L(\bar{X},\bar{X}^*)$ follows from the non-positivity of $P_k(t) \in L(\bar{X},\bar{X}^*)$ for $k = 1,\ldots$. □

Before going onto the infinite time, non-standard, linear quadratic problem it is important to obtain the differential version of (2.2.12).

Proposition (2.11)

Assume conditions (A1)-(A4)(A5)(ii)(A6)(iii) hold and $r < r_{\mathbb{C}}^3(A;B,C)$. Let $P(t) \in L(\bar{X},\bar{X}^*)$ be a non-positive, self adjoint, strongly continuous family of operators on $[t_0,T]$ and $\phi(s,t)$ the mild evolution operator given by (2.2.3) with $F(t) = - B^*P(t)$, then the following are equivalent:

i) $P(t)$ satisfies (2.2.12) for all $x \in \underline{X}$ and $t \in [t_0,T]$;

ii) for all $x \in \underline{X}$, $t \in [t_0,T]$

(2.2.16) $$P(t)x = -\int_t^T S^*(s-t)[C^*C + P(s)BB^*P(s)]S(s-t)x\,ds \quad .$$

iii) $P(t)x$ is continuously differentiable in $D_{\bar{X}}(A)^*$, on $[t_0,T]$, for all $x \in D_{\bar{X}}(A)$, where $D_{\bar{X}}(A)$ is endowed with the graph norm. Moreover

(2.2.17) $$\begin{cases} \frac{d}{dt} P(t)x + A^*P(t)x + P(t)Ax - P(t)BB^*P(t)x - r^2C^*Cx = 0 \\ P(T) \;=\; 0 \;. \end{cases}$$

Proof.

This result is independent of the positivity of the quadratic cost functional and follows therefore from proposition (2.7) Pritchard and Salamon [22]. □

Remark (2.12)

Statement (ii) gives confirmation of the non-positivity of $P(t) \in L(\bar{X},\bar{X}^*)$.

§3. The infinite time problem and the algebraic Riccati equation.

In order to study the infinite time problem introduced in §1 the following simplification of the finite time problem of §2 is studied.

Minimise $$J_T(x_0,v) = \int_0^T ||v(t)||_U^2 - r^2||Cx(t)||_Y^2 dt$$

subject to $v(\cdot) \in L^2(0,T;U)$ and

$$x(t) = S(t)x_0 + \int_0^t \bar{S}(t-s)Bv(s)ds \;, \qquad x_0 \in \bar{X} \;.$$

Lemma (3.1)

Assume (A1)-(A4)(A5)(ii)(A6)(iii) hold and $r < r^3_{\mathbb{C}}(A;B,C)$. Denote by $P_T(t) \in L(\bar{X},\bar{X}^*)$ the Riccati operator guaranteed by theorem (2.10) for $t_0 = 0$.

The following identity holds for all $0 \le t \le T-\alpha$ and $0 \le \alpha \le T$

$$(2.3.1) \qquad P_{T-\alpha}(t) \equiv P_T(t+\alpha) \ .$$

Proof.

$P_T(t+\alpha)$ satisfies for all $x \in \underline{X}$

$$P_T(t+\alpha)x = -\int_{t+\alpha}^{T} S^*(s-t-\alpha)[C^*C+P_T(s)BB^*P_T(s)]S(s-t-\alpha)xds$$

$$= -\int_{t}^{T-\alpha} S^*(s-t)[C^*C+P_T(s+\alpha)BB^*P_T(s+\alpha)]S(s-t)xds$$

$$= P_{T-\alpha}(t)x$$

by the equivalence of (2.2.12) and (2.2.16). Therefore by the uniqueness of the solution to the integral Riccati equation

$$P_T(t+\alpha) = P_{T-\alpha}(t) \quad .$$

Remark (3.2)

This result is useful in determining a limit for P_T as $T \to \infty$. Before considering this, the following corollary of proposition (2.11) is also important in carrying out this limiting procedure.

Corollary (3.3)

Assume conditions (A1)-(A4)(A5)(ii)(A6)(iii) hold. Let $P \in L(\bar{X},\bar{X}^*)$ be a non-positive selfadjoint operator. Denote by $S_P(t) \in L(\bar{X}) \cap L(\underline{X})$ the strongly continuous semigroup generated by $A-BB^*P : D_{\bar{X}}(A) \to \bar{X}$, defined by

$$(2.3.2) \qquad S_P(t)x = \bar{S}(t)x - \int_0^t \bar{S}(t-s)BB^*PS_P(s)xds$$

for all $x \in \bar{X}$ and $t \geq 0$. The following statements (i) - (iii) are equivalent.

i) $Px = S_P^*(t)PS_P(t)x - \int_0^t S_P^*(s)[C^*C - PBB^*P]S_P(s)xds$

for all $x \in \underline{X}$ and $t \geq 0$.

ii) $Px = \bar{S}^*(t)P\bar{S}(t)x - \int_0^t \bar{S}^*(s)[C^*C + PBB^*P]\bar{S}(s)xds$

for all $x \in \underline{X}$ and $t \geq 0$.

iii) $A^*Px + PAx - PBB^*Px - r^2C^*Cx = 0$ in $D_{\bar{X}}(A)^*$ for all $x \in D_{\bar{X}}(A)$. □

With this result it is possible to prove the main result concerning the non-standard linear quadratic problem and the stability radius $r^3_{\mathbb{C}}(A;B,C)$.

Theorem (3.4)

Assume conditions (A1)-(A4)(A5)(ii)(A6)(iii) hold. If $r < r^3_{\mathbb{C}}(A;B,C)$ then there exists an operator $P \in L(\bar{X},\bar{X}^*)$ satisfying for all $x \in D_{\bar{X}}(A)$

$$(2.3.3) \qquad A^*Px + PAx - PBB^*Px - r^2C^*Cx = 0 \text{ in } D_{\bar{X}}(A)^* .$$

Moreover such a solution is self adjoint and non-positive and unique amongst those solutions satisfying $S_P(t)$ is exponentially stable. In addition the optimisation problem $\inf_v J(x_0,v)$ is solved by $v(t) = - B^*Px(t)$, where $x(t)$ satisfies (2.2.2), and the cost is

$$\inf_{v\in L_U^2} J(x_0,v) = \langle x_0,Px_0\rangle_{\bar{X},\bar{X}^*}$$

Proof.

Let $T \geq \bar{T} \geq 0$ and let $P^T(\cdot)$ denote the Riccati operator guaranteed by theorem (2.10), $t_0 = 0$.

By lemma (3.1)

$$P_T(t) = P_{\bar{T}}(t-(T-\bar{T})) \leq P_{\bar{T}}(t) \quad .$$

Additionally

$$\langle x_0,P_T(0)x_0\rangle_{\bar{X},\bar{X}^*} = J_T(x_0,u_P)> - \infty \text{ , where } u_P = -B^*P_T(t)x(t)$$

and $x(t)$ satisfies (2.2.2). Therefore $\lim_{T\to\infty} \langle x_0,P_T(0)x_0\rangle_{\bar{X},\bar{X}^*}$ exists, for all $x_0 \in \bar{X}$. Hence there exists $P \in L(\bar{X},\bar{X}^*)$, self adjoint and non-positive such that $\lim_{T\to\infty}||P_T(0)x-Px||_{\bar{X}^*} = 0$, for all $x \in \bar{X}$. (Kato, [15], Theorem 3.3 p.452). Again using lemma (3.1)

$$\text{s-}\lim_{T\to\infty} P_T(t)x = \text{s-}\lim_{T\to\infty} P_{T-t}(0)x = Px \in \bar{X}^* \; .$$

Moreover this convergence is uniform in t on compact intervals.

Now, for all $t > 0$,

$$\begin{aligned} Px &= \underset{T\to\infty}{\text{s-lim}}\, P_T(0)x \\ &= -\underset{T\to\infty}{\text{s-lim}} \int_0^T \bar{S}^*(s)[C^*C + P_T(s)BB^*P_T(s)]\bar{S}(s)xds \\ &= -\underset{T\to\infty}{\text{s-lim}} \int_t^T \bar{S}^*(t)\bar{S}^*(s-t)\,[C^*C + P_T(s)BB^*P_T(s)]\bar{S}(s-t)\bar{S}(t)xds \\ &\quad -\underset{T\to\infty}{\text{s-lim}} \int_0^t \bar{S}^*(s)[C^*C + P_T(s)BB^*P_T(s)]\bar{S}(s)xds \quad . \end{aligned}$$

Hence by the uniform convergence of $P_T(t)$ on compact intervals

$$(2.3.4) \qquad Px = \bar{S}(t)^*P\bar{S}(t)x - \int_0^t \bar{S}^*(s)[C^*C + PBB^*P]\bar{S}(s)xds \; .$$

Therefore $P \in L(\bar{X},\bar{X}^*)$ satisfies the formulae i) ii) iii) in corollary (3.3). In particular Px satisfies (2.3.3) for all $x \in D_{\bar{X}}(A)$.

The next step is to establish that the feedback control $v(t) = -B^*Px(t)$ satisfies $v(\cdot) \in L^2_U$. Here again, $x(t)$ is given by (2.2.2) for this $v(\cdot)$. To this end let $\{t_n\}_{n=1}^{\infty}$, $t_n \to \infty$, be an increasing sequence of real numbers and consider the following sequence of control functions $v_n(\cdot) \in L^2_U$,

$$v_n(\cdot)\begin{cases} = -B^*P_{t_n}(t)x_{t_n}(t) & 0 \le t \le t_n \;,\quad n = 1,\ldots \\ = 0 & t > t_n \;, \end{cases}$$

where $x_{t_n}(\cdot)$ is the solution of (2.2.2) for $v = v_n$.

Then

$$J(x_0,v_n(\cdot)) = \int_0^{t_n} ||v_n(s)||_U^2 - r^2||Cx_{t_n}(s)||_Y^2 ds$$

$$- \int_{t_n}^{\infty} r^2||Cx_{t_n}(s)||_Y^2 ds$$

$$= <x_0,P_{t_n}(0)x_0>_{\bar{X},\bar{X}^*} - r^2\int_{t_n}^{\infty}|| \; Cx_{t_n}(s)||_Y^2 ds$$

by theorem (2.10). However

$$J(x_0,v_n(\cdot)) = \lim_{m\to\infty} J_{t_m}(x_0,v_n(\cdot))$$

$$\geq \lim_{n\to\infty} <x_0,P_{t_n}(0)x_0>_{\bar{X},\bar{X}^*}$$

$$= <x_0,Px_0>_{\bar{X},\bar{X}^*} \quad .$$

Consequently

$$\lim_{n\to\infty} \int_{t_n}^{\infty} ||Cx_{t_n}(s)||^2 ds = 0 \quad \text{and therefore}$$

$$\lim_{n\to\infty} J(x_0,v_n(\cdot)) = <x_0,Px_0>_{\bar{X},\bar{X}^*} \quad .$$

Also

$$<x_0,P_{t_n}(\cdot)x_0>_{\bar{X},\bar{X}^*} - r^2\int_{t_n}^{\infty} ||Cx_{t_n}(s)||_Y^2 ds$$

$$= \int_0^{\infty}||v_n(s)||_U^2 - r^2||CS(s)x_0 + (Lv_n)(s)||_Y^2 ds$$

$$\geq \left(1-\frac{r^2}{r_{\mathbb{C}}^2}(1+\alpha)\right)||v_n||^2_{L_U^2} - r^2(1+\frac{1}{\alpha})||CS(\cdot)x_0||^2_{L_Y^2} \quad .$$

Therefore, choosing α sufficiently small proves that v_n is a bounded sequence in L^2_U . Let v_{n_i} be a weakly convergent subsequence, converging to $\hat{v} \in L^2_U$. Now $J(x_0,v)$ is strictly convex and coercive, since $r < r_{\mathbb{C}}$, and therefore (see Ekeland and Temam, [58], p.35) $\hat{v}$ is the unique optimal control, so that

$$\inf_{v \in L^2_U} J(x_0,v) = J(x_0,\hat{v})$$

$$= \langle x_0, Px_0 \rangle_{\bar{X},\bar{X}^*} \quad .$$

Also $\hat{v}(\cdot) = -B^* Px(t)$ since for any $P = P^* \in L(\bar{X},\bar{X}^*)$ satisfying (2.3.3)

$$(2.3.5) \qquad J(x_0,\hat{v}) = \int_0^\infty ||\hat{v}(s) + B^* P\hat{x}(s)||^2_U ds + \langle x_0, Px_0 \rangle_{\bar{X},\bar{X}}$$

where $\hat{x}(t)$ is the solution of (2.2.2) for $v = \hat{v}$. As a result of this it follows also that $S_p(t)$ is exponentially stable on $X(\bar{X})$ since $S_p(t)x_0 = \hat{x}(t)$, and

$$||\hat{x}(\cdot)||_{L^2_X} \leq ||S(\cdot)x_0||_{L^2_X} + ||\int_0^{\cdot} \bar{S}(\cdot-s)B\hat{v}(s)ds||_{L^2_X}$$

$$(||\hat{x}(\cdot)||_{L^2_{\bar{X}}} \leq ||\bar{S}(\cdot)x||_{L^2_{\bar{X}}} + ||\int_0^{\cdot} \bar{S}(\cdot-s)B\hat{v}(s)ds||_{L^2_{\bar{X}}}).$$

Moreover $P \in L(\bar{X},\bar{X}^*)$ is the unique self adjoint solution of the algebraic Riccati equation with this property, since if $P \in L(\bar{X},\bar{X}^*)$ was another solution then $\tilde{v}(t) = B^*\tilde{P}\tilde{x}(t)$ would satisfy $\tilde{v}(\cdot) \in L^2_U$, and (2.3.5) yields

$$<x_0,\tilde{P}x_0>_{\bar{X},\bar{X}^*} = J(x_0,\tilde{v}) = \int_0^\infty ||\tilde{v}(s) + B^*P\tilde{x}(s)||_U^2 ds + <x_0,Px_0>_{\bar{X},\bar{X}^*}$$

showing that $<x_0,\tilde{P}x_0>_{\bar{X},\bar{X}^*} \geq <x_0,Px_0>_{\bar{X},\bar{X}^*}$.

Interchanging the roles of P and $\tilde{P}$ proves the required uniqueness of $P \in L(\bar{X},\bar{X}^*)$ with $S_P(t)$ exponentially stable. □

This proves the existence of a solution to the algebraic Riccati equation for all $r < r_{\mathbb{C}}$. As a partial converse

Proposition (3.5)

Assume there exists $P \in L(\bar{X},\bar{X}^*)$, a self-adjoint solution of the algebraic Riccati equation (2.3.3) and $0 < \sup_\omega ||G(i\omega)|| < \infty$ then $r \leq \dfrac{1}{\sup_\omega ||G(i\omega)||}$.

Proof.

Let $x \in D_{\bar{X}}(A)$ and $P = P^*$ then

$$(i\omega-A)^*Px + P(i\omega-A)x + r^2C^*Cx + PBB^*Px = 0 \text{ , for all } \omega \in \mathbb{R} .$$

Choose $x = (i\omega-A)^{-1}Bu$, $u \in U$, then

$$<(i\omega-A)^{-1}Bu,(i\omega-A)^*P(i\omega-A)^{-1}Bu>_{Z,Z^*} + <(i\omega-A)^{-1}Bu,PBu>_{Z,Z^*}$$
$$+ <(i\omega-A)^{-1}Bu, PBB^*P(i\omega-A)^{-1}Bu>_{Z,Z^*}$$
$$= -r^2<(i\omega-A)^{-1}Bu,C^*C(i\omega-A)^{-1}Bu>_{Z,Z^*} ,$$

where $Z = D_{\bar{X}}(A)$.

Therefore $||u||_U^2 - r^2||G(i\omega)u||_Y^2$

$$= <(I + B^*P(i\omega-A)^{-1}B)u,\ (I + B^*P(i\omega-A)^{-1}B)u>_U$$

for all $\omega \in \mathbb{R}$.

Thus

$$r^2 \le \frac{1}{\sup_\omega ||G(i\omega)||^2} \qquad . \qquad \square$$

Remark (3.6)

In Hinrichsen and Pritchard [8] it is shown, in the case of X , U and Y finite dimensional, that for $r = r_{\mathbb{C}}$ there exists a non-positive, self-adjoint solution, P , of the algebraic Riccati equation, such that $\sigma(A-BB^*P) \subseteq \bar{\mathbb{C}}_-$. Unfortunately the methods employed rely upon taking state transformations and considering the algebraic Riccati equation decomposed with respect to the observable and controllable parts. In this infinite dimensional case the methods employed in obtaining a solution of the algebraic Riccati equation are iterative and based upon the solution of corresponding quadratic cost problem. It is an open question whether there exists a solution of the algebraic Riccati equation when $r = r_{\mathbb{C}}$.

§4. Application - A non standard Liapunov functional.

The solution of the algebraic Riccati equation can be used as a tool for analysing the asymptotic stability of structured, unbounded, non-linear perturbed systems.

Let the nominal (unperturbed) system be as in Chapter 1, that is

$$(N) \qquad \dot{x}(t) = Ax(t) \ , \quad x(0) = x_0 \ .$$

Suppose the nominal system is subject to perturbations

$$(ND) \qquad \dot{x}(t) = Ax(t) + BN(Cx(t)) \ , \quad x(0) = x_0 \ .$$

As in Chapter 1, due to the unbounded nature of the perturbation term, the mild solution of (ND) is studied, that is the perturbed system is

$$(NP) \qquad x(t) = S(t)x_0 + \int_0^t \bar{S}(t-s)BN(Cx(s))ds \qquad x_0 \in X \ .$$

Again $S(t)$ is a strongly continuous semigroup on X, with generator $A : D_X(A) \to X$, X is a complex Hilbert space. The operators B and C are elements of $L(U,\bar{X})$ and $L(\underline{X},Y)$ respectively where $\underline{X}$, $\bar{X}$ are additional Hilbert spaces satisfying (A1) of Chapter 1. N is a non linear map $Y \to U$ and U and Y are Hilbert spaces. As in Chapter 1 it is possible to obtain existence for solutions to (NP) via a fixed point problem on L_Y^2.

<u>Proposition</u> (4.1)

Assume conditions (A1)-(A5)(A6)(iii) hold. Assume that there exists $\alpha > 0$ and $0 < k(\alpha) < r_{\mathbb{C}}^3(A;B,C)$ such that

$$||N(y_1)-N(y_2)||_U \leq k(\alpha)||y_1-y_2||_Y \quad \text{for all} \quad y_i \in Y \ , ||y_i|| \leq \alpha, \ i = 1,2 \ .$$

If $x_0 \in X(x_0 \in \bar{X})$, $||x_0||_{\bar{X}} \leq \dfrac{(1 - ||L||k(\alpha))\alpha}{c}$, where $c > 0$ is such that $||C\bar{S}(\cdot)x_0||_{L^2_Y} \leq c||x||_{\bar{X}}$ then there exists a unique continuous solution in X $(\bar{X})$ of (NP) such that $Cx(\cdot) \in L^2(0,T;Y)$.

Proof.

Consider the iterative procedure

$$y_{N+1} = y_0 + LN(y_N) \quad \text{where} \quad y_0 = C\bar{S}(\cdot)x_0 \ .$$

If $x_0 \in X$ $(x_0 \in \bar{X})$, $||x_0||_{\bar{X}} < \dfrac{(1 - ||L||k(\alpha))\alpha}{c}$, then $y_N \to y_\infty$ in L^2_Y and $||y_\infty||_{L^2_Y} \leq \alpha$.

As in Chapter 1 define $x(\cdot)$, a solution of (NP) , by

$$x(t) = \bar{S}(t)x_0 + \int_0^t \bar{S}(t-s)BN(y_\infty(s))ds \ .$$

Then $x(\cdot)$ is continuous in X $(\bar{X})$ and $Cx(\cdot) \in L^2(0,T;Y)$ for all $T > 0$. Suppose that $\tilde{x}(\cdot)$ is a second continuous solution of (NP) such that $C\tilde{x}(\cdot) \in L^2(0,T;Y)$ then

$$C\tilde{x}(\cdot) = CS(\cdot)x + C\int_0^t \bar{S}(t-s)BN(C\tilde{x}(\cdot))ds$$

and therefore by the guaranteed uniqueness for the solution of this equation on $L^2(0,T;Y)$, $C\tilde{x} = y_\infty$ and hence $\tilde{x}(\cdot) = x(\cdot)$. □

Note. If α is increased then $k(\alpha)$ increases.

Remark (4.2)

In addition to the existence of this solution, asymptotic stability of the solution follows by the estimates

$$||x(t)||_X \leq ||S(t)x||_X + b||N(Cx(\cdot)||_{L^2_U}$$

$$\leq \quad \bar{K}||x||_X \quad \text{where } \bar{K} \text{ is some constant}$$

and if $\varepsilon>0$ $$||x(t)||_X \leq ||S(t-T)x(T)||_X + ||\int_T^t \bar{S}(t-s)BN(Cx(s))ds||_X$$

$$< \varepsilon \quad \text{for } t,T \text{ sufficiently large.}$$

This gives a region of attraction for the origin of the non-linear system (NP). However the estimates in the proof of proposition (4.1) are rather crude and it would be expected that Liapunov methods would yield a larger region of attraction. In order to consider the problem of increasing the size of the region of asymptotic stability the following condition is imposed.

(NLE) Assume existence of solutions for all $x_0 \in X$ $(x_0 \in \bar{X})$ in the sense of proposition (4.1).

Remark (4.3)

Of course there exists more general theorems for the existence of solutions to (NP). However the important consideration here is the asymptotic stability of the perturbed system.

Theorem (4.4)

Suppose (A1)-(A4)(A5)(ii)(A6)(iii) hold. Let $0 < \rho < r^3_{\mathbb{C}}(A;B,C)$

and denote by $P_\rho \in (\bar{X},\bar{X}^*)$ the solution of (2.3.3), unique in the sense of Theorem (3.4). Assume that (N) is such that

(S) If $-\langle x,P_\rho x\rangle_{\bar{X},\bar{X}^*} < d$ and $Cx \in Y$, then $||Ny||_U \leq k||y||_Y$.

If $k < \rho$ then the solution of (NP) is asymptotically stable and $\Omega_\rho = \{x_0 \in \bar{X} \mid -\langle x_0,P_\rho x_0\rangle_{\bar{X},\bar{X}^*} < d\}$, $(\Omega_\rho \cap X)$ is a region of attraction of the origin for (NP) in $\bar{X},(X)$.

<u>Proof.</u>

Let $x_0 \in \Omega_\rho$ and let $x(\cdot)$, $y(\cdot)$ be the guaranteed solution pair of (NP) for x_0 . Since $x(\cdot)$ is continuous

$$-\langle x(t),P_\rho x(t)\rangle_{\bar{X},\bar{X}^*} < d \quad \text{for all} \quad t < T \text{ , } T \text{ sufficiently small.}$$

Also $Cx(\cdot) \in Y$ almost everywhere and therefore

$$||N(Cx(t))||_U \leq k||Cx(t)||_Y \quad \text{almost everywhere on} \quad [0,T] \text{ .}$$

Hence $N(Cx(\cdot)) \in L^2(0,T;U)$ and therefore

$$- \langle x(t),P_\rho x(t)\rangle_{\bar{X},\bar{X}^*} + \langle x_0,P_\rho x_0\rangle_{\bar{X},\bar{X}^*}$$

$$= -\int_0^t ||B^*P_\rho x(s)+N(Cx(s))||_U^2 ds - \int_0^t \rho^2||Cx(s)||_Y^2 - ||N(Cx(s))||_U^2 ds$$

$$\leq - (\rho^2-k^2)\int_0^t ||Cx(s)||_Y^2 dt \text{ , for all } t < T \text{ .}$$

Hence $-\langle x(t),P_\rho x(t)\rangle_{\bar{X},\bar{X}^*}$ decreases and $-\langle x(T),P_\rho x(T)\rangle_{\bar{X},\bar{X}^*} \leq -\langle x_0,P_\rho x_0\rangle_{\bar{X},\bar{X}^*}$.

Therefore as $t \to \infty$ $- <x(t), P_\rho x(t)>_{\bar{X},\bar{X}^*} \le \alpha < d$ for some $\alpha \in \mathbb{R}$. Consequently

$$(\rho^2-k^2)\int_0^t ||Cx(s)||^2 ds \le - <x_0,P_\rho x_0>_{\bar{X},\bar{X}^*}$$

and also $\int_0^t ||N(Cx(s))||_U^2 ds \le k^2\int_0^t ||Cx(s)||_Y^2 ds$.

Therefore

$$\begin{aligned}||x(t)||_{\bar{X}} &\le ||\bar{S}(t)x_0||_{\bar{X}} + bk||Cx(\cdot)||_{L_Y^2} \\ &\le \bar{k} \text{ for some } \bar{k} ,\end{aligned}$$

also given $\varepsilon > 0$

$$\begin{aligned}||x(t)||_{\bar{X}} &\le ||\bar{S}(t-T)x(T)||_{\bar{X}} + bk||Cx(\cdot)||_{L^2(T,\infty;Y)} \\ &< \varepsilon \quad \text{for sufficiently large } T > 0 \text{ and } t > T .\end{aligned}$$

Hence the origin is an asymptotically stable equilibrium point and $\Omega_\rho(\Omega_\rho \cap X)$ is a region of attraction of the origin in $\bar{X}$ (X) for (NP). □

Remark (4.5)

Let $0 < \rho < r_{\mathbb{C}}^3(A;B,C)$ and $x_0 \in \Omega_\rho$, with P_ρ as in theorem 4.4. Now

$$<x_0,P_\rho x_0>_{\bar{X},\bar{X}^*} - <\bar{S}(t)x_0,P_\rho\bar{S}(t)x_0>_{\bar{X},\bar{X}^*} + \int_0^t ||B^*P_\rho\bar{S}(s)x_0||_U^2 ds$$

$$= -\rho^2\int_0^t ||C\bar{S}(s)x_0||_Y^2 ds \quad .$$

Consequently $\rho^2\int_0^t ||C\bar{S}(s)x_0||_Y^2 ds \le <\bar{S}(t)x_0,P_\rho\bar{S}(t)x_0>_{\bar{X},\bar{X}^*} - <x_0,P_\rho x_0>_{\bar{X},\bar{X}^*}$

and $-<\bar{S}(t)_0, P_\rho \bar{S}(t)x_0>_{\bar{X},\bar{X}^*} \le -<x_0, Px_0>_{\bar{X},\bar{X}^*} = d$. Therefore

$$\rho^2\int_0^t ||C\bar{S}(s)x_0||_Y^2 ds \le d \quad \text{for all} \quad t \quad .$$

Thus for each $x_0 \in \Omega_\rho$, $\rho^2\int_0^t ||C\bar{S}(s)x_0||_Y^2 ds \le d$ for all t , and the region of attraction Ω_ρ is such that the "output" of the nominal system is bounded in the sense of L_Y^2 by the constant $\sqrt{d}/\rho$ When X is finite dimensional $C\bar{S}(\cdot)x_0$ is continuous and therefore $||Cx_0||_Y \le \alpha$ for some constant α . This remark says something about the region over which N is Lipschitz and therefore about the character of the non-linearity satisfying (S).

Conclusions.

In this chapter a certain non-standard linear quadratic problem has been studied via the abstract framework of Pritchard and Salamon [22]. The problem arises in the analysis of the robustness measures of Chapter 1, due to the characterisation of the measures, via the norm of "the map". The difference between this non-standard linear quadratic problem and the usual problem is the possibility that the infimum of the cost functional does not exist. It is here that the norm of the map appears as a bound on the range of parameter r . The drawback in the analysis is the question of existence of solutions to (2.3.3) when $r = r_{\mathbb{C}}^3(A;B,C)$. Unfortunately, the answer to this question is not known.

An application for the Riccati operator (of theorem 3.4) solving (2.3.3) for $r < r_{\mathbb{C}}^3(A;B,C)$, is the stability analysis of section (4), for a class of Lipschitz bounded non-linearities. This class includes those non-linearities satisfying a global Lipschitz bound k less than $r_{\mathbb{C}}^{3}(A,B,C)$. The analysis is completed without reference to compactness of orbits and the principle of La Salle and relies only on trapping the solution $x(\cdot,x_0)$ within a prescribed (unbounded) region in the state space. This in turn, bounds the "output" $Cx(\cdot)$ in the sense of L_Y^2 and finally, via the integral equation (NP), results in the asymptotic stability of the solution. Again, solution of the perturbed equation in L_Y^2 results in asymptotic stability of the perturbed solution. In fact, when the solution is given by a certain class of semigroups of nonlinear operators, the solution is exponentially stable, Ichikawa [19]. There will be a trade-off between the size of the allowed Lipschitz constant $k \leq \rho < r_{\mathbb{C}}^{3}(A;B,C)$ and the region of attraction Ω_ρ $(\Omega_\rho \cap X)$.

Apart from this application, of the Riccati operator as a non-standard Liapunov functional for non-linear equations, of greater importance is the associated problem of a non-standard linear quadratic game. In Szumko [53] study of the following algebraic Riccati equation

$$A^{*}K + KA + \frac{KDD^{*}K}{\varepsilon^{2}} - KBB^{*}K - r^{2}C^{*}C = 0 \ , \quad \varepsilon > 0$$

$A \in \mathbb{C}^{n\times n}$, $D \in \mathbb{C}^{n\times r}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$ leads to a feed-back operator $F \in \mathbb{C}^{r\times n}$ such that $r_{\mathbb{C}}(A+DF,B,C) \geq r_{\mathbb{C}}(A;B,C)$ and under suitable conditions the inequality is shown to be strict. The analysis relies considerably on the results of Willems [55] true in this finite-dimensional case. Applying these ideas to an infinite dimensional setting would require a deeper understanding of equations

$$A^{*}Px + PAx - r^{2}C^{*}Cx + PWPx = 0 \ , \quad \text{for} \quad x \in D_{\bar{X}}(A)$$

where $W^{*} = W \in L(\bar{X}^{*},\bar{X})$, with A and C as in Chapter 2.

CHAPTER 3. Resolvent and evolution operators: Generalisations and limitations of an L^2-stability approach to the robustness of linear systems.

§0. Introduction.

In this chapter the robustness concepts introduced in Chapter 1 for systems governed by semigroups of linear operators are extended considerably to encompass systems governed by resolvent operators and evolution operators. Recall from Chapter 1 that the robustness of the (formal) abstract differential equation

$$\text{(D)} \quad \dot{x}(t) = (A + BDC)x(t) \ , \qquad x(0) = x_0$$

with A the generator of an exponentially stable semigroup $S(t) \in \mathcal{L}(X)$ and $B \in \mathcal{L}(U,\bar{X})$, $C \in \mathcal{L}(\underline{X},Y)$ and $D \in \mathcal{L}(Y,U)$ is related closely to the solution of the auxiliary equation

$$\text{(Y)} \quad y(t) = CS(t)x_0 + C\int_0^t S(t-s)BDy(s)ds \ , \qquad x_0 \in X$$

for $y(\cdot) \in L_Y^2$. This equation results from the isolation of D in the mild equation

$$\text{(P)} \quad x(t) = S(t)x_0 + \int_0^t S(t-s)BDCx(s)ds \ , \qquad x_0 \in X \ .$$

The analysis of Chapter 1 for equations (Y) and (P) extends quite naturally to any class of systems for which there exists a variation of parameters formula, in particular for perturbations of the (formal) differential equations

$$(3.0.1) \qquad \dot{x}(t) = Ax(t) + \int_0^t K(t-s)x(s)ds + f(t) \ , \quad x(0) = x_0$$

and

$$(3.0.2) \qquad \dot{x}(t) = A(t)x(t) \ , \quad x(t_0) = x_0 \ .$$

As in Chapter 1, the solving of an equation like (Y) plays an important role, again due to the relationship between L^2-stability and exponential stability for systems governed by resolvent or evolution operators. In fact, for certain classes of evolution operators, these two properties are equivalent, Datko [18]. Due to the important role played by the auxiliary equation (Y), in section 1 an abstraction of this equation is considered. It is then possible to define a stability radius for this abstract system. The question then arises as to whether this abstract stability radius has implications for an abstract version of equation (P). This interplay between the equation (P) and equation (Y) is exploited in section 2 to obtain positive results for systems governed by resolvent operators. In section 3 the limitations of the abstract analysis is illustrated by a certain class of systems governed by evolution operators.

§1. L^2-stability : An abstract approach to robustness.

From the analysis of Chapter 1 it is clear that the following abstract equation is important in analysing the robustness of various linear infinite dimensional systems.

$(\mathcal{Y}) \qquad y = y_0 + CMD_y \quad , \quad y_0 \in L^2_Y \, ,$

where $C \in \mathcal{L}(L^2_{\underline{X}}, L^2_Y)$, $M \in \mathcal{L}(L^2_U, L_{\underline{X}})$, U,Y are Hilbert spaces and $\underline{X}$ is a Banach space. For example $C = C \in \mathcal{L}(\underline{X}, Y)$, $y_0(t) = CS(t)x_0$, $(Mu)(t) = \int_0^t S(t-s)Bu(s)ds$ as in Chapter 1, where $B \in \mathcal{L}(U,\bar{X})$ and $(S(t))_{t\geq 0}$ is a strongly continuous semigroup on $\underline{X}$, X , $\bar{X}$, where X , $\bar{X}$ are Banach spaces. The operator D represents the perturbation operator and belongs to a subset $\mathcal{D} \subseteq \mathcal{L}(L^2_Y, L^2_U)$ and the problem is to determine norm bounds (induced by the operator norm in $\mathcal{L}(L^2_Y, L^2_U)$) on the operator D such that the corresponding solution $y(\cdot, y_0)$ of $(\mathcal{Y})$ lies in L^2_Y .

Definition (1.1)

The stability radius for $(\mathcal{Y})$ is

$$(3.1.1) \qquad \rho(M,C,\mathcal{D}) := \sup\{d \mid D \in \mathcal{D} \, , \, r < d \text{ implies there exists } K \geq 0 \text{ with } \sup_{||D|| \leq r} ||y(\cdot \, ,D)||_{L^2_Y} < K||y_0||_{L^2_Y}\}.$$

Here $y(\cdot, D)$ is the solution of $(\mathcal{Y})$ for $D \in \mathcal{D}$ and $y_0 \in L^2_Y$.

Theorem (1.2)

Assume $M \in \mathcal{L}(L^2_U, L^2_{\underline{X}})$ and $C \in \mathcal{L}(L^2_{\underline{X}}, L^2_Y)$ then

$$\rho(M,C,\mathcal{L}(L^2_Y, L^2_U)) = ||CM||^{-1} \quad .$$

Proof.

If $||D|| < ||CM||^{-1}$ then $||CMD|| < 1$ and therefore CMD defines a contraction on L_Y^2 . Hence equation ( ) has a unique solution $y(\cdot,D) \in L_Y^2$. Moreover if $r < ||CM||^{-1}$ then

$$||y(\cdot,D)||_{L_Y^2} \le ||(I - CMD)^{-1}|| \; ||y_0||_{L_Y^2}$$

$$\le \frac{1}{1 - r||CM||} ||y_0||_{L_Y^2} \quad \text{for all } D \text{ , } ||D|| \le r \text{ .}$$

Therefore $\rho(M,C,\mathcal{L}(L_Y^2,L_U^2)) \ge ||CM||^{-1}$. In order to prove equality let $\{u_n\}_{n=1}^{\infty} \subset L_U^2$, $||u_n||_U = 1$ and $||CMu_n||_Y = \mu_n \uparrow \mu = ||CM||_Y$. Set $h_n = \mu^{-1}CMu_n$ and define

$$(3.1.2) \qquad y_n(\cdot) = y_0(\cdot) + \alpha_n h_n(\cdot)$$

where

$$(3.1.3) \qquad \alpha_n = (1-||h_n||_{L_Y^2}^2)^{-1}\{<y_0,h_n>_{L_Y^2} + (<y_0,h_n>_{L_Y^2}^2+(1-||h_n||_{L_Y^2}^2)||y_0||_{L_Y^2}^2)^{\frac{1}{2}}\} \text{ .}$$

It is easy to show that $\alpha_n = ||y_n||_{L_Y^2}$ and $y_n(\cdot) \in L_Y^2$ for all $n = 1,2, \ldots$

Define $D_n \in \mathcal{L}(L_Y^2,L_U^2)$ by

$$(3.1.4) \qquad D_n y = \frac{u_n<y_n,y>_{L_Y^2}}{\mu||y_n||_{L_Y^2}} \text{ .} \qquad \text{Then}$$

i) $||D_n|| = ||CM||^{-1}$ and

ii) $||CMD_n|| = \mu_n/\mu < 1$.

Consequently for each $n = 1,2,\ldots$ $(\mathcal{Y})$ with $D = D_n$ has a unique solution $y(\cdot,D_n) \in L^2_Y$. Moreover it is easy to show that $y(\cdot,D_n) = y_n(\cdot)$. Since u_n must satisfy $||CMu_n||_{L^2_Y} \uparrow \mu$ and $h_n = CMu_n/\mu$ it is possible to choose $<y_0,h_n>_{L^2_Y} \geq 0$. Therefore from (3.1.3)

$$||y_n||_{L^2_Y} \geq [1 - ||h_n||^2_{L^2_Y}]^{\frac{1}{2}} ||y_0||_{L^2_Y} \to \infty \quad \text{as} \quad n \to \infty .$$

Hence $\rho(C,M,\mathcal{L}(L^2_Y,L^2_U)) = ||CM||^{-1}$. □

As in Chapter 1 equation $(\mathcal{Y})$ becomes important when used as a tool for analysing a mild integral equation. The abstract version of this integral equation takes the form

$$(P) \quad x(\cdot) = x_0(\cdot) + MDCx(\cdot) \qquad x_0(\cdot) \in L^2_X$$

where $X \supseteq \underline{X}$ with continuous dense injection. As in Chapter 1 it is assumed that $x_0(\cdot)$ is continuous in X and $x_0(\cdot) \in D(C)$ where C is an unbounded map $L^2_X \to L^2_Y$. (e.g. $x_0(\cdot) \in \{S(\cdot)x \mid x \in X\}$.) Since the aim is, eventually, to analyse systems governed by evolution or resolvent operators, solutions of (P) are sought in the class of continuous functions. As in Chapter 1 a solution of equation (P) can be obtained via equation $(\mathcal{Y})$ by defining

$$x_y(\cdot) = x_0 + MDy \ , \quad y_0 = Cx_0 \qquad \text{where}$$

$y(\cdot)$ is the solution of $(\mathcal{Y})$ in L^2_Y, and this solution is continuous under suitable conditions on M. However, as in Chapter 1, the problem lies in proving the uniqueness of $x_y(\cdot)$. The problem is compounded in this abstract setting by the non causality of the system and in general it is not possible to prove uniqueness without an extra assumption of causality. To describe this:

Let ρ_T and σ_T be the operators introduced in Chapter 1, that is for $T > 0$

$$\begin{cases} \rho_T : L^2_Z \to L^2(0,T;Z) \\ \rho_T z(\cdot) = z(\cdot)|_{[0,T]} \end{cases} \text{, where } Z \text{ is a Banach space.}$$

σ_T is a right inverse defined by

$$\begin{cases} \sigma_T : L^2(0,T;Z) \to L^2_Z \\ \sigma_T z(\cdot) = \begin{cases} z(t) & t \in [0,T] \\ 0 & t > T \end{cases} \end{cases} .$$

Note. $\sigma_T\rho_T \in \mathcal{L}(L^2_Z)$ and $||\sigma_T\rho_T|| = 1$.

Definition (1.3)

L is said to be causal if

$\rho_T L = \rho_T L \sigma_T \rho_T$ for all $T > 0$. See Willems [56].

In order to establish the continuity and uniqueness of $x_y(\cdot)$ the following assumptions must be made about the abstract operators M,C and the subset $\mathcal{D} \subseteq \mathcal{L}(L^2_Y, L^2_U)$.

(H1) for all $T > 0$ $\rho_T M \in \mathcal{L}(L^2_U, C(0,T;X))$

(H2) there exists $r \in \mathbb{N}_0$ such that for all $T > 0$, $D \in \mathcal{D}$, $\underbrace{(CMD) \circ \ldots \circ (CMD)}_{r \text{ times}} C \sigma_T \in L(L^2(0,T;X),L^2_Y)$

Remark (1.4)

If (H2) holds for $r = 0$ then $\rho_T C \sigma_T \in L(L^2(0,T;X), L^2(0,T;Y))$. This is the abstract version of assumption (A6)(ii) in Chapter 1. Assumption (H2) for $r \in \mathbb{N}$ is the abstract version of assumption (A8) Chapter 1.

Theorem (1.5)

Assume $M \in L(L^2_U,L^2_{\underline{X}})$, $C \in L(L^2_{\underline{X}},L^2_Y)$ and $D \in \mathcal{D} \subseteq L(L^2_Y,L^2_U)$ are causal and satisfy assumptions (H1) and (H2). If $||D|| < ||CM||^{-1}$ and $x_0(\cdot) \in C(0,T;X) \cap D(C)$ then there exists a unique solution $x(\cdot)$ of ($\mathcal{P}$) in the class of continuous functions with values in X . Moreover $x(\cdot) \in L^2_X$.

Proof.

Since $||D|| < ||CM||^{-1}$ there exists a unique solution of ($\mathcal{Y}$) for $y(\cdot) \in L^2_Y$, with $y_0 = Cx_0$.

Therefore

$$x = x_0 + MDy$$

defines a function $x \in L^2_X$ such that $\rho_T x \in C(0,T;X)$ for all $T > 0$. Also $Cx \in L^2_Y$ and therefore $y = Cx$ and x satisfies ($\mathcal{P}$).

For the uniqueness assume that $\hat{x}$ is another solution of $(\mathcal{P})$ with $\rho_T\hat{x} \in C(0,T;X)$. Let $r \in \mathbb{N}_0$ be guaranteed by (H2) then since $x_0 \in D(C)$

$$(CMD)^r(C\sigma_T\rho_T\hat{x}) = (CMD)^r(C\sigma_T\rho_T x_0) + (CMD)^r(C\sigma_T\rho_T MDCx) \quad .$$

Thus

$$\sigma_T\rho_T(CMD)^r C\sigma_T\rho_T\hat{x} = \sigma_T\rho_T(CMD)^r C\sigma_T\rho_T x_0$$

$$+ \sigma_T\rho_T CMD\sigma_T\rho_T(CMD)^r C\sigma_T\rho_T\hat{x} \ .$$

But

$$\sigma_T\rho_T(CMD)^r y = \sigma_T\rho_T(CMD)^r C\sigma_T\rho_T x_0$$

$$+ \sigma_T\rho_T CMD\sigma_T\rho_T(CMD)^r y$$

and therefore since $||\sigma_T\rho_T|| = 1$ and $||CMD|| < 1$

$$\sigma_T\rho_T(CMD)^r y = \sigma_T\rho_T(CMD)^r C\sigma_T\rho_T\hat{x} \ .$$

Applying $\sigma_T\rho_T(CMD)^k C\sigma_T\rho_T$ to

$$\hat{x} = x_0 + MDC\hat{x} \quad \text{for} \quad k = r-1,\ldots, 0 \ ,$$

and noting at each stage that $\hat{x} \in D((CMD)^k\sigma_T\rho_T)$ as a map $L^2_X \to L^2_Y$, yields the identities

$$\sigma_T\rho_T(CMD)^k C\sigma_T\rho_T\hat{x} = \sigma_T\rho_T(CMD)^k C\sigma_T\rho_T x \ , \quad k = r-1,\ldots \ 0$$

and therefore uniqueness of x follows from the identity $\rho_T\hat{x} = \rho_T x$ for all $T > 0$. □

Remark (1.6)

In application of a contraction mapping principle quite often it is a power of the map and not the map itself that is used. The application of the contraction mapping theorem above is novel in that a power of a map is used to establish uniqueness, in this unbounded setting.

This theorem allows the definition of an abstract version of the stability radius of Chapter 1.

Definition (1.7)

The stability radius of system $(\mathcal{P})$ is

$$\rho_{\mathcal{P}}(M,C,\mathcal{D}) := \sup\{d \mid D \in \mathcal{D} \ , \ r < d \ \text{ implies there exists } \ K \geq 0$$

$$\text{with} \sup_{||D|| \leq r} ||x(\cdot,D)||_{L^2_X} < K||x_0||_{L^2_X}\} \ .$$

Here $x_0 \in D(C)$, $C : L^2_X \to L^2_Y$.

Unfortunately it is not possible to prove the abstract analogue of Theorem (3.7) Chapter 1 since the operators D_n , $n = 1,2,\ldots$ constructed in theorem (1.2) are not causal! It is the purpose of sections 2 and 3 to explore the interplay between causality on the one hand and the possibility that $||CM||^{-1}$ is the stability radius ρ_P on the other, when the maps C and M and the set $\mathcal{D}$ are specialised. In section 2 the case

$$C = C \in \mathcal{L}(\underline{X},Y) \quad \text{and} \quad (Mu)(t) = \int_0^t R(t-s)Bu(s)ds$$

is treated. $B \in \mathcal{L}(U,\bar{X})$, $X \subseteq \bar{X}$ and $(R(t))_{t\geq 0}$ is the resolvent operator for the integrodifferential equation

$$\dot{x}(t) = Ax(t) + \int_0^t K(t-s)x(s)ds \ , \quad x(0) = x_0 \ ,$$

and $\mathcal{D}$ reflects the possibility of perturbing either the operator A or the kernel $K(\cdot)$. The main result of section 2 is that the stability radius in both cases is $||CM||^{-1}$. In section 3 the analysis is applied to time varying systems where

$$(Cx)(t) = C(t)x(t) \quad , \quad (Mu)(t) = \int_{t_0}^t U(t,s)B(s)u(s)ds$$

and $U(\cdot,\cdot)$ is an evolution operator corresponding to the differential equation

$$\dot{x}(t) = A(t)x(t) \ , \quad x(t_0) = x_0 \quad \text{and}$$

$$\mathcal{D} = \{D \in \mathcal{L}(L_Y^2, L_U^2) \,|\, (Dy)(t) = D(t)y(t), D(\cdot) \in \mathcal{B}^\infty(t_0,\infty;\mathcal{L}(Y,U))\} \ .$$

For B_1 and B_2 Banach spaces $\mathcal{B}^\infty(t_0,\infty;\ (B_1,B_2)) = \{f:[t_0,\infty) \to \mathcal{L}(B_1,B_2)$ such that $f(\cdot)$ is strongly measurable and $\operatorname*{ess\,sup}_{\tau \ge t_0} ||f(\tau)|| < \infty\}$.

The main result of section 3 is that the abstract analysis of section 1 yields only a lower bound on the stability radius.

§2. Resolvent operators arising from integrodifferential equations.

This section deals with the important class of systems governed by resolvent operators arising from integrodifferential equations of the type

$$\text{(ID)} \qquad \dot{x}(t) = Ax(t) + \int_0^t K(t-s)x(s)ds + f(t) \ , \quad x(0) = x_0 \ .$$

Such classes of systems have received considerable interest in recent years by many authors, where well-posedness and stability of the system is considered (see Miller [33], Chen and Grimmer [30], Grimmer and Pritchard [32] concerning well-posedness results and Grimmer [31], Miller and Wheeler [34] for results concerning the stability of such systems.)

A is a closed operator with dense domain $D_X(A)$ on a complex Banach space X. $\{K(t)\}_{t\geq 0}$ is a family of closed linear operators on X with $D_X(K(t)) \supseteq D_X(A)$. A resolvent operator $(R(t))_{t\geq 0}$ for (ID) is a family of maps $R(t) \in L(X)$ such that

(i) $R(t)$ is strongly continuous on X, $R(0) = I_X$.

(ii) $R(t)$ is strongly continuous on $D_X(A)$.

(iii) For all $x \in D_X(A)$, $R(t)x$ is continuously differentiable and

$$\dot{R}(t)x = R(t)Ax + \int_0^t R(t-s)K(s)xds$$

(3.2.1)

$$= AR(t)x + \int_0^t K(t-s)R(s)xds \quad .$$

In part 1, perturbation of the operator A of the form $A + BDC$ is considered. In part 2, perturbation of the kernel $K(\cdot)$ of the form $K(\cdot) + BH(\cdot)C$ is considered. In part 3 joint perturbation of both the operator A and the kernel $K(\cdot)$ is treated, in the case $K(\cdot) = k(\cdot)A$ and is perturbed to $A + BDC$. The result obtained in part 3 for this special subset of systems defined by (ID) has a nice graphical interpretation in terms of the Nyquist plot of $\hat{k}(s) = \int_0^\infty e^{-st}k(t)dt$. This allows a comparison to be drawn between the robustness of systems with and without the memory term $K(\cdot)$.

Part 1 Perturbation of the A operator.

Let $B \in L(U,\bar{X})$, $C \in L(\underline{X},Y)$ be fixed and $D \in L(Y,U)$ be arbitrary, where U and Y are complex Hilbert spaces and $\underline{X}$, X , $\bar{X}$ are complex Banach spaces satisfying

(R1) $\underline{X} \subseteq X \subseteq \bar{X}$ with continuous dense injections.

Again, as in the analysis of Chapter 1, to allow for the possible unboundedness of the operators B and C the perturbed system is taken to be

$$\text{(PR)} \qquad x(t) = R(t)x_0 + \int_0^t R(t-s)f(s)ds + \int_0^t R(t-s)BDCx(s)ds \ , \quad x_0 \in X$$

the mild solution of

$$\text{(PD)} \qquad \begin{aligned} \dot{x}(t) &= Ax(t) + BDCx(t) + \int_0^t K(t-s)x(s)ds + f(t) \\ x(0) &= x_0 \ . \end{aligned}$$

As in Chapter 1, in order to make sense of (PR) the following conditions (R2)-(R7) must be imposed on the operators B , C , A and $(R(t))_{t\geq 0}$ and the spaces $\underline{X}$, X and $\bar{X}$.

(R2) $R(t)$ extends (restricts) to a resolvent operator $\bar{R}(t) \in L(\bar{X})$ $(\underline{R}(t) \in L(\underline{X}))$ for

$$\dot{x}(t) = \bar{A}x(t) + \int_0^t \bar{K}(t-s)x(s)ds$$

$$(\dot{x}(t) = \underline{A}x(t) + \int_0^t \underline{K}(t-s)x(s)ds) \quad \text{on} \quad \bar{X}(\underline{X}) ,$$

where $\bar{A}$, $\bar{K}(\cdot)$ ($\underline{A}$ and $\underline{K}(\cdot)$) are closed extensions (restrictions) of A , $K(\cdot)$.

(R3) The domain of $\bar{A}$ on $\bar{X}$ is contained in $\underline{X}$

$D_{\bar{X}}(\bar{A}) \subseteq \underline{X}$ with continuous dense injection and $D_{\bar{X}}(\bar{K}(t)) \supseteq D_{\bar{X}}(\bar{A}), D_{\bar{X}}(\bar{A})$ is endowed with the graph norm.

(R4) There exists $\alpha, M > 0$ such that

$||R(t)|| \le Me^{-\alpha t}$ on all three spaces $\underline{X}$, X , $\bar{X}$.

(R5) There exists $K \ge 0$ such that

$$||\int_0^{\cdot} \bar{R}(\cdot-s)f(s)ds||_{L^2_{\underline{X}}} \le K||f(\cdot)||_{L^2_{\bar{X}}}$$

for all $f(\cdot) \in L^2_{\bar{X}}$.

(R6) For all $T > 0$ and $f(\cdot) \in L^2(0,T;\bar{X})$, $\int_0^T \bar{R}(s)f(s)ds \in X$ and there exists $b \ge 0$ such that

$$||\int_0^T \bar{R}(s)f(s)ds||_X \le b||f(\cdot)||_{L^2(0,T;\bar{X})} \quad , \quad b \text{ independent of } T .$$

(R7)

(i) There exists $c \ge 0$ such that

$$||CR(\cdot)x||_{L^2_Y} \le c||x||_X , \quad \text{for all} \quad x \in \underline{X}$$

(ii) There exists $c' \ge 0$ such that

$$||CR(\cdot)x||_{L^2_Y} \leq c'||x||_{\bar{X}} \quad , \quad \text{for all} \quad x \in \underline{X} \; .$$

(R8) For all $x \in D_{\bar{X}}(\bar{A})$, $\bar{K}(\cdot)x$ is measurable in $\bar{X}$, and there exists $\beta > 0$, $k(\cdot)$ such that $e^{\beta \cdot}k(\cdot) \in L^1(R_+)$ and

$$||\bar{K}(t)x||_{\bar{X}} \leq k(t)(||x||_{\bar{X}} + ||\bar{A}x||_{\bar{X}}) \quad .$$

Remark (2.1)

a) For systems satisfying an exponential growth as required by (R4), Grimmer and Prüss [29] have a Hille-Yosida type theorem which would be useful in the verification of (R2).

b) In comparing the assumptions (R1)-(R8) with assumptions (A1)-(A6) of Chapter 1 the more restrictive assumptions here are R(5), (R6). This is due to the possibility of $f(\cdot)$ taking values in $\bar{X}$. Also this strengthening of the assumptions is required in obtaining the characterisation of the stability radius.

c) Again smoothing action of the resolvent operator is exploited. This smoothing action is again inherent within many classes of problems (see example 2.39).

d) Assumption (R8) is quite common in the literature, Grimmer [31], when exponential stability of the resolvent operator is investigated. It is imposed here because the Laplace transform of $\bar{K}(t)x$, $x \in D_{\bar{X}}(\bar{A})$ is required in the sequel.

Notation.

If (R5) holds denote by $M_R \in L(L^2_U, L^2_{\underline{X}})$ the map

(3.2.2) $\quad (M_R u)(t) = \int_0^t \bar{R}(t-s)Bu(s)ds \ , \quad u(\cdot) \in L_U^2 \ ,$ and

(3.2.3) $\quad x_0(\cdot) = R(\cdot)x_0 + \int_0^{\cdot} \bar{R}(\cdot-s)f(s)ds \ .$

Notice that $Cx_0(\cdot) \in L_Y^2$, for all $f(\cdot) \in L_{\bar{X}}^2$ and $x_0 \in X$, whenever (R5) and (R7)(i) hold. The function $x_0(\cdot)$ is of interest when studying the forced problem with $f \not\equiv 0$.

Theorem (2.2)

Assume conditions (R1)-(R7)(i) hold and $||D|| < ||CM_R||^{-1}$, then there exists a unique strongly continuous family of operators $R^D(t) \in L(X)$ such that

a) for all $x_0 \in \underline{X}$, $T > 0$ $\quad R^D(\cdot)x_0 \in L^2(0,T;\underline{X})$,

b) for all $x_0 \in X$ $\quad x(t,x_0) = R^D(t)x_0$ satisfies

$$x(t,x_0) = R(t)x_0 + \int_0^t \bar{R}(t-s)BDCx(s,x_0)ds \ .$$

Proof.

Existence follows from direct application of theorem (1.5) with $M = M_R$, $C = C$ (compare with Theorem (2.6) Chapter 1). Uniqueness of $(R^D(t))_{t\geq 0}$ satisfying a) and b) follows from the uniqueness of the solution to the corresponding (V) equation and the dense continuous injection of $\underline{X} \subseteq X$. □

Corollary (2.3)

Assume conditions (R1)-(R6)(R7)(i) hold, $M = M_R$, $C = C$ satisfy condition (H2) and $||D|| < ||CM_R||^{-1}$. Then there exists a family of strongly continuous operators $(R^D(t))_{t\geq 0}$ such that $x(t,x_0) = R^D(t)x_0$ is the unique continuous solution of

$$x(t,x_0) = R(t)x_0 + \int_0^t \bar{R}(t-s)BDCx(s,x_0)ds \ , \quad \text{for all} \quad x_0 \in X \ .$$

Proof.

Existence and uniqueness of $(R^D(t))_{t\geq 0}$ follow by direct application of theorem (1.5) for $M = M_R$, $C = C$. □

Before proving that $(R^D(t))_{t\geq 0}$ is a resolvent operator, satisfying the perturbed versions of (3.2.1), it is of interest to consider the forced version of (PR). In many applications $f(\cdot)$ represents the "memory" of the "history" of the state, i.e. $f(t) = \int_{-\infty}^0 K(t-s)x(s)ds$, where $x(\cdot)$ is defined on $(-\infty,0]$ and the abstract differential equation is

$$\dot{x}(t) = Ax(t) + \int_{-\infty}^t K(t-s)x(s)ds \ , \quad x(0) = x_0 \ .$$

If the history of the state $x(t)$, $-\infty < t < 0$ is sufficiently well behaved so that $f(\cdot) \in L^2_X$ then it is possible to prove the following theorem.

Theorem (2.4)

Assume condition (R1)-(R7)(i) hold, $||D|| < ||CM_R||^{-1}$ then there exists a map $R^D \in L(\underline{L^2_X})$ such that for $x_0 \in X$ and $f(\cdot) \in L^2_X$.

a) $x_0(\cdot) \in D(R^D:L^2_X \to L^2_X)$, $(R^Dx_0(\cdot))(t)$ is continous in X and

b) $x(t) = (R^Dx_0(\cdot))(t)$ solves (PR).

If in addition (H2) holds for some $r \in \mathbb{N}_0$, then $(R^D x_0(\cdot))(t)$ is the unique solution of (PR), and for $f(\cdot) \in L^2_X$

$$(3.2.4) \qquad (R^D x_0(\cdot))(t) = R^D(t)x_0 + \int_0^t R^D(t-s)f(s)ds \quad \text{for all} \quad x_0 \in X .$$

Proof.

Let $\hat{x}(\cdot) \in L^2_{\underline{X}}$ and for $||D|| < ||CM_R||^{-1}$ let $y_{\hat{x}}(\cdot) \in L^2_Y$ denote the unique solution of

$$(3.2.5) \qquad y(\cdot) = C\hat{x}(\cdot) + (CM_R Dy)(\cdot) .$$

Define $R^D\hat{x}(\cdot) = \hat{x}(\cdot) + (M_R D y_{\hat{x}})(\cdot)$. Clearly $R^D \in L(L^2_{\underline{X}})$. Let $x_0(\cdot) \in L^2_X$ be defined by (3.2.3) then $x_0(\cdot) \in D(R^D : L^2_X \to L^2_{\underline{X}})$ and also $(R^D x_0(\cdot))(t)$ is continuous and satisfies (PR). The uniqueness of $(R^D x_0(.))(t)$ solving (PR) follows from corollary (2.3). If $f(\cdot) \in L^2_X$ then the right hand side of (3.2.4) is well defined and (3.2.4) follows by direct manipulation. □

Remark (2.5)

It is not possible to prove that (3.2.4) holds for $f(\cdot) \in L^2_{\bar{X}}$ since as yet $R^D(t) \in L(X)$ does not extend to $\bar{R}^D(t) \in L(\bar{X})$. However R^D is the map which takes the pair $(x_0, f(\cdot)) \in X \times L^2_{\bar{X}}$ to the unique continuous solution $x(\cdot)$ of (PR).

Definition (2.6)

A stability radius for the inhomogeneous equation (PR) is

$$\rho_{\mathbb{C}}(A;B,C) = \sup_{d}\{||D|| < d \text{ implies there exists } R^D \in L(L^2_{\underline{X}}) \text{ such that } x(t) = (R^D x_0(\cdot))(t) \text{ solves (PR) for any } x_0 \in \underline{X} \text{ and } f(\cdot) \in L^2_{\bar{X}}\} \ .$$

Remark (2.7)

If $f(\cdot)$ is viewed as an input then this stability radius measures the robustness of L^2-input/L^2-state stability of the system defined by equation (PR) when $x_0 \in \underline{X}$ and $f(\cdot) \in L^2_{\bar{X}}$. (See Miller and Wheeler [34]). In order to establish a formula for $\rho_{\mathbb{C}}(A;B,C)$ the following lemma concerning the Laplace transform of $\bar{R}(t)x$ is required.

Lemma (2.8) (Grimmer and Prüss [29])

Assume (R1)(R2)(R3)(R4) and (R8) hold. Define, for all $x \in \bar{X}$ and $\operatorname{Re} \lambda > \max(-\beta,-\alpha)$, $\hat{\bar{R}}(\lambda) \in L(\bar{X}, D_{\bar{X}}(A))$ by

$$\hat{\bar{R}}(\lambda)x := \int_0^\infty e^{-\lambda t}\bar{R}(t)xdt \quad ,$$

then

$$\hat{\bar{R}}(\lambda)x = (\lambda I - \bar{A} - \hat{\bar{K}}(\lambda))^{-1}x$$

where

$$\hat{\bar{K}}(\lambda)x = \int_0^\infty e^{-\lambda t}\bar{K}(t)xdt \quad , \quad x \in D_{\bar{X}}(\bar{A}) \ .$$

Theorem (2.9)

Assume conditions (R1)-(R6)(R7)(i)(R8) and $\sup_{\omega}||C\hat{\bar{R}}(i\omega)B|| < \infty$

then $\rho_{\mathbb{C}}(A;B,C) = ||CM_R||^{-1} = \inf_{\omega} ||C\hat{R}(i\omega)B||^{-1}$.

Proof

Step 1.

The equality $||CM_R||^{-1} = \inf_{\omega}||C\hat{R}(i\omega)B||^{-1}$ follows by Plancherel's theorem analogously to proposition (3.6) Chapter 1.

Step 2.

If $||D|| < ||CM_R||^{-1}$ then existence of $R^D \in L(L^2_{\underline{X}})$ satisfying the conditions of definition (2.6) is guaranteed. Consequently $\rho(A;B,C) \geq ||CM_R||^{-1}$.

Step 3.

In order to show that $||CM_R||^{-1}$ is equal to ρ (A;B,C) it is required that for any $\varepsilon > 0$ there exists D , $||D|| < ||CM_R||^{-1} + \varepsilon$, $x_0 \in \underline{X}$ and $f(\cdot) \in L^2_{\underline{X}}$ such that the corresponding solution $x(\cdot) \notin L^2_{\underline{X}}$. Choose $\delta > 0$, $\omega_0 \in \mathbb{R}$, $u \in U$ $||u||_U = 1$ such that

$$\sup_{\omega}||C\hat{R}(i\omega)B|| \leq ||C\hat{R}(i\omega_0)Bu||_Y + \delta.$$

Define $D \in L(Y,U)$ by

$$Dy = \frac{u\langle y,v\rangle_Y}{||v||^2_Y} \quad \text{where} \quad v = C\hat{R}(i\omega_0)Bu \ .$$

It is easy to establish that

$$||D|| = ||C\hat{R}(i\omega_0)Bu||^{-1}_Y \leq [\sup_{\omega}||C\hat{R}(i\omega)B||-\varepsilon]^{-1} \ .$$

Also if $x_0 = \hat{\bar{R}}(i\omega_0)BDv$, then by lemma (2.8) $x_0 \in D_{\bar{X}}(\bar{A}) \subseteq \underline{X}$ by (R3). Moreover

$$(i\omega_0 I - \bar{A} - \bar{K}(i\omega_0))x_0 = BDv \ , \quad Dv = u$$

and hence $Cx_0 = v$. Therefore

$$i\omega_0 x_0 = (\bar{A} + BDC)x_0 + \hat{\bar{K}}(i\omega_0)x_0 \quad .$$

Set

$$f(t) = \int_{-\infty}^{0} \bar{K}(t-s)e^{i\omega_0 s}x_0 ds = e^{i\omega_0 t}\int_t^{\infty} e^{-i\omega_0 \rho}\bar{K}(\rho)x_0 d\rho \quad .$$

Using (R8) it is easy to show that $f(\cdot) \in L^2_{\bar{X}}$. Finally $x(t) = e^{i\omega_0 t}x_0$ satisfies $x(t) \in D_{\bar{X}}(\bar{A})$ for all t and therefore

$$\dot{x}(t) = (\bar{A}+BDC)x(t) + \int_0^t \bar{K}(t-s)x(s)ds + f(t) \ , \quad x(0) = x_0 \ .$$

Consequently $x(\cdot)$ satisfies (PR) and $x_0 \in \underline{X}$, $f(\cdot) \in L^2_{\bar{X}}$ but $x(\cdot) \notin L^2_{\underline{X}}$. Therefore

$$\rho_{\mathbb{C}}(A;B,C) \le [\sup_{\omega}||C\hat{\bar{R}}(i\omega)B|| - \delta]^{-1}$$

and the result follows. □

Remark (2.10)

This construction proves that if $||D|| > \rho_{\mathbb{C}}(A;B,C)$ and the "history" of the state is periodic then the perturbed system can stay in this periodic state.

Remark (2.11)

If in addition to (A1)-(A7)(i)(A8) holding (H2) holds then for $||D|| < \rho(A;B,C)$ the unique solution $x(\cdot)$ of (PR) guaranteed by theorem (2.4), for $x_0 \in X$, $f(\cdot) \in L^2_{\bar{X}}$, satisfies $||x(\cdot)||_{L^2_X} \leq k||x_0(\cdot)||_{L^2_X}$.

The following theorem is the first step in converting the L^2-input/L^2-state stability-radius of definition (2.6), theorem (2.9) to an exponential stability-radius for the family of operators $(R(t))_{t\geq 0}$.

Theorem (2.12)

Assume conditions (R1)-(R6), R(7)(ii) and (R8) hold and $||D|| < ||CM_R||^{-1}$, then there exists a unique resolvent operator $\bar{R}^D(t) \in L(\bar{X})$ for (PD) in the sense that

i) $\bar{R}^D(\cdot)$ is strongly continuous on $\bar{X}$, $\bar{R}^D(0) = I_{\bar{X}}$;

ii) $\bar{R}^D(\cdot)$ is strongly continuous on $D_{\bar{X}}(\bar{A}) = D_{\bar{X}}(\bar{A}+BDC)$

and

iii) $$\dot{\bar{R}}^D(t)x_0 = \bar{R}^D(t)(\bar{A}+BDC)x_0 + \int_0^t \bar{R}^D(t-s)\bar{K}(s)x_0 ds$$
$$= (\bar{A}+BDC)\bar{R}^D(t)x_0 + \int_0^t \bar{K}(t-s)\bar{R}^D(s)x_0 ds$$

for all $x_0 \in D_{\bar{X}}(\bar{A})$.

Proof.

The existence and strong continuity of $\bar{R}^D(t)$ follows just as in theorem (2.2), defined by the pair of equations,(using the additional assumption (R7)(ii))

$$(3.2.6)\quad \begin{cases} \text{(a)} \quad \bar{R}^D(t)\bar{x} = \bar{R}(t)\bar{x} + \int_0^t \bar{R}(t-s)BDy(s)ds \\ \\ \text{(b)} \quad y(t) = C\bar{R}(t)\bar{x} + C\int_0^t \bar{R}(t-s)BDy(s)ds \ , \ \bar{x} \in \bar{X} \ . \end{cases}$$

This proves condition (i).

For condition (ii) and the first part of (iii) let $x_0 \in D_{\bar{X}}(\bar{A})$. Then $(\bar{A} + BDC)x_0 \in \bar{X}$ and therefore $x_1(\cdot) = \bar{R}^D(t)(\bar{A}+BDC)x_0$ is continuous in $\bar{X}$, $y_1(\cdot) = Cx_1(\cdot) \in L_Y^2$ and $(x_1(\cdot), y_1(\cdot))$ satisfy their defining equations

$$(3.2.7)\quad \begin{cases} \text{(a)} \quad y(t) = C\bar{R}(t)(\bar{A} + BDC)x_0 + C\int_0^t \bar{R}(t-s)BDy(s)ds \\ \\ \text{(b)} \quad x(t) = \bar{R}(t)(\bar{A} + BDC)x_0 + \int_0^t \bar{R}(t-s)BDy(s)ds \ . \end{cases}$$

Moreover $y_1(\cdot)$ is the unique solution for $y(\cdot) \in L_Y^2$ of (3.2.7)(a).

Consider, now, the equation pair,

$$(3.2.8)\quad \begin{cases} \text{a)} \quad y(t) = C\int_0^t \bar{R}(t-s)\bar{K}(s)x_0 ds + C\int_0^t \bar{R}(t-s)BDy(s)ds \\ \\ \text{b)} \quad x(t) = \int_0^t \bar{R}(t-s)\bar{K}(s)x_0 ds + \int_0^t \bar{R}(t-s)BDy(s)ds \ . \end{cases}$$

Notice that $C\int_0^{\cdot} \bar{R}(\cdot-s)\bar{K}(s)x_0 ds \in L_Y^2$ and therefore, since $||D|| < \frac{1}{||CM_R||}$, $y(\cdot) \in L_Y^2$. Hence, $x(t)$ is continuous in $\bar{X}$. Denote these solutions by $y_2(\cdot)$ and $x_2(\cdot)$ respectively. Then by definition

$x_2(t) = \int_0^t \bar{R}^D(t-s)\bar{K}(s)x_0 ds$, and $y_2(\cdot)$ is the unique solution of (3.2.8)(a). Now the following equation has a unique solution for $y(\cdot) \in L_Y^2$,

$$(3.2.9)\quad y(t) = C\bar{R}(t)(\bar{A}+BDC)x_0+C\int_0^t \bar{R}(t-s)\bar{K}(s)x_0 ds+ C\int_0^t \bar{R}(t-s)BDy(s)ds \ .$$

Denote this solution by $y_3(\cdot)$ and define $x_3(\cdot)$, a continuous function in $\bar{X}$, by,

$$x_3(t) = \bar{R}(t)(\bar{A}+BDC)x_0 + \int_0^t \bar{R}(t-s)\bar{K}(s)x_0 ds + \int_0^t \bar{R}(t-s)BDy_3(s)ds \ .$$

As in corollary (2.9) Chapter 1, it is possible to show that

$$\int_0^t y_3(s)ds = C\bar{R}^D(t)x_0-Cx_0 \quad \text{and} \quad \int_0^t x_3(s)ds = \bar{R}^D(t)x_0-x_0 \quad .$$

Hence $C\bar{R}^D(t)x_0$ is differentiable, $C\dot{\bar{R}}^D(\cdot)x_0 \in L_Y^2$ and $\bar{R}^D(t)x_0$ is continuously differentiable. Now $y_1(\cdot)$ and $y_2(\cdot)$ defined by (3.2.7)(a) and (3.2.8)(a) respectively, satisfy

$$y_1(t) + y_2(t) = C\bar{R}(t)(\bar{A}+BDC)x_0 + C\int_0^t \bar{R}(t-s)\bar{K}(s)x_0 ds$$
$$+ C\int_0^t \bar{R}(t-s)BD(y_1(s) + y_2(s))ds$$

and therefore by the uniqueness of $y_3(\cdot)$ solving (3.2.9)

$$y_3(\cdot) = y_1(\cdot) + y_2(\cdot) \quad .$$

Hence $x_3(\cdot) = x_1(\cdot) + x_2(\cdot)$ and this proves

$$\dot{\bar{R}}^D(t)x_0 = \bar{R}^D(\bar{A}+BDC)x_0 + \int_0^t \bar{R}^D(t-s)\bar{K}(s)x_0 ds \ .$$

For the second part of (iii) note that

$$\bar{R}^D(t)x_0 = \bar{R}(t)x_0 + \int_0^t \bar{R}(t-s)BDy(s)ds \quad ,$$

with

$$y(t) = \int_0^t y_3(s)ds + Cx_0 \;, \quad y_3(\cdot) \in L^2_Y$$

and therefore (see Grimmer and Prüss [29]) $\bar{R}^D(t)x_0 \in D_{\bar{X}}(\bar{A})$ and

$$\dot{\bar{R}}^D(t)x_0 = \dot{\bar{R}}(t)x_0 + \bar{A}\int_0^t \bar{R}(t-s)BDy(s)ds + \int_0^t \bar{K}(t-s)\int_0^s \bar{R}(s-\rho)BDy(\rho)d\rho ds$$

$$+ BDC\bar{R}^D(t)x_0$$

$$= (\bar{A} + BDC)\bar{R}^D(t)x_0 + \int_0^t \bar{K}(t-s)\bar{R}^D(s)x_0 ds$$

and part (ii) follows also. Uniqueness follows as in corollary (2.9) Chapter 1. □

Remark (2.13)

When A is the generator of a semigroup assumptions (R1)-(R6)(R7)(i) (R8) require that the allowable unboundedness is of degree A. However (R7)(ii) restricts this degree to that of size $(-A)^{\frac{1}{2}}$, when A is negative.

The remainder of this part of section 3 is concerned with strengthening assumptions (R1)-(R8) in order that the L^2-input/L^2-state stability-radius becomes an exponential stability radius. These assumptions relate to the possibility of separating the smoothing action of the resolvent operator from the exponential stability of the resolvent operator (compare with assumption (A9) Chapter 1).

Let $\varepsilon^* > 0$, be guaranteed by lemma (2.8), such that

$$\hat{\bar{R}}(i\omega-\varepsilon) \in L(\bar{X},\underline{X}) \quad \text{for all} \quad \varepsilon \in [0,\varepsilon^*)$$

and assume that

(R9) $$\lim_{\varepsilon \downarrow 0} \sup_{\omega} ||C\hat{\bar{R}}(i\omega-\varepsilon)B|| = \sup_{\omega} ||C\hat{\bar{R}}(i\omega)B||$$

(R10) There exists $c \geq 0$ such that for all $\varepsilon \in [0,\varepsilon^*)$ and for all $x \in \underline{X}$,

$$||Ce^{\varepsilon \cdot}\bar{R}(\cdot)x||_{L^2_Y} \leq c||x||_{\bar{X}} \quad .$$

(R11) There exists $b \geq 0$ such that for all $\varepsilon \in [0,\varepsilon^*)$ and $u(\cdot) \in L^2_U$,

$$||\int_0^T e^{\varepsilon s}\bar{R}(s)Bu(s)ds||_X \leq b||u(\cdot)||_{L^2(0,T;U)} \quad .$$

Remark (2.14)

When $K(t) \equiv 0$ and A is the generator of a semigroup $(S(t))_{t\geq 0}$ satisfying (A1)-(A5)(A6)(iii), then (R10) and (R11) are immediate.

Definition (2.15)

A stability radius (for exponential stability) of the resolvent operator for (PD) is

$$\mu_{\mathbb{C}}(A;B,C) = \sup_d \{||D|| < d \text{ implies that there exists } (R^D(t))_{t\geq 0},(\bar{R}^D(t))_{t\geq 0} \text{ for (PR) and (PD) respectively, which are exponentially stable}\}.$$

Remark (2.16)

$R^D(t)$ solving (PR), $f(\cdot) \equiv 0$ is interpreted in the sense of corollary (2.3). $\bar{R}^D(t)$ solving (PD) is interpreted in the sense of theorem (2.12).

Theorem (2.17)

Assume (R1)-(R6)(R7)(ii)(R8)-(R11) and (H2) hold, then

$$\mu_{\mathbb{C}}(A;B,C) = ||CM_R||^{-1} = \inf_{\omega\in\mathbb{R}} ||C\hat{\bar{R}}(i\omega)B||^{-1} .$$

Proof.

Step 1. Assume $||D|| < ||CM_R||^{-1}$ then by corollary (2.3) and theorem (2.12) $R^D(t) \in L(X)$ and $\bar{R}^D(t) \in L(\bar{X})$ exist and also

$$||R^D(t)||_{L(X)} \le K , \quad ||\bar{R}^D(t)||_{L(\bar{X})} \le K , \qquad t \ge 0 .$$

By assumption (R9) choose $\varepsilon > 0$, sufficiently small, such that $||D|| < \inf_{\omega} ||C\hat{\bar{R}}(i\omega-\varepsilon)B||^{-1}$ and consider

$$\text{(a)} \quad x_\varepsilon(t) = e^{\varepsilon t}R(t)x_0 + \int_0^t e^{\varepsilon(t-s)}\bar{R}(t-s)BDCx(s)ds$$

(3.2.1)

$$\text{(b)} \quad y_\varepsilon(t) = e^{\varepsilon t}CR(t)x_0 + C\int_0^t e^{\varepsilon(t-s)}\bar{R}(t-s)BDy(s)ds .$$

Denote by $G_\varepsilon : L^2_U \to L^2_Y$ the map given by

$$(G_\varepsilon u(\cdot))(t) = C\int_0^t e^{\varepsilon(t-s)}\bar{R}(t-s)Bu(s)ds \quad .$$

As for $||CM_R||$, $||G_\varepsilon|| = \sup_{\omega} ||C\hat{\bar{R}}(i\omega-\varepsilon)B||$. Therefore under assumptions

(R9)-(R11) the equation (3.2.10)(a) has a solution $x_\varepsilon(\cdot)$. But then $e^{-\varepsilon t}x_\varepsilon(t)$ for $x_0 \in X$ $(x_0 \in \bar{X})$ satisfies (PR) ((PD)) and therefore by uniqueness of $x(t,x_0) = R^D(t)x_0 (x(t,x_0) = \bar{R}^D(t)x_0)$, $e^{-\varepsilon t}x_\varepsilon(t) = R^D(t)x_0$ $(e^{-\varepsilon t}x_\varepsilon(t) = \bar{R}^D(t)x_0)$ and therefore using (R10) and (R11) and the exponential stability of $R(t)$, $\bar{R}(t)$

$$||e^{\varepsilon t}R^D(t)||_{L(X)} \le K , \quad ||e^{\varepsilon t}\bar{R}^D(t)||_{L(\bar{X})} \le K$$

for $\varepsilon > 0$ sufficiently small.

Consequently $\mu_{\mathbb{C}}(A;B,C) \ge \inf_{\omega\in\mathbb{R}} ||C\hat{\bar{R}}(i\omega)B||^{-1}$.

Step 2. If $R^D(t)$ and $\bar{R}^D(t)$ exist, carrying out the construction of theorem (2.9), then for all $\hat{\varepsilon} > 0$ there exists $D \in L(Y,U)$, $||D|| \le \inf_{\omega} ||C\hat{\bar{R}}(i\omega)B||^{-1} + \hat{\varepsilon}$, $\omega_0 \in \mathbb{R}$ and $x_0 \in \underline{X}$ such that $x(t) = e^{i\omega_0 t}x_0$ satisfies

$$x(t) = R(t)x_0 + \int_0^t \bar{R}(t-s)BDCx(s)ds + \int_0^t \bar{R}(t-s)f(s)ds$$

where $f(t) = e^{i\omega_0 t}\int_t^\infty e^{i\omega_0 \rho}\bar{K}(\rho)x_0 d\rho$. If $R^D(t)$ and $\bar{R}^D(t)$ exist then $x(t) = R^D(t)x_0 + \int_0^t \bar{R}^D(t-s)f(s)ds$. This proves that both $R^D(t)$ and $\bar{R}^D(t)$ cannot be stable because if they both were, then necessarily $||R^D(t)x_0||_{\bar{X}} \to 0$ and $||\int_0^t \bar{R}^D(t-s)f(s)ds||_{\bar{X}} \to 0$ as $t \to \infty$, which contradicts $||x(t)||_{\bar{X}} = ||x_0||_{\bar{X}}$. □

Remark (2.18)

Thus under additional smoothing properties of the resolvent operator and also conditions involving the separation of the smoothing and exponential decaying of the semigroup, robustness of L^2-input/L^2-state stability is equal to robustness of exponential stability. Of course matters are simplified by the fact that $R(t)$ and $\bar{R}(t)$ are a priori exponentially stable.

Establishing (R1)-(R11) and performing the calculation $\inf_{\omega} ||C\hat{R}(i\omega)B||^{-1}$ would in general be quite complicated. However at the end of this section the analysis is applied to a very simple example.

Part 2 Perturbations of the kernel $K(\cdot)$.

In this part of section 2 perturbation of the kernel $K(\cdot)$ is considered. Due to the problems encountered in the perturbation of time varying systems (see section 3) the perturbation is assumed to take the form $K(t) + BH(t)C$ where $B \in L(U,\bar{X})$ and $C \in L(\underline{X},Y)$ and $H(\cdot) \in B^1_-(0,\infty;L(Y,U))$.

This condition upon $H(\cdot)$ is quite common in the literature when exponential stability of resolvent operators is analysed (see e.g. Grimmer [31]). Here $H(\cdot) \in B^1_-(0,\infty;L(Y,U))$ if $H(\cdot)$ is strongly measurable and there exists $\beta > 0$ such that

$$\int_0^\infty e^{\beta t}||H(t)||_{L(Y,U)}dt < \infty \quad .$$

Again due to the possibility that $\underline{X} \neq X \neq \bar{X}$ the perturbed system is, for $x_0 \in X$, (PK) $$x(t) = R(t)x_0 + \int_0^t \bar{R}(t-s)f(s)ds + \int_0^t \bar{R}(t-s)B\int_0^s H(s-\rho)Cx(\rho)d\rho ds \ .$$

Notation.

Denote by $L_H : L_Y^2 \to L_U^2$ the map defined by

$$(L_H y)(t) = \int_0^t H(t-s)y(s)ds \quad .$$

The following theorems are direct generalisations of theorems (2.2), (2.4) of part 1.

Theorem (2.19)

Assume conditions (R1)-(R7)(i) hold, $H(\cdot) \in \mathcal{B}_-^1(0,\infty;\mathcal{L}(Y,U))$ and $||L_H|| = \sup_{\omega\in\mathbb{R}}||\hat{H}(i\omega)|| < ||CM_R||^{-1}$, then there exists a unique strongly continuous family of operators $R^H(t) \in \mathcal{L}(X)$ satisfying

a) for all $x_0 \in \underline{X}$ $R^H(\cdot)x_0 \in L_{\underline{X}}^2$ and

b) for all $x_0 \in X$ $R^H(t)x_0$ satisfies

$$R^H(t)x_0 = R(t)x_0 + \int_0^t \bar{R}(t-s)B\int_0^s H(s-\rho)CR^H(\rho)x_0 d\rho ds$$

Corollary (2.20)

Assume conditions (R1)-(R7)(i) hold, $H(\cdot) \in \mathcal{B}_-^1(0,\infty;\mathcal{L}(Y,U))$ $||L_H|| < ||CM_R||^{-1}$ and additionally that (H2) holds for $C = C$, $M = M_R$ and $D = L_H$, then there exists a unique strongly continuous family of operators $R^H(t) \in \mathcal{L}(X)$ such that for all $x_0 \in X$,

$x(t,x_0) = R^H(t)x_0$ is the unique (continuous) solution of

$$x(t,x_0) = R(t)x_0 + \int_0^t \bar{R}(t-s)B\int_0^s H(s-\rho)Cx(\rho,x_0)d\rho ds \ .$$

Theorem (2.21)

Assume conditions (R1)-(R7)(i) hold, $H(\cdot) \in B^1_-(0,\infty;L(Y,U))$ and $||L_H|| < ||CM_R||^{-1}$, then there exists $R^H \in L(L^2_{\underline{X}})$ such that for all $x_0 \in \underline{X}$ and $f(\cdot) \in L^2_{\bar{X}}$, $R^H(x_0(\cdot))(t)$ is continuous with values in X. Also for $x_0 \in X$, $x_0(\cdot) \in D(R^H : L^2_X \to L^2_X)$ and if in addition (H2) holds then $(R^H x_0(\cdot))(t)$ is the unique solution, continuous in X of equation (PK) and if additionally $f(\cdot) \in L^2_X$ then

$$(3.2.11) \quad R^H(x_0(\cdot))(t) = R^H(t)x_0 + \int_0^t R^H(t-s)f(s)ds \ . \qquad \square$$

Remark (2.22)

As yet formula (3.2.11) does not hold for all $f(\cdot) \in L^2_{\bar{X}}$ because $R^H(t)$ is not necessarily defined on $\bar{X}$. The operator R^H does, however, give the unique continuous solution $x(\cdot)$ of (PK) when $x_0 \in X$ and $f(\cdot) \in L^2_{\bar{X}}$. This existence of the map R^H allows the definition of a stability radius for the inhomogeneous equation (PK).

Definition (2.23)

A stability radius for the inhomogeneous equation (PK) is

$$\rho_{\mathbb{C}}(K;B,C) = \sup_d\{||L_H|| < d \text{ implies there exists an operator } R^H \in L(L^2_{\underline{X}}) \text{ such that } (R^H x_0(\cdot))(t) \text{ is a continuous solution of (PK) for any } x_0(\cdot) \in L^2_{\underline{X}}\}.$$

Remark (2.24)

For $f(\cdot)$ viewed as an input, $\rho_{\mathbb{C}}(K;B,C)$ measures the robustness of L^2-input/L^2-state stability to perturbations of the kernel $K(\cdot)$. Of course, in applying $\rho_{\mathbb{C}}(K;B,C)$ in this context, uniqueness of the solution of the inhomogeneous equation (PK) is required.

Theorem (2.25)

Assume conditions (R1)-(R7)(i)(R8) hold and $H(\cdot) \in B^1_-(0,\infty;L(Y,U))$, then

$$\rho_{\mathbb{C}}(K;B,C) = \inf_{\omega} ||C\hat{R}(i\omega)B||^{-1} \quad .$$

Proof.

If $||L_H|| < \inf_{\omega} ||C\hat{R}(i\omega)B||^{-1}$ then existence of $\mathcal{R}^H \in L(L^2_{\underline{X}})$ is immediate from theorem (2.21).

As in the proof of theorem (2.9) the main step is to construct a destabilising perturbation $H(\cdot) \in B^1_-(0,\infty;L(Y,U))$. Let $\delta > 0$ and choose $\omega_0 \in \mathbb{R}$ and $u \in U$ such that

$$||C\hat{R}(i\omega_0)Bu|| \geq \sup_{\omega} ||C\hat{R}(i\omega)B|| - \delta \quad .$$

Define for $y \in Y$

$$\hat{H}(i\omega)y = (1 + i(\omega-\omega_0))^{-1}u \frac{<y,v>_Y}{||v||^2_Y}$$

where $v = C\hat{R}(i\omega_0)Bu$. It is easy to show that

$$H(t)y = e^{(-1+i\omega_0)t} u \frac{<y,v>}{||v||^2_Y} \text{ satisfies } \hat{H}(i\omega) = \int_0^\infty e^{-i\omega t}H(t)dt \ ,$$

and therefore $H(\cdot) \in B^1_-(0,\infty;L(Y,U))$ and $||L_H|| = \dfrac{1}{||C\hat{\bar{R}}(i\omega_0)Bu||}$.

Moreover $x_0 = \hat{\bar{R}}(i\omega_0)B\hat{H}(i\omega_0)v \in D_{\bar{X}}(\bar{A})$ and

$$(i\omega_0-\bar{A}-\hat{\bar{K}}(i\omega_0))x_0 = B\hat{H}(i\omega_0)v \ , \quad \hat{H}(i\omega_0)v = u \ .$$

Hence $Cx_0 = v$ and $i\omega_0 x_0 = \bar{A}x_0 + \hat{\bar{K}}(i\omega_0)x_0 + B\hat{H}(i\omega_0)Cx_0$.

Define $f(\cdot)$, $f(t) = e^{i\omega_0 t}[\int_t^\infty e^{-i\omega_0 s}\bar{K}(s)x_0 ds$

$$+ \int_t^\infty e^{-i\omega_0 s} BH(s)Cx_0 ds], t \geq 0 \ .$$

Using (R8) and the definition of $H(\cdot)$ it follows that $f(\cdot) \in L^2_{\bar{X}}$. ($f(\cdot)$ is the perturbed systems memory of the state history $x(t) = e^{i\omega_0 t}x_0$, $t \in (-\infty,0)$). For $t \in [0,\infty)$ set $x(t) = e^{i\omega_0 t}x_0$, then $x(t)$ satisfies

$$\dot{x}(t) = \bar{A}x(t) + \int_0^t \bar{K}(t-s)x(s)ds + B\int_0^t H(t-s)Cx(s)ds + f(t), \ x(0) = x_0 .$$

However $x(t) \in D_{\bar{X}}(\bar{A})$ for all $t \in [0,\infty)$ and therefore satisfies (PK). Also $x_0 \in \underline{X}$ and $f(\cdot) \in L^2_{\bar{X}}$ but $x(\cdot) \notin L^2_{\underline{X}}$. Consequently

$$\rho_{\mathbb{C}}(K;B,C) \leq (\sup_{\omega}||C\hat{\bar{R}}(i\omega)B|| - \delta)^{-1}$$

and the destabilisation result follows. □

Remark (2.26)

If (H2) in addition to assumptions (A1)-(R7)(i)(R8) hold, then for $H \in B^1_-(0,\infty;L(Y,U))$, such that $||L_H|| < \rho_{\mathbb{C}}(K;B,C)$, the unique solution, $x(\cdot)$, of (PK) guaranteed by theorem (2.21) for $x_0 \in X$, $f(\cdot) \in L^2_{\bar{X}}$ has the property that $x(\cdot) \in L^2_X$.

As in part 1 of this section, in order that this stability radius for L^2-stability of the inhomogeneous equation be converted into one for exponential stability of the resolvent operator, it is required that $R^H(t) \in L(X)$ extends to a strongly continuous family of operators on $L(\bar{X})$. A consequence of this is that the extension $\bar{R}^H(t)$ defines a resolvent operator for

$$(3.2.12) \qquad \dot{x}(t) = \bar{A}x(t) + \int_0^t \bar{K}(t-s)x(s)ds + B\int_0^t H(t-s)Cx(s)ds \quad \text{on} \quad \bar{X} .$$

By a resolvent operator for (3.2.12) it is meant that

i) $\bar{R}^H(0) = I_{\bar{X}}$

ii) $\bar{R}^H(\cdot)$ is continuous on $\bar{X}$ and $D_{\bar{X}}(\bar{A})$

iii) $\bar{R}^H(\cdot)x_0$ is continuously differentiable in $\bar{X}$ for all $x_0 \in D_{\bar{X}}(\bar{A})$ and

$$(3.2.13) \qquad \begin{aligned} \dot{\bar{R}}^H(t)x_0 &= \bar{R}^H(t)\bar{A}x_0 + \int_0^t \bar{R}^H(t-s)\bar{K}(s)x_0 ds + \int_0^t \bar{R}^H(t-s)BH(s)Cx_0 ds \\ &= \bar{A}\bar{R}^H(t)x_0 + \int_0^t \bar{K}(t-s)\bar{R}^H(s)x_0 ds + B\int_0^t H(t-s)C\bar{R}_H(s)x_0 ds \quad . \end{aligned}$$

Theorem (2.27)

Assume (R1)-(R6)(R7)(ii)-(R8) hold $||L_H|| < ||CM_R||^{-1}$, then there exists a family of operators $\bar{R}^H(t) \in L(\bar{X})$, extensions of $R^H(t) \in L(X)$ constructed in theorem (2.19). Moreover $\bar{R}^H(t)$ is the unique resolvent operator for (3.2.12).

Proof.

Existence of the strongly continuous family $\bar{R}^H(t) \in L(\bar{X})$ extending $R^H(t) \in L(X)$ follows by (R7)(ii) and solution of the equation pair,

$$(3.2.14) \begin{cases} y(t) = C\bar{R}(t)x_0 + C\int_0^t \bar{R}(t-s)B\int_0^s H(s-\rho)y(\rho)d\rho ds \\ x(t,x_0) = \bar{R}(t)x_0 + \int_0^t \bar{R}(t-s)B\int_0^s H(s-\rho)y(\rho)d\rho ds \end{cases}$$

on defining $\bar{R}^H(t)x_0 = x(t,x_0)$. The formulas (3.2.13) follow analogously to those in theorem (2.12). For the uniqueness let $\hat{R}^H(t)$ be a second resolvent operator for (3.2.12). Then uniqueness follows using (R3) and the identity

$$\bar{R}^H(t)x_0 - \hat{R}^H(t)x_0 = \int_0^t \frac{d}{ds} (\hat{R}^H(t-s)\bar{R}^H(s)x_0)ds$$

$$= 0 \quad \text{for all} \quad t \geq 0 \ , \ x_0 \in D_{\bar{X}}(\bar{A}) \ . \qquad \square$$

Definition (2.28)

A stability radius for (3.2.12) is

$\mu_{\mathbb{C}}(K,B,C) = \sup_d\{||L_H|| < d$ implies there exists $R^H(t)$ and $\bar{R}^H(t)$, a resolvent operator for (3.2.12), which are exponentially stable}.

Theorem (2.29)

Assume (R1)-(R6)(R7)(ii)(R8)-(R11)(H2) hold and $H(\cdot) \in B^1_-(0,\infty;L(Y,U))$ then

$$\mu_{\mathbb{C}}(K;B,C) = ||CM_R||^{-1} = \inf_{\omega\in\mathbb{R}} ||C\hat{\bar{R}}(i\omega)B||^{-1} .$$

Proof.

Existence of $R^H(t) \in L(X)$, $\bar{R}^H(t) \in L(\bar{X})$ for $||L_H|| < ||CM_R||^{-1}$ follow immediately from theorem (2.27). For the exponential stability of $R^H(t)$, $\bar{R}^H(t)$ consider the following equation pair

$$(3.2.15)\begin{cases} x(t) = e^{\varepsilon t}\bar{R}(t)x_0 + \int_0^t e^{\varepsilon(t-s)}\bar{R}(t-s)B\int_0^s e^{\varepsilon(s-\rho)}H(s-\rho)y(\rho)d\rho ds \\ \\ y(t) = e^{\varepsilon t}C\bar{R}(t)x_0 + C\int_0^t e^{\varepsilon(t-s)}\bar{R}(t-s)B\int_0^s e^{\varepsilon(s-\rho)}H(s-\rho)y(\rho)d\rho ds \end{cases} .$$

For $\varepsilon > 0$ sufficiently small $H_\varepsilon(.) = e^{\varepsilon\cdot}H(.) \in B^1_-(0,\infty;L(Y,U))$ and $||L_{H_\varepsilon}|| < ||CM_R||^{-1}$. Hence $||G_\varepsilon|| \, ||L_{H_\varepsilon}|| < 1$ and therefore (R10)(R11) imply there exists $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ solving (3.2.15). Additionally $||x_\varepsilon(t)||_X \leq K||x_0||_X$ for all $x_0 \in X$ ($||x_\varepsilon(t)||_{\bar{X}} \leq K||x_0||_{\bar{X}}$ for all $x_0 \in \bar{X}$). By the uniqueness of $R^H(t) \in L(X)$ ($\bar{R}^H(t) \in L(\bar{X})$) . $x_\varepsilon(\cdot) = e^{\varepsilon t}R^H(t)x_0$ ($= e^{\varepsilon t}\bar{R}^H(t)x_0$) for all $x_0 \in X$ ($x_0 \in \bar{X}$) and consequently

$$||R^H(t)||_{L(X)} \leq Ke^{-\varepsilon t} , \; (||\bar{R}^H(t)||_{L(\bar{X})} \leq Ke^{-\varepsilon t}) \qquad t \geq 0 .$$

For $\delta > 0$, choose $H(\cdot) \in \mathcal{B}^1_-(0,\infty;\mathcal{L}(Y,U))$, $\omega_0 \in \mathbb{R}$, $x_0 \in D_{\bar{X}}(\bar{A})$, $f(\cdot) \in L^2_{\bar{X}}$ as in theorem (2.25). Then $\mathcal{R}^H(x_0(\cdot)) \notin L^2_{\underline{X}}$ and $e^{i\omega_0 t}x_0 = \mathcal{R}^H(x_0(\cdot))(t) = R^H(t)x_0 + \int_0^t \bar{R}^H(t-s)f(s)ds$ and therefore $(\sup_\omega \|C\hat{\bar{R}}(i\omega)B\| - \delta)^{-1} \geq \mu_{\mathbb{C}}(K;B,C)$. Hence

$$\mu_{\mathbb{C}}(K;B,C) = \inf_\omega \|C\hat{\bar{R}}(i\omega)B\| \; . \qquad \square$$

Corollary (2.30)

Assume (R1)-(R11)(H2) hold then

$$\mu_{\mathbb{C}}(A;B,C) = \mu_{\mathbb{C}}(K;B,C) = \rho(M_R,C,\mathcal{L}(Y,U))$$
$$= \rho(M_R,C,\mathcal{M}(Y,U))$$

where $\mathcal{M} = \{D \in \mathcal{L}(L^2_Y,L^2_U) \mid (Dy)(t) = \int_0^t H(t-s)y(s)ds \; ,$

$H(\cdot) \in \mathcal{B}^1_-(0,\infty;\mathcal{L}(Y,U))\}$.

Remark (2.31)

Corollary (2.30) says that although the subset $\mathcal{D} \subseteq \mathcal{L}(L^2_Y,L^2_U)$ changes, the robustness of the system remains unchanged.

It would be very interesting to investigate the effect of joint perturbation of the operator A and the kernel $K(\cdot)$, that is the perturbed system takes the form

$$x(t) = R(t)x_0 + \int_0^t \bar{R}(t-s)f(s)ds + \int_0^t \bar{R}(t-s)B_1 D\ C_1 x(s)ds$$
$$+ \int_0^t \bar{R}(t-s)B_2 \int_0^s H(s-\rho)C_2 x(\rho)d\rho ds \ , \quad x_0 \in X \ .$$

Unfortunately, just as in the analysis of multiple neglected time delays in Chapter 1, this again leads to a problem of "multi perturbations" of infinite dimensional systems. However it is possible to treat the problem above for a very special class of integrodifferential equations.

Part 3 Perturbation of A when $K(t) = k(t)A$.

Consider the following nominal integrodifferential equation,

$$(3.2.16) \qquad \dot{x}(t) = Ax(t) + \int_0^t k(t-s)Ax(s)ds + f(t) \ , \ x(0) = x_0 \ .$$

Such classes of systems have received considerable interest in the literature as a source of applications for the general theory of resolvent operators (see e.g. Miller [33], also for more general systems when $K(t) = F(t)A$ see Grimmer [31]).

A is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X , a complex Banach space. $k(\cdot) \in \mathcal{F} \subseteq \mathcal{B}^1_-(0,\infty)$, where $\mathcal{F}$ is sufficiently smooth set of functions to guarantee the existence of $(R(t))_{t\geq 0}$ on $\underline{X}$, X , $\bar{X}$ for (3.2.16). Suppose that A is subject to unbounded perturbation BDC as in Part 1. This results in a perturbed (formal) integrodifferential equation

$$(3.2.17) \qquad \dot{x}(t) = (A+BDC)x(t) + \int_0^t k(t-s)(A+BDC)x(s)ds + f(t) \ , \ x(0) = x_0 \ .$$

Again due to this unboundedness, the perturbed system is taken to be

$$(3.2.18) \qquad x(t) = R(t)x_0 + \int_0^t \bar{R}(t-s)BDCx(s)ds + \int_0^t \bar{R}(t-s)B\int_0^\rho k(s-\rho)DCx(\rho)d\rho ds$$

$$+ \int_0^t \bar{R}(t-s)f(s)ds \ , \quad x_0 \in X \ .$$

Definition (2.32)

Denote by $L_D \in L(L^2_Y, L^2_U)$ the map

$$(L_D y)(t) = Dy(t) + \int_0^t k(t-s)Dy(s)ds \ .$$

Definition (2.33)

A stability radius for system (3.2.18), with respect to perturbation of A is

$$\rho_{\mathbb{C}}(A,kA;B,C) := \sup_d \{ ||D|| < d \text{ implies there exists } R^D \in L(L^2_{\underline{X}})$$
$$\text{such that for all } x_0(\cdot) \in L^2_{\underline{X}}$$
$$(R^D x_0(\cdot))(t) \text{ is the solution of } (3.2.18)\} \ .$$

Theorem (2.34)

Assume (R1)-(R7)(i),(H2) hold and $k(\cdot) \in B^1_-(0,\infty)$ then

$$\rho_{\mathbb{C}}(A,kA;B,C) = \inf_\omega ||C(\frac{i\omega}{1+\hat{k}(i\omega)} - A)^{-1}B||^{-1} \ .$$

Proof.

Follows identically to those of theorems (2.9) and (2.25).

It is worth noticing that the construction of the solution requires that

$$||L_D|| < \frac{1}{||CM_R||} \quad \text{but this follows from the bound}$$

$$||D|| < \inf_{\omega} ||C(\frac{i\omega}{1+\hat{k}(i\omega)} - A)^{-1}B||^{-1} . \qquad \square$$

Remark (2.35)

Of course, in some applications, it will be possible to establish (R1)-(R11) by treating the integrodifferential equation as a perturbation of the semigroup system. This is carried out in the detailed case study at the end of this section.

Definition (2.36)

A stability radius for exponential stability of system (3.2.17), (3.2.18), $f(\cdot) \equiv 0$ is

$\mu_{\mathbb{C}}(A,kA;B,C) := \sup_{d}\{||D|| < d$ implies there exists $R^D(t) \in L(X)$ and a resolvent operator $\bar{R}^D(t) \in L(\bar{X})$ such that $R^D(t)$ and $\bar{R}^D(t)$ are exponentially stable on X , $\bar{X}$ respectively .}

Theorem (2.37)

Assume conditions (R1)-(R6)(R7)(ii)(R9)-(R11)(H2) hold and

$k(\cdot) \in \mathcal{B}^1_-(0,\infty)$ then

$$\mu_{\mathbb{C}}(A,kA;B,C) = \inf_{\omega} ||C(\frac{i\omega}{1+\hat{k}(i\omega)} - A)^{-1}B||^{-1} \quad . \qquad \square$$

Remark (2.38)

If $r_{\mathbb{C}}(A;B,C)$ of Chapter 1 is considered as a "weighted distance of $\sigma(A)$ to the imaginary axis" then introduction of memory into the system, via the kernel $k(\cdot)A$, modifies the robustness to a "weighted distance of $\sigma(A)$ to the curve Γ", where $\Gamma = \{\frac{s}{1+\hat{k}(s)} \mid s \in i\mathbb{R}\}$. This change in the robustness of the system caused by the introduction of memory, is best illustrated by an example.

Example (2.39) A Case Study.

Let X be the separable Hilbert space of example (4.4) Chapter 1, with $\{\phi_n\}_{n=1}^{\infty}$ an orthonormal basis and $\{\lambda_n\}_{n=1}^{\infty}$ a set of real numbers $\ldots\lambda_n < \ldots < \lambda_1 < 0$.

Denote by $S(t) \in L(X)$ the semigroup defined on X by

$$(3.2.19) \quad S(t)x = \sum_{n=1}^{\infty} e^{\lambda_n t} <x,\phi_n>_X \phi_n \quad , \quad x \in X \; ,$$

with generator

$$Ax = \Sigma\, \lambda_n <x_n,\phi_n>_X \phi_n \; ,$$

and

$$D_X(A) = \{x \in X : \Sigma\, \lambda_n^2 <x,\phi_n>_X^2 < \infty\} \quad .$$

Let B and C be defined by

$$(3.2.20)\begin{cases} Bu = \sum_{n=1}^{\infty} b_n\langle x,\phi_n\rangle_X\phi_n & b_n \neq 0 \quad n = 1, \ldots \quad \text{and} \\ \\ Cx = \sum_{n=1}^{\infty} c_n\langle x,\phi_n\rangle\phi_n & c_n \neq 0 \quad n = 1, \ldots \end{cases}$$

as in example (4.4) Chapter 1.

$$(3.2.21)\begin{cases} \text{If} \quad \bar{X} = \{x = \Sigma\, x_n\phi_n \mid \Sigma\beta_n^2x_n^2 < \infty \;,\; \beta_n \neq 0 \quad n = 1,\ldots\ \} \\ \\ \quad \underline{X} = \{x = \Sigma\, x_n\phi_n \mid \Sigma\gamma_n^2x_n^2 < \infty \;,\; \gamma_n \neq 0 \quad n = 1, \ldots\} \end{cases}$$

then $B \in L(X,\bar{X})$ and $C \in L(\underline{X},X)$, $(||x||^2_{\bar{X}} = \Sigma\, \beta_n^2x_n^2 \;,\; ||x||^2_{\underline{X}} = \Sigma\, \gamma_n^2x_n^2)$ if $\beta_n^2b_n^2 \leq K$ and $c_n^2 \leq K\gamma_n^2$ for large n , and some constant K .

Case i) $k(\cdot) = 0$.

For perturbation of the operator A then as in Chapter 1 conditions (R1)-(R6)(R7)(i)(R8)-(R11) hold if $b_n^2 \leq K|\lambda_n|$, $c_n^2 \leq K|\lambda_n|$ for large n , whereas (H2) requires one of $c_n^2 \leq K|\lambda_n|$, $b_n^2 \leq K|\lambda_n|$ to be strict inequality. If these conditions hold then

$$\rho_{\mathbb{C}}(A;B,C) = r_{\mathbb{C}}^2(A;B,C) = \inf_n \frac{|\lambda_n|}{|b_nc_n|}$$

and when additionally $b_n^2c_n^2 \leq K|\lambda_n|$ for large n

$$\mu_{\mathbb{C}}(A;B,C) = r^3_{\mathbb{C}}\ (A;B,C) = \inf_n \frac{|\lambda_n|}{|b_n c_n|} \quad .$$

(Note however that in the semigroup case both stability radii refer to exponential stability of the perturbed semigroup.)

For perturbation of the kernel by $H(t) = h(t)I$ then

$$\sup_\omega |h(i\omega)| < \inf \frac{|\lambda_n|}{|b_n c_n|} \quad .$$

Unfortunately the abstract analysis of section 2 only guarantees a resolvent operator when $b_n^2 c_n^2 \leq K|\lambda_n|$. However if $b_n = -|\lambda_n|^{\frac{1}{2}}$ $c_n = |\lambda_n|^{\frac{1}{2}}$, then $R(t)x_0$ is given as the solution of the following perturbation of the semigroup system,

$$(3.2.22) \qquad x(t) = S(t)x_0 + \int_0^t S(t-s)\int_0^s k(s-\rho)Ax(\rho)d\rho \ , \qquad x_0 \in X \ .$$

(There is a great deal of flexibility in the domain of definition of the semigroup $S(t)$.) Then, due to the simplicity of the perturbation problem in this case, if $k(\cdot)$ is sufficiently smooth to guarantee existence of a resolvent operator on $\bar{X}_\lambda$, X , $\underline{X}_\lambda$ (defined by (3.2.21) for $\beta_n b_n = 1$, $\gamma_n = c_n$) , exponential stability of the resolvent operator for system (3.2.16) is guaranteed if

$$\sup_\omega |k(i\omega)| < 1 \ .$$

Case ii) $\quad k(\cdot) \neq 0$.

Let B , C , $\underline{X}$ and $\bar{X}$ be given by (3.2.20) and (3.2.21).

Assume that $k(\cdot)$ is sufficiently smooth to guarantee existence of resolvent operators in $\bar{X}$ and $\underline{X}$ for (3.2.16). Again due to the flexibility in the domain of definition of $S(t)$, exponential stability of the resolvent operator on $\bar{X}$ and $\underline{X}$ is guaranteed if

$$\sup_{\omega}|k(i\omega)| < 1 .$$

In order to establish that conditions (R5)-(R11) hold consider again (3.2.22). Since $S(t)$ defines a semigroup on $\underline{X}$ and $\bar{X}$ and (R5)-(R7)(i)(R8)-(R11) are satisfied for the semigroup system on these $\underline{X}$, $\bar{X}$ it follows from (3.2.22) and the analysis of section 2 that if $\sup_{\omega}|k(i\omega)| < 1$ then

$$||R(\cdot)x||_{L^2_{\underline{X}_\lambda}} \leq k'||S(\cdot)x||_{L^2_{\underline{X}_\lambda}} \leq k''||x||_X$$

for some constants k', k'' and all $x \in \underline{X}_\lambda$. However $c_n^2 \leq k|\lambda_n|$ for large n and therefore

$$||R(\cdot)x||_{L^2_{\underline{X}}} \leq k'''||R(\cdot)x||_{L^2_{\underline{X}_\lambda}} \quad \text{and (R7)(i) holds.}$$

It is also possible to establish the remainder of (R5)-(R11),(H2) using (3.2.22) and the fact that $S(t) \in L(X\ )$ defined via (3.2.19) yields a semigroup on any space X_β given by (3.2.21). For example

$$||C(\hat{\bar{R}}(i\omega-\varepsilon))Bu||_X^2 = \Sigma \frac{|\ b_nc_n\ |^2}{|(i\omega-\varepsilon-(1+\hat{k}(i\omega-\varepsilon))\lambda_n|^2} <u,\phi_n>_X^2$$

$$\leq \sup_n \frac{|b_nc_n|^2}{|i\omega-\varepsilon-(1+k(i\omega-\varepsilon))\lambda_n|^2} \quad ||u||_X^2 .$$

In the special case $k(t) = ae^{-t}$, it follows that

$$\sup_{\omega} ||C\hat{\hat{R}}(i\omega-\varepsilon)B|| \to \sup_{\omega} ||C\hat{\hat{R}}(i\omega)B|| \quad \text{as} \quad \varepsilon \to 0$$

providing that $a \in (-1,\infty)$, whereas the stability analysis for the resolvent operator as a perturbation of the semigroup system requires $|a| < 1$. In fact, a more direct analysis requires only $a \in (-1,\infty)$, for stability of the resolvent operator. Of course these norm based methods, whilst exact, will always be more conservative than a direct analysis.

If $b_n^2 < K|\lambda_n|$, $c_n^2 < K|\lambda_n|$ then (R1)-(R6)(R7)(i)(R8)-(R11),(H2) hold and

$$\rho_{\mathbb{C}}(A,kA;B,C) = \inf_{\omega} ||C(\frac{i\omega}{1+\hat{k}(i\omega)} \ I - A)^{-1}B||^{-1}$$

$$= \inf_{\omega} \inf_{n} \frac{|(\frac{i\omega}{1+\hat{k}(i\omega)} - \lambda_n)|}{| \ b_n c_n \ |} \quad .$$

The same formula holds for $\mu_{\mathbb{C}}(A,kA;B,C)$ when $b_n^2 c_n^2 \leq K|\lambda_n|$. In order to get specific formulae let $k(t) = ae^{-t}$, then

$$\frac{i\omega}{1+\hat{k}(i\omega)} = \alpha(\omega) + i\beta(\omega) \ , \quad \alpha(\omega) = \frac{-a\omega^2}{((1+a)^2+\omega^2)} \ , \quad \beta(\omega) = \frac{\omega^3+\omega(1+a)}{((1+a)^2+\omega^2)} \quad .$$

If $a \in (-1,0)$ then $\Gamma = \{\frac{s}{1+\hat{k}(s)} \mid s = i\mathbb{R}\} \subseteq \bar{\mathbb{C}}_+$ and therefore

$$\rho_{\mathbb{C}}(A,kA;B,C) = r^2_{\mathbb{C}}(A;B,C)$$

$$= \inf_{n} \frac{|\lambda_n|}{| \ b_n c_n|}$$

so that the robustness is not changed by the introduction of "memory" into the system. If however $a > 0$ then $\Gamma \subseteq \{\lambda \in \mathbb{C} | -a \le \text{Re}\,\lambda \le 0\}$ and a change in the robustness depends upon whether

$$(3.2.23) \quad |\lambda(\omega)-\lambda_n| < |\lambda_n| \quad \text{for} \quad \lambda(\omega) \in \Gamma \quad \text{and} \quad n = 1,2,\ldots \quad .$$

Now (3.2.23) holds if $(2\lambda_n a + 1 + \omega^2)\omega^2 < 0$ for some n .

If $\lambda_n \to -\infty$ (necessary to allow unbounded perturbation) then there exists $N(a)$ such that $|\lambda(\omega)-\lambda_n| < |\lambda_n|$ for all $n \ge N(a)$ and small ω . Hence

$$\rho_{\mathbb{C}}(A,kA;B,C) = \min\{\frac{|\lambda_n|}{|b_n c_n|} \, , \; n \le N(a) \; ; \; \inf_{\omega} \inf_{n>N(a)} \frac{|\frac{i\omega}{1+\hat{k}(i\omega)} - \lambda_n|}{|b_n c_n|} \}$$

and consequently the robustness may be reduced by the introduction of memory. In fact if $b_n = c_n = 1$ and $\lambda_1 < -\frac{1}{2a}$, then the robustness is reduced.

Remark (2.40)

If $a \in (-1,0]$ then $\inf_{\omega}||C(\frac{i\omega}{1+\hat{k}(i\omega)} - A)^{-1}B||$ occurs at $\omega = 0$ and so the perturbation may be real. However when $a \in (0,\infty)$ the destabilising perturbation may be complex.

Remark (2.41)

The robustness of the system with memory is exactly characterised by the weighted distance of the spectrum of A from the curve

$\Gamma^k = \{\frac{s}{1+\hat{k}(s)} \mid s \in i\mathbb{R}\}$. When X is finite dimensional, it is possible to define a robustness measure for $\sigma(A + BDC) \subseteq \mathbb{C}_g$ where $\mathbb{C}_g \subseteq \mathbb{C}$ represents a region of allowable eigenvalues of the perturbed system. If $\mathbb{C}_g$ is such that there exists $k(\cdot) \in B^1_-(0,\infty)$ with $\mathbb{C} \setminus \mathbb{C}_g \subseteq \Gamma^k_+$

$$\Gamma^k_+ = \{\lambda \in \mathbb{C} \mid \text{if } \mu \in \Gamma^k, \operatorname{Im}(\lambda-\mu) = 0 \text{ then } \operatorname{Re}(\lambda-\mu) \geq 0\}$$

then A could be termed robust with respect to $\mathbb{C}_g$ if $r_{\mathbb{C}}(A,kA;\, B,C) > 0$. This gives an infinite dimensional meaning to a class of "good" systems.

§3. Time varying systems.

In this section the abstract analysis of section 1 is applied to time-varying systems governed by a mild evolution operator. The main results are that the uniform asymptotic stability of a mild evolution operator is robust to unbounded perturbation, and that a lower bound on this robustness is the inverse of the norm of a suitable input-output map. However such a measure is shown (for a class of systems) to be strictly less than the "true" measure.

The nominal system is a mild evolution operator $U(\cdot,\cdot)$ on a complex Banach space, that is on $\Delta(t_0) = \{t_0 \leq s \leq t < \infty\}$, $U(t,s) \in L(X)$ and

(i) $U(t,t) = I_X$, $\quad t \in [t_0,\infty)$

(ii) $U(t,r)U(r,s) = U(t,s)$, $\quad t_0 \leq s \leq r \leq t < \infty$

(iii) $U(t,\cdot)$ is strongly continuous on $[t_0,t]$ and $U(\cdot,s)$ is strongly continuous on $[s,\infty)$. $U(t,t_0)x_0$ is the mild solution of the (formal) time varying differential equation

$$\text{(TN)} \qquad \dot{x}(t) = A(t)x(t) \qquad x(t_0) = x_0 .$$

The perturbed system defined on X is given by

$$\text{(TP)} \qquad x(t) = U(t,t_0)x_0 + \int_{t_0}^{t} U(t,s)B(s)D(s)C(s)x(s)ds \ , \quad x(t_0) = x_0$$

the mild solution of the (formal) perturbed time varying equation

$$\text{(TD)} \qquad \dot{x}(t) = (A(t) + B(t)D(t)C(t))x(t) \qquad x(t_0) = x_0 .$$

$B(\cdot) \in \mathcal{B}^\infty(t_0,\infty;\mathcal{L}(U,\bar{X}))$, $C(\cdot) \in \mathcal{B}^\infty(t_0,\infty;\mathcal{L}(\underline{X},Y))$ are fixed, representing both structure and unboundedness of the perturbation, and $D(\cdot) \in \mathcal{B}^\infty(t_0,\infty;\mathcal{L}(Y,U))$ is arbitrary, measuring the size of the allowable perturbations. $\underline{X}$, $\bar{X}$, U and Y are complex Banach spaces. (The Hilbert space structure of U and Y is not required in this section since Plancherel's theorem and Fourier transforms are redundant for time varying systems.) In order to make sense of (TP) in the space of continuous functions in X the following assumptions are required.

(TVA1) $\underline{X} \subseteq X \subseteq \bar{X}$, with continuous dense injections.

(TVA2) $U(t,s)$ extends (restricts) to a mild evolution operator $\bar{U}(t,s)(\underline{U}(t,s))$ on $\bar{X}$ $(\underline{X})$.

(TVA3) There exists $M,\alpha > 0$ such that

$$||U(t,s)|| \le Me^{-\alpha(t-s)} \ , \quad t \ge s \ge t_0 \text{ on } \underline{X} \ , X \ , \bar{X} \ .$$

(TVA4) For all $T > 0$, $s \geq t_0$, there exists $c_T \geq 0$ such that

$$||C(\cdot)U(\cdot,s)x||_{L^2(s,s+T;Y)} \leq c_T||x||_X \text{ for all } x \in \underline{X} .$$

(TVA5) For all $T > 0$, $s \geq t_0$, there exists $b_T \geq 0$ such that

for all $u(\cdot) \in L^2(s,s+T;U)$, $\int_s^{s+T} \bar{U}(s+T,\rho)B(\rho)u(\rho)d\rho \in X$

$$||\int_s^{s+T} \bar{U}(s+T,\rho)B(\rho)u(\rho)d\rho||_X \leq b_T||u(\cdot)||_{L^2(s,s+T;U)} .$$

(TV6) $$||\int_s^{\cdot} \bar{U}(\cdot,\tau)B(\tau)u(\tau)d\tau||_{L^2(s,\infty;\underline{X})} \leq K_s||u(\cdot)||_{L^2(s,\infty;u)}$$

for all $u(\cdot) \in L^2(s,\infty;u)$.

Remark (3.1)

a) There are no assumptions equivalent to (R8) and (R7)(ii) of section 2, since Laplace transform techniques are redundant and the problem of differentiability of solution is not considered.

b) Property (TVA3) is referred to as uniform asymptotic stability (U.A.S) of a mild evolution operator.

c) (TVA4) and (TVA5) might be thought of as saying that the unbounded/smoothing action depends only upon the length of the interval under consideration. This requirement becomes important when analysing the stability of the perturbed system. This dependence upon only the difference in the arguments is used also by Datko [18], Ichikawa [19]. In [18] evolution operators $U(\cdot,\cdot)$ are shown to be (U.A.S.) if, in addition to $\int_{t_0}^{\infty} ||U(t,t_0)||_X^2 dt \leq K||x_0||_X^2$ independent of t_0, they satisfy the type $C(0,e)$ condition, that is there exist, $M,\alpha \geq 0$ such that,

$$||U(t,s)|| \leq Me^{\alpha(t-s)} \quad \text{for all} \quad t_0 \leq s \leq t < \infty \ .$$

In fact (TVA3)-(TVA5) allow the proof of the following simple result concerning c_T and b_T in (TVA4) and (TVA5).

Lemma (3.2)

Assume that (TVA3)-(TVA5) hold, then the dependence upon T in c_T and b_T can be suppressed.

Proof.

Follows in the same way as lemma (2.2) Chapter 1.

Definition (3.3)

For all $s \geq t_0$, denote by $L_s : L^2(s,\infty;U) \rightarrow L^2(s,\infty;Y)$ the map

$$(L_s u)(t) = C(t)\int_s^t \bar{U}(t,\rho)B(\rho)u(\rho)d\rho \ .$$

Lemma (3.4)

Assume (TVA6) holds, then for all $s \geq t_0$, $L_s \in \mathcal{L}(L^2(s,\infty;U),L^2(s,\infty;Y))$ and $\mu(\cdot)$, defined by $\mu(s) = ||L_s||$, is non increasing for increasing s.

Definition (3.5)

Assume (TVA6) holds, define $\mu_L < \infty$ by

$$\mu_L \leq \lim_{s\uparrow\infty} ||L_s|| \ .$$

Remark (3.6)

This time dependence of $||L_s||$ is the first of many problems in this robustness analysis of time varying systems not present in either section 1, when D is allowed to be non causal, or in section 2, when time invariance of the system allows the analysis to be restricted to $[0,\infty)$. However direct application of the analysis of section 1 yields the existence result.

Theorem (3.7)

Assume (TVA1)-(TVA6) hold and $\operatorname*{ess\,sup}_{t\geq t_0} ||D(t)|| < \frac{1}{||L_{t_0}||}$

then there exists a unique mild evolution operator, $U^D(t,s) \in \mathcal{L}(X)$, defined on $\Delta(t_0)$, satisfying

a) for each $s \geq t_0$, $x_0 \in \underline{X}$, $U^D(\cdot,s)x_0 \in L^2_{\underline{X}}$

b) $U^D(t,s)x_0 = U(t,s)x_0 + \int_s^t \bar{U}(t,\rho)B(\rho)D(\rho)C(\rho)U^D(\rho,s)x_0 d\rho$

for all $x_0 \in X$.

Proof.

If $\operatorname*{ess\,sup}_{t\geq t_0} ||D(t)|| < \frac{1}{||L_{t_0}||}$ then for all $s \geq t_0$

$L_s \in \mathcal{L}(L^2(s,\infty;U),L^2(s,\infty;Y))$ satisfies $||L_sD|| < 1$

where $D \in \mathcal{L}(L^2(s,\infty;Y),L^2(s,\infty;U))$ is defined by $(Dy)(t) = D(t)y(t)$ for all $y(\cdot) \in L^2(s,\infty;Y)$. Therefore the following fixed point problem

$$y(t) = C(t)U(t,s)x_0 + C(t)\int_s^t \bar{U}(t,\tau)B(\tau)D(\tau)y\ (\tau)d\tau$$

has a unique solution $y_s(\cdot) \in L^2(s,\infty;Y)$. Define $U^D(t,s)$, $t_0 \le s \le t < \infty$ by

$$U^D(t,s)x_0 = U(t,s)x_0 + \int_s^t \bar{U}(t,\tau)B(\tau)D(\tau)y_s(\tau)d\tau \quad ,$$

then by lemma (3.2) $U^D(\cdot,s)x_0 \in L^2(s,\infty;X)$, $C(\cdot)U^D(\cdot,s)x_0 \in L^2(s,\infty;Y)$ and therefore $U^D(\cdot,s)$ satisfies (b) in $L^2(s,\infty;X)$. For the proofs of the strong continuity of $U^D(\cdot,s)$ on $[s,\infty)$ and $U^D(t,.)$ on $[t_0,t]$ and the evolution operator property on $\Delta(t_0)$ see Appendix. The uniqueness of $U^D(\cdot,\cdot)$ satisfying a) follows just as in the uniqueness proof for $S^D(\cdot)$ in theorem (2.6) Chapter 1. $\square$

Remark (3.8)

Again as in Chapter 1 and Chapter 3 section 1, 2 theorem (3.7) gives uniqueness under the condition that $C(\cdot)U^D(\cdot,s)x$ makes sense in $L^2(s,T;Y)$.

Corollary (3.9)

Assume conditions (TVA1)-(TVA6) hold, and for all $\tau \ge t_0$ there exists $r \in \mathbb{N}_0$ such that for $T > \tau$

$$\begin{aligned}&||C(t_r)\int_\tau^{t_r} \bar{U}(t_r,t_{r-1})B(t_{r-1})D(t_{r-1})\\ &\ldots C(t_1)\int_\tau^{t_1} \bar{U}(t_1,s)B(s)D(s)C(s)x(s)ds dt,\ldots\ dt_{r-1}||_{L^2(\tau,T;Y)}\\ &\le K||x(\cdot)||_{L^2(\tau,T;X)} \quad \text{some} \quad K \ge 0 \ .\end{aligned}$$

If $\operatorname{ess\,sup}_{t \geq t_0} ||D(t)|| < \frac{1}{||L_{t_0}||}$, then there exists a unique mild evolution operator, $U^D(t,s)$ on $\Delta(t_0)$, such for all $\alpha \geq t_0, x(t) = U^D(t,\alpha)x$ is the solution of

$$x(t) = U(t,\alpha)x + \int_\alpha^t \bar{U}(t,s)B(s)D(s)C(s)x(s)ds \ , \quad x(\alpha) = x \ .$$

Proof.

Existence is by theorem (3.7), uniqueness follows by direct generalisation of theorem (1.5).

Theorem (3.10)

Assume (TVA1)-(TVA7) hold and $||D(\cdot)||_{\mathcal{B}^\infty(t_0,\infty;\mathcal{L}(Y,U))} < \frac{1}{||L_{t_0}||}$

then there exists a unique uniformly asymptotically stable, mild evolution operator, $U^D(\cdot,\cdot)$ on $\Delta(t_0)$, such that

$$U^D(t,s)x_0 = U(t,s)x_0 + \int_s^t \bar{U}(t,\tau)B(\tau)D(\tau)D(\tau)U^D(\tau,s)x_0 d\tau$$

for all $x_0 \in X$ and $t_0 \leq s \leq t$.

Proof.

Existence of $U^D(\cdot,\cdot)$ on $\Delta(t_0)$ for $||D(\cdot)||_{\mathcal{B}^\infty(t_0,\infty;\mathcal{L}(Y,U))} < \frac{1}{||L_{t_0}||}$

is guaranteed by corollary (3.9). Also

$$U^D(t,s)x_0 = U(t,s)x_0 + \int_s^t \bar{U}(t,\rho)B(\rho)D(\rho)y_s(\rho)d\rho$$

for all $t_0 \leq s \leq t$, where $y_s(\cdot) \in L^2(s,\infty;Y)$ is the unique solution of

$$y_s(t) = C(t)U(t,s)x_0 + (L_s Dy(\cdot))(t) \ , \quad x_0 \in X \ .$$

Now

$$||y_s(\cdot)||_{L^2(s,\infty;Y)} \leq c_\infty||(I-L_s D)^{-1}|| \ ||x_0||_X$$

and therefore by lemma (3.2)

$$||U^D(t,s)x_0||_X \leq Me^{-\alpha(t-s)}||x_0||_X + b_\infty||D|| \ ||y_s(\cdot)||_{L^2(s,\infty;Y)}$$

$$\leq \bar{K}||x_0||_X \ , \quad \bar{K} > 0 \ .$$

Hence $U^D(\cdot,\cdot)$ is of type $C(0,e)$ with $\alpha = 0$. Additionally

$$||U^D(\cdot,s)x_0||_{L^2(s,\infty;X)} \leq \bar{\bar{K}}||x||_X \ , \quad \text{independent of } s \ .$$

Therefore, applying the results of Datko [18], there exists $M_D, \alpha_D > 0$ such that

$$||U^D(t,s)||_{L(X)} \leq M_D e^{-\alpha_D(t-s)} \quad \text{for all } t_0 \leq s \leq t < \infty \ .$$

Remark (3.11)

This is a direct application of the abstract stability radius characterisation of theorem (1.2), accounting only for the possibility

of $t_0 \neq 0$. However, if account is taken of the possible decrease in $\mu_L(s)$ as $s \to \infty$ then a larger class of
$D \in \mathcal{D} = \{D \in L(L^2(t_0,\infty;Y), L^2(t_0,\infty;U))$, $(Dy)(t) = D(t)y(t)\}$
can be allowed.

Corollary (3.12)

Assume (TVA1)-(TVA7) hold and $||D(\cdot)||_{B^\infty(t_0,\infty;L(Y,U))} \leq \frac{1}{\mu_L}$ then there exists $t_0 \leq \hat{t} < \infty$ and $U^D(\cdot,\cdot)$, a unique uniformly, asymptotically stable mild evolution operator on $\Delta(\hat{t})$ such that

$$U^D(t,s)x = U(t,s)x + \int_s^t \bar{U}(t,\tau)B(\tau)D(\tau)C(\tau)U^D(\tau,s)x d\tau$$

for all $x \in X$, $\hat{t} \leq s \leq t < \infty$. □

Remark (3.13)

If the perturbed mild evolution operator $U^D(t,s)$ is defined for all $t_0 \leq s \leq t < \infty$ and $||D(\cdot)||_{B^\infty(t_0,\infty;L(Y,U))} \leq \frac{1}{\mu_L}$ then the perturbed evolution operator $U^D(\cdot,\cdot)$, on $\Delta(t_0)$, is uniformly asymptotically stable. However this lack of existence of the perturbed solution for $t_0 \leq s \leq t < T$, makes it impossible to define a stability radius for system (TP) in this framework. This problem is similar to the problem encountered in Chapter 1, when considering the well posedness of the perturbed system for D , $||D|| > r_{\mathbb{C}}(A;B,C)$. The problem is resolved in this instance by imposing additional conditions on the system.

If a similar approach is adopted here, then it is required that for all $s \geq t_0$

$$\text{(TVA8)} \qquad \left|\left|\int_s^{\cdot} \bar{U}_R(\cdot,s)B(s)u(s)ds\right|\right|_{L^2(s,\infty;\underline{X})} \leq k_R||u(\cdot)||_{L^2(s,\infty;Y)}$$

where $k_R \to 0$ as $R \to \infty$ and $U_R(t,s) = e^{-R(t-s)}U(t,s)$.

This additional condition holds if there exists $\beta,\gamma \geq 0$ $\alpha > 0$ such that

$$||U(t,s)x||_{\underline{X}} \leq \frac{Me^{-\alpha(t-s)}}{(t-s)^{\beta}} ||x||_X \quad \text{for all} \quad x \in X$$

and

$$||\bar{U}(t,s)x||_X \leq \frac{Me^{-\alpha(t-s)}}{(t-s)^{\gamma}} ||x||_{\bar{X}} \quad \text{for all} \quad x \in \bar{X}$$

and $\gamma + \beta < 1$.

However, it is possible to enlarge the perturbation class still further.

<u>Corollary</u> (3.14)

Assume (TVA1)-(TVA8) hold, $D(\cdot) \in \mathcal{B}^{\infty}(t_0,\infty;\mathcal{L}(Y,U))$, $\lim_{t\to\infty} \operatorname{ess\,sup}_{\tau\geq t}||D(\tau)|| < \frac{1}{\mu_L}$, then there exists a unique, uniformly, asymptotically stable mild evolution operator $U^D(\cdot,\cdot)$ on $\Delta(t_0)$, such that

$$U^D(t,s)x = U(t,s)x + \int_s^t \bar{U}(t,\tau)B(\tau)D(\tau)U^D(\tau,s)xd\tau$$

for all $x \in X$, $t_0 \leq s \leq t < \infty$.

Definition (3.15)

$$r_{\mathbb{C}}(U;B,C) := \sup\{d \mid D(\cdot) \in B^{\infty}(t_0,\infty;L(Y,U)) \text{ and } \lim_{t\to\infty} \operatorname{ess\,sup}_{\tau\geq t} ||D(\tau)|| < d \text{ implies there exists a unique uniformly asymptotically stable mild evolution operator for (TP) on } \Delta(t_0)\} \;.$$

Remark (3.16)

Corollary (3.14) says that under suitable conditions $r_{\mathbb{C}}(U;B,C) \geq \frac{1}{\mu_L}$. Unfortunately, even when X is finite dimensional, $\frac{1}{\mu_L}$ is not, in general, the stability radius for (TP) as is demonstrated in the sequel.

Let $X = \bar{X} = \underline{X} = U = Y = \mathbb{C}^n$, $B(t) = C(t) = I$ and assume $A(.)$, $D(\cdot) \in PC_b(t_0,\infty;\mathbb{C}^{n\times n}) = \{M(.) : [t_0,\infty) \to \mathbb{C}^{n\times n}$, $M(\cdot)$ is piecewise continuous and bounded$\}$.

Then there exists $U(\cdot,\cdot)$ and $U^D(\cdot,\cdot)$ on $\Delta(t_0)$ such that $x(t) = U(t,t_0)x_0$ and $x^D(t) = U^D(t,t_0)x_0$ satisfy

$$\dot{x}(t) = A(t)x(t) \,,\; x(t_0) = x_0 \,,$$

$$\dot{x}^D(t) = (A(t)+D(t))x^D(t),\; x(t_0) = x_0 \text{ respectively.}$$

Proposition (3.17) (Hinrichsen et al [49], Ilchmann [48])

Let $A(\cdot) \in PC_b(t_0,\infty;\mathbb{C}^{n\times n})$ with $U(\cdot,\cdot)$ uniformly asymptotically stable, then

$$r_{\mathbb{C}}(U;I,I) = r_{\mathbb{C}}(U^{\mu};I,I)$$

where U^{μ} is the mild evolution operator for $A^{\mu}(\cdot), A^{\mu}(t) = A(t) + \frac{\dot{\mu}(t)}{\mu(t)} I$,

and

$$\mu(\cdot) \in PC_b^1(t_0,\infty;\mathbb{C}\backslash\{0\}) = \{\mu:[t_0,\infty) \to \mathbb{C}\backslash\{0\} \,\big|\, \mu(\cdot) \text{ is piecewise differentiable, } |\mu(\cdot)| \text{ is bounded above and away from } 0\} .$$

Corollary (3.18)

Let $n = 1$ and $a(\cdot) \in PC_b(t_0,\infty;\mathbb{C})$ be periodic, period $T > 0$ then

$$r_{\mathbb{C}}(U;1,1) = r_{\mathbb{C}}(\bar{a};1,1) \qquad \text{(see Chapter 1)}$$

where

$$\bar{a} = \frac{1}{T}\int_{t_0}^{t_0+T} a(t)dt .$$

Proof.

Define $\mu(\cdot) \in PC_b^1(t_0,\infty;\mathbb{C}\backslash\{0\})$ by $\mu(t) = e^{-\int_{t_0}^{t} a(s)ds}$. Then $a(t) + \frac{\dot{\mu}(t)}{\mu(t)} = \bar{a}$ and therefore

$$\begin{aligned} r_{\mathbb{C}}(U;1,1) &= r_{\mathbb{C}}(U^{\mu};1,1) \\ &= r_{\mathbb{C}}(\bar{a};1,1) . \end{aligned} \qquad \square$$

This simple corollary leads to an example of a system with $r_{\mathbb{C}}(U;I,I) > \frac{1}{\mu_L}$.

Counterexample (3.14) (Hinrichsen et al [49], Ilchmann [48])

Consider the scalar periodic system

$$(3.3.1) \qquad \dot{x}(t) = (-1 + ka(t))x(t) \qquad t \geq 0 \quad x(0) = x_0$$

where $a(\cdot) \in PC(0,\infty;\mathbb{R})$ is periodic, periodic $T > 0$ and $\int_0^T a(s)ds = 0$, $k \in \mathbb{R}$. For system (3.3.1)

$$U(t,s) = e^{-(t-s)}e^{k\int_s^t a(\rho)d\rho} \quad .$$

By corollary (3.18) $\qquad r_{\mathbb{C}}(U;I,I) = 1$.

Set $b(t) = k\int_0^t a(s)ds$ and $u(t) = e^{b(t)-2t}$ and choose $a(\cdot)$ such that

$$a(t) = \begin{cases} 0 & 0 \leq t < T/3 \\ -1 & \tau \leq t < 2T/3 \\ 1 & 2\tau \leq t < T . \end{cases}$$

Then it is easy to show that

$$||L_0u||^2_{L^2} - ||u||^2_{L^2} = \int_0^\infty e^{2b(t)-2t}[1 - 2e^{-t}]dt > 0$$

for k and T sufficiently large. Therefore for this system $||L_0|| > 1$. However, for any periodic system it is easy to show that $||L_s|| = ||L_0||$ for all $s \geq 0$. Consequently $r_{\mathbb{C}}(U,I,I) = 1 > \frac{1}{\mu_L}$.

For a more detailed analysis of this robustness of time-varying systems see Hinrichsen et al [49] where the stability radius concepts are related to the notions of Bohl and Liapunov exponent.

§4. Conclusion.

This chapter contains a mixture of extensions and limitations of the stability radius concepts introduced in Chapter 1. The first section concerns an abstract version of the problem, recognising the common framework of many perturbation problems, via a variation of parameters formula. In section 2 direct application of this abstract result yields stability radii for perturbations of systems governed by integrodifferential equations. In section 3, however, direct application of the results of section 1, whilst establishing the robustness of uniform asymptotic stability to unbounded perturbations does not yield exact stability radii. The problems are not only technical but are quite fundamental and well illustrated by a simple scalar periodic system. Probably the reason why the abstract analysis succeeds in the case of integrodifferential equations, but fails in the case of time-varying systems (and also in the real stability radius of Hinrichsen-Pritchard [7]) is that the former class of systems is amenable to a Fourier analysis. The use of the Fourier transform is essentially that of changing the system into one easier to analyse, without changing the topology on the set $\mathcal{D}$. In Hinrichsen et al some success in this direction has been attained by the use of Bohl transformations, which result in a change in the "input-output" map L . This might also be a possibility in the analysis of a real stability radius for infinite dimensional systems, using some norm preserving transformations.

In the analysis of section 2, joint perturbations of the operator A and the kernel K is possible only for the special case when $K(t) = k(t)A$.

Perturbation of both A and the kernel K results in multiperturbations, so that this class of perturbations would again be an interesting problem in this infinite dimensional setting. Another research area would be to resolve the problem of the difference between $\rho(A;B,C)$ and $\mu(A;B,C)$ $(\rho(H;B,C)$ and $\mu(H;B,C))$, which arises due to the requirement of a perturbed resolvent operator on $\bar{X}$ in the destabilisation analysis.

CHAPTER 4. Conclusion.

In Chapters 1 and 3 exponential stability of certain classes of infinite dimensional systems (typified by the examples of Chapter 1) is shown to be robust to highly structured, unbounded perturbation. The robustness margins or stability radii are shown to be exactly characterised by the inverse of the norm of a suitable input-output map, for classes of time invariant systems. In Chapter 2 a related problem of a non-standard linear quadratic problem is considered. This non-standard problem results in a non-standard Riccati equation, and this in turn results in a Liăpunov type stability analysis for a certain class of non-linear infinite dimensional systems. The framework used throughout Chapters 1, 2 and 3 is some derivative of that used by Pritchard and Salamon [22] to a great deal of success in their abstract theory for the linear quadratic control problem with unbounded input and output operators. This framework has also been applied to the problem of designing finite dimensional controllers for infinite dimensional systems, Curtain and Salamon [42] and also to the problems of balanced realisation and model reduction, Curtain [28], Curtain and Glover [27] Glover [26].

One of the major problems concerning the stability radii for linear state space systems (Hinrichsen and Pritchard [7], [8], Pritchard and Townley [9] and Hinrichsen et al [49]) is that of their computation and as yet no satisfactory algorithms exist. In the case of infinite dimensional systems it is clear, however, that some form of model reduction process will play an important role in any algorithm.

CHAPTER 5. Appendix.

Theorem (2.6) Chapter 1.

Proof of the continuity and semigroup property for the perturbed solution defined by (1.2.2). For the continuity let $0 \leq s \leq t$, $x_0 \in X$ then

$$
\begin{aligned}
||x(t,x_0) - x(s,x_0)||_X &\leq ||(S(t) - S(s))x_0||_X \\
&+ ||\int_0^t S(\rho)BDy(t-\rho)d\rho - \int_0^s S(\rho)BDy(s-\rho)d\rho||_X \\
&= ||(S(t) - S(s))x_0||_X \\
&+ ||\int_0^s S(\rho)BD(y(t-\rho)-y(s-\rho))d\rho + \int_s^t S(\rho)BDy(t-\rho)d\rho||_X \\
&\leq ||(S(t)-S(s))x_0||_X + ||\int_s^t S(\rho)BDy(t-\rho)d\rho||_X \\
&+ b||Dy(t-\cdot) - Dy(s-\cdot)||_{L^2(0,s;U)} \\
&\leq ||(S(t)-S(s))x_0||_X + Me^{-\alpha s} b||Dy(t-s-\cdot)||_{L^2(0,t-s;U)} \\
&+ b||Dy(t-\cdot)-Dy(s-\cdot)||_{L^2(0,s;U)} .
\end{aligned}
$$

A similar argument holds for $t \leq s$ and so the continuity of $x(\cdot,x_0)$ follows.

In order to establish the semigroup property for $S^D(t) \in L(X)$ let $0 \leq s,t$ and consider, for $x_0 \in X$,

$$C(S^D(\cdot+s)-S^D(\cdot)S^D(s))x_0 = LDC(S^D(\cdot+s)-S^D(\cdot)S^D(s))x_0 \ .$$

Then, since $||LD|| < 1$, $C(S^D(\cdot+s)-S^D(\cdot)S^D(s))x_0 = 0$ in L_Y^2 . But

$$(S^D(t+s)-S^D(t)S^D(s))x_0 = \int_0^t S(t-\rho)BDC(S^D(s+\rho)-S^D(s)S^D(\rho))x_0 d\rho \ .$$

Hence

$$[S^D(t+s)-S^D(t)S^D(s)]x_0 = 0 \quad \text{for all} \quad x_0 \in X \ . \qquad \square$$

Corollary (2.9) Chapter 1.

In order to establish $\int_0^t z(s)ds = C\bar{S}^D(t)x_0-Cx_0$ for $x_0 \in D_{\bar{X}}(A)$ consider

$$\int_0^t z(s)ds = C\bar{S}(t)x_0 - Cx_0 + C\int_0^t \bar{S}(s)BDCx_0 ds$$

$$+ \int_0^t C\int_0^s \bar{S}(s-\rho)BDz(\rho)d\rho ds \ , \quad \text{by (A2)}$$

$$= C\bar{S}(t)x_0 - Cx_0 + C\int_0^t \bar{S}(t-s)BDCx_0 ds$$

$$+ C\int_0^t \int_0^s \bar{S}(s-\rho)BDz(\rho)d\rho ds$$

by assumption (A1) and (A4).

$$= C\bar{S}(t)x_0 - Cx_0 + C\int_0^t \bar{S}(t-s)BDCx_0 ds$$

$$+ C\int_0^t \int_\rho^t \bar{S}(s-\rho)BDz(\rho)dsd\rho$$

(since $z(\cdot) \in L_Y^2$ and $\bar{S}(\cdot)$ is strongly continuous in $\bar{X}$).

Therefore

$$\int_0^t z(s)ds = C\bar{S}(t)x_0 - Cx_0 + C\int_0^t \bar{S}(t-s)BDCx_0 ds$$

$$+ C\int_0^t \bar{S}(t-\rho)BD\int_0^\rho z(s)dsd\rho \quad .$$

Hence $\int_0^t z(s)ds + Cx_0$ solves the (Y) equation and therefore by uniqueness $C\bar{S}^D(t)x_0 = \int_0^t z(s)ds + Cx_0$. □

Proposition (3.6) Chapter 1.

Let $u(\cdot) \in L_U^2$, then by (A4) , $y = Lu \in L_Y^2$. Therefore $\hat{y}(\cdot) \in L^2(-\infty,\infty;Y)$ where $\hat{y}(i\omega) = \int_0^\infty e^{-i\omega t}y(t)dt$.

$$\hat{y}(i\omega) = \int_0^\infty e^{-i\omega t}C\int_0^t S(t-s)Bu(s)dsdt$$

$$= C\int_0^\infty e^{-i\omega t}\int_0^t S(t-s)Bu(s)dsdt$$

$$= C\int_0^\infty\int_s^\infty e^{-i\omega t}S(t-s)Bu(s)dtds$$

$$= C\int_0^\infty\int_0^\infty e^{-i\omega\rho}S(\rho)Be^{-i\omega s}u(s)d\rho ds$$

$$= C(i\omega I - A)^{-1}B\hat{u}(i\omega)$$

where $\hat{u}(i\omega) = \int_0^\infty e^{-i\omega t}u(t)dt$.

By Plancherel's Theorem (valid for Hilbert spaces - see Kappel and Kunisch [36], Yosida [54])

$$||Lu||^2_{L^2_Y} = 1/2\pi\int_{-\infty}^{\infty} ||\hat{y}(i\omega)||^2_Y d\omega$$

$$\leq \sup_\omega ||G(i\omega)||^2 ||u||^2_{L^2_U} \quad .$$

Now

$$||G(i\omega)||-||G(i\bar{\omega})|| \leq ||C((i\omega I-A)^{-1}-(i\bar{\omega}I-A)^{-1})B)||$$

$$\leq ||C||_{L(\underline{X},Y)}||B||_{L(U,\bar{X})}||(i\omega I-A)^{-1}||_{L(\underline{X})}||(i\bar{\omega}I-A)^{-1}||_{L(\bar{X},\underline{X})}|\omega-\bar{\omega}|$$

$$= K(\omega,\bar{\omega})|\omega-\omega| \text{ , for all } \omega,\bar{\omega} \in \mathbb{R} \text{ ,}$$

and therefore $||G(i\cdot)||$ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Let $\varepsilon > 0$ and $\omega_0 \in \mathbb{R}$ be such that

$$||G(i\omega_0)|| \geq \sup_\omega ||G(i\omega)||- \varepsilon/3 \text{ .}$$

For this ω_0 choose $u \in U$, $||u|| = 1$ such that

$$||G(i\omega_0)u||_Y \geq ||G(i\omega_0)|| - \varepsilon/3 \text{ .}$$

Consider the following function $u_\delta(\cdot) \in L^2_U$ defined via its Fourier-Plancherel transform.

$$\hat{u}_\delta(i\omega) = (2\pi/\delta)^{\frac{1}{2}}u, \ \omega \in [\omega_0 - \delta/2, \omega_0 + \delta/2]$$

$$= 0 \qquad \text{otherwise.}$$

Then

$$||Lu_\delta||^2_{L^2_Y} = 1/2\pi \int_{-\infty}^{\infty} ||G(i\omega)\hat{u}_\delta(i\omega)||^2_Y d\omega$$

$$= 1/\delta \int_{\omega_0-\delta/2}^{\omega_0+\delta/2} ||G(i\omega)u||^2_Y d\omega \quad .$$

Thus

$$\inf_{\omega\in[\omega_0-\delta/2,\omega_0+\delta/2]} ||G(i\omega)u||_Y \le ||Lu_\delta||_{L^2_Y} \le \sup_{\omega\in[\omega_0-\delta/2,\omega_0+\delta/2]} ||G(i\omega)u||_Y \quad .$$

But $||G(i\cdot)u||$ is continuous, so for sufficiently small $\delta > 0$

$$||Lu_\delta||_Y \ge ||G(i\omega_0)u||_Y - \varepsilon/3 \quad .$$

Let $\varepsilon > 0$ then combining the above estimates it is easy to show that there exists $u(\cdot) \in L^2_U$, $||u(\cdot)||_{L^2_U} = 1$ such that

$$||Lu||_{L^2_U} \ge \sup_\omega ||G(i\omega)|| - \varepsilon \quad .$$

Therefore $||L|| = \sup_\omega ||G(i\omega)||$. □

Theorem (3.7) Chapter 3.

In order to establish the strong continuity of $U^D(\cdot,s)$ on $[s,\infty)$, defined by the equation pair

$$\begin{cases} U^D(t,s)x = U(t,s)x + \int_s^t \bar{U}(t,\tau)B(\tau)D(\tau)y_s(\tau)d\tau \\ \\ y_s(t) = C(t)U(t,s)x + C(t)\int_s^t \bar{U}(t,\tau)B(\tau)D(\tau)y_s(\tau)d\tau \end{cases}$$

for all $x \in X$, let $s \le t \le T$, then

$$||U^D(T,s)x-U^D(t,s)x||_X \le ||U(T,s)x-U(t,s)x||_X$$

$$+ ||\int_s^T \bar{U}(T,\tau)B(\tau)y_s(\tau)d\tau$$

$$- \int_s^t \bar{U}(t,\tau)B(\tau)y_s(\tau)d\tau||_X$$

$$\le ||U(T,s)x-U(t,s)x||_X + ||\int_s^t (\bar{U}(T,\tau)-\bar{U}(t,\tau))B(\tau)D(\tau)y_s(\tau)d\tau||_X$$

$$+ ||\int_t^T \bar{U}(T,\tau)B(\tau)D(\tau)y_s(\tau)d\tau||_X$$

$$\le ||(U(T,s)-U(t,s))x||_X + ||(U(T,t)-I)\int_s^t \bar{U}(t,\tau)B(\tau)D(\tau)y_s(\tau)d\tau||_X$$

$$+ b_{T-t}||D(\cdot)||\ ||y_s(\cdot)||_{L^2(t,T;Y)} \quad .$$

Therefore given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $T-t < \delta$, $||U^D(T,s)x-U^D(t,s)x||_X < \varepsilon$. A similar argument holds for $t \ge T \ge s$ and therefore strong continuity of $U(\cdot,s)$ on $[s,\infty)$ follows.

For the strong continuity of $U(t,\cdot)$ on $[\hat{t},t]$ let $t_0 \leq s \leq \sigma \leq t$. Then

$$||U^D(t,\sigma)x-U^D(t,s)x||_X \leq ||U(t,\sigma)x-U(t,s)x||_X$$

$$+ ||\int_\sigma^t \bar{U}(t,\tau)B(\tau)D(\tau)\{y_\sigma(\tau)-y_s(\tau)\}d\tau||_X$$

$$+ ||\int_s^\sigma \bar{U}(t,\tau)B(\tau)D(\tau)\{y_s(\tau)\}d\tau||_X \quad .$$

But

$$||y_\sigma(\cdot)-y_s(\cdot)||_{L^2(\sigma,\infty;Y)} \leq ||(I-L_\sigma D)^{-1}||\{||\hat{y}(\cdot)||_{L^2(\sigma,\infty;Y)}$$

$$+ ||\hat{\hat{y}}(\cdot)||_{L^2(\sigma,\infty;Y)}\}$$

where

$$\hat{y}(\cdot) = C(\cdot)(U(\cdot,\sigma) - U(\cdot,s))x \quad \text{on} \quad [\sigma,\infty)$$

$$\hat{\hat{y}}(\cdot) = C(\cdot)U(\cdot,\sigma)\int_s^\sigma \bar{U}(\sigma,\tau)B(\tau)D(\tau)y_s(\tau)d\tau \quad \text{on} \quad [\sigma,\infty) \ .$$

Now

$$||\hat{y}(\cdot)||_{L^2(\sigma,\infty;Y)} \leq c_\infty||(U(\sigma,s) - I)x||_X \quad \text{where}$$

c_∞ is c_T in (TVA4) taken independent of T, by lemma (3.2). Similarly

$$||\hat{\hat{y}}(\cdot)||_{L^2(\sigma,\infty;Y)} \leq c_\infty b_{\sigma-s}||D(\cdot)|| \ ||y_s(\cdot)||_{L^2(s,\sigma;Y)} \quad .$$

Therefore

$$||U^D(t,\sigma)x-U^D(t,s)x||_X \leq ||U(t,\sigma)x-U(t,s)x||_X$$

$$+ b_{t-\sigma}||D(\cdot)|| \; ||y_\sigma(\cdot)-y_s(\cdot)||_{L^2(\sigma,\infty;Y)}$$

$$+ b_{\sigma-s}||D(\cdot)|| \; ||y_s(\cdot)||_{L^2(s,\sigma;Y)} \quad .$$

Hence given $\varepsilon > 0$ there exists $\delta > 0$ such that $\sigma\text{-}s < \delta$ implies $||U^D(t,\sigma)x-U^D(t,s)x||_X < \varepsilon$. A similar argument holds for $t_0 \leq \sigma \leq s \leq t$ and therefore $U(t,\cdot)$ is strongly continuous on $[\hat{t},t]$. To prove the evolution operator property let $t_0 \leq s \leq \alpha \leq t$, then

$$U^D(t,\alpha)U^D(\alpha,s)x = U^D(t,\alpha)[U(\alpha,s)x + \int_s^\alpha \bar{U}(\alpha,\tau)B(\tau)D(\tau)y_s(\tau)d\tau]$$

$$= U(t,\alpha)[U(\alpha,s)x + \int_s^\alpha \bar{U}(\alpha,\tau)B(\tau)D(\tau)y_s(\tau)d\tau]$$

$$+ \int_\alpha^t \bar{U}(t,\rho)B(\rho)D(\rho)\hat{y}_\alpha(\rho)d\rho$$

where $\hat{y}_\alpha(\cdot)$ denotes the solution for $y(\cdot)$ of

$$y(t) = C(t)U(t,\alpha)z + C(t)\int_\alpha^t \bar{U}(t,\gamma)B(\gamma)D(\gamma)y(\gamma)d\gamma \quad \text{on} \quad [\alpha,t]$$

and $$z = U(\alpha,s)x + \int_s^\alpha \bar{U}(\alpha,\tau)B(\tau)D(\tau)y_s(\tau)d\tau \quad .$$

Therefore

$$\begin{aligned} U^D(t,\alpha)U^D(\alpha,s)x &= U(t,s)x + \int_s^\alpha \bar{U}(t,\tau)B(\tau)D(\tau)y_s(\tau)d\tau \\ &\quad + \int_\alpha^t \bar{U}(t,\rho)B(\rho)D(\rho)\hat{y}_\alpha(\rho)d\rho \\ &= U(t,s)x + \int_s^t \bar{U}(t,\tau)B(\tau)D(\tau)y_s(\tau)d\tau \\ &\quad + \int_\alpha^t \bar{U}(t,\rho)B(\rho)D(\rho)(\hat{y}_\alpha(\rho)-y_s(\rho))d\rho \ . \end{aligned}$$

However

$$\hat{y}_\alpha(\rho) - y_s(\rho) = C(\rho)\int_s^\rho \bar{U}(\rho,\gamma)B(\gamma)D(\gamma)[\hat{y}_\alpha(\gamma)-y_s(\gamma)]d\gamma$$

and $||L_s^{\hat{}}D|| < 1$ therefore $\hat{y}_\alpha(\rho) = y_s(\rho)$ for almost all $\rho \in [s,\infty)$ and consequently

$$\begin{aligned} U^D(t,\alpha)U^D(\alpha,s)x &= U(t,s)x + \int_s^t \bar{U}(t,\tau)B(\tau)D(\tau)y_s(\tau)d\tau \\ &= U^D(t,s)x \ . \end{aligned}$$

□

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