Castillo, K.; Costa, M. S.; Sri Ranga, A.; Veronese, D. O. (2014). "A Favard type theorem for orthogonal polynomials on the unit circle from a three term recurrence formula". *Journal of Approximation Theory*, v. 184, pp. 146-162.
DOI: 10.1016/j.jat.2014.05.007

A Favard type theorem for orthogonal polynomials on the unit circle from a three term recurrence formula ☆

K. Castillo[a], M.S. Costa[b], A. Sri Ranga[a,*], D.O. Veronese[c]

[a] *Departamento de Matemática Aplicada, IBILCE, UNESP – Universidade Estadual Paulista, 15054-000, São José do Rio Preto, SP, Brazil*
[b] *Faculdade de Matemática, Universidade Federal de Uberlândia, 38408-100, Uberlândia, MG, Brazil*
[c] *Instituto de Ciências Tecnológicas e Exatas, Universidade Federal do Triângulo Mineiro, 38064-200, Uberaba, MG, Brazil*

Communicated by Leonid Golinskii

Abstract

The objective of this manuscript is to study directly the Favard type theorem associated with the three term recurrence formula

$$R_{n+1}(z) = \big[(1+ic_{n+1})z + (1-ic_{n+1})\big]R_n(z) - 4d_{n+1}zR_{n-1}(z), \quad n \geq 1,$$

with $R_0(z) = 1$ and $R_1(z) = (1+ic_1)z + (1-ic_1)$, where $\{c_n\}_{n=1}^{\infty}$ is a real sequence and $\{d_n\}_{n=1}^{\infty}$ is a positive chain sequence. We establish that there exists a unique nontrivial probability measure μ on the unit circle for which $\{R_n(z) - 2(1-m_n)R_{n-1}(z)\}$ gives the sequence of orthogonal polynomials. Here, $\{m_n\}_{n=0}^{\infty}$ is the minimal parameter sequence of the positive chain sequence $\{d_n\}_{n=1}^{\infty}$. The element d_1 of the chain sequence, which does not affect the polynomials R_n, has an influence in the derived probability measure μ and hence, in the associated orthogonal polynomials on the unit circle. To be precise, if $\{M_n\}_{n=0}^{\infty}$ is the maximal parameter sequence of the chain sequence, then the measure μ is such that M_0 is the size of its mass at $z = 1$. An example is also provided to completely illustrate the results obtained.

☆ The works of all four authors have been supported by grants from CAPES, CNPq and FAPESP of Brazil. K. Castillo has also received support from Direcci´on General de Investigaci´on, Ministerio de Econom´ıa y Competitividad of Spain (Grant MTM2012–36732–C03–01) for his research.

* Corresponding author.
E-mail addresses: kenier@ibilce.unesp.br (K. Castillo), marisa@famat.ufu.br (M.S. Costa), ranga@ibilce.unesp.br (A. Sri Ranga), veronese@icte.uftm.edu.br (D.O. Veronese).

MSC: 42C05; 33C47

Keywords: Szegő polynomials; Kernel polynomials; Para-orthogonal polynomials; Chain sequences; Continued fractions

1. Introduction

Orthogonal polynomials on the unit circle (OPUC) have attracted a lot of interest in recent years. For some recent contributions on this topic we refer to [2,3,7,18,22–24,30]. Even though for many years a first hand text for an introduction to these polynomials has been the classical book [29] of Szegő, detailed accounts regarding the earlier research on these polynomials can be found, for example, in Geronimus [12], Freud [11] and Van Assche [31]. However, for recent and more up to date texts on this subject we refer to the two volumes of Simon [25,26]. There is also a nice chapter about these polynomials in Ismail [14].

Given a nontrivial measure $\mu(\zeta) = \mu(e^{i\theta})$ on the unit circle $\mathcal{C} = \{\zeta = e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, the associated sequence of monic OPUC $\{S_n\}$ are those polynomials with the property

$$\int_{\mathcal{C}} \overline{S_m(\zeta)} S_n(\zeta) d\mu(\zeta) = \int_0^{2\pi} \overline{S_m(e^{i\theta})} S_n(e^{i\theta}) d\mu(e^{i\theta}) = \delta_{mn} \kappa_n^{-2}.$$

The orthonormal polynomials on the unit circle are $s_n(z) = \kappa_n S_n(z)$, $n \geq 0$.

The monic OPUC satisfy the recurrence

$$\begin{aligned} S_n(z) &= z S_{n-1}(z) - \overline{\alpha}_{n-1}\, S_{n-1}^*(z), \\ S_n(z) &= (1 - |\alpha_{n-1}|^2) z S_{n-1}(z) - \overline{\alpha}_{n-1} S_n^*(z), \end{aligned} \quad n \geq 1, \tag{1.1}$$

where $\overline{\alpha}_{n-1} = -S_n(0)$ and $S_n^*(z) = z^n \overline{S_n(1/\overline{z})}$. Following Simon [25] (see also [27]) we will be referring to the numbers α_n as Verblunsky coefficients. It is well known that these coefficients are such that $|\alpha_n| < 1$, $n \geq 0$. It is also known that OPUC are completely characterized in terms of these coefficients as given by the following theorem (see, for example, [25, Theorem 1.7.11]).

Theorem A. *Given an arbitrary sequence of complex numbers $\{\alpha_n\}_{n=0}^{\infty}$, where $|\alpha_n| < 1$, $n \geq 0$, then associated with this sequence there exists a unique nontrivial probability measure μ on the unit circle such that the monic polynomials $\{S_n\}$ generated by (1.1) are the respective monic OPUC.*

A very nice and short constructive proof of this theorem can be found in Erdélyi, Nevai, Zhang and Geronimo [10]. Their knowledge that lead to this proof can be traced back to results found in [20,21] and references therein.

In almost all recent studies regarding OPUC, the Verblunsky coefficients and the recurrence relations (1.1) play a fundamental role. In this manuscript, however, the starting point of the analysis is the three term recurrence formula

$$R_{n+1}(z) = \big[(1 + ic_{n+1})z + (1 - ic_{n+1})\big] R_n(z) - 4d_{n+1} z R_{n-1}(z), \quad n \geq 1, \tag{1.2}$$

with $R_0(z) = 1$ and $R_1(z) = (1 + ic_1)z + (1 - ic_1)$, where

$\{c_n\}_{n=1}^{\infty}$ is a sequence of real numbers and

$\{d_n\}_{n=1}^{\infty}$ is a positive chain sequence.

For more details on positive chain sequences we refer to Chihara [4].

Although the first element d_1 of the chain sequence does not affect the sequence of polynomials $\{R_n\}$, its use will become apparent when we introduce the sequence of polynomials $\{Q_n\}$ in (3.1) and, in particular, the sequence of rational functions $\{A_n/B_n\}$ in Section 4.

The main objective of the present manuscript is to show that associated with the sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ there exists a unique nontrivial probability measure μ on the unit circle (which was also shown in [5] by a different method) and to show that the sequence of polynomials $\{R_n(z) - 2(1-m_n)R_{n-1}(z)\}$ are the sequence of OPUC with respect to this measure. Here, $\{m_n\}_{n=0}^{\infty}$ is the minimal parameter sequence of the positive chain sequence $\{d_n\}_{n=1}^{\infty}$. As also shown in [5], M_0, where $\{M_n\}_{n=0}^{\infty}$ is the maximal parameter sequence of $\{d_n\}_{n=1}^{\infty}$, gives the size of the mass at $z=1$ in the measure μ.

It was brought to our attention by one of the referees that the recurrence formula (1.2) has also been considered by Delsarte and Genin [9], in the following form

$$\tilde{R}_{n+1}(z) = (\overline{\beta}_n z + \beta_n)\tilde{R}_n(z) - z\tilde{R}_{n-1}(z), \quad n = 0, 1, \ldots, m.$$

The equivalence of this finite system of equations to (1.2) for $n = 1, 2, \ldots, m$ is obtained with

$$R_n(z) = \frac{\tilde{R}_n(z)}{\prod_{k=1}^{n} \mathcal{R}e(\beta_k)}, \qquad c_n = \frac{-\mathcal{I}m(\beta_n)}{\mathcal{R}e(\beta_n)} \quad \text{and} \quad d_{n+1} = \frac{1}{4\mathcal{R}e(\beta_{n-1})\mathcal{R}e(\beta_n)}.$$

In [9] the recurrence formulas and the resulting finite systems of OPUC were analyzed under the conditions that the tridiagonal matrices

$$J_n = \begin{bmatrix} 2\mathcal{R}e(\beta_0) & 1 & \cdots & 1 \\ 1 & 2\mathcal{R}e(\beta_1) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 2\mathcal{R}e(\beta_n) \end{bmatrix},$$

are positive definite for $n = 0, 1, \ldots, m-1$ and nonnegative definite for $n = m$. Note that using results found in Wall and Wetzel [33] (see also Wall [32, Theorem 16.2]) these conditions can also be expressed in terms of the sequence $\{d_{n+1}\}_{n=1}^{m-1} = \{1/(4\mathcal{R}e(\beta_{n-1})\mathcal{R}e(\beta_n))\}_{n=1}^{m-1}$ being a finite positive chain sequence.

In the present manuscript we consider (1.2) with an infinite set of formulas and, using a technique different from [9], completely describe the associated moments functionals, positive measure and OPUC.

2. Some preliminary results

The following result gives some information regarding the zeros of $R_n(z)$.

Lemma 2.1. *The polynomial $R_n(z)$ has all its n zeros simple and lying on the unit circle $|z| = 1$. Moreover, if one denotes the zeros of R_n by $z_{n,j} = e^{i\theta_{n,j}}$, $j = 1, 2, \ldots, n$, then*

$$0 < \theta_{n+1,1} < \theta_{n,1} < \theta_{n+1,2} < \cdots < \theta_{n,n} < \theta_{n+1,n+1} < 2\pi, \quad n \geq 1.$$

This lemma is part of a slightly more extensive result established in [8] with the use of the functions $G_n(x)$, defined on the interval $[-1, 1]$, by

$$G_n(x) = (4z)^{-n/2} R_n(z), \quad n \geq 0, \tag{2.1}$$

where $2x = z^{1/2} + z^{-1/2}$ and $z = e^{i\theta}$. Clearly, the zeros of the function $G_n(x)$ in $[-1, 1]$ are $x_{n,j} = \cos(\theta_{n,j}/2)$, $j = 1, 2, \ldots, n$ and Lemma 2.1 means that there holds the interlacing property

$$-1 < x_{n+1,n+1} < x_{n,n} < x_{n+1,n} < \cdots < x_{n+1,2} < x_{n,1} < x_{n+1,1} < 1,$$

for $n \geq 1$. As given in [8], these functions satisfy the recurrence formula

$$G_{n+1}(x) = \left(x - c_{n+1}\sqrt{1-x^2}\right) G_n(x) - d_{n+1}\, G_{n-1}(x), \quad n \geq 1,$$

with $G_0(x) = 1$ and $G_1(x) = x - c_1\sqrt{1-x^2}$. Moreover, the associated Christoffel–Darboux functions or Wronskians

$$W_n(x) = G_n'(x)G_{n-1}(x) - G_{n-1}'(x)G_n(x), \quad n \geq 1, \tag{2.2}$$

which not necessarily remain positive through out $[-1, 1]$, but satisfy at the zeros of $G_n(x)$

$$W_n(x_{n,j}) > 0, \qquad W_{n+1}(x_{n,j}) = d_{n+1} W_n(x_{n,j}) > 0, \quad j = 1, 2, \ldots, n, \tag{2.3}$$

for $n \geq 1$. Note that

$$W_n(x_{n,j}) = G_n'(x_{n,j})G_{n-1}(x_{n,j}) \quad \text{and} \quad W_{n+1}(x_{n,j}) = -G_n'(x_{n,j})G_{n+1}(x_{n,j}).$$

Now we consider the Wronskians

$$V_n(z) = R_n'(z)R_{n-1}(z) - R_{n-1}'(z)R_n(z), \quad n \geq 1, \tag{2.4}$$

associated with the polynomials $R_n(z)$. From (2.1)

$$G_n'(x) = (4z)^{-(n-1)/2}\left[2zR_n'(z) - nR_n(z)\right]\frac{1}{z-1}.$$

Hence, $W_n(x) = \frac{(4z)^{-(n-1)}}{z-1}\left[2zV_n(z) - R_{n-1}(z)R_n(z)\right]$, $n \geq 1$ and

$$\frac{z_{n,j}^{-(n-2)}}{z_{n,j}-1} V_n(z_{n,j}) = 2^{2n-3} W_n(x_{n,j}), \quad j = 1, 2, \ldots, n, \; n \geq 1. \tag{2.5}$$

From the recurrence formula for $\{R_n(z)\}$,

$$\frac{R_n(1)}{2R_{n-1}(1)}\left[1 - \frac{R_{n+1}(1)}{2R_n(1)}\right] = d_{n+1}, \quad n \geq 1.$$

Hence, $\{\hat{m}_n\}_{n=0}^{\infty}$, with

$$\hat{m}_n = 1 - \frac{R_{n+1}(1)}{2R_n(1)}, \quad n \geq 0,$$

is the minimal parameter sequence of the chain sequence $\{d_{1,n}\}_{n=1}^{\infty}$, where $d_{1,n} = d_{n+1}$, $n \geq 1$.

If we denote by $\{m_n\}_{n=0}^{\infty}$ and $\{M_n\}_{n=0}^{\infty}$ the minimal and maximal parameter sequences of $\{d_n\}_{n=1}^{\infty}$, respectively, then with $m_{1,n} = m_{n+1}$, $M_{1,n} = M_{n+1}$, $n \geq 0$, the sequences $\{m_{1,n}\}_{n=0}^{\infty}$

and $\{M_{1,n}\}_{n=0}^{\infty}$ are parameter sequences of $\{d_{1,n}\}_{n=1}^{\infty}$. Clearly, $\{m_{1,n}\}_{n=0}^{\infty}$ is such that $\hat{m}_n < m_{1,n}$, $n \geq 0$. However, $\{M_{1,n}\}_{n=0}^{\infty}$ is exactly the maximal parameter sequence of $\{d_{1,n}\}_{n=1}^{\infty}$. These and other interesting results on positive chain sequences can be found in Chapter III of Chihara [4].

Note that, the chain sequence $\{d_n\}_{n=1}^{\infty}$ can be such that $M_0 = m_0 = 0$. But, it is important to note that always $0 < m_{1,0} \leq M_{1,0} < 1$. The equality $m_{1,0} = M_{1,0}$ holds when the chain sequence $\{d_n\}$ has a unique parameter sequence.

3. Some correspondence and asymptotic properties

Let $\{Q_n\}$ be the sequence of polynomials given by the continued fraction expression

$$\frac{Q_n(z)}{R_n(z)} = \frac{2d_1}{(1+ic_1)z + (1-ic_1)} - \frac{4d_2 z}{(1+ic_2)z + (1-ic_2)} - \cdots - \frac{4d_n z}{(1+ic_n)z + (1-ic_n)}.$$

From the theory of continued fractions (see, for example, [6,17,19]) the polynomials Q_n are such that

$$Q_{n+1}(z) = \big[(1+ic_{n+1})z + (1-ic_{n+1})\big]Q_n(z) - 4d_{n+1}zQ_{n-1}(z), \quad n \geq 1, \tag{3.1}$$

with $Q_0(z) = 0$ and $Q_1(z) = 2d_1$.

First we look at an asymptotic result associated with the sequence $\{Q_n(1)/R_n(1)\}$. From the recurrence formulas for $\{R_n(z)\}$ and $\{Q_n(z)\}$ together with the theory of continued fractions

$$\begin{aligned}\frac{Q_n(1)}{R_n(1)} &= \frac{d_1}{1} - \frac{d_2}{1} - \frac{d_3}{1} - \cdots - \frac{d_n}{1}, \\ &= (1-M_0)\,\frac{M_{1,0}}{1} - \frac{(1-M_{1,0})M_{1,1}}{1} - \cdots - \frac{(1-M_{1,n-2})M_{1,n-1}}{1},\end{aligned}$$

for $n \geq 1$. Hence, one can write (see the proof of Lemma 3.2 in [4])

$$\frac{Q_n(1)}{R_n(1)} = (1-M_0)\,\frac{\sum_{k=1}^{n} \frac{M_{1,0}M_{1,1}\cdots M_{1,k-1}}{(1-M_{1,0})(1-M_{1,1})\cdots(1-M_{1,k-1})}}{1 + \sum_{k=1}^{n} \frac{M_{1,0}M_{1,1}\cdots M_{1,k-1}}{(1-M_{1,0})(1-M_{1,1})\cdots(1-M_{1,k-1})}}, \quad n \geq 1.$$

Therefore, Wall's characterization for the maximal parameter sequence (see [4, Chapter III, Theorem 6.2]) leads to the following lemma.

Lemma 3.1.

$$d_1 = \frac{Q_1(1)}{R_1(1)} < \frac{Q_2(1)}{R_2(1)} < \cdots < \frac{Q_{n-1}(1)}{R_{n-1}(1)} < \frac{Q_n(1)}{R_n(1)} < (1-M_0)$$

and

$$\lim_{n\to\infty} \frac{Q_n(1)}{R_n(1)} = (1-M_0).$$

Now we look at the series expansions of the rational functions $Q_n(z)/R_n(z)$. From the recurrence formula for $\{R_n\}$, if $R_n(z) = \sum_{j=0}^{n} r_{n,j}\, z^j$, then one can verify that

$$R_n(z) = \sum_{j=0}^{n} r_{n,j}\, z^j = \sum_{j=0}^{n} \overline{r}_{n,n-j}\, z^j = R_n^*(z), \quad n \geq 0 \tag{3.2}$$

and, in particular,

$$r_{0,0}=1 \quad \text{and} \quad r_{n,n}=\overline{r}_{n,0}=\prod_{k=1}^{n}(1+ic_k), \quad n\geq 1.$$

Here, $R_n^*(z) = z^n\overline{R_n(1/\overline{z})}$ is the reciprocal polynomial of $R_n(z)$. With the property (3.2) the polynomial R_n can be called a self-inversive polynomial. Likewise, the $n-1$ degree polynomial Q_n also satisfies the self inversive property

$$Q_n^*(z)=z^{n-1}\overline{Q_n(1/\overline{z})}=Q_n(z), \quad n\geq 1.$$

Applying the respective three term recurrence formulas in

$$U_n(z)=\begin{vmatrix} Q_n(z) & R_n(z) \\ Q_{n-1}(z) & R_{n-1}(z)\end{vmatrix} = Q_n(z)R_{n-1}(z)-Q_{n-1}(z)R_n(z), \quad n\geq 1,$$

there follows $U_1(z)=2d_1$ and

$$U_{n+1}(z)=4d_{n+1}zU_n(z)=2^{2n+1}d_1d_2\cdots d_{n+1}z^n, \quad n\geq 1. \tag{3.3}$$

Such formulas in the literature of continued fractions and orthogonal polynomials are known as determinant formulas.

Hence, considering the series expansions in terms of the origin and infinity,

$$\frac{Q_n(z)}{R_n(z)}-\frac{Q_{n-1}(z)}{R_{n-1}(z)}=\begin{cases}\dfrac{\overline{\gamma}_{n-1}}{\overline{r}_{n-1,n-1}}\,z^{n-1}+O\big(z^n\big), \\[2ex] \dfrac{\gamma_{n-1}}{r_{n-1,n-1}}\,\dfrac{1}{z^n}+O\big((1/z)^{n+1}\big),\end{cases} \quad n\geq 1, \tag{3.4}$$

where $\gamma_{n-1}=\frac{2^{2n-1}d_1d_2\cdots d_n}{r_{n,n}}$, $n\geq 1$. This means that there exist formal series expansions $E_0(z)$ and $E_\infty(z)$, respectively about the origin and about infinity, such that

$$E_0(z)-\frac{Q_n(z)}{R_n(z)}=\frac{\overline{\gamma}_n}{\overline{r}_{n,n}}\,z^n+O\big(z^{n+1}\big), \quad n\geq 0 \tag{3.5}$$

and

$$E_\infty(z)-\frac{Q_n(z)}{R_n(z)}=\frac{\gamma_n}{r_{n,n}}\,\frac{1}{z^{n+1}}+O\big((1/z)^{n+2}\big), \quad n\geq 0. \tag{3.6}$$

For reasons that will become clearer later we set

$$E_0(z)=-\nu_1-\nu_2z-\nu_3z^2-\nu_4z^3-\cdots$$

and

$$E_\infty(z)=\frac{\nu_0}{z}+\frac{\nu_{-1}}{z^2}+\frac{\nu_{-2}}{z^3}+\frac{\nu_{-3}}{z^4}+\cdots.$$

From (3.6), since R_n and Q_n are self inversive, we have

$$z^{-1}\overline{E_\infty(1/\overline{z})}-\frac{Q_n(z)}{R_n(z)}=\frac{\overline{\gamma}_n}{\overline{r}_{n,n}}\,z^n+O\big(z^{n+1}\big), \quad n\geq 0.$$

Comparing this with (3.5) we then have the symmetry property

$$z^{-1}\overline{E_\infty(1/\overline{z})} = E_0(z)$$

and

$$\nu_j = -\overline{\nu}_{-j+1}, \quad j = 1, 2, \ldots.$$

Considering the following systems of equations

$$\begin{array}{lllllllll}
-\nu_1 r_{n,0} & & & & -q_{n,0} & & & = & 0, \\
\vdots & \ddots & & & & \ddots & & \vdots & \\
-\nu_n r_{n,0} & \cdots & -\ \nu_1 r_{n,n-1} & & & & -\ q_{n,n-1} & = & 0, \\
-\nu_{n+1} r_{n,0} & \cdots & -\ \nu_2 r_{n,n-1} & -\ \nu_1 r_{n,n} & & & & = & \overline{\gamma}_n,
\end{array} \tag{3.7}$$

and

$$\begin{array}{llllllll}
\nu_0 r_{n,0} & +\ \nu_{-1} r_{n,1} & \cdots & +\ \nu_{-n} r_{n,n} & & & = & \gamma_n, \\
& \nu_0 r_{n,1} & \cdots & +\ \nu_{-n+1} r_{n,n} & -\ q_{n,0} & & = & 0, \\
& & \ddots & \vdots & & \ddots & \vdots & \\
& & & \nu_0 r_{n,n} & & -\ q_{n,n-1} & = & 0
\end{array} \tag{3.8}$$

in the coefficients of $Q_n(z) = \sum_{j=0}^{n-1} q_{n,j} z^j$ and $R_n(z) = \sum_{j=0}^{n} r_{n,j} z^j$, which follow from (3.5) and (3.6), we have

$$\gamma_n = (-1)^n \frac{H_{n+1}^{(-n)}}{H_n^{(-n+1)}} r_{n,n}, \quad n \geq 1,$$

where

$$H_n^{(m)} = \begin{vmatrix} \nu_m & \nu_{m+1} & \cdots & \nu_{m+n-1} \\ \nu_{m+1} & \nu_{m+2} & \cdots & \nu_{m+n} \\ \vdots & \vdots & & \vdots \\ \nu_{m+n-1} & \nu_{m+n} & \cdots & \nu_{m+2n-2} \end{vmatrix},$$

are the Hankel determinants associated with the double sequence $\{\nu_n\}_{n=-\infty}^{\infty}$. Moreover, the following lemma can also be stated.

Lemma 3.2. *Let $\{R_n\}$ and $\{Q_n\}$ be the sequences of polynomials obtained from the real sequence $\{c_n\}$, the positive chain sequence $\{d_n\}$ and the three term recurrence formulas (1.2) and (3.1). Then the rational functions Q_n/R_n satisfy the correspondence properties given by (3.5) and (3.6). Moreover, if the moment functional $\mathcal{N}$ is such that*

$$\mathcal{N}[\zeta^{-n}] = \nu_n, \quad n = 0, \pm 1, \pm 2, \ldots,$$

then the polynomials R_n satisfy

$$\mathcal{N}[\zeta^{-n+j} R_n(\zeta)] = \begin{cases} -\overline{\gamma}_n, & j = -1, \\ 0, & j = 0, 1, \ldots, n-1, \quad n \geq 1, \\ \gamma_n, & j = n \end{cases}$$

where $\gamma_0 = \nu_0 = \frac{2d_1}{1+ic_1}$ and $\gamma_n = \frac{4d_{n+1}}{(1+ic_{n+1})}\gamma_{n-1}$, $n \geq 1$.

The validity of the orthogonal relations in Lemma 3.2 follow from the linear system only in the coefficients of R_n obtained from (3.7) and (3.8).

In terms of $\gamma_n = \mathcal{N}[R_n(\zeta)]$, $n \geq 0$, the elements of $\{c_n\}$ and $\{d_n\}$ satisfy

$$\frac{1}{d_1} = \frac{1}{\gamma_0} + \frac{1}{\overline{\gamma}_0}, \qquad i\frac{c_1}{d_1} = \frac{1}{\gamma_0} - \frac{1}{\overline{\gamma}_0}$$

and

$$\frac{1}{2d_{n+1}} = \frac{\gamma_{n-1}}{\gamma_n} + \frac{\overline{\gamma}_{n-1}}{\overline{\gamma}_n}, \qquad i\frac{c_{n+1}}{2d_{n+1}} = \frac{\gamma_{n-1}}{\gamma_n} - \frac{\overline{\gamma}_{n-1}}{\overline{\gamma}_n}, \quad n \geq 1.$$

Observe also that one can write

$$\mathcal{N}\Big[\frac{1}{z-\zeta}\Big] = \begin{cases} E_0(z), & \text{for } z \sim 0, \\ E_\infty(z), & \text{for } z \sim \infty. \end{cases}$$

The referee who made us aware of the results in [9] also brought to our attention that given the recurrence formula (1.2) the existence of a moment functional $\mathcal{N}$ such that $\mathcal{N}[\zeta^{-n+j}R_n(\zeta)] = 0$, $0 \leq j < n$, also follows from [15, Theorem 2.1]. The authors of [15] consider a recurrence formula of the form

$$\hat{R}_n(z) = (z - \beta_n^{(1)})\hat{R}_{n-1}(z) - \beta_n^{(2)}(z - \beta_n^{(3)})\hat{R}_{n-2}(z), \quad n \geq 1,$$

with $\hat{R}_{-1}(z) = 0$ and $\hat{R}_1(z) = 1$, which reduces to the recurrence formula of the type considered in the current manuscript when $\{\beta_n^{(3)}\}$ is the zero sequence. To compare exactly with the recurrence formula (1.2), one must also impose further restrictions on the sequences $\{\beta_n^{(1)}\}$ and $\{\beta_n^{(2)}\}$. For example, the restriction on $\{\beta_n^{(1)}\}$ is $|\beta_n^{(1)}| = 1$, $n \geq 1$. From [15, Theorem 2.3], together with the fact that the zeros of R_n are all simple and lie on the unit circle $|z| = 1$, one may also deduce that the moment functional $\mathcal{N}$ can be given as $\mathcal{N}[f] = \int_{\mathcal{C}} f(\zeta)d\alpha(\zeta)$. With the specific conditions assumed in (1.2), we establish in Section 4 that $d\alpha(\zeta) = (1-\zeta)d\mu(\zeta)$, where $d\mu(\zeta)$ is a positive measure on the unit circle.

4. Associated moments and measure on the unit circle

Given the sequence of polynomials $\{R_n\}$ and $\{Q_n\}$, as defined in Sections 1 and 3, let the sequence of polynomials $\{A_n\}$ and $\{B_n\}$ be given by

$$A_n(z) = R_n(z) - Q_n(z) \quad \text{and} \quad B_n(z) = (z-1)R_n(z), \quad n \geq 0. \tag{4.1}$$

It is easily verified that

$$\frac{A_{n+1}(z)}{B_{n+1}(z)} - \frac{A_n(z)}{B_n(z)} = -\frac{1}{z-1}\left[\frac{Q_{n+1}(z)}{R_{n+1}(z)} - \frac{Q_n(z)}{R_n(z)}\right], \quad n \geq 0.$$

Hence, from (3.4)

$$\frac{A_{n+1}(z)}{B_{n+1}(z)} - \frac{A_n(z)}{B_n(z)} = \begin{cases} \dfrac{\overline{\gamma}_n}{\overline{r}_{n,n}} z^n + O\big(z^{n+1}\big), \\ -\dfrac{\gamma_n}{r_{n,n}}\dfrac{1}{z^{n+2}} + O\big((1/z)^{n+3}\big), \end{cases} \quad n \geq 0. \tag{4.2}$$

Thus, we can state the following lemma.

Theorem 4.1. *Associated with the real sequence $\{c_n\}$ and the positive chain sequence $\{d_n\}$ there exists a nontrivial probability measure μ on the unit circle. If $M_0 > 0$, where $\{M_n\}$ is the maximal parameter sequence of $\{d_n\}$, then μ has a pure point of mass M_0 at $z = 1$. Let $\mathcal{N}$ be the moment functional associated with $\{c_n\}$ and $\{d_n\}$ as given by Lemma 3.2. Then*

$$\mathcal{N}[\zeta^{-n}] = \int_{\mathcal{C}} \zeta^{-n}(1-\zeta)d\mu(\zeta), \quad n = 0, \pm 1, \pm 2, \ldots.$$

Proof. From (4.2) there exist series expansions $F_0(z)$ and $F_\infty(z)$ such that

$$F_0(z) - \frac{A_n(z)}{B_n(z)} = \frac{\overline{\gamma}_n}{\overline{r}_{n,n}} z^n + O\big(z^{n+1}\big), \quad n \geq 0 \tag{4.3}$$

and

$$F_\infty(z) - \frac{A_n(z)}{B_n(z)} = -\frac{\gamma_n}{r_{n,n}} \frac{1}{z^{n+2}} + O\big((1/z)^{n+3}\big), \quad n \geq 0. \tag{4.4}$$

Setting

$$F_0(z) = -\mu_1 - \mu_2 z - \mu_3 z^2 - \mu_4 z^3 - \cdots,$$

and

$$F_\infty(z) = \frac{\mu_0}{z} + \frac{\mu_{-1}}{z^2} + \frac{\mu_{-2}}{z^3} + \frac{\mu_{-3}}{z^4} + \cdots,$$

we then have from (3.5), (3.6), (4.1), (4.3) and (4.4) that the numbers μ_k satisfy

$$\left.\begin{aligned} \mu_n &= 1 + \sum_{j=1}^{n} \nu_j, \\ \mu_{-n} &= 1 - \sum_{j=1}^{n} \nu_{-j+1}, \end{aligned}\right\} \quad n = 1, 2, 3, \ldots, \tag{4.5}$$

with $\mu_0 = 1$. Since $\nu_j = -\overline{\nu}_{-j+1}$, $j \geq 1$, there hold

$$\mu_{-k} = \overline{\mu}_k, \quad k \geq 1.$$

If we define the moment functional $\mathcal{M}$ by

$$\mathcal{M}[\zeta^{-k}] = \mu_k, \quad k = 0, \pm 1, \pm 2, \ldots,$$

then

$$\mathcal{M}[\zeta^{-k}] = 1 - \mathcal{N}\Big[\frac{1-\zeta^{-k}}{1-\zeta}\Big], \quad k = 0, \pm 1, \pm 2, \ldots. \tag{4.6}$$

Since $\nu_{-k} = \mu_{-k} - \mu_{-k-1}$, $k = 0, \pm 1, \pm 2, \ldots$, which follows from (4.5), the moment functionals $\mathcal{M}$ and $\mathcal{N}$ also satisfy

$$\mathcal{N}[\zeta^k] = \mathcal{M}[\zeta^k(1-\zeta)], \quad k = 0, \pm 1, \pm 2, \ldots. \tag{4.7}$$

Now considering the partial decomposition of $A_n(z)/B_n(z)$, we have

$$\frac{A_n(z)}{B_n(z)} = \frac{R_n(z) - Q_n(z)}{(z-1)R_n(z)} = \frac{\lambda_{n,0}}{z-1} + \sum_{j=1}^{n} \frac{\lambda_{n,j}}{z - z_{n,j}}, \tag{4.8}$$

where $z_{n,j}$, $j = 1, 2, \ldots, n$, are the zeros of $R_n(z)$,

$$\lambda_{n,0} = 1 - \frac{Q_n(1)}{R_n(1)}$$

and

$$\lambda_{n,j} = \frac{Q_n(z_{n,j})}{(1 - z_{n,j})R_n'(z_{n,j})}, \quad j = 1, 2, \ldots, n.$$

Clearly, from Lemma 3.1,

$$1 - d_1 = \lambda_{1,0} > \lambda_{2,0} > \cdots > \lambda_{n,0} > \lambda_{n+1,0} > \cdots$$

and

$$\lim_{n\to\infty} \lambda_{n,0} = M_0.$$

Furthermore, since we can write

$$\lambda_{n,j} = \frac{U_n(z_{n,j})}{(1 - z_{n,j})V_n(z_{n,j})}, \quad j = 1, 2, \ldots, n,$$

where $V_n(z)$ and $U_n(z)$ are respectively given by (2.4) and (3.3), we have

$$\lambda_{n,j} = \frac{4d_1 d_2 \cdots d_n}{W_n(x_{n,j})} \frac{z_{n,j}}{(z_{n,j} - 1)(1 - z_{n,j})} > 0, \quad j = 1, 2, \ldots, n.$$

Here, $W_n(x)$ are the Wronskians defined in (2.2) and, with $z_{n,j} = e^{i\theta_{n,j}}$,

$$\frac{z_{n,j}}{(z_{n,j} - 1)(1 - z_{n,j})} = \frac{1}{4\sin^2(\theta_{n,j}/2)}.$$

In addition to the positiveness of the elements $\lambda_{n,j}$, $j = 0, 1, 2, \ldots, n$, by considering the limit of $zA_n(z)/B_n(z)$, as $z \to \infty$, we also have

$$\sum_{j=0}^{n} \lambda_{n,j} = 1.$$

Now if the step functions $\psi_n(e^{i\theta})$, $n \geq 1$, are defined on $[0, 2\pi]$ by

$$\psi_n(e^{i\theta}) = \begin{cases} 0, & \theta = 0, \\ \lambda_{n,0}, & 0 < \theta \leq \theta_{n,1}, \\ \displaystyle\sum_{j=0}^{k} \lambda_{n,j}, & \theta_{n,k} < \theta \leq \theta_{n,k+1}, \quad k = 1, 2, \ldots, n-1, \\ 1, & \theta_{n,n} < \theta \leq 2\pi \end{cases}$$

then from the definition of the Riemann–Stieltjes integrals

$$\frac{A_n(z)}{B_n(z)} = \int_{\mathcal{C}} \frac{1}{z - \zeta}\, d\psi_n(\zeta), \quad n \geq 1.$$

Hence, by the application of the Helly selection theorem (see [16]) there exists a subsequence $\{n_j\}$ such that $\psi_{n_j}(e^{i\theta})$ converges to a bonded non-decreasing function, say $\mu(e^{i\theta})$, in $[0, 2\pi]$.

From (4.3) and (4.4), since

$$\int_{\mathcal{C}} d\psi_n(\zeta) = 1 \quad \text{and} \quad \int_{\mathcal{C}} \zeta^k d\psi_n(\zeta) = \mu_{-k}, \quad k = \pm 1, \pm 2, \ldots, \pm n,$$

we also have that

$$\int_{\mathcal{C}} d\mu(\zeta) = 1 = \mathcal{M}[1] \quad \text{and} \quad \int_{\mathcal{C}} \zeta^k d\mu(\zeta) = \mu_{-k} = \mathcal{M}[\zeta^k], \quad k = \pm 1, \pm 2, \ldots.$$

Now, μ is the only probability measure that satisfies the above relations follows from known results on the moment problem on the unit circle. The measure has a jump M_0 at $z = 1$ is also confirmed by $\lim_{n\to\infty} \lambda_{n,0} = M_0$. ■

5. Further properties of $R_n(z)$ and the associated OPUC

With the probability measure μ obtained in the previous section we then have from Lemma 3.2 and Theorem 4.1 that

$$\nu_k = \mathcal{N}[\zeta^{-k}] = \int_{\mathcal{C}} \zeta^{-k}(1-\zeta) d\mu(\zeta), \quad k = 0, \pm 1, \pm 2, \ldots$$

and for $n \geq 1$,

$$\int_{\mathcal{C}} \zeta^{-n+k} R_n(\zeta)(1-\zeta) d\mu(\zeta) = 0, \quad 0 \leq k \leq n-1. \tag{5.1}$$

The following lemma provides information about the values of the integrals $\int_{\mathcal{C}} R_n(\zeta) d\mu(\zeta)$.

Lemma 5.1. *Let* $\widehat{\gamma}_n = \int_{\mathcal{C}} R_n(\zeta) d\mu(\zeta)$, $n \geq 0$. *Then*

$$\widehat{\gamma}_0 = 1 \quad \textit{and} \quad \widehat{\gamma}_n = 2(1 - m_n)\, \widehat{\gamma}_{n-1}, \quad n \geq 1, \tag{5.2}$$

where $\{m_n\}_{n=0}^{\infty}$ *is the minimal parameter sequence of the positive chain sequence* $\{d_n\}$.

Proof. First we observe from (5.1) that

$$\widehat{\gamma}_n = \int_{\mathcal{C}} \zeta^{-k} R_n(\zeta) d\mu(\zeta), \quad k = 0, 1, \ldots, n, \; n \geq 1. \tag{5.3}$$

Now by direct evaluations $\widehat{\gamma}_0 = \int_{\mathcal{C}} d\mu(\zeta) = 1$ and

$$\widehat{\gamma}_1 = (1 + ic_1)\mu_{-1} + (1 - ic_1).$$

Thus, from $\mu_{-1} = 1 - \nu_0$ and $\nu_0 = 2d_1/(1 + ic_1)$, there follows

$$\widehat{\gamma}_1 = 2(1 - d_1) = 2(1 - m_1),$$

proving (5.2) for $n = 1$.

Now from the three term recurrence formula (1.2), we have

$$\int_{\mathcal{C}} \zeta^{-1} R_{n+1}(\zeta) d\mu(\zeta) = \int_{\mathcal{C}} \zeta^{-1}(\zeta + 1) R_n(\zeta) d\mu(\zeta)$$
$$+ ic_{n+1} \int_{\mathcal{C}} \zeta^{-1}(\zeta - 1) R_n(\zeta) d\mu(\zeta) - 4\, d_{n+1} \int_{\mathcal{C}} R_{n-1}(\zeta) d\mu(\zeta),$$

for $n \geq 1$. Using (5.1) and (5.3) we then have

$$\widehat{\gamma}_{n+1} = 2\widehat{\gamma}_n - 4d_{n+1}\widehat{\gamma}_{n-1}, \quad n \geq 1.$$

From this we have

$$d_{n+1} = d_{n,1} = \frac{\widehat{\gamma}_n}{2\widehat{\gamma}_{n-1}}\big(1 - \frac{\widehat{\gamma}_{n+1}}{2\widehat{\gamma}_n}\big), \quad n \geq 1.$$

Since the minimal parameter sequence of the positive chain sequence $\{d_n\}$ is also a parameter sequence of the positive chain sequence $\{d_{n,1}\}$, we obtain from

$$\frac{\widehat{\gamma}_1}{2\widehat{\gamma}_0} = (1 - m_1),$$

that

$$\frac{\widehat{\gamma}_n}{2\widehat{\gamma}_{n-1}} = (1 - m_n), \quad n \geq 1,$$

which completes the proof of the lemma. ■

As an immediate consequence of the above results we have

$$\begin{aligned}\int_{\mathcal{C}} &\overline{\zeta^k}[R_n(\zeta) - 2(1 - m_n)R_{n-1}(\zeta)]d\mu(\zeta) \\ &= \int_{\mathcal{C}} \zeta^{-k}[R_n(\zeta) - 2(1 - m_n)R_{n-1}(\zeta)]d\mu(\zeta) = 0,\end{aligned}$$

for $k = 0, 1, \ldots, n-1$. Since $R_n(z) - 2(1-m_n)R_{n-1}(z)$ is a polynomial of exact degree n with leading coefficient $r_{n,n} = \prod_{k=1}^{n}(1 + ic_k)$, we can state the following.

Theorem 5.2. *If the sequence of polynomials $\{S_n\}$ is such that*

$$S_0(z) = 1 \quad \text{and} \quad S_n(z)\prod_{k=1}^{n}(1 + ic_k) = R_n(z) - 2(1 - m_n)R_{n-1}(z), \quad n \geq 1,$$

then $\{S_n\}$ is the sequence of monic OPUC with respect to the measure μ.

From the above theorem, together with the formula for $R_n(0)$ given after (3.2), the associated Verblunsky coefficients $\alpha_{n-1} = -\overline{S_n(0)}$, $n \geq 1$, are

$$\alpha_{n-1} = \frac{1 - 2m_n - ic_n}{1 + ic_n}\prod_{k=1}^{n}\frac{1 + ic_k}{1 - ic_k}, \quad n \geq 1.$$

6. An example

We now analyze the results obtained so far with the use of the following example.

Let the real sequences $\{c_n\}$ and $\{d_n\}$ be given by

$$\begin{aligned} &c_n = \frac{\eta}{\lambda + n}, \quad n \geq 1, \\ &d_1 = d_1(t) = \frac{1}{2}\frac{2\lambda + 1}{\lambda + 1}(1 - t), \qquad d_{n+1} = \frac{1}{4}\frac{n(2\lambda + n + 1)}{(\lambda + n)(\lambda + n + 1)}, \quad n \geq 1, \end{aligned} \tag{6.1}$$

where $\lambda, \eta \in \mathbb{R}$, $\lambda > -1/2$ and $0 \leq t < 1$.

Observe that, as verified in [5], $\{d_n\}_{n=1}^{\infty}$ is a positive chain sequence with its maximal parameter sequence $\{M_n^{(t)}\}_{n=0}^{\infty}$ given by

$$M_0^{(t)} = t, \qquad M_n^{(t)} = \frac{1}{2}\frac{2\lambda + n}{\lambda + n}, \quad n \geq 1. \tag{6.2}$$

It was first shown in [28] (see also [5]) that the polynomials R_n obtained from the above sequences $\{c_n\}$ and $\{d_n\}$, together with the recurrence formula (1.2), are

$$R_n(z) = \frac{(2\lambda+2)_n}{(\lambda+1)_n}\,{}_2F_1(-n, b+1;\ b+\overline{b}+2;\ 1-z), \quad n \geq 1,$$

where $b = \lambda + i\eta$. This actually follows from the contiguous relation

$$\begin{aligned}(c-a)\,{}_2F_1(a-1, b;\ c;\ z) &= \big(c - 2a - (b-a)z\big)\,{}_2F_1(a, b;\ c;\ z)\\ &\quad + a(1-z)\,{}_2F_1(a+1, b;\ c;\ z),\end{aligned}$$

of Gauss (see [1, Eq. (2.5.16)]), by letting $a = -n$, $b = b+1$, $c = b + \overline{b} + 2$ and $z = 1 - z$.

Now consider the rational functions Q_n/R_n, $n \geq 1$, given by the continued fraction expression

$$\begin{aligned}\frac{Q_n(z)}{R_n(z)} = \frac{2d_1}{(1+ic_1)z + (1-ic_1)} - \frac{4d_2 z}{(1+ic_2)z + (1-ic_2)} - \cdots\\ \cdots - \frac{4d_n z}{(1+ic_n)z + (1-ic_n)}.\end{aligned} \tag{6.3}$$

Clearly, by Lemma 3.1, the increasing sequence $\{Q_n(1)/R_n(1)\}$ satisfy

$$\lim_{n\to\infty} \frac{Q_n(1)}{R_n(1)} = 1 - t.$$

In order to obtain the power series expansions associated with the rational functions $Q_n(z)/R_n(z)$, we write the continued fraction expression (6.3) in the equivalent form

$$\frac{(1-ic_1)Q_n(z)}{2d_1\, R_n(z)} = \frac{1}{1+b_1 z} - \frac{a_2 z}{1+b_2 z} - \cdots - \frac{a_n z}{1+b_n z}, \tag{6.4}$$

where

$$b_n = \frac{1+ic_n}{1-ic_n} = \frac{b+n}{\overline{b}+n}, \qquad a_{n+1} = \frac{4d_{n+1}}{(1-ic_n)(1-ic_{n+1})} = \frac{n(b+\overline{b}+n+1)}{(\overline{b}+n)(\overline{b}+n+1)},$$
$$n \geq 1.$$

We thus can obtain the power series expansion about the origin of the rational function on the left hand side of (6.4) from

$$\Omega_0(b;\ z) = \frac{1}{1+b_1 z} - \frac{a_2 z}{1+b_2 z} - \cdots - \frac{a_{n-1} z}{1+b_{n-1} z} - \frac{a_n z}{1 + b_n z - a_{n+1} z\, \Omega_n(b;\ z)},$$

where $\Omega_n(b;\ z) = \frac{{}_2F_1(n+1, -b;\ \overline{b}+n+2;\ z)}{{}_2F_1(n, -b;\ \overline{b}+n+1;\ z)}$.

The above relation follows from the contiguous relation

$$
\begin{aligned}
{}_2F_1(a,b;c;z) &= \Big(1+\frac{a-b+1}{c}z\Big)\,{}_2F_1(a+1,b;c+1;z)\\
&\quad -\frac{(a+1)(c-b+1)}{c(c+1)}z\,{}_2F_1(a+2,b;c+2;z)
\end{aligned}
$$

of Gauss (see [1, Eq. (2.5.3)]), by substituting a, b and c with n, $-b$ and $\overline{b}+n+1$, respectively.

Since $\Omega_0(b;\ z) = {}_2F_1(1,-b;\ \overline{b}+2;\ z)$, we have

$$
\frac{Q_n(z)}{R_n(z)} \sim \frac{2d_1}{1-ic_1}\,{}_2F_1(1,-b;\ \overline{b}+2;\ z) = -\nu_1-\nu_2 z-\nu_3 z^2-\cdots,
$$

from which

$$
\nu_n = \frac{b+\overline{b}+1}{b+1}\frac{(-b-1)_n}{(\overline{b}+1)_n}(1-t), \quad n\geq 1. \tag{6.5}
$$

Since $\nu_n = -\overline{\nu}_{-n+1}$, with the convention $(a)_n = \Gamma(a+n)/\Gamma(a)$ for all integer values of n, the above expression for ν_n is also valid for $n \leq 0$.

Now we show that

$$
\nu_n = \mathcal{N}[\zeta^{-n}] = \int_{\mathcal{C}} \zeta^{-n}\, d\psi(t,b;\ \zeta), \quad n = 0,\pm1,\pm2,\ldots, \tag{6.6}
$$

where $d\psi(t,b;\ z) = \frac{i\,|\Gamma(b+1)|^2(1-t)}{2\pi\,\Gamma(b+\overline{b}+1)}(-z)^{-\overline{b}-1}(1-z)^{b+\overline{b}+1}dz$.

Here, the branch cuts in $(-z)^{-\overline{b}} = (e^{-i\pi}z)^{-\overline{b}}$ and $(1-z)^{b+\overline{b}} = \big(e^{-i\pi}(z-1)\big)^{b+\overline{b}}$ are along the positive real axis.

Before verifying (6.6), we observe that (6.6) can also be written in the equivalent form

$$
\nu_n = \frac{-i\,2^{b+\overline{b}+1}|\Gamma(b+1)|^2(1-t)}{2\pi\,\Gamma(b+\overline{b}+1)}\int_0^{2\pi} e^{-in\theta}e^{i\theta/2}e^{(\pi-\theta)\mathcal{I}m(b)}[\sin^2(\theta/2)]^{\mathcal{R}e(b)+1/2}d\theta.
$$

First we show (6.6) for ν_{-n} for those values of n such that $\mathcal{R}e(n-\overline{b}) > 0$. Since $\int_{\Lambda}(-z)^{n-\overline{b}-1}(1-z)^{b+\overline{b}+1}dz = 0$, where Λ is the contour given as in Fig. 1, we obtain for $\mathcal{R}e(n-\overline{b}) > 0$ that

$$
\nu_{-n} = \frac{-2\sin(\overline{b}\pi)\,|\Gamma(b+1)|^2(1-t)}{2\pi\,\Gamma(b+\overline{b}+1)}\int_0^1 x^{n-\overline{b}-1}(1-x)^{b+\overline{b}+1}dx.
$$

Hence, from the definitions of the gamma function, the beta function and Euler's reflection formula, we obtain the required result.

To obtain the result for other values of n, we note that for ν_n given by (6.6) there hold

$$
\nu_n = \frac{\overline{b}+1+n}{-b-1+n}\nu_{n+1} \quad \text{and} \quad \nu_n = -\overline{\nu}_{-n+1}, \quad n = 0,1,2,\ldots,
$$

the first of these results follows from integration by parts and other by simple conjugation.

The idea used here to calculate the integral (6.6) is the same employed in, for example, [13,28]. In [13] the authors consider a general set of parameters for the exponents of z and $1-z$, but restricting the values of the parameters to be real.

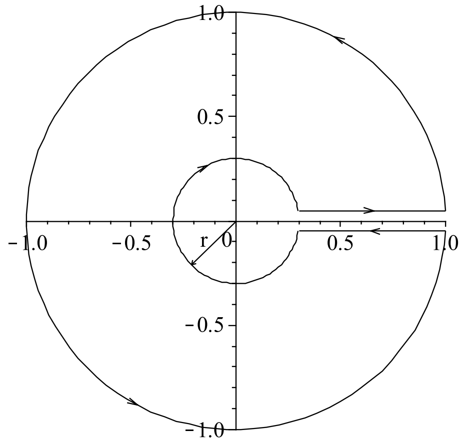

Fig. 1. Contour Λ.

Now from the representation (6.5) for the coefficients ν_n, we have from (4.5) that

$$\mathcal{M}[1] = 1, \qquad \overline{\mathcal{M}[\zeta^n]} = \mathcal{M}[\zeta^{-n}] = \mu_n = 1 + \frac{b+\overline{b}+1}{b+1}(1-t)\sum_{j=1}^{n}\frac{(-b-1)_j}{(\overline{b}+1)_j},$$
$$n \geq 1.$$

Hence,

$$\overline{\mu}_{-n} = \mu_n = t + (1-t)\frac{(-b)_n}{(\overline{b}+1)_n}, \quad n \geq 0,$$

which follows from the interesting summation formula

$$1 + \frac{b+\overline{b}+1}{b+1}\sum_{j=1}^{n}\frac{(-b-1)_j}{(\overline{b}+1)_j} = \frac{(-b)_n}{(\overline{b}+1)_n}, \quad n \geq 1,$$

verified easily by induction.

Now from the representation (6.6) for the coefficients ν_n, we have from (4.6) that

$$\mathcal{M}[\zeta^{-n}] = 1 - (1-t)\frac{i\,|\Gamma(b+1)|^2}{2\pi\,\Gamma(b+\overline{b}+1)}\int_{\mathcal{C}}(1-\zeta^{-n})\,(-\zeta)^{-\overline{b}-1}(1-\zeta)^{b+\overline{b}}d\zeta,$$

for $n = 0, \pm 1, \pm 2, \ldots$. Since we can verify using integration by parts

$$(1-t)\frac{i\,|\Gamma(b+1)|^2}{2\pi\,\Gamma(b+\overline{b}+1)}\int_{\mathcal{C}}(-\zeta)^{-\overline{b}-1}(1-\zeta)^{b+\overline{b}}d\zeta = -\frac{\overline{b}+1}{b+\overline{b}+1}\,\nu_1 = 1-t,$$

we can write

$$\begin{aligned}\mathcal{M}[\zeta^{-n}] &= \int_{\mathcal{C}}\zeta^{-n}d\mu(t,b;\ \zeta) \\ &= t(1^{-n}) + (1-t)\frac{i\,|\Gamma(b+1)|^2}{2\pi\,\Gamma(b+\overline{b}+1)}\int_{\mathcal{C}}\zeta^{-n}\,(-\zeta)^{-\overline{b}-1}(1-\zeta)^{b+\overline{b}}d\zeta, \qquad (6.7)\end{aligned}$$

for $n = 0, \pm 1, \pm 2, \ldots$. Equivalently, this can also be written as

$$\mu_n = \mathcal{M}[\zeta^{-n}]$$
$$= t\, e^{in0} + (1-t)\, \frac{2^{b+\bar{b}} |\Gamma(b+1)|^2}{2\pi\, \Gamma(b+\bar{b}+1)} \int_0^{2\pi} e^{-in\theta}\, [e^{(\pi-\theta)}]^{\mathcal{I}m(b)} [\sin^2(\theta/2)]^{\mathcal{R}e(b)} d\theta,$$

for $n = 0, \pm 1, \pm 2, \ldots$.

By Theorem 5.2, the monic OPUC and Verblunsky coefficients associated with the positive measure $\mu(t, b;\ z)$ given by (6.7) are

$$S_n^{(t)}(z) = \frac{(2\lambda+2)_n}{(b+1)_n} \Big[{}_2F_1(-n, b+1;\ b+\bar{b}+2;\ 1-z) - 2(1-m_n^{(t)}) \frac{(\lambda+n)}{2\lambda+n+1}\, {}_2F_1(-n+1, b+1;\ b+\bar{b}+2;\ 1-z) \Big]$$

and

$$\alpha_{n-1}^{(t)} = -\overline{S_n^{(t)}(0)} = -\frac{(b+1)_n}{(\bar{b}+1)_n} \Big[1 - 2(1-m_n^{(t)}) \frac{\lambda+n}{b+n} \Big],$$

for $n \geq 1$, where $\{m_n^{(t)}\}_{n=0}^{\infty}$, such that

$$m_0^{(t)} = 0, \qquad m_n^{(t)} = d_n/(1-m_{n-1}^{(t)}), \quad n \geq 1,$$

is the minimal parameter sequence of the positive chain sequence given in (6.1).

Two particular situation in which we have been able to give the Verblunsky coefficients explicitly are the following.

When $t = 0$, the minimal $\{m_n^{(t)}\}_{n=0}^{\infty}$ and maximal $\{M_n^{(t)}\}_{n=0}^{\infty}$ parameter sequences of $\{d_n\}$ coincide and, hence, we obtain from (6.2) that

$$\alpha_{n-1}^{(0)} = -\frac{(b)_n}{(\bar{b}+1)_n}, \quad n \geq 1.$$

In this case, see [28], the monic OPUC are $S_n^{(0)}(z) = \frac{(2\lambda+1)_n}{(b+1)_n}\, {}_2F_1(-n, b+1;\ b+\bar{b}+1;\ 1-z)$, $n \geq 1$.

The other situation is when $\lambda = 0$ (i.e., $b = i\eta$). One obtains (see [5]),

$$\alpha_{n-1}^{(t)} = -\frac{(i\eta+1)_{n-1}}{(-i\eta+1)_n} \Big[i\eta - \frac{nt}{1+(n-1)t} \Big], \quad n \geq 1.$$

Acknowledgments

The authors would like to thank the referees for many valuable comments and references.

References

[1] G.E. Andrews, R. Askey, R. Roy, Special Functions, in: Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2000.

[2] J. Breuer, E. Ryckman, B. Simon, Equality of the spectral and dynamical definitions of reflection, Comm. Math. Phys. 295 (2010) 531–550.

[3] K. Castillo, L. Garza, F. Marcellán, Perturbations on the subdiagonals of Toeplitz matrices, Linear Algebra Appl. 434 (2011) 1563–1579.

[4] T.S. Chihara, An Introduction to Orthogonal Polynomials, in: Mathematics and its Applications Series, Gordon and Breach, 1978.

[5] M.S. Costa, H.M. Felix, A. Sri Ranga, Orthogonal polynomials on the unit circle and chain sequences, J. Approx. Theory 173 (2013) 14–32.

[6] A. Cuyt, A.B. Petersen, B. Verdonk, H. Waadeland, W.B. Jones, Handbook of Continued Fractions for Special Functions, Springer, 2008.

[7] D. Damanik, P. Munger, W.N. Yessen, Orthogonal polynomials on the unit circle with Fibonacci Verblunsky coefficients, I. The essential support of the measure, J. Approx. Theory 173 (2013) 56–88.

[8] D.K. Dimitrov, A. Sri Ranga, Zeros of a family of hypergeometric para-orthogonal polynomials on the unit circle, Math. Nachr. 286 (2013) 1778–1791.

[9] P. Delsarte, Y. Genin, The tridiagonal approach to Szegő's orthogonal polynomials, Toeplitz linear system, and related interpolation problems, SIAM J. Math. Anal. 19 (1988) 718–735.

[10] T. Erdélyi, P. Nevai, J. Zhang, J. Geronimo, A simple proof of "Favard's theorem" on the unit circle, Atti Semin. Mat. Fis. Univ. Modena 39 (1991) 551–556. Also in "Trends in functional analysis and approximation theory" (Acquafredda di Maratea, 1989), 41–46, Univ. Modena Reggio Emilia, Modena, 1991.

[11] G. Freud, Orthogonal Polynomials, Pergamon Press, Oxford, 1971.

[12] Ya.L. Geronimus, Polynomials Orthogonal on a Circle and their Applications, in: Series and Approximations, Amer. Math. Soc. Transl., Series 1, vol. 3, Providence, RI, 1962, pp. 1–78.

[13] E. Hendriksen, H. van Rossum, Orthogonal Laurent polynomials, Indag. Math. Ser. A 48 (1986) 17–36.

[14] M.E.H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, Cambridge Univ. Press, Cambridge, 2005.

[15] M.E.H. Ismail, D.R. Masson, Generalized orthogonality and continued fractions, J. Approx. Theory 83 (1995) 1–40.

[16] W.B. Jones, O. Njåstad, W.J. Thron, Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle, Bull. Lond. Math. Soc. 21 (1989) 113–152.

[17] W.B. Jones, W.J. Thron, Continued Fractions Analytic Theory and Applications, in: Encyclopedia of Mathematics and Its Applications, vol. 11, Addison-Wesley, Reading, MA, 1980.

[18] A. Kheifets, L. Golinskii, F. Peherstorfer, P. Yuditskii, Scattering theory for CMV matrices: uniqueness, Helson–Szegő and strong Szegő theorems, Integral Equations Operator Theory 69 (2011) 479–508.

[19] L. Lorentzen, H. Waadeland, Continued Fractions with Applications, in: Studies in Computational Mathematics, vol. 3, North-Holland, Amsterdam, 1992.

[20] A. Máté, P. Nevai, V. Totik, Strong and week convergence of orthogonal polynomials, Amer. J. Math. 109 (1987) 239–282.

[21] A. Máté, P. Nevai, V. Totik, Strong and week convergence of orthogonal polynomials, Constr. Approx. 3 (1987) 73–96.

[22] F. Peherstorfer, Positive trigonometric quadrature formulas and quadrature on the unit circle, Math. Comp. 80 (2011) 1685–1701.

[23] F. Peherstorfer, A. Volberg, P. Yuditskii, CMV matrices with asymptotically constant coefficients. Szegő–Blaschke class, scattering theory, J. Funct. Anal. 256 (2009) 2157–2210.

[24] B. Simanek, Week convergence of CD kernels: a new approach on the circle and real line, J. Approx. Theory 164 (2012) 204–209.

[25] B. Simon, Orthogonal Polynomials on the Unit Circle. Part 1. Classical Theory, in: American Mathematical Society Colloquium Publications, vol. 54, American Mathematical Society, Providence, RI, 2004, Part 1.

[26] B. Simon, Orthogonal Polynomials on the Unit Circle. Part 2. Spectral Theory, in: American Mathematical Society Colloquium Publications, vol. 54, American Mathematical Society, Providence, RI, 2004, Part 2.

[27] B. Simon, OPUC on one foot, Bull. Amer. Math. Soc. 42 (2005) 431–460.

[28] A. Sri Ranga, Szegő polynomials from hypergeometric functions, Proc. Amer. Math. Soc. 138 (2010) 4259–4270.

[29] G. Szegö, Orthogonal Polynomials, fourth ed., in: Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI, 1975.

[30] S. Tsujimoto, A. Zhedanov, Elliptic hypergeometric Laurent biorthogonal polynomials with a dense point spectrum on the unit circle, SIGMA Symmetry Integrability Geom. Methods Appl. 5 (2009) 30p.

[31] W. Van Assche, Orthogonal polynomials in the complex plane and on the real line, in: M.E.H. Ismail, et al. (Eds.), Field Institute Communications 14: Special Functions, q-Series and Related Topics, Amer. Math. Soc., 1997, pp. 211–245.

[32] H.S. Wall, Analytic Theory of Continued Fractions, D. van Nostrand, 1948.

[33] H.S. Wall, M. Wetzel, Quadratic forms and convergence regions for continued fractions, Duke Math. J. 11 (1944) 89–102.