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Zeros of Orthogonal Polynomials Generated by the Geronimus Perturbation of Measures

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Abstract. This paper deals with monic orthogonal polynomial sequences (MOPS in short) generated by a Geronimus canonical spectral transformation of a positive Borel measure μ, i.e., $(x-c)^{-1}d\mu(x)+N\delta(x-c)$, for some free parameter $N \in \mathbb{R}_{+}$ and shift c. We analyze the behavior of the corresponding MOPS. In particular, we obtain such a behavior when the mass N tends to infinity as well as we characterize the precise values of N such the smallest (respectively, the largest) zero of these MOPS is located outside the support of the original measure μ. When μ is semi-classical, we obtain the ladder operators and the second order linear differential equation satisfied by the Geronimus perturbed MOPS, and we also give an electrostatic interpretation of the zero distribution in terms of a logarithmic potential interaction under the action of an external field. We analyze such an equilibrium problem when the mass point of the perturbation c is located outside the support of μ.

Keywords: Orthogonal polynomials, Canonical spectral transformations of measures, Zeros, Monotonicity, Laguerre polynomials, Asymptotic behavior, Electrostatic interpretation, Logarithmic potential.

1 Introduction

1.1 Geronimus Perturbation of a Measure

In the last years some attention has been paid to the so called canonical spectral transformations of measures. Some authors have analyzed them from the point of view of Stieltjes functions associated with such a kind of perturbations (see [23]) or from the relation between the corresponding Jacobi matrices (see [24]). The present contribution is focused on the behavior of zeros of monic orthogonal polynomial sequences (MOPS in the sequel) associated with a particular transformation of measures called the *Geronimus canonical transformation on the real line*. Let μ be an absolutely continuous measure with respect to the Lebesgue measure supported on a finite or infinite interval $E = \mathrm{supp}(\mu)$, such that $C_0(E) = [a, b] \subseteq \mathbb{R}$. The basic Geronimus perturbation of μ is defined as

$$d\nu_N(x) = \frac{1}{(x-c)} d\mu(x) + N\delta(x-c), \tag{1}$$

with $N \in \mathbb{R}_+$, $\delta(x-c)$ the Dirac delta function in $x = c$, and the shift of the perturbation verifies $c \notin E$. Observe that it is given simultaneously by a rational modification of μ by a positive linear polynomial whose real zero c is the point of transformation (also known as the *shift* of the transformation) and the addition of a Dirac mass at the point of transformation as well.

This transformation was introduced by Geronimus in the seminal papers [11,12] devoted to provide a procedure of constructing new families of orthogonal polynomials from other orthogonal families, and also was studied by Shohat (see [18]) concerning about mechanical quadratures. The problem was revisited by Maroni in [16], into a more general algebraic frame, who gives an expression for the MOPS associated with (1) in terms of the so called co-recursive polynomials of the classical orthogonal polynomials. In the past decade, Bueno and Marcellán reinterpreted this perturbation in the framework of the so called discrete Darboux transformations, LU and UL factorizations of shifted Jacobi matrices [5]. This interpretation as Darboux transformations, together with other canonical transformations (Christoffel and Uvarov), provide a link between orthogonal polynomials and discrete integrable systems (see [1,19,20]). More recently, in [4] the authors present a new computational algorithm for computing the Geronimus transformation with large shifts, and [7] concerns about a new revision of the Geronimus transformation in terms of symmetric bilinear forms in order to include certain Sobolev and Sobolev–type orthogonal polynomials into the scheme of Darboux transformations.

The purpose of this contribution is twofold. First, using a similar approach as was done in [13], we provide a new connection formula for the Geronimus perturbed MOPS, which will be crucial to obtain sharp limits (and the speed of convergence to them) of their zeros. We provide a comprehensive study of the zeros in terms of the free parameter of the perturbation N, which somehow determines how important the perturbation on the classical measure μ is. Second, from the aforementioned new connection formula we recover (from an alternative point of view) a connection formula already known in the literature (see [16]) in terms of two consecutive polynomials of the original measure μ. We also obtain explicit expressions for the ladder operators and the second order differential equation satisfied by the Geronimus perturbed MOPS. When the measure μ is semi-classical, we also obtain the corresponding electrostatic model for the zeros of the Geronimus perturbed MOPS, showing that they are the electrostatic equilibrium points of positive unit charges interacting according to a logarithmic potential under the action of an external field (see, for example, Szegő's book [21, Sect. 6.7], Ismail's book [15, Chap. 3] and the references therein).

The structure of the paper is as follows. The rest of this Section is devoted to introduce without proofs some relevant material about modified inner products and their corresponding MOPS. In Section 2 we provide our main results. We obtain a new connection formula for orthogonal polynomials generated by a basic Geronimus transformation of a positive Borel measure μ, sharp bounds and speed of convergence to them for their real zeros, and the ladder operators and the second linear differential equation that they satisfy. The results about the zeros follows from a lemma concerning the behavior of the zeros of a linear combination of two polynomials. In Section 3, we proof all the result provided in the former Section. Finally, in Section 4, we explore these results for the Geronimus perturbed Laguerre MOPS. For μ being semi-classical, we obtain the corresponding electrostatic model for the zeros of the Geronimus perturbed MOPS as equilibrium points in a logarithmic potential interaction of positive unit charges under the presence of an external field. We analyze such an equilibrium problem when the mass point is located outside the support of μ, and we provide explicit formulas for the Laguerre weight case.

1.2 Modified Inner Products and Notation

Let μ be a positive Borel measure μ, with existing moments of all orders, and supported on a subset $E \subseteq \mathbb{R}$ with infinitely many points. Given such a measure, we define the standard inner product $\langle \cdot, \cdot \rangle_\mu : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ by

$$\langle f, g \rangle_\mu = \int_E f(x)g(x)d\mu(x), \quad f, g \in \mathbb{P}, \tag{2}$$

where $\mathbb{P}$ is the linear space of the polynomials with real coefficients, and the corresponding norm $|| \cdot ||_\mu : \mathbb{P} \to [0, +\infty)$ is given by $||f||_\mu = \sqrt{\int_E |f(x)|^2 d\mu(x)}$. Let $\{P_n\}_{n \geq 0}$ be the MOPS associated with μ. It is well known that the former MOPS satisfy the recurrence formula

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad P_{-1} = 0, \, P_1 = 1. \tag{3}$$

According to the Christoffel-Darboux formula, for the n-th kernel polynomial corresponding to $\{P_n\}_{n\geq 0}$ we have, for every $n \in \mathbb{N}$

$$K_n(x, y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{||P_k||_\mu^2} = \frac{P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)}{(x - y)} \frac{1}{||P_n||_\mu^2}. \tag{4}$$

Here and subsequently, $\{P_n^{c,[k]}\}_{n\geq 0}$ denotes the MOPS with respect to

$$\langle f, g \rangle_{\mu,[k]} = \int_E f(x)g(x)(x - c)^k d\mu(x), \tag{5}$$

where $c \notin E = \operatorname{supp}(\mu)$. The polynomials $\{P_n^{c,[k]}\}_{n\geq 0}$ are orthogonal with respect to a polynomial modification of the measure μ called the *k-iterated Christoffel perturbation*. If $k = 1$ we have the *Christoffel canonical transformation of the measure* μ (see [23,24]). It is well known that $P_n^{c,[1]}(x)$ is the monic kernel polynomial which can be represented as (see [6, (7.3)])

$$P_n^{c,[1]}(x) = \frac{1}{(x-c)}\left(P_{n+1}(x) - \pi_n\, P_n(x)\right) = \frac{\|P_n\|_\mu^2}{P_n(c)} K_n(x,c), \tag{6}$$

$$\pi_n = \pi_n(c) = \frac{P_{n+1}(c)}{P_n(c)}, \quad P_n(c) \neq 0. \tag{7}$$

Next, let us consider the basic Geronimus perturbation of μ given in (1). Let $\{Q_n^c\}_{n\geq 0}$ be the MOPS associated with $\nu_N(x)$ when $N=0$. That is, they are orthogonal with respect to the measure

$$d\nu_{N=0}(x) = d\nu(x) = \frac{1}{(x-c)} d\mu(x). \tag{8}$$

This constitutes a linear rational modification of μ, and the corresponding MOPS $\{Q_n^c\}_{n\geq 0}$ with respect to

$$\langle f, g\rangle_\nu = \int_E f(x)g(x)d\nu(x) = \int_E f(x)g(x)\frac{1}{(x-c)}d\mu(x) \tag{9}$$

has been extensively studied in the literature (see, among others, [3], [10, §2.4.2], [15, §2.7], [22,23]). It is known that, for $n \geq 0$, $Q_n^c(x)$ can be expressed as

$$Q_n^c(x) = P_n(x) - r_{n-1}\, P_{n-1}(x), \quad Q_0^c = 1, \tag{10}$$

$$r_{n-1} = r_{n-1}(c) = \frac{F_n(c)}{F_{n-1}(c)}, \quad F_{-1}(c) = 1,\, c \notin E. \tag{11}$$

The functions $F_n(s) = \int_E \frac{P_n(x)}{x-s} d\mu(x), \quad s \in \mathbb{C} \setminus E$, are the Cauchy integrals of $\{P_n\}_{n\geq 0}$, or functions of the second kind associated with $\{P_n\}_{n\geq 0}$. For a proper way to compute the above Cauchy integrals, we refer the reader to [10, §2.3]. From the above, it is clear that

$$K_n^c(x,y) = \sum_{k=0}^{n} \frac{Q_k^c(x)Q_k^c(y)}{||Q_k^c||_\nu^2} = \frac{Q_{n+1}^c(x)Q_n^c(y) - Q_{n+1}^c(y)Q_n^c(x)}{(x-y)} \frac{1}{||Q_n^c||_\nu^2} \tag{12}$$

are the kernel polynomials corresponding to $\{Q_n^c\}_{n\geq 0}$, which also satisfies the corresponding reproducing property of polynomial kernels with respect to ν

$$\int_E f(x)\, K_n^c(x,c)\, d\nu(x) = f(c), \tag{13}$$

for any polynomial $f \in \mathbb{P}$ with $\deg f \leq n$. The so called confluent form of (12) is given by the positive quantity (see [6])

$$K_n^c(c,c) = \frac{[Q_{n+1}^c]'(c)Q_n^c(c) - [Q_n^c]'(c)Q_{n+1}^c(c)}{||Q_n^c||_\nu^2} = \sum_{k=0}^{n} \frac{[Q_k^c(c)]^2}{\|Q_k^c\|_\nu^2} > 0. \tag{14}$$

The key concept to find several of our results is that the polynomials $\{P_n\}_{n\geq 0}$ are the monic kernel polynomials of parameter c of the sequence $\{Q_n^c\}_{n\geq 0}$. According to this argument, the following expressions hold

$$P_n(x) = \frac{\|Q_n^c\|_\nu^2}{Q_n^c(c)} K_n^c(x,c) = \frac{1}{(x-c)} \left(Q_{n+1}^c(x) - \frac{Q_{n+1}^c(c)}{Q_n^c(c)} Q_n^c(x) \right). \tag{15}$$

Finally, let $\{Q_n^{c,N}\}_{n\geq 0}$ be the MOPS associated to $d\nu_N$ when $N > 0$. That is, $\{Q_n^{c,N}\}_{n\geq 0}$ are the Geronimus perturbed polynomials orthogonal with respect to the the inner product

$$\langle f, g\rangle_{\nu_N} = \int_E f(x)g(x)\frac{1}{(x-c)}d\mu(x) + Nf(c)g(c). \tag{16}$$

Note that this is a standard inner product in the sense that, for every $f, g \in \mathbb{P}$, we have $\langle xf, g\rangle_{\nu_N} = \langle f, xg\rangle_{\nu_N}$. From (9), (16), an easy computation shows that

$$\langle f, g\rangle_{\nu_N} = \langle f, g\rangle_\nu + Nf(c)g(c). \tag{17}$$

Is the aim of this contribution to analyze the asymptotic behavior of the zeros of $Q_n^{c,N}$ and provide as well an electrostatic model for these zeros when the original measure μ is semi-classical. To this end, we will use the remarkable fact that, for any f, $g \in \mathbb{P}$ the multiplication operator by $(x-c)$ is symmetric with respect to (9). That is, $\langle (x-c)f, g\rangle_\nu = \langle f, (x-c)g\rangle_\nu = \langle f, g\rangle_\mu$, which is a straightforward consequence of the inner products (2), (5), (9) and (16).

2 Statement of the Main Results

2.1 Connection Formulas

Next, we provide a new connection formula for $Q_n^{c,N}(x)$ in terms of $Q_n^c(x)$ and the monic Kernel polynomials $P_n^{c,[1]}(x)$. This representation will allow us to obtain the results about monotonicity, asymptotics, and speed of convergence (presented below in this Section) for the zeros of $Q_n^{c,N}(x)$ in terms of the parameter N present in the perturbation (1).

Theorem 1. *The Geronimus perturbed orthogonal polynomials of the sequence $\{\tilde{Q}_n^{c,N}\}_{n\geq 0}$, with $\tilde{Q}_n^{c,N}(x) = \kappa_n Q_n^{c,N}(x)$, can be represented as*

$$\tilde{Q}_n^{c,N}(x) = Q_n^c(x) + NB_n^c(x-c)P_{n-1}^{c,[1]}(x), \tag{18}$$

with $\kappa_n = 1 + NB_n^c$ and

$$B_n^c = \frac{-Q_n^c(c)P_{n-1}(c)}{\|P_{n-1}\|_\mu^2} = K_{n-1}^c(c,c) > 0. \tag{19}$$

Observe that one can even give another alternative expression for B_n^c, which only involves polynomials and functions of the second kind relative to the original measure μ, evaluated at the point of transformation c. Combining (10) with (19), we deduce that

$$B_n^c = K_{n-1}^c(c,c) = \frac{r_{n-1}\,P_{n-1}^2 - P_n(c)P_{n-1}(c)}{\|P_{n-1}\|_\mu^2}. \tag{20}$$

As a direct consequence of the above theorem, we can express $Q_n^{c,N}(x)$ in terms of only two consecutive elements of the initial MOPS $\{P_n\}_{n\geq 0}$. This expression of $Q_n^{c,N}$ was already studied in the literature (see [16, formula (1.4)] and [7, Sec. 1]). In fact, the original aim of Geronimus in its pioneer works on the subject was to find necessary and sufficient conditions for the existence of a sequence of coefficients Λ_n, such that the linear combination of polynomials $P_n(x) + \Lambda_n P_{n-1}(x)$, $\Lambda_n \neq 0$, $n \geq 0$ were, in turn, orthogonal with respect to some measure supported on $\mathbb{R}$. Here we rewrite the value of Λ_n in several new equivalent ways. Substituting (10) and (6) into (18) yields

$$\tilde{Q}_n^{c,N}(x) = \kappa_n Q_n^{c,N}(x) = P_n(x) - r_{n-1}P_{n-1}(x) + NB_n^c\left(P_n(x) - \pi_{n-1}P_{n-1}(x)\right).$$

Thus, having in mind that $\kappa_n = 1 + NB_n^c$, after some trivial computations we can state the following result.

Proposition 1. *The monic Geronimus perturbed orthogonal polynomials of the sequence* $\{Q_n^{c,N}\}_{n\geq 0}$ *can be represented as*

$$Q_n^{c,N}(x) = P_n(x) + \Lambda_n^c\,P_{n-1}(x), \tag{21}$$

$$\Lambda_n^c = \Lambda_n^c(N) = \frac{\pi_{n-1} - r_{n-1}}{1 + NB_n^c} - \pi_{n-1}, \tag{22}$$

with π_{n-1}, r_{n-1} *given in (7) and (11) respectively.*

Remark 1. The coefficient $\Lambda_n^c(N)$ can also be expressed only in terms of quantities relative to the original non-perturbed measure μ, the point of transformation c and the mass N. Thus, from (20) and (22), we obtain

$$\Lambda_n^c(N) = \left(\frac{1}{\pi_{n-1} - r_{n-1}} - N\frac{P_{n-1}^2(c)}{\|P_{n-1}\|_\mu^2}\right)^{-1} - \pi_{n-1}.$$

Also, observe that for $N = 0$, the coefficient $\Lambda_n^c(0)$ reduces to r_{n-1}, and we recover the connection formula (10).

2.2 Asymptotic Behavior and Sharp Limits of the Zeros

Let $x_{n,s}$, $x_{n,s}^{c,[k]}$, $y_{n,s}^c$, and $y_{n,s}^{c,N}$, $s = 1,\ldots,n$ be the zeros of $P_n(x)$, $P_n^{c,[k]}(x)$, $Q_n^c(x)$, and $Q_n^{c,N}(x)$, respectively, all arranged in an increasing order, and assume that $C_0(E) = [a,b]$. Next, we analyze the behavior of zeros $y_{n,s}^{c,N}$ as a function of the mass N in (1). We obtain such a behavior when N tends from zero to infinity as well as we characterize the exact values of N such the smallest (respectively, the largest) zero of $\{Q_n^{c,N}\}_{n\geq 0}$ is located outside of $E=\operatorname{supp}(\mu)$. In order to do that, we use a technique developed in [2, Lemma 1] and [8, Lemmas 1 and 2] concerning the behavior and the asymptotics of the zeros of linear combinations of two polynomials $h,g\in\mathbb{P}$ with interlacing zeros, such that $f(x)=h_n(x)+cg_n(x)$. From now on, we will refer to this technique as the *Interlacing Lemma*.

Taking into account that the positive constant B_n^c does not depend on N, we can use the connection formula (18) to obtain results about monotonicity, asymptotics, and speed of convergence for the zeros of $Q_n^{c,N}(x)$ in terms of the mass N. Indeed, let assume that $y_{n,k}^{c,N}$, $k=1,2,...,n$, are the zeros of $Q_n^{c,N}(x)$. Thus, from (18), the positivity of B_n^c, and Theorem 2, we are in the hypothesis of the Interlacing Lemma, and we immediately conclude the following results.

Theorem 2. *If $C_0(E)=[a,b]$ and $c<a$, then*

$$c<y_{n,1}^{c,N}<y_{n,1}^{c}<x_{n-1,1}^{c,[1]}<y_{n,2}^{c,N}<y_{n,2}^{c}<\cdots<x_{n-1,n-1}^{c,[1]}<y_{n,n}^{c,N}<y_{n,n}^{c}.$$

Moreover, each $y_{n,k}^{c,N}$ is a decreasing function of N and, for each $k=1,\ldots,n-1$,

$$\lim_{N\to\infty} y_{n,1}^{c,N}=c,\qquad \lim_{N\to\infty} y_{n,k+1}^{c,N}=x_{n-1,k}^{c,[1]},$$

as well as

$$\lim_{N\to\infty} N[y_{n,1}^{c,N}-c]=\frac{-Q_n^c(c)}{B_n^c P_{n-1}^{c,[1]}(c)},$$
$$\lim_{N\to\infty} N[y_{n,k+1}^{c,N}-x_{n-1,k}^{c,[1]}]=\frac{-Q_n^c(x_{n-1,k}^{c,[1]})}{B_n^c(x_{n-1,k}^{c,[1]}-c)[P_{n-1}^{c,[1]}]'(x_{n-1,k}^{c,[1]})}.$$

Theorem 3. *If $C_0(E)=[a,b]$ and $c>b$, then*

$$y_{n,1}^{c}<y_{n,1}^{c,N}<x_{n-1,1}^{c,[1]}<\cdots<y_{n,n-1}^{c}<y_{n,n-1}^{c,N}<x_{n-1,n-1}^{c,[1]}<y_{n,n}^{c}<y_{n,n}^{c,N}<c.$$

Moreover, each $y_{n,k}^{c,N}$ is an increasing function of N and, for each $k=1,\ldots,n-1$,

$$\lim_{N\to\infty} y_{n,n}^{c,N}=c,\qquad \lim_{N\to\infty} y_{n,k}^{c,N}=x_{n-1,k}^{c,[1]},$$

and

$$\lim_{N\to\infty} N[c-y_{n,n}^{c,N}]=\frac{Q_n^c(c)}{B_n^c P_{n-1}^{c,[1]}(c)},$$
$$\lim_{N\to\infty} N[x_{n-1,k}^{c,[1]}-y_{n,k}^{c,N}]=\frac{Q_n^c(x_{n-1,k}^{c,[1]})}{B_n^c(x_{n-1,k}^{c,[1]}-c)[P_{n-1}^{c,[1]}]'(x_{n-1,k}^{c,[1]})}.$$

Notice that the mass point c attracts one zero of $Q_n^{c,N}(x)$, i.e. when $N\to\infty$, it captures either the smallest or the largest zero, according to the location of the point c with respect to the support of the measure μ. When either $c<a$ or $c>b$, at most one of the zeros of $Q_n^{c,N}(x)$ is located outside of $[a,b]$. Next, give explicitly the value N_0 of the mass N, such that for $N>N_0$ one of the zeros is located outside $[a,b]$.

Corollary 1 (minimum mass). *If $C_0(E)=[a,b]$ and $c\notin[a,b]$, the following expressions hold. If $c<a$ (or $c>b$) then, the smallest zero $y_{n,1}^{c,N}$ (respectively, the greatest zero $y_{n,n}^{c,N}$) satisfies*

$$\begin{array}{ll} y_{n,1}^{c,N}>a \text{ (respectively } y_{n,n}^{c,N}<b), & \text{for} \quad N<N_0,\\ y_{n,1}^{c,N}=a \text{ (respectively } y_{n,n}^{c,N}=b), & \text{for} \quad N=N_0,\\ y_{n,1}^{c,N}<a \text{ (respectively } y_{n,n}^{c,N}>b), & \text{for} \quad N>N_0, \end{array}$$

where

$$N_0=N_0(n,c,a)=\frac{-Q_n^c(a)}{K_{n-1}^c(c,c)(a-c)P_{n-1}^{c,[1]}(a)}>0,$$

respectively

$$N_0=N_0(n,c,b)=\frac{-Q_n^c(b)}{K_{n-1}^c(c,c)(b-c)P_{n-1}^{c,[1]}(b)}>0.$$

Proof. (a) In order to deduce the location of $y_{n,1}^{c,N}$ with respect to the point $x=a$, it is enough to observe that $Q_n^{c,N}(a)=0$ if and only if $N=N_0$.

(b) Also, in order to find the location of $y_{n,n}^{c,N}$ with respect to the point $x=b$, notice that $Q_n^{c,N}(b)=0$ if and only if $N=N_0$.

2.3 Ladder Operators and 2nd Order Linear Differential Equation

Our next result concerns the *ladder (creation and annihilation) operators*, and the *second order linear differential equation* (also known as the *holonomic equation*) corresponding to $\{Q_n^{c,N}\}_{n\geq 0}$. We restrict ourselves to the case in which μ is a classical or semi-classical measure, and therefore satisfying a structure relation (see [9] and [17]) as

$$\sigma(x)[P_n(x)]'=a(x;n)P_n(x)+b(x;n)P_{n-1}(x) \tag{23}$$

where $a(x;n)$ and $b(x;n)$ are polynomials in the variable x, whose fixed degree do not depend on n. In order to obtain these results, we will follow a different approach as in [15, Ch. 3]. Our technique is based on the connection formula (21) given in Proposition 1, the three term recurrence relation (3) satisfied by $\{P_n\}_{n\geq 0}$, and the structure relation (23). The results are presented here and will be proved in Section 3.

Theorem 4. *Let $\mathfrak{a}_n=-\xi_1^c(x;n)\mathrm{I}+\mathrm{D}_x$ and $\mathfrak{a}_n^{\dagger}=-\eta_2^c(x;n)\mathrm{I}+\mathrm{D}_x$ be differential operators, where I, D_x are the identity and x-derivative operator respectively, satisfying*

$$\mathfrak{a}_n[Q_n^{c,N}(x)]=\eta_1^c(x;n)\,Q_{n-1}^{c,N}(x), \tag{24}$$

$$\mathfrak{a}_n^{\dagger}[Q_{n-1}^{c,N}(x)]=\xi_2^c(x;n)\,Q_n^{c,N}(x), \tag{25}$$

where, for $k=1,2$

$$\xi_k^c(x;n)=\frac{C_k(x;n)B_2(x;n)\,\gamma_{n-1}+D_k(x;n)\Lambda_{n-1}^c}{\Delta(x;n)\,\gamma_{n-1}},\ \eta_k^c(x;n)=\frac{D_k(x;n)-C_k(x;n)\Lambda_n^c}{\Delta(x;n)}.$$

In turn, all the above expressions are given only in terms of the coefficients in (3), (23), and (21) as follows

$$B_2(x;n) = \Lambda^c_{n-1}\left(\tfrac{1}{\Lambda^c_{n-1}} + \tfrac{(x-\beta_{n-1})}{\gamma_{n-1}}\right), \ C_1(x;n) = \tfrac{1}{\sigma(x)}\left(a(x;n) - \Lambda^c_n \tfrac{b(x;n)}{\gamma_{n-1}}\right),$$

$$D_1(x;n) = \tfrac{1}{\sigma(x)}\left(b(x;n) + \Lambda^c_n\, b(x;n-1)\left(\tfrac{a(x;n-1)}{b(x;n-1)} + \tfrac{(x-\beta_{n-1})}{\gamma_{n-1}}\right)\right),$$

$$C_2(x;n) = \tfrac{-\Lambda^c_{n-1}}{\sigma(x)}\left(\tfrac{a(x;n)}{\gamma_{n-1}} + \tfrac{b(x;n-1)}{\gamma_{n-1}}\left(\tfrac{1}{\Lambda^c_{n-1}} + \tfrac{(x-\beta_{n-1})}{\gamma_{n-1}}\right)\right),$$

$$D_2(x;n) = \tfrac{\Lambda^c_{n-1}}{\sigma(x)} \quad \tfrac{\sigma(x)-b(x;n)+a(x;n-1)\gamma_{n-1}+b(x;n-1)(x-\beta_{n-1})}{\gamma_{n-1}}\left(\tfrac{1}{\Lambda^c_{n-1}} + \tfrac{(x-\beta_{n-1})}{\gamma_{n-1}}\right)\Big],$$

$$\Delta(x;n) = B_2(x;n) + \tfrac{\Lambda^c_n\,\Lambda^c_{n-1}}{\gamma_{n-1}}, \quad \deg \Delta(x;n) = 1.$$

Thus, $\mathfrak{a}_n$ *and* $\mathfrak{a}^\dagger_n$ *are respectively lowering and raising operators associated to the Geronimus perturbed MOPS* $\{Q^{c,N}_n\}_{n\geq 0}$.

For a deeper discussion of raising and lowering operators we refer the reader to [15, Ch. 3]. We next provide the second order linear differential equation satisfied by the MOPS $\{Q^{c,N}_n\}_{n\geq 0}$ when the measure μ is semi-classical (for definition of a semi-classical measure see [17]). This is the main tool for the further electrostatic interpretation of zeros.

Theorem 5. *The Geronimus perturbed MOPS* $\{Q^{c,N}_n\}_{n\geq 0}$ *satisfies the holonomic equation (second order linear differential equation)*

$$[Q^{c,N}_n(x)]'' + \mathcal{R}(x;n)[Q^{c,N}_n(x)]' + \mathcal{S}(x;n)Q^{c,N}_n(x) = 0,$$

where

$$\mathcal{R}(x;n) = -\left(\xi^c_1(x;n) + \eta^c_2(x;n) + \tfrac{[\eta^c_1(x;n)]'}{\eta^c_1(x;n)}\right),$$

$$\mathcal{S}(x;n) = \xi^c_1(x;n)\eta^c_2(x;n) - \eta^c_1(x;n)\xi^c_2(x;n) + \tfrac{\xi^c_1(x;n)[\eta^c_1(x;n)]' - [\xi^c_1(x;n)]'\eta^c_1(x;n)}{\eta^c_1(x;n)}.$$

3 Proofs of the Main Results

3.1 Proof of Theorem 1 and the Positivity of B^c_n

First, we need to prove the following lemma concerning a first way to represent the Geronimus perturbed polynomials $Q^{c,N}_n(x)$, using the kernels (12).

Lemma 1. *Let* $\{Q^{c,N}_n\}_{n\geq 0}$ *and* $\{Q^c_n\}_{n\geq 0}$ *be the MOPS corresponding to the measures* ν_N *and* ν *respectively. Then, the following connection formula holds*

$$Q^{c,N}_n(x) = Q^c_n(x) - NQ^{c,N}_n(c)K^c_{n-1}(x,c), \tag{26}$$

$$Q^{c,N}_n(c) = \frac{Q^c_n(c)}{1+NK^c_{n-1}(c,c)} = \kappa_n^{-1}Q^c_n(c), \tag{27}$$

where $\kappa_n = 1 + NB^c_n$ *, and* $K^c_{n-1}(c,c)$ *is given in (14).*

Proof. From (17) we can express $Q_n^{c,N}(x)$ in terms of the polynomials $Q_n^c(x)$ with coefficients

$$b_{n,k} = \frac{Q_i^c(x), Q_n^{c,N}(x) \quad \nu}{\|Q_k^c\|_\nu^2}, \quad 0 \leq k \leq n-1,$$

and hence we have

$$Q_n^{c,N}(x) = Q_n^c(x) - NQ_n^{c,N}(c) \sum_{k=0}^{n-1} \frac{Q_k^c(x) Q_k^c(c)}{\|Q_k^c\|_\nu^2}.$$

Next, taking into account (4) for the sequence $\{Q_k^c\}_{n\geq 0}$, we get

$$Q_n^{c,N}(x) = Q_n^c(x) - NQ_n^{c,N}(c) K_{n-1}^c(x,c).$$

In order to find $Q_n^{c,N}(c)$, we evaluate (26) in $x = c$, which yields (27). This completes the proof.

Next, in order to prove the orthogonality of the polynomials defined by (18), we deal with the basis $\mathcal{B}^n = \{1, (x-c), (x-c)^2, \ldots, (x-c)^n\}$ of the space of polynomials of degree at most n. Thus

$$\begin{aligned}
\langle 1, \tilde{Q}_n^{c,N} \rangle_{\nu_N} &= \langle 1, Q_n^c \rangle_\nu + NB_n^c \langle 1, (x-c) P_{n-1}^{c,[1]} \rangle_\nu + NQ_n^c(c) = 0, \\
\langle (x-c), \tilde{Q}_n^{c,N} \rangle_{\nu_N} &= \langle (x-c), Q_n^c \rangle_\nu + NB_n^c \langle 1, P_{n-1}^{c,[1]} \rangle_{\mu,[1]} = 0, \\
&\ \vdots \\
\langle (x-c), \tilde{Q}_n^{c,N} \rangle_{\nu_N} &= \langle (x-c)^{n-1}, Q_n^c \rangle_\nu + NB_n^c \langle (x-c)^{n-2}, P_{n-1}^{c,[1]} \rangle_{\mu,[1]} = 0, \\
\langle (x-c), \tilde{Q}_n^{c,N} \rangle_{\nu_N} &= \|Q_n^c\|_\nu^2 + NB_n^c \|P_{n-1}^{c,[1]}\|_{\mu,[1]}^2 > 0.
\end{aligned}$$

In order to prove (19), from (6), (15), (13) we deduce

$$\begin{aligned}
\langle (x-c), P_{n-1}^{c,[1]}(x) \rangle_\nu &= \textstyle\int_E (x-c) \frac{1}{(x-c)} \left(P_{n+1}(x) - \frac{P_{n+1}(c)}{P_n(c)} P_n(x) \right) d\nu(x) \\
&= \tfrac{\|Q_n^c\|_\nu^2}{Q_n^c(c)} \textstyle\int_E K_n^c(x,c) d\nu(x) \\
&\quad - \tfrac{Q_{n-1}^c(c)}{\|Q_{n-1}^c\|_\nu^2} \tfrac{\|Q_n^c\|_\nu^2}{Q_n^c(c)} \tfrac{K_n^c(c,c)}{K_{n-1}^c(c,c)} \tfrac{\|Q_{n-1}^c\|_\nu^2}{Q_{n-1}^c(c)} \textstyle\int_E K_{n-1}^c(x,c) d\nu(x) \\
&= \tfrac{\|Q_n^c\|_\nu^2}{Q_n^c(c)} - \tfrac{Q_{n-1}^c(c)}{\|Q_{n-1}^c\|_\nu^2} \tfrac{\|Q_n^c\|_\nu^2}{Q_n^c(c)} \tfrac{K_n^c(c,c)}{K_{n-1}^c(c,c)} \tfrac{\|Q_{n-1}^c\|_\nu^2}{Q_{n-1}^c(c)} \\
&= \tfrac{-Q_n^c(c)}{K_{n-1}^c(c,c)}.
\end{aligned}$$

Hence, taking into account (14), we have

$$B_n^c = \frac{-\kappa_n Q_n^{c,N}(c)}{\langle 1, (x-c) P_{n-1}^{c,[1]}(x) \rangle_\nu} = \frac{-\kappa_n \frac{Q_n^c(c)}{1+NK_{n-1}^c(c,c)}}{\frac{-Q_n^c(c)}{K_{n-1}^c(c,c)}} = K_{n-1}^c(c,c) > 0.$$

3.2 Proofs of Theorems 2 and 3

To apply the Interlacing Lemma and get the results of Theorems 2 and 3, we need to show that we satisfy the hypotheses of the Interlacing Lemma. To do this, we first prove that the zeros of $Q_n^c(x)$ and $(x-c)P_{n-1}^{c,[1]}(x)$ interlace.

Lemma 2. *Let $y_{n,k}^c$ and $x_{n,k}^{c,[1]}$ be the zeros of $Q_n^c(x)$ and $P_n^{c,[1]}(x)$, respectively, all arranged in an increasing order. The inequalities*

$$y_{n+1,1}^c < x_{n,1}^{c,[1]} < y_{n+1,2}^c < x_{n,2}^{c,[1]} < \cdots < y_{n+1,n}^c < x_{n,n}^{c,[1]} < y_{n+1,n+1}^c$$

hold for every $n \in \mathbb{N}$.

Proof. Combining (15) with (6) yields

$$(x-c)^2 P_n^{c,[1]}(x) = Q_{n+2}^c(x) - d_n^c Q_{n+1}^c(x) + e_n^c Q_n^c(x), \tag{28}$$

where

$$\begin{aligned} e_n^c &= \tfrac{P_{n+1}(c)}{P_n(c)} \tfrac{Q_{n+1}^c(c)}{Q_n^c(c)} &= \tfrac{\|Q_{n+1}^c\|_\nu^2}{\|Q_n^c\|_\nu^2} \tfrac{K_{n+1}^c(c,c)}{K_n^c(c,c)} > 0, \\ d_n^c &= \tfrac{Q_{n+2}^c(c)}{Q_{n+1}^c(c)} + \tfrac{P_{n+1}(c)}{P_n(c)} &= \tfrac{Q_{n+2}^c(c)+Q_n^c(c)e_n^c}{Q_{n+1}^c(c)}. \end{aligned}$$

On the other hand, the sequence $\{Q_n^c\}_{n\geq 0}$ satisfies the three term recurrence relation

$$Q_n^c(x) = (x-\beta_n^c)Q_{n-1}^c(x) - \gamma_n^c Q_{n-2}^c(x), \quad n = 1, 2, \ldots \tag{29}$$

The coefficients β_n^c, and γ_n^c are given in several works. A particularly clear discussion about how to obtain β_n^c, γ_n^c from those β_n, γ_n of the initial μ is given in [10, §2.4.4]. From (11), for $n \geq 1$, the modified coefficients are given by $\beta_n^c = \beta_n + r_n - r_{n-1}$, and $\gamma_n^c = \gamma_{n-1}\frac{r_{n-1}}{r_{n-2}}$, with the initial conditions for $n = 0$,

$$\beta_0^c = \beta_0 + r_0, \quad \gamma_0^c = \int_E d\nu(x) = \int_E \frac{1}{x-c} d\mu(x) = -F_0(c).$$

Combining (28) with (29) yields

$$(x-c)^2 P_n^{c,[1]}(x) = \left(x - \beta_{n+2}^c - d_n^c\right) Q_{n+1}^c(x) + \left(e_n^c - \gamma_{n+2}^c\right) Q_n^c(x). \tag{30}$$

Being μ a positive definite measure, the modified measure ν is also positive definite, because $c \notin E =$ *supp*(μ) and therefore $(x-c)^{-1}$ do not change sign in E. Hence, by [6, Th. 4.2(a)], the coefficient of $Q_n^c(x)$ in (30) can be expressed by

$$e_n^c - \gamma_{n+2}^c = \tfrac{||Q_{n+1}^c||_\nu^2}{||Q_n^c||_\nu^2} \left(\tfrac{K_{n+1}^c(c,c)}{K_n^c(c,c)} - 1 \right) = \tfrac{1}{||Q_n^c||_\nu^2} \tfrac{[Q_{n+1}^c(c)]^2}{K_n^c(c,c)} > 0, \tag{31}$$

which is positive for every $n \geq 0$, no matter the position of c with respect to the interval E. Finally, evaluating $P_n^{c,[1]}(x)$ at the zeros $y_{n+1,k}$, from (30) and (31), we get $(x-c)^2 P_n^{c,[1]}(y_{n+1,k}) = \left(e_n^c - \gamma_{n+2}^c\right) Q_n^c(y_{n+1,k})$, for every $k = 1, \ldots, n+1$, so it is clear that

$$sign\,(P_n^{c,[1]}(y_{n+1,k})) = sign\,(Q_n^c(y_{n+1,k})), \quad k = 1, \ldots, n+1. \tag{32}$$

Thus, from (32) and the very well known fact that the zeros of $Q_{n+1}^c(x)$ interlace with the zeros of $Q_n^c(x)$, we conclude that $P_n^{c,[1]}(x)$ has at least one zero in every interval $(y_{n+1,k}, y_{n+1,k+1})$ for every $k = 1, \ldots n$. This completes the proof.

3.3 Proofs of Theorems 4 and 5

Next, we prove Theorem 4. Shifting the index in (23) as $n \to n-1$, and using (3) we obtain

$$[P_{n-1}(x)]' = \tfrac{-b(x;n-1)}{\sigma(x)\,\gamma_{n-1}} P_n(x) + \quad \tfrac{a(x;n-1)}{\sigma(x)} + \tfrac{b(x;n-1)(x-\beta_{n-1})}{\sigma(x)\,\gamma_{n-1}} \Big) P_{n-1}(x). \tag{33}$$

Next, taking x derivative in (21), we get $[Q_n^{c,N}(x)]' = [P_n(x)]' + \Lambda_n^c\,[P_{n-1}(x)]'$. Substituting (23) and (33) into the above expression, we obtain the relation

$$[Q_n^{c,N}(x)]' = C_1(x;n)P_n(x) + D_1(x;n)P_{n-1}(x), \tag{34}$$

with the explicit expressions for $C_1(x;n)$ and $D_1(x;n)$ in the statement of Theorem 4. Observe that the sequences of monic polynomials $\{Q_n^{c,N}\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ are also related by

$$Q_{n-1}^{c,N}(x) = A_2(n)P_n(x) + B_2(x;n)P_{n-1}(x), \tag{35}$$

$$[Q_{n-1}^{c,N}(x)]' = C_2(x;n)P_n(x) + D_2(x;n)P_{n-1}(x), \tag{36}$$

where $A_2(n)$, $B_2(x;n)$, $C_2(x;n)$, and $D_2(x;n)$ are given in the statement of Theorem 4. The above two expressions are a straightforward consequence of (21), (23), (34), and the three term recurrence relation (3) for the MOPS $\{P_n\}_{n\geq 0}$.

We next provide the converse relation of (35)–(36) for the polynomials $P_n(x)$ and $P_{n-1}(x)$. That is, we express these two consecutive polynomials of $\{P_n\}_{n\geq 0}$ in terms of only two consecutive Geronimus perturbed polynomials of the MOPS $\{Q_n^{c,N}\}_{n\geq 0}$, as follows

$$\begin{aligned} P_n(x) &= \tfrac{B_2(x;n)}{\Delta(x;n)} Q_n^{c,N}(x) - \tfrac{\Lambda_n^c}{\Delta(x;n)} Q_{n-1}^{c,N}(x), \\ P_{n-1}(x) &= \tfrac{\Lambda_{n-1}^c}{\Delta(x;n)\,\gamma_{n-1}} Q_n^{c,N}(x) + \tfrac{1}{\Delta(x;n)} Q_{n-1}^{c,N}(x). \end{aligned} \tag{37}$$

where

$$\Delta(x;n) = \tfrac{\Lambda_{n-1}^c}{\gamma_{n-1}} \quad x - \beta_{n-1} + \Lambda_n^c + \tfrac{\gamma_{n-1}}{\Lambda_{n-1}^c} \Big), \ \deg \Delta(x;n) = 1.$$

To obtain the above expressions, note that (35)–(36) can be interpreted as a system of two linear equations with two polynomial unknowns, namely $P_n(x)$ and $P_{n-1}(x)$, hence from Cramer's rule (37) follows. Finally, replacing (37) in (34) and (36) one obtains the ladder equations

$$\begin{aligned} [Q_n^{c,N}(x)]' &= \tfrac{C_1(x;n)B_2(x;n)\,\gamma_{n-1} + D_1(x;n)\Lambda_{n-1}^c}{\Delta(x;n)\,\gamma_{n-1}} Q_n^{c,N}(x) + \tfrac{D_1(x;n) - C_1(x;n)\Lambda_n^c}{\Delta(x;n)} Q_{n-1}^{c,N}(x) \\ [Q_{n-1}^{c,N}(x)]' &= \tfrac{C_2(x;n)B_2(x;n)\,\gamma_{n-1} + D_2(x;n)\Lambda_{n-1}^c}{\Delta(x;n)\,\gamma_{n-1}} Q_n^{c,N}(x) + \tfrac{D_2(x;n) - C_2(x;n)\Lambda_n^c}{\Delta(x;n)} Q_{n-1}^{c,N}(x), \end{aligned}$$

which are fully equivalent to (24)–(25). This completes the proof of Theorem 4.

The proof of Theorem 5 comes directly from the ladder operators provided in Theorem 4. The usual technique (see, for example [15, Th. 3.2.3]) consists in applying the raising operator to both sides of the equation satisfied by the lowering operator, i.e.

$$\mathfrak{a}_n^\dagger \; \frac{1}{\eta_1^c(x;n)}\mathfrak{a}_n[Q_n^{c,N}(x)]\Big] = \mathfrak{a}_n^\dagger \; Q_{n-1}^{c,N}(x)\Big] = \xi_2^c(x;n)\, Q_n^{c,N}(x)$$

is a second order differential equation for $Q_n^{c,N}(x)$. After some doable computations, Theorem 5 easily follows.

Remark 2. Observe that the coefficients $A_2(n)$, $B_2(x;n)$, $C_1(x;n)$, $D_1(x;n)$, $C_2(x;n)$, $D_2(x;n)$, and $\Delta(x;n)$ can be given strictly in terms of the following known quantities: the coefficient Λ_n^c in (22), the coefficients β_{n-1}, γ_{n-1} of the three term recurrence relation (3) and $\sigma(x)$, $a(x;n)$, $b(x;n)$ of the structure relation (23) satisfied by $\{P_n\}_{n\geq 0}$.

4 Zero Behavior and Electrostatic Model for the Laguerre Case

Once we have the second order differential equation satisfied by the MOPS $\{Q_n^{c,N}\}_{n\geq 0}$ it is easy to obtain an electrostatic model for their zeros (see [13,14,15] among others). In this Section we shall derive the electrostatic model for the zeros in case μ is the Laguerre classical measure.

Let $\{L_n^\alpha\}_{n\geq 0}$ be the monic Laguerre polynomials orthogonal with respect to the Laguerre classical measure $d\mu_\alpha(x) = x^\alpha e^{-x}dx$, $\alpha > -1$, supported on $[0,+\infty)$. We will denote by $\{Q_n^{\alpha,c,N}\}_{n\geq 0}$ and $\{Q_n^{\alpha,c}\}_{n\geq 0}$ the MOPS corresponding to (1) and (8) when μ is the Laguerre classical measure, and $\{y_{n,s}^{\alpha,c,N}\}_{s=1}^n$, $\{y_{n,s}^{\alpha,c}\}_{s=1}^n$ their corresponding zeros.

The structure relation (23) for the monic classical Laguerre polynomials is

$$\sigma(x)[L_n^\alpha(x)]' = a(x;n)L_n^\alpha(x) + a(x;n)L_{n-1}^\alpha(x),$$

and therefore $\sigma(x) = x$, $a(x;n) = n$, and $b(x;n) = n(n+\alpha)$. Their three term recurrence relation is $xL_n^\alpha(x) = L_{n+1}^\alpha(x) + \beta_n L_n^\alpha(x) + \gamma_n L_{n-1}^\alpha(x)$, with $\beta_n = \beta_n^\alpha = 2n+\alpha+1$, $\gamma_n = \gamma_n^\alpha = n(n+\alpha)$, and the connection formula (21) for $Q_n^{\alpha,c,N}(x)$ in terms of $\{L_n^\alpha\}_{n\geq 0}$ reads $Q_n^{\alpha,c,N}(x) = L_n^\alpha(x) + \Lambda_n^{\alpha,c} L_{n-1}^\alpha(x)$. Taking into account exclusively the coefficients in the above three expressions, we obtain the explicit expressions for the ladder operators and the coefficients in the holonomic equation for this example. After some cumbersome computations, for $\Lambda_n^{\alpha,c} = \Lambda_n^{\alpha,c}(N)$ in (21) we have

$$C_1^\alpha(x;n) = \frac{n-\Lambda_n^{\alpha,c}}{x}, \qquad D_1^\alpha(x;n) = \frac{n(n+\alpha)+(x-(n+\alpha))\Lambda_n^{\alpha,c}}{x},$$

$$A_2^\alpha(n) = \frac{-\Lambda_n^{\alpha,c}}{(n-1)(n+\alpha-1)}, \quad B_2^\alpha(x;n) = 1+\Lambda_{n-1}^{\alpha,c}\frac{(x+1-2n+\alpha)}{(n-1)(n+\alpha-1)},$$

$$C_2^\alpha(x;n) = \frac{-1}{x} - \Lambda_{n-1}^{\alpha,c}\frac{x+1-(n+\alpha)}{x(n-1)(n-1+\alpha)},$$

$$D_2^\alpha(x;n) = \frac{x-(n+\alpha)}{x} + \Lambda_{n-1}^{\alpha,c}\frac{(x+1-2n+\alpha)(x-(n+\alpha))+(x-n(n+\alpha))}{x(n-1)(n-1+\alpha)}.$$

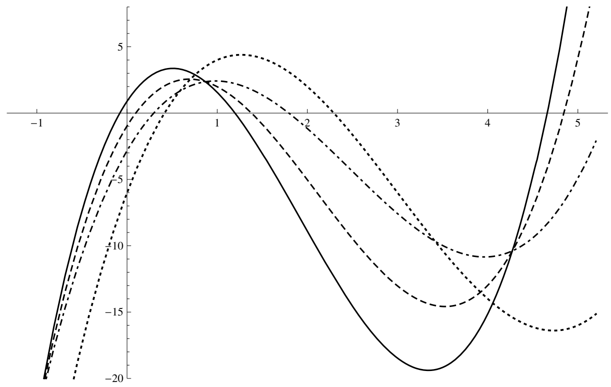

Fig. 1. The graphs of $L_3^0(x)$ (dotted) and $Q_3^{0,-1,N}(x)$ for some values of N

Hence, they satisfy the holonomic equation

$$[Q_n^{\alpha,c,N}(x)]'' + \mathcal{R}_L(x;n)[Q_n^{\alpha,c,N}(x)]' + \mathcal{S}_L(x;n)Q_n^{\alpha,c,N}(x) = 0, \tag{38}$$

with coefficients

$$\mathcal{R}_L(x;n) = \frac{-\Lambda_n^{\alpha,c}}{\Lambda_n^{\alpha,c}x + (n-\Lambda_n^{\alpha,c})(n+\alpha-\Lambda_n^{\alpha,c})} + \frac{\alpha+1}{x} - 1,$$

$$\mathcal{S}_L(x;n) = \frac{\Lambda_n^{\alpha,c}x + (n+\alpha)(n-\Lambda_n^{\alpha,c})}{x\left(\Lambda_n^{\alpha,c}x + (n-\Lambda_n^{\alpha,c})(n+\alpha-\Lambda_n^{\alpha,c})\right)} + \frac{n-1}{x}.$$

Now we evaluate (38) at the zeros $\{y_{n,s}^{\alpha,c,N}\}_{s=1}^n$, yielding

$$\frac{[Q_n^{\alpha,c,N}]''}{[Q_n^{\alpha,c,N}]'} = -\mathcal{R}_L(y_{n,s}^{\alpha,c,N};n) = \frac{\Lambda_n^{\alpha,c}}{\Lambda_n^{\alpha,c}y_{n,s}^{\alpha,c,N} + (n-\Lambda_n^{\alpha,c})(n+\alpha-\Lambda_n^{\alpha,c})} - \frac{\alpha+1}{y_{n,s}^{\alpha,c,N}} + 1.$$

The above reads as the electrostatic equilibrium condition for $\{y_{n,s}^{\alpha,c,N}\}_{s=1}^n$. Taking $u_L(n;x) = \Lambda_n^{\alpha,c}x + (n-\Lambda_n^{\alpha,c})(n+\alpha-\Lambda_n^{\alpha,c})$, it can be rewritten as

$$\sum_{j=1,\, j\neq k}^{n} \frac{1}{y_{n,j}^{\alpha,c,N} - y_{n,k}^{\alpha,c,N}} + \frac{1}{2}\frac{[u_L]'(n;y_{n,k}^{\alpha,c,N})}{u_L(n;y_{n,k}^{\alpha,c,N})} - \frac{1}{2}\frac{\alpha+1}{y_{n,s}^{\alpha,c,N}} + \frac{1}{2} = 0,$$

which means that the set of zeros $\{y_{n,s}^{\alpha,c,N}\}_{s=1}^n$ are the critical points of the gradient of the total energy. Hence, the electrostatic interpretation of the distribution of zeros means that we have an equilibrium position under the presence of an external potential

$$V_L^{ext}(x) = \frac{1}{2}\ln u_L(x;n) - \frac{1}{2}\ln x^{\alpha+1}e^{-x}, \tag{39}$$

where the first term represents a *short range potential* corresponding to a unit charge located at the unique real zero $z_L(n;x) = \frac{-1}{\Lambda_n^{\alpha,c}}(n-\Lambda_n^{\alpha,c})(n+\alpha-\Lambda_n^{\alpha,c})$

of the linear polynomial $u_L(x;n)$, and the second one is a *long range potential* associated with the Laguerre weight function.

To illustrate the results of Theorem 2, we consider the Geronimus perturbation (1) on the Laguerre measure with $\alpha = 0$ and $c = -1$, and we obtain the behavior of the zeros $\{y_{n,s}^{0,-1,N}\}_{s=1}^{n}$ as N increases. We enclose in Fig. 1 the graphs of $L_3^0(x)$ (dotted line), $Q_3^{0,-1}(x)$ (dash-dotted line), and $Q_3^{0,-1,N}(x)$ for some N, to show the monotonicity of their zeros as a function of the mass N.

Table 1. Zeros of $Q_3^{0,-1,N}(x)$ and $z(0,-1,3,N;x)$ for some values of N

N	*1st*	*2nd*	*3rd*	*z(N)*
0	0.296771	1.794881	5.327153	-1.27309
0.0125	0.096936	1.381317	4.846199	-0.039345
0.025	$\mathbf{-0.079531}$	1.196907	4.66079	-0.015274
0.05	$\mathbf{-0.324373}$	1.050055	4.50679	-0.156362
5	$\mathbf{-0.988481}$	0.87094	4.276644	-0.700057

Table 1 shows the behavior of the zeros of $Q_3^{0,-1,N}(x)$ for several choices of N. Observe that the smallest zero converges to $c = -1$ and the other two zeros converge to the zeros of the monic kernel polynomial $L_2^{\alpha,c,[1]}(x)$, in accordance with Theorem 2. That is, they converge to $x_{2,1}^{0,-1,[1]} = 0.869089$ and $x_{2,2}^{0,-1,[1]} = 4.273768$. Notice that all the zeros decrease as N increases. The zeros outside the interval $[0,+\infty)$, namely the support of the classical Laguerre measure, appear in bold.

References

1. Barrios, D., Branquinho, A.: Complex high order Toda and Volterra lattices. J. Difference Equ. Appl. 15(2), 197–213 (2009)
2. Bracciali, C.F., Dimitrov, D.K., Sri Ranga, A.: Chain sequences and symmetric generalized orthogonal polynomials. J. Comput. Appl. Math. 143, 95–106 (2002)
3. Branquinho, A., Marcellán, F.: Generating new classes of orthogonal polynomials. Int. J. Math. Math. Sci. 19(4), 643–656 (1996)
4. Bueno, M.I., Deaño, A., Tavernetti, E.: A new algorithm for computing the Geronimus transformation with large shifts. Numer. Alg. 54, 101–139 (2010)
5. Bueno, M.I., Marcellán, F.: Darboux tranformations and perturbation of linear functionals. Linear Algebra Appl. 384, 215–242 (2004)
6. Chihara, T.S.: An Introduction to Orthogonal Polynomials. Mathematics and its Applications Series. Gordon and Breach, New York (1978)
7. Derevyagin, M., Marcellán, F.: A note on the Geronimus transformation and Sobolev orthogonal polynomials. Numer. Algorithms, 1–17 (2013), doi: 10.1007/s11075-013-9788-6

8. Dimitrov, D.K., Mello, M.V., Rafaeli, F.R.: Monotonicity of zeros of Jacobi-Sobolev type orthogonal polynomials. Appl. Numer. Math. 60, 263–276 (2010)
9. Dini, J., Maroni, P.: La multiplication d'une forme linéaire par une forme rationnelle. Application aux polynômes de Laguerre-Hahn. Ann. Polon. Math. 52, 175–185 (1990)
10. Gautschi, W.: Orthogonal Polynomials: Computation and Approximation. Numerical Mathematics and Scientific Computation Series. Oxford University Press, New York (2004)
11. Geronimus, Y.L.: On the polynomials orthogonal with respect to a given number sequence and a theorem. In: Hahn, W., Nauk, I.A. (eds.), vol. 4, pp. 215–228 (1940) (in Russian)
12. Geronimus, Y.L.: On the polynomials orthogonal with respect to a given number sequence. Zap. Mat. Otdel. Khar'kov. Univers. i NII Mat. i Mehan. 17, 3–18 (1940) (in Russian)
13. Huertas, E.J., Marcellán, F., Rafaeli, F.R.: Zeros of orthogonal polynomials generated by canonical perturbations of measures. Appl. Math. Comput. 218(13), 7109–7127 (2012)
14. Ismail, M.E.H.: More on electrostatic models for zeros of orthogonal polynomials. Numer. Funct. Anal. Optimiz. 21, 191–204 (2000)
15. Ismail, M.E.H.: Classical and Quantum Orthogonal Polynomials in One Variable. Encyclopedia of Mathematics and its Applications, vol. 98. Cambridge University Press, Cambridge (2005)
16. Maroni, P.: Sur la suite de polynômes orthogonaux associée à la forme $u = \delta_c + \lambda(x-c)^{-1}L$. Period. Math. Hungar. 21(3), 223–248 (1990)
17. Maroni, P.: Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques. In: Brezinski, C., et al. (eds.) Orth. Pol. and Their Applications. Annals. Comput. Appl. Math., vol. 9, pp. 95–130 (1991)
18. Shohat, J.: On mechanical quadratures, in particular, with positive coefficients. Trans. Amer. Math. Soc. 42(3), 461–496 (1937)
19. Spiridonov, V., Zhedanov, A.: Discrete Darboux transformations, the discrete-time Toda lattice, and the Askey-Wilson polynomials. Methods Appl. Anal. 2(4), 369–398 (1995)
20. Spiridonov, V., Zhedanov, A.: Discrete-time Volterra chain and classical orthogonal polynomials. J. Phys. A: Math. Gen. 30, 8727–8737 (1997)
21. Szegő, G.: Orthogonal Polynomials, 4th edn., vol. 23. Amer. Math. Soc. Coll. Publ., Providence RI, (1975)
22. Uvarov, V.B.: The connection between systems of polynomials orthogonal with respect to different distribution functions. USSR Compt. Math. Phys. 9(6), 25–36 (1969)
23. Zhedanov, A.: Rational spectral transformations and orthogonal polynomials. J. Comput. Appl. Math. 85, 67–83 (1997)
24. Yoon, G.J.: Darboux transforms and orthogonal polynomials. Bull. Korean Math. Soc. 39, 359–376 (2002)