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Perturbations on the antidiagonals of Hankel matrices

K. Castillo [a], D.K. Dimitrov [a], L.E. Garza [b,*], F.R. Rafaeli [a]

[a] *Departamento de Matemática Aplicada, Universidade Estadual Paulista – IBILCE/UNESP, Rua Cristóvão Colombo, 2265 – Jardim Nazareth, 15054 000 São José do Rio Preto, SP, Brazil*

[b] *Facultad de Ciencias, Universidad de Colima, Bernal Díaz del Castillo No. 340, Colima, Mexico*

A B S T R A C T

Given a strongly regular Hankel matrix, and its associated sequence of moments which defines a quasi definite moment linear functional, we study the perturbation of a fixed moment, i.e., a perturbation of one antidiagonal of the Hankel matrix. We define a linear functional whose action results in such a perturbation and establish necessary and sufficient conditions in order to preserve the quasi definite character. A relation between the corresponding sequences of orthogonal polynomials is obtained, as well as the asymptotic behavior of their zeros. We also study the invariance of the Laguerre Hahn class of linear functionals under such perturbation, and determine its relation with the so called canonical linear spectral transformations.

Keywords: Hankel matrix, Linear moment functional, Orthogonal polynomials, Laguerre-Hahn class, Zeros

1. Introduction

1.1. Hankel matrices and orthogonal polynomials

Given a sequence of complex numbers $\{\mu_n\}_{n\geqslant 0}$, one can define a linear functional $\mathcal{M}$ in the linear space of polynomials with complex coefficients $\mathbb{P}$ such that

$$\langle \mathcal{M}, x^n\rangle \quad \mu_n. \tag{1}$$

In the literature (see [9,17], among others), $\mathcal{M}$ is said to be a moment linear functional, and the complex numbers $\{\mu_n\}_{n\geqslant 0}$ are called the moments associated with $\mathcal{M}$. The semi infinite matrix

$$\mathbf{H} \quad \left[\langle \mathcal{M}, x^{i+j}\rangle\right]_{i,j \quad 0,1,\ldots} \quad \left[\mu_{i+j}\right]_{i,j \quad 0,1,\ldots} \quad \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n & \cdots \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2}$$

is the Gram matrix associated with the bilinear form of the linear functional (1) in terms of the canonical basis $\{x^n\}_{n\geqslant 0}$ of $\mathbb{P}$. I f there exist a family of monic polynomials such that $\deg(P_n) \quad n$ and

$$\langle \mathcal{M}, P_n(x)P_m(x)\rangle \quad \gamma_n^{\;2}\delta_{n,m},\ \gamma_n \neq 0,\ n, m \geqslant 0,$$

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* Corresponding author.

E-mail addresses: kcastill@math.uc3m.es (K. Castillo), dimitrov@ibilce.unesp.br (D.K. Dimitrov), garzaleg@gmail.com (L.E. Garza), rafaeli@ibilce.unesp.br (F.R. Rafaeli).

where $\delta_{n,m}$ is the Kronecker delta, then $\{P_n\}_{n\geqslant 0}$ is called the monic orthogonal polynomials sequence (MOPS) associated with $\mathcal{M}$.

The Hankel matrices and their determinants play an important role in the study of moment functionals. The linear functional (1) is called quasi-definite if the moments matrix is strongly regular or, equivalently, the determinants of the principal leading submatrices $\mathbf{H}_n$ of order $(n+1)\times(n+1)$ are all different from 0. In this case, there exists a unique MOPS associated with $\mathcal{M}$.

On the other hand, a linear functional $\mathcal{M}$ is called positive definite if and only if its moments are all real and $\det \mathbf{H}_n > 0, n \geqslant 0$. In such a case, there exist a unique sequence of real polynomials $\{p_n\}_{n\geqslant 0}$ orthonormal with respect to $\mathcal{M}$, i.e., the following condition holds

$$\langle \mathcal{M}, p_n(x)p_m(x)\rangle = \delta_{n,m},$$

where

$$p_n(x) = \gamma_n x^n + \delta_n x^{n-1} + (\text{lower degree terms}), \quad \gamma_n > 0, \quad n \geqslant 0.$$

From the Riesz representation theorem, we know that every positive definite linear functional $\mathcal{M}$ has an integral representation (not necessarily unique)

$$\langle \mathcal{M}, x^n\rangle = \int_I x^n d\mu(x),$$

where μ denotes a nontrivial measure supported on some infinite subset I of the real line.

One of the most important characteristics of orthonormal polynomials on the real line is the fact that any three consecutive polynomials are connected by the simple recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x), \quad n \geqslant 0, \tag{3}$$

with initial conditions $p_{-1} \equiv 0,\ p_0 \equiv {\mu_0}^{-1/2}$, and recurrence coefficients given by

$$a_n = \int_I xp_{n-1}(x)p_n(x)d\mu(x) = \frac{\gamma_{n-1}}{\gamma_n} > 0,$$
$$b_n = \int_I xp_n^2(x)d\mu(x) = \frac{\delta_n}{\gamma_n} - \frac{\delta_{n+1}}{\gamma_{n+1}}.$$

There are explicit formulae for orthonormal polynomials in terms of the determinants of the corresponding Hankel matrix. The n-th degree orthonormal polynomial is given by the Heine's formula

$$p_n(x) = \frac{1}{\sqrt{\det \mathbf{H}_n \det \mathbf{H}_{n-1}}}\begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \\ 1 & x & x^2 & \ldots & x^n \end{vmatrix},$$

while its leading coefficient is given by a ratio of two Hankel determinants

$$\gamma_n = \sqrt{\frac{\det \mathbf{H}_{n-1}}{\det \mathbf{H}_n}}.$$

The n-th order reproducing kernel associated with $\{p_n\}_{n\geqslant 0}$ is defined by

$$K_n(x,y) = \sum_{k=0}^{n} p_k(x)p_k(y), \quad n \geqslant 0.$$

The name comes from the fact that, for any polynomial q_n of degree at most n, we have

$$q_n(y) = \int_I q_n(x)K_n(x,y)d\mu(x).$$

The reproducing kernel can be represented in a simple way in terms of the polynomials p_n and p_{n+1} using the Christoffel–Darboux formula (see [9,17], among others)

$$K_n(x,y) = a_{n+1}\frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}, \quad x \neq y,$$

which can be deduced in a straightforward way from the three-term recurrence relation (3). We will denote by $K_n^{(i,j)}(x,y)$ the i-th (resp. j-th) partial derivative of $K_n(x,y)$ with respect to the variable x (resp. y). For the quasi-definite case, the reproducing kernel is defined as

$$K_n(x,y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\gamma_k^2}.$$

When polynomials are studied, one of the most important quantities to be considered are their zeros. The fundamental theorem of algebra asserts that any polynomial of degree n has exactly n zeros (counting multiplicities). When dealing with orthogonal polynomials on the real line, one can say much more about their localization. Two of the most relevant properties of their zeros are the following:

(i) The zeros of p_n are all real, simple and lie in the convex hull of I.
(ii) Suppose $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ are the zeros of p_n, then

$$x_{n,k} < x_{n-1,k} < x_{n,k+1}, \quad 1 \leqslant k \leqslant n-1.$$

Our interest in perturbations of Hankel matrices is motivated by their many applications in mathematical and physical problems. Their relations with moment problems ([7]), integrable systems ([23]), Padé approximation ([20]), as well as their applications in coding theory and combinatorics (see [18,22] and references therein), constitute an illustrative sample of their impact.

1.2. Perturbations of Hankel matrices

Before introducing the problem to be analyzed in this contribution, let us briefly discuss two rather straightforward but interesting examples where the moments are modified in a natural way. First, instead of taking the canonical basis of $\mathbb{P}$, consider the basis $\{1, (x-a), (x-a)^2, \ldots\}$, where $a \in \mathbb{R}$. Then, the new sequence of moments $\{\upsilon_n\}_{n\geqslant 0}$ is given by

$$\upsilon_n = \langle \mathcal{M}, (x-a)^n \rangle = \left\langle \mathcal{M}, \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} a^{n-j} x^j \right\rangle = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} a^{n-j} \mu_j. \quad (4)$$

As a consequence, the $(n+1)\times(n+1)$ principal leading submatrix of the corresponding Hankel matrix is

$$\widetilde{\mathbf{H}}_n = [\upsilon_{i+j-2}]_{1\leqslant i,j\leqslant n} = \begin{bmatrix} \mu_0 & \mu_1 + m_1 & \cdots & \mu_n + m_n \\ \mu_1 + m_1 & \mu_2 + m_2 & \cdots & \mu_{n+1} + m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n + m_n & \mu_{n+1} + m_{n+1} & \cdots & \mu_{2n} + m_{2n} \end{bmatrix},$$

where

$$m_n = \sum_{j=1}^{n-1} \binom{n}{j} (-1)^{n-j} a^{n-j} \mu_j.$$

Thus, if $\mathcal{M}$ is a quasi-definite moment linear functional, then the polynomials

$$Q_n(x) = \frac{1}{\det \widetilde{\mathbf{H}}_{n-1}} \begin{vmatrix} \upsilon_0 & \upsilon_1 & \upsilon_2 & \cdots & \upsilon_n \\ \upsilon_1 & \upsilon_2 & \upsilon_3 & \cdots & \upsilon_{n+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \upsilon_{n-1} & \upsilon_n & \upsilon_{n+1} & \cdots & \upsilon_{2n-1} \\ 1 & (x-a) & (x-a)^2 & \cdots & (x-a)^n \end{vmatrix}, \quad n \geqslant 0, \quad (5)$$

constitute a sequence of monic polynomials orthogonal with respect to $\mathcal{M}$, using the new basis, with $Q_n(x-a) = P_n(x)$. Notice that this simple change of the basis resulted in a perturbation on the antidiagonals of (2). Namely, each of the $(j+1)$-th antidiagonal is perturbed by the addition of the constant m_j. In the remaining of the manuscript, we will use the basis $\{1, (x-a), (x-a)^2, \ldots\}$, since most of the required calculations can be performed in a simpler way.

The second example is given by the Uvarov's spectral transformation (see [21,24]), whose action results in a perturbation of the first moment υ_0, while leaving the others unaffected. This perturbation (the most simple case that we can consider) is closely related with the Uvarov–Chihara integrable system (see [23]). In order to define it, we introduce the real Dirac delta functional $\delta(x-a)$ supported at $x = a$, which acts in the following way

$$\langle \delta(x-a), P(x) \rangle = P(a), \quad P \in \mathbb{P}.$$

Then, Uvarov's transformation is defined by

$$\langle \mathcal{M}_U, p(x) \rangle = \langle \mathcal{M}, p(x) \rangle + m\langle \delta(x-a), p(x) \rangle = \langle \mathcal{M}, p(x) \rangle + mp(a),$$

i.e., a perturbation on the first antidiagonal on the Hankel matrix.

Now a natural question arises: Is there a linear functional $\widehat{\mathcal{M}}$ such that its action results on a perturbation of (only) the moment υ_j or, equivalently, the $(j+1)$-th antidiagonal of the Hankel matrix $\widetilde{\mathbf{H}}$? In other words, we are interested in the properties of a functional $\widehat{\mathcal{M}}$ whose moments are given by

$$\widetilde{\upsilon}_n = \langle \widehat{\mathcal{M}}, (x-a)^n \rangle = \begin{cases} \upsilon_n, & n \neq j, \\ \upsilon_n + m_j, & n = j, \end{cases} \tag{6}$$

for some $m \in \mathbb{R}$, i.e., its corresponding Hankel matrix is

$$\mathbf{H}(m_j) = \begin{bmatrix} \upsilon_0 & \cdots & \upsilon_j + m_j & \upsilon_{j+1} & \cdots \\ \vdots & \iddots & \vdots & \vdots & \cdots \\ \upsilon_j + m_j & \cdots & \upsilon_{2j} & \upsilon_{2j+1} & \cdots \\ \upsilon_{j+1} & \cdots & \upsilon_{2j+1} & \upsilon_{2j+2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{7}$$

An analogous problem for linear functionals defined in the linear space of Laurent polynomials has been analyzed recently in [6]. There, the corresponding moments matrix is a Toeplitz matrix and the perturbation studied is one that modifies two symmetric subdiagonals of such matrix by means of a modification of the Lebesgue measure, supported on the unit circle. The main objective of this manuscript is to study equivalent perturbations of Hankel matrices, which appear in the theory of orthogonal polynomials on the real line. In Section 2, a linear functional with the desired properties, denoted by $\mathcal{M}_j$, is defined in terms of $\mathcal{M}$, and we obtain necessary and sufficient conditions for the quasi-definiteness of $\mathcal{M}_j$, provided that $\mathcal{M}$ is quasi-definite. Under those conditions, we obtain an expression that relates their corresponding families of orthogonal polynomials. In Section 3, we analyze the connection between $\mathcal{M}_j$ and the so-called canonical spectral transformations, and the invariance of the Laguerre-Hahn class under this perturbation is studied in Section 4. Section 5 deals with an asymptotic analysis of the zeros of the corresponding orthogonal polynomials. Finally, in Section 6, we pose some related open problems that will be considered in future contributions.

2. A perturbation on the antidiagonal of a Hankel matrix

In order to state our main result, we will need some definitions. Given a moment linear functional $\mathcal{M}$, the usual distributional derivative $D\mathcal{M}$ (see [19]) is given by

$$\langle D\mathcal{M}, p \rangle = -\langle \mathcal{M}, p' \rangle, \quad p \in \mathbb{P}.$$

In particular, if j is a nonnegative integer, then

$$\langle D^{(j)}\delta(x-a), p(x) \rangle = (-1)^j p^{(j)}(a).$$

Now, we introduce the linear functional $\mathcal{M}_j$, given by

$$\langle \mathcal{M}_j, p(x) \rangle = \langle \mathcal{M}, p(x) \rangle + (-1)^j \frac{m_j}{j!} \langle D^{(j)}\delta(x-a), p(x) \rangle = \langle \mathcal{M}, p(x) \rangle + \frac{m_j}{j!} p^{(j)}(a), \tag{8}$$

where m_j and a are real constants.

It is easy to see that all the moments associated with $\mathcal{M}_j$ are equal to the moments υ_n of $\mathcal{M}$, except for the j-th one, which is equal to $\upsilon_j + m_j$. That is, if we denote by $\widetilde{\upsilon}_k$ the moments of $\mathcal{M}_j$, then they are given by (6) and their corresponding Hankel matrix is (7). Notice that this perturbation is the simplest one that preserves the Hankel structure of the moment matrix.

We are ready to state our main result, which establishes necessary and sufficient conditions under which the linear functional $\mathcal{M}_j$ preserves the quasi-definite character, and provides the relation between the corresponding MOPS. We refer to [2,4,5,12] for other contributions about some other kinds of perturbations of moment functionals.

Proposition 1. *Let $\mathcal{M}$ be a quasi-definite moment linear functional and $\{P_n\}_{n\geqslant 0}$ its corresponding MOPS. Then the following statements are equivalent:*

(i) The moment linear functional $\mathcal{M}_j$, defined as in (8), is quasi-definite.
(ii) For every $n \geqslant 0$, the matrix $\mathbf{I}_j + \mathbf{K}_j\mathbf{D}_j$, with

$$\mathbf{D}_j = \frac{m_j}{j!} \begin{bmatrix} \binom{j}{j} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \binom{j}{0} \end{bmatrix}, \quad \mathbf{K}_j = \begin{bmatrix} K_{n-1}^{(j,0)}(a,a) & K_{n-1}^{(j-1,0)}(a,a) & \cdots & K_{n-1}^{(0,0)}(a,a) \\ K_{n-1}^{(j,1)}(a,a) & K_{n-1}^{(j-1,1)}(a,a) & \cdots & K_{n-1}^{(0,1)}(a,a) \\ \vdots & \vdots & \ddots & \vdots \\ K_{n-1}^{(j,j)}(a,a) & K_{n-1}^{(j-1,j)}(a,a) & \cdots & K_{n-1}^{(0,j)}(a,a) \end{bmatrix},$$

and $\mathbf{I}_j$ denotes the $(j+1)\times(j+1)$ identity matrix, is non singular and

$$\langle \mathcal{M}, P_n^2(x)\rangle + \begin{bmatrix} P_n^{(j)}(a) \\ P_n^{(j-1)}(a) \\ \vdots \\ P_n(a) \end{bmatrix}^T \mathbf{D}_j(\mathbf{I}_j + \mathbf{K}_j\mathbf{D}_j)^{-1} \begin{bmatrix} P_n(a) \\ P_n^{(1)}(a) \\ \vdots \\ P_n^{(j)}(a) \end{bmatrix} \neq 0.$$

Moreover, if $\mathcal{M}_j$ is quasi definite and we denote by $\{P_n(j;\cdot)\}_{n\geqslant 0}$ its corresponding MOPS, then

$$P_n(j;x) = P_n(x) - \begin{bmatrix} K_{n-1}^{(j,0)}(a,x) \\ K_{n-1}^{(j-1,0)}(a,x) \\ \vdots \\ K_{n-1}^{(0,0)}(a,x) \end{bmatrix}^T \mathbf{D}_j(\mathbf{I}_j + \mathbf{K}_j\mathbf{D}_j)^{-1} \begin{bmatrix} P_n(a) \\ P_n^{(1)}(a) \\ \vdots \\ P_n^{(j)}(a) \end{bmatrix}. \tag{9}$$

Proof. Suppose that $\mathcal{M}_j$ is a quasi definite moment linear functional. Since $\{p_n\}_{n\geqslant 0}$ is the MOPS associated with $\mathcal{M}$, there exist constants $\lambda_{n,0},\ldots,\lambda_{n,n-1}$ such that

$$P_n(j;x) = P_n(x) + \sum_{k=0}^{n-1}\lambda_{n,k}P_k(x). \tag{10}$$

Thus, using the orthogonality property, we have

$$\lambda_{n,k} = \frac{\langle \mathcal{M}, P_n(j;x)P_k(x)\rangle}{\langle \mathcal{M}, P_k^2(x)\rangle} = -\frac{m_j}{j!}\frac{\sum_{l=0}^{\min(k,j)}\binom{j}{l}P_n^{(j-l)}(j;a)P_k^{(l)}(a)}{\langle \mathcal{M}, P_k^2(x)\rangle}, \quad 0\leqslant k\leqslant n-1.$$

Substituting the above expression in (10), we obtain

$$\begin{aligned} P_n(j;x) &= P_n(x) - \frac{m_j}{j!}\sum_{k=0}^{n-1}\frac{\sum_{l=0}^{\min(k,j)}\binom{j}{l}P_n^{(j-l)}(j;a)P_k^{(l)}(a)}{\langle \mathcal{M}, P_k^2(x)\rangle}P_k(x) \\ &= P_n(x) - \frac{m_j}{j!}\sum_{l=0}^{j}\binom{j}{l}P_n^{(j-l)}(j;a)K_{n-1}^{(l,0)}(a,x). \end{aligned} \tag{11}$$

In particular, for $0\leqslant i\leqslant j$, we get the following linear system of $j+1$ equations and $j+1$ unknowns

$$P_n^{(i)}(j;x) = P_n^{(i)}(x) - \frac{m_j}{j!}\sum_{l=0}^{j}\binom{j}{l}P_n^{(j-l)}(j;a)K_{n-1}^{(l,i)}(a,x), \tag{12}$$

which, setting $x=a$, reads as

$$(\mathbf{I}_j + \mathbf{K}_j\mathbf{D}_j)\begin{bmatrix} P_n(j;a) \\ P_n^{(1)}(j;a) \\ \vdots \\ P_n^{(j)}(j;a) \end{bmatrix} = \begin{bmatrix} P_n(a) \\ P_n^{(1)}(a) \\ \vdots \\ P_n^{(j)}(a) \end{bmatrix}.$$

From (11), if $P_n^{(l)}(j;a) = 0$ for all $0\leqslant l\leqslant j$, then we have $P_n(j;x) = P_n(x)$, which contradicts the uniqueness of P_n. Thus $[P_n(j;a), P_n^{(1)}(j;a),\ldots,P_n^{(j)}(j;a)]^t$ is not zero, and it constitutes the unique solution of the linear system. Therefore the $(j+1)\times(j+1)$ matrix $\mathbf{I}_j + \mathbf{K}_j\mathbf{D}_j$ is non singular. Furthermore, (11) reduces to (9).

On the other hand,

$$\begin{aligned} 0 \neq \langle \mathcal{M}_j, P_n(j;x)P_n(x)\rangle &= \langle \mathcal{M}, P_n(j;x)P_n(x)\rangle + \frac{m_j}{j!}[P_n(j;x)P_n(x)]^{(j)}|_{x=a} \\ &= \left\langle \mathcal{M}, \left[P_n(x) - \frac{m_j}{j!}\sum_{l=0}^{j}\binom{j}{l}P_n^{(j-l)}(j;a)K_{n-1}^{(l,0)}(a,x)\right]P_n(x)\right\rangle + \frac{m_j}{j!}[P_n(j;x)P_n(x)]^{(j)}|_{x=a} \\ &= \langle \mathcal{M}, P_n(x)P_n(x)\rangle + \begin{bmatrix} P_n^{(j)}(a) \\ P_n^{(j-1)}(a) \\ \vdots \\ P_n(a) \end{bmatrix}^T \mathbf{D}_j(\mathbf{I}_j + \mathbf{K}_j\mathbf{D}_j)^{-1} \begin{bmatrix} P_n(a) \\ P_n^{(1)}(a) \\ \vdots \\ P_n^{(j)}(a) \end{bmatrix}. \end{aligned}$$

For the converse, assume that (*ii*) holds, and define $\{P_n(j;\ )\}_{n\geqslant 0}$ as in (9). Then it is straightforward to show that $\{P_n(j;\ )\}_{n\geqslant 0}$ is the MOPS with respect to $\mathcal{M}_j$, and its quasi-definite character is proved. □

3. Canonical linear spectral transformations

In this section, we will assume $\mathcal{M}$ is a positive definite linear functional, with an associated positive Borel measure μ supported in some interval I of the real line. The corresponding Stieltjes function is then defined by

$$S(x) = \int_I \frac{d\mu(y)}{x-y},$$

and admits the following equivalent representation as a series expansion at infinity

$$S(x) = \sum_{n=0}^{\infty} \frac{\mu_n}{x^{n+1}}, \tag{13}$$

i.e., it is a generating function of the sequence of moments for the measure $d\mu$ (questions about convergence are not considered here). In many problems, (13) has more simple analytical and transformation properties than the spectral measure μ and hence dealing with $S(x)$ is often much more convenient for analysis.

A generic linear spectral transformation of a Stieltjes function $S(x)$ is another Stieltjes function of the form

$$\widetilde{S}(x) = \frac{A(x)S(x)+B(x)}{C(x)S(x)+D(x)}, \tag{14}$$

where $A(x), B(x), C(x)$ and $D(x)$ are polynomials. If $C(x) = 0$, then the transformation is called linear.

If the linear functional $\mathcal{M}_j$ is quasi-definite and we denote by $\widetilde{S}$ its corresponding Stieltjes function, then we can connect S and $\widetilde{S}$ by a simple relation. In fact, since only the j-th moment is perturbed, then it is clear that

$$\widetilde{S}(x) = S(x) + \frac{m_j}{(x-a)^{j+1}}, \tag{15}$$

that is, the Stieltjes function is modified by the addition of a rational function with a pole of order $j+1$ at $x = a$. In this way, we conclude

$$\widetilde{S}(x) = \frac{(x-a)^{j+1}S(x) + m_j}{(x-a)^{j+1}}$$

and therefore the transformation is linear.

Definition 1. The moment linear functional $\mathcal{M}$ belongs to the Laguerre–Hahn class if its corresponding Stieltjes function satisfies the Ricatti equation

$$\Phi(x)S'(x) = B(x)S^2(x) + C(x)S(x) + D(x), \tag{16}$$

where $\Phi(x), B(x), C(x)$ and $D(x)$ are polynomials with complex coefficients such that $\Phi(x) \neq 0$ and $D(x) = [(D\mathcal{M})\theta_0\Phi](x) + (\mathcal{M}\theta_0 C)(x) - [(\mathcal{M}\odot\mathcal{M})\theta_0 B](x)$.

The invariance of the Laguerre–Hahn class under rational spectral transformations was established in [24]. Therefore, in our case we can conclude

Proposition 2. *Let $\mathcal{M}$ be a linear functional that belongs to the Laguerre–Hahn class. Then $\mathcal{M}_j$ also belongs to the Laguerre–Hahn class and its corresponding Stieltjes function satisfies*

$$\widetilde{\Phi}(x)\widetilde{S}'(x) = \widetilde{B}(x)\widetilde{S}^2(x) + \widetilde{C}(x)\widetilde{S}(x) + \widetilde{D}(x),$$

with

$$\begin{aligned}
\widetilde{\Phi}(x) &= (x-a)^{2j+2}\Phi(x),\\
\widetilde{B}(x) &= (x-a)^{2j+2}B(x),\\
\widetilde{C}(x) &= (x-a)^{2j+2}C(x) - m_j(x-a)^{j+1}B(x),\\
\widetilde{D}(x) &= m_j^2 B(x) - m_j(x-a)^{j+1}C(x) + m_j(j+1)(x-a)^{j}\Phi(x) + (x-a)^{2j+2}D(x).
\end{aligned}$$

Proof. From the previous discussion, it is clear that $\mathcal{M}_j$ belongs to the Laguerre–Hahn class. Now, let S be the Stieltjes function associated with $\mathcal{M}$. Then,

$$\Phi(x)S'(x) = B(x)S^2(x) + C(x)S(x) + D(x)$$

holds for some polynomials Φ, B, C and D. Moreover, from (15) we obtain

$$\Phi(x)\left(\widetilde{S}(x) - \frac{m_j}{(x-a)^{j+1}}\right)' = B(x)\left(\widetilde{S}(x) - \frac{m_j}{(x-a)^{j+1}}\right)^2 + C(x)\left(\widetilde{S}(x) - \frac{m_j}{(x-a)^{j+1}}\right) + D(x).$$

Rearranging the terms, we can complete the proof. □

Remark 1. If B (x)=0 in (16), then the corresponding linear functional is said to be semiclassical. This is an important class of linear functionals, since they extend some well known characterizations of the classical orthogonal polynomials. In [24], the authors also prove that the semiclassical class is invariant under linear spectral transformations. Thus, if $\mathcal{M}$ is a semiclassical linear functional, then $\mathcal{M}_j$ is also a semiclassical linear functional.

4. Zeros

In this section, we will continue with the assumption that the linear functional $\mathcal{M}$ is positive definite, and we will assume that $\mathcal{M}_j$ is quasi-definite. We will show some properties regarding the zeros of its corresponding MOPS. Let $x_1, \ldots, x_r$ be the zeros of $P_n(j;x)$ on I with odd multiplicity and define $Q_r(x) = (x-x_1)\cdots(x-x_r)$. Then, $P_n(j;x)Q_r(x)(x-a)^{2k}$, where k is the smallest integer such that $k \geqslant (j+1)/2$, is a polynomial that does not change sign on I and, furthermore, we have

$$\langle \mathcal{M}_j, P_n(j;x)Q_r(x)(x-a)^{2k}\rangle = \langle \mathcal{M}, P_n(j;x)Q_r(x)(x-a)^{2k}\rangle \neq 0.$$

From the orthogonality of $P_n(j;x)$ with respect to $\mathcal{M}_j$, for $n > 2k$ we have

Proposition 3. *$P_n(j;x)$ has at least $n-2k$ zeros with odd multiplicity on I.*

Now, we analyze the asymptotic behavior of the zeros of $\{P_n(j;\cdot)\}_{n\geqslant 0}$ when the mass m_j tends to infinity. Notice that, from (5),

$$P_n(j;x) = \frac{1}{\det \mathbf{H}_{n-1}(m_j)}\begin{vmatrix} \upsilon_0 & \cdots & \upsilon_j + m_j & \cdots & \upsilon_n \\ \vdots & & \vdots & & \vdots \\ \upsilon_j + m_j & & \upsilon_{2j} & & \upsilon_{n+j} \\ \vdots & & & & \vdots \\ \upsilon_{n-1} & & \upsilon_{n+j-1} & & \upsilon_{2n-1} \\ 1 & \cdots & (x-a)^j & \cdots & (x-a)^n \end{vmatrix}. \tag{17}$$

On the other hand, let $\{R_n^k(a;\cdot)\}_{n\geqslant 0}$, where k is a positive integer, be the MOPS with respect to the linear functional

$$\langle \widehat{\mathcal{M}}, p(x)\rangle := \langle \mathcal{M}, (x-a)^k p(x)\rangle,$$

i.e., k iterations of the Christoffel perturbation. Then,

$$R^{2(j+1)}_{n-j-1}(a;x) = \frac{1}{\det \mathbf{H}^{(2j+2)}_{n-j-2}}\begin{vmatrix} \upsilon_{2j+2} & \upsilon_{2j+3} & \cdots & \upsilon_{n+j} & \upsilon_{n+j+1} \\ \upsilon_{2j+3} & \upsilon_{2j+4} & \cdots & \upsilon_{n+j+1} & \upsilon_{n+j+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \upsilon_{n+j} & \upsilon_{n+j+1} & \cdots & \upsilon_{2n-2} & \upsilon_{2n-1} \\ 1 & (x-a) & \cdots & (x-a)^{n-j-2} & (x-a)^{n-j-1} \end{vmatrix}, \quad n > j+1,$$

where $\mathbf{H}^{(k)}$ denotes the Hankel matrix associated with $\{R_n^k(a;\cdot)\}_{n\geqslant 0}$. Now, if the matrix in (17) is block partitioned into

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix},$$

where $\mathbf{A}$ is a $(j+1)\times(j+1)$ matrix, then

$$\det\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det(\mathbf{D})\det(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}).$$

This equation is usually called Schur's identity. It is clear that $\det(\mathbf{D}) = \det \mathbf{H}^{(2j+2)}_{n-j-2}(x-a)^{j+1}R^{2(j+1)}_{n-j-1}(a;x)$. Moreover, $\mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ is a $(j+1)\times(j+1)$ matrix that does not depend on m_j, and thus

$$P_n(j;x) = \frac{\det \mathbf{H}_{n-j-2}^{(2j+2)}(x-a)^{j+1}R_{n-j-1}^{2(j+1)}(a;x)Q(m_j)}{\det \mathbf{H}_{n-j-2}^{(2j+2)}R(m_j)}, \tag{18}$$

where $Q(m_j)$ and $R(m_j)$ are monic polynomials in m_j of degree $j+1$. Therefore,

$$\lim_{m_j\to\infty} P_n(j;x) = (x-a)^{j+1}R_{n-(j+1)}^{2(j+1)}(a;x), \tag{19}$$

and, by Hurwitz's theorem, we conclude

Proposition 4. *The zeros* $x_{n,k}(j;m_j)$, $k=1,\ldots,n$, *of the polynomial* $P_n(j;x)$ *converge to the zeros of the polynomial* $(x-a)^{j+1}R_{n-(j+1)}^{2(j+1)}(a;x)$ *when* m_j *tends to infinity.*

Observe that the mass point a attracts $j+1$ zeros of $P_n(j;x)$ when m_j tends to infinity.

A rather natural question is if the $x_{n,k}(j;m_j)$, considered as functions of m_j, tends to the zeros of $(x-a)^{j+1}R_{n-(j+1)}^{2(j+1)}(a;x)$ in a monotonic way. For the particular case when $j=0$, it was proved (see [10,11]) that the zeros of the so called Laguerre and Jacobi type orthogonal polynomials, which are particular cases of the Uvarov's perturbation, do behave monotonically with respect to m_j. Unfortunately, this phenomenon does not occur for every positive integer j. We have performed some numerical experiments with specific classical measures. For example, if the initial measure is the one associated with the Laguerre polynomials $L_n^{\alpha}(x), j=1$ and $a=0$, then the zeros $x_{n,k}(1;m_j)$ of the corresponding polynomials $L_n^{\alpha}(1;x)$ do converge to those of $x^2L_{n-2}^{\alpha+4}(x)$, although they are not monotonic functions of m_j, when it varies in $(0,\infty)$.

We present some tables that show the behavior of the zeros of $P_n(j;x)$ with respect to m_j, when the initial measure is $d\mu(x)=1/\sqrt{1-x^2}dx$ (Chebyshev polynomials of the first kind) for $j=1$ and $n=2,3$. (See Tables 1 and 2).

Notice the existence of complex zeros depending the values of the parameter m_j. This is a consequence of the non-positive definite character of $\mathcal{M}_1$, since $\det H_1(m_1)$ is negative for m_1 large enough. In general, for a given moment functional $\mathcal{M}_j$ with m_j large enough, the determinant of $H_j(m_j)$ is negative. It is also observed that two zeros of the polynomial approach the point $x=a$ as m_j increases, as established in Proposition 4.

5. Further open questions

In this section, we formulate some natural open questions which arose during the study of the main problem stated in the paper. First of all, it is natural to ask if, given $j\in\mathbb{N}$, the corresponding perturbation functional $\mathcal{M}_j$ preserves the positive definiteness of $\mathcal{M}$. Of course, the necessary and sufficient conditions are given by (*ii*) in Proposition 1, replacing $\langle\mathcal{M}_j, P_n(j;x)P_n(x)\rangle \neq 0$ by $\langle\mathcal{M}_j, P_n(j;x)P_n(x)\rangle > 0,\ n\geqslant 0$. However, if one is interested in the existence of a neighborhood

Table 1
Zeros of $P_2(j;x)$ for $a=3, j=1$ and some values of m_1.

m_1	$x_{2,1}(j;m_j)$	$x_{2,2}(j;m_j)$
0	−0.707107	0.707107
0.1	−0.4034	1.61713
0.5	−0.317089	3.16098
1	−0.355225	3.74581
5	−0.95446	4.54373
10	−2.39228	4.881
10^2	$3.18582+1.08651i$	$3.18582-1.08651i$
10^3	$3.01522+0.317155i$	$3.01522-0.317155i$
10^4	$3.0015+0.0995593i$	$3.0015-0.0995593i$
10^5	$3.00015+0.0314605i$	$3.00015-0.0314605i$

Table 2
Zeros of $P_3(j;x)$ for $a=3, j=1$ and some values of m_1.

m_1	$x_{3,1}(j;m_1)$	$x_{3,2}(j;m_1)$	$x_{3,3}(j;m_1)$
0	−0.866025	0	0.866025
0.1	−0.801321	0.51227	3.61105
0.5	−1.18576	0.0479705	3.7638
1	−3.70458	−0.305553	3.86415
5	−0.510437	$3.38703+0.805069i$	$3.38703-0.805069i$
10	−0.525903	$3.15823+0.549961i$	$3.15823-0.549961i$
10^2	−0.538469	$3.01358+0.167363i$	$3.01358-0.167363i$
10^3	−0.539661	$3.00134+0.052716i$	$3.00134-0.052716i$
10^4	−0.539779	$3.00013+0.0166637i$	$3.00013-0.0166637i$
10^5	−0.539791	$3.00001+0.0052693i$	$3.00001-0.0052693i$

(τ_1, τ_2) such that the functional $\mathcal{M}_j$ is positive definite for every $m_j \in (\tau_1, \tau_2)$, then to determine such interval from (ii) might be very complicated. An open problem is to analyze if there exists a different approach that allows one to determine the val ues of m_j such that $\mathcal{M}_j$ is positive definite. Certainly, the interval (τ_1, τ_2) should depend essentially on the initial functional $\mathcal{M}$ and the point a.

Another question that might be of interest is if, given two positive definite moment functionals $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, there exists a sequence of perturbations $\mathcal{M}_{j_k}$ such that we can obtain $\widetilde{\mathcal{M}}$ from the consecutive application of those perturbations to $\mathcal{M}$, i.e.,

$$\mathcal{M} \overset{\mathcal{M}_{j_1}}{\rightarrow} \mathcal{M}^{(1)} \overset{\mathcal{M}_{j_2}}{\rightarrow} \mathcal{M}^{(2)} \overset{\mathcal{M}_{j_3}}{\rightarrow} \cdots \overset{\mathcal{M}_{j_k}}{\rightarrow} \mathcal{M}^{(k)} \rightarrow \widetilde{\mathcal{M}}$$

with the condition that positiveness must be preserved in each step.

As an example, consider the linear functionals $\mathcal{M}^1$ and $\mathcal{M}^2$, associated with the Chebyshev polynomials of the first and second kind, respectively. When using the basis $\{1, (x \quad 1), (x \quad 1)^2, \dots\}$, it is easy to see that one of the sequences of mo ments can be obtained from the other by means of a shift. Thus, one can go from $\mathcal{M}^1$ to $\mathcal{M}^2$ applying a sequence of pertur bations $\mathcal{M}_{j_k}, k \quad 0, 1, \dots$, with $a \quad 1$, and appropriated masses m_{j_k}. However, proceeding in such a way, positive definiteness would be lost after second step. Nevertheless, another sequence $\mathcal{M}_{j_k}$ which preserves the positive definiteness may still exist.

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