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CASORATI TYPE DETERMINANTS OF SOME $\mathfrak{q}$-CLASSICAL ORTHOGONAL POLYNOMIALS

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Abstract. Some symmetries for Casorati determinants whose entries are $\mathfrak{q}$-classical orthogonal polynomials are studied. Special attention is paid to the symmetry involving Big $\mathfrak{q}$-Jacobi polynomials. Some limiting situations, for other related $\mathfrak{q}$-classical orthogonal polynomial families in the $\mathfrak{q}$-Askey scheme, namely $\mathfrak{q}$-Meixner, $\mathfrak{q}$-Charlier, and $\mathfrak{q}$-Laguerre polynomials, are considered.

1. Introduction

In the last years, several papers have appeared on identities of Wronskian and Casorati determinants whose entries are orthogonal polynomials belonging to the Askey and $\mathfrak{q}$-Askey schemes ([2], [3], [13], [14]). In fact, determinants whose entries are orthogonal polynomials is a long-studied subject. One can mention Turán inequality for Legendre polynomials [15] and its generalizations, especially that of Karlin and Szegő on Hankel determinants whose entries are ultraspherical, Laguerre, Hermite, Charlier, Meixner, Krawtchouk, and other families of orthogonal polynomials [10]. Karlin and Szegő's strategy was precisely to express these Hankel determinants in terms of the Wronskian of certain orthogonal polynomials of another class (see, also, [1], [4], [5], [6], [7], [8], [9]).

The approach used by S. Odake and R. Sasaki in [13], [14] is based on the equivalence between eigenstate adding and deleting Darboux transformations for solvable (discrete) quantum mechanical systems.

On the other hand, the approach used by one of us in [3] is based on certain purely algebraic transformations of a Wronskian type determinant whose entries are orthogonal polynomials. These Wronskian type determinants are of the form

$$\det\left(T^{i-1}(p_{m+j-1}(x))\right)_{i,j=1}^{n}, \tag{1.1}$$

where $m \in \mathbb{N}$, $(p_n)_n$ is a sequence of orthogonal polynomials with respect to a measure μ and T is a linear operator acting on the linear space of polynomials $\mathbb{P}$ and satisfying that $\deg(T(p)) = \deg(p) - 1$, for all polynomials p (see Theorem 2.1 in Section 2).

Key words and phrases. Orthogonal polynomials, q-classical polynomials, Wronskian determinant, Casorati determinant.

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Using this approach, some nice symmetries for Casorati determinants whose entries are classical discrete orthogonal polynomials have been found. Here is one of these symmetries for the Hahn polynomials. For $\alpha+c-N\neq -1,-2,\cdots$, we write $(h_n^{\alpha,c,N})_n$ for the sequence of Hahn polynomials normalized by taking its leading coefficient equal to $1/n!$ (see [11], p. 204) and $H_{n,m,x}^{\alpha,c,N}$, $n,m\geq 0$, for the Casorati Hahn determinant

$$H_{n,m,x}^{\alpha,c,N}=\det\left(h_{m+j-1}^{\alpha,c,N}(x+i-1)\right)_{i,j=1}^{n}.$$

Then, for $n,m\geq 0$, we have

$$H_{n,m,x}^{\alpha,c-n-m,N+n+m}=(-1)^{nm}H_{m,n,-x}^{-\alpha,2-c,-N} \tag{1.2}$$

(see [3], Corollary 5.4). Notice that the determinant in the left-hand side of the previous identity is of size $n\times n$ while the determinant in the right-hand side is of size $m\times m$.

The purpose of this paper is to find such symmetries for the Big $\mathfrak{q}$-Jacobi polynomials and, passing to the limit, for other related $\mathfrak{q}$-classical families in the $\mathfrak{q}$-Askey scheme. In the sequel the terms 'Wronskian type determinant' and 'Casorati determinant' are used indistinctly.

The content of this paper is as follows. The algebraic transformation of a Wronskian type determinant of the form (1.1) developed in [3] will be recalled in Section 2. In Section 3, we study it for the particular case of the $\mathfrak{q}$-derivative: $T=D_{\mathfrak{q}}$ where

$$D_{\mathfrak{q}}(p)=\begin{cases}\dfrac{p(x)-p(\mathfrak{q}x)}{x(1-\mathfrak{q})}, & x\neq 0,\\ p'(0), & x=0,\end{cases} \tag{1.3}$$

and $\mathfrak{q}$ is a real number $\mathfrak{q}\neq 1$.

In Section 4, we prove the following symmetry for the $\mathfrak{q}$-Wronskian type determinant associated with the Big $\mathfrak{q}$-Jacobi polynomials. Consider the Big $\mathfrak{q}$-Jacobi polynomials $(P_n^{a,b,c;\mathfrak{q}})_n$ with leading coefficient equals $\mathfrak{q}^{n^2}/(\mathfrak{q};\mathfrak{q})_n$ (see (4.1) below). Define the $\mathfrak{q}$-Casorati Big $\mathfrak{q}$-Jacobi determinant

$$\mathcal{P}_{n,m,x}^{a,b,c;\mathfrak{q}}=x^{\binom{m}{2}}\det\left(P_{m+j-1}^{a,b,c;\mathfrak{q}}(x/\mathfrak{q}^{i-1})\right)_{i,j=1}^{n}.$$

Then we have the following symmetry.

Theorem 1.1. *For $n,m\geq 0$ and $a\neq 0,\mathfrak{q}\neq 1$, there holds*

$$\mathcal{P}_{n,m,x}^{a,b,c;\mathfrak{q}}=(-1)^{nm}\mathfrak{q}^{mn^2+nm^2-mn}\mathcal{P}_{m,n,x}^{a\mathfrak{q}^{n+m},b\mathfrak{q}^{n+m},c\mathfrak{q}^{n+m};1/\mathfrak{q}}. \tag{1.4}$$

Notice that as in (1.2), the determinant in the left-hand side of the previous identity is of size $n\times n$ while the determinant in the right-hand side is of size $m\times m$.

Finally, in Section 5 we consider the analogous symmetries for other $\mathfrak{q}$-classical orthogonal polynomials which can be reached from the Big $\mathfrak{q}$-Jacobi family by taking limits in the $\mathfrak{q}$-Askey scheme.

2. Preliminaries

Consider a linear operator T acting on the linear space of polynomials $\mathbb{P}$ and satisfying that $\deg(T(p)) = \deg(p) - 1$, for all polynomial p. We associate two sequences of polynomials $(r_n)_n$ and $(s_n)_n$ with this operator T. The sequence $(r_n)_n$ satisfies that

$$\left\{ \begin{array}{l} r_0 = 1 \text{ and the degree of } r_n \text{ is } n,\ n \geq 0, \\ T(r_n) = r_{n-1},\ n \geq 0\ (r_{-1} = 0). \end{array} \right. \tag{2.1}$$

It is proved in [3], p. 65, that this sequence of polynomials always exists and it is unique if we fix the values of r_n, $n \geq 1$, at a given number x_0.

The sequence $(s_n)_n$ is now defined recursively by $s_0 = 1$ and

$$\sum_{j=0}^{n} s_j(x) r_{n-j}(x) = 0. \tag{2.2}$$

It is easy to see that if we write $\Psi_r(x,t)$, $\Psi_s(x,t)$ for the (formal) generating functions of the sequences $(r_n)_n$, $(s_n)_n$, respectively,

$$\Psi_r(x,t) = \sum_{n=0}^{\infty} r_n(x) t^n, \quad \Psi_s(x,t) = \sum_{n=0}^{\infty} s_n(x) t^n,$$

we have $\Psi_r(x,t)\Psi_s(x,t) = 1$.

In [3], one of us has proved the following algebraic transformation of a Wronskian type determinant of the form (1.1).

Theorem 2.1 (Theorem 1.2 of [3]). *Consider a linear operator T acting on the linear space of polynomials $\mathbb{P}$ and satisfying that $\deg(T(p)) = \deg(p) - 1$, for all polynomial p. We associate it with the two sequences of polynomials $(r_n)_n$ and $(s_n)_n$ as above. Let μ be a measure and consider a sequence $(p_n)_n$ of orthogonal polynomials with respect to μ. For a given sequence of polynomials $(\psi_i)_i$, ψ_i of degree i, we write μ_j^i, $i, j \geq 0$, for the numbers*

$$\mu_j^i = \int r_j \bar{\psi}_i d\mu. \tag{2.3}$$

We now consider the polynomials q_n^i, $i, n \geq 0$, defined by

$$q_n^i(x) = \sum_{j=0}^{n} \mu_j^i s_{n-j}(x). \tag{2.4}$$

We then have

$$\Omega_{m-1} \det \left(T^{i-1}(p_{m+j-1}(x))\right)_{i,j=1}^{n} = C_{n,m} \det \left(q_{n+i-1}^{j-1}(x)\right)_{i,j=1}^{m}, \tag{2.5}$$

where Ω_{m-1} and $C_{n,m}$ are independent of x,

$$\Omega_{m-1} = \det(\mu_{m-i}^{j-1})_{i,j=1}^{m}, \quad C_{n,m} = (-1)^{mn+\binom{m}{2}} \prod_{j=0}^{n-1} \frac{\xi_{m+j}}{\sigma_{m+j}},$$

and ξ_n and σ_n are the coefficients of x^n in the power expansion of p_n and r_n, respectively.

Let us note that we have some degrees of freedom in the polynomials q_n^i, $n, i \geq 0$. Indeed, for a given number x_0, they depend on the numbers $r_n(x_0)$, $n \geq 1$, and on the sequence of polynomials $(\psi_i)_i$.

As explained in [3], p. 62, the determinants Ω_{m-1} can be computed from the ℓ^2 norm of the monic orthogonal polynomials $(\hat{p}_n)_n$ with respect to μ as follows:

$$\Omega_n = (-1)^{n(n+1)/2} \prod_{j=0}^{n} \sigma_j \bar{\upsilon}_j \|\hat{p}_j\|^2,$$

where υ_n denotes the leading coefficient of ψ_n.

Remark 2.2. The strategy that we will follow to prove the symmetry (1.4) is the following.

First step. In Section 3, we will identify the sequences $(r_n)_n$ and $(s_n)_n$ for the $\mathfrak{q}$-derivative operator $D_{\mathfrak{q}}$ defined by (1.3). By then taking $\psi_n = r_n$, $n \geq 0$, we find a useful expression for the polynomials q_n^i, $i, n \geq 0$, in terms of the sequence of numbers $(\mu_n^0)_n$ and the polynomials $(r_n)_n$ (see (3.26)).

Second step. For the particular case when the orthogonal polynomials $(p_n)_n$ are the Big $\mathfrak{q}$-Jacobi polynomials, $p_n = P_n^{a,b,c;\mathfrak{q}}(x)$ (see (4.1) below), that expression (3.26) will allow us (in Section 4) to identify the polynomials q_n^i also as Big $\mathfrak{q}$-Jacobi polynomials but with different parameters, namely

$$q_n^i(x) = (1-\mathfrak{q})^n \mathfrak{q}^{-in+\binom{n}{2}} P_n^{a\mathfrak{q}^{n+1+i}, b\mathfrak{q}^{n+1}, c q^{n+1+i}; 1/\mathfrak{q}}(\mathfrak{q}^i x) \tag{2.6}$$

(see (4.4)).

Third step. Using Theorem 2.1, we will transform the $n \times n$ Wronskian type determinant associated with the Big $\mathfrak{q}$-Jacobi polynomials $P_n^{a,b,c;\mathfrak{q}}$ in an $m \times m$ determinant whose entries are the Big $\mathfrak{q}$-Jacobi polynomials $P_n^{a\mathfrak{q}^{n+1+i}, b\mathfrak{q}^{n+1}, cq^{n+1+i}; 1/\mathfrak{q}}$. Finally, the symmetry (1.4) follows by performing suitable combinations of columns and rows in this $m \times m$ determinant (see proof of Theorem 1.1 in Section 4).

3. The role of linear operator T as $\mathfrak{q}$-derivative

In this section, we proceed with the first step above.

For a number $\mathfrak{q} \neq 1$, as linear operator T, we consider the $\mathfrak{q}$-derivative ($T = D_{\mathfrak{q}}$) defined by (1.3).

Recall that for classical $\mathfrak{q}$-orthogonal polynomials their $\mathfrak{q}$-derivatives constitute an orthogonal polynomial family [12]. This property, among others, characterizes the classical $\mathfrak{q}$-orthogonal polynomials. Indeed, they satisfy a hypergeometric $\mathfrak{q}$-difference equation; they can be expressed by a Rodrigues-type formula and their associated orthogonalizing weights satisfy a Pearson-type $\mathfrak{q}$-difference equation. As any other family of orthogonal polynomials they also satisfy a three-term recurrence relation. In addition, these polynomials can be represented in terms of the basic hypergeometric series [11, 12]

$${}_r\varphi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; \mathfrak{q}\,,\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; \mathfrak{q})_k}{(b_1, \dots, b_s; \mathfrak{q})_k} \frac{z^k}{(\mathfrak{q}; \mathfrak{q})_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{s-r+1}, \tag{3.1}$$

where $(a_1, \dots, a_r; \mathfrak{q})_k = (a_1; \mathfrak{q})_k \cdots (a_r; \mathfrak{q})_k$ and

$$(a; \mathfrak{q})_0 = 1, \quad (a; \mathfrak{q})_k = \prod_{m=0}^{k-1} (1 - a\mathfrak{q}^m), \quad k \in \mathbb{N},$$

denotes the $\mathfrak{q}$-analogue of the Pochhammer symbol (also called $\mathfrak{q}$-shifted factorial [11], p. 11). This latter formula implies that

$$(a; \mathfrak{q})_k = \frac{(a; \mathfrak{q})_\infty}{(a\mathfrak{q}^k; \mathfrak{q})_\infty}, \quad 0 < |q| < 1. \tag{3.2}$$

Among several interesting possibilities for the sequence of polynomials $(r_n)_n$, we consider

$$r_n^{\mathfrak{q}}(x) = \frac{(\mathfrak{q}-1)^n (x; \mathfrak{q})_n}{(\mathfrak{q}; \mathfrak{q})_n}, \tag{3.3}$$

$$\psi_i(x) = r_i^{\mathfrak{q}}(x). \tag{3.4}$$

A simple computation then shows that

$$\begin{aligned} s_n(x) &= (-1)^n \mathfrak{q}^{\binom{n}{2}} r_n^{\mathfrak{q}}(x/\mathfrak{q}^{n-1}) \\ &= (-1)^n r_n^{1/\mathfrak{q}}(x); \end{aligned} \tag{3.5}$$

see (2.1) and (2.2) for the definition of the polynomials r_n, s_n, $n \geq 0$.

Lemma 3.1. *The following relations are valid:*

$$r_i^{\mathfrak{q}}(x) = \sum_{l=0}^{i} (1-\mathfrak{q})^{i-l} \mathfrak{q}^{\binom{i-l}{2} - ij} \begin{bmatrix} j \\ i-l \end{bmatrix} r_l^{\mathfrak{q}}(x\mathfrak{q}^j), \tag{3.6}$$

$$\begin{aligned} \sum_{j=g-i}^{\min(n,g)} (\mathfrak{q}-1)^{i-g+j} \mathfrak{q}^{\binom{i-g+j}{2} + \binom{n-j}{2} - ij} \begin{bmatrix} j \\ i-g+j \end{bmatrix} \begin{bmatrix} g \\ g-j \end{bmatrix} r_{n-j}^{\mathfrak{q}}(x/\mathfrak{q}^{n-j-1}) \\ = \mathfrak{q}^{\binom{g-i+1}{2} + \binom{n}{2} - gn} \begin{bmatrix} g \\ i \end{bmatrix} r_{n-g+i}^{\mathfrak{q}}(x/\mathfrak{q}^{n-g-1}). \end{aligned} \tag{3.7}$$

Proof. We first prove (3.6). Indeed, by using (3.3) in the variable $x\mathfrak{q}^j$ as well as relations (1.8.16) and (1.9.4) given in [11] (for $\mathfrak{q}$-shifted factorial and $\mathfrak{q}$-binomial coefficient, respectively) the right-hand term of (3.6) can be rewritten as follows:

$$\sum_{l=0}^{i} (1-\mathfrak{q})^{i-l} \mathfrak{q}^{\binom{i-l}{2} - ij} \begin{bmatrix} j \\ i-l \end{bmatrix} r_l^{\mathfrak{q}}(x\mathfrak{q}^j) = \frac{(1-\mathfrak{q})^i \mathfrak{q}^{\binom{i}{2}}}{(\mathfrak{q}; \mathfrak{q})_i} \sum_{l=0}^{i} \beta_{l,j}^i \tau_{i-l,j}^{\mathfrak{q}}(x),$$

where

$$\beta_{l,j}^i = \frac{\mathfrak{q}^l (\mathfrak{q}^{-i}; \mathfrak{q})_l (\mathfrak{q}^{-j}; \mathfrak{q})_l}{(\mathfrak{q}; \mathfrak{q})_l}, \quad l = 0, 1, \dots, i, \tag{3.8}$$

$$\tau_{i,j}^{\mathfrak{q}}(x) = \prod_{k=0}^{i-1} \left(x - \frac{1}{\mathfrak{q}^{j+k}} \right) = (-1)^i \mathfrak{q}^{-ij - \binom{i}{2}} (x\mathfrak{q}^j; \mathfrak{q})_i, \quad i, j = 0, 1, 2, \dots. \tag{3.9}$$

Using the identity

$$\tau^{\mathfrak{q}}_{i,0}(x) = \sum_{l=0}^{i} \beta^{i}_{l,j} \tau^{\mathfrak{q}}_{i-l,j}(x), \tag{3.10}$$

and expression (3.3), there holds the following relation:

$$\begin{aligned}\sum_{l=0}^{i} (1-\mathfrak{q})^{i-l} \mathfrak{q}^{\binom{i-l}{2}-ij} \begin{bmatrix} j \\ i-l \end{bmatrix} r^{\mathfrak{q}}_{l}(x\mathfrak{q}^{j}) &= \frac{(1-\mathfrak{q})^{i} \mathfrak{q}^{\binom{i}{2}}}{(\mathfrak{q};\mathfrak{q})_{i}} \tau^{\mathfrak{q}}_{i,0}(x) \\ &= r^{\mathfrak{q}}_{i}(x),\end{aligned}$$

which proves (3.6).

Observe that (3.10) can be easily proved by induction. Indeed, it holds for $i = 1$. Using the induction hypothesis for $i \in \mathbb{N}$ (formula (3.10)) and relation

$$\tau^{\mathfrak{q}}_{i+1,j}(x) = \tau^{\mathfrak{q}}_{i,j}(x) \left(x - \frac{1}{\mathfrak{q}^{i+j}} \right), \tag{3.11}$$

between two consecutive elements from the basis $\{\tau^{\mathfrak{q}}_{i+1,j}(x), \ldots, \tau^{\mathfrak{q}}_{0,j}(x)\}$ of the linear subspace $\mathbb{P}_{i+1}$, the coefficients in the (formal) series expansion

$$\tau^{\mathfrak{q}}_{i+1,0}(x) = \sum_{l=0}^{i+1} \alpha^{i+1}_{l,j} \tau^{\mathfrak{q}}_{i+1-l,j}(x), \tag{3.12}$$

can be straightforwardly computed, getting $\alpha^{i+1}_{l,j} = \beta^{i+1}_{l,j}$, $l = 0, 1, \ldots, i+1$, as shown below. In fact,

$$\begin{aligned}\tau^{\mathfrak{q}}_{i+1,0}(x) &= \left(x - \mathfrak{q}^{-i}\right) \tau^{\mathfrak{q}}_{i,0}(x) \\ &= \sum_{l=0}^{i} \left[(x - \mathfrak{q}^{-i+l-j}) - \mathfrak{q}^{-i}(1 - \mathfrak{q}^{l-j}) \right] \beta^{i}_{l,j} \tau^{\mathfrak{q}}_{i-l,j}(x).\end{aligned} \tag{3.13}$$

Notice that for coefficients (3.8) we have

$$\begin{aligned}\beta^{i+1}_{i+1,j} &= (\mathfrak{q}^{-j} - \mathfrak{q}^{-i}) \beta^{i}_{i,j}, \quad \beta^{i+1}_{0,j} = \beta^{i}_{0,j} = 1, \\ \beta^{i+1}_{l,j} &= \left(\beta^{i}_{l,j} + \frac{(\mathfrak{q}^{l-j-1} - 1)}{\mathfrak{q}^{i}} \beta^{i}_{l-1,j} \right), \quad l = 1, \ldots, i.\end{aligned} \tag{3.14}$$

Hence, taking into account formula (3.11) and relations (3.14), one transforms expression (3.13) into

$$\tau^{\mathfrak{q}}_{i+1,0}(x) = \sum_{l=0}^{i+1} \beta^{i+1}_{l,j} \tau^{\mathfrak{q}}_{i+1-l,j}(x),$$

which gives the series expansion (3.12). This completes the proof of (3.6).

Now we proceed with the proof of (3.7), in which identity (3.10) will play a key factor when we replace parameter $\mathfrak{q}$ by $1/\mathfrak{q}$. Let us start by performing some combinatorial operations on equation (3.7). Observe that

$$\begin{bmatrix} j \\ i-g+j \end{bmatrix} \begin{bmatrix} g \\ g-j \end{bmatrix} = (-1)^j \mathfrak{q}^{-\binom{j}{2}+gj} \frac{(\mathfrak{q}^{-g};\mathfrak{q})_j}{(\mathfrak{q};\mathfrak{q})_{g-i}(\mathfrak{q};\mathfrak{q})_{j-g+i}}, \tag{3.15}$$

$$\begin{bmatrix} g \\ i \end{bmatrix} = \frac{(\mathfrak{q};\mathfrak{q})_g}{(\mathfrak{q};\mathfrak{q})_i(\mathfrak{q};\mathfrak{q})_{g-i}} \tag{3.16}$$

(see relation (1.9.4) in [11]). Moreover, from (3.3) and (3.9) we have that

$$r^{\mathfrak{q}}_{k+i}(x/\mathfrak{q}^{k-1}) = \mathfrak{q}^{\binom{k+i}{2}} \frac{(1-\mathfrak{q})^{k+i}}{(\mathfrak{q};\mathfrak{q})_{k+i}} \tau^{\mathfrak{q}}_{k+i,0}(x/\mathfrak{q}^{k-1}), \quad i=0,1,\ldots, \quad k\in\mathbb{N}. \tag{3.17}$$

In addition, from (3.9) we also have the following useful expressions:

$$\tau^{1/\mathfrak{q}}_{m,0}(x) = q^{m(m-1)} \tau^{\mathfrak{q}}_{m,0}(x/\mathfrak{q}^{m-1}) \tag{3.18}$$

$$= \mathfrak{q}^{-\alpha m} \tau^{1/\mathfrak{q}}_{m,\alpha}(\mathfrak{q}^{\alpha} x), \tag{3.19}$$

where α denotes any number.

Hence, substituting relations (3.15)-(3.16) into equation (3.7) and using (3.17)-(3.19) we get that equation (3.7) is equivalent to

$$\sum_{j=g-i}^{m} \frac{(\mathfrak{q}^{-g};\mathfrak{q})_j}{(\mathfrak{q};\mathfrak{q})_{j-g+i}(\mathfrak{q};\mathfrak{q})_{n-j}} \tau^{1/\mathfrak{q}}_{n-j,0}(x) = \frac{(-1)^{g-i}\mathfrak{q}^{\binom{n}{2}-gn-\binom{n-g+i}{2}}(\mathfrak{q};\mathfrak{q})_g}{(\mathfrak{q};\mathfrak{q})_i(\mathfrak{q};\mathfrak{q})_{n-g+i}} \tau^{1/\mathfrak{q}}_{n-g+i,0}(\mathfrak{q}^i x), \tag{3.20}$$

where $m=\min(n,g)$. Indeed, taking into account again (3.19), the left-hand term of (3.20) transforms into

$$\sum_{l=0}^{m-g+i} \frac{\mathfrak{q}^{-i(n+i-g-l)}(\mathfrak{q}^{-g};\mathfrak{q})_{g-i+l}}{(\mathfrak{q};\mathfrak{q})_l(\mathfrak{q};\mathfrak{q})_{n-g+i-l}} \tau^{1/\mathfrak{q}}_{n-g+i-l,i}(\mathfrak{q}^i x). \tag{3.21}$$

Notice that, by using formulas (1.8.10) and (1.8.16) from [11], we have that

$$(\mathfrak{q}^{-g};\mathfrak{q})_{g-i+l} = (\mathfrak{q}^{-g};\mathfrak{q})_{g-i}(\mathfrak{q}^{-i};\mathfrak{q})_l, \tag{3.22}$$

$$(\mathfrak{q};\mathfrak{q})_{n-g+i-l} = \frac{(\mathfrak{q};\mathfrak{q})_n}{(\mathfrak{q}^{-n};\mathfrak{q})_{g-i}(\mathfrak{q}^{g-i-n};\mathfrak{q})_l} (-1)^{g-i+l}\mathfrak{q}^{\binom{g-i+l}{2}-n(g-i+l)}. \tag{3.23}$$

Thus, substituting (3.22)-(3.23) into expression (3.21), it is equal to

$$\frac{(-1)^{g-i}(\mathfrak{q}^{-g};\mathfrak{q})_{g-i}(\mathfrak{q}^{-n};\mathfrak{q})_{g-i}}{(\mathfrak{q};\mathfrak{q})_n} \sum_{l=0}^{n-g+i} \frac{(-1)^l(\mathfrak{q}^{g-n-i};\mathfrak{q})_l(\mathfrak{q}^{-i};\mathfrak{q})_l \tau^{1/\mathfrak{q}}_{n-g+i-l,i}(\mathfrak{q}^i x)}{\mathfrak{q}^{i(n+i-g-l)+\binom{g-i+l}{2}-n(g-i+l)}(\mathfrak{q};\mathfrak{q})_l}.$$

Finally, if we replace $\mathfrak{q}$ by $1/\mathfrak{q}$ in all $\mathfrak{q}$-shifted factorials $(a;\mathfrak{q})_l$, but not in polynomials $\tau^{1/\mathfrak{q}}_{n-g+i-l,i}(\mathfrak{q}^i x)$, and use formula (1.8.7) from [11] as well as identity (3.10), we obtain

$$\frac{(-1)^{g-i}\mathfrak{q}^{\binom{n}{2}-gn-\binom{n-g+i}{2}}(\mathfrak{q};\mathfrak{q})_g}{(\mathfrak{q};\mathfrak{q})_i(\mathfrak{q};\mathfrak{q})_{n-g+i}}\sum_{l=0}^{n-g+i}\frac{\mathfrak{q}^{-l}(\mathfrak{q}^{g-n-i};1/\mathfrak{q})_l(\mathfrak{q}^i;1/\mathfrak{q})_l}{(1/\mathfrak{q};1/\mathfrak{q})_l}\tau^{1/\mathfrak{q}}_{n-g+i-l,i}(\mathfrak{q}^i x),$$

which is nothing else than the right-hand term of equation (3.20). $\square$

Two important applications of the above Lemma 3.1 deal with the computations of the numbers μ^i_j, $i,j\geq 0$ (see formula (2.3)), and the polynomials $q^i_n(x)$, $n,i\geq 0$ (see formula (2.6)). First we compute the numbers μ^i_j. Indeed, using (3.4), (3.6), and relation $(\mathfrak{q};\mathfrak{q})_j(\mathfrak{q}^j a;\mathfrak{q})_l=(a;\mathfrak{q})_{j+l}$, we have

$$\mu^i_j=\int r_j\bar{\psi}_i d\mu=\int r^{\mathfrak{q}}_j r^{\mathfrak{q}}_i d\mu \tag{3.24}$$

$$=\sum_{l=0}^{i}(1-\mathfrak{q})^{i-l}\mathfrak{q}^{\binom{i-l}{2}-ij}\begin{bmatrix} j\\ i-l\end{bmatrix}\int\frac{(\mathfrak{q}-1)^{l+j}}{(\mathfrak{q};\mathfrak{q})_j(\mathfrak{q};\mathfrak{q})_l}(x;\mathfrak{q})_{l+j}d\mu$$

$$=\sum_{l=0}^{i}(1-\mathfrak{q})^{i-l}\mathfrak{q}^{\binom{i-l}{2}-ij}\begin{bmatrix} j\\ i-l\end{bmatrix}\begin{bmatrix} j+l\\ l\end{bmatrix}\mu^0_{j+l}. \tag{3.25}$$

Finally, from (3.5) by noticing that

$$r^{\mathfrak{q}}_i(x/\mathfrak{q}^{i-1})=\mathfrak{q}^{-\binom{i}{2}}r^{1/\mathfrak{q}}_i(x),$$

we rewrite (2.4) as follows:

$$q^i_n(x)=\sum_{k=0}^{n}(-1)^k\mathfrak{q}^{\binom{k}{2}}\mu^i_{n-k}r_k(x/\mathfrak{q}^{k-1}).$$

Hence, from (3.6) and (3.25), after cumbersome computation, we get

$$q^i_n(x)=(-1)^n\mathfrak{q}^{\binom{n}{2}-in}\sum_{g=0}^{n}(-1)^g\mu^0_{g+i}\mathfrak{q}^{\binom{g+1}{2}-gn}\begin{bmatrix} g+i\\ i\end{bmatrix}r^{\mathfrak{q}}_{n-g}(x/\mathfrak{q}^{n-g-i-1}). \tag{3.26}$$

4. Symmetry for $\mathfrak{q}$-Casorati Big $\mathfrak{q}$-Jacobi determinants

Here we use the normalized Big $\mathfrak{q}$-Jacobi polynomials $(P^{a,b,c;\mathfrak{q}}_n)_n$ with leading coefficient equals $\mathfrak{q}^{n^2}/(\mathfrak{q};\mathfrak{q})_n$ defined by

$$P^{a,b,c;\mathfrak{q}}_n(x)=\frac{\mathfrak{q}^{n^2}(a\mathfrak{q},c\mathfrak{q};\mathfrak{q})_n}{(ab\mathfrak{q}^{n+1},\mathfrak{q};\mathfrak{q})_n}\,{}_3\varphi_2\left(\begin{matrix}\mathfrak{q}^{-n},ab\mathfrak{q}^{n+1},x\\ a\mathfrak{q},c\mathfrak{q}\end{matrix}\ ;\mathfrak{q}\ ,\ \mathfrak{q}\right). \tag{4.1}$$

See (3.1) and [11], pp. 438-443, for some known relations involved Big $\mathfrak{q}$-Jacobi polynomials contained in this section.

According to the strategy explained in Remark 2.2, we have to identify the polynomials q^i_n in Theorem 2.1 when the orthogonal polynomials $(p_n)_n$ are the Big $\mathfrak{q}$-Jacobi polynomials. To do that, we first calculate the sequence $(\mu^0_n)_n$ (see (2.3)).

For $0 < a\mathfrak{q} < 1$, $0 \leq b\mathfrak{q} < 1$, and $c < 0$ the polynomial sequence $(P_n^{a,b,c;\mathfrak{q}})_n$ verifies the orthogonality relation

$$\int_{c\mathfrak{q}}^{a\mathfrak{q}} P_n^{a,b,c;\mathfrak{q}}(x)P_m^{a,b,c;\mathfrak{q}}(x)\omega^{a,b,c;\mathfrak{q}}(x)d_{\mathfrak{q}}x = \frac{(1-ab\mathfrak{q})(-ac)^n}{(1-ab\mathfrak{q}^{2n+1})}\frac{(a\mathfrak{q},b\mathfrak{q},c\mathfrak{q},abc^{-1}\mathfrak{q};\mathfrak{q})_n}{(\mathfrak{q},ab\mathfrak{q};\mathfrak{q})_n(ab\mathfrak{q}^{n+1};\mathfrak{q})_n^2}\mathfrak{q}^{\binom{n}{2}+2n(n+1)}\delta_{n,m}, \tag{4.2}$$

with respect to the normalized weight function

$$\omega^{a,b,c;\mathfrak{q}}(x) = \frac{(a\mathfrak{q},b\mathfrak{q},c\mathfrak{q},abc^{-1}\mathfrak{q};\mathfrak{q})_\infty}{a\mathfrak{q}(1-\mathfrak{q})(\mathfrak{q},ab\mathfrak{q}^2,a^{-1}c,ac^{-1}\mathfrak{q};\mathfrak{q})_\infty}\frac{(a^{-1}x,c^{-1}x;\mathfrak{q})_\infty}{(x,bc^{-1}x;\mathfrak{q})_\infty}.$$

By using (3.2), the numbers (3.24) associated with this weight function for $i=0$ and $j=0,1,2,\ldots$, are given by

$$\begin{aligned}\mu_j^0 &= \frac{(\mathfrak{q}-1)^j}{(\mathfrak{q};\mathfrak{q})_j}\int_{c\mathfrak{q}}^{a\mathfrak{q}}(x;q)_j\omega^{a,b,c;\mathfrak{q}}(x)d_{\mathfrak{q}}x \\ &= \frac{(-1)^j(1-\mathfrak{q})^{j-1}(a\mathfrak{q},b\mathfrak{q},c\mathfrak{q},abc^{-1}\mathfrak{q};\mathfrak{q})_\infty}{a\mathfrak{q}^{j+1}(\mathfrak{q};\mathfrak{q})_j(q,ab\mathfrak{q}^2,a^{-1}c,ac^{-1}\mathfrak{q};\mathfrak{q})_\infty}\int_{c_j\mathfrak{q}}^{a_j\mathfrak{q}}\frac{(a_j^{-1}y,c_j^{-1}y;\mathfrak{q})_\infty}{(y,bc_j^{-1}y;\mathfrak{q})_\infty}d_{\mathfrak{q}}y,\end{aligned}$$

where $a_j = a\mathfrak{q}^j$, $c_j = c\mathfrak{q}^j$, and $y = x\mathfrak{q}^j$. Hence, from (4.2) we obtain the following explicit expression:

$$\mu_j^0 = \frac{(\mathfrak{q}-1)^j(a\mathfrak{q},c\mathfrak{q};\mathfrak{q})_j}{(ab\mathfrak{q}^2,\mathfrak{q};\mathfrak{q})_j}, \quad j=0,1,2,\ldots. \tag{4.3}$$

We can now show that the polynomials q_n^i, $n,i \geq 0$, in Theorem 2.1, formula (2.4), are up to a change of variable, again Big $\mathfrak{q}$-Jacobi polynomials. Indeed, by substituting (4.3) into (3.26) and expanding the resulting expression in the basis $(x;\mathfrak{q})_k$, $k=0,1,\ldots,n$, one gets, after a careful identification of the given coefficients in such a series expansion and comparing them with those from (4.1), the following relation:

$$q_n^i(x) = (1-\mathfrak{q})^n\mathfrak{q}^{-in+\binom{n}{2}}\mu_i^0 P_n^{a\mathfrak{q}^{n+1+i},b\mathfrak{q}^{n+1},c\mathfrak{q}^{n+1+i};1/\mathfrak{q}}(\mathfrak{q}^i x). \tag{4.4}$$

In the sequel we will need some formulas involving the Big $\mathfrak{q}$-Jacobi polynomials which will be used in the proof of Theorem 1.1. Let us start with an important consequence of the orthogonality relation (4.2), namely the three-term recurrence relation

$$\begin{aligned}(x-1)P_n^{a,b,c;\mathfrak{q}}(x) = A_nP_{n+1}^{a,b,c;\mathfrak{q}}(x) - (B_n+C_n)P_n^{a,b,c;\mathfrak{q}}(x) \\ + A_{n-1}^{-1}B_{n-1}C_nP_{n-1}^{a,b,c;\mathfrak{q}}(x),\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}A_n &= \frac{(1-\mathfrak{q}^{n+1})}{\mathfrak{q}^{2n+1}}, \\ B_n := B_n^{a,b,c;\mathfrak{q}} &= \frac{(1-a\mathfrak{q}^{n+1})(1-ab\mathfrak{q}^{n+1})(1-c\mathfrak{q}^{n+1})}{(1-ab\mathfrak{q}^{2n+1})(1-ab\mathfrak{q}^{2n+2})}, \\ C_n := C_n^{a,b,c;\mathfrak{q}} &= -ac\mathfrak{q}^{n+1}\frac{(1-\mathfrak{q}^n)(1-abc^{-1}\mathfrak{q}^n)(1-b\mathfrak{q}^n)}{(1-ab\mathfrak{q}^{2n})(1-ab\mathfrak{q}^{2n+1})}.\end{aligned}$$

Among other relations for the Big $\mathfrak{q}$-Jacobi polynomials, we have that for the $\mathfrak{q}$-derivative (1.3) there holds

$$D_{\mathfrak{q}}\left(P_n^{a,b,c;\mathfrak{q}}(x/\mathfrak{q})\right) = \frac{q^n}{(q-1)} P_{n-1}^{a\mathfrak{q},b\mathfrak{q},c\mathfrak{q};\mathfrak{q}}(x),$$

or equivalently (see the forward shift operator in [11] p. 439),

$$P_n^{a,b,c;\mathfrak{q}}(x) - P_n^{a,b,c;\mathfrak{q}}(x/\mathfrak{q}) = -\mathfrak{q}^{n-1} x P_{n-1}^{a\mathfrak{q},b\mathfrak{q},c\mathfrak{q};\mathfrak{q}}(x). \tag{4.6}$$

Another useful relation is the following forward shift-type operator for the $\mathfrak{q}$-Big Jacobi polynomials:

$$\mathfrak{q}^{n+1} P_{n+1}^{a/\mathfrak{q},b,c/\mathfrak{q};\mathfrak{q}}(x/\mathfrak{q}) - P_{n+1}^{a,b,c;\mathfrak{q}}(x) = \frac{a\mathfrak{q}^{3n+2}(b\mathfrak{q}^{n+1}-1)(ab\mathfrak{q}^{n+1}-c)}{(ab\mathfrak{q}^{2n+1};\mathfrak{q})_2} P_n^{a,b,c;\mathfrak{q}}(x). \tag{4.7}$$

For showing (4.7) we consider the Fourier expansion

$$P_{n+1}^{a/\mathfrak{q},b,c/\mathfrak{q};\mathfrak{q}}(x/\mathfrak{q}) = \sum_{k=0}^{n+1} \alpha_k^{a,b,c;\mathfrak{q}} P_k^{a,b,c;\mathfrak{q}}(x),$$

where

$$\alpha_k^{a,b,c;\mathfrak{q}} \|P_k^{a,b,c;\mathfrak{q}}\|^2 = \int_{cq}^{aq} P_{n+1}^{a/\mathfrak{q},b,c/\mathfrak{q};\mathfrak{q}}(x/\mathfrak{q}) P_k^{a,b,c;\mathfrak{q}}(x) \omega^{a,b,c;\mathfrak{q}}(x) d_{\mathfrak{q}}x.$$

By using the relation

$$\omega^{a,b,c;\mathfrak{q}}(x) = \frac{(1-ab\mathfrak{q})}{\mathfrak{q}(1-a)(1-c)}(1-x/\mathfrak{q})\omega^{a/\mathfrak{q},b,c/\mathfrak{q};\mathfrak{q}}(x/\mathfrak{q}), \tag{4.8}$$

and (4.5), we get that $\alpha_k^{a,b,c;\mathfrak{q}}$ vanishes for $k = 0, \ldots, n-1$.

Observe that the leading coefficients of $P_n^{a/\mathfrak{q},b,c/\mathfrak{q};\mathfrak{q}}(x/\mathfrak{q})$ and $P_n^{a,b,c;\mathfrak{q}}(x)$ differ in a factor $\mathfrak{q}^{-n}$; hence, for $k = n+1$ we have

$$\begin{aligned}
\alpha_{n+1}^{a,b,c;\mathfrak{q}} &= \|P_{n+1}^{a,b,c;\mathfrak{q}}\|^{-2} \int_{cq}^{aq} P_{n+1}^{a/\mathfrak{q},b,c/\mathfrak{q};\mathfrak{q}}(x/\mathfrak{q}) P_{n+1}^{a,b,c;\mathfrak{q}}(x) \omega^{a,b,c;\mathfrak{q}}(x) d_{\mathfrak{q}}x \\
&= \mathfrak{q}^{-n-1}.
\end{aligned}$$

For $k = n$ we use once more (4.8) and (4.5), i.e.,

$$\begin{aligned}
\alpha_n^{a,b,c;\mathfrak{q}} &= \|P_n^{a,b,c;\mathfrak{q}}\|^{-2} \int_{cq}^{aq} P_{n+1}^{a/\mathfrak{q},b,c/\mathfrak{q};\mathfrak{q}}(x/\mathfrak{q}) P_n^{a,b,c;\mathfrak{q}}(x) \omega^{a,b,c;\mathfrak{q}}(x) d_{\mathfrak{q}}x \\
&= \frac{\|P_n^{a/\mathfrak{q},b,c/\mathfrak{q};\mathfrak{q}}(x/\mathfrak{q})\|^2}{\|P_n^{a,b,c;\mathfrak{q}}\|^2} \frac{\mathfrak{q}^{n+1}(1-ab\mathfrak{q})}{\mathfrak{q}(1-a)(1-c)} A_n^{-1} B_n^{a/\mathfrak{q},b,c/\mathfrak{q};\mathfrak{q}} C_{n+1}^{a/\mathfrak{q},b,c/\mathfrak{q};\mathfrak{q}} \\
&= \frac{\mathfrak{q}^{2n+1} a(ab\mathfrak{q}^{n+1}-c)(b\mathfrak{q}^{n+1}-1)}{(1-ab\mathfrak{q}^{2n+1})(1-ab\mathfrak{q}^{2n+2})},
\end{aligned}$$

which completes the proof of (4.7).

We are now ready to prove the main result of this paper, that is, Theorem 1.1 in the Introduction.

Proof of Theorem 1.1. Consider the $\mathfrak{q}$-Casorati Big $\mathfrak{q}$-Jacobi determinant

$$\mathcal{P}_{n,m,x}^{a,b,c;\mathfrak{q}} = x^{\binom{m}{2}} \det\left(P_{m+j-1}^{a,b,c;\mathfrak{q}}(x/\mathfrak{q}^{i-1})\right)_{i,j=1}^n.$$

We will use Theorem 2.1. We have already identified the polynomials q_n^i, $n, i \geq 0$, as Big q-Jacobi polynomials (4.4) (with different parameters and up to a change of variable),

$$q_n^i(x) = (1-\mathsf{q})^n \mathsf{q}^{-in+\binom{n}{2}} \mu_i^0 P_n^{a\mathsf{q}^{n+1+i}, b\mathsf{q}^{n+1}, c\mathsf{q}^{n+1+i}; 1/\mathsf{q}}(\mathsf{q}^i x). \tag{4.9}$$

This gives that the determinant in the right-hand side of (2.5) is, except for a constant depending on n and m but not on x, equal to the determinant of the matrix

$$(\mathsf{q}^{-ij} P_{n+i-1}^{a\mathsf{q}^{n+i+j-1}, b\mathsf{q}^{n+i}, c\mathsf{q}^{n+i+j-1}; 1/\mathsf{q}}(\mathsf{q}^{j-1}x))_{i,j=1}^m. \tag{4.10}$$

We now start with the matrix

$$(P_{n+i-1}^{a\mathsf{q}^{n+m}, b\mathsf{q}^{n+m}, c\mathsf{q}^{n+m}; 1/\mathsf{q}}(\mathsf{q}^{j-1}x))_{i,j=1}^m, \tag{4.11}$$

which coincides with the transpose of the matrix in the determinant of the right-hand side of (1.4). Now, aiming to transform matrix (4.11), we will perform on it some elementary column operations (summing up columns) and use formula (4.6) in which we replace q by $1/\mathsf{q}$, a by $a\mathsf{q}^{n+m}$, b by $b\mathsf{q}^{n+m}$, and c by $c\mathsf{q}^{n+m}$. Thus, addition of -1 times column $m-1$ of (4.11) to column m produces, by using (4.6), a new column m as follows:

$$(x\mathsf{q}^{-n+m-i} P_{n+i-2}^{a\mathsf{q}^{n+m-1}, b\mathsf{q}^{n+m-1}, c\mathsf{q}^{n+m-1}; 1/\mathsf{q}}(\mathsf{q}^{m-2}x))_{i=1}^m.$$

This is a column replacement operation. Similarly, addition of -1 times column $m-2$ of (4.11) to column $m-1$ produces

$$(x\mathsf{q}^{-n+m-i-1} P_{n+i-2}^{a\mathsf{q}^{n+m-1}, b\mathsf{q}^{n+m-1}, c\mathsf{q}^{n+m-1}; 1/\mathsf{q}}(\mathsf{q}^{m-3}x))_{i=1}^m.$$

Then, we continue with column replacement operations from left to right on the matrix (4.11) ending with the addition of -1 times column 1 of (4.11) to column 2, which produces

$$(x\mathsf{q}^{-(n+i-2)} P_{n+i-2}^{a\mathsf{q}^{n+m-1}, b\mathsf{q}^{n+m-1}, c\mathsf{q}^{n+m-1}; 1/\mathsf{q}}(x))_{i=1}^m.$$

Now, we repeat the above process of column replacement operations involving columns $m, m-1, \ldots, 2$, as follows: Add $-\mathsf{q}$ times column $m-1$ to column m, add $-\mathsf{q}$ times column $m-2$ to column $m-1$, ending with the addition of $-\mathsf{q}$ times column 2 to column 3. After $\binom{m}{2}$ operations we end up with the determinant of matrix (4.11) equal to (up to a power of q depending on n and m)

$$x^{\binom{m}{2}} \det(\mathsf{q}^{-ij} P_{n+i-j}^{a\mathsf{q}^{n+m-j+1}, b\mathsf{q}^{n+m-j+1}, c\mathsf{q}^{n+m-j+1}; 1/\mathsf{q}}(x))_{i,j=1}^m. \tag{4.12}$$

The next transformation of (4.12) will consist of a successive use of formula (4.7) and summing up rows. Similarly to the above process, we perform $\binom{m}{2}$ row replacement operations. Indeed, we end up with the determinant of matrix (4.12), up to a constant factor depending on q, n, and m, equal to

$$\det(\mathsf{q}^{-ij} P_{n+m-j}^{a\mathsf{q}^{n+2m-i-j+1}, b\mathsf{q}^{n+m-j+1}, c\mathsf{q}^{n+2m-i-j+1}; 1/\mathsf{q}}(x\mathsf{q}^{m-i}))_{i,j=1}^m.$$

More precisely, this determinant is, up to a power of q depending on n and m, equal to

$$\det(\mathsf{q}^{-(m-i+1)(m-j+1)} P_{n+m-j}^{a\mathsf{q}^{n+2m-i-j+1}, b\mathsf{q}^{n+m-j+1}, c\mathsf{q}^{n+2m-i-j+1}; 1/\mathsf{q}}(x\mathsf{q}^{m-i}))_{i,j=1}^m.$$

The matrix of this last determinant is equal to (4.10) by changing i by $m-j+1$ and j by $m-i+1$. Up to a sign, this change does not modify the value of the determinant.

A careful computation of all aforementioned constant factors yields (1.4), which completes the proof of Theorem 1.1. □

5. Limit relations

It is well known that under some restrictions on the parameters of the Big $\mathfrak{q}$-Jacobi polynomials we relate these polynomials to other families of classical orthogonal polynomials (see [11], pp. 441-443). Thus, analogous symmetry relations to those given in Theorem 1.1 involving the corresponding $\mathfrak{q}$-Casorati determinant can be obtained. Indeed, if we set $b = -a^{-1}cd^{-1}$ (with $d > 0$) in the expression (4.1) of the Big $\mathfrak{q}$-Jacobi polynomials and take the limit $c \to -\infty$ we obtain the $\mathfrak{q}$-Meixner polynomials

$$\lim_{c\to-\infty} P_n^{a,c/ad,c;\mathfrak{q}}(\mathfrak{q}^{-x}) = M_n^{a,d;\mathfrak{q}}(\mathfrak{q}^{-x}). \tag{5.1}$$

In particular, the $\mathfrak{q}$-Meixner polynomials (see relations in [11], pp. 488-490) are defined by

$$M_n^{a,d;\mathfrak{q}}(x) = \frac{(-d)^n}{(\mathfrak{q};\mathfrak{q})_n}\sum_{j=0}^{n}\frac{(\mathfrak{q}^{-n};\mathfrak{q})_j(x;\mathfrak{q})_j}{(a\mathfrak{q};\mathfrak{q})_j(\mathfrak{q};\mathfrak{q})_j}\left(\frac{-\mathfrak{q}^{n+1}}{d}\right)^j, \quad d > 0. \tag{5.2}$$

Define the $\mathfrak{q}$-Casorati $\mathfrak{q}$-Meixner determinant

$$\mathcal{M}_{n,m,x}^{a,d;\mathfrak{q}} = x^{\binom{m}{2}}\det\left(M_{m+j-1}^{a,d;\mathfrak{q}}(x/\mathfrak{q}^{i-1})\right)_{i,j=1}^{n}.$$

Hence, taking into account (1.4), (5.1), and (5.2) the symmetry property

$$\mathcal{M}_{n,m,x}^{a,d;\mathfrak{q}} = (-1)^{nm}\mathfrak{q}^{mn^2+nm^2-mn}\mathcal{M}_{m,n,x}^{a\mathfrak{q}^{n+m},d/\mathfrak{q}^{n+m};1/\mathfrak{q}}, \quad n,m \geq 0, \quad \mathfrak{q} \neq 1,$$

holds.

The $\mathfrak{q}$-Charlier polynomials (see relations in [11], pp. 530-533) given by

$$C_n^{d;\mathfrak{q}}(x) = \frac{(-d)^n}{(\mathfrak{q};\mathfrak{q})_n}\sum_{j=0}^{n}\frac{(\mathfrak{q}^{-n};\mathfrak{q})_j(x;\mathfrak{q})_j}{(\mathfrak{q};\mathfrak{q})_j}(-\mathfrak{q}^{n+1}/d)^j, \quad d > 0,$$

can easily be obtained from the $\mathfrak{q}$-Meixner polynomials as follows:

$$M_n^{0,d;\mathfrak{q}}(x) = C_n^{d;\mathfrak{q}}(x). \tag{5.3}$$

Therefore, from the above limit (5.1) and settings in (5.3), we get the following symmetry,

$$\mathcal{C}_{n,m,x}^{d;\mathfrak{q}} = (-1)^{nm}\mathfrak{q}^{mn^2+nm^2-mn}\mathcal{C}_{m,n,x}^{d/\mathfrak{q}^{n+m};1/\mathfrak{q}}, \quad n,m \geq 0, \quad \mathfrak{q} \neq 1,$$

for the $\mathfrak{q}$-Casorati $\mathfrak{q}$-Charlier determinant

$$\mathcal{C}_{n,m,x}^{d;\mathfrak{q}} = x^{\binom{m}{2}}\det\left(C_{m+j-1}^{d;\mathfrak{q}}(x/\mathfrak{q}^{i-1})\right)_{i,j=1}^{n}.$$

Furthermore, the $\mathfrak{q}$-Laguerre polynomials (see relations in [11], pp. 522-525) defined by

$$L_n^{\alpha;\mathfrak{q}}(x) = \frac{(-1)^n(\mathfrak{q}^{\alpha+1};\mathfrak{q})_n}{q^{\alpha n}(\mathfrak{q};\mathfrak{q})_n}\sum_{j=0}^{n}\frac{(\mathfrak{q})^{\binom{j}{2}}(\mathfrak{q}^{-n};\mathfrak{q})_j}{(\mathfrak{q}^{\alpha+1};\mathfrak{q})_j(\mathfrak{q};\mathfrak{q})_j}(\mathfrak{q}^{n+\alpha+1}x)^j, \quad \alpha \neq -1,-2,\cdots,$$

can be obtained from the $\mathfrak{q}$-Meixner polynomials by setting $a = \mathfrak{q}^\alpha$ and substituting $\mathfrak{q}^{-x}$ by $d\mathfrak{q}^\alpha x$ in relation (5.3) and then taking the limit $d \to \infty$. Hence, under these settings the following symmetry relation,

$$\mathcal{L}^{\alpha;\mathfrak{q}}_{n,m,x} = (-1)^{nm}\mathfrak{q}^{m^2n+mn^2-nm}\mathcal{L}^{-\alpha-n-m;1/\mathfrak{q}}_{m,n,x}, \quad n, m \geq 0, \quad \mathfrak{q} \neq 1,$$

yields, where

$$\mathcal{L}^{\alpha;\mathfrak{q}}_{n,m,x} = x^{\binom{m}{2}} \det \left(L^{\alpha;\mathfrak{q}}_{m+j-1}(x/\mathfrak{q}^{i-1}) \right)^n_{i,j=1}$$

is the $\mathfrak{q}$-Casorati $\mathfrak{q}$-Laguerre determinant.

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