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Zeros of para–orthogonal polynomials and linear spectral transformations on the unit circle

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Abstract We study the interlacing properties of zeros of para–orthogonal polynomi-als associated with a nontrivial probability measure supported on the unit circle $d\mu$ and para–orthogonal polynomials associated with a modification of $d\mu$ by the addi-tion of a pure mass point, also called Uvarov transformation. Moreover, as a direct consequence of our approach, we present some results related with the Christoffel transformation.

Keywords Para–orthogonal polynomials · Interlacing of zeros · Uvarov transformation · Christoffel transformation

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1 Introduction

We denote by $\mathbb{P} := \mathbb{C}[z]$ the linear space of polynomials with complex coefficients and $\mathbb{P}_n$ the linear subspace of polynomials of degree, at most, n, while $\mathbb{P}_{-1} \equiv \{0\}$ is the trivial subspace. Let $d\mu$ be a nontrivial (i.e., with infinite support) probability measure supported on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C}; |z| = 1\}$ parametrized by $z = e^{i\theta}$, $\theta \in [0, 2\pi)$. By using the Gram–Schmidt orthogonalization procedure we obtain a sequence of orthonormal polynomials, $\{\varphi_n\}_{n\geq 0}$, with respect to $d\mu$, that is, satisfying

$$\int \varphi_n(z)\overline{\varphi_m(z)}d\mu(z) = \delta_{n,m},$$

where $\varphi_n \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}$ is given by

$$\varphi_n(z) := \kappa_n z^n + \text{lower degree terms}, \quad \kappa_n > 0.$$

Here $\delta_{n,m}$ denotes the Kronecker delta. The orthogonality conditions determine the orthonormal sequence up to an unimodular factor and, finally, the conditions $\kappa_n > 0$ uniquely determine the sequence. The associated monic orthogonal polynomials are

$$\Phi_n(z) = \kappa_n^{-1}\varphi_n(z) = z^n + \text{lower degree terms}.$$

Note that $\mathbf{k}_n := \|\Phi_n\|^2 = \kappa_n^{-2}$, where $\|\cdot\|$ is the $\mathrm{L}^2_{d\mu}$–norm. For obvious reasons, the above polynomials are known as *orthogonal polynomials on the unit circle* (OPUC, in short); see [23, 25, 40, 42, 43].

One of the most important algebraic properties of the OPUC, $\{\Phi_n\}_{n\geq 0}$, is the Szegő forward recurrence formula (named after [43, Thm. 11.4.2]). This means that

$$\Phi_0 :\equiv 1, \quad \Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n\Phi_n^*(z), \quad n \geq 0, \tag{1}$$

where $\{\alpha_n\}_{n\geq 0} \in \mathbb{D}$, $\alpha_n = -\overline{\Phi_{n+1}(0)}$, $\mathbb{D} := \{z \in \mathbb{C}; |z| < 1\}$, are the so–called Verblunsky coefficients (comments about this notation can be found in [40]) and for $f \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}$, $f^*(z) := z^n\overline{f}(z^{-1})$.

The polynomials

$$K_n(z, \zeta) := \sum_{k=0}^{n} \varphi_k(z)\overline{\varphi_k(\zeta)}, \quad n \geq 0,$$

are the reproducing kernels associated with the orthogonality measure $d\mu$. From the Szegő recurrence (1), one obtains the Christoffel–Darboux formula (named after [13, 14]), i.e.,

$$K_n(z, \zeta) = \frac{\Phi_{n+1}^*(z)\overline{\Phi_{n+1}^*(\zeta)} - \Phi_{n+1}(z)\overline{\Phi_{n+1}(\zeta)}}{\mathbf{k}_{n+1}(1 - \overline{\zeta}z)}. \tag{2}$$

In the last years, a weakened form of orthogonality, called para–orthogonality, has been introduced in the literature in the framework of quadrature rules with nodes on the unit circle. We say that the polynomials $\{\Phi_n(\cdot, \beta_n)\}_{n\geq 0}$ are invariant *para–orthogonal polynomials on the unit circle* (POPUC, in short) associated with the sequence of OPUC $\{\Phi_n\}_{n\geq 0}$, if there exist complex numbers $\eta_n \in \mathbb{C}$ and $\beta_n \in \partial\mathbb{D}$ such that

$$\Phi_n(z, \beta_n) := \eta_n(\Phi_n(z) + \beta_n\Phi_n^*(z)). \tag{3}$$

This definition is supported by the characterization of the POPUC (see [31, Thm. 6.1]). Notice that the zeros of the polynomial defined by (3) are located on $\partial\mathbb{D}$. According to our definition the POPUC are not necessarily monic. Since we are interested only in the behavior of the zeros, henceforth we will assume that $\eta_n \equiv 1$.

The earliest reference to the existence of invariant POPUC is found in a paper by Geronimus [24, Thm. III]. In a more general setting the POPUC (not necessarily invariant) were introduced by Jones, Njåstad, and Thron at the end of the 1980's. From now on, we will consider only invariant POPUC, although we refer to them as POPUC.

After their formal introduction, the para–orthogonality theory was significantly enriched from both, the theoretical and practical points of view. The essential role in the development of quadrature formulas (see [31, 38] among others), their relation with discrete systems analysis, digital signal processing and linear least-squares estimation (see [17–20]), the use of their zeros instead of zeros of OPUC in frequency analysis problems (see [16]) and their appearance as in the isometric Arnoldi minimization problem (see [29]) represent some of the best known applications of POPUC. On the other hand, the works of Cantero, Moral, and Velázquez (see [8]), and Golinskii (see [28]) reveal the similarity between the behavior of zeros of POPUC and zeros of *orthogonal polynomials on the real line* (OPRL, in short), see also [44]. In the previous results, the basic tool is the Christoffel–Darboux formula (2). Some of these results are studied by Simon (see [41]) using the connection of POPUC with CMV matrices and the theory of rank one perturbation of unitary matrices.

The main aim of this contribution is to show that under some perturbations of the measure, the similarities between the behavior of zeros of certain POPUC and zeros of OPRL do not always hold. More specifically, the zeros of POPUC associated with the Uvarov and Christoffel transformations are not necessarily interlacing with the zeros of the original sequence of POPUC. In this contribution, we obtain conditions in order to preserve the interlacing of zeros. One of the motivations for studying the interlacing property under transformations of the measure concerns the fact that given two monic consecutive polynomials $\Psi_n(z)$ and $\Psi_{n+1}(z)$ whose zeros are simple and strictly interlacing on $\partial\mathbb{D}$, there exists a measure $d\widetilde{\mu}$ supported on $\partial\mathbb{D}$ for which they are POPUC (see [9]). Moreover, if these polynomials have at most one zero in common, the previous statement is also true. All such measures have the same $\{\Psi_j\}_{j=0}^{n}$. In particular, any polynomial Ψ_n with distinct zeros on $\partial\mathbb{D}$ is a POPUC for some measure $d\mu$ and $\Psi_{n-1}(z), \Psi_{n-2}(z), \ldots$ are not determined uniquely. In the context of the works of Delsarte and Genin [17–20], the interlacing property says that we can obtain a new sequence of *singular predictor polynomials* (special POPUC given by a three term recurrence relation) using the perturbed and original POPUC. The singular predictor polynomials are used to replace the OPUC in several signal problems and provide new techniques for the interpolation problem, the retrieval of harmonics problem and Toeplitz systems. Moreover, these polynomials are related with unitary Hessenberg matrices and, therefore, our interlacing conditions could be used also to obtain new results in the perturbation and interlace theory of unitary eigenvalues problems (see [4, 22]).

The manuscript is organized as follows. Section 2 presents some preliminaries and basic background concerning POPUC. In Section 3 we proceed with the study of the Uvarov transformation. Section 4 is devoted to the study of some special cases of para–orthogonal polynomials associated with the Christoffel transformation. We have included some numerical examples associated with the Bernstein–Szegő measure and rational modifications of the Lebesgue measure in order to illustrate our results.

2 Preliminary results

First, we need to define what we mean by 'zeros interlace on $\partial\mathbb{D}$ and, hence, to introduce the concept of ordered cycle for a set of points on the unit circle. If ω is a fixed real number, then a vector of different complex numbers on $\partial\mathbb{D}$, $(e^{i\theta_1}, \dots, e^{i\theta_n})$, is said to be *cyclicly ordered* if

$$\omega < \theta_1 < \cdots < \theta_n < \omega + 2\pi.$$

This means that for two different points on $\partial\mathbb{D}$, $e^{i\theta_1}$ and $e^{i\theta_2}$ with $\theta_1, \theta_2 \in [\omega, \omega + 2\pi)$, we have an order relation such that

$$e^{i\theta_1} \prec e^{i\theta_2} \quad \text{if and only if} \quad \theta_1 < \theta_2.$$

Let $(e^{i\theta_{n,1}}, \dots, e^{i\theta_{n,n}})$ and $(e^{i\psi_{n,1}}, \dots, e^{i\psi_{n,n}})$ be two cyclicly ordered sets of zeros corresponding to the polynomials $f_n(z)$ and $g_n(z)$, respectively. We say that the zeros of $f_n(z)$ and $g_n(z)$ *strictly interlace* on $\partial\mathbb{D}$ if they can be numbered such that there exists a number $\widetilde{\omega}$, so that

$$\widetilde{\omega} < \theta_{n,1} < \psi_{n,1} < \cdots < \theta_{n,n} < \psi_{n,n} < \widetilde{\omega} + 2\pi. \tag{4}$$

Note that the previous definition also includes the case when the role of $\theta_{n,k}$ and $\psi_{n,k}$, $k = 1, 2, \dots, n$, is reversed. This definition can be naturally extended to two cyclicly ordered sets of zeros with different number of elements.

Let us now state and prove the main results to be used in the sequel.

Lemma 1 *Let $f_n(z)$ be an arbitrary polynomial with simple zeros on $\partial\mathbb{D}$, then $f_n(z)$ is a POPUC with respect to some nontrivial probability measure supported on $\partial\mathbb{D}$.*

Proof As the zeros of $f_n(z)$ lie on $\partial\mathbb{D}$, then $f_n(z) = \sigma z^n \overline{f}_n(1/z)$, $\sigma \in \partial\mathbb{D}$. By differentiation, we get

$$f_n(z) = \frac{1}{n} z f_n'(z) + \sigma \frac{1}{n} z^{n-1} \overline{f'}_n(1/z).$$

Set $h_{n-1}(z) = (1/n) f_n'(z)$, the above expression can be written as

$$f_n(z) = z h_{n-1}(z) + \sigma h_{n-1}^*(z).$$

Combining the Gauss–Lucas theorem [36, Thm. 2.1.1] and the Bonsall–Marden lemma [3], we conclude that the zeros of $h_{n-1}(z)$ lie on $\mathbb{D}$. Moreover, by Geronimus' theorem [24, Thm. I.], $h_{n-1}(z)$ is a OPUC with respect to some nontrivial probability measure supported on $\partial\mathbb{D}$, and, consequently, $f_n(z)$ is the corresponding POPUC [31, Thm. 6.1]. □

Lemma 2 *Let $f_n(z)$ and $g_n(z)$ be two polynomials of exact degree n whose zeros strictly interlace on $\partial\mathbb{D}$. If the polynomial*

$$f_n(z) + c g_n(z), \quad c \in \mathbb{R} \setminus \{0\}, \tag{5}$$

has n zeros on $\partial\mathbb{D}$, then they are strictly interlacing with the zeros of $f_n(z)$ and $g_n(z)$.

Proof As we are interested in the zeros, there is no loss of generality if we consider an appropriated normalization of the polynomials $f_n(z)$ and $g_n(z)$. From Lemma 1, $f_n(z)$ and $g_n(z)$ are POPUC with respect to two different nontrivial probability measures supported on $\partial\mathbb{D}$. Hence, we can assume that

$$\begin{aligned} f_n(z) &= \overline{\beta} P_n(z) - \beta P_n^*(z), \quad \beta \in \mathbb{C} \setminus \{0\}, \\ g_n(z) &= \overline{\alpha} Q_n(z) - \alpha Q_n^*(z), \quad \alpha \in \mathbb{C} \setminus \{0\}, \end{aligned}$$

where $P_n(z)$ and $Q_n(z)$ are the OPUC associated with $f_n(z)$ and $g_n(z)$, respectively. That is, we consider sequences of normalized to (-1)-invariant POPUC, i.e., $f_n^*(z) = -f_n(z)$ and $g_n^*(z) = -g_n(z)$. Note that $f_n(z)$ and $g_n(z)$ are not just "any polynomial with simple zeros on the unit circle".

Let us introduce two auxiliary functions

$$\widetilde{f}_n(\theta) := \frac{f_n(z)}{i z^{n/2}}, \quad \widetilde{g}_n(\theta) := \frac{g_n(z)}{i z^{n/2}},$$

where $(re^{i\theta})^{1/2} = \sqrt{r} e^{i\theta/2}$, $r > 0$, and $\theta \in (\widetilde{\omega}, \widetilde{\omega} + 2\pi)$. Clearly, $\widetilde{f}_n(\theta)$ and $\widetilde{g}_n(\theta)$ are real–valued C^∞ functions defined on $(\widetilde{\omega}, \widetilde{\omega} + 2\pi)$ and, by definition they have the same number of zeros on $(\widetilde{\omega}, \widetilde{\omega} + 2\pi)$ as $f_n(z)$ and $g_n(z)$ on $\partial\mathbb{D}$, respectively. Moreover, if one denotes the zeros of $\widetilde{f}_n(\theta)$ (resp. $\widetilde{g}_n(\theta)$) by $x_{n,k}$ (resp. $y_{n,k}$), on the account of the interlacing property of the zeros of $f_n(z)$ and $g_n(z)$ on $\partial\mathbb{D}$, we have that the zeros of $\widetilde{f}_n(\theta)$ and $\widetilde{g}_n(\theta)$ satisfy

$$\widetilde{\omega} < y_{n,n} < x_{n,n} < \cdots < y_{n,1} < x_{n,1} < \widetilde{\omega} + 2\pi, \tag{6}$$

or in the reverse order.

Now, let us define a function $\widetilde{h}_n(\theta)$ as follows

$$\widetilde{h}_n(\theta) := \frac{f_n(z) + c g_n(z)}{i z^{n/2}} = \widetilde{f}_n(\theta) + c\widetilde{g}_n(\theta),$$

where its zeros are denoted by $t_{n,k}$, $k = 1, 2, \ldots, n$. Notice that $f_n(z) + c g_n(z)$ is a polynomial of degree at most n and the number of their zeros on $\partial\mathbb{D}$ is exactly the same as the number of zeros of $\widetilde{h}_n(\theta)$ in $(\widetilde{\omega}, \widetilde{\omega} + 2\pi)$. Since $f_n(z) + c g_n(z)$ cannot have more than n zeros, the number of zeros of $\widetilde{h}_n(\theta)$ in $(\widetilde{\omega}, \widetilde{\omega} + 2\pi)$ cannot exceed n. Without restriction of generality we can also assume

$$\widetilde{f}_n(\widetilde{\omega} + 2\pi) > 0, \quad \widetilde{g}_n(\widetilde{\omega} + 2\pi) > 0, \quad c > 0.$$

Thus, if (6) holds,

$$\operatorname{sgn} \widetilde{h}_n(y_{n,k}) = \operatorname{sgn} \widetilde{f}_n(y_{n,k}) = (-1)^k,$$
$$\operatorname{sgn} \widetilde{h}_n(x_{n,k}) = \operatorname{sgn} \widetilde{g}_n(x_{n,k}) = (-1)^{k+1},$$

and the lemma is proved. □

Remark 1 An illustration of the comments about the normalization in the proof of Lemma 2 can be shown through a simple example. Set $f_2(z) = (z-1)(z-i)$, a (-i)-invariant polynomial, i.e., $f_2^*(z) = -i f_2(z)$. It is clear that $\widetilde{f_2}(\theta)$ is not a real-valued function. Since we are interested in the zeros, there is no loss of generality if we consider the polynomial $f_2(z)$ normalized to (-1)-invariant as

$$\widehat{f_2}(z) = (1-i)^{-1} f_2(z).$$

There holds

$$\widetilde{f_2}(\theta) = \frac{\widehat{f_2}(e^{i\theta})}{ie^{i\theta}} = \sin\theta + \cos\theta - 1\,,$$

which is real-valued. Hence, an appropriated normalization of $f_n(z)$ is found so that $\widetilde{f_n}(\theta)$ is real-valued. Notice that for any other possible example of a polynomial with simple zeros on the unit circle, the same process as above can be applied.

For polynomials with real zeros, the previous lemma is closely related to the Hermite-Kakeya theorem [36, Thm. 6.3.8] and sometimes called Obrechkoff's theorem [37]. Extensions of this idea are mainly consider by Driver and coauthors (see among others [1, 2, 21]).

Lemma 3 *Set $\zeta \in \partial\mathbb{D}$ and let a and b be arbitrary nonzero complex numbers. Then,*

$$a\ K_{n-1}(z,\zeta) + b\ z\ K_{n-1}^*(z,\zeta) = r(z)\Phi_n(z,\omega_n),$$

where $r(z)$ and ω_n are given by

$$r(z) = -\frac{1}{\mathbf{k}_n}\frac{a + b\zeta^{n-1}z}{1-\overline{\zeta}z}\ \overline{\Phi_n(\zeta)}, \quad \omega_n = -\frac{\Phi_n(\zeta)}{\Phi_n^*(\zeta)}.$$

Proof From [40, Lemma 2.2.8], we have

$$K_{n-1}^*(z,\zeta) = \zeta^{n-1} K_{n-1}(z,\zeta),$$

where the *-transform is assumed to operate only on the variable z. The result follows after an elementary calculation. □

An immediate consequence of the above lemma is the following.

Corollary 1 *Under the hypothesis of Lemma 3, the polynomial*

$$aK_{n-1}(z,\zeta) + bzK_{n-1}^*(z,\zeta)$$

is an invariant POPUC of exact degree n associated with the measure $d\mu$ if and only if

$$a = -b\,\zeta^n, \quad \zeta \in \partial\mathbb{D}.$$

From the above results, it is natural to expect that the interlacing properties of zeros of OPRL under modifications of the orthogonality measure do not always hold for arbitrary POPUC. In the next section we will study when these similarities still hold for the Uvarov transformation.

3 The Uvarov transformation

The so–called canonical Uvarov transformation of a nontrivial probability measure, $d\mu$, supported on the unit circle appears by the addition to such a measure of a positive mass point on the support of the orthogonality measure, i.e,

$$d\mu(z) + m\ \delta_\alpha, \quad \alpha \in \partial\mathbb{D}. \tag{7}$$

In order to (7) be positive definite (see [15, Prop. 4.1]), we will consider real numbers m such that

$$1 + m\ K_{n-1}(\alpha, \alpha) > 0,$$

for every $n \geq 1$. Note that for $m > 0$, the previous inequality always holds.

The Uvarov transformation has been investigated by both, the mathematical physics and the orthogonal polynomials communities. An early reference is due to Von Neumann and Wigner (see [39]). The name of Uvarov transformation, frequently used by the orthogonal polynomials communities, as well as with the Christoffel and Geronimus transformations, is probably due to Zhedanov (see [46]). Nevertheless, in the theory of orthogonal polynomials, this transformation has a long history whose origins can be traced back to Geronimus (see [23, 25]). For a more recent contribution with historical references the reader may consult [45].

The following result was first obtained by Geronimus (see for example [23, Eq. 3.30]), and rediscovered and extended by Cachafeiro and Marcellán (see [5–7], among others). Furthermore, for OPRL it was rediscovered by Nevai (see [35]).

Theorem 1 ([25]) *Let $\{U_n\}_{n\geq 0}$ be the sequence of polynomials associated with the Uvarov transformation (7). Then,*

$$U_n(z) = \Phi_n(z) - M_n\ K_{n-1}(z, \alpha),$$

where

$$M_n = \frac{m\Phi_n(\alpha)}{1 + m\ K_{n-1}(\alpha, \alpha)}.$$

Taking into account their potential applications, the relation between the POPUC associated with the Uvarov transformation and the unperturbed ones deserves attention, especially regarding their zeros.

Let us define by

$$U_n(z, \beta_n) := U_n(z) + \beta_n\ U_n^*(z), \tag{8}$$

the POPUC associated with the Uvarov transformation (7). Using the previous theorem one can obtain an analog result for POPUC. Note that the POPUC (3) and (8) are defined by using the same parameter.

Proposition 1 *The following relation holds:*

$$U_n(z, \beta_n) = \Phi_n(z, \beta_n) + s_n(z)\Phi_n(z, \tau_n),$$

where

$$s_n(z) = \frac{1}{\mathbf{k}_n} \frac{M_n + \beta_n \overline{M}_n \alpha^{n-1} z}{1 - \overline{\alpha} z} \overline{\Phi_n(\alpha)}, \quad \tau_n = -\frac{\Phi_n(\alpha)}{\Phi_n^*(\alpha)}. \tag{9}$$

Proof This result follows from (8), Theorem 1 and Lemma 3. □

From the above proposition and Lemma 2 it is clear that for arbitrary parameters β_n, the zeros of $U_n(z, \beta_n)$ and $\Phi_n(z, \beta_n)$ do not necessarily interlace.

Theorem 2 *Let $\Phi_n(z, \beta_n)$, $s_n(z)$, and $\Phi_n(z, \tau_n)$ be given as in Proposition 1. Let $(e^{i\theta_1}, \ldots, e^{i\theta_n})$ be the cyclicly ordered set of zeros of the POPUC $\Phi_n(z, \beta_n)$ such that*

$$\omega < \theta_1 < \cdots < \theta_n < \omega + 2\pi,$$

and let l be a positive integer number such that

$$\theta_l < \arg(\alpha) < \theta_{l+1}.$$

Then, the zeros of the polynomials $\Phi_n(z, \beta_n)$ and $s_n(z)\Phi_n(z, \tau_n)$, $\beta_n \neq \tau_n$, strictly interlace on $\partial\mathbb{D}$ if and only if

$$\theta_l < \arg\left(\alpha \, \frac{\tau_n}{\beta_n}\right) < \theta_{l+1} \quad (mod\ (\omega, \omega + 2\pi]) \,. \tag{10}$$

Proof Since $\beta_n \neq \tau_n$, then $\Phi_n(z, \beta_n)$ and $\Phi_n(z, \tau_n)$ interlace zeros on $\partial\mathbb{D}$ (see for example [41, Thm. 1.3]). Note that α is a zero of $\Phi_n(z, \tau_n)$. It is easy to check that the zeros of $\Phi_n(z, \beta_n)$ strictly interlace with the zeros of $s_n(z)\Phi_n(z, \tau_n)$ if and only if (10) holds. □

Now, we are in a position to state our first results related to the interlacing properties.

Theorem 3 *The zeros of $U_n(z, \beta_n)$ strictly interlace on $\partial\mathbb{D}$ with the zeros of $\Phi_n(z, \beta_n)$ and $\Phi_n(z, \tau_n)$ if and only if the conditions of Theorem 2 hold.*

Proof Since we assume that $\Phi_n(z, \beta_n)$ and $\Phi_n(z, \tau_n)$ interlace zeros on $\partial\mathbb{D}$ and the zeros of $U_n(z, \beta)$ are on $\partial\mathbb{D}$, then the theorem follows as a direct consequence of Lemma 2. □

For discrete Sobolev OPRL, analogous results are proved in [11, 33]. Concerning orthogonality on the unit circle, namely on the distribution of zeros of OPUC, these results naturally move to the corresponding POPUC. To the best of the authors' knowledge, the present contribution is the first one to be devoted to the study of interlacing of zeros of POPUC under spectral transformations of the corresponding para-orthogonality measure. Theorem 2 states unknown differences between the behavior of zeros of OPRL and POPUC, while Theorem 3 shows that under certain conditions such differences can be avoided.

3.1 Bernstein–Szegő case

Let us consider the following modification of the Bernstein–Szegő measure (see [40]),

$$\frac{1-|\lambda|^2}{|e^{i\theta}-\lambda|^2}\frac{d\theta}{2\pi}+m\delta_\alpha, \quad \lambda\in\mathbb{D}, \quad \alpha\in\partial\mathbb{D}. \tag{11}$$

It is well known that $\Phi_n(z)=z^n-\lambda z^{n-1}$ is the OPUC of degree n with respect to the Bernstein–Szegő measure, thus,

$$\Phi_n(z,\tau_n)=z^n-\lambda z^{n-1}+\tau_n(-\overline{\lambda}z+1), \tag{12}$$

where

$$\tau_n=-\frac{\alpha-\lambda}{\overline{\alpha}-\overline{\lambda}}\alpha^{n-2}. \tag{13}$$

Moreover, $s_n(z)$ is given as in (9) with $\mathbf{k}_0=1$, $\mathbf{k}_n=1/(1-|\lambda|^2)$ for every $n\geq 1$, and

$$M_n=m\frac{\alpha^{n-1}(\alpha-\lambda)}{1+n\dfrac{|\alpha-\lambda|}{1-|\lambda|^2}}.$$

In order to illustrate Theorem 3, we consider for $n=7$ and the following choices of the parameters: $\lambda=-1/2i$, $m=1$, $\alpha=-1$ and $\beta_7=-i$. In this case, (13) yields $\tau_7=3/5-4/5i$. In order to check the interlacing stated in Theorem 3, we compute the zeros of the polynomial $\Phi_7(z,-i)=z^7+1/2iz^6-1/2z-i$, which are $-i$, $-0.995218+0.0976748i$, $-0.686236-0.727379i$, $-0.475521+0.879704i$, $0.475521+0.879704i$, $0.686236-0.727379i$ and $0.995218+0.0976748i$. It is easy to see that

$$-0.995218+0.0976748i\prec -1\prec -0.686236-0.727379i. \tag{14}$$

Finally, since $\alpha\tau_n/\beta_n=4/5+3/5i$ and

$$-0.995218+0.0976748i\prec -\frac{4}{5}-\frac{3}{5}i\prec -0.686236-0.727379i, \tag{15}$$

according to Theorem 2, $\Phi_7(z,-i)$ and $s_n(z)\Phi_7(z,3/5-4/5i)$ interlace zeros on $\partial\mathbb{D}$. Thus, from Theorem 3, $U_7(z,-i)$, $\Phi_7(z,-i)$ and $s_n(z)\Phi_7(z,3/5-4/5i)$ interlace zeros on $\partial\mathbb{D}$. Similarly, it is an easy exercise to check that for the same values of the parameter and $\beta_7=i$, we do not have interlacing.

Figure 1 is obtained by using Wolfram Mathematica® 9.0[1] and shows the interlacing property of the zeros of $U_7(z,-i)$ (blue discs), $\Phi_7(z,-i)$ (purple squares) and $s_n(z)\Phi_7(z,3/5-4/5i)$ (yellow diamonds). Figure 2 shows the behavior of the zeros when $\beta_7=i$.

[1] Wolfram Mathematica is a registered trademark of Wolfram Research, Inc.

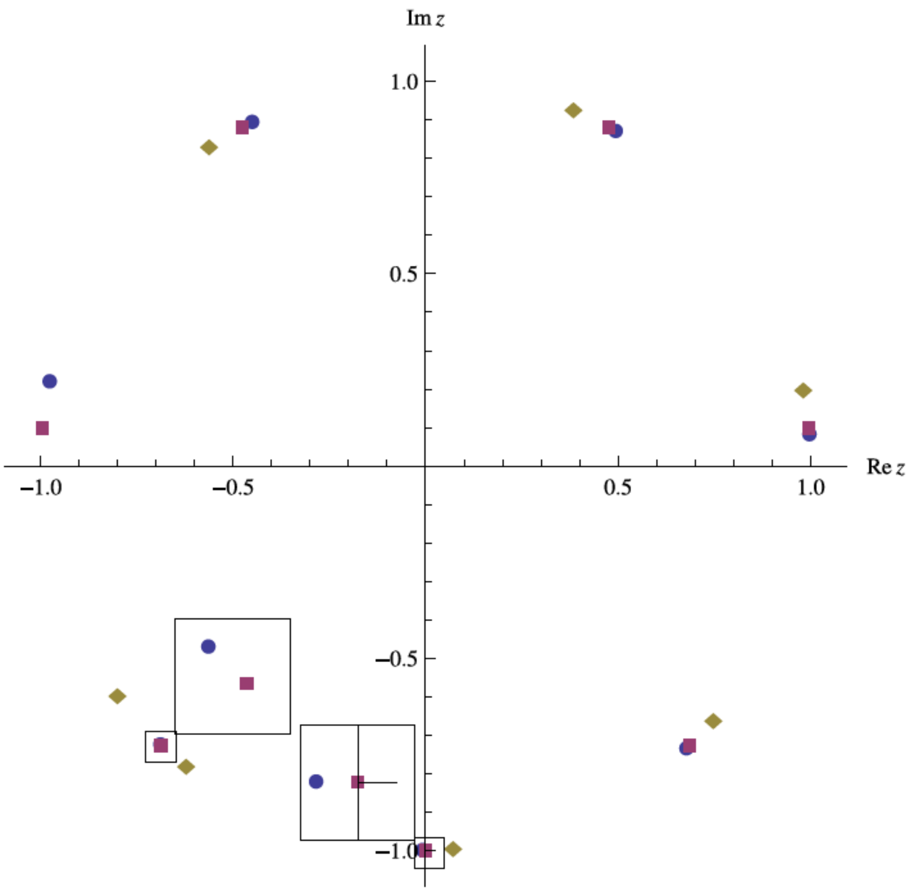

Fig. 1 Zeros of POPUC associated with the Uvarov transformation for $\beta_7 = -i$

4 Further results

Notice that from (3) the difference between POPUC and OPUC is that they have the same orthogonality conditions except that POPUC are not orthogonal to the constants while this fact holds for OPUC. This one–dimension lowered condition makes possible to get good properties for the zeros of POPUC in comparison with OPUC. The counterpart to the deficiency of this one less orthogonality condition is the fact that POPUC are not unique, and basically depend on a unimodular free parameter. In the previous section, we have deduced conditions in order to find para–orthogonal polynomials related to the same parameter such that their zeros under the Uvarov transformation strictly interlace on $\partial\mathbb{D}$. In this section, we will see how our approach apply to other kind of transformations when the para–orthogonality parameters are not necessarily the same.

Let us consider the so–called Christoffel transformation on the unit circle. This transformation has the effect of multiply the orthogonality measure by a Laurent

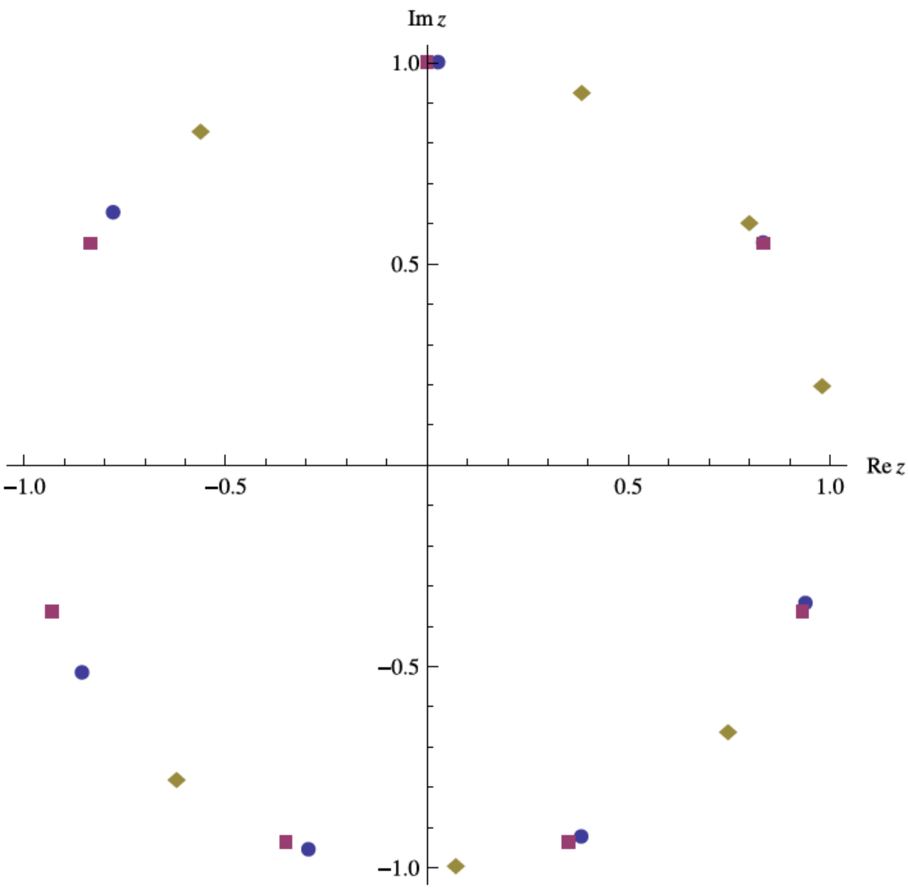

Fig. 2 Zeros of POPUC associated with the Uvarov transformation for $\beta_7 = i$

polynomial that is nonnegative on $\partial\mathbb{D}$. In this section, we will consider a class of Christoffel transformation of the form

$$|z-\alpha|^2 d\mu(z), \quad \alpha \in \partial\mathbb{D}. \tag{16}$$

Notice that (16) is always positive definite (see for example [15, Prop. 2.4]).

The Christoffel transformation (16) was defined for OPRL by Szegő in his classical monograph (see [43]). This transformation leads to kernel polynomials (see [12, Ch. 1, Sec. 7] and [43, Thm. 3.1.4]) playing an important role in the spectral theory of orthogonal polynomials. For OPUC, the Christoffel transformation and their extensions are mainly considered by Marcellán and co–authors (see, among others, [10, 15, 26, 32, 34]). Fast algorithms to compute the QR step corresponding to the Hessenberg matrices associated with Christoffel transformations have been recently presented in [30, Sec. 5].

Theorem 4 ([26]) *Let* $\{C_n\}_{n\geq 0}$ *be the sequence of orthogonal polynomials associated with the Christoffel transformation* (16)*, then*

$$(z-\alpha)C_{n-1}(z) = \Phi_n(z) - N_n\, K_{n-1}(z,\alpha),$$

where

$$N_n = \frac{\Phi_n(\alpha)}{K_{n-1}(\alpha,\alpha)}.$$

Let us define by

$$C_{n-1}(z,\beta_{n-1}) := C_{n-1}(z) + \beta_{n-1}C^{*}_{n-1}(z), \quad \beta_{n-1} \in \partial\mathbb{D}. \tag{17}$$

the POPUC associated with the Christoffel transformation (16). Now, we can obtain the analog of Proposition 1.

Proposition 2 *The following relation holds:*

$$(z-\alpha)C_{n-1}(z,\beta_{n-1}) = \Phi_n(z,-\alpha\beta_{n-1}) + t_n(z)\Phi_n(z,\tau_n),$$

where

$$t_n(z) = \frac{1}{\mathbf{k}_n}\frac{N_n - \beta_{n-1}\overline{N}_n\alpha^n z}{1-\overline{\alpha}z}\,\overline{\Phi_n(\alpha)}, \tag{18}$$

and τ_n *given in* (9)*.*

Proof From (17), we get

$$(1-\overline{\alpha}z)C_{n-1}(z,\beta_n) = (z-\alpha)C_{n-1}(z) + \beta_n(1-\overline{\alpha}z)C^{*}_{n-1}(z).$$

Since,

$$((z-\alpha)C_{n-1}(z))^{*} = (1-\overline{\alpha}z)C^{*}_{n-1}(z),$$

the rest of the proof follows as in the proof of Proposition 1. □

Now, we can state without proof analogous results to those presented in Theorem 2 and Theorem 3.

Theorem 5 *Let* $\Phi_n(z,-\alpha\beta_{n-1})$*,* $t_n(z)$*, and* $\Phi_n(z,\tau_n)$ *be given as in Proposition 1. Let* $(e^{i\theta_1},\ldots,e^{i\theta_n})$ *be the cyclicly ordered set of zeros of the POPUC* $\Phi_n(z,-\alpha\beta_{n-1})$ *such that*

$$\omega < \theta_1 < \cdots < \theta_n < \omega + 2\pi,$$

and let l *be a positive integer number such that*

$$\theta_l < \arg(\alpha) < \theta_{l+1}.$$

Then, the zeros of the polynomials $\Phi_n(z,-\alpha\beta_{n-1})$ *and* $t_n(z)\Phi_n(z,\tau_n)$*,* $-\alpha\beta_{n-1} \neq \tau_n$*, strictly interlace on* $\partial\mathbb{D}$ *if and only if*

$$\theta_l < \arg\left(-\frac{\tau_n}{\beta_{n-1}}\right) < \theta_{l+1} \quad (\textit{mod}\ (\omega,\omega+2\pi]). \tag{19}$$

Theorem 6 *The zeros of* $(z-\alpha)C_{n-1}(z,\beta_{n-1})$ *strictly interlace on* $\partial\mathbb{D}$ *with the zeros of* $\Phi_n(z,-\alpha\beta_{n-1})$ *and* $\Phi_n(z,\tau_n)$ *if and only if the conditions of Theorem 2 hold.*

The previous result for OPRL are contained in [12, Ch. 1, Sec. 7].

4.1 Rational case

Let us consider the following Christoffel transformation of the Bernstein-Szegő measure (see [27, 40]),

$$\frac{|e^{i\theta}-\alpha|^2}{|e^{i\theta}-\lambda|^2}\frac{d\theta}{2\pi}, \quad \lambda\in\mathbb{D}, \quad \alpha\in\partial\mathbb{D}. \tag{20}$$

Notice that in this case, as in the Bernstein–Szegő case, $\Phi_n(z)=z^n-\lambda z^{n-1}$ is the OPUC of degree n, thus, $\Phi_n(z,\tau_n)$ and τ_n are given by (12) and (13), respectively.

On the other hand, $t_n(z)$ is given as in (18) with $\mathbf{k}_0=1/(1-|\lambda|^2)$, $\mathbf{k}_n=1$ for every $n\geq 1$, and

$$N_n=\frac{\alpha^{n-1}(\alpha-\lambda)}{1+|\lambda|^2+n|\alpha-\lambda|^2}.$$

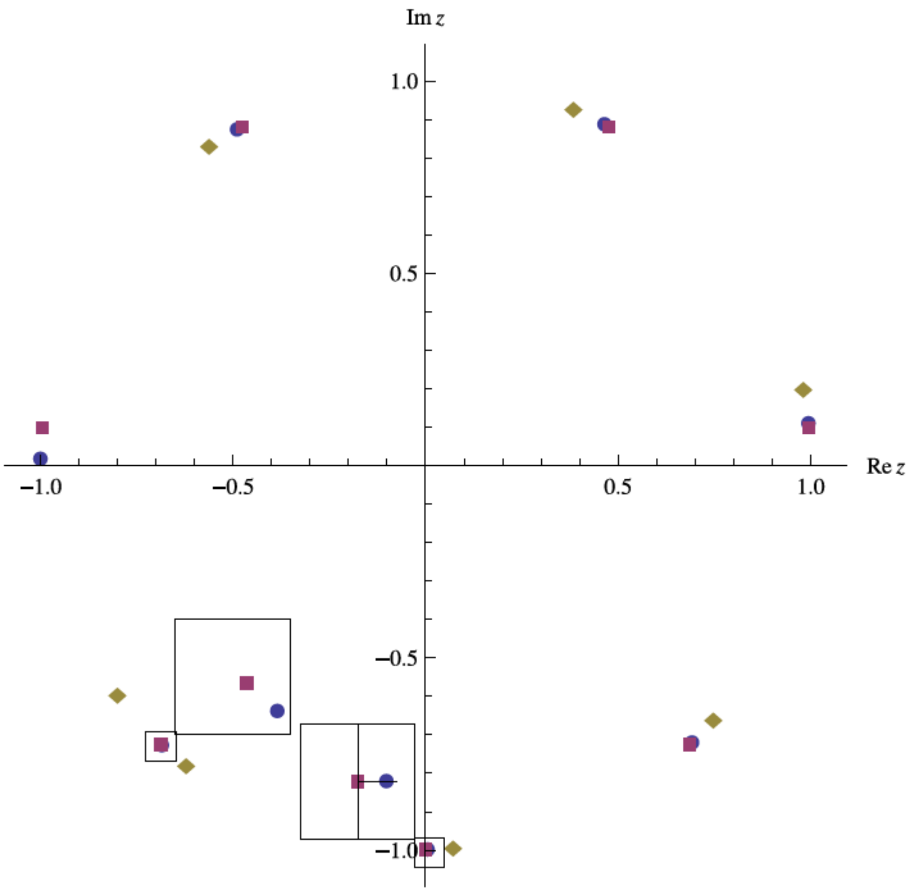

Fig. 3 Zeros of POPUC associated with the Christoffel transformation for $\beta_7=-i$

Comparing (10) and (19), we can see that for the same values of the parameters the result shown in Figures 1 and 2 also holds for this case. In order to illustrate Theorem 3, we consider for $n = 7$ and the same parameters as in the Bernstein-Szegő case. Hence, τ_7 and $\Phi_7(z, -i)$ are the same as in the Bernstein–Szegő case. Finally, as $\tau_n/\beta_{n-1} = 4/5 + 3/5i$, (14) and (15) hold. According to Theorem 2, $\Phi_7(z, -i)$ and $t_n(z)\Phi_7(z, 3/5 - 4/5i)$ interlace zeros on $\partial\mathbb{D}$. Thus, from Theorem 3, $C_7(z, -i)$, $\Phi_7(z, -i)$ and $t_n(z)\Phi_7(z, 3/5 - 4/5i)$ interlace zeros on $\partial\mathbb{D}$. Similarly, it is a straightforward exercise to check that for the same values of the parameter and $\beta_7 = i$, we do not have interlacing for their zeros.

Figure 3 is obtained by using Wolfram Mathematica® 9.0 and shows the interlacing property of the zeros of $C_7(z, -i)$ (blue discs), $\Phi_7(z, -i)$ (purple squares) and $t_n(z)\Phi_7(z, 3/5 - 4/5i)$ (yellow diamonds). In Fig. 4, we consider the case in which $\beta_7 = i$.

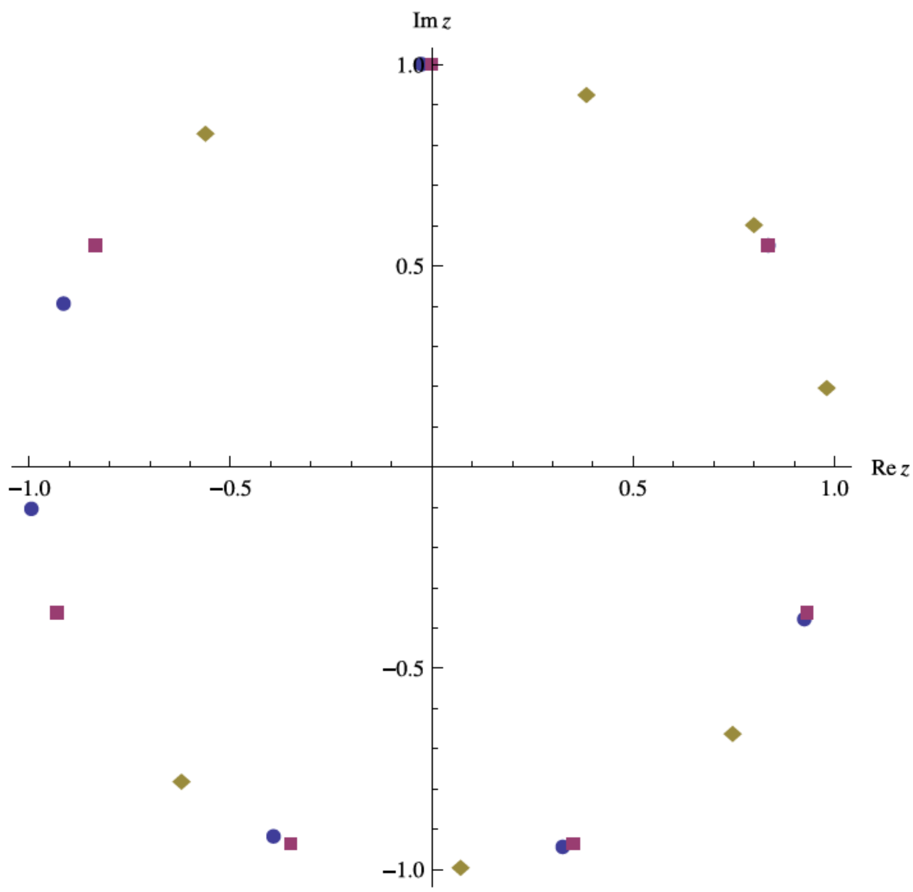

Fig. 4 Zeros of POPUC associated with the Christoffel transformation for $\beta_7 = i$

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