This is a postprint version of the following published document:

Marcellán, M.; Shayanfar, N. (2015). "OPUC, CMV matrices and perturbations of measures supported on the unit circle". *Linear Algebra and its Applications*, v. 485, November, pp. 305-344.
DOI: 10.1016/j.laa.2015.07.026

Proyecto MTM2012-36732-C03-01

OPUC, CMV matrices and perturbations of measures supported on the unit circle

Francisco Marcellán[a], Nikta Shayanfar[b], *

[a] *Instituto de Ciencias Matemáticas (ICMAT) and Departamento de Matemáticas, Universidad Carlos III de Madrid, 28911 Leganés, Madrid, Spain*
[b] *Departamento de Matemáticas, Universidad Carlos III de Madrid, 28911 Leganés, Madrid, Spain*

A B S T R A C T

Let us consider a Hermitian linear functional defined on the linear space of Laurent polynomials with complex coefficients. In the literature, canonical spectral transformations of this functional are studied. The aim of this research is focused on perturbations of Hermitian linear functionals associated with a positive Borel measure supported on the unit circle. Some algebraic properties of the perturbed measure are pointed out in a constructive way. We discuss the corresponding sequences of orthogonal polynomials as well as the connection between the associated Verblunsky coefficients. Then, the structure of the Θ matrices of the perturbed linear functionals, which is the main tool for the comparison of their corresponding CMV matrices, is deeply analyzed.
From the comparison between different CMV matrices, other families of perturbed Verblunsky coefficients will be considered. We introduce a new matrix, named Fundamental matrix, that is a tridiagonal symmetric unitary matrix, containing basic information about the family of orthogonal polynomials. However, we show that it is connected to another family of orthogonal polynomials through the Takagi decomposition.

Keywords:
Orthogonal polynomials on the unit circle
GGT matrix
CMV matrix
Fundamental matrix
Canonical linear spectral transformations

* Corresponding author.
E-mail addresses: pacomarc@ing.uc3m.es (F. Marcellán), nikta.shayanfar@gmail.com (N. Shayanfar).

1. Introduction

The Jacobi matrices come from the representation of the multiplication operator in terms of orthogonal polynomials on the real line (OPRL), taking into account the three term recurrence relation they satisfy. The spectral analysis of these matrices yields an accurate information about the zeros of OPRL. In the case of orthogonal polynomials on the unit circle (OPUC) the representation of the multiplication operator in terms of the OPUC yields a Hessenberg matrix, that is known in the literature as GGT matrix. The GGT representation has several limitations, in particular, the matrix is not unitary. Moreover, its complicated structure yields some difficulties in the study of the spectral theory [30]. By using Laurent orthogonal polynomials instead of orthogonal polynomials, the representation of the multiplication operator gives a five-diagonal matrix, called CMV matrix. From a linear algebraic point of view, it is a remarkable simplification that the eigenvalue problem for certain Hessenberg matrices reduces to the eigenvalue problem of a five-diagonal matrix [55]. In fact, the CMV matrices in the theory of orthogonal polynomials on the unit circle constitute a unitary analogue of Jacobi matrices for orthogonal polynomials on the real line.

To begin with, let $\mathcal{L}$ be a positive definite Hermitian linear functional, and let us define its associated family of orthonormal polynomials, while for a quasi-definite Hermitian linear functional we will have a family of monic orthogonal polynomials. Throughout this paper, *when the linear functional is not explicitly specified, we assume it is a positive definite linear functional*. We characterize the GGT matrix and CMV matrix for nontrivial probability measures supported on the unit circle. Their entries are given in terms of the so-called Verblunsky coefficients. They provide a qualitative information about the family of orthogonal polynomials.

Our first goal is to analyze the effect of different perturbations of the measure on the corresponding families of orthogonal polynomials as well as the relation between their CMV matrices. Suppose $\mathcal{C}$ is the CMV matrix for the Verblunsky sequence $\{\Phi_n(0)\}_{n\geqslant 1}$ associated with a measure. We will denote by $\widetilde{\mathcal{C}}$ and $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$ the CMV matrix and the sequence of Verblunsky coefficients for the perturbed measure. Since $\mathcal{C}$ includes 2×2 block matrices, called Θ matrices, for the study of $\widetilde{\mathcal{C}}$ we will need the connection between the corresponding Θ matrices as an initial step.

The relation of Θ matrices can be done via several equivalence relations, generally called *congruence relations*. Roughly speaking, the situation for the unitary matrix case is as follows.

- On one hand, we focus our attention on the *unitary *congruence*, which is the finest unitary relation for changing orthonormal bases in a linear pace, and it is shown to be equivalent to **congruence*. A family of unitary matrices, which provides the unitary similarity relation between Θ matrices, can be obtained by using the spectral decomposition and it is shown to hold for some particular families of Verblunsky coefficients.

- On the other hand, the simplified relation of congruence is applied for general Verblunsky coefficients which is gained by the Takagi decomposition. The families of unitary matrices are among those that are likely to play, in the case of congruence, the role of the **congruence* for the unitary similarity relation.

Later on, we indicate how the change of the Verblunsky sequence can affect the CMV matrix, and finally, we analyze the canonical linear spectral transformations and their corresponding CMV matrices.

Our next goal is to introduce a new perturbed measure by putting gaps between Verblunsky coefficients. This family yields a novel matrix, named *Fundamental matrix.* It includes all the information about the Verblunsky coefficients. The novelty of this matrix allows us to make the connection with the CMV matrix. At this point, the study of the measure of the Fundamental matrix remains an open problem, as far as we know.

The structure of this paper is as follows. In Section 2, after providing a basic background of matrix analysis, we will deal with CMV matrices in the framework of the theory of orthogonal polynomials on the unit circle. The aim of Section 3 is to begin the development of the structural formulas for the unitary similarities. In Section 4, the basic role of equivalence relations for Θ matrices is established. From these facts, in Section 5 we deal with the relationship between the corresponding CMV matrices. In Section 6, we analyze the above results for canonical linear spectral transformations. The effect of a particular perturbed sequence of Verblunsky coefficients yields a new structured Fundamental matrix, which is introduced in Section 7. We guess that the Fundamental matrix will play, with an undetermined different basis, an important role in the same way as the CMV matrix in the Laurent orthonormal polynomial basis. Some properties of this new matrix in terms of the CMV matrix have been also studied in Section 7. Particular examples of sequences of Verblunsky coefficients are given in Section 8 in order to analyze how CMV and Fundamental matrices are reflected by the canonical linear spectral transformations. The preservation of the structure of the Fundamental matrix under the Takagi decomposition is discussed in Section 9. Eventually, the analysis of the spectral measure for the corresponding Fundamental matrix remains an open problem and it is explained in Section 10, where we have summarized the main results of our contribution.

2. Basic background on matrix analysis and OPUC

2.1. Matrix analysis

To explain the origin of similarities of CMV matrices, it is necessary to give some background from matrix analysis. For a more complete description of what follows, the reader should turn to [32].

Let $M_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices. Note that a complex symmetric matrix does not retain the desirable properties of real symmetric or complex Hermitian matrices. In fact any real or complex square matrix is similar to a complex symmetric matrix [23], but there exist several equivalence relations between complex square matrices. The most important relations related to similarity transformations of matrices are as follows.

First, we consider the **congruence equivalence relation*, $A = SBS^*$ for some nonsingular matrix S (note that S^* denotes the complex conjugate transpose of S), and the finer relation *unitary *congruence* (unitary similar) $A = UBU^*$ for some unitary matrix U. Next, we restrict ourself to the *congruence relation*, $A = SBS^T$ for some nonsingular matrix S, and the finer relation *unitary congruence* $A = UBU^T$ for some unitary matrix U. It is easy to check that the above mentioned relations are equivalence relations, and canonical forms for the square complex matrices are given in [33,34].

Similarity relation defines similar matrices that correspond to the same linear transformation in different bases, whereas congruent matrices, which have been obtained by congruent relation, correspond to equivalent bilinear forms. The last decade has witnessed a growing interest in the field of congruence, see [3,4,16,17,37–39] in the literature.

2.1.1. Unitary *congruence and *congruence

Unitary similarity is a natural equivalence relation in the study of normal matrices: UAU^* is normal if U is unitary and A is normal. Our consistent point of view is that unitary *congruence is a special kind of *congruence, rather than a special kind of similarity $A = SBS^{-1}$, that is to be analyzed with methods from the general theory of *congruence. In order to verify the unitary similarity, one can use the classical *Specht's Theorem* [32].

Theorem 2.1. *Matrices A and B are unitary similar if and only if $trW(A, A^*) = trW(B, B^*)$, for every monomial $W(s, t)$ in the noncommutating variables s and t.*

Specht's theorem requires that an infinite set of conditions must be verified in terms of the trace of the matrices, reminding that $tr(A)$ is the sum of the diagonal elements of the matrix A. However, Pearcy made this theorem an effective criterion [56]. Particularly, a refinement of Pearcy idea ensures that it suffices to check three trace identities, for 2×2 matrices [40,42].

Lemma 2.2. *Let $A, B \in M_2(\mathbb{C})$. Then A and B are unitary similar if and only if $tr(A) = tr(B)$, $tr(A^2) = tr(B^2)$, and $tr(A^*A) = tr(B^*B)$.*

Note that the condition $tr(A^*A) = tr(B^*B)$ may be removed if it is known in advance that A and B are normal. From the similarity point of view, we state the following lemma.

Lemma 2.3. *Two normal matrices of any size are similar if and only if they are unitary similar.*

Since Θ matrices have symmetric unitary structure, intertwining identities involving unitary matrices lead to characterizations for unitary *congruences and *congruences as follows.

Theorem 2.4. *(See [35].) Let A and B be both unitary complex matrices, then A and B are unitary *congruent if and only if they are *congruent.*

The preceding argument shows that for Θ matrices, the study of unitary *congruent reduces to *congruent, for which the following lemma is applicable.

Lemma 2.5. *Two normal matrices in $M_n(\mathbb{C})$ are *congruent if and only if they have the same rank and the same number of eigenvalues on each ray $\{re^{i\theta} : r > 0\}$ from the origin.*

Before proceeding to the congruence, we present one of the most fundamental facts about the normal matrices, *Spectral Theorem*, which will be used for obtaining *congruence relation.

Theorem 2.6. *A matrix $N \in M_n(\mathbb{C})$ is normal if and only if there exists a unitary matrix U such that U^*NU is diagonal. The representation*

$$A = UDU^*, \tag{2.1}$$

for unitary U and diagonal matrix D is called Spectral decomposition of A.

That is, the class of normal matrices of a given size is closed under unitary similarity.

2.1.2. Unitary congruence and congruence

Unitary congruence is an equivalence relation in the study of complex symmetric matrices: UAU^T is symmetric if U is unitary and A is symmetric. In a parallel development of *congruent, we deal with unitary congruence as a special kind of congruence, rather than as a special kind of consimilarity, $A = SB\bar{S}^{-1}$, where $\bar{S}$ denotes the complex conjugate of S. Comparatively to Theorem 2.4 for congruence, we have:

Lemma 2.7. *(See [35].) Two unitary complex matrices are unitary congruent if and only if they are congruent.*

The unitary structure of Θ matrices assures us that the study of unitary congruent relation is identical to congruent, for which the following lemma guarantees its existence.

Lemma 2.8. *Two symmetric matrices of any size are unitarily congruent if and only if they have the same singular values (i.e. the same square roots of eigenvalues of A^*A and AA^*).*

Remark 2.9. The singular values of a unitary matrix are equal to unity. Indeed, the eigenvalues are unimodular.

Noticing Remark 2.9 together with Lemma 2.8, as an immediate corollary, we get the desired criterion for the congruence between two complex matrices.

Corollary 2.10. *Two symmetric unitary matrices in $M_n(\mathbb{C})$ are unitary congruent.*

In order to compute the unitary congruence factors, we need to introduce the *Takagi decomposition* which is a less known diagonalization method for complex symmetric matrices. It is an analog of the eigenvalue decomposition of Hermitian matrices. Actually, it combines the concepts of singular values and exploitation of structure with respect to complex bilinear forms. The symmetry of A is exploited by choosing unitary factors for the singular value decomposition.

Theorem 2.11 *(Takagi decomposition). Let $A \in M_n(\mathbb{C})$. If A is symmetric, there is a unitary $U \in M_n(\mathbb{C})$ such that*

$$A = U\Sigma U^T, \tag{2.2}$$

in which Σ is a nonnegative diagonal matrix, whose diagonals are the singular values of A, in any desired order.

The columns of U are called the *Takagi vectors* of A and the diagonal elements of Σ are its *Takagi values*, which are exactly its singular values [32]. Since $U^T = \bar{U}^*$, the Takagi decomposition is a symmetric form of the singular value decomposition (SVD), that is the reason why Takagi decomposition is also called *symmetric singular value decomposition* (SSVD) in some contexts [5]. An algorithm for computing the Takagi decomposition has been introduced in [5], and another algorithm, which applies to complex symmetric tridiagonal matrices, has been presented in [65]. Furthermore, the Takagi decomposition for that provides the Jordan canonical forms, has been developed in [50]. Finally, if a complex matrix is not only symmetric but is also unitary, then its Takagi decomposition can be found by arithmetic operations and quadratic radicals [36].

According to Remark 2.9, Σ in (2.2) is the identity matrix, and the Takagi decomposition of unitary symmetric matrices, take the form

$$A = UU^T, \tag{2.3}$$

for unitary matrix U.

Recently, an effective way for computing the Takagi decomposition of Θ matrices has been proposed in [36]. The following method demonstrates how to get the Takagi decomposition of the unitary symmetric matrix $A \in M_2(\mathbb{C})$.

Lemma 2.12. *For unitary symmetric matrix $A \in M_2(\mathbb{C})$, assume that the nonzero normalized vector $y \in \mathbb{C}^2$ satisfies $A\bar{y} = y$. This vector can be chosen as*

$$y = A\bar{x} + x,$$

for an arbitrary nonzero unit vector $x \in \mathbb{C}^2$. Let $V_1 \in M_2(\mathbb{C})$ be a unitary matrix that has the vector y as its first column, and let $V_2 = Diag[1, \upsilon]$ for some entry υ of modulus 1. Then $U = V_1 V_2$ is the desired Takagi decomposition of (2.3).

In attention to the Takagi decomposition, determination of the unitary congruence relation can be done as follows.

Theorem 2.13. *Let $A, B \in M_n(\mathbb{C})$ be unitary symmetric matrices. There exist unitary matrices U_A and U_B such that $U = U_A U_B^*$ makes the unitary congruence relation*

$$UBU^T = A. \tag{2.4}$$

Proof. Consider the Takagi decomposition (2.3) for unitary symmetric matrices A and B. There exist unitary matrices U_A and U_B such that

$$A = U_A U_A^T, \qquad B = U_B U_B^T.$$

Then we can substitute the above relations in (2.4). The congruence obviously holds if we choose U satisfying $UU_B = U_A$. This completes the proof. $\square$

2.2. Orthogonal polynomials on the unit circle

We begin with an overview of orthogonal polynomials on the unit circle, for a more complete description of what follows, the reader may refer to quiet prominent monographs and papers such as [24,25,59,60,64]. Let $\mathcal{L}$ be a Hermitian linear functional in the linear space of Laurent polynomials $\Lambda := span\{z^n\}_{n\in\mathbb{Z}}$ with complex coefficients such that

$$c_n = \langle \mathcal{L}, z^n \rangle = \overline{\langle \mathcal{L}, z^{-n} \rangle} = \bar{c}_{-n}, \qquad n \in \mathbb{Z}.$$

A bilinear functional associated with $\mathcal{L}$ in the linear space $\mathbb{P}$ of polynomials with complex coefficients is introduced as follows [24,43]

$$\langle p, q \rangle_{\mathcal{L}} = \langle \mathcal{L}, p(z)\bar{q}(z^{-1}) \rangle, \qquad p, q \in \mathbb{P}.$$

The Gram matrix of the above bilinear functional with respect to the canonical basis $\{z^n\}_{n\geqslant 0}$ is the following Hermitian Toeplitz matrix [29]:

$$T = \begin{pmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix}.$$

If the principal leading $n \times n$ submatrices T_n of T are nonsingular for every $n \geqslant 1$, the linear functional $\mathcal{L}$ is said to be *quasi-definite*. Then a unique family of monic orthogonal polynomials $\{\Phi_n(z)\}_{n\geqslant 0}$ can be introduced such that

$$\left\langle \Phi_n, \Phi_m \right\rangle_{\mathcal{L}} = \mathbf{k}_n \delta_{n,m}, \tag{2.5}$$

where $\mathbf{k}_n \neq 0$ for every $n \geqslant 0$. The linear functional $\mathcal{L}$ is called *positive definite* if all $T_n, n \geqslant 1$, have positive determinant. If $c_0 = 1$, then $\mathcal{L}$ has an integral presentation

$$\left\langle \mathcal{L}, f \right\rangle = \int_{\mathbb{T}} f(z) d\mu(z), \qquad f \in \mathbb{P}, \tag{2.6}$$

for a unique nontrivial probability measure on $\mathbb{T}$. In such a case, there exists a family $\{\varphi_n(z)\}_{n\geqslant 0}$ of orthonormal polynomials

$$\varphi_n(z) = \kappa_n z^n + \cdots, \qquad \kappa_n > 0,$$

such that

$$\int_{\mathbb{T}} \varphi_n(z)\overline{\varphi_m(z)} d\mu(z) = \delta_{n,m}, \qquad m, n \geqslant 0, \tag{2.7}$$

and its relation to the family of monic orthogonal polynomials is $\Phi_n(z) = \frac{\varphi_n(z)}{\kappa_n}$.

Considering the *reversed polynomial* $\Phi_n^*(z) = z^n \overline{\Phi}_n(z^{-1})$, monic orthogonal polynomials satisfy the following forward and backward recurrence relations due to Szegő [24, 59,64]:

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \qquad n \geqslant 0, \tag{2.8}$$

$$\Phi_{n+1}(z) = (1 - |\Phi_{n+1}(0)|^2) z\Phi_n(z) + \Phi_{n+1}(0)\Phi_{n+1}^*(z), \qquad n \geqslant 0, \tag{2.9}$$

where $\Phi_0(z) = 1$.

Remark 2.14. Since we have

$$\frac{\left\langle \Phi_n, \Phi_n \right\rangle_{\mathcal{L}}}{\left\langle \Phi_{n-1}, \Phi_{n-1} \right\rangle_{\mathcal{L}}} = 1 - |\Phi_n(0)|^2, \qquad n \geqslant 1,$$

the so-called *Verblunsky coefficients* $\{\Phi_n(0)\}_{n\geqslant 1}$ satisfy

- quasi-definite linear functionals: $|\Phi_n(0)| \neq 1$, $n \geqslant 1$,
- positive definite linear functionals: $|\Phi_n(0)| < 1$, $n \geqslant 1$.

These coefficients characterize the corresponding family of orthogonal polynomials. In fact, there is a one-to-one correspondence between a linear functional and its sequence of Verblunsky coefficients [59]. More precisely:

Remark 2.15. For any sequence of Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$ where $|\Phi_n(0)| < 1$, $n \geqslant 1$, the relation (2.8) gives a family of orthonormal polynomials on $\mathbb{T}$ and the associated measure $d\mu$ is unique. The same holds for quasi-definite linear functionals.

Throughout this paper, in the positive definite case the Verblunsky coefficient $\Phi_n(0), n \geqslant 1$, is always accompanied by a real number

$$\rho_n := \left(1 - |\Phi_n(0)|^2\right)^{\frac{1}{2}}, \tag{2.10}$$

where the square root is assumed to be positive.

Definition 2.16. The *functions of second kind associated with $d\mu$* are defined as

$$q_j(t) = \int_{\mathbb{T}} \frac{\overline{\varphi_j(z)}}{t-z} d\mu(z), \qquad t \notin \mathbb{T}, \quad j \geqslant 0,$$

and we denote

$$Q_j(t) = \int_{\mathbb{T}} \frac{\overline{\Phi_j(z)}}{t-z} d\mu(z), \qquad t \notin \mathbb{T}, \quad j \geqslant 0.$$

Theorem 2.17 *(The Christoffel–Darboux Formula). For any $n \geqslant 0$ and $y, z \in \mathbb{C}$ with $\bar{y}z \neq 1$, the nth polynomial kernel $K_n(z,y)$ associated with $\{\Phi_n(z)\}_{n\geqslant 0}$ is defined as*

$$K_n(z,y) = \sum_{j=0}^{n} \frac{\overline{\Phi_j(y)}\Phi_j(z)}{\mathbf{k}_j} = \frac{\overline{\Phi^*_{n+1}(y)}\Phi^*_{n+1}(z) - \overline{\Phi_{n+1}(y)}\Phi_{n+1}(z)}{\mathbf{k}_{n+1}(1-\bar{y}z)},$$

and for every polynomial p of degree at most n, it satisfies the so-called reproducing property

$$\int_{\mathbb{T}} K_n(z,y)\overline{p(z)} d\mu(z) = \overline{p(y)}.$$

Remark 2.18. It can be easily observed that the Christoffel–Darboux Formula for a family of orthonormal polynomials becomes

$$K_n(z,y) = \frac{\overline{\varphi^*_{n+1}(y)}\varphi^*_{n+1}(z) - \overline{\varphi_{n+1}(y)}\varphi_{n+1}(z)}{1-\bar{y}z}.$$

For $|\alpha| \neq 1$, the confluent Christoffel–Darboux Formula is

$$K_n(\alpha,\alpha) = \frac{|\Phi^*_n(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2}{\mathbf{k}_n(1-|\alpha|^2)}.$$

Moreover, for the reversed polynomial we get

$$\Phi^*_n(z) = \mathbf{k}_n K_n(z,0). \tag{2.11}$$

2.2.1. GGT Matrix

The GGT matrix, from the initials of Geronimus, Gragg and Teplyaev, is one of the most interesting topics in the theory of OPUC [59]. It was studied by several authors, taking into account some applications as the distribution of zeros of the nth orthogonal polynomial $\Phi_n(z)$ [18,27,59], quadrature formulas on the unit circle [14,43], the frequency analysis problem [44,54], and integrable systems related to the complex semi-discrete modified KdV equation, namely, the Schur flow [1,51,62], among others.

The connection between the GGT matrix associated with a probability measure and the perturbed linear functional, respectively, in terms of their QR factorization instead of the LU factorization has been analyzed in [15,20,49]. More precisely, in [21], explicit expressions for polynomials orthogonal with respect to the perturbed measure have been obtained in terms of the orthogonal polynomials with respect to the initial probability measure.

Let $\mathcal{H} = \mathbb{L}^2(\mathbb{T}, d\mu)$ be the Hilbert space of functions on the unit circle such that the norm associated with the inner product (2.6) as

$$\| f \|^2_{2,\mu} = \int_{\mathbb{T}} |f(e^{i\theta})|^2 d\mu,$$

is finite. If we choose a basis for $\mathbb{P}$, then every linear operator $g : \mathbb{P} \to \mathbb{P}$ has a matrix representation in terms of such a basis. In this subsection, we consider first the orthonormal basis $\{\varphi_n(z)\}_{n\geqslant 0}$, whose matrix representation is the GGT matrix.

From (2.9) and (2.11), one gets

$$z\varphi_i(z) = \sum_{j=0}^{i+1} \mathbb{H}_{i,j}\varphi_j(z),$$

which shows the multiplication operator $h : \mathbb{P} \to \mathbb{P}$ on $\mathcal{H}$, $(h(p))(z) = zp(z)$, with respect to $\{\varphi_n(z)\}_{n\geqslant 0}$. The matrix representation is given by $z\varphi(z) = \mathbb{H}\varphi(z)$, where

$$\varphi(z) = [\varphi_0(z), \varphi_1(z), \cdots]^T. \tag{2.12}$$

The entries of the matrix $\mathbb{H}$ can be obtained via

$$\mathbb{H}_{i,j} := \left\langle z\varphi_i, \varphi_j \right\rangle_{\mathcal{L}}, \qquad i, j \geqslant 0,$$

and therefore $\mathbb{H}$ is a semi-infinite irreducible lower Hessenberg matrix whose entries are given in terms of the Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$ as follows:

$$\mathbb{H}_{i,j} = \begin{cases} \frac{-\kappa_j}{\kappa_i}\Phi_{i+1}(0)\overline{\Phi_j(0)}, & j \leqslant i, \\ \frac{\kappa_i}{\kappa_{i+1}}, & j = i+1, \\ 0, & j > i+1. \end{cases} \tag{2.13}$$

The characteristic polynomial of the principal leading $n \times n$ submatrix $\mathbb{H}_n$ of $\mathbb{H}$ is the nth monic orthogonal polynomial [18,27]. That is, the zeros of $\Phi_n(z)$ are the eigenvalues of $\mathbb{H}_n$. Hence, the spectral theory of $\mathbb{H}$ can provide useful relations between the zeros of orthogonal polynomials and the Verblunsky coefficients. The increasing attention to the analysis of the zeros of OPUC, according to their extensive applications, makes the spectral study of the multiplication operator of special interest. The recurrence relation of the orthogonal polynomials on the unit circle (unlike the scalar case) does not conclude a spectral representation for their zeros. In fact, such a representation can be obtained by computing the matrix corresponding to the multiplication operator in the linear space of complex polynomials when orthogonal polynomials are chosen as a basis, but the result is the irreducible Hessenberg matrix (2.13) with much more complicated structure than the Jacobi matrix on the real line.

We now revisit the Szegő class $\mathcal{S}$, which is an important class of measures characterized in [59].

Proposition 2.19. *Let $d\mu$ be a nontrivial probability measure, the following statements are equivalent:*

i. *The measure belongs to the Szegő class $\mathcal{S}$.*
ii. $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty$.
iii. $\lambda_\infty(0) > 0$, *where* $\lambda_\infty(z) = lim_{n\to\infty} \frac{1}{K_n(z,z)}$ *is the Christoffel function associated with $d\mu$.*
iv. *The linear space of polynomials is not dense in $\mathcal{H}$, i.e. the OPUC orthonormal sequence does not constitute a complete orthonormal system in $\mathcal{H}$.*

We can deduce that if the measure $d\mu$ belongs to $\mathcal{S}$, the matrix $\mathbb{H}$ is not unitary, but has orthonormal rows. It is called almost unitary, in the sense that

$$\mathbb{H}\mathbb{H}^* = I, \qquad \mathbb{H}^*\mathbb{H} = I - \lambda_\infty(0)\overline{\Phi(0)}\Phi(0)^T,$$

where I is the semi-infinite identity matrix, $\lambda_\infty(0)$ was introduced in Proposition 2.19, $\Phi(0) = [\Phi_0(0), \Phi_1(0), \cdots]^T$.

Thus, the matrix $\mathbb{H}$ is unitary if and only if the measure does not belong to the Szegő class. This constraint is one of the main deficiencies of the GGT representation. Taking into account the linear space of Laurent polynomials Λ is dense in $\mathcal{H}$ independently of the measure μ, then it is natural to work with orthogonal Laurent polynomials instead of OPUC. The matrix representation of the multiplication operator in terms of the orthonormal Laurent polynomial basis given in [12] yields a five-diagonal matrix, called CMV matrix. It shows more convenient spectral representations for OPUC, and will be explained in the following subsection.

2.2.2. CMV Matrix

According to the latter description, GGT matrix does not seem a promising way to study properties of orthogonal polynomials on the unit circle. A more pertinent presentation of orthogonal polynomials on the unit circle is through CMV matrix. This matrix representation gives a spectral interpretation for the zeros of orthogonal polynomials which is much simpler than the one given by the GGT matrices according to its special structure. The initials CMV honor Cantero, Moral and Velázquez [12], and hence the name of the special unitary matrices.

The CMV matrices came to the theory of orthogonal polynomials on the unit circle as a unitary analogue of Jacobi matrices for orthogonal polynomials on the real line. In fact, it comes from the representation of the multiplication operator in the linear space of Laurent polynomials, when a suitable orthonormal basis related to the orthogonal polynomials is chosen. These matrices play a similar role among unitary matrices as Jacobi matrices among all Hermitian matrices, for instance, see [45,61]. This analogy is illustrated in many fields of application such as random matrix theory and integrable systems [52], Dirichlet data of a circular periodic problem [53] and scattering problem [58]. The spectral analysis of CMV matrices has also attracted much attention in the last years [2,12,13,31,60].

The CMV basis $\{\chi_n(z)\}_{n\geqslant 0}$ is obtained by orthonormalization of the basis $\{1, z, z^{-1}, z^2, z^{-2}, \cdots\}$ using the Gram–Schmidt process.

Definition 2.20. Given a positive definite Hermitian functional $\mathcal{L}$ on Λ, we denote by $\{\chi_n(z)\}_{n\geqslant 0}$ the sequence of orthonormal CMV polynomials defined by

$$\begin{aligned} \chi_{2n}(z) &= z^{-n}\varphi^*_{2n}(z), && n \geqslant 0, \\ \chi_{2n+1}(z) &= z^{-n}\varphi_{2n+1}(z), && n \geqslant 0, \end{aligned}$$

where $\{\varphi_n(z)\}_{n\geqslant 0}$ is the corresponding sequence of orthonormal polynomials [12].

The entries of the matrix representation of the multiplication operator in terms of the basis $\{\chi_n(z)\}_{n\geqslant 0}$ of Λ (the CMV matrix) are

$$\mathcal{C}_{i,j} := \langle z\chi_i, \chi_j\rangle_{\mathcal{L}},$$

see [59,61] for more details. Taking into account the above sequence of orthonormal CMV polynomials satisfies a five-term recurrence relation that follows in a straightforward way from the recurrence relations for the corresponding orthonormal OPUC sequence, we get (see [59,61])

$$\mathcal{C} = \begin{pmatrix} -\Phi_1(0) & \rho_1 & 0 & 0 & 0 & \cdots \\ -\Phi_2(0)\rho_1 & -\Phi_2(0)\overline{\Phi_1(0)} & -\Phi_3(0)\rho_2 & \rho_3\rho_2 & 0 & \cdots \\ \rho_2\rho_1 & \overline{\Phi_1(0)}\rho_2 & -\Phi_3(0)\overline{\Phi_2(0)} & \overline{\Phi_2(0)}\rho_3 & 0 & \cdots \\ 0 & 0 & -\Phi_4(0)\rho_3 & -\Phi_4(0)\overline{\Phi_3(0)} & -\Phi_5(0)\rho_4 & \cdots \\ 0 & 0 & \rho_4\rho_3 & \overline{\Phi_3(0)}\rho_4 & -\Phi_5(0)\overline{\Phi_4(0)} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Thus,

Theorem 2.21. *(See [12].) For the CMV orthonormal basis $\{\chi_n(z)\}_{n\geqslant 0}$ of $L^2(d\mu)$, the multiplication operator $f(z) \to zf(z)$ in the linear space of Laurent polynomials is represented by the matrix $\mathcal{C}$ whose entries are given in terms of the Verblunsky coefficients of the measure $d\mu$.*

There is an efficient way of writing $\mathcal{C}$ which is useful for computational purposes.

Definition 2.22. If

$$\Theta_n := \begin{pmatrix} -\Phi_{n+1}(0) & \rho_{n+1} \\ \rho_{n+1} & \overline{\Phi_{n+1}(0)} \end{pmatrix}, \tag{2.14}$$

then the Θ factorization of the CMV matrix is

$$\mathcal{C} = \mathcal{M}\mathcal{L}, \tag{2.15}$$

where $\mathcal{M}$ is a tridiagonal matrix with a single 1×1 block, denoted by 1, followed by 2×2 blocks, and $\mathcal{L}$ is a tridiagonal matrix of 2×2 blocks, as follows:

$$\mathcal{M} := 1 \oplus \Theta_1 \oplus \Theta_3 \oplus \Theta_5 \oplus \cdots \tag{2.16}$$

$$\mathcal{L} := \Theta_0 \oplus \Theta_2 \oplus \Theta_4 \oplus \cdots \tag{2.17}$$

This definition reveals the appropriate name of Θ factorization. The CMV matrix is a para-tridiagonal matrix and according to (2.10), Θ matrices are unitary which cause the CMV matrix (2.15) to be unitary.

3. Congruence for Θ matrices

We have now completed all the steps required to get the relations of the matrices Θ_n and $\widetilde{\Theta}_n$ under unitary *congruence and unitary congruence. It is worth stressing the importance of viewing Θ matrices as the basic feature of CMV matrices.

*3.1. *Congruence*

To ensure that a unitary similarity relation exists for Θ matrices, some conditions have to be imposed on the complex Verblunsky coefficients.

Theorem 3.1. *Let Θ_n and $\widetilde{\Theta}_n$ be the Θ matrices defined in (2.14), corresponding to the Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$ and $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$, respectively. Θ_n is unitarily *congruent to $\widetilde{\Theta}_n$ if and only if*

$$\mathfrak{Im}(\Phi_n(0)) = \mathfrak{Im}(\widetilde{\Phi}_n(0)), \qquad n \geqslant 1. \tag{3.1}$$

Proof. The proof proceeds through four different approaches, depending on various properties of unitary similar matrices.

First approach: Applying Lemma 2.2, Θ_n is unitarily *congruent to $\widetilde{\Theta}_n$ if and only if $tr(\Theta_n) = tr(\widetilde{\Theta}_n)$, $tr(\Theta_n^2) = tr(\widetilde{\Theta}_n^2)$ which leads directly to the condition (3.1).

Second approach: We deal with the problem of unitary similarity as a particular case of similarity. Considering Lemma 2.3, Θ_n is unitary *congruent to $\widetilde{\Theta}_n$ if and only if they are similar. We already know similar matrices have the same eigenvalues, which satisfy

$$t^2 + 2i\mathfrak{Im}(\Phi_n(0))t - 1 = 0, \tag{3.2}$$

$$t^2 + 2i\mathfrak{Im}(\widetilde{\Phi}_n(0))t - 1 = 0, \tag{3.3}$$

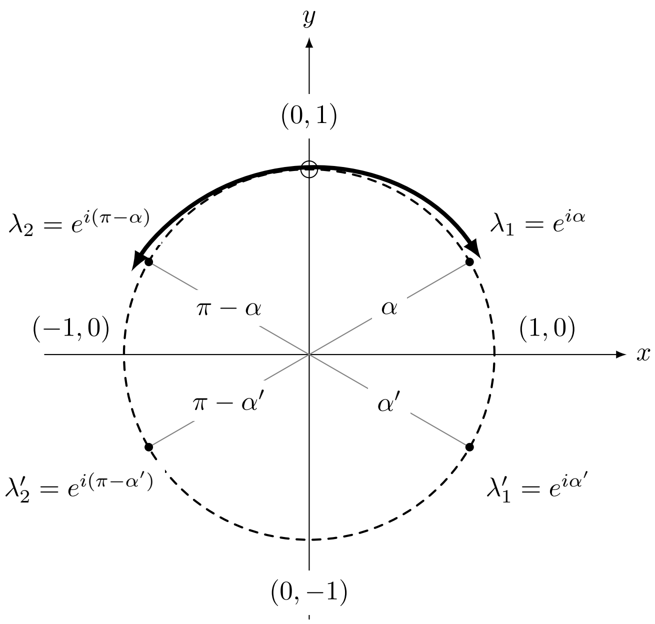
and it is obvious to see that equality (3.1) has to be hold.

Third approach: Theorem 2.4 shows that unitary similarity is identical to *congruence. On the other hand, Lemma 2.5 acclaims that for Θ_n and $\widetilde{\Theta}_n$, the necessary and sufficient condition for the unitary *congruent is the same eigenvalues which follows immediately from the second approach. In this case, the same rank is the clear characteristic of invertible Θ matrices.

Fourth approach: An alternative approach to this problem can be given from a geometrical point of view. For the Θ matrix Θ_n, its eigenvalues are the zeros of the characteristic polynomial (3.2). Then, there exists $0 < \alpha \neq \frac{\pi}{2}$, such that $\lambda_1 = e^{i\alpha}$ and $\lambda_2 = -\bar{\lambda}_1$. Without loss of generality, suppose that $0 < \alpha < \frac{\pi}{2}$, then $\lambda_2 = e^{i(\pi-\alpha)}$. Following the same process for $\widetilde{\Theta}_n$, there exists $0 < \beta < \frac{\pi}{2}$ such that the eigenvalues of $\widetilde{\Theta}_n$ are $\vartheta_1 = e^{i\beta}$ and $\vartheta_2 = e^{i(\pi-\beta)}$. β can be chosen as

$$\alpha < \beta \leqslant \pi - \alpha, \qquad \alpha \leqslant \beta < \pi - \alpha. \tag{3.4}$$

The following figure shows the location of the eigenvalues on the thick arc.



Next, we turn to $\widetilde{\Theta}_n$, and changing the role of the matrices, we have the same plot with β replaced by α. From an algebraic point of view

$$\beta < \alpha \leqslant \pi - \beta, \qquad \beta \leqslant \alpha < \pi - \beta, \tag{3.5}$$

and combining the inequalities of (3.4) and (3.5), yields simultaneously

$$0 < \alpha \leqslant \beta < \frac{\pi}{2}, \qquad 0 < \beta \leqslant \alpha < \frac{\pi}{2},$$

which gives $\beta = \alpha$. The same procedure can be done for eigenvalues below the vertical axis, namely α' for the argument of the first eigenvalue. This proves that the matrices are constrained to have the same eigenvalues. $\square$

According to Theorem 2.4, for Θ matrices, unitary *congruence is equivalent to *congruence. Obtaining these relations needs the equation $XA - AX = 0$ to be solved. This equation is a special case of linear matrix equation $AX - XB = C$ called *Sylvester's equation*, which is well-studied in [23]. Besides, a matrix identity $AX = XB$ is known as an *intertwining relation*. The following theorem solves a Sylvester's equation in the case of unitary similarity of Θ matrices.

Theorem 3.2. *For the Verblunsky coefficients* $\{\Phi_n(0)\}_{n\geqslant 1}$ *and* $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$, *let us assume that condition* (3.1) *holds. Then there exist unitary matrices* U_n *such that*

$$\Theta_n = U_n \widetilde{\Theta}_n U_n^*, \qquad n \geqslant 0. \tag{3.6}$$

Proof. For fixed n, (3.6) can be written as

$$\Theta U = U \widetilde{\Theta}, \tag{3.7}$$

which is a kind of Sylvester's equation. One may be able to discover the special structure by replacing Θ and $\widetilde{\Theta}$ by canonical forms and studying the resulting intertwining relation involving the canonical forms and a transformed U. Since Θ matrices are normal, the spectral decomposition (2.1) gives

$$\Theta := U_\Theta D_\Theta U_\Theta^*, \qquad \widetilde{\Theta} := U_{\widetilde{\Theta}} D_{\widetilde{\Theta}} U_{\widetilde{\Theta}}^*, \tag{3.8}$$

for unitary matrices U_Θ and $U_{\widetilde{\Theta}}$. Note that the distinct eigenvalues of Θ (which are equal to eigenvalues of $\widetilde{\Theta}$) occur with modulus one on the diagonal of $D_\Theta = D_{\widetilde{\Theta}}$. Replacing (3.8) in (3.7) and some manipulation get the transformed Sylvester's equation

$$D_\Theta \widetilde{U} = \widetilde{U} D_{\widetilde{\Theta}}, \tag{3.9}$$

for $\widetilde{U} := U_\Theta^* U U_{\widetilde{\Theta}}$. The trivial commutator of the intertwining relation (3.9) is the identity matrix. Afterward

$$U = U_\Theta U_{\widetilde{\Theta}}^{-1}, \tag{3.10}$$

is the desired unitary matrix displayed in (3.6). $\square$

We have already proved that unitary *congruent and *congruent can be presented for the Θ matrices under the condition (3.1). A straightforward observation is that unitary similarity corresponds to a change of basis, but of a special type. Indeed, it corresponds to a change from an orthonormal basis to another. It is worthy to mention that (3.10) shows the matrix of change of basis.

3.2. Congruence

The remaining natural equivalence relation is unitary congruence which, according to Corollary 2.10 holds for Θ matrices associated with any sequence of Verblunsky coefficients. Lemma 2.7 yields the equivalence of unitary congruence and congruence for Θ matrices. In this case, the condition (3.1) is not required to be imposed on the Verblunsky coefficients.

Corollary 3.3. *For the Verblunsky coefficients* $\{\Phi_n(0)\}_{n\geqslant 1}$ *and* $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$*, there exist unitary matrices* U_n *such that*

$$\Theta_n = U_n \widetilde{\Theta}_n U_n^T. \tag{3.11}$$

Proof. Ignoring the index n in the Takagi decomposition of Θ_n and $\widetilde{\Theta}_n$, we have

$$\Theta = U_\Theta U_\Theta^T, \qquad \widetilde{\Theta} = U_{\widetilde{\Theta}} U_{\widetilde{\Theta}}^T, \tag{3.12}$$

for unitary matrices U_Θ and $U_{\widetilde{\Theta}}$. Then Theorem 2.13 gives

$$U = U_\Theta U_{\widetilde{\Theta}}^*, \tag{3.13}$$

for the congruence relation (3.11). $\square$

Remark 3.4. It is surprising to note that, according to Theorem 3.2 and Corollary 3.3, both unitary *congruence and unitary congruence factor of Θ_n and $\widetilde{\Theta}_n$ is $U_\Theta U_{\widetilde{\Theta}}^*$, look at formulas (3.10) and (3.13), respectively. Aside from the fact that the unitary matrices of unitary *congruence in (3.8) have been obtained from the Spectral decomposition (2.1), while for unitary congruent in (3.12), they have been given by Takagi decomposition (2.2).

4. Computation of Θ matrices congruence

Since real Verblunsky coefficients trivially satisfy the condition (3.1), using Theorem 3.2, we can obtain the *unitary similarity relation.* Concerning any pair of complex Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$ and $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$, Corollary 3.3 connects the Θ matrices by a *unitary congruent relation.*

4.1. Real Verblunsky coefficients

The following theorem describes in an explicit formula how, given a pair of real Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$ and $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$, unitary factors can be obtained. To distinguish the real case, let us denote the unitary matrix in (3.6) by Δ_n.

Theorem 4.1. *For the real Verblunsky coefficients* $\{\Phi_n(0)\}_{n\geqslant 1}$ *and* $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$*, the* Θ_n *matrices satisfy*

$$\Theta_n = \Delta_n \widetilde{\Theta}_n \Delta_n^*, \tag{4.1}$$

where the unitary matrix Δ_n *is defined as*

$$\Delta_n = \frac{1}{\gamma_{n+1}} \begin{pmatrix} \mu_{n+1} & \sigma_{n+1} \\ -\sigma_{n+1} & \mu_{n+1} \end{pmatrix}, \tag{4.2}$$

and

$$\mu_n := (\Phi_n(0)+1)(\widetilde{\Phi}_n(0)+1)+\rho_n\widetilde{\rho}_n, \tag{4.3}$$

$$\sigma_n := \rho_n(\widetilde{\Phi}_n(0)+1)-\widetilde{\rho}_n(\Phi_n(0)+1), \tag{4.4}$$

$$\gamma_n^2 := 4(\Phi_n(0)+1)(\widetilde{\Phi}_n(0)+1). \tag{4.5}$$

Proof. The matrix Θ_n for $\{\Phi_n(0)\}_{n\geqslant 1}$ is spectrally decomposed, see (3.8), by the diagonal matrix of eigenvalues 1, -1 and the unitary matrix:

$$U_\Theta = \Big(\frac{\Phi_{n+1}(0)+1}{2}\Big)^{\frac{1}{2}} \begin{pmatrix} \frac{\rho_{n+1}}{\Phi_{n+1}(0)+1} & 1 \\ 1 & -\frac{\rho_{n+1}}{\Phi_{n+1}(0)+1} \end{pmatrix}.$$

The final statement (4.1) follows from (3.10) by some tedious calculations. □

Since the Verblunsky coefficients are real, the unitary matrix Δ_n can be simplified as follows:

Corollary 4.2. *Let $\omega_{n,i,j}$ for $i,j=0,1,$ be defined as follows*

$$\omega_{n,i,j}^2 := \big(1+(-1)^i\Phi_n(0)\big)\big(1+(-1)^j\widetilde{\Phi}_n(0)\big), \tag{4.6}$$

and

$$\eta_n := \omega_{n,0,0}+\omega_{n,1,1}, \qquad \beta_n := \omega_{n,1,0}-\omega_{n,0,1}. \tag{4.7}$$

Then

$$\Delta_n = \frac{1}{2}\begin{pmatrix} \eta_{n+1} & \beta_{n+1} \\ -\beta_{n+1} & \eta_{n+1} \end{pmatrix}. \tag{4.8}$$

The latter corollary invokes a similarity relation for Θ matrices that condenses formulas (4.3)–(4.5).

4.2. Complex Verblunsky coefficients

Our final result, discussing congruence relations, deals with the unitary congruence of general sequences of complex Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$ and $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$.

Theorem 4.3. *Let $\{\Phi_n(0)\}_{n\geqslant 1}$ and $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$ be two sequences of Verblunsky coefficients. The congruence relation*

$$\Theta_n = U_n\widetilde{\Theta}_n U_n^T, \tag{4.9}$$

holds for the unitary matrix U_n, which is defined as

$$U_n = \frac{1}{\lambda_{n+1}} \begin{pmatrix} \tau_{n+1} & \bar{\xi}_{n+1} \\ -\xi_{n+1} & \bar{\tau}_{n+1} \end{pmatrix}, \tag{4.10}$$

where

$$\tau_n := (\Phi_n(0) - 1)(\overline{\widetilde{\Phi}_n(0)} - 1) + \rho_n \widetilde{\rho}_n, \tag{4.11}$$

$$\xi_n := \rho_n(\overline{\widetilde{\Phi}_n(0)} - 1) - \widetilde{\rho}_n(\overline{\Phi_n(0)} - 1), \tag{4.12}$$

$$\lambda_n^2 := 4\big(1 - \mathfrak{Re}(\Phi_n(0))\big)\big(1 - \mathfrak{Re}(\widetilde{\Phi}_n(0))\big). \tag{4.13}$$

Proof. To begin with, let us first compute the Takagi decomposition of Θ_n as formula (2.3). Following Lemma 2.12, let us deal with the most trivial choice for $x = [1, 0]^T$, then $y = \Theta_n \bar{x} + x = [1 - \Phi_{n+1}(0), \rho_{n+1}]^T$. After normalizing, y takes the form

$$y_n = \frac{1}{l_{n+1}} \begin{pmatrix} 1 - \Phi_{n+1}(0) \\ \rho_{n+1} \end{pmatrix},$$

where l_n is the 2-norm of y as $l_n^2 = |1 - \Phi_n(0)|^2 + \rho_n^2 = 2\big(1 - \mathfrak{Re}(\Phi_n(0))\big)$. The structure of V_1 shows that

$$V_1 = \frac{1}{l_{n+1}} \begin{pmatrix} 1 - \Phi_{n+1}(0) & \dfrac{\rho_{n+1}}{\overline{\Phi_{n+1}(0)} - 1} \\ \rho_{n+1} & \end{pmatrix},$$

and put

$$V_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

Then $U_\Theta := V_1 V_2$ satisfies

$$\Theta_n = U_\Theta U_\Theta^T. \tag{4.14}$$

The essential observation is that U_Θ is a unitary matrix. Simultaneously, $\widetilde{\Theta}_n$ is decomposed as $\widetilde{\Theta}_n = U_{\widetilde{\Theta}} U_{\widetilde{\Theta}}^T$, where

$$U_{\widetilde{\Theta}} = \frac{1}{\widetilde{l}_{n+1}} \begin{pmatrix} 1 - \widetilde{\Phi}_{n+1}(0) & \dfrac{\widetilde{\rho}_{n+1}}{\overline{\widetilde{\Phi}_{n+1}(0)} - 1} \\ \widetilde{\rho}_{n+1} & \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$

coupled with $\widetilde{l}_n^2 = 2\big(1 - \mathfrak{Re}(\widetilde{\Phi}_n(0))\big)$.

It only remains to follow Corollary 3.3 to obtain U_n from (3.13). Some simple manipulation yields:

$$U_n = \frac{1}{l_{n+1}\widetilde{l}_{n+1}} \begin{pmatrix} 1 - \Phi_{n+1}(0) & \dfrac{i\rho_{n+1}}{i(\overline{\Phi_{n+1}(0)} - 1)} \\ \rho_{n+1} & \end{pmatrix} \begin{pmatrix} 1 - \overline{\widetilde{\Phi}_{n+1}(0)} & \widetilde{\rho}_{n+1} \\ -i\widetilde{\rho}_{n+1} & -i(\widetilde{\Phi}_{n+1}(0) - 1) \end{pmatrix}.$$

All things considered, the unitary matrix U_n in (4.10) is the desired congruence factor. $\square$

As a summary, Theorem 4.3 establishes a beautiful and substantial decomposition (4.14) for Θ matrices, and (4.9) is the most pleasant relation that can connect Θ matrices, Θ_n and $\widetilde{\Theta}_n$ corresponding to complex Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$ and $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$.

The results motivated the adoption of two conventions:

- For real Verblunsky coefficients, we follow Theorem 4.1 to get unitary *congruence (4.1) for Θ matrices.
- Once the Verblunsky coefficients are complex, we bring the unitary congruence (4.9) from Theorem 4.3 into play.

A subtle point worth mentioning is that *congruence can be also a successful relation for the complex Verblunsky coefficients with identical imaginary parts.

5. Characterization of CMV matrices

The results of the previous section enable us to characterize the relation between CMV matrices. Considering two families of Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$ and $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$, the following theorem shows this relation.

Theorem 5.1. *Let $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ be CMV matrices of Verblunsky sequences $\{\Phi_n(0)\}_{n\geqslant 1}$ and $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$, respectively. There exist unitary matrices V and W such that*

$$V\mathcal{C}W = \widetilde{\mathcal{C}}, \tag{5.1}$$

where

$$\begin{aligned} V &:= 1 \oplus V_1 \oplus V_2 \oplus V_3 \oplus \cdots \\ W &:= W_0 \oplus W_1 \oplus W_2 \oplus \cdots \end{aligned}$$

and V_n and W_n are 2×2 matrices

$$V_n := \widetilde{\Theta}_{2n-1}\Theta_{2n-1}^{-1}, \qquad W_n := \Theta_{2n}^{-1}\widetilde{\Theta}_{2n}. \tag{5.2}$$

Proof. Given $\widetilde{\mathcal{C}} := \widetilde{\mathcal{M}}\widetilde{\mathcal{L}}$ similar to (2.15), the matrices

$$V = \widetilde{\mathcal{M}}\mathcal{M}^{-1}, \qquad W = \mathcal{L}^{-1}\widetilde{\mathcal{L}},$$

present an instructive approach to get the result. $\square$

The above theorem shows that, knowing $\widetilde{\Theta}_n$ for any class of $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$, the formula (5.1) yields the relation between $\mathcal{C}$ and $\widetilde{\mathcal{C}}$, so the congruence notions of Θ matrices are crucial in the investigation of CMV matrices.

To conclude this section, let us consider the sequence of Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$ and its associated CMV matrix, $\mathcal{C}$, likely to $\widetilde{\mathcal{C}}$ for $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$. We are now in a position to summarize the relation between CMV matrices. Half of this assertion is characterized by the relation of Θ matrices which is studied in Section 4. The foregoing observations suggest that if one thinks of unitary to be preserved, then it is wise to suppose that the Verblunsky coefficients lie on the real line. Theorem 4.1 provides unitary similarity relation (4.1) and Theorem 4.3 presents unitary congruence (4.9). The other half relies on Theorem 5.1 which augmented with identities that yield the relation of CMV matrices.

6. Perturbed measures

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc on the complex plane and let $d\mu$ be a non-trivial probability measure supported on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. For a class of perturbations of the measure $d\mu$, algebraic and analytic properties of the sequences of polynomials orthogonal with respect to the perturbed measure $d\tilde{\mu}$ have been extensively studied, see [9–11,15,19–21,26,28,41,46,48,49] among others. We will focus our attention on three examples of perturbations:

(1) $d\widetilde{\mu}_C = |z-\alpha|^2 d\mu$, $\alpha \in \mathbb{C}$, $|z| = 1$,
(2) $d\widetilde{\mu}_U = d\mu + m\delta(z-\alpha)$, $|\alpha| = 1$, $m \in \mathbb{R}_+$,
(3) $d\widetilde{\mu}_G = \frac{1}{|z-\alpha|^2} d\mu + m\delta(z-\alpha) + \overline{m}\delta(z-\bar{\alpha}^{-1})$, $|z| = 1$, $|\alpha| > 1$, $m \in \mathbb{C}\setminus\{0\}$.

They are known in the literature as canonical Christoffel ($\mathcal{F}_C$), Uvarov ($\mathcal{F}_U$), and Geronimus ($\mathcal{F}_G$) transformations. They are related by

- $\mathcal{F}_C \circ \mathcal{F}_G = \mathcal{I}$ (identity transformation).
- $\mathcal{F}_G \circ \mathcal{F}_C = \mathcal{F}_U$.

The above perturbations are examples of linear spectral transformations (see [21]). On the other hand, rational spectral transformations for Hermitian Toeplitz matrices are introduced in [57] and they have been analyzed in [15,22], among others. The canonical spectral transformations on the real line have been investigated by several authors. The matrix approach to the canonical transformations of a Jacobi matrix (the representation of the multiplication operator in terms of an orthogonal basis of polynomials in the real line) as well as the analysis of the generators of the family of linear and rational spectral functions, respectively, have been done in [7,63,66,67].

In this section, first, the relations between the corresponding families of orthogonal polynomials of the new measure $d\tilde{\mu}$, that appears as a perturbation of the nontrivial probability measure $d\mu$ will be presented. In the case of quasi-definite linear functionals, necessary and sufficient conditions for the quasi-definite character of the perturbed linear functional will be explained. Then, the behavior of the families of Verblunsky coefficients of the perturbed family will be studied for each of the canonical linear transformations (Christoffel, Uvarov, and Geronimus). We particularly point out the unitary similarity relation of CMV matrices according to the perturbation of the measure.

6.1. Christoffel transformation

For $\alpha \in \mathbb{C}$, a Hermitian bilinear functional is defined as

$$\langle p, q \rangle_{\mathcal{L}_C} = \langle (z-\alpha)p, (z-\alpha)q \rangle_{\mathcal{L}}, \qquad p, q \in \mathbb{P}.$$

In [15,48] the connection between the associated Hessenberg matrices using the QR factorization has been studied. In the real case, the iteration of the canonical Christoffel transformation yields a connection between the corresponding Jacobi matrices, and using the QR factorization of the Jacobi associated matrix, it has been analyzed in several papers [6,8,26,41,46].

Since the Verblunsky coefficients are the values of monic orthogonal polynomials at $z = 0$, one of the most important results about the perturbed measure is the explicit formula of the corresponding family of orthogonal polynomials for Christoffel transformation [48].

Proposition 6.1. *The family of monic orthogonal polynomials with respect to $\mathcal{L}_C$ is given by*

$$\widetilde{\Phi}_n(z) = \frac{1}{z-\alpha}\Big(\Phi_{n+1}(z) - \frac{\Phi_{n+1}(\alpha)}{K_n(\alpha,\alpha)}K_n(z,\alpha)\Big), \quad n \geqslant 0.$$

Since the Verblunsky coefficients characterize the family of orthogonal polynomials, the following lemma illustrates them [21].

Lemma 6.2. *The sequence of Verblunsky coefficients for the Christoffel transformation is*

$$\widetilde{\Phi}_{C,n}(0) = A_C(\alpha,n)\Phi_{n+1}(0) + B_C(\alpha,n), \qquad n \geqslant 1, \tag{6.1}$$

where

$$A_C(\alpha,n) = \frac{\bar{\alpha}\Big(|\Phi_n(\alpha)|^2 - |\Phi_n^*(\alpha)|^2\Big)}{|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2}, \tag{6.2}$$

$$B_C(\alpha,n) = \frac{\Phi_n(\alpha)\overline{\Phi_n^*(\alpha)}(1-|\alpha|^2)}{|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2}. \tag{6.3}$$

Remark 6.3. The Christoffel canonical transformation preserves positive definiteness of the measure. That is, assuming $d\mu$ to be positive definite, $d\widetilde{\mu}$ is also positive definite, but $\mathcal{L}_C$ is quasi-definite if and only if $K_n(\alpha,\alpha) \neq 0$, $n \geqslant 1$.

For convenient references, we restate the Θ matrix for this measure:

$$\widetilde{\Theta}_{C,n} := \begin{pmatrix} -\widetilde{\Phi}_{C,n+1}(0) & \dfrac{\widetilde{\rho}_{C,n+1}}{\overline{\widetilde{\Phi}}_{C,n+1}(0)} \\ \widetilde{\rho}_{C,n+1} & \end{pmatrix}, \tag{6.4}$$

where $\widetilde{\rho}^2_{C,n} := 1 - |\widetilde{\Phi}_{C,n}(0)|^2$. Note that the index C changes to U and G for the Uvarov and Geronimus measure, respectively, in the next two subsections.

6.2. The Uvarov transformation

In this section, we focus our attention on the Uvarov transformation. Consider the Hermitian bilinear functional

$$\langle p, q \rangle_{\mathcal{L}_U} = \langle p, q \rangle_{\mathcal{L}} + mp(\alpha)\overline{q(\alpha)}, \quad |\alpha| = 1, \quad m \in \mathbb{R}_+, \; p, q \in \mathbb{P}.$$

In [15], the connection between the corresponding sequences of monic orthogonal polynomials as well as the associated Hessenberg matrices using the LU and QR factorization has been studied. The iteration of this transformation has been discussed in [24,46], and asymptotic properties for the corresponding family of orthogonal polynomials have been analyzed in [68].

Proposition 6.4. *The family of monic orthogonal polynomials with respect to $\mathcal{L}_U$ is given by*

$$\widetilde{\Phi}_n(z) = \Phi_n(z) - \frac{m\Phi_n(\alpha)}{1 + mK_{n-1}(\alpha, \alpha)} K_{n-1}(z, \alpha).$$

We make use of the above proposition to calculate Verblunsky coefficients.

Lemma 6.5. *The sequence of Verblunsky coefficients*

$$\widetilde{\Phi}_{U,n}(0) = A_U(\alpha, n)\Phi_n(0) + B_U(\alpha, n), \tag{6.5}$$

with

$$A_U(\alpha, n) = 1 - \frac{m|\Phi^*_{n-1}(\alpha)|^2}{\mathbf{k}_{n-1}\big(1 + mK_{n-1}(\alpha, \alpha)\big)}, \tag{6.6}$$

$$B_U(\alpha, n) = \frac{-m\alpha\Phi_{n-1}(\alpha)\overline{\Phi^*_{n-1}(\alpha)}}{\mathbf{k}_{n-1}\big(1 + mK_{n-1}(\alpha, \alpha)\big)}, \tag{6.7}$$

characterizes the family of orthogonal polynomials with Uvarov perturbed measure.

Remark 6.6. For fixed α, the necessary and sufficient conditions about the choices of $m \in \mathbb{R}_+$ such that the linear functional $\mathcal{L}_U$ is quasi-definite have been obtained in [21, 48], which is $1+mK_{n-1}(\alpha,\alpha) \neq 0$, $n \geqslant 1$, in contrast to the fact that it is unconditionally true for positive definite measures.

6.3. Geronimus transformation

There are many solutions to the inverse problem of Christoffel transformation

$$\big\langle (z-\alpha)p, (z-\alpha)q \big\rangle_{\mathcal{L}_G} = \big\langle p, q \big\rangle_{\mathcal{L}}, \qquad p, q \in \mathbb{P},$$

which are defined up to the addition of a trivial linear functional $m\delta(z-\alpha)+\overline{m}\delta(z-\bar{\alpha}^{-1})$. Here for $m \in \mathbb{C}\backslash\{0\}$, $|\alpha| > 1$, we consider

$$\big\langle p, q \big\rangle_{\mathcal{L}_G} = \int_{\mathbb{T}} \frac{1}{|z-\alpha|^2} p(z)\overline{q(z)}d\mu + mp(\alpha)\overline{q(\bar{\alpha}^{-1})} + \overline{m}p(\bar{\alpha}^{-1})\overline{q(\alpha)},$$

where $\mathcal{L}_G$ is called the Geronimus transformation of $\mathcal{L}$. The relation between the corresponding family of monic orthogonal polynomials and the associated Hessenberg matrices is stated in [47]. A more general framework is presented in [28]. Besides, a special case of the Geronimus transformation has been analyzed in [20].

Proposition 6.7. *(See [20].) The family of monic orthogonal polynomials with respect to $\mathcal{L}_G$ is*

$$\widetilde{\Phi}_{n+1}(z) = (z-\alpha)\Phi_n(z) + \frac{\overline{A}_n}{\varepsilon_{n-1}(\alpha)}\Big(1 + (z-\alpha)\sum_{j=0}^{n-1} \frac{A_j}{\mathbf{k}_j}\Phi_j(z)\Big),$$

where

$$A_j = Q_j(\alpha) + m(\bar{\alpha} - \alpha^{-1})\overline{\Phi_j(\bar{\alpha}^{-1})}, \qquad 0 \leqslant j \leqslant n, \tag{6.8}$$

and

$$\varepsilon_n(\alpha) := \|\mu_G\| - \sum_{j=0}^{n} |q_j(\alpha) + m(\bar{\alpha} - \alpha^{-1})\overline{\varphi_j(\bar{\alpha}^{-1})}|^2, \tag{6.9}$$

where $\|\mu_G\| = \int_{\mathbb{T}} \frac{d\mu}{|z-\alpha|^2} + m + \overline{m}$.

Lemma 6.8. *The sequence of Verblunsky coefficients associated with $\mathcal{L}_G$ is given by:*

$$\widetilde{\Phi}_{G,n}(0) = \frac{\overline{A}_{n-1}}{\varepsilon_{n-2}(\alpha)} + \sum_{k=0}^{n-1} D(n,k)\Phi_k(0), \qquad n \geqslant 1, \tag{6.10}$$

where

$$D(n,k) := \begin{cases} -\frac{\overline{A}_{n-1}A_k}{\varepsilon_{n-2}(\alpha)\mathbf{k}_k}\alpha, & 0 \leqslant k \leqslant n-2, \\ -\alpha, & k = n-1. \end{cases} \tag{6.11}$$

Remark 6.9. The linear functional $\mathcal{L}_G$ is quasi-definite if and only if $\varepsilon_n(\alpha) \neq 0$, $n \geqslant 0$.

6.4. CMV matrices corresponding to the canonical spectral transformations

Let us remind that our aim is to study the unitary similarity relation between CMV matrices. First, the matrix (4.8) characterizes the unitary relation of Θ matrices Θ_n and $\widetilde{\Theta}_{C,n}$, introduced in (2.14) and (6.4), respectively. (The same relation is required for Uvarov and Geronimus measure, when the index C changes to U and G, respectively.) We remark that the comparison has been made for real Verblunsky coefficients by applying Theorem 4.1, and for complex Verblunsky parameters, Theorem 4.3 gives the desired relation, although the complicated structure of the transformed Verblunsky coefficients does not let us simplify the unitary matrix (4.10). Then formula (5.2) uses the relation of Θ matrices, and finally, (5.1) shows the explicit relation between $\mathcal{C}$ and $\widetilde{\mathcal{C}}$.

In short, the desired relation of CMV matrices reduces to the computation of $\omega_{n,i,j}$ in (4.6). In the following theorem, we indicate an explicit formula of $\omega_{n,i,j}$ for the above mentioned transformations.

Theorem 6.10. *The coefficients $\omega^2_{n,i,j}$ for Christoffel, Uvarov and Geronimus transformations are respectively given by:*

$$\omega^2_{n,i,j}(C) := \Big(1+(-1)^i\Phi_n(0)\Big)A_C(\alpha,n)\Big(\frac{1+(-1)^jB_C(\alpha,n)}{A_C(\alpha,n)} + (-1)^j\Phi_{n+1}(0)\Big),$$

$$\omega^2_{n,i,j}(U) := \Big(1+(-1)^i\Phi_n(0)\Big)A_U(\alpha,n)\Big(\frac{1+(-1)^jB_U(\alpha,n)}{A_U(\alpha,n)} + (-1)^j\Phi_n(0)\Big),$$

$$\omega^2_{n,i,j}(G) := \Big(1+(-1)^i\Phi_n(0)\Big)\Big(1+\frac{(-1)^j\overline{A}_{n-1}}{\varepsilon_{n-2}(\alpha)} + \sum_{k=0}^{n-1}(-1)^jD(n,k)\Phi_k(0)\Big).$$

The proof is the matter of computation from (6.1), (6.5) and (6.10) being substituted in (4.6). In fact, $\omega_{n,i,j}$ gives a useful formula for making the technical relationship for Θ matrices.

7. The Fundamental matrix

In this section, we are in a position to address the main novel issue of this paper, called *Fundamental matrix*. The idea has been developed by making some periodic gaps between the Verblunsky coefficients. If we put zeros between every two consecutive Verblunsky coefficients in $\{\Phi_n(0)\}_{n\geqslant 1}$, then we have a new family of Verblunsky coefficients $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$, and deriving the Θ factorization of the new family introduces a new linear functional. Let us consider the general case of this special perturbation, for which we interrupt the Verblunsky coefficients by putting $k-1$ zeros between every two consecutive Verblunsky coefficients.

Definition 7.1. Let $\{\Phi_n(0)\}_{n\geqslant 1}$ be the Verblunsky coefficients associated with orthogonal polynomials $\{\Phi_n(z)\}_{n\geqslant 0}$. Assume k a positive integer, k-*Verblunsky coefficients* are defined as:

$$\widetilde{\Phi}_{k,n}(0) := \begin{cases} \Phi_{\frac{n}{k}}(0), & k \mid n, \\ 0, & k \nmid n. \end{cases} \tag{7.1}$$

Since Θ matrices play an important role in $\widetilde{\mathcal{C}}$, first let us compute $\widetilde{\Theta}_{k,n}$ in terms of Θ_n.

Proposition 7.2. *The Θ matrices associated with the Verblunsky coefficients $\{\widetilde{\Phi}_{k,n}(0)\}_{n\geqslant 1}$ are given by*

$$\widetilde{\Theta}_{k,n} := \Lambda_{\frac{n+1}{k}}, \qquad n \geq 0, \tag{7.2}$$

where

$$\Lambda_j := \begin{cases} \Theta_{j-1}, & j \in \mathbb{Z}, \\ J, & j \notin \mathbb{Z}, \end{cases} \tag{7.3}$$

and

$$J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.4}$$

Proof. Let us define the Θ matrix of $\{\widetilde{\Phi}_{k,n}(0)\}_{n\geqslant 1}$ similar to (2.14),

$$\widetilde{\Theta}_{k,n} := \begin{pmatrix} -\Phi_{k,n+1}(0) & \dfrac{\rho_{k,n+1}}{\overline{\Phi_{k,n+1}(0)}} \\ \rho_{k,n+1} & \end{pmatrix}.$$

We will consider two cases:

i. If $k \mid n+1$, then (7.1) gives

$$\widetilde{\Theta}_{k,n} := \begin{pmatrix} -\Phi_{\frac{n+1}{k}}(0) & \dfrac{\rho_{k,n+1}}{\overline{\Phi_{\frac{n+1}{k}}(0)}} \\ \rho_{k,n+1} & \end{pmatrix} = \Theta_{\frac{n+1}{k}-1}.$$

ii. If $k \nmid n+1$, then $\widetilde{\Phi}_{k,n+1}(0) = 0$ and (2.10) gives $\widetilde{\Theta}_{k,n} = J$.

Taking Λ_j as in (7.3) thus we complete the proof. $\square$

Particularly, we obtain the CMV matrix of k-Verblunsky coefficients for $k = 2$.

Theorem 7.3. *Let us consider the family of orthogonal polynomials corresponding to the 2-Verblunsky coefficients defined in (7.1). The Θ factorization of the CMV matrix can be given as:*

$$\widetilde{\mathcal{M}} := 1 \oplus \Theta_0 \oplus \Theta_1 \oplus \Theta_2 \oplus \Theta_3 \oplus \cdots \tag{7.5}$$

$$\widetilde{\mathcal{L}} := J \oplus J \oplus J \oplus J \oplus \cdots \tag{7.6}$$

Proof. By definition, we have

$$\widetilde{\mathcal{M}} := 1 \oplus \widetilde{\Theta}_{2,1} \oplus \widetilde{\Theta}_{2,3} \oplus \widetilde{\Theta}_{2,5} \oplus \widetilde{\Theta}_{2,7} \oplus \cdots$$

$$\widetilde{\mathcal{L}} := \widetilde{\Theta}_{2,0} \oplus \widetilde{\Theta}_{2,2} \oplus \widetilde{\Theta}_{2,4} \oplus \widetilde{\Theta}_{2,6} \oplus \cdots$$

Formulas (7.2) and (7.3) show that

$$\widetilde{\Theta}_{2,n} = \begin{cases} \Theta_{\frac{n-1}{2}}, & n: \ odd, \\ J, & n: \ even. \end{cases} \tag{7.7}$$

(7.5) and (7.6) are the direct conclusion of the above formulas. □

This observation has an important consequence. Actually, looking carefully at relation (7.5), Theorem 7.3 is the motivation for defining a new matrix namely *Fundamental matrix*.

Definition 7.4. The matrix

$$\mathcal{F} := 1 \oplus \Theta_0 \oplus \Theta_1 \oplus \Theta_2 \oplus \Theta_3 \oplus \cdots \tag{7.8}$$

is said to be the Fundamental matrix associated with the sequence of Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$, where 1 is a 1×1 matrix followed by 2×2 block Θ matrices.

Since Θ matrices give extensive information about the family of orthogonal polynomials, it is wise to collect all these matrices. It is noticed that the Fundamental matrix is not only one of the factors of Θ factorization in (7.5).

We have already introduced the Fundamental matrix, and now we are ready to invoke its connection with CMV matrix. The ease of the relation can be seen through uncomplicated calculations.

Proposition 7.5. *The CMV matrix $\mathcal{C}$ and the Fundamental matrix $\mathcal{F}$ are related by*

$$\mathcal{C} = \mathcal{F}\mathcal{T}, \tag{7.9}$$

where $\mathcal{T}$ can be decomposed as $\mathcal{T} = \mathcal{S}\mathcal{L}$

$$\mathcal{S} := 1 \oplus S_0 \oplus S_1 \oplus S_2 \oplus S_3 \oplus \cdots \tag{7.10}$$

where

$$S_n = \Theta_n^{-1}\Theta_{2n+1}, \qquad n \geqslant 0, \tag{7.11}$$

and $\mathcal{L}$ is introduced in (2.17).

The connection of Fundamental matrices for a pair of Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$ and $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$ is stated as follows.

Theorem 7.6. *Let us assume that $\{\Phi_n(0)\}_{n\geqslant 1}$ and $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$ have $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, respectively, as Fundamental matrices. Let*

$$\begin{aligned} P &:= 1 \oplus P_0 \oplus I \oplus P_1 \oplus I \oplus P_2 \oplus I \oplus \cdots \\ Q &:= 1 \oplus I \oplus Q_1 \oplus I \oplus Q_2 \oplus I \oplus Q_3 \cdots \end{aligned}$$

where P_n and Q_n are block matrices:

$$P_n := \widetilde{\Theta}_{2n}\Theta_{2n}^{-1}, \qquad Q_n := \Theta_{2n-1}^{-1}\widetilde{\Theta}_{2n-1}. \tag{7.12}$$

Then

$$P\mathcal{F}Q = \widetilde{\mathcal{F}}. \tag{7.13}$$

Proof. The matrix $\mathcal{F}$ in (7.8) can be decomposed as $\mathcal{F} = \mathcal{F}_+\mathcal{F}_-$, where

$$\begin{aligned} \mathcal{F}_+ &:= 1 \oplus \Theta_0 \oplus I \oplus \Theta_2 \oplus I \oplus \Theta_4 \oplus I \oplus \cdots \\ \mathcal{F}_- &:= 1 \oplus I \oplus \Theta_1 \oplus I \oplus \Theta_3 \oplus I \oplus \Theta_5 \oplus \cdots \end{aligned}$$

To verify our claim, it is enough to follow similar arguments to those given for the CMV matrices in the proof of Theorem 5.1. $\square$

Now, we compare the CMV matrix of the transformed Verblunsky coefficients $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$ with the Fundamental matrix of Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$.

Theorem 7.7. *The CMV matrix of $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$ and Fundamental matrix of $\{\Phi_n(0)\}_{n\geqslant 1}$ are connected through*

$$\widetilde{\mathcal{C}} = \mathcal{F}\widetilde{\mathcal{T}}, \tag{7.14}$$

where $\widetilde{T}$ can be decomposed as $\widetilde{T} = \widetilde{S}\widetilde{\mathcal{L}}$, in which $\widetilde{S} := 1 \oplus \widetilde{S}_0 \oplus \widetilde{S}_1 \oplus \widetilde{S}_2 \oplus \cdots$ where the blocks are

$$\widetilde{S}_n = \Theta_n^{-1}\widetilde{\Theta}_{2n+1}, \tag{7.15}$$

and $\widetilde{\mathcal{L}}$ is (2.17) for $\{\widetilde{\Phi}_n(0)\}$.

Proof. The matrices $\mathcal{M}, \mathcal{L}$ become $\widetilde{\mathcal{M}}, \widetilde{\mathcal{L}}$ for the transformed Verblunsky coefficients. Proposition 7.5 shows that by choosing $\widetilde{S}_n$ as (7.15) and $\widetilde{\mathcal{L}}_n = \widetilde{\Theta}_n$, we complete the proof. $\square$

The relation between the matrices $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{F}}$, the CMV and Fundamental matrices of the transformed Verblunsky coefficients, can be shown in a very similar way as in Proposition 7.5.

Corollary 7.8. *Let $\widetilde{\mathcal{T}} := \widetilde{\mathcal{S}}\widetilde{\mathcal{L}}$, and $\widetilde{\mathcal{S}}$ be defined likely to the matrix (7.10) with entries*

$$\widetilde{\mathcal{S}}_n = \widetilde{\Theta}_n^{-1}\widetilde{\Theta}_{2n+1}, \tag{7.16}$$

and $\widetilde{\mathcal{L}}$ contains even Θ matrices of $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$. The CMV and Fundamental matrices are connected through

$$\widetilde{\mathcal{C}} = \widetilde{\mathcal{F}}\widetilde{\mathcal{T}}. \tag{7.17}$$

Remark 7.9. The above computation shows that, knowing $\widetilde{\Theta}_n$ for any family of transferred orthogonal polynomials, formulas (5.1), (7.13), (7.9), (7.14), and (7.17) give the relations $\mathcal{C} \sim \widetilde{\mathcal{C}}$, $\mathcal{F} \sim \widetilde{\mathcal{F}}$, $\mathcal{F} \sim \mathcal{C}$, $\mathcal{F} \sim \widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{F}} \sim \widetilde{\mathcal{C}}$, respectively.

Corollary 7.10. *Let $\mathcal{C}, \mathcal{F}$ and $\widetilde{\mathcal{C}}, \widetilde{\mathcal{F}}$ be CMV and Fundamental matrices of Verblunsky sequences of $\{\Phi_n(0)\}_{n\geqslant 1}$ and k-Verblunsky coefficients $\{\widetilde{\Phi}_{k,n}(0)\}_{n\geqslant 1}$ defined in (7.1), respectively. The following relations hold*

(i). $\Big(1 \oplus V_1 \oplus V_2 \oplus V_3 \oplus \cdots\Big)\mathcal{C}\Big(W_0 \oplus W_1 \oplus W_2 \oplus \cdots\Big) = \widetilde{\mathcal{C}},$

$$V_n := \Lambda_{\frac{2n}{k}}\Theta_{2n-1}^{-1}, \qquad W_n := \Theta_{2n}^{-1}\Lambda_{\frac{2n+1}{k}}.$$

(ii). $\widetilde{\mathcal{C}} = \mathcal{F}\widetilde{T}$, *where* $\widetilde{T} = \Big(1 \oplus \widetilde{S}_0 \oplus \widetilde{S}_1 \oplus \widetilde{S}_2 \oplus \cdots\Big)\Big(\widetilde{\mathcal{L}}_0, \widetilde{\mathcal{L}}_1, \widetilde{\mathcal{L}}_2, \cdots\Big),$

$$\widetilde{S}_n = \Theta_n^{-1}\Lambda_{\frac{2n+2}{k}}, \qquad \widetilde{\mathcal{L}}_n = \Lambda_{\frac{2n+1}{k}}.$$

(iii). $\Big(1 \oplus P_0 \oplus I \oplus P_1 \oplus I \oplus P_2 \oplus I \oplus \cdots\Big)\mathcal{F}\Big(1 \oplus I \oplus Q_1 \oplus I \oplus Q_2 \oplus I \oplus Q_3 \cdots\Big) = \widetilde{\mathcal{F}},$

$$P_n := \Lambda_{\frac{2n+1}{k}}\Theta_{2n}^{-1}, \qquad Q_n := \Theta_{2n-1}^{-1}\Lambda_{\frac{2n}{k}}.$$

8. Examples

In this section, we will show how the behavior of the Verblunsky coefficients can affect the CMV matrices. The examples are meant to illustrate the behavior of the CMV and Fundamental matrices.

8.1. Example 1

The first example corresponds to the trivial sequence of Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1} = \{0\}$. Since it satisfies the second condition of Proposition 2.19, the corresponding measure belongs to the Szegő class.

Observe that the Θ matrix of this family is equal to (7.4) and, consequently the CMV and Fundamental matrices are obtained as follows:

$$\mathcal{C} = \mathcal{E}_{1,2} + \sum_{i:even} \mathcal{E}_{i,i+2} + \sum_{3\leqslant i:odd} \mathcal{E}_{i,i-2},$$
$$\mathcal{F} = \mathcal{E}_{1,1} + \sum_{i:even} \mathcal{E}_{i,i+1} + \sum_{3\leqslant i:odd} \mathcal{E}_{i,i-1},$$

where $\mathcal{E}_{i,j}$ stands for the matrix with entry (i,j) equal to 1, and all other entries equal to zero.

Proposition 8.1. *If we perturb the sequence of Verblunsky coefficients* $\{\Phi_n(0)\}_{n\geqslant 1} = \{0\}$ *to get the transformed one* $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$*, the* Θ *matrix* $\widetilde{\Theta}_n$ *can be written as*

$$\widetilde{\Theta}_n = \frac{1}{4}\begin{pmatrix} -2\beta_{n+1}\eta_{n+1} & \eta_{n+1}^2 - \beta_{n+1}^2 \\ \eta_{n+1}^2 - \beta_{n+1}^2 & 2\beta_{n+1}\eta_{n+1} \end{pmatrix}, \tag{8.1}$$

where

$$\eta_n := \big(1 + \widetilde{\Phi}_n(0)\big)^{\frac{1}{2}} + \big(1 - \widetilde{\Phi}_n(0)\big)^{\frac{1}{2}}, \tag{8.2}$$
$$\beta_n := \big(1 + \widetilde{\Phi}_n(0)\big)^{\frac{1}{2}} - \big(1 - \widetilde{\Phi}_n(0)\big)^{\frac{1}{2}}. \tag{8.3}$$

Proof. From Corollary 4.2, we get:

$$\omega_{n,0,0} = \omega_{n,1,0} = \big(1 + \widetilde{\Phi}_n(0)\big)^{\frac{1}{2}},$$
$$\omega_{n,0,1} = \omega_{n,1,1} = \big(1 - \widetilde{\Phi}_n(0)\big)^{\frac{1}{2}}.$$

Formula (4.8) gives the unitary matrix Δ_n that satisfies the unitary similarity relation (4.1). The rest of the proof is a simplification of the unitary matrix $\widetilde{\Theta}_n$. $\square$

Remark 8.2. It is easy to prove that $\widetilde{\Theta}_n$ presented in (8.1) is equal to

$$\widetilde{\Theta}_n = \begin{pmatrix} -\widetilde{\Phi}_{n+1}(0) & \dfrac{\widetilde{\rho}_{n+1}}{\overline{\widetilde{\Phi}_{n+1}(0)}} \\ \widetilde{\rho}_{n+1} & \end{pmatrix}. \tag{8.4}$$

It will be advantageous to work with (8.1), since it is computed with direct substitution of $\widetilde{\Phi}_n(0)$, without calculation of $\widetilde{\rho}_n$. The computation here is both simpler and more insightful.

Now, let Γ_n denote the following matrix associated with $\widetilde{\Theta}_n$

$$\Gamma_n = \frac{1}{4}\begin{pmatrix} \eta_{n+1}^2 - \beta_{n+1}^2 & 2\beta_{n+1}\eta_{n+1} \\ -2\beta_{n+1}\eta_{n+1} & \eta_{n+1}^2 - \beta_{n+1}^2 \end{pmatrix} = \begin{pmatrix} \widetilde{\rho}_{n+1} & \overline{\widetilde{\Phi}_{n+1}(0)} \\ -\widetilde{\Phi}_{n+1}(0) & \widetilde{\rho}_{n+1} \end{pmatrix}. \tag{8.5}$$

It can be checked that

$$J\widetilde{\Theta}_n = \Gamma_n, \qquad \widetilde{\Theta}_n J = \Gamma_n^T. \tag{8.6}$$

We are now in a position to make the similarity relation of CMV and Fundamental matrices. According to formulas (5.2), (7.12), and considering (8.6), we have

$$V_n = \Gamma_{2n-1}^T, \qquad W_n = \Gamma_{2n},$$
$$P_n = \Gamma_{2n}^T, \qquad Q_n = \Gamma_{2n-1}.$$

As a conclusion, we get

Corollary 8.3. *Let $\mathcal{C}, \mathcal{F}$ and $\widetilde{\mathcal{C}}, \widetilde{\mathcal{F}}$ be CMV and Fundamental matrices of Verblunsky coefficients of $\{\Phi_n(0)\}_{n\geqslant 1} = \{0\}$ and transformed Verblunsky coefficients $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$, and put Γ_n from (8.5). The following relations*

$$\widetilde{\mathcal{C}} = \Big(1 \oplus \Gamma_1 \oplus \Gamma_3 \oplus \Gamma_5 \oplus \cdots\Big)^T \mathcal{C}\Big(\Gamma_0 \oplus \Gamma_2 \oplus \Gamma_4 \oplus \cdots\Big),$$
$$\widetilde{\mathcal{F}} = \Big(1 \oplus \Gamma_0 \oplus I \oplus \Gamma_2 \oplus I \oplus \Gamma_4 \oplus I \oplus \cdots\Big)^T \mathcal{F}\Big(1 \oplus I \oplus \Gamma_1 \oplus I \oplus \Gamma_3 \oplus I \oplus \Gamma_5 \cdots\Big),$$

connect $\mathcal{C}, \widetilde{\mathcal{C}}$ and $\mathcal{F}, \widetilde{\mathcal{F}}$, respectively.

Our remaining task is to make the CMV matrices and Fundamental matrices of the canonical spectral transformation of the measure corresponding to the Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1} = \{0\}$. By substituting the transformed Verblunsky coefficients $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$ in (8.2) and (8.3), we get η_n and β_n, respectively. Then, following Corollary 8.3 we get the desired matrices.

Since η_n and β_n require $\widetilde{\Phi}_n(0)$, at this point, we only need to present the Verblunsky coefficients $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$ of the transformed measure, from Lemmas 6.2, 6.5 and 6.8 respectively.

Proposition 8.4. *The Verblunsky coefficients for Christoffel, Uvarov and Geronimus transformations of* $\{\Phi_n(0)\}_{n\geqslant 1} = \{0\}$ *are respectively given by*

$$\widetilde{\Phi}_{n,C}(0) = B_C(\alpha, n),$$
$$\widetilde{\Phi}_{n,U}(0) = B_U(\alpha, n),$$
$$\widetilde{\Phi}_{n,G}(0) = \frac{\overline{A}_{n-1}}{\varepsilon_{n-2}(\alpha)},$$

for the introduced values of (6.3), (6.7), (6.8), *and* (6.9).

8.2. Example 2

Consider the sequence of Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1} = \{a\}$ where a is a constant complex number satisfying $0 < |a| < 1$. The divergence of the series of part ii of Proposition 2.19 shows that the measure corresponding to $\{\Phi_n(0)\}_{n\geqslant 1}$ does not belong to the Szegő class.

The Θ matrix of this family is

$$\Theta_n = \begin{pmatrix} -a & r \\ r & \bar{a} \end{pmatrix}, \tag{8.7}$$

where $r = (1-|a|^2)^{\frac{1}{2}}$. The CMV and Fundamental matrices are obtained via formulas (2.15) and (7.8).

Note that for the constant Verblunsky coefficients $\Phi_n(0) = a$, $n \geqslant 1$, Theorem 4.3 gives the unitary congruent relation. The unitary matrix $\widetilde{\Theta}_n$ can be obtained from (4.9), while the formulas (4.11)–(4.13) are simplified by substituting $\Phi_n(0) = a$ and $\rho_n = r$ as follows:

$$\tau_n = (a-1)(\overline{\widetilde{\Phi}_n(0)} - 1) + r\widetilde{\rho}_n,$$
$$\xi_n = r(\overline{\widetilde{\Phi}_n(0)} - 1) - \widetilde{\rho}_n(\bar{a} - 1),$$
$$\lambda_n^2 = 4\big(1 - \mathfrak{Re}(a)\big)\big(1 - \mathfrak{Re}(\widetilde{\Phi}_n(0))\big).$$

Proposition 8.5. *The similarity factors of relation of CMV and Fundamental matrices are*

$$V_n = \Upsilon_{2n-1}^T, \qquad W_n = \Upsilon_{2n},$$
$$P_n = \Upsilon_{2n}^T, \qquad Q_n = \Upsilon_{2n-1},$$

where

$$\Upsilon_n = \begin{pmatrix} s_{n+1} & \bar{t}_{n+1} \\ -t_{n+1} & \bar{s}_{n+1} \end{pmatrix}, \tag{8.8}$$

and

$$s_n := \widetilde{\Phi}_n(0)\bar{a} + r\widetilde{\rho}_n, \tag{8.9}$$

$$t_n := \widetilde{\Phi}_n(0)r - \widetilde{\rho}_n a. \tag{8.10}$$

Proof. Define $\Upsilon_n := \Theta_n^{-1}\widetilde{\Theta}_n$. According to (8.4) and (8.7), we have

$$\Theta_n^{-1}\widetilde{\Theta}_n = (\widetilde{\Theta}_n\Theta_n^{-1})^T = \begin{pmatrix} \widetilde{\Phi}_{n+1}(0)\bar{a} + r\widetilde{\rho}_{n+1} & \overline{\widetilde{\Phi}_{n+1}(0)}r - \widetilde{\rho}_{n+1}\bar{a} \\ \widetilde{\rho}_{n+1}a - \widetilde{\Phi}_{n+1}(0)r & \overline{\widetilde{\Phi}_{n+1}(0)}a + r\widetilde{\rho}_{n+1} \end{pmatrix},$$

and relations (5.2) and (7.12) complete the proof. □

Corollary 8.6. *Let $\mathcal{C}, \mathcal{F}$ be CMV and Fundamental matrices of Verblunsky sequences of $\{\Phi_n(0)\}_{n\geqslant 1} = \{a\}$, for the complex number a with $0 < |a| < 1$, respectively. Consider Υ_n as (8.8) for the defined values of (8.9) and (8.10). We have*

$$\widetilde{\mathcal{C}} = \Big(1 \oplus \Upsilon_1 \oplus \Upsilon_3 \oplus \Upsilon_5 \oplus \cdots\Big)^T \mathcal{C}\Big(\Upsilon_0 \oplus \Upsilon_2 \oplus \Upsilon_4 \oplus \cdots\Big),$$

$$\widetilde{\mathcal{F}} = \Big(1 \oplus \Upsilon_0 \oplus I \oplus \Upsilon_2 \oplus I \oplus \Upsilon_4 \oplus I \oplus \cdots\Big)^T \mathcal{F}\Big(1 \oplus I \oplus \Upsilon_1 \oplus I \oplus \Upsilon_3 \oplus I \oplus \Upsilon_5 \cdots\Big),$$

where $\widetilde{\mathcal{C}}, \widetilde{\mathcal{F}}$ are CMV and Fundamental matrices of the transformed Verblunsky coefficients $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$.

Now, we study the CMV matrices and Fundamental matrices of the canonical spectral transformation of the measure corresponding to the Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1} = \{a\}$. We replace the transformed Verblunsky coefficients $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$ in (8.9) and (8.10). Then, we follow Proposition 8.5 to get the congruence factors. Therefore, if we present the Verblunsky coefficients $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$ of the transformed measure, Corollary 8.6 shows the relation of CMV and Fundamental matrices.

Proposition 8.7. *The Verblunsky coefficients for Christoffel, Uvarov and Geronimus transformations of the Verblunsky sequence $\{\Phi_n(0)\}_{n\geqslant 1} = \{a\}$ are respectively given by:*

$$\widetilde{\Phi}_{n,C}(0) = A_C(\alpha, n)a + B_C(\alpha, n),$$

$$\widetilde{\Phi}_{n,U}(0) = B_U(\alpha, n)a + B_U(\alpha, n),$$

$$\widetilde{\Phi}_{n,G}(0) = \frac{\overline{A}_{n-1}}{\varepsilon_{n-2}(\alpha)} + a\sum_{k=0}^{n-1} D(n,k),$$

for the introduced values of (6.2)–(6.3), (6.6)–(6.7) and (6.11), by considering (6.8) and (6.9).

8.3. Forward and backward Verblunsky coefficients

In this section, we introduce two groups of transformed Verblunsky coefficients, based on the sequence of Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$:

- forward-Verblunsky coefficients

$$\widetilde{\Phi}_{n,f}(0) := \Phi_{n+1}(0), \qquad n \geqslant 1, \tag{8.11}$$

- backward-Verblunsky coefficients

$$\widetilde{\Phi}_{n,b}(0) := \Phi_{n-1}(0), \quad n > 1, \qquad \widetilde{\Phi}_1(0) := c. \tag{8.12}$$

Since the backward coefficients push the Verblunsky coefficients back, the first transformed Verblunsky coefficient is needed to be imposed to the problem, where c is a constant complex $0 < |c| < 1$.

Remark 8.8. Notice that these two families of Verblunsky coefficients have the same behavior as Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$, in the sense that if the measure $d\mu$ belongs to the Szegő class, it is obvious that $d\widetilde{\mu}$ does too. We have an invariance property that also holds for measures that do not belong to the Szegő class.

Since $\widetilde{\Theta}_n$ appears in $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{F}}$, the first step is to find the expression of $\widetilde{\Theta}_n$ in terms of Θ_n. For the above mentioned classes, it can be easily seen that

$$\widetilde{\Theta}_{f,n} := \Theta_{n+1}, \qquad n \geq 0, \tag{8.13}$$

$$\widetilde{\Theta}_{b,n} := \Theta_{n-1}, \qquad n \geq 1. \tag{8.14}$$

Note that $\widetilde{\Theta}_{b,0} := \begin{pmatrix} -c & \rho \\ \rho & \bar{c} \end{pmatrix}$ is defined according to the constant complex c in (8.12).

Similar to the other cases, formula (5.2) yields the matrices V and W for each class of the Verblunsky coefficients. For the above mentioned classes, we have

$$\begin{aligned} V_{f,n} &:= \Theta_{2n}\Theta_{2n-1}^{-1}, \qquad n \geqslant 1, \\ W_{f,n} &:= \Theta_{2n}^{-1}\Theta_{2n+1}, \qquad n \geqslant 0, \end{aligned}$$

and

$$\begin{aligned} V_{b,n} &:= \Theta_{2n-2}\Theta_{2n-1}^{-1}, \qquad n \geqslant 1, \\ W_{b,n} &:= \Theta_{2n}^{-1}\Theta_{2n-1}, \qquad n \geqslant 1, \quad W_{b,0} := \Theta_0^{-1}\widetilde{\Theta}_{b,0}. \end{aligned}$$

We also compare the Fundamental matrices. Taking into account (8.13)–(8.14), (7.12) gives

$$P_{f,n} := \Theta_{2n+1}\Theta_{2n}^{-1}, \qquad n \geqslant 0,$$
$$Q_{f,n} := \Theta_{2n-1}^{-1}\Theta_{2n}, \qquad n \geqslant 1,$$

and similarly

$$P_{b,n} := \Theta_{2n-1}\Theta_{2n}^{-1}, \qquad n \geqslant 1, \qquad P_{b,0} := \widetilde{\Theta}_{b,0}\Theta_0^{-1},$$
$$Q_{b,n} := \Theta_{2n-1}^{-1}\Theta_{2n-2}, \qquad n \geqslant 1.$$

Finding the connection of CMV and Fundamental matrices is formally the same as previous examples.

9. A decomposition for Fundamental matrix

Notice that the Fundamental matrix $\mathcal{F}$ is associated with the sequence of Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$ that characterizes the family of orthogonal polynomials. This matrix has to give comprehensive information about properties of the family of orthogonal polynomials. More precisely, knowing the Fundamental matrix, we should be able to establish the family of monic orthogonal polynomials through the Szegő recurrence relation (2.8).

For orthogonal polynomials on the real line, the matrix interpretation of the polynomial perturbation with a first degree polynomial is given in terms of the LU factorization, and the matrix interpretation of a second order polynomial perturbation is given in terms of the QR factorization. In [8], the authors established a relation between the Hessenberg matrices associated with the initial and the perturbed functionals using LU and QR factorizations. Similar to the so-called spectral transformation of the real line, the tridiagonal Fundamental matrices associated with the respective functionals should be decomposed to unitary matrices.

In the case that the LU factorization of the Fundamental matrix associated with $\{\Phi_n(0)\}_{n\geqslant 1}$ is considered, although UL matrix presents a tridiagonal block matrix, but the blocks do not follow the structure of the Θ matrices. On the other hand, since Θ matrices are unitary, QR factorization does not make sense. In this section, an important aspect of the structure of the Fundamental matrix is captured by its Takagi decomposition. Our aim is to find a factorization for the Fundamental matrix such that it can simultaneously present another measure. We show how to obtain such a factorization such that it can provide the same structure.

Theorem 9.1. *Let $\mathcal{F}$ defined in (7.8) be the Fundamental matrix associated with the Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$. This family is coupled with the Verblunsky coefficients*

$$\widetilde{\Phi}_n(0) = \frac{\Phi_n(0)\big(i\mathfrak{Im}(\Phi_n(0)) - 1\big) + 1}{\mathfrak{Re}(\Phi_n(0)) - 1}. \tag{9.1}$$

Proof. For convenience of the reader, we begin by recalling the Takagi decomposition

$$U_\Theta = \frac{1}{l_{n+1}} \begin{pmatrix} 1-\Phi_{n+1}(0) & \dfrac{\rho_{n+1}}{\Phi_{n+1}(0)-1} \\ \rho_{n+1} & \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$

with ${l_n}^2 = 2\big(1-\mathfrak{Re}(\Phi_n(0))\big)$, obtained in the proof of Theorem 4.3. Observe that Θ_n satisfies (4.14). The classical approach to the spectral theory suggests us to consider replacement of $U^T U$, which gives

$$\widetilde{\Theta}_n = U^T U = \begin{pmatrix} -\widetilde{\Phi}_{n+1}(0) & \dfrac{\rho_{n+1}}{\widetilde{\overline{\Phi}}_{n+1}(0)} \\ \rho_{n+1} & \end{pmatrix},$$

where $\widetilde{\Phi}_n(0)$ is defined in (9.1). $\square$

Remark 9.2. An intrinsic limitation of Theorem 9.1 is that it does not give information for real Verblunsky coefficients. Here, the situation is different. The unitary matrix U that satisfies $UU^T = \Theta_n$ with real Verblunsky coefficients, it turns out to

$$U^T U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Although it has the Θ matrix structure, but it carries the trouble of $\Phi_n(0) = 1$, which does not obey Remark 2.14.

Proposition 9.3. *The linear functional with Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$ is positive definite (quasi-definite) if and only if the linear functional corresponding to $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$ defined in (9.1) is positive definite (quasi-definite).*

Proof. The proof is a straightforward consequence of the fact that $\Phi_n(0) = \mathfrak{Re}(\Phi_n(0)) + i\mathfrak{Im}(\Phi_n(0))$ and hence will be omitted. $\square$

The seemingly minor observation made in Proposition 9.3 has a brilliant impact on the perturbed measure $d\widetilde{\mu}$ to be well-defined. Remark 2.15 guarantees the existence of another family of Verblunsky coefficients $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 1}$. Moreover, due to the structure of the Fundamental matrix, it arises as a natural question to characterize the measures $d\mu$ corresponding to this matrix. It is an open problem to obtain the family of orthogonal polynomials deduced from the spectral measure associated with the Fundamental matrix.

10. Summary and open problems

In this paper, we have stated a connection between the CMV matrices associated with a positive measure and a perturbation of it, respectively. The main relation we have used is the relation for the Θ matrices, appeared in the structure of the CMV matrix. Using

Verblunsky coefficients to find the relation, we observe that all the formulas can work directly for different classes of measures, requiring only substitution of formulas.

We have emphasized the analysis of connecting Θ matrices, whose explicit expressions have been derived using the Verblunsky coefficients. The main results of the equivalence relations can be summarized in the following:

A *congruence relation has been obtained using a kind of eigenvalue decomposition which happens to be hold for special sequences of Verblunsky coefficients with the equal imaginary parts. Specifically, for real Verblunsky sequences, we have constructed the unitary similarity relation and we have compared the CMV matrices of the known canonical transformed measures, Christoffel, Uvarov and Geronimus. In the general case of complex Verblunsky coefficients, there exists a congruence relation which has been established explicitly with combination of eigenvalue and singular value decompositions.

We have followed the results for different classes of Verblunsky coefficients, and one of the considered classes has motivated us to present a new unitary symmetric matrix which contains all the necessary information of the family of orthogonal polynomials. We have tried to solve the spectral problem of the Fundamental matrix. In particular, the following question is open: *What is the spectral measure of the Fundamental matrix?*

Acknowledgements

The authors would like to thank Professor Bernhard Beckermann and Professor Roger A. Horn for valuable and insightful discussions about congruence relations. We also thank the suggestions by the referees which have contributed to improve substantially the presentation of the manuscript. The work of the first author (FM) was partially supported by Dirección General de Política Científica y Tecnológica, Ministerio de Economía y Competitividad (MINECO) of Spain, under grant MTM2012-36732-C03-01. The second author (NS) thanks Alexander von Humboldt Foundation for the support and the Department of Mathematics, Universidad Carlos III de Madrid, for its constant support and friendly atmosphere during the period January–July 2014 when the manuscript was finished.

References

[1] G.S. Ammar, W.G. Gragg, Schur flows for orthogonal Hessenberg matrices, Fields Inst. Commun. 3 (1994) 27–34.

[2] Y. Arlinskii, L. Golinskii, E. Tsekanovskii, Contractions with rank one defect operators and truncated CMV matrices, J. Funct. Anal. 254 (1) (2008) 154–195.

[3] A.K. Abdikalykov, Kh.D. Ikramov, Simultaneous decomplexification of a pair of complex matrices via a unitary similarity transformation, J. Math. Sci. 182 (6) (2012) 745–747.

[4] Yu.A. Al'pin, Kh.D. Ikramov, A criterion for unitary congruence between matrices, Dokl. Math. 83 (2) (2011) 141–142.

[5] A. Bunse-Gerstner, W.B. Gragg, Singular value decompositions of complex symmetric matrices, J. Comput. Appl. Math. 21 (1988) 41–54.

[6] M. Buhmann, A. Iserles, On orthogonal polynomials transformed by the QR algorithm, J. Comput. Appl. Math. 43 (1992) 117–134.

[7] M.I. Bueno, F. Marcellán, Darboux transformation and perturbation of linear functionals, Linear Algebra Appl. 384 (2004) 215–242.
[8] M.I. Bueno, F. Marcellán, Polynomial perturbations of bilinear functionals and Hessenberg matrices, Linear Algebra Appl. 414 (2006) 64–83.
[9] K. Castillo, Spectral problems and orthogonal polynomials on the unit circle, Ph.D. thesis, Universidad Carlos III de Madrid, Spain, 2012.
[10] K. Castillo, L. Garza, F. Marcellán, A new linear spectral transformation associated with derivatives of Dirac linear functionals, J. Approx. Theory 163 (2011) 1834–1853.
[11] K. Castillo, L. Garza, F. Marcellán, Linear spectral transformations, Hessenberg matrices, and orthogonal polynomials, Rend. Circ. Mat. Palermo (2) Suppl. 82 (2010) 3–26.
[12] M.J. Cantero, L. Moral, L. Velázquez, Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle, Linear Algebra Appl. 362 (2003) 29–56.
[13] M.J. Cantero, L. Moral, L. Velázquez, Minimal representations of unitary operators and orthogonal polynomials on the unit circle, Linear Algebra Appl. 408 (2005) 40–65.
[14] L. Daruis, P. González-Vera, Szegő polynomials and quadrature formulas on the unit circle, Appl. Numer. Math. 36 (1999) 79–112.
[15] L. Daruis, J. Hernández, F. Marcellán, Spectral transformations for Hermitian Toeplitz matrices, J. Comput. Appl. Math. 202 (2007) 155–176.
[16] F. de Terán, Canonical forms for congruence of matrices: a tribute to H.W. Turnbull and A.C. Aitken, in: 2nd Meeting on Linear Algebra, Matrix Analysis and Applications, ALAMA 2010, Valencia, Spain, June 2010, pp. 2–4.
[17] H. Fassbender, Kh.D. Ikramov, Conjugate-normal matrices: a survey, Linear Algebra Appl. 429 (2008) 1425–1441.
[18] P. García, F. Marcellán, On zeros of regular orthogonal polynomials on the unit circle, Ann. Polon. Math. 58 (1993) 287–298.
[19] L. Garza, Transformaciones Espectrales, Funciones de Carathéodory y Polinomios Ortogonales en la Circunferencia Unidad, Ph.D. thesis, Universidad Carlos III de Madrid, Spain, 2008 (in Spanish).
[20] L. Garza, J. Hernández, F. Marcellán, Orthogonal polynomials and measures on the unit circle. The Geronimus transformations, J. Comput. Appl. Math. 233 (2010) 1220–1231.
[21] L. Garza, F. Marcellán, Verblunsky parameters and linear special transformations, Methods Appl. Anal. 16 (1) (2009) 69–86.
[22] L. Garza, F. Marcellán, Szegő transformations and rational spectral transformations for associated polynomials, J. Comput. Appl. Math. 233 (2009) 730–738.
[23] F.R. Gantmacher, The Theory of Matrices, Chelsea Publishing Company, New York, 1960.
[24] Ya.L. Geronimus, Polynomials Orthogonal on a Circle and Their Applications, Amer. Math. Soc. Transl. Ser. 1, vol. 3, 1962, pp. 1–78.
[25] Ya.L. Geronimus, Orthogonal polynomials, in: Two Papers on Special Functions, in: Amer. Math. Soc. Transl. Ser. 2, vol. 108, Amer. Math. Soc., Providence, RI, 1977, English translation of the appendix to the Russian translation of Szegő's book, Fizmatzig, Moscow, 1961.
[26] E. Godoy, F. Marcellán, An analog of the Christoffel formula for polynomial modification of a measure on the unit circle, Boll. Unione Mat. Ital. A (7) 5 (1991) 1–12.
[27] E. Godoy, F. Marcellán, Orthogonal polynomials on the unit circle: distribution of zeros, J. Comput. Appl. Math. 37 (1991) 195–208.
[28] E. Godoy, F. Marcellán, Orthogonal polynomials and rational modifications of measures, Canad. J. Math. 45 (5) (1993) 930–943.
[29] U. Grenander, G. Szegő, Toeplitz Forms and their Applications, University of California Press, Berkeley, 1958; 2nd edition, Chelsea, New York, 1984.
[30] G.H. Golub, C.F. Van Loan, Matrix Computations, 3rd edition, The Johns Hopkins University Press, Baltimore, 1996.
[31] F. Gesztesy, M. Zinchenko, Weyl–Titchmarsh theory for CMV operators associated with orthogonal polynomials on the unit circle, J. Approx. Theory 139 (2006) 172–213.
[32] R.A. Horn, C.R. Johnson, Matrix Analysis, 2nd edition, Cambridge University Press, Cambridge, UK, 2012.
[33] R.A. Horn, V.V. Sergeichuk, Canonical forms for complex matrix congruence and *congruence, Linear Algebra Appl. 416 (2006) 1010–1032.
[34] R.A. Horn, V.V. Sergeichuk, Congruence of a square matrix and its transpose, Linear Algebra Appl. 389 (2004) 347–353.
[35] R.A. Horn, V.V. Sergeichuk, Canonical forms for unitary congruence and *congruence, Linear Multilinear Algebra 57 (8) (2009) 777–815.

[36] Kh.D. Ikramov, Takagi's decomposition of a symmetric unitary matrix as a finite algorithm, Comput. Math. Math. Phys. 52 (1) (2012) 1–3.
[37] Kh.D. Ikramov, Simultaneous decomplexification of a pair of complex matrices by a unitary congruence transformation, Dokl. Math. 84 (3) (2011) 767–769.
[38] Kh.D. Ikramov, Finite algorithms for verifying unitary similarity and unitary congruence for a pair of complex matrices, Dokl. Math. 83 (2) (2011) 158–160.
[39] Kh.D. Ikramov, Unitary congruence classes of 2-by-2 complex matrices, Dokl. Math. 81 (2) (2010) 171–175.
[40] Kh.D. Ikramov, Criteria for unitary congruence of complex 2×2 matrices, Math. Notes 88 (2) (2010) 228–237.
[41] M.E.H. Ismail, R.W. Ruedemann, Relation between polynomials orthogonal on the unit circle with respect to different weights, J. Approx. Theory 71 (1992) 39–60.
[42] C.R. Johnson, S. Furtado, A generalization of Sylvester's law of inertia, Linear Algebra Appl. 338 (2001) 287–290.
[43] W.B. Jones, O. Njastad, W.J. Thron, Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle, Bull. Lond. Math. Soc. 21 (1989) 113–152.
[44] W.B. Jones, O. Njastad, W.J. Thron, H. Waadeland, Szegő polynomials applied to frequency analysis, J. Comput. Appl. Math. 46 (1993) 217–228.
[45] R. Killip, I. Nenciu CMV, The unitary analogue of Jacobi matrices, Comm. Pure Appl. Math. 60 (8) (2007) 1148–1188.
[46] X. Li, F. Marcellán, Representations of orthogonal polynomials for modified measures, Commun. Anal. Theory Contin. Fract. 7 (1999) 9–22.
[47] F. Marcellán, Polinomios ortogonales no estándar. Aplicaciones en Análisis Numerico y Teoria de Aproximacion, Rev. Acad. Colombiana Cienc. Exact. Fis. Natur. 30 (117) (2006) 563–579 (in Spanish).
[48] F. Marcellán, J. Hernández, Christoffel transforms and Hermitian linear functionals, Mediterr. J. Math. 2 (2005) 451–458.
[49] F. Marcellán, J. Hernández, Geronimus spectral transforms and measures on the complex plane, J. Comput. Appl. Math. 217 (2008) 441–456.
[50] C. Mehl, V. Mehrmann, H. Xu, Singular-value-like decomposition for complex matrix triples, J. Comput. Appl. Math. 233 (2010) 1245–1276.
[51] A. Mukaihira, Y. Nakamura, Schur flow for orthogonal polynomials on the unit circle and its integrable discretization, J. Comput. Appl. Math. 139 (2002) 75–94.
[52] I. Nenciu, CMV matrices in random matrix theory and integrable systems: a survey, J. Phys. A 39 (28) (2006) 8811–8822.
[53] I. Nenciu, A note on circular trace formula, Proc. Amer. Math. Soc. 136 (8) (2008) 2785–2792.
[54] O. Njastad, H. Waadeland, Generalized Szegő theory in frequency analysis, J. Math. Anal. Appl. 206 (1997) 289–307.
[55] V. Olshevsky, G. Strang, P. Zhlobich, Green's matrices, Linear Algebra Appl. 432 (2010) 218–241.
[56] C. Pearcy, A complete set of unitary invariants for operators generating finite W^*-algebras of type I, Pacific J. Math. 12 (1962) 1405–1416.
[57] F. Peherstorfer, A special class of polynomials orthogonal on the unit circle including the associated polynomials, Constr. Approx. 12 (2) (1996) 161–185.
[58] F. Peherstorfer, A. Volberg, P. Yuditskii, CMV matrices with asymptotically constant coefficients. Szegő over Blaschke class, scattering theory, J. Funct. Anal. 256 (7) (2009) 2157–2210.
[59] B. Simon, Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory, Amer. Math. Soc. Colloq. Publ., vol. 54, Amer. Math. Soc., Providence, RI, 2005.
[60] B. Simon, Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory, Amer. Math. Soc. Colloq. Publ., vol. 54, Amer. Math. Soc., Providence, RI, 2005.
[61] B. Simon, CMV matrices: five years after, J. Comput. Appl. Math. 208 (2007) 120–154.
[62] B. Simon, Zeros of OPUC and long time asymptotics of Schur and related flows, Inverse Probl. Imaging 1 (2007) 189–215.
[63] V. Spiridonov, L. Vinet, A. Zhedanov, Spectral transformations, self-similar reductions and orthogonal polynomials, J. Phys. A: Math. Gen. 30 (1997) 7621–7637.
[64] G. Szegő, Orthogonal Polynomials, 4th edition, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI, 1975.
[65] W. Xu, S. Qiao, A Divide-and-Conquer method for the Takagi factorization, SIAM J. Matrix Anal. Appl. 30 (2008) 142–153.

[66] G.J. Yoon, Darboux transforms and orthogonal polynomials, Bull. Korean Math. Soc. 39 (2002) 359–376.
[67] A. Zhedanov, Rational spectral transformations and orthogonal polynomials, J. Comput. Appl. Math. 85 (1997) 67–83.
[68] M.L. Wong, Generalized bounded variation and inserting point masses, Constr. Approx. 30 (1) (2009) 1–15.