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Direct and Inverse Results on Row Sequences of Hermite–Padé Approximants

J. Cacoq · B. de la Calle Ysern · G. López Lagomasino

Abstract We give necessary and sufficient conditions for the convergence with ge-ometric rate of the common denominators of simultaneous rational interpolants with a bounded number of poles. The conditions are expressed in terms of intrinsic prop-erties of the system of functions used to build the approximants. Exact rates of con-vergence for these denominators and the simultaneous rational approximants are pro-vided.

Keywords Montessus de Ballore theorem · Simultaneous approximation · Hermite–Padé approximation · Rate of convergence · Inverse results

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In memory of A.A. Gonchar. He passed away on October 10, 2012 at the age of 80. See [2] a n d [3] for a brief account on his fruitful life and important contributions in approximation theory.

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J. Cacoq · G. López Lagomasino (✉)
Dpto. de Matemáticas, Escuela Politécnica Superior, Universidad Carlos III de Madrid, Universidad 30, 28911 Leganés, Spain
e-mail: lago@math.uc3m.es

J. Cacoq
e-mail: jcacoq@math.uc3m.es

B. de la Calle Ysern
Dpto. de Matemática Aplicada, E. T. S. de Ingenieros Industriales, Universidad Politécnica de Madrid, José G. Abascal 2, 28006 Madrid, Spain
e-mail: bcalle@etsii.upm.es

1 Introduction

Let $\mathbf{f} = (f_1, \ldots, f_d)$ be a system of d formal or convergent Taylor expansions about the origin; that is, for each $k = 1, \ldots, d$, we have

$$f_k(z) = \sum_{n=0}^{\infty} \phi_{n,k} z^n, \quad \phi_{n,k} \in \mathbb{C}. \tag{1}$$

Let $\mathbf{D} = (D_1, \ldots, D_d)$ be a system of domains such that, for each $k = 1, \ldots, d$, f_k is meromorphic in D_k. We say that the point ξ is a pole of $\mathbf{f}$ in $\mathbf{D}$ of order τ if there exists an index $k \in \{1, \ldots, d\}$ such that $\xi \in D_k$ and it is a pole of f_k of order τ, and for $j \neq k$ either ξ is a pole of f_j of order less than or equal to τ or $\xi \notin D_j$. When $\mathbf{D} = (D, \ldots, D)$, we say that ξ is a pole of $\mathbf{f}$ in D.

Let $R_0(\mathbf{f})$ be the radius of the largest disk $D_0(\mathbf{f})$ in which all the expansions $f_k, k = 1, \ldots, d$ correspond to analytic functions. If $R_0(\mathbf{f}) = 0$, we take $D_m(\mathbf{f}) = \emptyset, m \in \mathbb{Z}_+$; otherwise, $R_m(\mathbf{f})$ is the radius of the largest disk $D_m(\mathbf{f})$ centered at the origin to which all the analytic elements $(f_k, D_0(f_k))$ can be extended so that $\mathbf{f}$ has at most m poles counting multiplicities. The disk $D_m(\mathbf{f})$ constitutes for systems of functions the analog of the m-th disk of meromorphy defined by J. Hadamard in [9] for $d = 1$. Moreover, in that case both definitions coincide.

By $\mathcal{Q}_m(\mathbf{f})$, we denote the monic polynomial whose zeros are the poles of $\mathbf{f}$ in $D_m(\mathbf{f})$ counting multiplicities. The set of distinct zeros of $\mathcal{Q}_m(\mathbf{f})$ is denoted by $\mathcal{P}_m(\mathbf{f})$.

Definition 1.1 Let $\mathbf{f} = (f_1, \ldots, f_d)$ be a system of d formal Taylor expansions as in (1). Fix a multi-index $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$, where $\mathbf{0}$ denotes the zero vector in $\mathbb{Z}_+^d$. Set $|\mathbf{m}| = m_1 + \cdots + m_d$. Then, for each $n \geq \max\{m_1, \ldots, m_d\}$, there exist polynomials $Q, P_k, k = 1, \ldots, d$, such that

(a.1) $\deg P_k \leq n - m_k, k = 1, \ldots, d, \deg Q \leq |\mathbf{m}|, Q \not\equiv 0,$
(a.2) $Q(z) f_k(z) - P_k(z) = A_k z^{n+1} + \cdots$.

The vector rational function $\mathbf{R}_{n,\mathbf{m}} = (P_1/Q, \ldots, P_d/Q)$ is called an $(n, \mathbf{m})$ (type II) Hermite–Padé approximation of $\mathbf{f}$.

This vector rational approximation, in general, is not uniquely determined, and hereafter we assume that given $(n, \mathbf{m})$, one particular solution is taken. For that solution, we write

$$\mathbf{R}_{n,\mathbf{m}} = (R_{n,\mathbf{m},1}, \ldots, R_{n,\mathbf{m},d}) = (P_{n,\mathbf{m},1}, \ldots, P_{n,\mathbf{m},d})/Q_{n,\mathbf{m}}, \tag{2}$$

where $Q_{n,\mathbf{m}}$ is a monic polynomial that has no common zero simultaneously with all the $P_{n,\mathbf{m},k}$. Sequences $\{\mathbf{R}_{n,\mathbf{m}}\}$ for which $|\mathbf{m}|$ remains fixed when n varies are called row sequences, and when $m_1 = \cdots = m_d = m$, $n = (d+1)m, m \in \mathbb{Z}_+$ (or nearby configurations of multi-indices), diagonal sequences.

There is another construction called type I Hermite–Padé approximation which is intimately connected with the type II Hermite–Padé approximants we have introduced above, see [18, Chap. 4] for details. However, throughout the paper we restrict our attention to the type II and, to abbreviate, we simply call them Hermite–Padé approximations.

The study of simultaneous Hermite–Padé approximations of systems of functions has a long tradition (see [10–13]), and they have been subject to renewed interest in the recent past (see, for instance, [5] and the references therein). Many papers deal with diagonal sequences and their applications in different fields (number theory, random matrices, Brownian motions, Toda lattices, to name a few). At the same time, few papers study row sequences. In this second direction, a significant contribution is due to Graves-Morris/Saff in [15], where they prove an analog of the Montessus de Ballore theorem which plays a central role in the classical theory of Padé approximation. See also [16, 17] for different approaches to the same type of results as well as [19] and references therein for least-squares versions.

Before going into details, let us briefly describe the scalar case ($d = 1$) corresponding to classical Padé approximation, which is well understood. When $d = 1$, we write $\mathbf{f} = f$, $\mathbf{m} = m \in \mathbb{N}$, and $\mathbf{R}_{n,\mathbf{m}} = R_{n,m}$. Given a compact set $K \subset \mathbb{C}$, $\|\cdot\|_K$ denotes the sup norm on K. We summarize what we need in the following statement.

Gonchar's theorem *Let f be a formal Taylor expansion about the origin, and fix $m \in \mathbb{N}$. Then the following two assertions are equivalent:*

(a) $R_0(f) > 0$ *and f has exactly m poles in $D_m(f)$ counting multiplicities.*
(b) *There is a polynomial Q_m of degree m, $Q_m(0) \neq 0$, such that the sequence of denominators $\{Q_{n,m}\}_{n \geq m}$ of the Padé approximations of f satisfies*

$$\limsup_{n\to\infty} \|Q_m - Q_{n,m}\|^{1/n} = \theta < 1,$$

where $\|\cdot\|$ denotes the coefficient norm in the space of polynomials.

Moreover, if either (a) or (b) takes place, then $Q_m \equiv \mathcal{Q}_m(f)$,

$$\theta = \frac{\max\{|\xi| : \xi \in \mathcal{P}_m(f)\}}{R_m(f)}, \tag{3}$$

and

$$\limsup_{n\to\infty} \|f - R_{n,m}\|_K^{1/n} = \frac{\|z\|_K}{R_m(f)}, \tag{4}$$

where K is any compact subset of $D_m(f) \setminus \mathcal{P}_m(f)$.

From this result, it follows that if ξ is a pole of f in $D_m(f)$ of order τ, then for each $\varepsilon > 0$, there exists n_0 such that for $n \geq n_0$, $Q_{n,m}$ has exactly τ zeros in $\{z : |z - \xi| < \varepsilon\}$. We say that each pole of f in $D_m(f)$ attracts as many zeros of $Q_{n,m}$ as its order when n tends to infinity.

So stated, Gonchar's theorem first appears as a remark in Sect. 3, Sect. 4, in [6] (see also [8, Sect. 2]). Under assumptions (a), in [14] Montessus de Ballore proved that

$$\lim_{n\to\infty} Q_{n,m} = \mathcal{Q}_m(f), \qquad \lim_{n\to\infty} R_{n,m} = f,$$

with uniform convergence on compact subsets of $D_m(f) \setminus \mathcal{P}_m(f)$ in the second limit. In essence, Montessus proved that (a) implies (b) with $Q_m = \mathcal{Q}_m(f)$, s h o w e d that $\theta \leq \max\{|\xi| : \xi \in \mathcal{P}_m(f)\}/R_m(f)$, and proved (4) with equality replaced by $\leq$. These are the so-called direct statements of the theorem. The inverse statements, (b) implies (a), $\theta \geq \max\{|\xi| : \xi \in \mathcal{P}_m(f)\}/R_m(f)$, and the inequality $\geq$ in (4) a r e i m - mediate consequences of [7, Theorem 1]. The study of inverse problems of Padé approximation was suggested by A.A. Gonchar in [7, Sect. 12], where he presented some interesting conjectures. Some of them were solved in [20] and [21].

In [15], Graves-Morris and Saff proved an analog of the direct part of Gonchar's theorem for simultaneous approximation with the aid of the concept of polewise independence of a system of functions.

For each $r > 0$, set $D_r = \{z \in \mathbb{C} : |z| < r\}$, $\Gamma_r = \{z \in \mathbb{C} : |z| = r\}$, and $\overline{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$.

Definition 1.2 Let $\mathbf{f} = (f_1, \ldots, f_d)$ be a system of meromorphic functions in the disk D_r and let $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$. We say that the system $\mathbf{f}$ is polewise independent with respect to $\mathbf{m}$ in D_r if there do not exist polynomials $p_1, \ldots, p_d$, at least one of which is non-null, such that

(b.1) $\deg p_k < m_k$ if $m_k \geq 1, k = 1, \ldots, d$,
(b.2) $p_k \equiv 0$ if $m_k = 0, k = 1, \ldots, d$,
(b.3) $\sum_{k=1}^{d} p_k f_k$ is analytic on D_r.

Graves-Morris and Saff also established in [15] upper bounds for the convergence rates corresponding to (3) and (4). These results were refined and complemented in [4, Theorem 4.4] by weakening the assumption of polewise independence, improving the upper bound given in [15] for the rate (3), and giving the exact one for (4). Until now, results of inverse type for row sequences of Hermite–Padé approximants have not been available.

Our purpose is to obtain an analog of Gonchar's theorem for simultaneous Hermite–Padé approximants, characterizing the exact rates of convergence of the $Q_{n,\mathbf{m}}$ and $\mathbf{R}_{n,\mathbf{m}}$.

The underlying idea in inverse-type results is that a polynomial which is the limit of the denominators of the approximants must have as zeros the poles of the function being approximated, provided that the rate of convergence is geometric. However, the actual situation in simultaneous approximation may be rather complicated, as the following example shows. Take $\mathbf{f} = (f_1, f_2)$, where

$$f_1 = \frac{1}{1-2z} + \sum_{n=0}^{\infty} z^{n!} + \frac{1}{z-2}, \qquad f_2 = \frac{1}{1-2z} + \sum_{n=0}^{\infty} z^{n!}, \tag{5}$$

and $\mathbf{m} = (1, 1)$. It is clear that the unit circle is a natural boundary of definition for both functions f_1 and f_2, and thus $z = 2$ cannot be a pole of $\mathbf{f}$ in any system of domains. However, results contained in [4] show that the denominators $Q_{n,\mathbf{m}}$ of the simultaneous Hermite–Padé approximants converge with geometric rate to the polynomial $(z - 1/2)(z - 2)$.

This kind of example leads us to introduce the following concept, which is actually inspired by Definition 1.2.

Definition 1.3 Given $\mathbf{f} = (f_1, \ldots, f_d)$ and $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$, we say that $\xi \in \mathbb{C} \setminus \{0\}$ is a system pole of order τ of $\mathbf{f}$ with respect to $\mathbf{m}$ if τ is the largest positive integer such that for each $s = 1, \ldots, \tau$, there exists at least one polynomial combination of the form

$$\sum_{k=1}^{d} p_k f_k, \quad \deg p_k < m_k,\ k = 1, \ldots, d, \tag{6}$$

which is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole at $z = \xi$ of exact order s. If some component m_k equals zero, the corresponding polynomial p_k is taken identically equal to zero.

The great advantage of this definition with respect to that of polewise independence is that we have liberated it from establishing a priori a region where the property should be verified. This turns out to be crucial.

We wish to underline that if some component m_k equals zero, that component places no restriction on Definition 1.1 and does not provide any benefit in finding system poles; therefore, without loss of generality, we can restrict our attention to multi-indices $\mathbf{m} \in \mathbb{N}^d$, and we will do so hereafter, except in reference to the convergence of the approximants themselves.

Notice that the definition of system pole strongly depends on the multi-index $\mathbf{m}$ and that a system $\mathbf{f}$ cannot have more than $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ counting their order (see Lemma 3.5 below). During the proof of Theorem 1.4 below, carried out in Sect. 3, we give a procedure for finding in a finite number of steps all the system poles of $\mathbf{f}$ with respect to a multi-index $\mathbf{m}$ under appropriate conditions.

It is easy to see that a system pole may not be a pole of $\mathbf{f}$ or vice versa. For example, let $\mathbf{f}$ be the system given by (5) and $\mathbf{m} = (1, 1)$. The point $z = 2$, which lies beyond the natural boundary of definition of f_1 and f_2, is not a pole; however, it is a system pole of $\mathbf{f}$ since $f_1 - f_2$ has a pole at $z = 2$.

On the other hand, take $\mathbf{f} = (f_1, f_2)$, with

$$f_1 = \frac{1}{z-1} + \frac{1}{z-2}, \qquad f_2 = \frac{1}{z-3},$$

and $\mathbf{m} = (1, 1)$. Then the points $z = 1$ and $z = 3$ are poles and system poles of $\mathbf{f}$ but $z = 2$ is only a pole because there is no way of eliminating the pole at $z = 1$ through linear combinations of f_1 and f_2 without eliminating the pole at $z = 2$.

To each system pole ξ of $\mathbf{f}$ with respect to $\mathbf{m}$ we associate several characteristic values. Let τ be the order of ξ as a system pole of $\mathbf{f}$. For each $s = 1, \ldots, \tau$, denote by $r_{\xi,s}(\mathbf{f}, \mathbf{m})$ the largest of all the numbers $R_s(g)$ (the radius of the largest disk containing at most s poles of g), where g is a polynomial combination of type (6) that is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole at $z = \xi$ of order s. Then

$$R_{\xi,s}(\mathbf{f},\mathbf{m}) = \min_{k=1,\dots,s} r_{\xi,k}(\mathbf{f},\mathbf{m}),$$

$$R_{\xi}(\mathbf{f},\mathbf{m}) = R_{\xi,\tau}(\mathbf{f},\mathbf{m}) = \min_{s=1,\dots,\tau} r_{\xi,s}(\mathbf{f},\mathbf{m}).$$

Obviously, if $d = 1$ and $(\mathbf{f},\mathbf{m}) = (f, m)$, system poles and poles in $D_m(f)$ coincide. Also, $R_\xi(\mathbf{f},\mathbf{m}) = R_m(f)$ for each pole ξ of f in $D_m(f)$.

By $\mathcal{Q}(\mathbf{f},\mathbf{m})$, we denote the monic polynomial whose zeros are the system poles of $\mathbf{f}$ with respect to $\mathbf{m}$ taking account of their order. The set of distinct zeros of $\mathcal{Q}(\mathbf{f},\mathbf{m})$ is denoted by $\mathcal{P}(\mathbf{f},\mathbf{m})$.

The following theorem constitutes our main result.

Theorem 1.4 *Let* $\mathbf{f}$ *be a system of formal Taylor expansions as in* (1), *and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$. *Then the following two assertions are equivalent*:

(a) $R_0(\mathbf{f}) > 0$ *and* $\mathbf{f}$ *has exactly* $|\mathbf{m}|$ *system poles with respect to* $\mathbf{m}$ *counting multiplicities.*

(b) *The denominators* $Q_{n,\mathbf{m}}$, $n \geq |\mathbf{m}|$, *of simultaneous Padé approximations of* $\mathbf{f}$ *are uniquely determined for all sufficiently large* n, *and there exists a polynomial* $Q_{|\mathbf{m}|}$ *of degree* $|\mathbf{m}|$, $Q_{|\mathbf{m}|}(0) \neq 0$, *such that*

$$\limsup_{n\to\infty} \| Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}} \|^{1/n} = \theta < 1.$$

Moreover, if either (a) *or* (b) *takes place, then* $Q_{|\mathbf{m}|} \equiv \mathcal{Q}(\mathbf{f},\mathbf{m})$ *and*

$$\theta = \max\left\{ \frac{|\xi|}{R_\xi(\mathbf{f},\mathbf{m})} : \xi \in \mathcal{P}(\mathbf{f},\mathbf{m}) \right\}. \tag{7}$$

If $d = 1$, $R_{n,m}$ and $Q_{n,m}$ are uniquely determined. Therefore, Theorem 1.4 implies Gonchar's theorem except for (4), whose analog will be presented in Sect. 3.2 to avoid introducing new notation at this stage.

The paper is structured as follows. In Sect. 2, we continue with the study of incomplete Padé approximants initiated in [4], proving results of inverse type. Section 3 is dedicated to the proof of Theorem 1.4 and the analog of (4).

2 Incomplete Padé Approximants

Let

$$f(z) = \sum_{n=0}^{\infty} \phi_n z^n, \quad \phi_n \in \mathbb{C}, \tag{8}$$

denote a formal or convergent Taylor expansion about the origin.

Definition 2.1 Let f denote a formal Taylor expansion as in (8). Fix $m \geq m^* \geq 1$. Let $n \geq m$. We say that the rational function $r_{n,m}$ is an incomplete Padé approximation of type (n, m, m^*) corresponding to f if $r_{n,m}$ is the quotient of any two polynomials p and q that verify

(c.1) $\deg p \leq n - m^*, \deg q \leq m, q \not\equiv 0,$
(c.2) $q(z)f(z) - p(z) = Az^{n+1} + \cdots.$

Notice that given $(n, m, m^*), n \geq m \geq m^*$, any of the Padé approximants $R_{n,m^*}, \ldots, R_{n,m}$ can be regarded an incomplete Padé approximation of type (n, m, m^*) of f. From Definition 1.1 and (2), it follows that $R_{n,\mathbf{m},k}, k = 1, \ldots, d$, is an incomplete Padé approximation of type $(n, |\mathbf{m}|, m_k)$ with respect to f_k.

Hereafter, for each $n \geq m \geq m^*$, we choose one candidate. After canceling out common factors between q and p, we write $r_{n,m} = p_{n,m}/q_{n,m}$, where, additionally, $q_{n,m}$ is normalized to be monic. Suppose that q and p have a common zero at $z = 0$ of order λ_n. Notice that $0 \leq \lambda_n \leq m$. From (c.1)–(c.2), it follows that

(c.3) $\deg p_{n,m} \leq n - m^* - \lambda_n, \deg q_{n,m} \leq m - \lambda_n,\ q_{n,m} \not\equiv 0,$
(c.4) $q_{n,m}(z)f(z) - p_{n,m}(z) = Az^{n+1-\lambda_n} + \cdots,$

where A is, in general, a different constant from the one in (c.2).

The first difficulty encountered in dealing with inverse-type results is to justify in terms of the data that the formal series corresponds to an analytic element which does not reduce to a polynomial. In our aid comes the next result, which provides such information in terms of whether the zeros of the polynomials $q_{n,m}$ remain away or not from 0 and/or ∞ as n grows. Let

$$\mathcal{P}_{n,m} = \{\zeta_{n,1}, \ldots, \zeta_{n,m_n}\}, \quad n \geq m,\ m_n \leq m,$$

denote the collection of zeros of $q_{n,m}$ repeated according to their multiplicity, where $\deg q_{n,m} = m_n$. Set

$$S = \sup_{N \geq m} \inf\left\{|\zeta_{n,k}| : n \geq N, m_n \geq 1, 1 \leq k \leq m_n\right\}$$

and

$$G = \inf_{N \geq m} \sup\left\{|\zeta_{n,k}| : n \geq N, m_n \geq 1, 1 \leq k \leq m_n\right\}.$$

Finally, set

$$\tau_n = \min\left\{n - m^* - \lambda_n - \deg p_{n,m}, m - \lambda_n - m_n\right\}, \quad n \geq m.$$

From (c.3), we know that $0 \leq \tau_n \leq m, n \geq m$.

Theorem 2.2 *Let f be a formal power series as in* (8). *Fix $m \geq m^* \geq 1$. The following assertions hold*:

(i) *If $|\lambda_n - \lambda_{n-1}| \leq m^* - 1, n \geq n_0$, and $S > 0$, then $R_0(f) > 0$.*
(ii) *If $|(m_n + \lambda_n + \tau_n) - (m_{n-1} + \lambda_{n-1} + \tau_{n-1})| \leq m^* - 1, n \geq n_0$, and $G < \infty$, then either f is a polynomial or $R_0(f) < \infty$. If, additionally, there exists a sequence of indices Λ such that $\deg q_{n,m} \geq 1, n \in \Lambda$, then $R_0(f) < \infty$.*

Proof By definition,

$$(q_{n,m}f - p_{n,m})(z) = Az^{n+1-\lambda_n} + \cdots, \tag{9}$$

and $q_{n,m}(0) \neq 0$.

We may suppose that $\inf\{|\zeta_{n,k}| : n \geq n_0, m_n \geq 1, 1 \leq k \leq m_n\} > 0$ and $|\lambda_n - \lambda_{n-1}| \leq m^* - 1, n \geq n_0$. Normalize $q_{n,m}$ as follows. If $m_n \geq 1$, take

$$q_{n,m}(z) = \prod_{k=1}^{m_n}\left(1 - \frac{z}{\zeta_{n,k}}\right) = a_{n,0} + a_{n,1}z + \cdots + a_{n,m_n}z^{m_n}, \quad a_{n,0} = 1.$$

Otherwise, $q_{n,m}(z) \equiv 1 = a_{n,0}$.

Using the Vieta formulas connecting the coefficients of a polynomial and its zeros, it follows that there exists $C_1 \geq 1$ such that

$$\sup\big\{|a_{n,k}| : 0 \leq k \leq m_n, n \geq n_0\big\} \leq C_1 < \infty. \tag{10}$$

The coefficient corresponding to $z^k, k \in \{n - m^* - \lambda_n + 1, \ldots, n - \lambda_n\}$, in the left-hand side of (9) equals

$$\phi_k + a_{n,1}\phi_{k-1} + \cdots + a_{n,m_n}\phi_{k-m_n} = 0, \tag{11}$$

since $\deg p_{n,m} \leq n - m^* - \lambda_n$.

If $m_n \geq 1$, (10) and (11) imply that

$$|\phi_k| \leq C_1\big(|\phi_{k-1}| + \cdots + |\phi_{k-m_n}|\big).$$

Therefore, for each $k \in \{n - m^* - \lambda_n + 1, \ldots, n - \lambda_n\}$, there exists $k' \in \{k-1, \ldots, k-m\}$ $(m_n \leq m)$ such that

$$|\phi_k| \leq C_1 m |\phi_{k'}|. \tag{12}$$

Should $m_n = 0$, for the same values of k, we have $\phi_k = 0$, and (12) is trivially verified. Substituting n by $n-1$, we deduce that for each $k \in \{n - m^* - \lambda_{n-1}, \ldots, n - \lambda_{n-1} - 1\}$, there exists $k' \in \{k-1, \ldots, k-m\}$ such that

$$|\phi_k| \leq C_1 m |\phi_{k'}|. \tag{13}$$

As $n \geq n_0$, we have

$$n - \lambda_{n-1} \geq n - \lambda_n - m^* + 1$$

and

$$n - \lambda_{n-1} - m^* \leq n - \lambda_n - 1,$$

because $|\lambda_n - \lambda_{n-1}| \leq m^* - 1$. Consequently, the range of values taken by k due to relations (12) and (13) are either contiguous or overlapping for $n \geq n_0$. Since $n - \lambda_n$ tends to ∞ as n goes to ∞, we conclude that for all $n \geq n_0$, there exists $n' \in \{n-1, \ldots, n-m\}$ such that

$$|\phi_n| \leq C_1 m |\phi_{n'}|. \tag{14}$$

Let Λ be a sequence of indices such that

$$\lim_{n\in\Lambda} |\phi_n|^{1/n} = \limsup_{n\to\infty} |\phi_n|^{1/n} = 1/R_0(f).$$

Choose $n \in \Lambda$. Due to (14), there exist indices $n_1 > n_2 > \cdots > n_{r_n}, n_{r_n} \le n_0$, where $r_n \le n - n_0$, such that

$$|\phi_n| \le C_1 m |\phi_{n_1}| \le \cdots \le (C_1 m)^{r_n} |\phi_{n_{r_n}}|.$$

Consequently,

$$1/R_0(f) = \lim_{n\in\Lambda} |\phi_n|^{1/n} \le \limsup_{n\to\infty} (C_1 m)^{r_n/n} \le C_1 m.$$

Therefore, $R_0(f) \ge (C_1 m)^{-1} > 0$, which proves (i).

As for (ii), assume that $\sup\{|\zeta_{n,k}| : n \ge n_0, m_n \ge 1, 1 \le k \le m_n\} < \infty$ and $|(m_n + \lambda_n + \tau_n) - (m_{n-1} + \lambda_{n-1} + \tau_{n-1})| \le m^* - 1, n \ge n_0$. Set $t_n(z) = (z-1)^{\tau_n}$. Define $\tilde{q}_{n,m} = t_n q_{n,m}$ and $\tilde{p}_{n,m} = t_n p_{n,m}$. Normalize $\tilde{q}_{n,m}$ as follows. If $m_n + \tau_n \ge 1$, take

$$\tilde{q}_{n,m}(z) = \prod_{k=1}^{m_n+\tau_n} (z - \zeta_{n,k}) = b_{n,0} z^{m_n+\tau_n} + \cdots + b_{n,m_n+\tau_n-1} z + b_{n,m_n+\tau_n},$$

where $b_{n,0} = 1$. Should $m_n + \tau_n = 0$, we set $\tilde{q}_{n,m} \equiv 1 = b_{n,0}$. Using the Vieta formulas, it follows that there exists $C_2 \ge 1$ such that

$$\sup\big\{|b_{n,k}| : 0 \le k \le m_n, n \ge n_0\big\} \le C_2 < \infty. \tag{15}$$

The coefficient corresponding to z^k, $k \in \{n - m^* - \lambda_n + 1, \ldots, n - \lambda_n\}$, in the left-hand side of (9) equals

$$\phi_{k-m_n-\tau_n} + b_{n,1}\phi_{k-m_n-\tau_n+1} + \cdots + b_{n,m_n+\tau_n}\phi_k = 0, \tag{16}$$

since $\deg \tilde{p}_{n,m} \le n - m^* - \lambda_n$.

Should $m_n + \tau_n \ge 1$, (15) and (16) imply that

$$|\phi_{k-m_n-\tau_n}| \le C_2\big(|\phi_{k-m_n-\tau_n+1}| + \cdots + |\phi_k|\big),$$

or, which is the same, for each $k \in \{n - m^* - \lambda_n - m_n - \tau_n + 1, \ldots, n - \lambda_n - m_n - \tau_n\}$, we have

$$|\phi_k| \le C_2\big(|\phi_{k+1}| + \cdots + |\phi_{k+m_n+\tau_n}|\big).$$

Therefore, for each $k \in \{n - m^* - \lambda_n - m_n - \tau_n + 1, \ldots, n - \lambda_n - m_n - \tau_n\}$, there exists $k' \in \{k+1, \ldots, k+m\}$ $(m_n + \tau_n \le m)$ such that

$$|\phi_{k'}| \ge \frac{|\phi_k|}{C_2 m}. \tag{17}$$

In the case that $m_n + \tau_n = 0$, we have $\phi_k = 0$ for the same values of k, and (17) is also true.

Using the assumption that $|\lambda_n + m_n + \tau_n - \lambda_{n-1} - m_{n-1} - \tau_{n-1}| \le m^* - 1$, it is easy to check, similarly to the previous case, that the range of values taken by the parameter k for consecutive values of n are either contiguous or overlapping. Also,

$n - \lambda_n - m_n - \tau_n$ tends to ∞ as n goes to ∞. Consequently, from (17), we have that for all $n \geq n_0$, there exists $n' \in \{n+1, \dots, n+m\}$ such that

$$|\phi_{n'}| \geq \frac{|\phi_n|}{C_2 m}. \tag{18}$$

Using (18), we can find an increasing sequence of indices $\{n_s\}_{s \in \mathbb{Z}_+}$, $n_{s+1} \in \{n_s + 1, \dots, n_s + m\}$ and $n_1 \in \{n_0, \dots, n_0 + m\}$ such that

$$|\phi_{n_{s+1}}| \geq \frac{|\phi_{n_1}|}{(C_2 m)^s}.$$

Should f be a polynomial, there is nothing to prove. Otherwise, changing the value of n_0 if necessary, without loss of generality, we can assume that $\phi_{n_1} \neq 0$. Then

$$\liminf_{s\to\infty} |\phi_{n_{s+1}}|^{1/n_{s+1}} \geq \frac{1}{\limsup_{s\to\infty} (C_2 m)^{s/n_{s+1}}} \geq \frac{1}{C_2 m},$$

since

$$\limsup_{s\to\infty} \frac{s}{n_{s+1}} \leq \limsup_{s\to\infty} \frac{s}{n_1 + s} = 1.$$

It follows that

$$R_0(f) = \frac{1}{\limsup_{n\to\infty} |\phi_n|^{1/n}} \leq \frac{1}{\liminf_{s\to\infty} |\phi_{n_{s+1}}|^{1/n_{s+1}}} \leq C_2 m < \infty,$$

as we needed to prove.

Finally, if f is a polynomial, say of degree N, we would have that for all $n \geq N + m$, $f \equiv p_{n,m}/q_{n,m}$ and $q_{n,m} \equiv 1$. Consequently, if there exists Λ such that $\deg q_{n,m} \geq 1, n \in \Lambda$, f cannot be a polynomial and, therefore, only $R_0(f) < \infty$ is possible. □

Lemma 2.3 *A sufficient condition to have* $|\lambda_n - \lambda_{n-1}| \leq m^* - 1$ *and* $|(m_n + \lambda_n + \tau_n) - (m_{n-1} + \lambda_{n-1} + \tau_{n-1})| \leq m^* - 1$ *is that*

$$\min\{m_n + \tau_n, m_{n-1} + \tau_{n-1}\} \geq m - m^* + 1.$$

Proof In fact, for $k = n-1$ and $k = n$, if $m_k + \tau_k \geq m - m^* + 1$, then $0 \leq \lambda_k \leq m^* - 1$ because $\lambda_k + m_k + \tau_k \leq m$, and the first inequality readily follows. On the other hand,

$$\begin{aligned} &\left|(m_n + \lambda_n + \tau_n) - (m_{n-1} + \lambda_{n-1} + \tau_{n-1})\right| \\ &\quad = \left|\left(m_n + \lambda_n + \tau_n - m + m^* - 1\right) - \left(m_{n-1} + \lambda_{n-1} + \tau_{n-1} - m + m^* - 1\right)\right|, \end{aligned}$$

and $0 \leq m_k + \lambda_k + \tau_k - m + m^* - 1 \leq m^* - 1$ for $k = n-1$ and $k = n$. Therefore, the second inequality also holds. □

Applied to Padé approximation ($m^* = m$), Theorem 2.2 and Lemma 2.3 imply that if $\deg Q_{n,m} \geq 1$ and its zeros remain uniformly bounded away from 0 and ∞, for sufficiently large n, then $0 < R_0(f) < \infty$. This result has not been stated elsewhere.

Let us see some consequences of Theorem 2.2 and Lemma 2.3 on the extendability of a formal power series and the location of some of its poles in terms of the behavior of the zeros of the approximants. First we call attention to some results from [4].

Let B be a subset of the complex plane $\mathbb{C}$. By $\mathcal{U}(B)$, we denote the class of all coverings of B by at most a numerable set of disks. Set

$$\sigma(B) = \inf\left\{\sum_{i=1}^{\infty} |U_i| : \{U_i\} \in \mathcal{U}(B)\right\},$$

where $|U_i|$ stands for the radius of the disk U_i. The quantity $\sigma(B)$ is called the 1-dimensional Hausdorff content of the set B.

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence of functions defined on a domain $D \subset \mathbb{C}$ and φ another function defined on D. We say that $\{\varphi_n\}_{n\in\mathbb{N}}$ converges in σ-content to the function φ on compact subsets of D if for each compact subset K of D and for each $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \sigma\left\{z \in K : \left|\varphi_n(z) - \varphi(z)\right| > \varepsilon\right\} = 0.$$

We denote this by writing $\sigma\text{-}\lim_{n\to\infty} \varphi_n = \varphi$ inside D.

We define the number $R_m^*(f)$ as the radius of the largest disk centered at the origin on compact subsets of which the sequence $\{r_{n,m}\}_{n\geq m}$ converges to f in σ-content. In [4], we gave a formula to produce this number and showed that it depends on the specific sequence of incomplete Padé approximants considered. Set $D_m^*(f) = \{z \in \mathbb{C} : |z| < R_m^*(f)\}$.

Among other direct-type results, we proved that

$$R_{m^*}(f) \leq R_m^*(f) \leq R_m(f), \tag{19}$$

that $R^*{}_m(f) > 0$ implies $R_0(f) > 0$, and that each pole of the function f in $D^*{}_m(f)$ attracts, with geometric rate, at least as many zeros of $q_{n,m}$ as its order (see [4, Theorem 3.5]). Therefore, Theorem 2.2 and Lemma 2.3 imply:

Corollary 2.4 *Let f be a formal power series as in* (8). *Fix $m \geq m^* \geq 1$. Assume that there exists a polynomial q_m of degree greater than or equal to $m - m^* + 1$, $q_m(0) \neq 0$, such that $\lim_{n\to\infty} q_{n,m} = q_m$. Then $0 < R_0(f) < \infty$, and the zeros of q_m contain all the poles, counting multiplicities, that f has in $D_m^*(f)$.*

We need a relaxed version of Corollary 2.4 for the proof of Theorem 1.4.

Lemma 2.5 *Let f be a formal power series as in* (8) *that is not a polynomial. Fix $m \geq m^* \geq 1$. Let $r_{n,m} = \tilde{p}_{n,m}/\tilde{q}_{n,m}$ be an incomplete Padé approximant of type (n, m, m^*) corresponding to f, where $\tilde{p}_{n,m}$ and $\tilde{q}_{n,m}$ are obtained from Definition* 2.1 *and common factors between them are allowed. Assume that there exists a polynomial $\tilde{q}_m$ of degree m, $\tilde{q}_m(0) \neq 0$, such that $\lim_{n\to\infty} \tilde{q}_{n,m} = \tilde{q}_m$. Then $0 < R_0(f) < \infty$, and the zeros of $\tilde{q}_m$ contain all the poles, counting multiplicities, that f has in $D_m^*(f)$.*

Proof Let us show that the assumptions of Lemma 2.3 are verified for the incomplete approximant $r_{n,m}$. Let $r_{n,m} = p_{n,m}/q_{n,m}$, where the polynomials $p_{n,m}$ and $q_{n,m}$ are relatively prime. Since $\tilde{q}_m(0) \neq 0$, then $\tilde{q}_{n,m}(0) \neq 0$, $n \geq n_0$. Thus, $\tilde{p}_{n,m}$ and $\tilde{q}_{n,m}$ do not have a common zero at $z = 0$, and $\lambda_n = 0$ for all $n \geq n_0$. As before, set $m_n = \deg q_{n,m}$ and

$$\tau_n = \min\{n - m^* - \deg p_{n,m}, m - m_n\}, \quad n \geq n_0.$$

Notice that $\tau_n = m - m_n$, $n \geq n_0$, because the polynomials $q_{n,m}$ and $p_{n,m}$ are obtained eliminating possible common factors between $\tilde{q}_{n,m}$ and $\tilde{p}_{n,m}$ and by assumption

$$\min\{n - m^* - \deg \tilde{p}_{n,m}, m - \deg \tilde{q}_{n,m}\} = 0, \quad n \geq n_0.$$

Therefore, we have

$$m_n + \tau_n = m \geq m - m^* + 1, \quad n \geq n_0,$$

and Lemma 2.3 is applicable.

From Theorem 2.2, we obtain $0 < R_0(f) < \infty$. Now, from the fact that each pole of f in $D_m^*(f)$ attracts as many zeros of $q_{n,m}$ as its order, it follows that the zeros of $\tilde{q}_m$ contain all the poles, counting multiplicities, that f has in $D_m^*(f)$. □

In the case that there exists $R > R_{m^*}(f)$ inside of which f is meromorphic, then D_R contains at least $m^* + 1$ poles of f since $D_{m^*}(f)$ is the largest disk where f is meromorphic with at most m^* poles. We can prove the following inverse-type result.

Theorem 2.6 *Fix $m \geq m^* \geq 1$. Let f be a formal power series as in (8) that is not a rational function with at most $m^* - 1$ poles. Let $r_{n,m} = \tilde{p}_{n,m}/\tilde{q}_{n,m}$ be an incomplete Padé approximant of type (n, m, m^*) corresponding to f, where $\tilde{p}_{n,m}$ and $\tilde{q}_{n,m}$ are obtained from Definition 2.1 and common factors between them are allowed. Suppose that there exists a polynomial $\tilde{q}_m$, of degree m, $\tilde{q}_m(0) \neq 0$, such that*

$$\limsup_{n\to\infty} \|\tilde{q}_{n,m} - \tilde{q}_m\|^{1/n} = \theta < 1. \tag{20}$$

Then, either f has exactly m^* poles in $D_{m^*}(f)$, which are zeros of $\tilde{q}_m$ counting multiplicities, or $R_0(\tilde{q}_m f) > R_{m^*}(f)$.

Proof From Lemma 2.5, we have $R_0(f) > 0$. So, f is analytic in a neighborhood of $z = 0$. We also know that $R_0(\tilde{q}_m f) \geq R_{m^*}(f)$ since the zeros of $\tilde{q}_m$ contain all the poles that f has in $D_{m^*}(f)$. Assume that $R_0(\tilde{q}_m f) = R_{m^*}(f)$. Let us show that then f has exactly m^* poles in $D_{m^*}(f)$. To the contrary, suppose that f has in $D_{m^*}(f)$ at most $m^* - 1$ poles. Then there exists a polynomial q_{m^*}, with $\deg q_{m^*} < m^*$, such that

$$R_0(q_{m^*} f) = R_{m^*}(f) = R_0(q_{m^*}\tilde{q}_m f).$$

Let

$$q_{m^*}(z)\tilde{q}_m(z) f(z) = \sum_{n=0}^{\infty} a_n z^n;$$

then

$$R_{m^*}(f) = R_0(q_{m^*}\tilde{q}_m f) = 1/\limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

The n-th Taylor coefficient of $q_{m^*}[\tilde{q}_{n,m} f - \tilde{p}_{n,m}]$ is equal to zero. Therefore, the n-th Taylor coefficients of $q_{m^*}\tilde{q}_m f$ and $q_{m^*}\tilde{q}_m f - q_{m^*}\tilde{q}_{n,m} f + q_{m^*}\tilde{p}_{n,m}$ coincide. Take $0 < r < R_{m^*}(f)$, and recall that $\Gamma_r = \{z \in \mathbb{C} : |z| = r\}$. Hence

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_{\Gamma_r} \frac{[q_{m^*}\tilde{q}_m f - q_{m^*}\tilde{q}_{n,m} f + q_{m^*}\tilde{p}_{n,m}](\omega)}{\omega^{n+1}}\, d\omega \\ &= \frac{1}{2\pi i}\int_{\Gamma_r} \frac{[\tilde{q}_m - \tilde{q}_{n,m}](\omega) q_{m^*}(\omega) f(\omega)}{\omega^{n+1}}\, d\omega. \end{aligned}$$

Making use of (20), it readily follows that

$$\frac{1}{R_{m^*}(f)} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \le \frac{\theta}{r}.$$

Letting r tend to $R_{m^*}(f)$, we have

$$\frac{1}{R_{m^*}(f)} \le \frac{\theta}{R_{m^*}(f)}, \quad \theta < 1,$$

which implies that $R_{m^*}(f) = \infty$. Let us show that this is not possible.

In fact,

$$[q_{m^*}\tilde{q}_{n,m} f - q_{m^*}\tilde{p}_{n,m}](z) = A_n z^{n+1} + \cdots,$$

and $\deg q_{m^*}\tilde{p}_{n,m} \le n-1$. It follows that $(q_{m^*}\tilde{p}_{n,m})/\tilde{q}_{n,m} = (q_{m^*}p_{n,m})/q_{n,m}$ is an incomplete Padé approximant of the function $q_{m^*} f$ of type $(n, m, 1)$, where the polynomials $p_{n,m}$ and $q_{n,m}$ are relatively prime. As $\tilde{q}_{n,m}(0) \ne 0, n \ge n_0$, the polynomials $q_{m^*}\tilde{p}_{n,m}$ and $\tilde{q}_{n,m}$ do not have a common zero at $z = 0$ and $\lambda_n = 0$ for all $n \ge n_0$. Again, set $m_n = \deg q_{n,m}$ and

$$\tau_n = \min\{n - 1 - \deg p_{n,m}, m - m_n\}.$$

Notice that $\tau_n = m - m_n, n \ge n_0$, because

$$\min\{n - 1 - \deg q_{m^*}\tilde{p}_{n,m}, m - \deg \tilde{q}_{n,m}\} = 0, \quad n \ge n_0.$$

Thus, $m_n + \tau_n = m, n \ge n_0$. Using Lemma 2.3 (for $m^* = 1$) and Theorem 2.2, we conclude that either $R_0(q_{m^*} f) < \infty$ or $q_{m^*} f$ is a polynomial. However, the latter is not possible by hypotheses. On the other hand, $R_0(q_{m^*} f) < \infty$ contradicts $R_{m^*}(f) = \infty$. As claimed, f has exactly m^* poles in $D_{m^*}(f)$. □

We wish to underline that the two possibilities stated in the thesis of Theorem 2.6 should not be understood as mutually exclusive. It may well happen that $R_0(\tilde{q}_m f) > R_{m^*}(f)$ and f has exactly m^* poles in $D_{m^*}(f)$, which are zeros of $\tilde{q}_m$ counting multiplicities. A similar remark applies to Corollary 3.4 below.

3 Simultaneous Approximation

Throughout this section, $\mathbf{f} = (f_1, \ldots, f_d)$ denotes a system of formal power expansions as in (1), and $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ is a fixed multi-index. We are concerned with the simultaneous approximation of $\mathbf{f}$ by sequences of vector rational functions defined according to Definition 1.1 taking account of (2). That is, for each $n \in \mathbb{N}, n \geq |\mathbf{m}|$, let $(R_{n,\mathbf{m},1}, \ldots, R_{n,\mathbf{m},d})$ be a Hermite–Padé approximation of type $(n, \mathbf{m})$ corresponding to $\mathbf{f}$.

As we mentioned earlier, $R_{n,\mathbf{m},k}$ is an incomplete Padé approximant of type $(n, |\mathbf{m}|, m_k)$ with respect to $f_k, k = 1, \ldots, d$. Thus, from (19), we have

$$D_{m_k}(f_k) \subset D^*_{|\mathbf{m}|}(f_k) \subset D_{|\mathbf{m}|}(f_k), \quad k = 1, \ldots, d.$$

Definition 3.1 A vector $\mathbf{f} = (f_1, \ldots, f_d)$ of formal power expansions is said to be polynomially independent with respect to $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ if there do not exist polynomials $p_1, \ldots, p_d$, at least one of which is non-null, such that

(d.1) $\deg p_k < m_k, k = 1, \ldots, d,$
(d.2) $\sum_{k=1}^{d} p_k f_k$ is a polynomial.

In particular, polynomial independence implies that for each $k = 1, \ldots, d,\ f_k$ is not a rational function with at most $m_k - 1$ poles. Notice that polynomial independence may be verified solely in terms of the coefficients of the formal Taylor expansions defining the system $\mathbf{f}$.

Given $\mathbf{f} = (f_1, \ldots, f_d)$ and $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, we consider the associated system $\bar{\mathbf{f}}$ of formal power expansions

$$\bar{\mathbf{f}} = \left(f_1, \ldots, z^{m_1-1} f_1, f_2, \ldots, z^{m_d-1} f_d\right) = (\bar{f}_1, \ldots, \bar{f}_{|\mathbf{m}|}).$$

We also define an associated multi-index $\overline{\mathbf{m}}$ given by $\overline{\mathbf{m}} = (1, 1, \ldots, 1)$ with $|\overline{\mathbf{m}}| = |\mathbf{m}|$. The systems $\mathbf{f}$ and $\bar{\mathbf{f}}$ share most properties. In particular, poles of $\mathbf{f}$ and $\bar{\mathbf{f}}$ coincide and $R_m(\mathbf{f}) = R_m(\bar{\mathbf{f}}), m \in \mathbb{Z}_+$.

From the definition, it readily follows that $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$ if and only if there do not exist constants $c_k, k = 1, \ldots, |\mathbf{m}|$, not all zero, such that

$$\sum_{k=1}^{|\mathbf{m}|} c_k \bar{f}_k$$

is a polynomial. That is, $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$ if and only if $\bar{\mathbf{f}}$ is polynomially independent with respect to $\overline{\mathbf{m}}$. By the same token, the system poles of $\mathbf{f}$ with respect to $\mathbf{m}$ are the same as the system poles of $\bar{\mathbf{f}}$ with respect to $\overline{\mathbf{m}}$.

Finally, it is very easy to check that, for all $n \geq |\mathbf{m}|$, the equations that define the common denominator $Q_{n,\mathbf{m}}$ for $(\mathbf{f}, \mathbf{m})$ are the same as those defining $Q_{n,\overline{\mathbf{m}}}$ for $(\bar{\mathbf{f}}, \overline{\mathbf{m}})$ and, consequently, both classes of polynomials coincide.

Lemma 3.2 *Let* $\mathbf{f} = (f_1, \ldots, f_d)$ *be a system of formal Taylor expansions as in* (1), *and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$. *Suppose that for all* $n \geq n_0$, *the polynomial* $Q_{n,\mathbf{m}}$

is unique and $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$. *Then the system* $\mathbf{f}$ *is polynomially independent with respect to* $\mathbf{m}$.

Proof Because of what was said just before the statement of Lemma 3.2, we can assume without loss of generality that $\mathbf{m} = (1, 1, \ldots, 1)$ and $d = |\mathbf{m}|$. We argue by contradiction. Suppose that there exist constants $c_k, k = 1, \ldots, d$, not all zero, such that $\sum_{k=1}^{d} c_k f_k$ is a polynomial. Should $d = 1$, $Q_{n,\mathbf{m}} \equiv 1$ for all n sufficiently large and $\deg Q_{n,\mathbf{m}} < 1 = |\mathbf{m}|$. If $d > 1$, without loss of generality, we can assume that $c_1 \neq 0$. Then

$$f_1 = p - \sum_{k=2}^{d} c_k f_k,$$

where p is a polynomial, say of degree N.

On the other hand, for each $n \geq d - 1$, there exist polynomials $Q_n, P_{n,k}, k = 2, \ldots, d$, such that

- $\deg P_{n,k} \leq n - 1, k = 2, \ldots, d, \deg Q_n \leq d - 1, Q_n \not\equiv 0$,
- $Q_n(z) f_k(z) - P_{n,k}(z) = A_k z^{n+1} + \cdots, k = 2, \ldots, d.$

Therefore,

$$Q_n(z)\left(p(z) - \sum_{k=2}^{d} c_k f_k(z)\right) - \left(Q_n(z)p(z) - \sum_{k=2}^{d} c_k P_{n,k}(z)\right) = Az^{n+1} + \cdots$$

and, for $n \geq d + N$, the polynomial $P_{n,1} = Q_n p - \sum_{k=2}^{d} c_k P_{n,k}$ verifies $\deg P_{n,1} \leq n - 1$. Thus, for all n sufficiently large, the polynomials $P_{n,k}, k = 1, \ldots, d$, satisfy Definition 1.1 with respect to $\mathbf{f}$ and $\mathbf{m}$. Naturally, Q_n gives rise to a polynomial $Q_{n,\mathbf{m}}$ with $\deg Q_{n,\mathbf{m}} < d = |\mathbf{m}|$ against our assumption on $Q_{n,\mathbf{m}}$. □

Set

$$\mathbf{D}_{\mathbf{m}}^*(\mathbf{f}) = \left(D_{|\mathbf{m}|}^*(f_1), \ldots, D_{|\mathbf{m}|}^*(f_d)\right).$$

The following corollaries are straightforward consequences of Lemma 2.5 and Theorem 2.6, respectively, together with the fact that, for each $k = 1, \ldots, d$, $R_{n,\mathbf{m},k} = P_{n,\mathbf{m},k}/Q_{n,\mathbf{m}}$ is an incomplete Padé approximant of type $(n, |\mathbf{m}|, m_k)$ with respect to f_k.

Corollary 3.3 *Let* $\mathbf{f} = (f_1, \ldots, f_d)$ *be a system of formal Taylor expansions as in* (1), *and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$. *Assume that* $\mathbf{f}$ *is polynomially independent with respect to* $\mathbf{m}$ *and there exists a polynomial* $Q_{|\mathbf{m}|}$ *of degree* $|\mathbf{m}|$, $Q_{|\mathbf{m}|}(0) \neq 0$, *such that* $\lim_{n\to\infty} Q_{n,\mathbf{m}} = Q_{|\mathbf{m}|}$. *Then* $R_0(\mathbf{f}) > 0$, *the zeros of* $Q_{|\mathbf{m}|}$ *contain all the poles that* $\mathbf{f}$ *has in* $\mathbf{D}_{\mathbf{m}}^*(\mathbf{f})$, *and* $R_0(f_k) < \infty$ *for each* $k = 1, \ldots, d$.

Corollary 3.4 *Let* $\mathbf{f} = (f_1, \ldots, f_d)$ *be a system of formal Taylor expansions as in* (1), *and fix a multi-index* $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$. *Assume that* $\mathbf{f}$ *is polynomially independent with respect to* $\mathbf{m}$ *and there exists a polynomial* $Q_{|\mathbf{m}|}$ *of degree* $|\mathbf{m}|$, $Q_{|\mathbf{m}|}(0) \neq 0$, *such that*

$$\limsup_{n\to\infty} \| Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}} \|^{1/n} = \theta < 1.$$

Then, for each $k = 1, \ldots, d$, *either* f_k *has exactly* m_k *poles in* $D_{m_k}(f_k)$ *or* $R_0(Q_{|\mathbf{m}|} f_k) > R_{m_k}(f_k)$.

Before proving Theorem 1.4, we wish to describe some properties of system poles.

Lemma 3.5 *Given* $\mathbf{f} = (f_1, \ldots, f_d)$ *and* $\mathbf{m} \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$, $\mathbf{f}$ *can have at most* $|\mathbf{m}|$ *system poles with respect to* $\mathbf{m}$ *(counting their order). Moreover, if the system has exactly* $|\mathbf{m}|$ *system poles with respect to* $\mathbf{m}$ *and* ξ *is a system pole of order* τ, *then for all* $s > \tau$ *there can be no polynomial combination of the form* (6) *analytic on a neighborhood of* $\overline{D}_{|\xi|}$ *except for a pole at* $z = \xi$ *of exact order* s.

Proof Notice that the polynomial combinations of the form (6) generate a vector space of dimension less than or equal to $|\mathbf{m}|$. On the other hand, the set of functions which determine the system poles and their order are linearly independent. Consequently, there may be at most $|\mathbf{m}|$ such functions. Thus, the number of system poles counting their order is at most $|\mathbf{m}|$.

Assume that there are exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$, and let ξ be one of them of order τ. Take $s > \tau$. Obviously, for $s = \tau + 1$, there can be no polynomial combination of the form (6) analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole at $z = \xi$ of exact order s because the order of the system pole would be at least $\tau + 1$. For $s \geq \tau + 2$, no such combination can exist either because that would give another function which is linearly independent to the rest of the functions which determine the system poles and their order, which by assumption are already $|\mathbf{m}|$. □

3.1 Proof of Theorem 1.4

Let us prove first that (b) implies (a). From Lemma 3.2, it follows that $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$ and, in turn, from Corollary 3.3, we know that $R_0(\mathbf{f}) > 0$. So, it is enough to prove that $\mathbf{f}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ and without loss of generality we can assume that $\mathbf{m} = (1, 1, \ldots, 1)$.

We divide the proof into two parts. First, we collect a set of $|\mathbf{m}|$ candidates to be system poles of $\mathbf{f}$ and prove that they are the zeros of $Q_{|\mathbf{m}|}$. In the second part we prove that all these points previously collected are actually system poles of $\mathbf{f}$.

Notice that for each $k = 1, \ldots, d$, by Corollaries 3.4 and 3.3, either the disk $D_1(f_k)$ contains exactly one pole of f_k and it is a zero of $Q_{|\mathbf{m}|}$, or $R_0(Q_{|\mathbf{m}|} f_k) > R_1(f_k)$. Therefore, $D_0(\mathbf{f}) \neq \mathbb{C}$ and $Q_{|\mathbf{m}|}$ contains as zeros all the poles of f_k on the boundary of $D_0(f_k)$ counting their order for $k = 1, \ldots, d = |\mathbf{m}|$. Moreover, the functions f_k cannot have on the boundary of $D_0(f_k)$ singularities other than poles.

According to this, the poles of $\mathbf{f}$ on the boundary of $D_0(\mathbf{f})$ are all zeros of $Q_{|\mathbf{m}|}$ counting multiplicities and the boundary contains no other singularity except poles. Let us call them candidate system poles of $\mathbf{f}$ and denote them by $a_1, \ldots, a_{n_1}$ taking account of their order.

Since $\deg Q_{|\mathbf{m}|} = |\mathbf{m}|$, we have $n_1 \le |\mathbf{m}|$. Should $n_1 = |\mathbf{m}|$, we have found all the candidates we were looking for. Let us assume that $n_1 < |\mathbf{m}|$. We can find coefficients $c_1, \ldots, c_{|\mathbf{m}|}$ such that

$$\sum_{k=1}^{|\mathbf{m}|} c_k f_k$$

is analytic in a neighborhood of $\overline{D_0(\mathbf{f})}$. Finding the coefficients c_k reduces to solving a linear homogeneous system of n_1 equations with $|\mathbf{m}|$ unknowns. In fact, if $z = a$ is a candidate system pole of $\mathbf{f}$ with multiplicity τ, we obtain τ equations choosing the coefficients c_k so that

$$\int_{|\omega-a|=\delta} (\omega - a)^i \left(\sum_{k=1}^{|\mathbf{m}|} c_k f_k(\omega) \right) d\omega = 0, \quad i = 0, \ldots, \tau - 1, \tag{21}$$

where δ is sufficiently small. We do the same with each distinct candidate on the boundary of $D_0(\mathbf{f})$. The linear homogeneous system of equations so obtained has at least $|\mathbf{m}| - n_1$ linearly independent solutions, which we denote by $\mathbf{c}_j^1$, $j = 1, \ldots, |\mathbf{m}| - n_1^*$, where $n_1^* \le n_1$ denotes the rank of the system of equations.

Set

$$\mathbf{c}_j^1 = \left(c_{j,1}^1, \ldots, c_{j,|\mathbf{m}|}^1\right), \quad j = 1, \ldots, |\mathbf{m}| - n_1^*.$$

Construct the $(|\mathbf{m}| - n_1^*) \times |\mathbf{m}|$ dimensional matrix

$$C^1 = \begin{pmatrix} \mathbf{c}_1^1 \\ \vdots \\ \mathbf{c}_{|\mathbf{m}|-n_1^*}^1 \end{pmatrix}.$$

Define the system $\mathbf{g}_1$ of $|\mathbf{m}| - n_1^*$ functions by means of

$$\mathbf{g}_1^t = C^1 \mathbf{f}^t = (g_{1,1}, \ldots, g_{1,|\mathbf{m}|-n_1^*})^t,$$

where $(\cdot)^t$ means taking transpose. We have

$$g_{1,j} = \sum_{k=1}^{|\mathbf{m}|} c_{j,k}^1 f_k, \quad j = 1, \ldots, |\mathbf{m}| - n_1^*.$$

As the rows of C^1 are non-null, none of the functions $g_{1,j}$ are polynomials because of the polynomial independence of $\mathbf{f}$ with respect to $\mathbf{m} = (1, 1, \ldots, 1)$.

Consider the region

$$D_0(\mathbf{g}_1) = \bigcap_{j=1}^{|\mathbf{m}|-n_1^*} D_0(g_{1,j}).$$

Obviously, by construction, $D_0(\mathbf{f})$ is strictly included in $D_0(\mathbf{g}_1)$.

It is easy to see that

$$\sum_{k=1}^{|\mathbf{m}|} c_{j,k}^1 \frac{P_{n,\mathbf{m},k}}{Q_{n,\mathbf{m}}}$$

is an $(n, |\mathbf{m}|, 1)$ incomplete Padé approximant of $g_{1,j}$. Using Theorem 2.6 with $m^* = 1$, for each $j = 1, \ldots, |\mathbf{m}| - n_1^*$, either the disk $D_1(g_{1,j})$ contains exactly one pole of $g_{1,j}$ and it is a zero of $Q_{|\mathbf{m}|}$, or $R_0(Q_{|\mathbf{m}|} g_{1,j}) > R_1(g_{1,j})$. In particular, $D_0(\mathbf{g}_1) \neq \mathbb{C}$, and all the singularities of $\mathbf{g}_1$ on the boundary of $D_0(\mathbf{g}_1)$ are poles which are zeros of $Q_{|\mathbf{m}|}$ counting their order. They constitute the next layer of candidate system poles of $\mathbf{f}$ (now, it is possible that some candidates are not poles of $\mathbf{f}$ since the functions f_k intervene in the linear combination as we saw in example (5)).

Let us denote these new candidates by $a_{n_1+1}, \ldots, a_{n_1+n_2}$. Of course, $n_1 + n_2 \leq |\mathbf{m}|$. Should $n_1 + n_2 = |\mathbf{m}|$, we are done. Otherwise, $n_2 < |\mathbf{m}| - n_1 \leq |\mathbf{m}| - n_1^*$, and we can repeat the process. In order to eliminate the n_2 poles, we have $|\mathbf{m}| - n_1^*$ functions which are analytic on $D_0(\mathbf{g}_1)$ and meromorphic on a neighborhood of $\overline{D_0(\mathbf{g}_1)}$. The corresponding homogeneous linear system of equations, similar to (21), has at least $|\mathbf{m}| - n_1^* - n_2$ linearly independent solutions $\mathbf{c}_j^2$, $j = 1, \ldots, |\mathbf{m}| - n_1^* - n_2^*$, where $n_2^* \leq n_2$ is the rank of the new system. Set

$$\mathbf{c}_j^2 = \left(c_{j,1}^2, \ldots, c_{j,|\mathbf{m}|-n_1^*}^2\right), \quad j = 1, \ldots, |\mathbf{m}| - n_1^* - n_2^*.$$

Construct the $(|\mathbf{m}| - n_1^* - n_2^*) \times (|\mathbf{m}| - n_1^*)$ dimensional matrix

$$C^2 = \begin{pmatrix} \mathbf{c}_1^2 \\ \vdots \\ \mathbf{c}_{|\mathbf{m}|-n_1^*-n_2^*}^2 \end{pmatrix}.$$

Define the system $\mathbf{g}_2$ of $|\mathbf{m}| - n_1^* - n_2^*$ functions by means of

$$\mathbf{g}_2^t = C^2 \mathbf{g}_1^t = C^2 C^1 \mathbf{f}^t = (g_{2,1}, \ldots, g_{2,|\mathbf{m}|-n_1^*-n_2^*})^t.$$

The rows of $C^2 C^1$ are of the form $\mathbf{c}_j^2 C^1$, $j = 1, \ldots, |\mathbf{m}| - n_1^* - n_2^*$, where C^1 has rank $|\mathbf{m}| - n_1^*$ and the vectors $\mathbf{c}_k^2$ are linearly independent. Therefore, the rows of $C^2 C^1$ are linearly independent; in particular, they are non-null. Consequently, the components of $\mathbf{g}_2$ are not polynomials because of the polynomial independence of $\mathbf{f}$ with respect to $\mathbf{m} = (1, 1, \ldots, 1)$. Thus, we can again apply Theorem 2.6. The proof is completed using finite induction.

Notice that the numbers $n_1, n_2, \ldots$ which arise are greater than or equal to 1, and on each iteration their sum is less than or equal to $|\mathbf{m}|$. Therefore, in a finite number of steps, say $N - 1$, their sum must equal $|\mathbf{m}|$. Consequently, the number of candidate system poles of $\mathbf{f}$ in some disk, counting their multiplicities, is exactly equal to $|\mathbf{m}|$, and they are precisely the zeros of $Q_{|\mathbf{m}|}$ as we wanted to prove.

Summarizing, in the $N - 1$ steps we have taken, we have produced N layers of candidate system poles. Each layer contains n_k candidates, $k = 1, \ldots, N$. At the same time, at each step k, $k = 1, \ldots, N - 1$, we have solved a linear system of n_k equations, of rank n_k^*, with $|\mathbf{m}| - n_1^* - \cdots - n_k^*$, $n_k^* \le n_k$, linearly independent solutions. We find ourselves on the N-th layer with n_N candidates.

Let us try to eliminate these poles. As before, we write the corresponding system of linear homogeneous equations as in (21), and we get

$$n_N = |\mathbf{m}| - n_1 - \cdots - n_{N-1} \le |\mathbf{m}| - n_1^* - \cdots - n_{N-1}^* =: \overline{n}_N$$

equations with $\overline{n}_N$ unknowns. For each candidate system pole a of multiplicity τ on the N-th layer, we impose the equations

$$\int_{|\omega - a| = \delta} (\omega - a)^i \left(\sum_{k=1}^{\overline{n}_N} c_k g_{N-1,k}(\omega) \right) d\omega = 0, \quad i = 0, \dots, \tau - 1, \tag{22}$$

where δ is sufficiently small and the $g_{N-1,k}$, $k = 1, \dots, \overline{n}_N$, are the functions associated with the linearly independent solutions produced on step $N - 1$.

Let n_N^* be the rank of this last homogeneous linear system of equations. Assume that $n_k* < n_k$ for at least one $k \in \{1, \dots, N\}$. In this case, there exists at least one nontrivial solution of the system. The corresponding function g can be written as a linear combination of the components of $\mathbf{f}$, and it cannot reduce to a polynomial because $\mathbf{f}$ is polynomially independent. Using Theorem 2.6, we obtain that g has on the boundary of its disk of analyticity a pole which is a zero of $Q_{|\mathbf{m}|}$, but this is clearly impossible because all the zeros of $Q_{|\mathbf{m}|}$ are strictly contained in that disk. Consequently, $n_k = n_k^*$, $k = 1, \dots, N$.

What we have proved implies that in all the N homogeneous systems which we have solved (including the last one), there are no redundant equations. In turn, this implies that if in any one of those systems of equations we equate one of its equations to 1, instead of zero (see (21) or (22)), the corresponding nonhomogeneous linear system of equations has a solution. Applying the definition of a system pole, this means that each candidate system pole is a system pole of order at least equal to its multiplicity as zero of $Q_{|\mathbf{m}|}$. But, as we saw in Lemma 3.5, $\mathbf{f}$ can have at most $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$; therefore, all candidate system poles are indeed system poles, and their order coincides with the multiplicity of that point as a zero of $Q_{|\mathbf{m}|}$.

Thus, the proof of the inverse-type result is complete, and we have $Q_{|\mathbf{m}|} = \mathcal{Q}(\mathbf{f}, \mathbf{m})$ as well.

Let us prove now that (a) implies (b). Except for some details related to the numbers $R_\xi(\mathbf{f}, \mathbf{m})$, where ξ is a system pole of $\mathbf{f}$, the arguments are similar to those employed in [15]. In spite of this, for completeness, we give the entire proof.

For each $n \ge |\mathbf{m}|$, let $q_{n,\mathbf{m}}$ be the polynomial $Q_{n,\mathbf{m}}$ normalized so that

$$\sum_{k=1}^{|\mathbf{m}|} |\lambda_{n,k}| = 1, \quad q_{n,\mathbf{m}}(z) = \sum_{k=1}^{|\mathbf{m}|} \lambda_{n,k} z^k. \tag{23}$$

Due to this normalization, the polynomials $q_{n,\mathbf{m}}$ are uniformly bounded on each compact subset of $\mathbb{C}$.

Let ξ be a system pole of order τ of $\mathbf{f}$ with respect to $\mathbf{m}$. Consider a polynomial combination g_1 of type (6) that is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a simple pole at $z = \xi$ and verifies that $R_1(g_1) = R_{\xi,1}(\mathbf{f}, \mathbf{m}) (= r_{\xi,1}(\mathbf{f}, \mathbf{m}))$. Then we have

$$g_1 = \sum_{k=1}^{|\mathbf{m}|} p_{k,1} f_k, \quad \deg p_{k,1} < m_k, \ k = 1, \ldots, |\mathbf{m}|,$$

and

$$q_{n,\mathbf{m}}(z) h_1(z) - (z - \xi) \sum_{k=1}^{|\mathbf{m}|} p_{k,1}(z) P_{n,\mathbf{m},k}(z) = A z^{n+1} + \cdots,$$

where $h_1(z) = (z - \xi) g_1(z)$. Hence, the function

$$\frac{q_{n,\mathbf{m}}(z) h_1(z)}{z^{n+1}} - \frac{z - \xi}{z^{n+1}} \sum_{k=1}^{|\mathbf{m}|} p_{k,1}(z) P_{n,\mathbf{m},k}(z)$$

is analytic on $D_1(g_1)$. Take $0 < r < R_1(g_1)$, and set $\Gamma_r = \{z \in \mathbb{C} : |z| = r\}$. Using Cauchy's formula, we obtain

$$q_{n,\mathbf{m}}(z) h_1(z) - (z - \xi) \sum_{k=1}^{|\mathbf{m}|} p_{k,1}(z) P_{n,\mathbf{m},k}(z) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{z^{n+1}}{\omega^{n+1}} \frac{q_{n,\mathbf{m}}(\omega) h_1(\omega)}{\omega - z} \, d\omega,$$

for all z with $|z| < r$, since $\deg \sum_{k=1}^{|\mathbf{m}|} p_{k,1} P_{n,\mathbf{m},k} < n$. In particular, taking $z = \xi$ in the above formula, we arrive at

$$q_{n,\mathbf{m}}(\xi) h_1(\xi) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{\xi^{n+1}}{\omega^{n+1}} \frac{q_{n,\mathbf{m}}(\omega) h_1(\omega)}{\omega - \xi} \, d\omega. \tag{24}$$

Straightforward calculations lead to

$$\limsup_{n \to \infty} \left| h_1(\xi) q_{n,\mathbf{m}}(\xi) \right|^{1/n} \le \frac{|\xi|}{r}.$$

Using that $h_1(\xi) \neq 0$ and making r tend to $R_1(g_1)$, we obtain

$$\limsup_{n \to \infty} \left| q_{n,\mathbf{m}}(\xi) \right|^{1/n} \le \frac{|\xi|}{R_{\xi,1}(\mathbf{f}, \mathbf{m})} < 1.$$

Now, we employ induction. Suppose that

$$\limsup_{n \to \infty} \left| q_{n,\mathbf{m}}^{(j)}(\xi) \right|^{1/n} \le \frac{|\xi|}{R_{\xi,j+1}(\mathbf{f}, \mathbf{m})}, \quad j = 0, 1, \ldots, s-2 \tag{25}$$

(recall that $R_{\xi,j+1}(\mathbf{f}, \mathbf{m}) = \min_{k=1,\ldots,j+1} r_{\xi,k}(\mathbf{f}, \mathbf{m})$), with $s \le \tau$, and let us prove that formula (25) holds for $j = s - 1$.

Consider a polynomial combination g_s of the type (6) that is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole of order s at $z = \xi$ and verifies that $R_s(g_s) = r_{\xi,s}(\mathbf{f}, \mathbf{m})$. Then we have

$$g_s = \sum_{k=1}^{|\mathbf{m}|} p_{k,s} f_k, \quad \deg p_{k,s} < m_k, \ k = 1, \ldots, |\mathbf{m}|.$$

Set $h_s(z) = (z - \xi)^s g_s(z)$. Reasoning as in the previous case, the function

$$\frac{q_{n,\mathbf{m}}(z) h_s(z)}{z^{n+1}(z - \xi)^{s-1}} - \frac{z - \xi}{z^{n+1}} \sum_{k=1}^{|\mathbf{m}|} p_{k,s}(z) P_{n,\mathbf{m},k}(z)$$

is analytic on $D_s(g_s) \setminus \{\xi\}$. Set $P_s = \sum_{k=1}^{|\mathbf{m}|} p_{k,s} P_{n,\mathbf{m},k}$. Fix an arbitrary compact set $K \subset (D_s(g_s) \setminus \{\xi\})$. Take $\delta > 0$ sufficiently small and $0 < r < R_s(g_s)$ with $K \subset D_r$. Using Cauchy's integral formula and the residue theorem, for all $z \in K$, we have

$$\frac{q_{n,\mathbf{m}}(z) h_s(z)}{(z - \xi)^{s-1}} - (z - \xi) P_s(z) = I_n(z) - J_n(z), \tag{26}$$

where

$$I_n(z) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{z^{n+1}}{\omega^{n+1}} \frac{q_{n,\mathbf{m}}(\omega) h_s(\omega)}{(\omega - \xi)^{s-1}(\omega - z)} \, d\omega$$

and

$$J_n(z) = \frac{1}{2\pi i} \int_{|\omega - \xi| = \delta} \frac{z^{n+1}}{\omega^{n+1}} \frac{q_{n,\mathbf{m}}(\omega) h_s(\omega)}{(\omega - \xi)^{s-1}(\omega - z)} \, d\omega.$$

We have used in (26) that $\deg P_s < n$. The first integral I_n is estimated as in (24) to obtain

$$\limsup_{n \to \infty} \| I_n(z) \|_K^{1/n} \leq \frac{\|z\|_K}{R_s(g_s)} = \frac{\|z\|_K}{r_{\xi,s}(\mathbf{f}, \mathbf{m})}. \tag{27}$$

As for J_n, write

$$q_{n,\mathbf{m}}(\omega) = \sum_{j=0}^{|\mathbf{m}|} \frac{q_{n,\mathbf{m}}^{(j)}(\xi)}{j!} (\omega - \xi)^j.$$

Then

$$J_n(z) = \sum_{j=0}^{s-2} \frac{1}{2\pi i} \int_{|\omega - \xi| = \delta} \frac{z^{n+1}}{\omega^{n+1}} \frac{q_{n,\mathbf{m}}^{(j)}(\xi)}{j!(\omega - z)} \frac{h_s(\omega)}{(\omega - \xi)^{s-1-j}} \, d\omega. \tag{28}$$

Using the inductive hypothesis (25), estimating the integral in (28), and making ε tend to zero, we obtain

$$\limsup_{n \to \infty} \| J_n(z) \|_K^{1/n} \leq \frac{\|z\|_K}{|\xi|} \frac{|\xi|}{R_{\xi,s-1}(\mathbf{f}, \mathbf{m})} = \frac{\|z\|_K}{R_{\xi,s-1}(\mathbf{f}, \mathbf{m})},$$

which, together with (27) and (26), gives

$$\limsup_{n\to\infty}\left\|q_{n,\mathbf{m}}(z)h_s(z)-(z-\xi)^s P_s(z)\right\|_K^{1/n}\le\frac{\|z\|_K}{R_{\xi,s}(\mathbf{f},\mathbf{m})}. \tag{29}$$

As the function inside the norm in (29) is analytic in $D_s(g_s)$, inequality (29) also holds for any compact set $K\subset D_s(g_s)$. Besides, we can differentiate $s-1$ times that function and the inequality still holds true by virtue of Cauchy's integral formula. So, taking $z=\xi$ in (29) for the differentiated version, we obtain

$$\limsup_{n\to\infty}\left|(q_{n,\mathbf{m}}h_s)^{(s-1)}(\xi)\right|^{1/n}\le\frac{|\xi|}{R_{\xi,s}(\mathbf{f},\mathbf{m})}.$$

Using the Leibnitz formula for higher derivatives of a product of two functions and the induction hypothesis (25), we arrive at

$$\limsup_{n\to\infty}\left|q_{n,\mathbf{m}}^{(s-1)}(\xi)\right|^{1/n}\le\frac{|\xi|}{R_{\xi,s}(\mathbf{f},\mathbf{m})},$$

since $h_s(\xi)\neq 0$. This completes the induction.

Let $\xi_1,\dots,\xi_p$ be the distinct system poles of $\mathbf{f}$, and let τ_i be the order of ξ_i as a system pole, $i=1,\dots,p$. By assumption, $\tau_1+\cdots+\tau_p=|\mathbf{m}|$. We have proved that, for $i=1,\dots,p$ and $j=0,1,\dots,\tau_i-1$,

$$\limsup_{n\to\infty}\left|q_{n,\mathbf{m}}^{(j)}(\xi_i)\right|^{1/n}\le\frac{|\xi_i|}{R_{\xi_i,j+1}(\mathbf{f},\mathbf{m})}\le\frac{|\xi_i|}{R_{\xi_i}(\mathbf{f},\mathbf{m})}. \tag{30}$$

Recall that $\mathcal{Q}(\mathbf{f},\mathbf{m})$ is the monic polynomial whose zeros are the system poles of $\mathbf{f}$ with respect to $\mathbf{m}$. Denote by $L_{i,j}$, $i=1,\dots,p$; $j=0,1,\dots\tau_i-1$, the fundamental interpolating polynomials at the zeros of $\mathcal{Q}(\mathbf{f},\mathbf{m})$; that is, for each $i=1,\dots,p$ and $j=0,1,\dots\tau_i-1$, $\deg L_{i,j}\le|\mathbf{m}|-1$ and

$$L_{i,j}^{(\nu)}(\xi_\kappa)=\delta_{i\kappa}\delta_{j\nu},\quad \kappa=1,\dots,p,\ \nu=0,1,\dots,\tau_i-1.$$

Then

$$q_{n,\mathbf{m}}(z)=\lambda_{n,|\mathbf{m}|}\mathcal{Q}(\mathbf{f},\mathbf{m})+\sum_{i=1}^{p}\sum_{j=0}^{\tau_i-1}q_{n,\mathbf{m}}^{(j)}(\xi_i)L_{i,j}(z). \tag{31}$$

From (30) and (31), it follows that

$$\limsup_{n\to\infty}\left\|q_{n,\mathbf{m}}-\lambda_{n,|\mathbf{m}|}\mathcal{Q}(\mathbf{f},\mathbf{m})\right\|_K^{1/n}\le\theta<1$$

for any compact $K\subset\mathbb{C}$, where

$$\theta=\max\left\{\frac{|\xi|}{R_\xi(\mathbf{f},\mathbf{m})}:\xi\in\mathcal{P}(\mathbf{f},\mathbf{m})\right\}. \tag{32}$$

As all norms in finite dimensional spaces are equivalent, we obtain

$$\limsup_{n\to\infty} \left\| q_{n,\mathbf{m}} - \lambda_{n,|\mathbf{m}|} \mathcal{Q}(\mathbf{f},\mathbf{m}) \right\|^{1/n} \leq \theta < 1. \tag{33}$$

Now, necessarily we have

$$\liminf_{n\to\infty} |\lambda_{n,|\mathbf{m}|}| > 0, \tag{34}$$

since if there exists a subsequence $\Lambda \subset \mathbb{N}$ such that $\lim_{n\in\Lambda} \lambda_{n,|\mathbf{m}|} = 0$, then from (33), we have $\lim_{n\in\Lambda} \|q_{n,\mathbf{m}}\| = 0$, contradicting (23).

As $q_{n,\mathbf{m}} = \lambda_{n,|\mathbf{m}|} Q_{n,\mathbf{m}}$, we have proved

$$\limsup_{n\to\infty} \left\| Q_{n,\mathbf{m}} - \mathcal{Q}(\mathbf{f},\mathbf{m}) \right\|^{1/n} \leq \theta < 1, \tag{35}$$

where θ is given by (32). In particular, for $n \geq n_0$, $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$. The difference of any two noncollinear solutions Q_1 and Q_2 of Definition 1.1 with the same degree and equal leading coefficient produces a new solution of smaller degree, but we have proved that any solution must have degree $|\mathbf{m}|$ for all sufficiently large n. Hence, the polynomial $Q_{n,\mathbf{m}}$ is uniquely determined for all sufficiently large n. With this we have concluded the proof of the direct result.

Let us prove that the upper bound in (35) actually gives the exact rate of convergence to obtain (7). To the contrary, suppose that

$$\limsup_{n\to\infty} \left\| Q_{n,\mathbf{m}} - \mathcal{Q}(\mathbf{f},\mathbf{m}) \right\|^{1/n} = \theta' < \theta. \tag{36}$$

Let ζ be a system pole of $\mathbf{f}$ such that

$$\frac{|\zeta|}{R_\zeta(\mathbf{f},\mathbf{m})} = \theta = \max\left\{ \frac{|\xi|}{R_\xi(\mathbf{f},\mathbf{m})} : \xi \in \mathcal{P}(\mathbf{f},\mathbf{m}) \right\}.$$

Naturally, if there is inequality in (36), then $R_\zeta(\mathbf{f},\mathbf{m}) < \infty$.

Choose a polynomial combination

$$g = \sum_{k=1}^{d} p_k f_k, \quad \deg p_k < m_k, \ k = 1, \ldots, d, \tag{37}$$

that is analytic on a neighborhood of $\overline{D}_{|\zeta|}$ except for a pole of order s at $z = \zeta$ with $R_s(g) = R_\zeta(\mathbf{f},\mathbf{m})$. On the boundary of $D_s(g)$, the function g must have a singularity which is not a system pole. In fact, if all the singularities were of this type we could find a different polynomial combination g_1 of type (37) for which $R_s(g_1) > R_s(g) = R_\zeta(\mathbf{f},\mathbf{m})$ against our definition of $R_\zeta(\mathbf{f},\mathbf{m})$. For short, write $\mathcal{Q}(\mathbf{f},\mathbf{m}) = Q_{|\mathbf{m}|}$. Consequently, the function $Q_{|\mathbf{m}|} g$ can be represented as a power series $\sum_{j=0}^{\infty} c_j z^j$ with radius of convergence $R_\zeta(\mathbf{f},\mathbf{m})$. So

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 1/R_\zeta(\mathbf{f},\mathbf{m}). \tag{38}$$

On the other hand, by virtue of (37), we have

$$H_n(z) := Q_{n,\mathbf{m}}(z)g(z) - \sum_{k=1}^{d} p_k(z)P_{n,\mathbf{m},k}(z) = B_n z^{n+1} + \cdots,$$

and this function is analytic at least in $D_{|\zeta|}$ with a zero of multiplicity at least $n+1$ at $z=0$. Taking $r < |\zeta|$, we obtain

$$\frac{1}{2\pi i}\int_{\Gamma_r} \frac{H_n(\omega)}{\omega^{n+1}}\,d\omega = 0.$$

Set $P_n = \sum_{k=1}^{d} p_k P_{n,\mathbf{m},k}$. Clearly, $Q_{|\mathbf{m}|}g \equiv (Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}})g + P_n + H_n$ and, since $\deg P_n \le n-1$, we arrive at

$$c_n = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{Q_{|\mathbf{m}|}(\omega)g(\omega)}{\omega^{n+1}}\,d\omega = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{[Q_{|\mathbf{m}|}(\omega) - Q_{n,\mathbf{m}}(\omega)]g(\omega)}{\omega^{n+1}}\,d\omega.$$

Taking (38) and (36) into consideration, estimating the integral, and letting r tend to $|\zeta|$, it follows that

$$\frac{1}{R_\zeta(\mathbf{f},\mathbf{m})} = \limsup_{n\to\infty} \sqrt[n]{|c_n|} \le \frac{\theta'}{|\zeta|} < \frac{\theta}{|\zeta|} = \frac{1}{R_\zeta(\mathbf{f},\mathbf{m})},$$

which is absurd. We have completed the proof of Theorem 1.4. □

3.2 Convergence of the Hermite–Padé Approximants

The following result is in some sense the analog of the formula displayed just after (58) in [7] written in different terms.

Theorem 3.6 *Assume that either* (a) *or* (b) *in Theorem* 1.4 *takes place. If* ξ *is a system pole of order* τ *of* $\mathbf{f}$ *with respect to* $\mathbf{m}$, *then*

$$\max_{j=0,\ldots,\overline{s}} \limsup_{n\to\infty} \left|Q_{n,\mathbf{m}}^{(j)}(\xi)\right|^{1/n} = \frac{|\xi|}{R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})}, \quad \overline{s} = 0,1,\ldots,\tau-1. \tag{39}$$

Proof Let ξ be as indicated. From (30) and (34), we have

$$\max_{j=0,\ldots,\overline{s}} \limsup_{n\to\infty} \left|Q_{n,\mathbf{m}}^{(j)}(\xi)\right|^{1/n} \le \frac{|\xi|}{R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})}, \quad \overline{s} = 0,1,\ldots,\tau-1.$$

Assume that there is strict inequality for some $\overline{s} \in \{0,\ldots,\tau-1\}$ and fix $\overline{s}$.

Choose a polynomial combination

$$g = \sum_{k=1}^{d} p_k f_k, \quad \deg p_k < m_k,\ k = 1,\ldots,d,$$

that is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole of order $s(\leq \bar{s}+1)$ at $z=\xi$ with $R_s(g)=R_{\xi,\bar{s}+1}(\mathbf{f},\mathbf{m})$. As before, on the boundary of $D_s(g)$ the function g must have a singularity which is not a system pole. Set $\mathcal{Q}(\mathbf{f},\mathbf{m})=\mathcal{Q}_{|\mathbf{m}|}$. Consequently, the function $\mathcal{Q}_{|\mathbf{m}|}g$ can be represented as a power series $\sum_{j=0}^{\infty} c_j z^j$ with radius of convergence $R_{\xi,\bar{s}+1}(\mathbf{f},\mathbf{m})$. So

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 1/R_{\xi,\bar{s}+1}(\mathbf{f},\mathbf{m}). \tag{40}$$

On the other hand, by virtue of (37), we have

$$H_n(z) := Q_{n,\mathbf{m}}(z)g(z) - \sum_{k=1}^{d} p_k(z)P_{n,\mathbf{m},k}(z) = B_n z^{n+1} + \cdots,$$

and this function is analytic in $D_s(g)\setminus\{\xi\}$. Take r smaller than but sufficiently close to $R_{\xi,\bar{s}+1}(\mathbf{f},\mathbf{m})$ and $\delta>0$ sufficiently small. Let $\Gamma_{\delta,r}$ be the positively oriented curve determined by $\gamma_\delta=\{\omega : |\omega-\xi|=\delta\}$ and Γ_r. We have

$$\frac{1}{2\pi i}\int_{\Gamma_{\delta,r}} \frac{H_n(\omega)}{\omega^{n+1}}\,d\omega = 0.$$

Set $P_n=\sum_{k=1}^{d} p_k P_{n,\mathbf{m},k}$ and $h(\omega)=(\omega-\xi)^s g(\omega)$. Obviously,

$$\mathcal{Q}_{|\mathbf{m}|}g \equiv (\mathcal{Q}_{|\mathbf{m}|}-Q_{n,\mathbf{m}})g + P_n + H_n,$$

and, since $\deg P_n \leq n-1$, we obtain

$$\begin{aligned}
c_n &= \frac{1}{2\pi i}\int_{\Gamma_{\delta,r}} \frac{\mathcal{Q}_{|\mathbf{m}|}(\omega)g(\omega)}{\omega^{n+1}}\,d\omega = \frac{1}{2\pi i}\int_{\Gamma_{\delta,r}} \frac{[\mathcal{Q}_{|\mathbf{m}|}-Q_{n,\mathbf{m}}](\omega)h(\omega)}{(\omega-\xi)^s\omega^{n+1}}\,d\omega \\
&= \frac{1}{2\pi i}\int_{\Gamma_r} \frac{[\mathcal{Q}_{|\mathbf{m}|}-Q_{n,\mathbf{m}}](\omega)h(\omega)}{(\omega-\xi)^s\omega^{n+1}}\,d\omega - \sum_{\nu=0}^{|\mathbf{m}|}\frac{1}{2\pi i}\int_{\gamma_\delta} \frac{[\mathcal{Q}_{|\mathbf{m}|}^{(\nu)}-Q_{n,\mathbf{m}}^{(\nu)}](\xi)h(\omega)}{\nu!(\omega-\xi)^{s-\nu}\omega^{n+1}}\,d\omega \\
&= \frac{1}{2\pi i}\int_{\Gamma_r} \frac{[\mathcal{Q}_{|\mathbf{m}|}-Q_{n,\mathbf{m}}](\omega)h(\omega)}{(\omega-\xi)^s\omega^{n+1}}\,d\omega + \sum_{\nu=0}^{s-1}\frac{1}{2\pi i}\int_{\gamma_\delta} \frac{Q_{n,\mathbf{m}}^{(\nu)}(\xi)h(\omega)}{\nu!(\omega-\xi)^{s-\nu}\omega^{n+1}}\,d\omega.
\end{aligned}$$

Estimating these integrals, using (7) and the temporary assumption that

$$\max_{j=0,\ldots,\bar{s}}\ \limsup_{n\to\infty}\left|Q_{n,\mathbf{m}}^{(j)}(\xi)\right|^{1/n} = \frac{|\xi|}{\kappa} < \frac{|\xi|}{R_{\xi,\bar{s}+1}(\mathbf{f},\mathbf{m})},$$

we obtain

$$\limsup_{n\to\infty}|c_n|^{1/n} \leq \max\left\{\frac{1}{\kappa}, \frac{\theta}{R_{\xi,\bar{s}+1}(\mathbf{f},\mathbf{m})}\right\} < \frac{1}{R_{\xi,\bar{s}+1}(\mathbf{f},\mathbf{m})},$$

which contradicts (40). Hence, (39) takes place. □

Now we are ready to give the analog of (4) for simultaneous approximation. We need to introduce some notation. Fix $k \in \{1, \ldots, d\}$. Let $D_k(\mathbf{f}, \mathbf{m})$ be the largest disk centered at $z = 0$ in which all the poles of f_k are system poles of $\mathbf{f}$ with respect to $\mathbf{m}$, their order as poles of f_k does not exceed their order as system poles, and f_k has no other singularity. By $R_k(\mathbf{f}, \mathbf{m})$, we denote the radius of this disk. Let $\xi_1, \ldots, \xi_N$ be the poles of f_k in $D_k(\mathbf{f}, \mathbf{m})$. For each $j = 1, \ldots, N$, let $\widetilde{\tau}_j$ be the order of ξ_j as a pole of f_k and τ_j its order as a system pole. By assumption, $\widetilde{\tau}_j \leq \tau_j$. Set

$$R_k^*(\mathbf{f}, \mathbf{m}) = \min\left\{R_k(\mathbf{f}, \mathbf{m}), \min_{j=1,\ldots,N} R_{\xi_j, \widetilde{\tau}_j}(\mathbf{f}, m)\right\},$$

and let $D_k^*(\mathbf{f}, \mathbf{m})$ be the disk centered at $z = 0$ with this radius.

Recall that $\sigma(B)$ stands for the 1-dimensional Hausdorff content of the set B and $D_{|\mathbf{m}|}^*(f_k)$ is the largest disk centered at the origin inside of which $\sigma\text{-}\lim_{n\to\infty} R_{n,\mathbf{m},k} = f_k$. Its radius is denoted by $R_{|\mathbf{m}|}^*(f_k)$.

We say that a compact set $K \subset \mathbb{C}$ is σ-regular if for each $z_0 \in K$ and for each $\delta > 0$, it holds that $\sigma\{z \in K : |z - z_0| < \delta\} > 0$.

Theorem 3.7 *Let* $\mathbf{f}$ *be a system of formal Taylor expansions as in* (1) *and fix a multi-index* $\mathbf{m} \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$. *Suppose that either* (a) *or* (b) *in Theorem* 1.4 *takes place. Then*

$$\limsup_{n\to\infty} \| f_k - R_{n,\mathbf{m},k}\|_K^{1/n} \leq \frac{\|z\|_K}{R_{|\mathbf{m}|}^*(f_k)}, \quad k = 1, \ldots, d, \tag{41}$$

where K *is any compact subset of* $D_{|\mathbf{m}|}^*(f_k) \setminus \mathcal{P}(\mathbf{f}, \mathbf{m})$. *If, additionally,* K *is* σ*-regular, then we have equality in* (41). *Moreover,*

$$R_{|\mathbf{m}|}^*(f_k) = R_k^*(\mathbf{f}, \mathbf{m}), \quad k = 1, \ldots, d.$$

Proof Let us fix $k \in \{1, \ldots, d\}$ and maintain the notation introduced above. Let K be a compact subset contained in $D_k^*(\mathbf{f}, \mathbf{m}) \setminus \mathcal{P}(\mathbf{f}, \mathbf{m})$. Take r smaller than but sufficiently close to $R_k^*(\mathbf{f}, \mathbf{m})$, and $\delta > 0$ sufficiently small so that K is in the region bounded by Γ_r and the circles $\{z : |z - \xi_j| = \delta\}$, $j = 1, \ldots, N$. Let $\Gamma_{\delta,r}$ be the curve with positive orientation determined by Γ_r and those circles. On account of Definition 1.1, using Cauchy's integral formula, we have

$$(Q_{n,\mathbf{m}} f_k - P_{n,\mathbf{m},k})(z) = \frac{1}{2\pi i} \int_{\Gamma_{\delta,r}} \frac{z^{n+1}}{\omega^{n+1}} \frac{(Q_{n,\mathbf{m}} f_k)(\omega)}{\omega - z}\, d\omega.$$

Since $\lim_n Q_{n,\mathbf{m}} = Q_{|\mathbf{m}|}$, using (39) and standard arguments, we obtain

$$\limsup_{n\to\infty} \| f_k - R_{n,\mathbf{m},k}\|_K^{1/n} \leq \frac{\|z\|_K}{R_k^*(\mathbf{f}, \mathbf{m})}. \tag{42}$$

This last relation implies that $\sigma\text{-}\lim_{n\to\infty} R_{n,\mathbf{m},k} = f_k$ inside $D_k^*(\mathbf{f}, \mathbf{m})$. Since $R_{|\mathbf{m}|}^*(f_k)$ is the largest disk inside of which such convergence takes place, it readily follows that $R_k^*(\mathbf{f}, \mathbf{m}) \leq R_{|\mathbf{m}|}^*(f_k)$. Should $D_k^*(\mathbf{f}, \mathbf{m})$ contain on its boundary some singularity which is not a system pole, then necessarily $R_k^*(\mathbf{f}, \mathbf{m}) = R_{|\mathbf{m}|}^*(f_k)$ because

σ-convergence implies that all singularities inside must be zeros of $Q_{|\mathbf{m}|}$, but the zeros of this polynomial are all system poles as we proved in Theorem 1.4. Assume that $R^*_{|\mathbf{m}|}(f_k) > R^*_k(\mathbf{f}, \mathbf{m})$. Then, we have $R^*_{|\mathbf{m}|}(f_k) > \min_{j=1,\ldots,N} R_{\xi_j,\widetilde{\tau}_j}(\mathbf{f}, m)$. From the proof of [4, Theorem 3.6], we know that for each pole ξ of order $\widetilde{\tau}$ of f_k inside $D^*_{m_k}(f_k)$,

$$\limsup_{n\to\infty} \left| Q^{(j)}_{n,\mathbf{m}}(\xi) \right|^{1/n} \le \frac{|\xi|}{R^*_{|\mathbf{m}|}(f_k)}, \qquad j = 0, 1, \ldots, \widetilde{\tau} - 1.$$

This contradicts (39). Consequently $R^*_{|\mathbf{m}|}(f_k) = R^*_k(\mathbf{f}, \mathbf{m})$ as claimed.

Due to (42), we have also proved (41). In order to show that this formula is exact for σ-regular compact subsets, one must argue as in the corresponding part of the proof of [4, Theorem 4.4]. □

As compared with [4, Theorem 4.4], Theorem 3.7 offers weaker assumptions and a characterization of the values $R^*_{|\mathbf{m}|}(f_k)$ in terms of the analytic properties of the functions in the system instead of the coefficients of their Taylor expansion. An open question is to obtain an analogous characterization when the assumptions of Theorem 3.7 do not take place.

It would be interesting to study inverse problems for row sequences of Hermite–Padé approximation when only the limit behavior of some of the zeros of the polynomials $Q_{n,\mathbf{m}}$ is known, in the spirit of the conjectures proposed by A.A. Gonchar in [7] (see also Sect. 6.3 of Chap. 1 in [1]).

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