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"Equilibrium Growth in a Model with Economic Obsolescence of Machines"

by

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EQUILIBRIUM GROWTH IN A MODEL WITH ECONOMIC OBSOLESCENCE OF MACHINES

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I

Leif Johansen [6] formalized the idea of ex post rigidity of factor proportions in a 'vintage-capital' growth model with technical progress embodied only in new equipments. Edmund Phelps [10], and very recently, Murray Kemp and Pham Chí Thánh [7] have investigated the properties of steady-state growth equilibrium in such a model for the special case of ex ante Cobb-Douglas production functions.[1] In view of the extremely complicated relationships in this model even in the steady state, the simplification of the Cobb-Douglas assumption is very helpful. But, as is shown in this paper, the Cobb-Douglas assumption obscures many of the important properties of this model. For example, the relationship between economic life of capital and rate of interest, the uniqueness of growth equilibrium,[2] the needed concavity of the present-value function of profits, etc. are all very sensitive to the assumption about the ex ante elasticity of substitution of factors, and by assuming it to be unity one tends to overlook a rich variety of issues and problems in this model. The present paper is an attempt at a fuller generalization of this model.

We have a model in which machines embody the technology of their date of construction, and once a machine is built there is not scope for altering its labour requirement. Technical progress is completely embodied and Harrod-neutral at a constant rate g. Physical depreciation is ignored (it should be

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[1] Solow, Tobin, von Weizsäcker and Yaari [12] extensively discuss the properties of a vintage-capital model where there is no choice of substitution either ex ante or ex post. The popular nickname for their model is 'clay-clay,' whereas ours is a 'putty-clay' model.

[2] As mentioned in f.n. 7, this has not been proved by Phelps [10] or Kemp and Thánh [7] even for the Cobb-Douglas case.

the easiest thing to introduce a fixed rate of depreciation in this model) and obsolescence takes its toll when rising wages absorb all the revenues from a particular machine. The total labour force grows at constant rate μ and gross investment is taken to be a fixed proportion s of gross output. As for our expectations assumption, we assume that the entrepreneurs (correctly) expect the wage rate, W, to grow at rate g--the case of 'perfect foresight.' Kemp and Thánh [7] also consider the other extreme case of 'zero foresight' where entrepreneurs persist in their erroneous expectation of a constant wage, an assumption which we find rather unattractive.

One easy criticism of this paper--as of the papers by Phelps [10] and Kemp and Thánh [7]--is that throughout we confine ourselves to the long-run equilibrium growth path. This is certainly a matter of mathematical convenience. But there is probably much more justification for analyzing the steady-state properties for this model than for the usual neo-classical growth model, because due to the complex structure of this model some of these properties are not even now well-known or well-understood in the literature and, as the elaborate calculations in the Appendix to this paper would testify, the answers to some very simple but important questions asked in terms of this model are quite complicated indeed, even when we are in the relatively comfortable world of the steady state.

For each vintage of equipment there is a production relationship of the form

$$F_v(t) = F_v[I_v, e^{gv} \cdot L_v] \tag{1}$$

where $F_v(t)dv$ stands for the rate of output at time t produced on machines of vintage v, i.e., capital installed during a period (v, v+dv) with $t \geq v$, $I_v dv$ is the number of machines installed in the period (v, v+dv), and $L_v dv$ is the labour employed on capital of vintage v. Since this is a one-sector model, we measure capital goods in units identical with the unit of output. Total output

at time t accrues at a rate

$$F(t) = \int_{t-T}^{t} F_v(t)dv \qquad (2)$$

where T is the <u>economic</u> life of the oldest machine in use. In the steady state T is a constant.

If r is the constant rate of interest at which future quasi-rents are discounted, the present value of the output of capital of vintage t over its economic life is given by

$$F_t \frac{[1-e^{-rt}]}{r} \quad \text{for } r \neq 0.^3 \qquad (3)$$

The present value of the cost of labour employed with capital of vintage t over its life is likewise

$$W(t) \cdot L_t \frac{[1-e^{-(r-g)T}]}{r-g} \quad \text{for } r \neq g \qquad (4)$$

since the wage rate, W, is expected to grow at rate g.

The excess of (3) over (4) is the present value of current investment. As entrepreneurs expect the wage rate to rise they will not push hiring of labour to the point where the marginal product of labour is equal to the wage rate, but will stop at the point when extra labour does not add to the present value of current investment. This implies that

$$\frac{\partial F_t}{\partial L_t} = W(t) \cdot \frac{r[1-e^{-(r-g)T}]}{(r-g)[1-e^{-rT}]} \qquad (5)$$

i.e., the present value of the discounted marginal product of labour is equal to the discounted value of the wage stream.

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[3] In the main text of the paper we shall be assuming that $r \neq 0$ and $r \neq g$. In Appendix (<u>J</u>) we refer the cases of $r = 0$ and $r = g$.

Capital of vintage t is scrapped when its output is absorbed in labour costs. This means

$$W(t+T) = W(t)e^{gT} = \frac{F_t}{L_t} \tag{6}$$

Substituting for W(t) from (5),

$$\frac{F_t}{L_t \cdot \frac{\partial F_t}{\partial L_t}} = \frac{[e^{rT}-1](r-g)}{[e^{(r-g)T}-1]r} \tag{7}$$

The left-hand side on equation (7) is the labour elasticity of output in the <u>ex ante</u> production function.

Let us now introduce some new notations that will be helpful. Since the production function in (1) is assumed to be homogeneous of degree one, labour productivity on capital of vintage v is

$$\frac{F_v}{L_v} = e^{gv} f_v[k_v e^{-gv}] \tag{8}$$

where $k_v = \frac{I_v}{L_v}$. For our purpose of investigating the properties of long-run equilibrium, it is enough to study the equilibrium at time zero. At t = 0, labour productivity on current machines is $\frac{F_0}{L_0} = f_0(k_0)$, and the marginal productivity of labour on current machines is $\frac{F_0}{L_0} = f_0(k_0) - f_0'(k_0) \cdot k_0$. Just for convenience, from now on we shall drop the subscript 0, a variable without the vintage label will denote the value of the variable at t = 0.

Equation (7) may now be rewritten as

$$\frac{f(k)}{f(k)-f'(k)\cdot k} = \frac{[e^{rT}-1](r-g)}{[e^{(r-g)T}-1]r}. \tag{9}$$

With given g, this gives us a relationship among r, T and k. In the case of <u>ex ante</u> Cobb-Douglas production function, as assumed by Phelps [10] and Kemp and Thánh [7], the left-hand side of (9) is a constant and one can directly find out the relationship between r and T. As has been noted by these authors and also

explicitly proved in Bardhan [3], Appendix, $\frac{dT}{dr} < 0$ in this case; i.e., the economic life of machines is <u>longer</u> in the equilibrium with the lower rate of interest. For finding out the sign of $\frac{dT}{dr}$ in the more general case we shall now need some more information.

In competitive equilibrium the present value of net profit is zero (i.e., the present value of current investment = I_t), so that from (3) and (4) and with the use of our new notations,

$$V = f(k) \cdot \frac{[1-e^{-rT}]}{r} - W \cdot \frac{[1-e^{-(r-g)T}]}{(r-g)} - k = 0, \tag{10}$$

where V might be called the present value of profits from investment designed to employ one man. Now the entrepreneur would choose k, the capital-intensity on new machines, in such a way that V is maximized. Totally differentiating[4] V with respect to k, we get as a condition for maximizing V

$$f'(k) = \frac{r}{1-e^{-rT}} \tag{11}$$

Christopher Bliss has pointed out to me that the second-order condition for maximizing V depends on the value of the elasticity of substitution, σ, along the <u>ex ante</u> production function and that if σ is very large we may have a non-negative second derivative. But we have checked[5] that the second-order condition for

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[4]In totally differentiating V, we take account of (6) which in our new notation means $We^{gT} = f(k)$ and also that this implies $\frac{\partial T}{\partial K} = \frac{f'(k)}{gf(k)}$.

[5] $\frac{d^2V}{dk^2} = f''(k)\,\frac{[1-e^{-rT}]}{r} + \frac{[f'(k)]^2}{gf(k)}e^{-rT}$. Since the elasticity of <u>ex ante</u> substitution, $\sigma = -\frac{f'(k)[f(k)-f'(k)k]}{f''(k)\cdot f(k)k}$, this implies that for the second derivative to be negative,

$$\frac{[e^{rT}-1]g}{r} \cdot \frac{[f(k) - f'(k)k]}{f'(k)k} > \sigma. \tag{12}$$

Footnote 5 continued on next page.

maximum is satisfied under the sufficient condition of $\sigma \leqq 1$.

Why some kind of a restriction like this is needed may be explained in the following intuitive way. An important feature of a vintage-capital model like ours is that capital in such a model has two dimensions, one <u>intensive</u>, represented by k, the technique on current machines and the other <u>extensive</u>, represented by T, the economic lifetime of machines. "Capital-deepening", through concavity of the <u>ex ante</u> production function, tends to make the present value function for net profits concave, but, at the same time, it leads to "capital lengthening" (since for a given W, the more capital-intensive machines are scrapped later) which, in its turn, tends to detract from concavity of the present-value function. For ensuring that the present-value function is concave, we have to assume that the forces of diminishing returns generated by capital-deepening are strong enough (one way of securing that is not to have too high σ) to outweigh the lengthening effect.

Now equations (9) and (11) give us two relationships among r, T and k. Totally differentiating both the equations with respect to r we work out in Appendix (E) and (D) the values of $\frac{dk}{dr}$ and $\frac{dT}{dr}$.

As proved in Appendix (E), $\frac{dk}{dr}$ is always negative. This means that the (long-run) equilibrium with the higher rate of interest is characterized by a more labour-intensive technique for current equipment.

More complicated is the relationship between T and r. From Appendix (D) we find out that

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5 (cont'd) It is shown in Appendix (B) that the value of L. H. S. of (12) is greater than 1, so $\frac{d^2V}{dk^2} < 0$ for $\sigma \leqq 1$.

It is also indicated in Appendix (B) that this sufficient condition on σ can be significantly weakened if r is positive: In that case $\frac{d^2V}{dk^2} < 0$ for $\sigma \leqq 2$. This is useful to know since Britto [5] has shown that at least for C. E. S. <u>ex ante</u> production functions, equilibrium r must have a lower positive bound for the case of $\sigma > 1$.

$$\frac{dT}{dr}\frac{r}{T} = \frac{[z(m)-1][\sigma z(n)+(1-\sigma)z(m)] - z(m)\frac{m}{n}[z(n)-1]}{z(m)z(n)(m-n) - \sigma[z(m) - z(n)]} \tag{13}$$

where $z(x) \equiv \frac{e^x-1}{x}$, $x = m,n$, $x = m,n$, $m \equiv rT$, $n \equiv (r-g)T$.

As proved in Appendix (B), the denominator in equation (13) is positive if $\sigma \leqq 1$. It is the sign of the numerator that is more difficult to determine. It is, however, easy to check, as is done in Appendix (D), the numerator is positive for σ near or equal to unity. In other words, the (long-run) equilibrium with higher rate of interest will have longer economic life of equipment if the elasticity of substitution along the *ex ante* production function is near zero, and a shorter life of equipment if the elasticity of substitution is near or equal to unity.[6] When σ is significantly below unity but above zero, one is not sure of the sign of $\frac{dT}{dr}$.

Since the long-run equilibrium total output is growing at an exponential rate of $\lambda(=\mu+g)$, output from new equipment at time t is related to total output as follows:

$$F(t) = \frac{F_t[1-e^{-\lambda T}]}{\lambda}. \tag{14}$$

If s is the constant fraction of total gross output saved and currently invested,

$$sF(t) = I_t. \tag{15}$$

Using (8) and (14), (15) may be rewritten as

$$s = \frac{k}{f(k)} \cdot \frac{\lambda}{[1-e^{-\lambda T}]}. \tag{16}$$

We now try to find out if the growth equilibrium as characterized by equation (16) is unique for a given gross saving ratio s. This is important since most of

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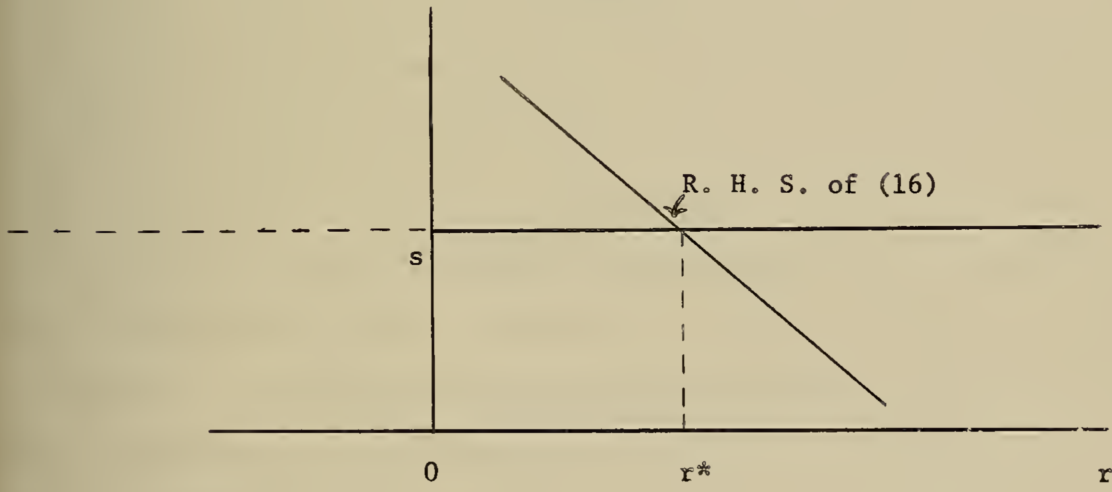
[6] These results regarding $\frac{dT}{dr}$ are stated and interpreted, but not correctly proved, in Matthews [9]. He also overlooks the necessity of some kind of an upper bound on σ as implicit in our sufficient condition for maximizing the present-value function. Britto [5] has independently reached the same results as ours for the special case of C. E. S. production functions. Christopher Bliss has also shown me similar results in unpublished notes.

the comparative-dynamic propositions which have been derived in terms of such vintage-capital models in the literature are of limited usefulness unless, among other things, uniqueness of growth equilibrium is proved.[7]

In Appendix (F) we show, after considerable manipulation, that the R. H. S. of (16) is a decreasing function of r under the sufficient condition of $\sigma \lesseqgtr 1$. Figure 1 shows the uniqueness of growth equilibrium under this sufficient condition.

FIGURE 1



[7] Phelps observes, "We are able to find a golden-age solution to the equations. The difficulty lies in showing that it is the only asymptotic solution possible" (italics mine), [10], p. 276. He then goes on to assume the problem away. Kemp and Thánh [7], p. 269 seem to assert uniqueness, but their Figure 2, to which they refer, has two downward sloping curves for their equations (4.8) and (4.9), and there is nothing on the face of it which precludes multiple intersection. The limit of those two curves seem to ensure existence but not uniqueness of equilibrium. Murray Kemp has confirmed this in correspondence.

The crucial problem of asymptotic stability of growth equilibrium in such models (with 'perfect foresight') has so far defied our attempt to prove. The major difficulty is the awfully complicated nature of non-steady state behavior in such models.

Solow, Tobin, von Weizsäcker and Yaari [12] have proved asymptotic stability for $\sigma = 0$ case. Following on their work, Sheshinski [11] has proved asymptotic stability for the case of $0 < \sigma \leqq 1$, but under the naive assumption of 'zero foresight' on the part of the entrepreneurs.

A rough intuitive explanation of why some condition of this type is needed may be given. Let us go back to equations (15) and (16). A fall in the interest r increases k, i.e., current gross investment per man, in other words, the investment function is a negatively sloped curve corresponding to changes in r. Now let us look at the savings function. With a constant fraction of gross output being saved, it depends on how output itself behaves corresponding to changes in r. A fall in r through "capital deepening" means a larger output on the current machine, but depending on the sign of $\frac{dT}{dr}$, it also affects the economic lifetime of capital. If $\frac{dT}{dr} < 0$, a fall in r implies "capital lengthening," and that tends to increase total output, so that on both counts savings tend to be a declining function of r as well. With both the saving and investment functions negatively sloped, the problem of multiple equilibria arises. What in effect we have shown above is that if σ is not very large, at the intersection of the two curves the absolute value of the slope of the investment function exceeds that of the savings function and the equilibrium is unique. There are two factors that bring this about: (a) because of the concavity of the <u>ex ante</u> production function, the effect of capital deepening on output of current machines in the savings function is swamped by capital deepening itself in the investment function, and (b) as long as σ is not very large, the impact of a change in r on output, and therefore savings, through capital lengthening is relatively small, since, as may be checked from (13), the <u>absolute</u> value of the elasticity of T with respect to r is an increasing function of σ, for given r and T. If $\frac{dT}{dr} > 0$ (which we know is the case at least for very small σ), then, of course, the capital-shortening effect of a fall in r on output reinforces the cause of uniqueness of growth equilibrium.

II

Having proved the uniqueness result, one might also use this model to derive some comparative-dynamic propositions. For example, as the R. H. S. of equation (16) is a decreasing function of r for $\sigma \leqq 1$, it is immediately seen that $\frac{dr}{ds} < 0$, i.e., comparing between two steady-state equilibria, the equilibrium with the <u>higher</u> gross saving ratio should have the <u>lower</u> rate of interest under our elasticity of substitution condition.

How about distributive shares? With exponential growth of the labour force at rate μ, labour assigned to new machines is related to the total labour force L(t) in the following way:

$$L_t = \frac{L(t) \cdot \mu}{[1-e^{-\mu T}]} \tag{17}$$

Using (6), (14), and (17), the total wage-share in the economy is

$$Q(t) = \frac{W(t) \cdot L(t)}{F(t)} = \frac{[e^{\mu T}-1]\lambda}{[e^{\lambda T}-1]\mu} \tag{18}$$

It is easy to check that the extreme R. H. S. of (18) is a decreasing function of T. We have already seen that when σ is very near or equal to unity, $\frac{dT}{dr} < 0$; in this case, therefore, we may say $\frac{dQ}{ds} < 0$, i.e., between two steady-state equilibria the one with the <u>higher</u> gross saving ratio has a <u>lower</u> wage share. We get the opposite result when σ is very near zero and $\frac{dT}{dr} > 0$.

In Appendix (<u>C</u>) it is shown that $\frac{dW}{dr} < 0$, i.e., the wage rate and the rate of interest are always inversely related in this model. It is also shown that the absolute value of the elasticity of what might be called the 'factor-price frontier' in this model, $\frac{dW}{dr}\frac{r}{W}$, is <u>less</u> than (whereas in usual neo-classical models it is equal to) the ratio of investment elasticity of output to the labour elasticity of output along the <u>ex ante</u> production function.

Another interesting comparative-dynamic result, as shown in Appendix (G), is that in this model the elasticity of the average productivity of labour with respect to the wage rate

(i.e., $\frac{d\frac{F(t)}{L(t)}}{dW(t)} \cdot \frac{W(t)}{\frac{F(t)}{L(t)}}$) is higher than the elasticity of substitution (for $\sigma \leqq 1$ and $r \geqq \lambda$) along the *ex ante* production function, whereas in usual neo-classical models the former is always equal to the latter. This implies that the well-known method of estimating elasticity of factor substitution *à la* Arrow, Chenery, Minhas and Solow [2]--i.e., by estimating the coefficient of regression of the logarithm of observed output per unit of labour on that of observed wage rate--may give an overestimate, if the data-generating model is not static, but has the properties of a vintage capital model of the type analyzed here.[8]

A corollary of the above result is that the average productivity of labour is always an increasing function of the wage rate. This implies that comparing countries in steady states, the country with the higher wage rate will have a higher productivity of labour.[9] Labour productivity in this model depends not merely on the capital-intensity of the machines in use but also on the economic life of the machines, and both are affected by the wage-rate.

Finally, given the gross saving ratio, does a higher rate of (Harrod-neutral)

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[8] In [3] we pointed to this overestimation bias in the Arrow-Chenery-Minhas-Solow procedure for the cases of *ex ante* production functions of Cobb-Douglas and fixed-coefficients types. In this paper we generalize the result for all $\sigma \leqq 1$.

[9] In [4] we have a model which explains inter-country productivity differentials, and therefore trade, in terms of differences in the economic life of equipment as determined by factor prices. Even when the stream of new technical knowledge (embodied in new machines) is the same for all countries, their rate of utilization of this knowledge, as reflected in the economic life of machines, is different, since different wage rates lead them to scrap machines at different dates.

technical progress imply a higher or a lower rate of interest? It is proved in Appendix (I) that given the gross saving ratio and the rate of growth of population, $\frac{dr}{dg} > 0$ for $r \geqq \lambda$[10] and $\sigma \leqq 1$.

Reviewing the whole of our analysis in this paper it seems imperative to underline the important role of σ, the elasticity of factor substitution, in shaping the different types of interrelationships in the model and that this tends to be obscured by the usual Cobb-Douglas assumption. We have seen how very large σ might cause problems in maximizing the present-value function of net profits as given by our equation (10). We have also found some restriction like $\sigma \leqq 1$ as a (sufficient) condition for uniqueness of growth equilibrium. Then again we have noticed how the value of σ is important in finding out the type of relationship that holds in equilibrium between T, the economic lifetime of capital and r, the rate of interest. As it turns out, the sign of $\frac{dT}{dr}$ is one of the most important items of information one needs for deriving all sorts of comparative-dynamic propositions in such vintage-capital models. The latter, therefore, are very sensitive to the particular value of σ.

Before ending we may also note that all the results of this paper carry over to a generalized model of Arrow-type 'learning by doing' [1] with *ex ante* factor substitutability and *ex post* rigidity. (This is essentially because despite differences in the origin of technical progress the structures of the two types of models are very similar particularly in their steady-state solutions.) Levhari in his extension [8] of Arrow's model has been able to avoid some of the problems mentioned in this paper by assuming the far easier case of *ex post* factor substitutability.

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[10] A faster rate of technical progress may imply a lower rate of return when $\lambda > r$. This is in contrast to the case of *ex ante* fixed-coefficients production functions discussed in Solow, Tobin von Weizäcker and Yaari [12].

APPENDIX

(A). In the subsequent sections we shall define

$$z(x) = \frac{e^x - 1}{x} > 0 \text{ for } x \gtrless 0 \tag{19}$$

We shall also define

$$m = rT, \text{ so that } \frac{e^{rT}-1}{rT} = z(m)$$

$$n = (r-g)T, \text{ so that } \frac{e^{(r-g)T}-1}{(r-g)T} = z(n)$$

$$q = \lambda T, \text{ so that } \frac{e^{\lambda T}-1}{\lambda T} = z(q)$$

$$\text{and } \rho = \mu T, \text{ so that } \frac{e^{\mu T}-1}{\mu T} = z(\rho)$$

It is easy to check that

$$z'(x) = \frac{1 + [x-1]z(x)}{x} > 0 \text{ for } x \gtrless 0. \tag{20}$$

(B). We prove that hte L. H. S. of inequality (12) is greater than 1.

Let us rewrite (9) as

$$\frac{f(k)}{f(k)-f'(k)k} = \frac{z(m)}{z(n)} . \tag{21}$$

This implies that

$$\frac{f(k)-f'(k)k}{f'(k)k} = \frac{z(n)}{z(m)-z(n)}. \tag{22}$$

So in terms of our new notation the L. H. S. of (12) may be rewritten as

$$\frac{z(m)z(n)(m-n)}{z(m) - z(n)} . \tag{23}$$

We show that this expression is greater than unity, i.e.

$$m + \frac{1}{z(m)} > n + \frac{1}{z(n)} .$$

If we define $y(x) = x + \frac{1}{z(x)}$ which is, of course, positive for x either positive or negative, all we have to show is that $y'(x) > 0$, since $m > n$.

Using (20) it is easily seen that $y'(x)$ has the same sign as that of

$$\frac{1}{x}\left[xz^2(x) - (x-1)z(x) - 1\right] = \frac{[z(x)-1][1+xz(x)]}{x} \tag{24}$$

which is positive for x either positive or negative. It may also be mentioned here that if the rate of interest, r, is positive, i.e., $m > 0$, then one can show that

$$\frac{z(m)z(n)(m-n)}{z(m) - z(n)} > 2. \tag{25}$$

(<u>C</u>). Totally differentiating (10) and using (6) and (11)

$$\frac{dW}{dr}\frac{r}{W} = \frac{[e^{(r-g)T}-1-(r-g)T]r}{[e^{(r-g)T}-1](r-g)} - \frac{[e^{rT}-1-rT](r-g)}{[e^{(r-g)T}-1]r} = \frac{\frac{m}{n}[z(n)-1] - [z(m)-1]}{z(n)}. \tag{26}$$

If m and n are positive, (26) is negative if we can show that

$$\frac{[z(m)-1]}{m} > \frac{[z(n)-1]}{n}. \tag{27}$$

If we now define $y(x) = \frac{[z(x)-1]}{x}$, all we have to show is that $y'(x) > 0$, since $m > n$. Using (20), $y'(x)$ has the same sign as that of $xz(x) - 2[z(x)-1] = \frac{1}{x}[xe^x - 2e^x + x + 2]$. Define $P(x) = xe^x - 2e^x + x + 2$, so that $P(0) = 0$, and $P'(x) = xe^x - e^x + 1$ which is positive, since $P'(0) = 0$, and $P''(x) > 0$. Therefore $P(x)$ and $y'(x)$ are positive, proving inequality (27).

The reader can easily check from (26) that $\frac{dw}{dr} < 0$ even when $m < 0$, $n < 0$, or $m > 0$, $n < 0$.

We can also prove that the absolute value of (26) is <u>less</u> than the ratio of investment elasticity of output to the labour elasticity of output along the <u>ex ante</u> production function. From (8), the latter may be expressed as

$\frac{kf'(k)}{f(k)-f'(k)k}$ which, from (22), is equal to $\frac{z(m) - z(n)}{z(n)}$.

This means, using (26) we have to prove that

$$[z(m)-1] - \frac{m}{n} [z(n)-1] - [z(m)-z(n)] < 0. \tag{28}$$

The L. H. S. of (28) is equal to $-[z(n)-1] \frac{(m-n)}{n} < 0$.

(D). Totally differentiating (11),

$$\frac{dk}{dr} = - \frac{[f'(k)]^2}{f''(k)} e^{-rT} [\frac{dT}{dr} - \frac{e^{rT}-1-rT}{r^2}] \tag{29}$$

Totally differentiating in (9), using (29) for the value of $\frac{dk}{dr}$, using the definition of σ, and after simplification with the use of (9) and (11),

$$\frac{dT}{dr}\frac{r}{T} = \frac{(1-\sigma)[z(m)-1][z(m)-z(n)] - z(m)z(n)[\frac{z'(m)}{z(m)} - \frac{z'(n)}{z(n)}]m}{(1-\sigma)[z(m)-z(n)] + z(m)z(n)[m\frac{z'(m)}{z(m)} - n\frac{z'(n)}{z(n)}]}. \tag{30}$$

Using the value of $\frac{z'(x)}{z(x)}$ from (20) and after simplification (30) is equal to

$$\frac{[z(m)-1][\sigma z(n) + (1-\sigma)z(m)] - z(m)\frac{m}{n} [z(n)-1]}{z(m)z(n)(m-n) - \sigma[z(m)-z(n)}. \tag{31}$$

From Appendix (B) we already know that the denominator of (31) is positive for $\sigma \leqq 1$. (If $m > 0$, the denominator is positive for $\sigma \leqq 2$.) What about the numerator?

$[\sigma z(n) + (1-\sigma)z(m)]$ is a weighted average of z(m) and z(n). When σ is equal to or very near zero, this approximates z(m), and inequality (27) of Appendix (C) ensures that the numerator, and therefore, $\frac{dT}{dr}$ is positive. When σ is equal to or very near unity, the weighted average tends to approximate z(n), and the numerator, and therefore, $\frac{dT}{dr}$ is negative, since we can prove that

$$\frac{[z(m)-1]}{mz(m)} < \frac{[z(n)-1]}{nz(n)} \tag{32}$$

Define $y(x) = \frac{z(x)-1}{xz(x)}$. Using (20) the sign of $y'(x)$ is the same as that of

$$[-z^2(x) + 1 + xz(x)]$$

$$= - [\frac{(e^x-1)^2}{x^2} - e^x].$$

Since by expansion in Taylor series it can be shown that $\frac{e^x-1}{x} > e^{\frac{x}{2}}$, $y'(x) < 0$.

(E). Here we prove that $\frac{dk}{dr} < 0$.

From (29) in Appendix (D) all we have to prove is that

$$\frac{dT}{dr} < \frac{e^{rT}-1-rT}{r^2} \tag{33}$$

If $r > 0$, using (31) in Appendix (D), this means we have to show that

$$[z(m)-1][\sigma z(n)+(1-\sigma)z(m)] - z(m)\frac{m}{n}[z(n)-1] < [z(m)-1]\Big\{z(m)z(n)(m-n)-\sigma[z(m)-z(n)]\Big\}.$$

On simplification, all we have to show is that

$$1 - \frac{m}{n}\frac{[z(n)-1]}{[z(m)-1]} < z(n)(m-n). \tag{34}$$

From Appendix (B), we know that $z(n)(m-n) > 1 - \frac{z(n)}{z(m)}$. Using this, and inequality (32) in Appendix (D), it is easy to prove (34). For $r < 0$, we have to show that $\frac{dT}{dr}\frac{r}{T} > [z(m)-1]$.

Given (32) of Appendix (D), the proof follows easily and is left to the reader.

(F). The R. H. S. of (16) is $\frac{k}{f(k)} \cdot \frac{\lambda}{[1-e^{-\lambda T}]} = \frac{A(k)}{B(t)}$ (35)

where $A(k) = \frac{k}{f(k)}$ and $B(T) = \frac{1-e^{-\lambda T}}{\lambda}$.

Since from Appendix (E), $\frac{dk}{dr} < 0$, A(k) is a declining function of r. B(T) is an increasing function of T, and if $\frac{dT}{dr} \gtreqless 0$, we immediately have our desired result, viz., the R. H. S. of (16) is a declining function of r.

Let us now consider the case when $\frac{dT}{dr} < 0$. We show that even in this case--the case when both A and B are declining functions of r--$\frac{dA}{dr}\frac{1}{A} < \frac{dB}{dr}\frac{1}{B}$.

From (29) in Appendix (D) and (11),

$$\frac{dA}{dr}\frac{1}{A} = \frac{\sigma r}{e^{rT}-1}\left[\frac{dT}{dr} - \frac{e^{rT}-1-rT}{r^2}\right] \tag{36}$$

$$\frac{dB}{dr}\frac{1}{B} = \frac{dT}{dr} \cdot \frac{\lambda}{[e^{\lambda T}-1]} \tag{37}$$

$$\frac{dA}{dr}\frac{1}{A} - \frac{dB}{dr}\frac{1}{B} = \frac{dT}{dr}\left[\frac{\sigma r}{e^{rT}-1} - \frac{\lambda}{e^{\lambda T}-1}\right] - \frac{\sigma[e^{rT}-1-rT]}{r[e^{rT}-1]}. \tag{38}$$

Rewritten, (38) is negative if

$$\frac{\sigma}{r}[z(m)-1] > \frac{dT}{dr}\frac{1}{T}\left[\sigma - \frac{z(m)}{z(q)}\right]. \tag{39}$$

Since the L. H. S. of (39) is positive, and $\frac{dT}{dr} < 0$, (39) is immediately proved if $\sigma \geqq \frac{z(m)}{z(q)}$. Let us therefore assume that $\sigma < \frac{z(m)}{z(q)}$. If $r > 0$, all we have to prove is that

$$\frac{\sigma[z(m)-1]z(q)}{[z(m)-\sigma z(q)]} > -\frac{dT}{dr}\frac{r}{T}. \tag{40}$$

Now deducting 1 from both sides, using (31) for $\frac{dT}{dr}\frac{r}{T}$, dividing through by $-z(m)$

and reversing inequality, we have to prove that

$$\frac{\frac{(m-n)}{n}\,[1+(n-1)z(n)] + (1-\sigma)[z(m)-z(n)]}{z(m)z(n)(m-n) - \sigma[z(m)-z(n)]} > \frac{1-\sigma z(q)}{[z(m)-1] + [1-\sigma z(q)]}. \tag{41}$$

Since L. H. S. is positive, the inequality is immediately proved if $\sigma z(q) \geqq 1$. So let us assume $1 > \sigma z(q)$.

Define $a = (m-n)z(m)z(n) > 0$

$b = z(m) - z(n) > 0$

$c = \frac{(m-n)}{n}\,[1+(n-1)z(n)] > 0$

The denominators on both sides of (41) are positive. On cross multiplication and simplification we have to show that

$$[1-\sigma z(q)][a-b-c] < [z(m)-1][b(1-\sigma)+c]. \tag{42}$$

Since $z(q) > 1$, it is enough to show that

$$(1-\sigma)(a-b-c) < [z(m)-1][b(1-\sigma)+c]. \tag{43}$$

Since $a = z(m)[c + \frac{(m-n)}{n}\{(z(m)-1)-b\}]$,

$$a-b-c = [z(m)-1][b+c+z(m)\,\frac{(m-n)}{n}] - b\cdot\frac{m}{n}\cdot z(m). \tag{44}$$

Using the value of (a-b-c) on (43) and with simplification we have to prove that

$$[z(m)-1]\sigma c > \frac{(1-\sigma)}{n}\,z(m)[(m-n)\{z(m)-1\} - m\cdot b]. \tag{45}$$

The L. H. S. of (45) is positive. On the R. H. S., the bracketed expression is negative, using the value of b and inequality (27) of Appendix (C). Therefore the R. H. S. is non-positive if $\sigma \leqq 1$, and that proves inequality (45). The proof for $r < 0$ follows essentially the same line and is left to the reader.

(G). From (18), the average productivity of labour in the economy is

$$\frac{F(t)}{L(t)} = W(t) \cdot \frac{z(q)}{z(\rho)}. \tag{46}$$

So the elasticity of average productivity with respect to the wage rate is equal to

$$1 + E_{Tr} \cdot E_{rw} \left[q\frac{z'(q)}{z(q)} - \frac{z'(\rho)}{z(\rho)}\rho\right] \tag{47}$$

where $E_{Tr} = \frac{dT}{dr}\frac{r}{T}$

and $E_{rw} = \frac{dr}{dW}\frac{W}{r}$.

We have to show that (47) is greater than σ. Using (20) from Appendix (A),

$$q \cdot \frac{z'(q)}{z(q)} - \rho \cdot \frac{z'(\rho)}{z(\rho)} = (q-\rho) - \frac{[z(q)-z(\rho)]}{z(q)z(\rho)} \tag{48}$$

which is positive since $q > \rho$, following the same proof as in (23) of Appendix (B).

Since $E_{rw} < 0$, we can immediately say that (47) is greater than σ for $\sigma \leqq 1$ if $E_{Tr} \leqq 0$. So let us suppose $E_{Tr} > 0$.

Using (26) from Appendix (C) and (31) from Appendix (D), and after simplification, we have to show that

$$\frac{(1-\sigma)[\{z(m)-1\} - \frac{m}{n}\{z(n)-1\}][z(m)z(n)(m-n)-\sigma\{z(m)-z(n)\}]}{z(n)[\{z(m)-1\}\{\sigma z(n)+(1-\sigma)z(m)\} - z(m)\frac{m}{n}\{z(n)-1\}]} \tag{49}$$

$$> (q-\rho) - \frac{[z(q)-z(\rho)]}{z(q)z(\rho)}.$$

In Appendix (H) it is shown that

$$\frac{[z(q) - z(\rho)]}{z(q)z(\rho)} \geqq \frac{[z(m) - z(n)]}{z(m)\cdot z(n)}, \text{ for } m \geqq q \text{ or } r \geqq \lambda \tag{50}$$

which we assume now.

With $\sigma \leqq 1$ and $q-\rho = gT - (m-n)$, this means that the R. H. S. of (49) is smaller than

$$(m-n) - \frac{\sigma[z(m)-z(n)]}{z(m)z(n)}. \tag{51}$$

So it is enough for us to show that the L. H. S. of (49) is greater than (51). This means we have to show that

$$(1-\sigma)z(m)[\{z(m)-1\} - \frac{m}{n}\{z(n)-1\}] > [\{z(m)-1\}\{\sigma z(n)+(1-\sigma)z(m)\}-z(m)\frac{m}{n}\{z(n)-1\}]. \tag{52}$$

It is easy to check that this inequality is valid, once we take into account inequality (32) proved in Appendix (D).

(H). Define $\alpha = \frac{r}{g}$, $\beta = \frac{\lambda}{g}$, and $\theta = gT$.

Since we take $r \geqq \lambda$ and $\lambda = \mu + g$, $\alpha \geqq \beta > 1$.

Then,

$$\frac{z(m)-z(n)}{z(m)z(n)} = \frac{z[\alpha\theta] - z[(\alpha-1)\theta]}{z[\alpha\theta]\cdot z[(\alpha-1)\theta]} \tag{53}$$

and

$$\frac{z(q)-z(\rho)}{z(q)z(\rho)} = \frac{z[\beta\theta] - z[(\beta-1)\theta]}{z[\beta\theta]\cdot z[(\beta-1)\theta]} \tag{54}$$

If $r = \lambda$, then, of course, $\alpha = \beta$, and (53) = (54). Let us assume $r > \lambda$, or $\alpha > \beta$.

Define

$$J(y) = \frac{z[y\theta]-z[(y-1)\theta]}{z[y\theta]\cdot z[(y-1)\theta]}, \quad y = \alpha, \beta. \tag{55}$$

We have to prove that $J'(y) < 0$, since that will imply (53) smaller than (54).

Working out the value of $\frac{J'(y)}{J(y)}$ from (55), and after cross multiplication and simplification what remains for us to prove is that

$$\frac{z'[y\theta]}{z^2[y\theta]} < \frac{z'[(y-1)\theta]}{z^2[(y-1)\theta]}. \tag{56}$$

Noting the value of $z'(x)$ from (20) of Appendix (A), all we have to prove is that

$$p'(x) < 0,$$

where

$$p(x) = \frac{1 + [x-1]z(x)}{x\cdot z^2(x)}, \quad x = y\theta, (y-1)\theta. \tag{57}$$

Using (20) of Appendix (A) again, and upon simplification $p'(x)$ has the same sign as that of

$$2[z(x)-1] - xz(x)[3+\{x-2\}z(x)]. \tag{58}$$

We have to show that this is negative.

From Appendix (C) we know that

$$xz(x) > 2[z(x)-1]. \tag{59}$$

It also immediately follows from (59) that

$$3 + [x-2]z(x) > 1. \tag{60}$$

Using (59) and (60), it is easy to see that (58) is negative.

(I). If we define the R. H. S. of (16) as N, then

$$\frac{dr}{dg} = -\frac{\frac{\partial N}{\partial g}}{\frac{\partial N}{\partial r}} \tag{61}$$

In Appendix (F) we have proved that $\frac{\partial N}{\partial r} < 0$ for $\sigma \leqq 1$. Here we prove that $\frac{\partial N}{\partial g} > 0$ for $r \geqq \lambda$ and $\sigma \leqq 1$, so that $\frac{dr}{dg} > 0$.

From (16),

$$\frac{\partial N}{\partial g}\frac{1}{N} = [\frac{f(k)-f'(k)k}{f(k)k} - \frac{\lambda}{e^{\lambda T}-1} \cdot \frac{dT}{dk}] \frac{dk}{dg} + \frac{e^{\lambda T}-1-\lambda T}{\lambda[e^{\lambda T}-1]} \tag{62}$$

where $\frac{dT}{dk}$ and $\frac{dk}{dg}$ are to be evaluated with r constant.

From (11), $$\frac{dT}{dk} = \frac{[e^{rT}-1]}{r} \cdot \frac{[f(k)-f'(k)k]}{\sigma f(k)k} > 0. \tag{63}$$

From (9), one can work out the value of $\frac{dk}{dg}$, the numerator of which is positive, since $z'(n) > 0$, and the denominator is negative, since $\sigma \leqq 1$, $\frac{dT}{dk} > 0$ and

$$\frac{z'(m)}{z(m)}m - \frac{z'(n)}{z(n)}n > 0. \tag{64}$$

(64) is correct, since its L. H. S. is, using (20) of Appendix (A), equal to

$$(m-n) - \frac{[z(m)-z(n)]}{z(m)z(n)} \tag{65}$$

which we have already proved to be positive in Appendix (B).

(62) may be rewritten as

$$\frac{[f(k)-f'(k)k]}{f(k)k}\left[1 - \frac{(e^{rT}-1)\lambda}{r(e^{\lambda T}-1)\sigma}\right]\frac{dk}{dg} + \frac{e^{\lambda T}-1-\lambda T}{\lambda[e^{\lambda T}-1]}. \tag{66}$$

Since $\frac{e^{rT}-1}{r} \gtreqq \frac{e^{\lambda T}-1}{\lambda}$ for $r \gtreqq \lambda$, and with $\sigma \leqq 1$ and $\frac{dk}{dg} < 0$, (66) is positive as we had set out to prove.

(J). We have assumed so far that $r \neq 0$ or $r \neq g$. If $r = 0$, we have the following changes in the pattern of our basic equations. In place of (9), we have

$$\frac{f(k)}{f(k)-f'(k)k} = \frac{gT}{1-e^{-gT}}. \tag{67}$$

In place of (10), we have

$$V = f(k)\cdot T - W\frac{[e^{gT}-1]}{g} - k = 0. \tag{68}$$

In place of (11), we have

$$f'(k) = \frac{1}{T}. \tag{69}$$

If $r = g > 0$, we have the following changes.

In place of (9), we have

$$\frac{f(k)}{f(k)-f'(k)k} = \frac{e^{rT}-1}{rT} \tag{70}$$

In place of (10), we have

$$V = f(k)\frac{[1-e^{-rT}]}{r} - W\circ T - k = 0. \tag{71}$$

In either of these two cases the model becomes much more simplified, and the analysis is left to the reader. Let us only note that with $r = 0$, $\frac{dT}{dW} < 0$ for $\sigma \leqq 1$.

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