Coherent Pricing

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Abstract Recent literature proved the existence of an unbounded market price of risk (MPR) or maximum generalized Sharpe ratio (GSR) if one combines the most important Brownian-motion-linked arbitrage free pricing models with a coherent and expectation bounded risk measure. Furthermore, explicit sequences of portfolios with a theoretical (risk, return) diverging to $(-\infty, +\infty)$ were constructed and their performance tested. The empirical evidence revealed that the divergence to $(-\infty, +\infty)$ is only theoretical (not real), but the MPR is much larger than the GSR of the most important international stock indices. The natural question is how to modify the available pricing models so as to prevent the caveat above. The theoretical MPR cannot equal inf nity but must be large enough (consistent with the empirical findings) and this will be the focus of this paper. It will be shown that every arbitrage free pricing model can be improved in such a manner that the new stochastic discount factor (SDF) satisfie the two requirements above, and the new MPR becomes bounded but large enough. This is important for several reasons; Firstly, if the existent models predict unrealistic price evolutions then these mistakes may imply important capital losses to practitioners and theoretical errors to researchers. Secondly, the lack of an unbounded MPR is much more coherent and consistent with equilibrium. Finally, the major discrepancies between the initial pricing model and the modifie one will affect the tails of their SDF, which seems to justify several empirical caveats of previous literature. For instance, it has been pointed out that it is not easy to explain the real quotes of many deeply OTM options with the existing pricing models.

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1. Introduction

Risk measures are becoming more and more studied in actuarial and financia mathematics. Among many others, important examples are the actuarial risk measures (Goovaerts and Laeven, 2008), coherent risk measures (Artzner *et al.*, 1999), convex risk measures (Follmer and Schied, 2002), expectation bounded risk measures (Rockafellar *et al.*, 2006), riskiness indices (Aumann and Serrano, 2008), satisfying measures (Brown and Sim, 2009), dynamic risk measures (Cheridito and Tianhui, 2009), maturity-independent risk measures (Zariphopoulou and Gordan, 2010), set valued risk measures (Hamel and Heyde, 2010) conditional risk measures (Filipovic *et al.*, 2012), etc. The introduction of new risk measures has generated new looks for many classical financia problems. Among many other revisited topics, interesting examples are pricing and hedging issues (Wang, 2000, Nakano, 2004, etc.), risk management methods (Basak and Shapiro, 2001) and asset allocation problems (Dupacová and Kopa, 2014, Zhao and Xiao, 2016, etc.).

Regarding portfolio selection and asset allocation, Balbás *et al.* (2010) proved the existence of an unbounded generalized Sharpe ratio (*i.e.*, return/risk ratio) if one combines the most important Brownian-motion-linked arbitrage free pricing models (Black and Scholes, stochastic volatility, etc.) with a coherent and expectation bounded risk measure.[1] The same authors proved that several pricing models reflec similar shortcomings when combined with some deviations. For instance, the Black and Scholes model predicts the existence of sequences of self-fina cing strategies with an expected return diverging to $+\infty$ and an absolute deviation converging to zero.

These theoretical finding were more deeply studied in Balbás *et al. (*2016a), where explicit sequences of portfolios with a $(risk, return)$ diverging to $(-\infty, \infty)$ were constructed. The risk was measured with coherent and expectation bounded risk measures, and the constructed sequence of portfolios had a slight sensitivity with respect to the selected risk measure, *i.e.*, significan risk measure changes did not modify any portfolio in the sequence. These portfolios (henceforth, desirable strategies, DS) were composed of derivatives and their performance was empirically tested in several American and European markets. The empirical evidence revealed that the divergence to $(-\infty, \infty)$ was only theoretical (not real), but the generalized Sharpe ratio (GSR) of

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[1]See also Stoica and Li (2010) and Assa (2015) for interesting discussions about this caveat.

every DS was much larger than the value of this ratio for its underlying asset.

The theoretical and empirical finding of Balbás *et al.* (2010) and (2016a) suggest three potential implications. On the one hand, the existing theoretical models for pricing derivatives are provoking the drawback above, since modification of the risk measure do not overcome the problem, and some deviations lead to similar conclusions. On the other hand, the existing pricing models are not able to explain the stochastic price evolution of many derivatives, because the empirical optimal GSR is systematically lower than that predicted by the model. Finally, however, the error committed by the theoretical pricing models seems to be moderate, since they are able to inspire the construction of strategies of derivatives outperforming their underlying assets, and the superiority of these DS may be empirically validated. These three implications motivate the natural Question Q below, which is the focus of this paper;

Question Q; Can a slight/moderate modificatio of an existing pricing model recover the absence of DS and simultaneously lead to an optimal GSR consistent with the empirical evidence?

As will be seen, the answer to Question Q will be "yes". This is important for several reasons. First, the existence of DS in arbitrage-free pricing models is a serious drawback of the model from a theoretical point of view. Indeed, traders trying to implement DS should buy some "underpriced" securities and sell the "overpriced" ones, and this additional trading should modify the price process (the model). Second, if the existent models predict an unrealistic price evolution of a DS, then they may also make mistakes when pricing and hedging some other derivatives, provoking errors that may affect practitioners and researchers. In this sense, the improvement of the available pricing models may be interesting from theoretical and practical viewpoints.

The outline of the paper is as follows. Section 2 will present the general framework, the problem and some basic properties. We will pose the problem in the abstract setting of Hilbert spaces. The firs reason is that Hilbert spaces will allow us to integrate many different particular cases in a single formulation. In particular, Examples 1 and 2 will illustrate how our setting can involve complete pricing models, incomplete models, uncertainty free models and ambiguous pricing models. As a second reason justifying the use of Hilbert spaces, let us point out that the existence of DS is closely related to many geometric properties of the classical spaces of random variables, and therefore the caveat may be overcome with "geometric solutions", which are easily detected with the abstract approach. In fact, the price modificatio preventing DS will be given by means

of projections and orthogonal projections.[2]

Sections 3 and 4 contain the most important results of this paper. In particular, Theorems 5 and 8 (Section 3) and Theorem 11 (Section 4) provide us with the positive answer to Question Q. These results, along with their corollaries, yield the lowest modificatio of a pricing model such that the new prices prevent the existence of DS and generate an optimal GSR compatible with the empirical evidence. All the given statements apply for risk measures. Similar results (with essentially similar proofs) could be given for deviations, though we will omit them in order to shorten the exposition.

Many risk measurement linked papers frequently present examples constructed with the Conditional Value at Risk ($CVaR$) of Rockafellar and Uryasev (2000). This coherent and expectation bounded risk measure may be easily optimized and interpreted in terms of potential capital losses (Rockafellar and Uryasev, 2000), outperforms the standard deviation with respect to the second order stochastic dominance (Ogryczak and Ruszczynski, 1999 and 2002) and has natural extensions to ambiguous frameworks (Zhu and Fukushima, 2009), among many other good properties. Section 5 ends this paper with some $CVaR$ linked properties and toy examples illustrating and interpreting the main theorems above. Under the $CVaR$, we will see that the modificatio of prices will modify the stochastic discount factor (SDF) in such a manner that its logarithm becomes essentially bounded. The major differences between the initial SDF and the modifie one will be concentrated on their tails. This findin seems to be consistent with the empirical evidence, which suggests that many classical pricing models have problems to explain the real quotations of OTM options (Bondarenko, 2014). Overall, the examples of Section 5 will present concrete arbitrage free pricing models overcoming the existence of DS, and they will illustrate the theoretical finding of this paper. Unbounded GSR and market price of risk (MPR) can be overcome in such a manner that the new ones become finit and equal a desired target. The desired MPR can be respected, and then the price modificatio will be "as small as possible". The modifie price process will reflec lower volatilities and will moderate the tail behavior of the SDF. Anyway, it may be worthwhile to point out that the objective of this paper is theoretical. The modifie pricing model will depend on desired MPR. This value may be calibrated to market but we will not address this issue, which is beyond our scope. Section 6 will conclude the paper.

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[2]Previous literature has proved that Hilbert space linked approaches may be interesting to solve deep problems in Financial Mathematics (see, for instance, Schachermayer, 1992).

2. Preliminaries and notations

Consider a Hilbert space Y endowed with the inner product $\langle ., . \rangle$ and the norm $\|.\|$, an element $y_\Pi \in Y$ and a linear and continuous function $\Pi : Y \to \mathbb{R}$ given by $\Pi(y) = \langle y, y_\Pi \rangle$ for every $y \in Y$ (Choquet, 1966). A closed subspace $U \subset Y$ will be called stable, and we will fi $u_0 \in U$ with

$$\langle u_0, y_\Pi \rangle = \Pi(u_0) = 1. \tag{1}$$

Consider two convex and weakly compact sets $\mathcal{K} \subset \Delta \subset Y$ and the function

$$\rho(y) = Max \ \{- \langle y, z \rangle \, ; \, z \in \Delta\} \tag{2}$$

for every $y \in Y$. If (2) holds then it may be proved that Δ is unique, and $z \in \Delta$ if and only if $- \langle y, z \rangle \leq \rho(y)$ for every $y \in Y$. (2) implies that ρ is continuous, sub-additive ($\rho(y_1 + y_2) \leq \rho(y_1) + \rho(y_2)$ for $y_i \in Y$, $i = 1, 2$) and positively homogeneous ($\rho(\lambda y) = \lambda \rho(y)$ for $\lambda \geq 0$ and $y \in Y$). We will also assume that

$$u_0 \in \mathcal{K} \subset \Delta, \tag{3}$$

and therefore $\rho(y) \geq - \langle y, u_0 \rangle$ for every $y \in Y$. Finally, let us assume that

$$\langle z, u_0 \rangle = 1 \tag{4}$$

holds for every $z \in \Delta$, which is equivalent to

$$\rho(y + ku_0) = \rho(y) - k \tag{5}$$

for every $k \in \mathbb{R}$ and every $y \in Y$. Notice that (3) and (4) lead to

$$\|u_0\|^2 = \langle u_0, u_0 \rangle = 1. \tag{6}$$

Consider Problem

$$\begin{cases} Min \ \rho(y) \\ \langle y, y_\Pi \rangle \leq 1; \ \langle y, f \rangle \geq R, \ \forall f \in \mathcal{K} \end{cases} \tag{7}$$

$y \in Y$ being the decision variable and $R > 1$. Straightforward extensions of several arguments

in Balbás *et al.* (2016*b*) can show that the dual problem of (7) is

$$
Max \ (R-1)\lambda - 1, \quad
\begin{cases}
z + \lambda f = (\lambda + 1)\, y_\Pi \\
f \in \mathcal{K};\ z \in \Delta;\ \lambda \geq 0
\end{cases}
\tag{8}
$$

$(\lambda, f, z) \in \mathbb{R} \times Y \times Y$ being the decision variable. Obviously, the solution of (8) (if it exists) does not depend on $R > 1$ and may be found by solving

$$
Max \ \lambda, \quad
\begin{cases}
z + \lambda f = (\lambda + 1)\, y_\Pi \\
f \in \mathcal{K};\ z \in \Delta;\ \lambda \geq 0
\end{cases}
\tag{9}
$$

Though we are dealing with an abstract approach, the relationships between (7) and (9) are quite similar to those found in Balbás *et al.* (2016*b*) for a general portfolio choice problem (see Examples 1 and 2 below), and therefore their proofs will be omitted. Moreover, in order to improve the reader intuition, ρ will be called "robust risk", and for $y \in Y$ such that $\Pi(y) = 1$ the value

$$
Min \ \{\langle y, f \rangle\ ;\ f \in \mathcal{K})\}
\tag{10}
$$

will be called "robust return relative to $\mathcal{K}$".

The change of variable

$$
\beta = \frac{1}{1+\lambda}, \quad or \quad \lambda = \frac{1-\beta}{\beta}
\tag{11}
$$

leads to Problem

$$
Min \ \beta, \quad
\begin{cases}
\beta z + (1-\beta) f = y_\Pi \\
f \in \mathcal{K},\ z \in \Delta,\ 0 \leq \beta \leq 1
\end{cases}
\tag{12}
$$

which is equivalent to (9) and has a solution (β^*, f^*, z^*) if it is feasible. Remark 1 below shows that there may be four disjoint and complementary situations.

Remark 1 *Case 1. $y_\Pi \in \mathcal{K}$. In such a case (7) is not feasible if $R > 1$. We will say that the market is $\mathcal{K}-$risk-neutral. Besides, (λ, y_Π, y_Π) is obviously (9)$-$feasible for every $\lambda \geq 0$, and both (8) and (9) are feasible and unbounded (the (9)-optimal value becomes $\lambda^* = \infty$). $(0, y_\Pi, y_\Pi)$ is (12)$-$feasible and therefore $\beta^* = 0$ is the optimal value of (12). We will say that the $(\Pi, \mathcal{K}, \Delta)-$capital market line (($\Pi, \mathcal{K}, \Delta) - CML$) is horizontal, and the market price of risk (MPR) $M_{(\Pi, \mathcal{K}, \Delta)} = 0$ will vanish (under the framework of Examples 1 and 2 below every robust*

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expected return equals 1 *and does not depend on the portfolio risk).*

Case 2. $y_\Pi \notin \mathcal{K}$, $y_\Pi \in \Delta$ *and* $0 < \beta^* < 1$ *(and therefore* $\infty > \lambda^* > 0$*). Taking into account the objective function of (8), the optimal risk is* $\rho = (R-1)\lambda^* - 1$ *or, equivalently,*

$$R = \frac{1}{\lambda^*}(\rho + 1) + 1 = \frac{\beta^*}{1-\beta^*}(\rho + 1) + 1 \tag{13}$$

yields the relationship between the optimal risk and the guaranteed (or robust) expected return. We will say that

$$M_{(\Pi,\mathcal{K},\Delta)} = \frac{\beta^*}{1-\beta^*} \tag{14}$$

is the MPR*, and (13) will be called the* $(\Pi, \mathcal{K}, \Delta) - CML$.

Case 3. $y_\Pi \notin \mathcal{K}$, $y_\Pi \in \Delta$ *and* $\beta^* = 1$ *(and therefore* $\lambda^* = 0$*). (7) (8) and (9) are feasible, but now the null optimal value of (9) implies that the optimal robust risk satisfie (see (8))*

$$\rho = \lambda^*(R-1) - 1 = -1$$

and it does not depend on R*. In fact, one can construct sequences of portfolios (or (7)-feasible elements) whose robust expected return is as large as desired (* R *tends to* $+\infty$*) while their risk is as close to* -1 *as desired. There is no* MPR *because every guaranteed expected return is reached with a similar risk level. The* $(\Pi, \mathcal{K}, \Delta) - CML$ *is vertical, and we will accept the convention* $M_{(\Pi,\mathcal{K},\Delta)} = \infty$.[3]

Case 4. $y_\Pi \notin \Delta$*. Then, (7) is unbounded for every* $R > 1$ *and one can construct a sequence of portfolios guaranteeing an expected return as large as desired (* R *tends to* $+\infty$*) and whose risk is as negative as desired (* ρ *tends to* $-\infty$*). Once again, we will accept the convention above* $M_{(\Pi,\mathcal{K},\Delta)} = \infty$. $\qquad\square$

Cases 1 and 2 above may be natural and, extending the approach of Balbás *et al.* (2010), we will say that $(\Pi, \mathcal{K}, \Delta)$ is strongly compatible (and compatible) if they hold. If Case 3 holds then we will say that $(\Pi, \mathcal{K}, \Delta)$ is compatible but it is not strongly compatible. If Case 4 holds we will say that $(\Pi, \mathcal{K}, \Delta)$ is not compatible.

Remark 2 *(Recovering compatibility) Though our framework is more general, following a proof*

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[3]Throughout this paper we will accept the convention $\dfrac{\beta}{1-\beta} = \infty$ if $\beta = 1$.

of Balbás and Balbás (2009) one can replace ρ with a new "quite similar risk measure $\tilde{\rho}$" given by

$$\tilde{\rho}(y) := Max \ \{\rho(y), -\Pi(y)\} \tag{15}$$

for every $y \in Y$. It is easy to see that (2), (3) and (5) still hold if $\tilde{\rho}$ replaces ρ and

$$\tilde{\Delta} = \{wz + (1 - w) y_\Pi; \ z \in \Delta, \ 0 \le w \le 1\} \supset \Delta \tag{16}$$

replaces $\tilde{\Delta}$. $\left(\Pi, \mathcal{K}, \tilde{\Delta}\right)$ is compatible ((16) trivially shows that $y_\Pi \in \tilde{\Delta}$) and will be called the compatible modificatio of $(\Pi, \mathcal{K}, \Delta)$. Moreover, $\tilde{\rho} = \rho$ if and only if $(\Pi, \mathcal{K}, \Delta)$ is compatible, and if and only if $\tilde{\Delta} = \Delta$. $\qquad\square$

Suppose that $(\Pi, \mathcal{K}, \Delta)$ is not compatible. $\tilde{\rho}$ permits us to recover compatibility. However, Proposition 1 below shows that there are no "minor" modification of ρ allowing us to reach strong compatibility. Besides, dealing with particular cases of Example 1 below, Balbás *et al.* (2010) have shown that for many important pricing models (Black and Scholes, Heston, other stochastic volatility models, etc.) and risk measures ($CVaR$, spectral risk measures, etc.) "minor" modifi cations of $\tilde{\rho}$ will not solve this caveat either.[4]

Proposition 1 *a) $M_{(\Pi,\mathcal{K},\Delta)} = 0$ if and only if $M_{\left(\Pi,\mathcal{K},\tilde{\Delta}\right)} = 0$. In other words, $(\Pi, \mathcal{K}, \Delta)$ satisfie Case 1 of Remark 1 if and only if $\left(\Pi, \mathcal{K}, \tilde{\Delta}\right)$ satisfie Case 1 of Remark 1.*

b) $0 < M_{(\Pi,\mathcal{K},\Delta)} < \infty$ if and only if $0 < M_{\left(\Pi,\mathcal{K},\tilde{\Delta}\right)} < \infty$. In other words, $(\Pi, \mathcal{K}, \Delta)$ satisfie Case 2 of Remark 1 if and only if $\left(\Pi, \mathcal{K}, \tilde{\Delta}\right)$ satisfie Case 2 of Remark 1. If so, $M_{\left(\Pi,\mathcal{K},\tilde{\Delta}\right)} = M_{(\Pi,\mathcal{K},\Delta)}$.

c) $M_{(\Pi,\mathcal{K},\Delta)} = \infty$ if and only if $M_{\left(\Pi,\mathcal{K},\tilde{\Delta}\right)} = \infty$. In other words, $(\Pi, \mathcal{K}, \Delta)$ satisfie Cases 3 or 4 of Remark 1 if and only if $\left(\Pi, \mathcal{K}, \tilde{\Delta}\right)$ satisfie Case 3 of Remark 1.

d) $0 \le M_{(\Pi,\mathcal{K},\Delta)} = M_{\left(\Pi,\mathcal{K},\tilde{\Delta}\right)} \le \infty$. Consequently, $\left(\Pi, \mathcal{K}, \tilde{\Delta}\right)$ is strongly compatible if and only if $(\Pi, \mathcal{K}, \Delta)$ is strongly compatible.

Proof. *a*) According to Remark 1, both $M_{(\Pi,\mathcal{K},\Delta)} = 0$ and $M_{\left(\Pi,\mathcal{K},\tilde{\Delta}\right)} = 0$ are equivalent to $y_\Pi \in \mathcal{K}$.

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[4]See also Stoica and Lib (2010).

b) If $0 < M_{(\Pi,\mathcal{K},\Delta)} < \infty$ then $(\Pi,\mathcal{K},\Delta)$ is compatible, and therefore $\tilde{\Delta} = \Delta$, $\tilde{\rho} = \rho$ and $M_{(\Pi,\mathcal{K},\tilde{\Delta})} = M_{(\Pi,\mathcal{K},\Delta)}$. Conversely, suppose that $0 < M_{(\Pi,\mathcal{K},\tilde{\Delta})} < \infty$. $M_{(\Pi,\mathcal{K},\Delta)} = 0$ cannot hold due to $a)$. If we show that $M_{(\Pi,\mathcal{K},\Delta)} = \infty$ cannot hold either the proof will be ended. Suppose that $M_{(\Pi,\mathcal{K},\Delta)} = \infty$ and consider $0 < \beta < 1$ solving (12) when $\tilde{\Delta}$ replaces Δ. There exist $0 \leq w \leq 1$, $f \in \mathcal{K}$ and $z \in \Delta$ such that (see(16))

$$\beta \left[wz + (1-w)\, y_\Pi \right] + (1-\beta)\, f = y_\Pi.$$

$1 - \beta > 0$ and $w\beta \geq 0$ imply that $1 - \beta + w\beta > 0$. Manipulating,

$$\frac{w\beta}{1-\beta+w\beta} z + \frac{1-\beta}{1-\beta+w\beta} f = y_\Pi.$$

Thus, $\left(\frac{w\beta}{1-\beta+w\beta}, f, z \right)$ is (12)-feasible, and therefore (12) has feasible solutions and its optimal value cannot be higher than $\frac{w\beta}{1-\beta+w\beta} < 1$.

c) It is an obvious consequence of $a)$ and $b)$, along with the property $y_\Pi \in \tilde{\Delta}$ (see(16)).

d) It is an obvious consequence of $a)$, $b)$ and $c)$. $\qquad\square$

Remark 3 *Suppose that $V \subset Y$ is a closed subspace containing U. Consider the classical projection $\varphi_V : Y \to V$.[5] The restriction of Π to V is given by $\Pi(v) = \langle v, \varphi_V(y_\Pi) \rangle$ and the role of $\mathcal{K}$ and Δ may be played by $\varphi_V(\mathcal{K})$ and $\varphi_V(\Delta)$ (see (20)). In other words, if (7) and (12) are*

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[5]If $C \subset Y$ is convex and closed then there exists a continuous projection $\varphi_C : Y \to C$. If $y \in Y$ then $\varphi_C(y)$ is characterized by two equivalent conditions (Choquet, 1966);
 1) $\varphi_C(y) \in C$ and
$$\|y - \varphi_C(y)\| < \|y - c\| \tag{17}$$
for every $c \in C$ such that $c \neq \varphi_C(y)$.
 2) $\varphi_C(y) \in C$ and
$$\langle y - \varphi_C(y), \varphi_C(y) \rangle \geq \langle y - \varphi_C(y), c \rangle \tag{18}$$
for every $c \in C$.
   If C is a closed subspace then φ_C is linear, it is called orthogonal projection, and for every $y \in Y$ we have that $\varphi_C(y)$ is characterized by;
 3) $\varphi_C(y) \in C$ and
$$y - \varphi_C(y) \in C^\perp, \tag{19}$$
$C^\perp$ denoting subspace orthogonal to C. In particular, $\langle y - \varphi_C(y), c \rangle = 0$ and therefore
$$\langle y, c \rangle = \langle \varphi_C(y), c \rangle \tag{20}$$
for every $y \in Y$ and every $c \in C$.
   If C is a closed subspace and $a \in Y$ then for the affin manifold $a + C$ we have that
$$\varphi_{a+C}(y) = a + \varphi_C(y - a) \tag{21}$$

restricted to V then they become

$$\begin{cases} Min \; \rho\left(v\right) \\ v \in V; \; \left\langle v, \varphi_V\left(y_\Pi\right)\right\rangle \leq 1; \; \left\langle v, f\right\rangle \geq R, \; \forall f \in \varphi_V\left(\mathcal{K}\right) \end{cases} \tag{23}$$

and

$$Min \; \beta, \; \begin{cases} \beta z + \left(1 - \beta\right) f = \varphi_V\left(y_\Pi\right) \\ f \in \varphi_V\left(\mathcal{K}\right); \; z \in \varphi_V\left(\Delta\right); \; 0 \leq \beta \leq 1 \end{cases} \tag{24}$$

respectively, and the remarks above still apply. If $\left(\beta, f, z\right)$ is (12)-feasible we have that

$$\varphi_V\left(\beta z + \left(1 - \beta\right) f\right) = \varphi_V\left(y_\Pi\right),$$

and $\left(\beta, \varphi_V(f), \varphi_V(z)\right)$ is (24)-feasible. The obvious consequence is that the optimal value of (24) is equal to or lower than the optimal value of (12), and therefore $M_{\left(\Pi, \varphi_V(\mathcal{K}), \varphi_V(\Delta)\right)} \leq M_{\left(\Pi, \mathcal{K}, \Delta\right)}$.[6] In particular, $M_{\left(\Pi, \varphi_U(\mathcal{K}), \varphi_U(\Delta)\right)} \leq M_{\left(\Pi, \mathcal{K}, \Delta\right)}$. $\qquad \square$

Remark 4 *If $V \subset Y$ is a closed subspace containing U then (16) leads to*

$$\varphi_V(\tilde{\Delta}) = \{wz + \left(1 - w\right)\varphi_V(y_\Pi); \; z \in \varphi_V(\Delta), \; 0 \leq w \leq 1\}$$

which implies that $\left(\Pi, \varphi_V(\mathcal{K}), \varphi_V(\tilde{\Delta})\right)$ is the compatible modificatio of $\left(\Pi, \varphi_V(\mathcal{K}), \varphi_V(\Delta)\right)$. In particular, $M_{\left(\Pi, \varphi_V(\mathcal{K}), \varphi_V(\tilde{\Delta})\right)} = M_{\left(\Pi, \varphi_V(\mathcal{K}), \varphi_V(\Delta)\right)}$. $\qquad \square$

Example 1 (Non-ambiguous and ambiguous complete pricing models) *The abstract approach above contains the standard framework of every complete pricing model. Indeed, consider a probability space $\left(\Omega, \mathcal{F}, \mathbb{P}\right)$ composed of the set of states of nature Ω that may arise at a future date T, the $\sigma-$algebra $\mathcal{F}$ reflectin the information available at T, and the probability measure $\mathbb{P}$. $Y = L^2\left(\mathcal{F}\right)$ will represent the Hilbert space of random variables on $\left(\Omega, \mathcal{F}, \mathbb{P}\right)$ whose expectation and variance are finit , endowed with the usual inner product and Euclidean norm. There will exist a linear and continuous function providing investors with the current price $\Pi\left(y\right) \in \mathbb{R}$*

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for every $y \in Y$. Furthermore, if $\tilde{C} \subset a + C$ is a closed convex set, then

$$\varphi_{\tilde{C}} = \varphi_{\tilde{C}} \circ \varphi_{a+C}. \tag{22}$$

[6]In order to simplify several expressions, we still denote by Π the restriction of Π to V.

of every marketed claim $y \in L^2(\mathcal{F})$. If $\mathbb{E}(.)$ represents mathematical expectation with respect to $\mathbb{P}$, then the stochastic discount factor (SDF) is the unique $y_\Pi \in L^2(\mathcal{F})$ such that

$$\Pi(y) = \mathbb{E}(y_\Pi y) \tag{25}$$

for every $y \in L^2(\mathcal{F})$.

A closed subspace $U \subset Y$ will be composed of those pay-offs at T whose current price does not depend on the pricing model we are dealing with. U may be said to be the space of price-invariant marketed claims (or price-invariant assets). For instance, in practice we could consider that $\Pi(u)$ is directly given by a market quotation, for every $u \in U$.

The role of u_0 will be played by a riskless asset $1 \in U$, and (1) will imply the existence of a null riskless rate (if the interest rate r_f is non null then we will assume that every pay-off has been multiplied by the discount factor $e^{-r_f T}$, and therefore every $y \in L^2(\mathcal{F})$ actually represents the present value of a related pay-off at T).

In order to introduce $\mathcal{K}$ we can consider ambiguous investors whose set of priors contains $\mathbb{P}$, and then deal with the framework of Balbás et al. (2016b). Therefore, $\mathcal{K} \subset L^2(\mathcal{F})$ will be a convex and weakly compact set containing the zero variance random variable $u_0 = 1$ (so $u_0 \in \mathcal{K}$) and such that $\mathbb{P}(f \geq 0) = 1$ and $\mathbb{E}(f) = 1$ for every $f \in \mathcal{K}$. The set of priors (or ambiguity set) will be composed of the $\mathbb{P}-$continuous probability measures p whose Radon-Nikodym derivative $\frac{dp}{d\mathbb{P}}$ with respect to $\mathbb{P}$ belongs to $\mathcal{K}$ ($\frac{dp}{d\mathbb{P}} \in \mathcal{K}$).

The ambiguous setting above contains many particular cases. For instance, due to the Alaoglu's theorem and the Hahn-Banach theorem (Kopp, 1984), $\mathcal{K}$ may be the intersection of a closed interval $[S_1, S_2] \subset L^2(\mathcal{F})$ and the subspace of random variables whose expectation equals 1, i.e.,

$$\mathcal{K} = \mathcal{K}[S_1, S_2] = \left\{ f \in L^2(\mathcal{F}); \ \mathbb{E}(f) = 1, \ S_1 \leq f \leq S_2 \right\}, \tag{26}$$

where $0 \leq S_1 \leq 1 \leq S_2$ and S_1 and S_2 are two arbitrary random variables of $L^2(\mathcal{F})$. Firstly, if $S_1 = 1$ or $S_2 = 1$ then the non-ambiguous (or uncertainty free) case will be included in our framework, because $\mathcal{K}[S_1, S_2] = \{1\}$ will become a singleton, and the probabilities of the states of nature will be known and given by $\mathbb{P}$. Secondly, if $S_1 \neq 1$ and $S_2 \neq 1$ then $\mathbb{P}$ may be interpreted as an estimated probability measure containing possible errors, which makes the investor incorporate the "spread" $S_1 \leq 1 \leq S_2$ indicating the estimation accuracy (the accuracy

increases as $S_2 - 1 \geq 0$ and $1 - S_1 \geq 0$ decrease).

With respect to ρ and Δ, they become natural if we follow the approach of Balbás et al. (2016b) in order to measure risks under ambiguity. A risk measure will be a function $\rho : L^2(\mathcal{F}) \longrightarrow \mathbb{R}$ such that there exists a convex and weakly compact subset Δ of $L^2(\mathcal{F})$ with

$$\rho(y) = Max \ \{-\mathbb{E}(yz); \ z \in \Delta\} \tag{27}$$

for every $y \in L^2(\mathcal{F})$.

In the ambiguity free case ($\mathcal{K} = \{1\}$) there are many risk measures. For instance, every expectation bounded risk measure (Rockafellar et al., 2006) and many coherent risk measures (Artzner et al., 1999). In more general cases we can defin "the robust extension of a risk measure relative to the set of priors $\mathcal{K}$". For instance, the robust $CVaR$ with confidenc level $0 \leq \mu < 1$ relative to the set of priors $\mathcal{K}$ will be given by

$$RCVaR_{(\mathcal{K},\mu)}(y) := Max \ \left\{CVaR_{(\{p\},\mu)}(y); \ \frac{dp}{d\mathbb{P}} \in \mathcal{K}\right\} \tag{28}$$

for every $y \in L^2(\mathcal{F})$, $CVaR_{(\{p\},\mu)}(y)$ denoting the usual $CVaR$ of y if p is the selected probability measure and μ is the level of confidenc . If there exists an upper bound of $\mathcal{K}$ (an element $S \in L^2(\mathcal{F})$ such that $f \leq S$ holds for every $f \in \mathcal{K}$) then Balbás et al. (2016b) show that (28) is well define and satisfie the required conditions with

$$\mathcal{K} \subset \Delta = \left\{z \in L^2(\mathcal{F}); \mathbb{E}(z) = 1 \ and \ \exists f \in \mathcal{K} \ with \ 0 \leq z \leq \frac{f}{1 - \mu}\right\}. \tag{29}$$

Analogously, bearing in mind (15) and (25), we can defin the compatible robust $CVaR$ with confidenc level $0 \leq \mu < 1$ relative to the set of priors $\mathcal{K}$ by means of

$$CRCVaR_{(\mathcal{K},\mu)}(y) := Max \ \left\{RCVaR_{(\mathcal{K},\mu)}(y), -\mathbb{E}(y_\Pi y)\right\}$$

for every $y \in L^2(\mathcal{F})$, and (2) and (27) will hold again if $\tilde{\Delta}$ replaces Δ, where Δ is given by (29) and $\tilde{\Delta}$ is given by (16).

Problem (7) may be understood as a classical portfolio choice involving both, the (robust) risk measure ρ and the robust expected return relative to the set set of priors $\mathcal{K}$, which particularizes

(10) and is given by

$$\mathbb{E}_{\mathcal{K}}(y) := Min \ \{ \ \mathbb{E}(yf) \, ; \ f \in \mathcal{K}) \} \tag{30}$$

for every priced one $y \in L^2(\mathcal{F})$. □

Example 2 (Incomplete pricing models) *The abstract setting also applies for incomplete pricing models. Indeed, under the notations of Example 1, the role of $L^2(\mathcal{F})$ may be played by a proper closed subspace $Y \subset L^2(\mathcal{F})$ containing reachable marketed claims. The pricing rule will be still given by (25) for some random variable $y_\Pi \in Y$ still called SDF. The existence of an available riskless asset $u_0 = 1 \in Y$ and a null interest rate must be imposed too, as well as the property $1 \in U \subset Y$. The risk measure (27) must be define as a function whose domain is $L^2(\mathcal{F})$, but (19) and (20) trivially imply that*

$$\rho(y) = Max \ \{- \, \mathbb{E}(yz) \, ; \ z \in \varphi_Y(\Delta) \}$$

for every $y \in Y$, and therefore the role of Δ may be played by $\varphi_Y(\Delta)$. Similarly, with respect to the ambiguity set we have that $\mathbb{E}(fy) = \mathbb{E}(\varphi_Y(f)y)$ for every $y \in Y$ and every $f \in \mathcal{K}$, so the robust expected return may be given by $\mathbb{E}_{\mathcal{K}}(y) = Min \ \{ \, \mathbb{E}(yf) \, ; \ f \in \varphi_Y(\mathcal{K}) \}$ for every priced one $y \in Y$, and therefore $\varphi_Y(\mathcal{K})$ may play the role of $\mathcal{K}$. Finally, according to (12) and Remark 1, if $(\Pi, \varphi_Y(\mathcal{K}), \varphi_Y(\Delta))$ is strongly compatible then there exists $(f, z, \beta) \in \mathcal{K} \times \Delta \times [0,1)$ such that $\beta \varphi_Y(z) + (1 - \beta)\varphi_Y(f) = y_\Pi$. Hence, (20) and (25) imply that $\Pi(y) = \mathbb{E}(y_\Pi y) = \beta \mathbb{E}(\varphi_Y(z)y) + (1 - \beta)\mathbb{E}(\varphi_Y(f)y) = \beta \mathbb{E}(zy) + (1 - \beta)\mathbb{E}(fy)$. If

$$\mathbb{P}(z \geq 0) := 1 \quad and \quad \mathbb{P}(f > 0) = 1 \tag{31}$$

then the implication

$$\mathbb{P}(y \geq 0) = 1, \ \mathbb{P}(y > 0) > 0 \Longrightarrow \Pi(y) > 0$$

becomes obvious because $1 - \beta > 0$. In other words, the absence of arbitrage is guaranteed under (31) and strong compatibility. Furthermore, this argument also applies for Example 1, which is a particular case of Example 2. Condition (31) holds for every coherent risk measure (Artzner et al., 1999) and many ambiguity sets such as $\mathcal{K} = \{1\}$ (uncertainty free case) or, more generally, $\mathcal{K}$ given by (26) with $\mathbb{P}(S_1 > 0) = 1$. □

3. Recovering strong compatibility

According to Proposition 1 above, if Case 3 or Case 4 hold then a modificatio of ρ will be insuffi cient to recover strong compatibility. The only way to prevent this caveat will be the modificatio of y_Π, and therefore the modificatio of Π. This is the focus of this section. We will modify Π so as to recover strong compatibility, though the price of every strategy in the stable subspace U will remain the same. If $M_{(\Pi,\varphi_U(\mathcal{K}),\varphi_U(\Delta))} = \infty$ and we do not modify the restriction of Π to U then the MPR on U will still equal infinit , and Remark 3 will show that Case 3 or Case 4 will still hold on Y. Thus, we will impose;

Assumption 1 $M_{(\Pi,\varphi_U(\mathcal{K}),\varphi_U(\Delta))} < \infty$, *i.e.,* $(\Pi, \varphi_U(\mathcal{K}), \varphi_U(\Delta))$ *is strongly compatible.* □

Proposition 2 *If* $U = \mathcal{L}\{u_0\}$ *then* $M_{(\Pi,\varphi_U(\mathcal{K}),\varphi_U(\Delta))} = 0$*, and therefore Assumption* 1 *holds and* $M_{\left(\Pi,\varphi_U(\mathcal{K}),\varphi_U(\tilde{\Delta})\right)} = 0.$

Proof. Expression (1) gives $\langle u_0, y_\Pi \rangle = 1$, and (20) leads to $\langle u_0, \varphi_U(y_\Pi)\rangle = 1$. Since $\varphi_U(y_\Pi) = \alpha u_0$ for some real number α, $\alpha = \langle u_0, \alpha u_0 \rangle = \langle u_0, \varphi_U(y_\Pi)\rangle = 1$, and therefore $\varphi_U(y_\Pi) = u_0$. Since $u_0 \in \mathcal{K}$, we have that $u_0 = \varphi_U(u_0) \in \varphi_U(\mathcal{K})$, *i.e.,* $\varphi_U(y_\Pi) \in \varphi_U(\mathcal{K})$. Thus, the rest of the proof trivially follows from Remarks 1, 3 and 4. □

For every $0 \leq \beta \leq 1$ we will consider the (obviously continuous) function

$$Y \times Y \ni (f, z) \longmapsto \Phi_\beta(f, z) := \beta z + (1 - \beta) f \in Y, \tag{32}$$

and the sets

$$\begin{cases} \Delta^{(\beta} := \Phi_\beta(\mathcal{K} \times \Delta) = \{\Phi_\beta(f, z)\,;\ f \in \mathcal{K},\ z \in \Delta\} \subset Y, \\ C^{(\beta} := \Delta^{(\beta} \cap (y_\Pi + U^\perp) \subset y_\Pi + U^\perp, \\ \tilde{\Delta}^{(\beta} := \Phi_\beta\left(\mathcal{K} \times \tilde{\Delta}\right) = \left\{\Phi_\beta(f, z)\,;\ f \in \mathcal{K},\ z \in \tilde{\Delta}\right\} \subset Y, \\ \tilde{C}^{(\beta} := \tilde{\Delta}^{(\beta} \cap (y_\Pi + U^\perp) \subset y_\Pi + U^\perp. \end{cases} \tag{33}$$

Proposition 3 *a) If* $0 \leq \alpha \leq 1$ *and* $0 \leq \beta \leq 1$ *then* $\Phi_\alpha(f, \Phi_\beta(f, z)) = \Phi_{\alpha\beta}(f, z)$ *for every* $(f, z) \in Y \times Y$.

b) $\Delta^{(\beta}$ *is convex and weakly compact for every* $0 \leq \beta \leq 1$*. Furthermore,* $\Delta^{(0} = \mathcal{K}$ *and* $\Delta^{(1} = \Delta$.

c) $\mathcal{K} \subset \Delta^{(\beta} \subset \Delta$ for every $0 \leq \beta \leq 1$. In particular, $\langle z, u_0 \rangle = 1$ for every $0 \leq \beta \leq 1$ and for every $z \in \Delta^{(\beta}$.

d) The net $\left(\Delta^{(\beta} \right)_{0 \leq \beta \leq 1}$ is increasing, i.e., $\Delta^{(\beta_1} \subset \Delta^{(\beta_2}$ if $0 \leq \beta_1 \leq \beta_2 \leq 1$.

Proof. a) The proof is straightforward and therefore omitted.

b) The convexity of $\Delta^{(\beta}$ trivially follows from the convexity of $\Delta \times \mathcal{K}$. The weak compactness of $\Delta^{(\beta}$ follows from the weak compactness of $\Delta \times \mathcal{K}$ and the weak continuity of Φ_β. Equalities $\Delta^{(0} = \mathcal{K}$ and $\Delta^{(1} = \Delta$ are trivial.

c) (3) states that $\mathcal{K} \subset \Delta$. Therefore, if $f \in \mathcal{K} \subset \Delta$ we have that $f = \Phi_\beta(f, f) \in \Phi_\beta(\mathcal{K} \times \mathcal{K}) \subset \Phi_\beta(\mathcal{K} \times \Delta) = \Delta^{(\beta}$. Besides, if $\Phi_\beta(f, z) \in \Delta^{(\beta}$ with $z \in \Delta$ and $f \in \mathcal{K} \subset \Delta$ then $\Phi_\beta(f, z)$ is a convex combination of two elements of Δ, and therefore $\Phi_\beta(f, z) \in \Delta$ because this set is convex. Finally, the equality $\langle z, u_0 \rangle = 1$ for every $0 \leq \beta \leq 1$ and every $z \in \Delta^{(\beta}$ trivially follows from $\Delta^{(\beta} \subset \Delta$ and (4).

d) If $\beta_2 = 0$ then $\beta_1 = 0$, and the result is obvious.

Suppose that $0 < \beta_2$ and take $\alpha = \dfrac{\beta_1}{\beta_2}$. Suppose that $\tilde{z} = \Phi_{\beta_1}(f, z) \in \Delta^{(\beta_1}$ with $f \in \mathcal{K}$ and $z \in \Delta$. Statement a) and the inclusion $\Delta^{(\alpha} \subset \Delta$ imply that

$$\tilde{z} = \Phi_{\beta_2}(f, \Phi_\alpha(f, z)) \in \Phi_{\beta_2}(\mathcal{K} \times \Delta_\alpha) \subset \Phi_{\beta_2}(\mathcal{K} \times \Delta) = \Delta^{(\beta_2}.$$

$\square$

Proposition 4 If $0 \leq \beta \leq 1$ and $M_{(\Pi_U, \varphi_U(\mathcal{K}), \varphi_U(\Delta))} \leq \dfrac{\beta}{1 - \beta} \leq \infty$ then $C^{(\beta} \neq \varnothing$ (see (33)). Furthermore, if $U = \mathcal{L}\{u_0\}$ then $\Delta^{(\beta} \subset \Delta \subset y_\Pi + U^\perp$ and $C^{(\beta} = \Delta^{(\beta}$.

Proof. Consider $V = U$ and Problem (24). Remark 1 and Assumption 1 imply that this problem is feasible and solvable, and its optimal value is lower than one. If $(\beta_U, f_U, z_U) \in [0, 1) \times \mathcal{K} \times \Delta$ is such that $(\beta_U, \varphi_U(f_U), \varphi_U(z_U))$ solves (24), then (14) shows that

$$M_{(\Pi, \varphi_U(\mathcal{K}), \varphi_U(\Delta))} = \dfrac{\beta_U}{1 - \beta_U}.$$

Obviously, $\beta_U \leq \beta$ because $M_{(\Pi, \varphi_U(\mathcal{K}), \varphi_U(\Delta))} \leq \dfrac{\beta}{1-\beta}$, and therefore (see Proposition $3d$) $\Delta^{(\beta_U} \subset \Delta^{(\beta} \Rightarrow C^{(\beta_U} \subset C^{(\beta}$. Thus, it is sufficien to see that $C^{(\beta_U} \neq \varnothing$. We have that

$$\varphi_U(\beta_U z_U + (1 - \beta_U) f_U) = \varphi_U(y_\Pi) \tag{34}$$

owing to the constraints of (24). Therefore, (19) and (34) imply that

$$\beta_U z_U + (1 - \beta_U) f_U - \varphi_U(y_\Pi) \in U^\perp.$$

Similarly, (19) implies that $y_\Pi - \varphi_U(y_\Pi) \in U^\perp$. Therefore

$$\beta_U z_U + (1 - \beta_U) f_U - y_\Pi = (\beta_U z_U + (1 - \beta_U) f_U - \varphi_U(y_\Pi)) + (\varphi_U(y_\Pi) - y_\Pi) \in U^\perp,$$

and $\beta_U z_U + (1 - \beta_U) f_U \in y_\Pi + U^\perp$. Besides, $\beta_U z_U + (1 - \beta_U) f_U \in \Delta^{(\beta_U}$ is obvious.

Next, suppose that $U = \mathcal{L}\{u_0\}$. In order to prove that $\Delta \subset y_\Pi + U^\perp$ consider $z \in \Delta$. (1) and (4) show that $\langle u_0, z - y_\Pi \rangle = 0$, and therefore $z - y_\Pi \in U^\perp$ and $z \in y_\Pi + U^\perp$. $\qquad\square$

Since $C^{(\beta}$ is obviously convex and weakly compact, Proposition 4 allows us to defin

$$\begin{cases} y^{(\beta} = \varphi_{C^{(\beta}}(y_\Pi), \\ \tilde{y}^{(\beta} = \varphi_{\tilde{C}^{(\beta}}(y_\Pi) \end{cases} \tag{35}$$

for every $0 \leq \beta \leq 1$ such that $M_{(\Pi, \varphi_U(\mathcal{K}), \varphi_U(\Delta))} \leq \dfrac{\beta}{1-\beta} \leq \infty$.

As said above, the main purpose of this section is to yield an adequate modificatio of the pricing rule. Suppose that y' replaces y_Π (or Π' replaces Π). The Cauchy-Schwarz inequality leads to

$$|\langle y_\Pi - y', y \rangle| \leq \|y\| \, \|y_\Pi - y'\| \tag{36}$$

for every $y \in Y$. This suggests that the price modificatio will be as small as possible if $\|y_\Pi - y'\|$ is minimized. Next, let us prove that the lack of strong compatibility may be overcome in such a manner that $\|y_\Pi - y'\|$ is minimized and prices remain the same in the stable subspace.

Theorem 5 *Take $\beta \in [0, 1)$ such that $M_{(\Pi, \varphi_U(\mathcal{K}), \varphi_U(\Delta))} \leq \dfrac{\beta}{1-\beta}$. Consider the pricing rule $\Pi^{(\beta}(y) = \langle y, y^{(\beta} \rangle$ for every $y \in Y$.*

a) $\left(\Pi^{(\beta}, \mathcal{K}, \Delta\right)$ *is strongly compatible and* $M_{\left(\Pi^{(\beta}, \mathcal{K}, \Delta\right)} \leq \dfrac{\beta}{1 - \beta}$.

b) $\Pi^{(\beta}(u) = \Pi(u)$ *for every* $u \in U$, *and therefore* $M_{(\Pi, \varphi_U(\mathcal{K}), \varphi_U(\Delta))}$ *remains the same if* $\Pi^{(\beta}$ *replaces* Π.

c) *Suppose that* $y' \in Y$ *and* $Y \ni y \to \Pi'(y) = \langle y, y' \rangle \in \mathbb{R}$ *is a linear and continuous function such that* $\Pi'(u) = \Pi(u)$ *for every* $u \in U$ *and* $M_{(\Pi', \mathcal{K}, \Delta)} \leq \dfrac{\beta}{1 - \beta}$. *Then* $\left\| y_\Pi - y^{(\beta} \right\| \leq \left\| y_\Pi - y' \right\|$, *and* $\left\| y_\Pi - y^{(\beta} \right\| = \left\| y_\Pi - y' \right\|$ *if and only if* $y' = y^{(\beta}$.

Proof. If $y^{(\beta}$ replaces y_Π then Problem (12) becomes

$$
Min \; \gamma, \quad
\begin{cases}
\gamma z + (1 - \gamma) f = y^{(\beta} \\
f \in \mathcal{K}; \; z \in \Delta; \; 0 \leq \gamma \leq 1
\end{cases}
\tag{37}
$$

$(\gamma, f, z) \in \mathbb{R} \times Y \times Y$ being the decision variable.

a) Since $y^{(\beta} \in C^{(\beta} \subset \Delta^{(\beta}$, there exists $(f, z) \in \mathcal{K} \times \Delta$ such that $y^{(\beta} = \beta z + (1 - \beta) f$. Hence, (β, f, z) is $(37)-$feasible and the optimal value γ_0 of this problem exists (Remark 1) and will satisfy $\gamma_0 \leq \beta$. According to Remark 1,

$$
M_{\left(\Pi^{(\beta}, \mathcal{K}, \Delta\right)} = \frac{\gamma_0}{1 - \gamma_0} \leq \frac{\beta}{1 - \beta}.
$$

b) Since $y^{(\beta} \in C_\beta \subset y_\Pi + U^\perp$, we have that $\langle u, y^{(\beta} - y_\Pi \rangle = 0$ holds for every $u \in U$.

c) Suppose that $y' \in Y$ satisfie the required conditions, and take $\gamma \in [0, 1)$ such that $M_{(\Pi', \mathcal{K}, \Delta)} = \dfrac{\gamma}{1 - \gamma} \leq \dfrac{\beta}{1 - \beta}$. Take $(\gamma, f, z) \in \mathbb{R} \times \mathcal{K} \times \Delta$ solving (12) when y' replaces y_Π. $y' = \gamma z + (1 - \gamma) f$ must hold, and therefore $y' \in \Delta^{(\gamma} \subset \Delta^{(\beta}$ (see Proposition 3d). Besides, $\Pi'(u) = \Pi(u)$ for every $u \in U$ implies that $\langle y, y' - y_\Pi \rangle = 0$ for every $u \in U$, and therefore $y' \in y_\Pi + U^\perp$. Consequently, $y' \in C^{(\beta}$, and (17) leads to $\left\| y_\Pi - y^{(\beta} \right\| \leq \left\| y_\Pi - y' \right\|$ and $\left\| y_\Pi - y^{(\beta} \right\| = \left\| y_\Pi - y' \right\|$ if and only if $y' = y^{(\beta}$. $\quad\square$

Example 3 (Counter-example) *It is worth to pointing out that the inequality* $M_{\left(\Pi^{(\beta}, \mathcal{K}, \Delta\right)} \leq \dfrac{\beta}{1 - \beta}$ *of Theorem 5a is not an equality in general. Similarly, under the conditions of Theorem 5c, the implication*

$$
M_{(\Pi', \mathcal{K}, \Delta)} \leq \frac{\beta}{1 - \beta} \implies M_{(\Pi', \mathcal{K}, \Delta)} \leq M_{\left(\Pi^{(\beta}, \mathcal{K}, \Delta\right)}
\tag{38}
$$

will not necessarily hold. Indeed, consider the framework of Example 1 *with a set of states of nature of three elements* $\Omega = \{\omega_1, \omega_2, \omega_3\}$ *and the probabilities* $\mathbb{P}(\omega_i) = 1/3$, $i = 1, 2, 3$. *Suppose that* $y_\Pi = (1,25; 0,5; 1,25)$, $u_0 = (1; 1; 1)$, $U = \mathcal{L}\{u_0\}$, $\mathcal{K} = \{u_0\}$ *and finall* $\Delta = Co\{3/2; 3/2; 0\}, (0; 0; 3)\}$, *"Co" denoting "convex hull". Proposition* 2 *shows that Assumption* 1 *holds. It is easy to see that* $y_\Pi \notin \Delta$, *and Remark* 1 *leads to* $M_{(\Pi,\mathcal{K},\Delta)} = \infty$. *Furthermore, for every* $\beta \in (0,1)$ *we have that (Proposition* 4)

$$C^{(\beta} = \Delta^{(\beta} = Co\{(1 + \beta/2; 1 + \beta/2, 1 - \beta); (1 - \beta, 1 - \beta, 1 + 2\beta)\}.$$

Then, $y^{(\beta} = \varphi_{C^{(\beta}}(y_\Pi) = u_0$ *and therefore* $M_{(\Pi^{(\beta},\mathcal{K},\Delta)} = 0 < \dfrac{\beta}{1 - \beta}$. *Besides, if* $y' = (1 - \beta/2; 1 - \beta/2; 1 + \beta) \in C^{(\beta}$ *then* $M_{(\Pi',\mathcal{K},\Delta)} \le \dfrac{\beta}{1 - \beta}$ *and* $M_{(\Pi',\mathcal{K},\Delta)} > 0 = M_{(\Pi^{(\beta},\mathcal{K},\Delta)}$ *because* $y' \notin \mathcal{K} = \{(1; 1; 1)\}$ *(Remark* 1). $\square$

Despite the counter-example above, under additional conditions the inequality $M_{(\Pi^{(\beta},\mathcal{K},\Delta)} \le \dfrac{\beta}{1 - \beta}$ of Theorem 5 will become an equality, and the implication (38) will hold too. Let us prove that an important particular case arises if $\beta > 0$, $\dfrac{\beta}{1 - \beta} < M_{(\Pi,\mathcal{K},\Delta)}$, $\mathcal{K} = \{u_0\}$ and $y_\Pi \in \Delta$.

Lemma 6 *Consider a convex and closed subset* $C \subset Y$ *and the projection* $\varphi_C : Y \to C$. *If* $y \in Y$, $y \notin C$, $0 \le w \le 1$ *and* $wy + (1 - w)\varphi_C(y) \in C$ *then* $w = 0$.

Proof. The assumptions and (18) imply that

$$\langle y - \varphi_C(y), wy + (1 - w)\varphi_C(y)\rangle \le \langle y - \varphi_C(y), \varphi_C(y)\rangle,$$

and straightforward manipulations lead to $w\langle y - \varphi_C(y), y\rangle \le w\langle y - \varphi_C(y), \varphi_C(y)\rangle$. If $w > 0$ one can simplify, and one trivially gets $\|y - \varphi_C(y)\|^2 \le 0$. Since the opposite inequality is obvious, we have that $y = \varphi_C(y) \in C$, contradicting the assumptions. $\square$

Lemma 7 *Suppose that* $\mathcal{K} = \{u_0\}$, $0 < \beta' < \beta \le 1$, $y_\Pi \notin \Delta^{(\beta}$ *and* $y_\beta \in \Delta^{(\beta'}$. *If* $0 \le w \le 1$ *and* $wy_\Pi + (1 - w)u_0 \in \Delta^{(\beta}$ *then* $w = 0$.

Proof. Take $\alpha = \beta'/\beta$. Proposition 3a implies that $\Phi_\alpha(u_0, \Phi_\beta(u_0, z)) = \Phi_{\beta'}(u_0, z)$ for every

$z \in Y$. Expression $y_\beta \in \Delta^{(\beta'}$ implies the existence of $\tilde{z} \in \Delta$ such that

$$y^{(\beta} = \Phi_{\beta'} (u_0, \tilde{z}) = \Phi_\alpha (u_0, \Phi_\beta (u_0, \tilde{z})) = \Phi_\alpha (u_0, z),$$

where $z = \Phi_\beta (u_0, \tilde{z}) \in \Delta^{(\beta}$. Hence,

$$y^{(\beta} = \alpha z + (1 - \alpha) u_0,$$
$$z = \frac{1}{\alpha} y^{(\beta} - \frac{1 - \alpha}{\alpha} u_0.$$

Suppose that $0 \le w \le 1$ and $w y_\Pi + (1 - w) u_0 \in \Delta^{(\beta}$. Then, since $\Delta^{(\beta}$ is convex, if $0 \le w' \le 1$,

$$\Delta^{(\beta} \ni w' z + (1 - w') [w y_\Pi + (1 - w) u_0] =$$
$$w' \left[\frac{1}{\alpha} y^{(\beta} - \frac{1 - \alpha}{\alpha} u_0 \right] + (1 - w') [w y_\Pi + (1 - w) u_0] =$$
$$w' \frac{1}{\alpha} y^{(\beta} + (1 - w') w y_\Pi + \left((1 - w') (1 - w) - w' \frac{1 - \alpha}{\alpha} \right) u_0.$$

Taking $w' = \dfrac{1 - w}{1 - w + \dfrac{1 - \alpha}{\alpha}}$ we have that $0 \le w' < 1$ and $(1 - w') (1 - w) - w' \dfrac{1 - \alpha}{\alpha} = 0$, so

$$w' \frac{1}{\alpha} y^{(\beta} + (1 - w') w y_\Pi \in \Delta^{(\beta}.$$

Besides, $y^{(\beta} = \varphi_{C^{(\beta}} (y_\Pi) \in y_\Pi + U^\perp$ (see (33) and (35)), and $y_\Pi \in y_\Pi + U^\perp$ is obvious. Since $w' \frac{1}{\alpha} \ge 0$, $(1 - w') w \ge 0$, $w' \frac{1}{\alpha} + (1 - w') w = 1$ and $y_\Pi + U^\perp$ is convex,

$$w' \frac{1}{\alpha} y^{(\beta} + (1 - w') w y_\Pi \in C^{(\beta},$$

so Lemma 6 implies that $(1 - w') w = 0$, and therefore $w = 0$ because $w' < 1$. $\qquad \square$

Theorem 8 *Take $\beta \in (0, 1)$ such that $M_{(\Pi, \varphi_U(\mathcal{K}), \varphi_U(\Delta))} \le \dfrac{\beta}{1 - \beta} < M_{(\Pi, \mathcal{K}, \Delta)} \le \infty$. Consider the pricing rule $\Pi^{(\beta} (y) = \langle y, y^{(\beta} \rangle$ for every $y \in Y$. If $\mathcal{K} = \{u_0\}$ and $y_\Pi \in \Delta$ then $M_{(\Pi^{(\beta}, \mathcal{K}, \Delta)} = \dfrac{\beta}{1 - \beta}$.*

Proof. Inequality $\dfrac{\beta}{1-\beta} < M_{(\Pi,\mathcal{K},\Delta)}$ implies that

$$y_\Pi \notin \Delta^{(\beta}. \tag{39}$$

Suppose that

$$M_{\left(\Pi^{(\beta},\mathcal{K},\Delta\right)} = \frac{\beta'}{1-\beta'} \tag{40}$$

for some $0 \le \beta' < \beta$. $y_\Pi \in \Delta$ implies that

$$\beta' y_\Pi + (1-\beta')\, u_0 \in \Delta^{(\beta'} \subset \Delta^{(\beta} \tag{41}$$

(Proposition 3d). The optimal value of (37) equals β', and therefore there exists $z \in \Delta$ such that $\beta' z + (1-\beta')\, u_0 = y^{(\beta}$. Consequently,

$$y_\beta \in \Delta^{(\beta'}. \tag{42}$$

If $\beta' > 0$ then (39), (42), (41) and Lemma 7 would imply that $\beta' = 0$, and we would have a contradiction. Thus, $\beta' = 0$, and Proposition 3b and (42) imply that $y^{(\beta} = u_0$. Hence,

$$\varphi_{C^{(\beta}}(y_\Pi) = u_0. \tag{43}$$

Let us show that

$$\beta y_\Pi + (1-\beta)\, u_0 \in C^{(\beta}. \tag{44}$$

Indeed, we only have to show that $\beta y_\Pi + (1-\beta)\, u_0 \in y_\Pi + U^\perp$. (43) implies that $u_0 \in C^{(\beta} \subset y_\Pi + U^\perp$, and $y_\Pi \in y_\Pi + U^\perp$ is obvious, so $\beta y_\Pi + (1-\beta)\, u_0 \in y_\Pi + U^\perp$ follows from the convexity of $y_\Pi + U^\perp$.

(17), (43) and (44) lead to $\|y_\Pi - u_0\| \le \|y_\Pi - (\beta y_\Pi + (1-\beta)\, u_0)\| = (1-\beta)\, \|y_\Pi - u_0\|$, and $\|y_\Pi - u_0\| > 0$ would imply $1 \le 1 - \beta$ and $\beta \le 0$, contradiction with the assumptions. Hence, $y_\Pi = u_0$ and therefore $y_\Pi \in \mathcal{K}$, which contradicts $0 < M_{(\Pi,\mathcal{K},\Delta)}$ (see Remark 1). The contradiction is provoked by (40). $\qquad\square$

Corollary 9 *Consider the pricing rule $\tilde{\Pi}^{(\beta}(y) = \langle y, \tilde{y}^{(\beta} \rangle$ for every $y \in Y$. If $\beta \in (0,1)$, $\mathcal{K} = \{u_0\}$ and $M_{\left(\Pi,\varphi_U(\mathcal{K}),\varphi_U(\tilde{\Delta})\right)} \le \dfrac{\beta}{1-\beta} < M_{(\Pi,\mathcal{K},\tilde{\Delta})} \le \infty$ then $M_{\left(\tilde{\Pi}^{(\beta},\mathcal{K},\tilde{\Delta}\right)} = \dfrac{\beta}{1-\beta}$.*

Proof. Expression (16) shows that $y_\Pi \in \tilde{\Delta}$. Hence, the result follows from Theorem 8. □

Remark 5 *Suppose that $M_{(\Pi,\mathcal{K},\Delta)} = \infty$. Theorems 5 and 8 and Corollary 9 show that prices in the stable sub-space may be respected and the strong compatibility may be recovered if Π is replaced by $\Pi^{(\beta)}$, where β may be estimated so as to obtain an upper bound $\dfrac{\beta}{1-\beta}$ of the MPR. Moreover, if $\mathcal{K} = \{u_0\}$ (non ambiguous case, in the framework of Examples 1 and 2) then $\dfrac{\beta}{1-\beta}$ becomes the exact MPR if $y_\Pi \in \Delta$ or the risk measure ρ is replaced by its compatible modificatio $\tilde{\rho}$.* □

4.   The two steps approach

Suppose that Y represents "pay-offs at T" and we modify the pricing rule Π so as to recover strong compatibility. For maturities T' longer than T we could face "pathological properties again", and the pricing rule connecting 0 and T' could again reflec the lack of strong compatibility. This section will be devoted to analyzing this problem.

First of all, let us present an instrumental lemma. Since this result is abstract and only involves Hilbert spaces, *i.e.*, it does not contain any idea related to Financial Mathematics, the proof will be omitted.

Lemma 10 *Suppose that F is a Hilbert space, $E \subset F$ is a closed subspace and $L : F \to \mathbb{R}$ is a linear and continuous function such that $L(x_0) = 1$ for some $x_0 \in E$. Denote by $Ker(L)$ the kernel of L. Then;*

a) $L(x)x_0 \in (x + Ker(L)) \cap E$ and therefore $(x + Ker(L)) \cap E \neq \varnothing$ for every $x \in F$.

b) Defin $L_{E,F}^{(*}: F \to E$ by means of

$$L_{E,F}^{(*}(x) := \varphi_{(x+Ker(L))\cap E}(x) \tag{45}$$

for every $x \in F$. Then, $L_{E,F}^{(*}(x) = x$ for every $x \in E$ and $L\left(L_{E,F}^{(*}(x)\right) = L(x)$ for every $x \in F$.

c) If $x, y \in F$ then $L_{E,F}^{(*}(x) = y$ if and only if $y \in E$, $x - y \in Ker(L)$ and $x - y \in (Ker(L) \cap E)^\perp$.

d) $L_{E,F}^{(*}$ *is linear and continuous.*

e) *If there exists* $x_L \in E$ *such that* $L(x) = \langle x, x_L \rangle$ *for every* $x \in F$, *then* $L_{E,F}^{(*} = \varphi_E$.

f) *Suppose that* $D \subset E \subset F$ *is a closed subspace such that* $L(x_1) \neq 0$ *for some* $x_1 \in D$. *With the obvious notations,* $L_{D,F}^{(*} = L_{D,E}^{(*} \circ L_{E,F}^{(*}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 4 *In general, under the notations of Lemma* 10, *there are many linear and continuous functions* $L_{E,F} : F \to E$ *satisfying* $L_{E,F}(x) = x$ *for every* $x \in E$ *and* $L(L_{E,F}(x)) = L(x)$ *for every* $x \in F$. *Moreover, the example* $L_{E,F}^{(*}$ *given in (45) is not necessarily the usual one in asset pricing. Indeed, consider the (toy) binomial two periods (three dates) pricing model*

$$
\begin{array}{ccccc}
 & & & \nearrow & 16 \\
 & & \nearrow \quad 8 & & \\
4 & & & \searrow & 4 \\
 & \searrow \quad 2 & \nearrow & & \\
 & & & \searrow & 1
\end{array}
\tag{46}
$$

Under the notations of this paper, the space Y *is* $L^2(\mathcal{F})$, $\mathcal{F}$ *being the discrete* $\sigma-$*algebra of the set of states of nature* $\Omega = \{(u,u), (u,d), (d,u), (d,d)\}$, *where "u means up in (46)" and "d means down" (see Example* 1*). Obviously,* Y *may be identifie with* $\mathbb{R}^4$ *endowed with*

$$
\left\langle (y_j)_{j=1}^4, (z_j)_{j=1}^4 \right\rangle = y_1 z_1 p^2 + y_2 z_2 p(1-p) + y_3 z_3 p(1-p) + y_4 z_4 (1-p)^2,
$$

$p \in (0,1)$ *being the (physical) probability of "up" in (46). The pricing rule* $\Pi : Y \to \mathbb{R}$ *is*

$$
\Pi(y) = \frac{y_1}{9} + \frac{2y_2}{9} + \frac{2y_3}{9} + \frac{4y_4}{9}.
\tag{47}
$$

Consider the subspace $E \subset Y$ *indicating "information after one period", i.e.,*

$$
E = \{y \in Y;\ y_2 = y_1\ and\ y_4 = y_3\}.
$$

Obviously, E *may be identifie with* $\mathbb{R}^2$. *If* $p = 1/3$ *(i.e., if* p *equals the risk-neutral probability)*

then $\Pi_{E,Y}^{(*}:Y\to E$ may be denoted by $\Pi_{E,Y,1/3}^{(*}$, and Lemma $10e)$ implies that

$$\Pi_{E,Y,1/3}^{(*}(y)=\left(\frac{y_1+2y_2}{3};\frac{y_3+2y_4}{3}\right),\qquad(48)$$

which equals the usual price after one period. Nevertheless, if $p=1/2$, then Lemma $10c)$ leads to

$$\Pi_{E,Y,1/2}^{(*}(y)=\left(\frac{7y_1+8y_2-y_3+y_4}{15};\frac{-2y_1+2y_2+11y_3+19y_4}{30}\right).$$

More generally, for every $0<p<1$ we can fin $\Pi_{E,Y,p}^{(*}$ such that $\Pi_{E,Y,p}^{(*}(y)=y$ for every $y\in E$ and $\Pi\left(\Pi_{E,Y,p}^{(*}(y)\right)=\Pi(y)$ for every $y\in Y$. $\qquad\square$

Suppose that Y is a closed subspace of a Hilbert space $X\supset Y$, and suppose also that Π may be extended to a new pricing rule still denoted by $\Pi:X\to\mathbb{R}$. If $x_\Pi\in X$ is such that $\Pi(x)=\langle x,x_\Pi\rangle$ for every $x\in X$, then (20) shows that $y_\Pi=\varphi_Y(x_\Pi)$. Let us assume the existence of convex and weakly-compact sets $\mathcal{K}_X\subset\Delta_X\subset X$ such that $\mathcal{K}=\varphi_Y(\mathcal{K}_X)$ and $\Delta=\varphi_Y(\Delta_X)$. (2), (10) and (20) show that Problem (7) has an obvious extension from Y to X. Next, let us show that once the restriction of Π to Y has been modifie to $\Pi^{(\beta}$ so as to recover a finit $M_{\left(\Pi^{(\beta},\mathcal{K},\Delta\right)}$ consistent with the desired target, one can also extend $\Pi^{(\beta}$ to X and then modify the extension so as to reach a new $M_{\left(\Pi^{(\alpha},\mathcal{K}_X,\Delta_X\right)}$ still consistent with new finit desired targets and such that $M_{\left(\Pi^{(\beta},\mathcal{K},\Delta\right)}\le M_{\left(\Pi^{(\alpha},\mathcal{K}_X,\Delta_X\right)}$.

Theorem 11 *Fix $\Pi_{Y,X}:X\to Y$ linear, continuous, and such that $\Pi_{Y,X}(y)=y$ for every $y\in Y$ and $\Pi(\Pi_{Y,X}(x))=\Pi(x)$ for every $x\in X$. Take $\beta\in(0,1)$ such that $M_{(\Pi,\varphi_U(\mathcal{K}),\varphi_U(\Delta))}\le\frac{\beta}{1-\beta}<M_{(\Pi,\mathcal{K},\Delta)}\le\infty$, and for $y\in Y$ consider the pricing rule $\Pi^{(\beta}(y)=\langle y,y^{(\beta}\rangle$ of Theorems 5 and 8. Extend $\Pi^{(\beta}$ to $\Pi^{(\beta}:X\to\mathbb{R}$ by means of $\Pi^{(\beta}(x)=\Pi^{(\beta}(\Pi_{Y,X}(x))$ for every $x\in X$. Take $x^{(\beta}\in X$ such that $\Pi^{(\beta}(x)=\langle x,x^{(\beta}\rangle$ for every $x\in X$. Fix $\alpha\in[\beta,1)$ and a closed subspace $W\subset X$ such that $Y+W$ is closed, $\Pi_{Y,X}(W)=U$, and $M_{\left(\Pi^{(\beta},\varphi_{Y+W}(\mathcal{K}_X),\varphi_{Y+W}(\Delta_X)\right)}\le\frac{\alpha}{1-\alpha}<\infty$. With the obvious notation, consider $C_X^{(\alpha}=\Delta_X^{(\alpha}\cap\left(x^{(\beta}+(Y+W)^\perp\right)$. Then;*

a) $C_X^{(\alpha}$ is non void, convex and weakly compact, and there exists $x^{(\alpha}=\varphi_{C_X^{(\alpha}}\left(x^{(\beta}\right)$.

b) Consider the pricing rule $\Pi^{(\alpha}(x)=\langle x,x^{(\alpha}\rangle$ for every $x\in X$. Then, $M_{\left(\Pi^{(\alpha},\mathcal{K}_X,\Delta_X\right)}\le\frac{\alpha}{1-\alpha}<\infty$, and $\left(\Pi^{(\alpha},\mathcal{K}_X,\Delta_X\right)$ is strongly compatible.

c) $\Pi^{(\alpha}(x)=\Pi^{(\beta}(x)$ for every $x\in Y+W$. Moreover, $\Pi^{(\alpha}(w)=\Pi^{(\beta}(w)=\Pi(w)$ for every

$w \in W$ *and for every* $w \in U$.

d) If $\mathcal{K}_X = \{u_0\}$, $x^{(\beta} \in \Delta_X$ *and* $x_\Pi \in \Delta_X$ *then* $M_{\left(\Pi^{(\beta},\mathcal{K},\Delta\right)} = \dfrac{\beta}{1-\beta} \leq \dfrac{\alpha}{1-\alpha} = M_{\left(\Pi^{(\alpha},\mathcal{K}_X,\Delta_X\right)} < \infty$.

Proof. *a)* $\Delta_X^{(\alpha} \cap \left(x_\Pi + (Y+W)^\perp\right)$ is convex and weakly compact with the same arguments as in Proposition 3b. It is also non-void because $M_{\left(\Pi^{(\beta},\varphi_{Y+W}(\mathcal{K}_X),\varphi_{Y+W}(\Delta_X)\right)} \leq \dfrac{\alpha}{1-\alpha}$ and the proof of Proposition 4 applies.

b) Once again, it is an obvious particular case of Theorem 5a.

c) As in Theorem 5b, $\Pi^{(\alpha}(x) = \Pi^{(\beta}(x)$ for every $x \in Y + W$. Furthermore, if $w \in W$, then $\Pi^{(\alpha}(w) = \Pi^{(\beta}(w) = \Pi^{(\beta}(\Pi_{Y,X}(w)) = \Pi(\Pi_{Y,X}(w))$ because $\Pi_{Y,X}(w) \in U$. Thus, $\Pi^{(\beta}(w) = \Pi(\Pi_{Y,X}(w)) = \Pi(w)$. Finally, if $u \in U$ we will have that $\Pi^{(\alpha}(u) = \Pi^{(\beta}(u)$ because $u \in Y$, and $\Pi^{(\beta}(u) = \Pi(u)$ due to Theorem 5b.

d) If $x_\Pi \in \Delta_Z$ then $y_\Pi = \varphi_Y(x_\Pi) \in \varphi_Y(\Delta_X) = \Delta$, and the result follows from Theorem 8. $\square$

Corollary 12 *Under the notations of Theorem* 11, *consider*

$$\tilde{\Delta}_X = \{wz + (1-w)\,x_\Pi;\ z \in \Delta_X,\ 0 \leq w \leq 1\} \supset \Delta_X,$$

along with the associated risk function $X \ni x \to \tilde{\rho}_X(x) := Max\,\{\rho(x), -\Pi(x)\} \in \mathbb{R}$ *(see (15) and (16)) extending* $\tilde{\rho}$. *Take* $\tilde{C}_X^{(\alpha} = \tilde{\Delta}_X^{(\alpha} \cap \left(x^{(\beta} + (Y+W)^\perp\right)$, $\tilde{x}^{(\alpha} = \varphi_{\tilde{C}_X^{(\alpha}}(x^{(\beta})$, *and* $\tilde{\Pi}^{(\alpha}(x) = \langle x, \tilde{x}^{(\alpha}\rangle$ *for every* $x \in X$. *If* $\mathcal{K}_X = \{u_0\}$ *and* $x^{(\beta} \in \tilde{\Delta}_X$ *then* $M_{\left(\Pi^{(\beta},\mathcal{K},\tilde{\Delta}\right)} = \dfrac{\beta}{1-\beta} \leq \dfrac{\alpha}{1-\alpha} = M_{\left(\Pi^{(\alpha},\mathcal{K}_X,\tilde{\Delta}_X\right)} < \infty$. $\square$

Corollary 13 *Under the notations of Theorem* 11, *consider*

$$\tilde{\Delta}_{X,x^{(\beta}} = \{wz + (1-w)\,x^{(\beta};\ z \in \Delta_X,\ 0 \leq w \leq 1\} \supset \Delta_X,$$

along with the associated risk function $X \ni x \to \tilde{\rho}_{X,x^{(\beta}}(x) := Max\,\{\rho(x), -\Pi^{(\beta}(x)\} \in \mathbb{R}$ *(see (15) and (16)) extending* $\tilde{\rho}$. *Take* $\tilde{C}_{X,x^{(\beta}}^{(\alpha} = \tilde{\Delta}_{X,x^{(\beta}}^{(\alpha} \cap \left(x^{(\beta} + (Y+W)^\perp\right)$, $\tilde{x}^{(\alpha} = \varphi_{\tilde{C}_{X,x^{(\beta}}^{(\alpha}}(x^{(\beta})$, *and* $\tilde{\Pi}^{(\alpha}(x) = \langle x, \tilde{x}^{(\alpha}\rangle$ *for every* $x \in X$. *If* $\mathcal{K}_X = \{u_0\}$ *then* $M_{\left(\Pi^{(\beta},\mathcal{K},\tilde{\Delta}\right)} = \dfrac{\beta}{1-\beta} \leq \dfrac{\alpha}{1-\alpha} = M_{\left(\Pi^{(\alpha},\mathcal{K}_X,\tilde{\Delta}_{X,x^{(\beta}}\right)} < \infty$. $\square$

Remark 6 *Theorem* 11 *(and therefore Corollaries* 12 *and* 13*) requires* W *to satisfy three non-trivial properties;* $Y + W$ *must be closed,* $M_{\left(\Pi^{(\beta)}, \varphi_{Y+W}(\mathcal{K}_X), \varphi_{Y+W}(\Delta_X)\right)} \leq \dfrac{\alpha}{1-\alpha}$ *must hold and* $\Pi_{Y,X}(W) = U$ *must hold too. Notice that the three properties trivially hold if* $W = U$*, since in such a case* $W \subset Y$*,* $Y + W = Y$ *and* $M_{\left(\Pi^{(\beta)}, \varphi_{Y+W}(\mathcal{K}_X), \varphi_{Y+W}(\Delta_X)\right)} \leq \dfrac{\beta}{1-\beta} \leq \dfrac{\alpha}{1-\alpha}$*. A second important case also satisfying that* $Y + W$ *is closed arises if the dimension of* W *is finit , as shown in Proposition* 14 *below.[7] As Lemma* 10*, Proposition* 14 *is a general result about Hilbert spaces, and therefore its proof will be omitted too.* □

Proposition 14 *If* V_1 *is a finite-dimensiona subspace of a Hilbert space* F*, and* V_2 *is a closed subspace of* F*, then* $V_1 + V_2$ *is also closed.* □

Remark 7 *Notice that Theorem* 11 *and its corollaries may be trivially extended in order to incorporate more than two maturities* T *and* $T' > T$*. In general, we can consider (at least) every finit family of maturities.* □

Example 5 (Counter-example) *Corollaries* 9*,* 12 *and* 13 *show that the desired market price of risk may be reached if one deals with "compatible modificatio of usual (non-robust) risk measures" ((see (15) and (16)). One might wonder whether condition* $x^{(\beta} \in \tilde{\Delta}_X$ *(Corollary* 12*) must be really imposed, but it is easy to provide examples illustrating that this condition may fail. Indeed, consider again the binomial tree of Example* 4 *with* $p = 1/2$ *as the probability of "up" in (46). Let us introduce a minor modificatio in the notation. The "information after two periods" will be represented by* X*, which may be identifie with* $\mathbb{R}^4$*. The subspace* $Y \subset X$ *indicating "information after one period" may be identifie with* $\mathbb{R}^2$*. If* $\mathcal{K} = \{1\}$ *and* $\rho = CVaR_{(p,0,2)}$*, then (29) shows that after one period*

$$\Delta = \{(z_1; z_2) \, ; \, z_1 + z_2 = 2, \, 0 \leq z_1, z_2 \leq 1{,}25\} \subset Y = \mathbb{R}^2.$$

Hence, it is easy to see that Δ *is "the segment" of the straight line* $z_1 + z_2 = 2$ *contained between the points* $(0{,}75; 1{,}25)$ *and* $(1{,}25; 0{,}75)$*. Consequently, since the restriction* $\Pi(y) = \frac{y_1}{3} + \frac{2y_2}{3}$ *of (47) to* Y *clearly implies that*

$$y_\Pi = \left(\frac{1/3}{p}; \frac{2/3}{1-p}\right) = (2/3; 4/3) \, ,$$

---

[7] In practice, the stable subspace W may be given by available quotations. Since the number of available quotations will be always finit in practice, it is not at all restrictive to assume that the dimension of W is finite

we have that $y_\Pi \notin \Delta$, $(\Pi, \mathcal{K}, \Delta)$ is not compatible, and moreover $\tilde{\Delta}$ is "the segment" of the straight line $z_1 + z_2 = 2$ contained between $(2/3; 4/3)$ and $(1{,}25; 0{,}75)$. Let us consider that "an infinit market price of risk" is not realistic and suppose that $M_{(\Pi^{(\beta}, \mathcal{K}, \Delta)} = 1$ is more reasonable. According to (14), take $\beta = 1/2$. Bearing in mind that $\beta (2/3; 4/3) + (1 - \beta)(1; 1) = (5/6; 7/6)$, if the stable subspace is the subspace generated by the riskless asset then (33), Proposition 4 and (35) show that $\tilde{y}^{(\beta} = (5/6; 7/6)$ and

$$\tilde{\Pi}^{(\beta}(y) = \frac{5y_1 + 7y_2}{12}.$$

Under the assumptions of Theorem 11 and its corollaries, we can consider the "factor" $\Pi_{Y,X} : X \to Y$ given by (48)

$$\Pi_{Y,X}(x) = \left(\frac{x_1 + 2x_2}{3}; \frac{x_3 + 2x_4}{3} \right),$$

and the extension of $\tilde{\Pi}^{(\beta}$ will become

$$\tilde{\Pi}^{(\beta}(x) = \frac{5\left(\frac{x_1 + 2x_2}{3}\right) + 7\left(\frac{x_3 + 2x_4}{3}\right)}{12} = \frac{5x_1 + 10x_2 + 7x_3 + 14x_4}{36}.$$

Since the (physical) probabilities of the four states of nature equal $0{,}25$, with the notations of Theorem 11 we have that

$$x^{(\beta} = \left(\frac{5}{9}; \frac{10}{9}; \frac{7}{9}; \frac{14}{9} \right),$$

and

$$x_\Pi = \left(\frac{4}{9}; \frac{8}{9}; \frac{8}{9}; \frac{16}{9} \right).$$

With the notations of Corollary 12, let us show that $x^{(\beta} \notin \tilde{\Delta}_X$. Indeed, (29) shows that after two periods

$$\Delta_X = \left\{ (z_1; z_2; z_3; z_4); \ \sum z_i = 4, \ 0 \leq z_i \leq 1{,}25 \right\} \subset X = \mathbb{R}^4.$$

Hence, $x^{(\beta} \in \tilde{\Delta}_X$ holds if and only if there exists a solution (z, γ), $0 \leq \gamma \leq 1$, of (see (16))

$$\begin{cases} \frac{5}{9} = \gamma z_1 + (1 - \gamma)\frac{4}{9} \\ \frac{10}{9} = \gamma z_1 + (1 - \gamma)\frac{8}{9} \\ \frac{7}{9} = \gamma z_1 + (1 - \gamma)\frac{8}{9} \\ \frac{14}{9} = \gamma z_1 + (1 - \gamma)\frac{16}{9} \\ 4 = z_1 + z_2 + z_3 + z_4 \end{cases}$$

With the change of variable $\gamma_i = \gamma z_i$, $\gamma_0 = 1 - \gamma$ we have the equivalent linear system

$$\begin{cases} \frac{5}{9} = \frac{4}{9}\gamma_0 + \gamma_1 \\ \frac{10}{9} = \frac{8}{9}\gamma_0 + \gamma_2 \\ \frac{7}{9} = \frac{8}{9}\gamma_0 + \gamma_3 \\ \frac{14}{9} = \frac{16}{9}\gamma_0 + \gamma_4 \\ 4 = z_1 + z_2 + z_3 + z_4 \end{cases}$$

which has no solution. Thus, $x^{(\beta} \notin \tilde{\Delta}_X$. □

5. CVaR-consistent pricing

Many risk measurement linked papers present examples constructed with the $CVaR$ of Rockafellar and Uryasev (2000). This coherent (Artzner *et al.*, 1999) and expectation bounded (Rockafellar *et al.*, 2006) risk measure may be easily optimized and interpreted in terms of potential capital losses (Rockafellar and Uryasev, 2000), outperforms the standard deviation with respect to the second order stochastic dominance (Ogryczak and Ruszczynski, 1999 and 2002), its compatible modificatio is quite tractable (Balbás and Balbás, 2009) and has natural extensions to ambiguous frameworks (Zhu and Fukushima, 2009), among many other good properties. Let us end this paper with some $CVaR$ linked properties and examples illustrating and interpreting the main theorems above. First of all let us see that the modifie SDF has a bounded logarithm under the $CVaR$.

Proposition 15 *Consider Example 1 in a non-ambiguous setting ($K = \{1\}$). Suppose that $\beta \in (0,1)$ and $M_{(\Pi, \varphi_U(\mathcal{K}), \varphi_U(\Delta))} \leq \dfrac{\beta}{1-\beta}$. Take $0 < \mu < 1$ and $\rho = CVaR_{(\{\mathbf{P}\}, \mu)}$. Then,*

$$\mathbb{P}\left(1 - \beta \leq y^{(\beta} \leq 1 - \beta + \frac{\beta}{1-\mu}\right) = 1. \tag{49}$$

Proof. (33) and (35) imply that $y^{(\beta} \in \Delta^{(\beta}$, and therefore (29) and (32) lead to the existence of $z \in L^2(\mathcal{F})$ such that $y^{(\beta} = \beta z + (1 - \beta)$ and $0 \leq z \leq \frac{1}{1-\mu}$. Hence, (49) becomes trivial. □

Proposition 15 above may be "a little bit improved" under an additional assumption about duality gaps in linear programming (Anderson and Nash, 1987). Actually, the new SDF may become a call spread under the $CVaR$.

Proposition 16 *Consider Example 1 in a non-ambiguous setting ($K = \{1\}$). Suppose that $\beta \in (0,1)$ and $M_{(\Pi,\varphi_U(K),\varphi_U(\Delta))} \leq \dfrac{\beta}{1 - \beta}$. Take $0 < \mu < 1$ and $\rho = CVaR_{(\{\mathbf{P}\},\mu)}$. Then;*

a) $y^{(\beta}$ is the unique solution of the following linear optimization problem

$$Min \; \mathbb{E}\left(\left(y^{(\beta} - y_\Pi\right) y\right) \quad \begin{cases} \varphi_U\left(y - y_\Pi\right) = 0 \\ 1 - \beta \leq y \leq \dfrac{\beta}{1 - \mu} + (1 - \beta) \end{cases} \tag{50}$$

$y \in L^2\left(\mathcal{F}\right)$ being the decision variable.

b) If (50) and its dual do not present any duality gap then there exists $u^{(\beta} \in U$ such that $y^{(\beta}$ is the call-spread

$$y^{(\beta} = \begin{cases} 1 - \beta, & y_\Pi + u^{(\beta} < 1 - \beta \\ y_\Pi + u^{(\beta}, & 1 - \beta \leq y_\Pi + u^{(\beta} \leq \dfrac{\beta}{1 - \mu} + (1 - \beta) \\ \dfrac{\beta}{1 - \mu} + (1 - \beta), & \dfrac{\beta}{1 - \mu} + (1 - \beta) < y_\Pi + u^{(\beta} \end{cases} \tag{51}$$

where $u^{(\beta}$ is characterized by $u^{(\beta} \in U$ (or, equivalently, $\varphi_U\left(u^{(\beta)}\right) = u^{(\beta)}$) and $\varphi_U\left(y^{(\beta} - y_\Pi\right) = 0$.

c) If (50) and its dual do not present any duality gap then $y^{(\beta}$ is characterized by the existence of $(\lambda_m, \lambda_M) \in L^2\left(\mathcal{F}\right) \times L^2\left(\mathcal{F}\right)$ such that

$$\begin{cases} y^{(\beta} = y_\Pi + \varphi_U\left(\lambda_M - \lambda_m\right) + \lambda_m - \lambda_M \\ \lambda_m\left(y^{(\beta} - (1 - \beta)\right) = \lambda_M\left(\dfrac{\beta}{1 - \mu} + (1 - \beta) - y^{(\beta}\right) = 0 \\ \lambda_m \geq 0, \; \lambda_M \geq 0, \; 1 - \beta \leq y^{(\beta} \leq \dfrac{\beta}{1 - \mu} + 1 - \beta \end{cases} \tag{52}$$

Proof. (29) and (33) easily imply that

$$\Delta^{(\beta} = \left\{y \in L^2\left(\mathcal{F}\right); \mathbb{E}\left(y\right) = 1, \; 1 - \beta \leq y \leq \dfrac{\beta}{1 - \mu} + (1 - \beta)\right\}. \tag{53}$$

Expressions (18) and (35) show that $y^{(\beta} \in C^{(\beta}$ is characterized by the inequality

$$\mathbb{E}\left(\left(y^{(\beta} - y_\Pi\right) y^{(\beta)}\right) \le \mathbb{E}\left(\left(y^{(\beta} - y_\Pi\right) y\right)$$

for every $y \in C^{(\beta}$. Therefore, $a)$ will become obvious if we show that $y \in C^{(\beta}$ if and only if y satisfie the constraints of (50), which trivially follows from (33) and (53).

Let us prove $b)$ and $c)$. Since U coincides with its dual space, under the absence of duality gap the solution $y^{(\beta}$ of (50) must be feasible and is characterized by the existence of $u^{(\beta} \in U$ and $(\lambda_m, \lambda_M) \in L^2(\mathcal{F}) \times L^2(\mathcal{F})$ such that

$$
\begin{cases}
y^{(\beta} - y_\Pi - u^{(\beta} - \lambda_m + \lambda_M = 0 \\
\lambda_m\left(y^{(\beta} - (1 - \beta)\right) = \lambda_M\left(\dfrac{\beta}{1 - \mu} + (1 - \beta) - y^{(\beta}\right) = 0 \\
\lambda_m \ge 0, \ \lambda_M \ge 0
\end{cases}
$$

(Anderson and Nash, 1987). Hence, the proof of $b)$ becomes trivial because the random variable $y^{(\beta}$ given by (51) satisfie these conditions if one takes

$$
\lambda_m =
\begin{cases}
1 - \beta - \left(y_\Pi + u^{(\beta)}\right), & y_\Pi + u^{(\beta} < 1 - \beta \\
0, & otherwise
\end{cases}
$$

and

$$
\lambda_M =
\begin{cases}
y_\Pi + u^{(\beta} - \left(\dfrac{\beta}{1 - \mu} + (1 - \beta)\right), & \dfrac{\beta}{1 - \mu} + (1 - \beta) < y_\Pi + u^{(\beta} \\
0, & otherwise
\end{cases}
$$

Moreover, $\varphi_U\left(y^{(\beta} - y_\Pi\right) = 0$ (or, equivalently, $\varphi_U\left(\lambda_M - \lambda_m\right) = u^{(\beta)}$) is an obvious consequence of $y^{(\beta} \in y_\Pi + U^\perp$. Therefore, to prove $c)$, notice that (51) is equivalent to

$$
\begin{cases}
y^{(\beta} = y_\Pi + u^{(\beta} + \lambda_m - \lambda_M \\
\varphi_U\left(u^{(\beta)}\right) = u^{(\beta} \\
\varphi_U\left(\lambda_M - \lambda_m\right) = u^{(\beta} \\
\lambda_m\left(y^{(\beta} - (1 - \beta)\right) = \lambda_M\left(\dfrac{\beta}{1 - \mu} + (1 - \beta) - y^{(\beta}\right) = 0 \\
\lambda_m \ge 0, \ \lambda_M \ge 0, \ 1 - \beta \le y^{(\beta} \le \dfrac{\beta}{1 - \mu} + 1 - \beta
\end{cases}
$$

which equivalent to (52) because $\varphi_U \circ \varphi_U = \varphi_U$ (see (22)). $\qquad\square$

Remark 8 *(Interpretation of Propositions 15 and 16). Actually, bearing in mind Proposition 3d, with a similar proof as that of Proposition 15 one can show that $M_{(\Pi, \mathcal{K}=\{1\}, \Delta)} \leq \frac{\beta}{1-\beta}$ would require an initial SDF $y_\Pi > 0$ such that*

$$\mathbb{P}\left(1 - \beta \leq y_\Pi \leq 1 - \beta + \frac{\beta}{1-\mu}\right) = 1. \tag{54}$$

Brownian motion linked pricing models will rarely satisfy (54). For instance, (54) will fail if y_Π has the log-normal distribution (Black and Scholes) or a heavier tailed one (most of the stochastic volatility models). Since Brownian motion linked pricing models do not satisfy (54) and they are not strongly compatible with the $CVaR$ and other coherent risk measures, a suitable interpretation of (54) might be as follows; "Brownian motion linked pricing models incorporate practical pricing errors on tails, which provoke the absence of strong compatibility". This statement seems to be consistent with the usual theoretical justificatio for Brownian motions, which is closely related to the central limit theorem, and the converge of the central limit theorem is very inefficien on tails. Moreover, Balbás et al. (2016a) have dealt with the $CVaR$ and have constructed portfolios of derivatives whose theoretical marked price of risk should be enormous. They showed that these portfolios should concentrate their highest and lowest pay-offs on states of nature closely related to the tails of the underlying asset. In other words, the major discrepancies between y_Π and $y^{(\beta}$ must be concentrated on tails, as indicated in (51).[8]

Ideas above about the practical pricing errors of Brownian motions on tails may be is also consistent with the empirical evidence. It is famous the $OTM-$put price puzzle, provoked because $OTM-$puts are often "too expensive". Bondarenko (2014) also shows that it is not easy to explain the real market price of many $OTM-$puts with the classical option pricing models. In general, the tail of the SDF will have a critical influenc on the price of $OTM-$options. Errors on the $SDF-$tail will imply significan pricing errors for $OTM-$options. In this sense, the modificatio of the SDF proposed in (51) may be useful, since, as said above, the major discrepancies between y_Π and $y^{(\beta}$ will affect their tails. □

Remark 9 *(Computing $y^{(\beta)}$). Both (51) and (52) permit us to develop algorithms generating $y^{(\beta}$. For instance, according to Proposition 15b), $y = y^{(\beta}$ minimizes the quadratic expression $\|\varphi_U (y - y_\Pi)\|^2$ if $(u, y) \in L^2(\mathcal{F}) \times L^2(\mathcal{F})$ is the decision variable and it is restricted by*

---

[8]Proposition 16b implies that $u^{(\beta}$ in (51) will be a constant if the stable subspace U is only composed of riskless assets. Consequently, the out of tails close relationship between y_Π and $y^{(\beta}$ becomes even clearer.

$\varphi_U(u) = u$ *and (51)*. □

Proposition 15 also makes it easy to recover arbitrage free stochastic price processes given by martingales.[9] We can consider the (new risk-neutral) probability measure $Q^{(\beta}$ given by

$$d Q^{(\beta} = y^{(\beta} d \mathbb{P}. \tag{55}$$

We can also consider the Hilbert space $L^2\left(Q^{(\beta}, \mathcal{F}\right)$ composed of those random variables whose square is $Q^{(\beta}-$integrable. Then, $L^2(\mathcal{F})$ and $L^2\left(Q^{(\beta}, \mathcal{F}\right)$ contain the same random variables and have the same topology, *i.e.*, the norms of $L^2(\mathcal{F})$ and $L^2\left(Q^{(\beta}, \mathcal{F}\right)$ are equivalent, and the only difference between both spaces is given by the inner product (Rudin, 1987). Indeed, we have;

Proposition 17 *Under the assumptions and notations of Proposition* 15, *consider the* $\sigma-$*additive measure* $Q^{(\beta}$ *given by (55).*

a) $Q^{(\beta}$ *is a probability measure equivalent to* $\mathbb{P}$.

b) If y *is a* $\mathcal{F}-$*measurable random variable, then* $y \in L^2\left(Q^{(\beta}, \mathcal{F}\right)$ *if and only if* $y \in L^2(\mathcal{F})$. *In other words,* $L^2\left(Q^{(\beta}, \mathcal{F}\right) = L^2(\mathcal{F})$.

c) The natural topologies of $L^2(\mathcal{F})$ *and* $L^2\left(Q^{(\beta}, \mathcal{F}\right)$ *coincide.*

Proof. *a*) In order to prove that $Q^{(\beta}$ is a probability measure we only have to prove that $\mathbb{E}\left(y^{(\beta)}\right) = 1$, which trivially follows from (1), $u_0 = 1 \in U$ (see Example 1) and $\Pi^{(\beta}(u) = \Pi(u)$ for every $u \in U$ (Theorem 5c). Furthermore, (55) obviously implies the $\mathbb{P}-$continuity of $Q^{(\beta}$. Finally, Proposition 15 leads to $y^{(\beta} \geq 1 - \beta > 0$, and therefore

$$d \mathbb{P} = \frac{1}{y^{(\beta}} d Q^{(\beta} \tag{56}$$

implies the $Q^{(\beta}-$continuity of $\mathbb{P}$.

b) Proposition 15, (55) and (56) lead to

$$\int_\Omega y^2 Q^{(\beta}(d\omega) \leq \int_\Omega y^2 \left(\frac{\beta}{1 - \mu} + (1 - \beta)\right) \mathbb{P}(d\omega)$$

---

[9]Recall that the existence of strong compatibility prevents the existence of arbitrage (Example 2).

and

$$\int_\Omega y^2 \mathbb{P}\,(d\omega) \leq \int_\Omega y^2 \left(\frac{1}{1-\beta}\right) \mathsf{Q}^{(\beta}\,(d\omega)\,.$$

Consequently, $\int_\Omega y^2 \mathsf{Q}^{(\beta}\,(d\omega) < \infty$ if and only if $\int_\Omega y^2 \mathbb{P}\,(d\omega) < \infty$.

c) Consider the identity map from $L^2\,(\mathcal{F})$ and $L^2\,(\mathsf{Q}^{(\beta},\mathcal{F})$. In order to show its continuity, and according to the closed graph theorem (Rudin, 1987), we have to see that a sequence $(y_n)_{n=1}^\infty$ converging to 0 in $L^2\,(\mathcal{F})$ and converging to y in $L^2\,(\mathsf{Q}^{(\beta},\mathcal{F})$ will satisfy $y = 0$. Since there exists a sub-sequence $(y_{n_k})_{n_k}$ ($\mathbb{P}$ and Q) almost surely converging to 0 and y (Rudin, 1987), the required equality becomes trivial. With a similar argument, one can prove that the identity map is also continuous from $L^2\,(\mathsf{Q}^{(\beta},\mathcal{F})$ to $L^2\,(\mathcal{F})$. $\qquad\square$

Remark 10 *(On the martingale property). Consider the assumptions and notations of Proposition 15. Consider also a set of trading dates $\mathcal{T} \subset [0,T]$ (see Example 1) such that $\{0,T\} \subset \mathcal{T} \subset [0,T]$. Suppose that the filt ation $(\mathcal{F}_\tau)_{\tau \in \mathcal{T}}$ represents the arrival of information and satisfie $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. As in Proposition 17, and under the obvious notation, it is easy to prove that $L^2\,(\mathsf{Q}^{(\beta},\mathcal{F}_\tau) = L^2\,(\mathcal{F}_\tau)$ holds for every $\tau \in \mathcal{T}$ and both spaces have the same topology. Actually, (55) and (56) easily lead to*

$$d\mathsf{Q}_\tau^{(\beta} = \mathbb{E}\left(y^{(\beta}\,\middle|\,\mathcal{F}_\tau\right) d\mathbb{P}_\tau \quad and \quad d\mathbb{P}_\tau = \mathbb{E}_{\mathsf{Q}^{(\beta}}\left(\frac{1}{y^{(\beta}}\,\middle|\,\mathcal{F}_\tau\right) d\mathsf{Q}_\tau^{(\beta}$$

for every $\tau \in \mathcal{T}$, $\mathbb{E}\,(.|\,\mathcal{F}_\tau)$ denoting the conditional expectation.

For every fina pay-off $y \in L^2\,(\mathcal{F}) = L^2\,(\mathsf{Q}^{(\beta},\mathcal{F})$ one can defin the adapted price process $(y_\tau)_{\tau \in \mathcal{T}}$ by means of

$$y_\tau := \mathbb{E}_{\mathsf{Q}^{(\beta}}\,(y|\,\mathcal{F}_\tau) = \frac{\mathbb{E}\left(y^{(\beta}y\,\middle|\,\mathcal{F}_\tau\right)}{\mathbb{E}\left(y^{(\beta}\,\middle|\,\mathcal{F}_\tau\right)}. \tag{57}$$

Obviously, $(y_\tau)_{\tau \in \mathcal{T}}$ is a martingale under $\mathsf{Q}^{(\beta}$ and therefore this price process is arbitrage-free. Moreover, $y_0 = \mathbb{E}\left(y^{(\beta}y\right) = \Pi^{(\beta}\,(y)$ for $y \in L^2\,(\mathcal{F})$ and $y_0 = \Pi\,(y)$ for $u \in U$. Thus, the price process $(y_\tau)_{\tau \in \mathcal{T}}$ is the modificatio of the whole price trajectory for pay-off y once the initial SDF, y_Π, is replaced by $y^{(\beta}$. In other words, Theorems 5, 8 and 11 also allow us to modify the whole price process in such a manner that initial prices remain the same for pay-offs belonging to the stable subspace, the strong compatibility is recovered, and the absence of arbitrage is guaranteed by (57). $\qquad\square$

Remark 11 *(The ambiguous framework) The non-ambiguous setting ($K = \{1\}$) of this section may be relaxed and every result will remain true (under minor and obvious modification and with the same proof) except Proposition* 16 *and Remark* 9. *In fact, one only have to replace the assumption $K = \{1\}$ with the weaker one $\mathbb{P}\left(\Gamma \geq f \geq \gamma\right) = 1$ for some $\Gamma \geq \gamma > 0$ and every $f \in \mathcal{K}$. If so, then $1 - \beta$ must be replaced by $\gamma(1 - \beta) > 0$ and $1 - \beta + \frac{\beta}{1-\mu}$ must be replaced by $\Gamma\left(1 - \beta + \frac{\beta}{1-\mu}\right)$ in several expressions.* $\square$

Remark 12 *(Incomplete markets) Similarly, if adequate assumptions are imposed, the (maybe ambiguous) incomplete market framework of Example* 2 *also satisfie the results of this section except Proposition* 16 *and Remark* 9. *For instance, an appropriate assumption is as follows; "* $\mathbb{P}\left(\Gamma \geq \varphi_Y\left(f\right) \geq \gamma\right) = 1$ *for some $\Gamma \geq \gamma > 0$ and every $f \in \mathcal{K}$".* $\square$

Example 6 (Binomial model) *Let us present a toy example illustrating Propositions* 15, 16 *and* 17 *above. Consider the binomial model of Example* 4 *and (46), and suppose that the physical probability of "up" is $p = 0{,}9$. Expression (25) easily leads to the SDF*

$$y_\Pi = (0{,}137174211 \qquad 2{,}469135802 \qquad 2{,}469135802 \qquad 44{,}44444444) \,. \qquad (58)$$

Let us draw on the (non ambiguous) $CVaR$ risk measure with the confidenc level $90\,\%$. Bearing in mind Problem (12), Expression (29) and Remark 1, *it is easy to see that we are facing lack of compatibility and a market price of risk $M = \infty$.*

Let us deal with three potential stable subspaces. If $U_1 = \mathcal{L}\left\{(1; 1; 1; 1)\right\}$ is the linear manifold generated by the riskless security, then Problem (24) leads to a market price of risk $M_1 = 0$ on U_1, while for

$$U_2 = \mathcal{L}\left\{(1; 1; 1; 1)\,, (8; 8; 2; 2)\right\}$$

and

$$U_3 = \mathcal{L}\left\{(1; 1; 1; 1)\,, (8; 8; 2; 2)\,, (16; 4; 4; 1)\right\}$$

the same optimization problem gives $M_2 = 1{,}7$ on U_2 and $M_3 = 32{,}3$ on U_3. The main results of Section 3 *imply that for the three subspaces we can recover strong compatibility if y_Π is modified If we select U_1 then every desired new MPR may be reached. Suppose that we choose $M_1' = 1$.*

Then, Proposition 16*, Remark* 9 *and (55) lead to*

$$
\begin{aligned}
y^{(\beta} &= (0{,}530562775 \qquad 2{,}8624741 \qquad 2{,}8624741 \qquad 5{,}499881447) \\
Q^{(\beta} &= (0{,}429755847 \qquad 0{,}257622669 \qquad 0{,}257622669 \qquad 0{,}054998814)
\end{aligned}
\tag{59}
$$

and the proportion between (58) and (59) equals

$$
(0{,}258544734 \qquad 0{,}862587998 \qquad 0{,}862587998 \qquad 8{,}080982267) \tag{60}
$$

Similarly, for U_2*,* $M_2' = 5$*, (due to Theorem* 5*,* M_2' *must be higher than* $M_2 = 1{,}7$*),* U_3 *and* $M_3' = 100$ *(higher than* $M_3 = 32{,}3$*) we have*

$$
\begin{aligned}
y^{(\beta} &= (0{,}166792688 \qquad 2{,}200798628 \qquad 6{,}464019728 \qquad 8{,}506427078) \\
Q^{(\beta} &= (0{,}135102077 \qquad 0{,}198071877 \qquad 0{,}581761776 \qquad 0{,}085064271)
\end{aligned}
\tag{61}
$$

and

$$
\begin{aligned}
y^{(\beta} &= (0{,}010112615 \qquad 3{,}470330287 \qquad 6{,}425019751 \qquad 10{,}12272782) \\
Q^{(\beta} &= (0{,}008191218 \qquad 0{,}312329726 \qquad 0{,}578251778 \qquad 0{,}101227278)
\end{aligned}
\tag{62}
$$

respectively, and the proportions between (58) and (61) and (62) become

$$
(0{,}822423411 \qquad 1{,}121927182 \qquad 0{,}381981477 \qquad 5{,}224807553) \tag{63}
$$

and

$$
(13{,}5646624 \qquad 0{,}711498791 \qquad 0{,}384300111 \qquad 4{,}390560057) \tag{64}
$$

respectively. (60), (63) and (64) confir   the interpretation given in Remark 8*, since their components become far from one for the states of nature* $\{(u, u), (d, d)\}$*, associated with the tails of the binomial model.*

With regard to the martingale property of Remark 10, *bearing in mind (57), (59), (61) and (62), it may be easily seen that the price process (46) will be modifie according to*

$$
\begin{array}{ccccccc}
& & & \nearrow & 16 & & & & \nearrow & 7{,}117379366 \\
& \nearrow & 11{,}50251869 & & & & \nearrow & 5{,}116736826 & & \\
8{,}992073726 & & & \searrow & 4 & \quad 4 & & & \searrow & 1{,}779344842 \\
& \searrow & 3{,}472216555 & \nearrow & & & \searrow & 1{,}544567654 & \nearrow & \\
& & & \searrow & 1 & & & & \searrow & 0{,}44483621
\end{array}
$$

$$\tag{65}$$

if U_1 is the stable subspace. On the left we conserve the fina pay-off $(16; 4; 4; 1)$ and create the price process. On the right we normalize the process so as to get a proportional one with current value equaling 4 (the current price in (46)). Similarly, (46) becomes

$$
\begin{array}{ccccccc}
& & & \nearrow & 16 & & & & \nearrow & 11{,}92687607 \\
& \nearrow & 8{,}866001402 & & & & \nearrow & 6{,}608981247 & & \\
5{,}366032113 & & & \searrow & 4 & \quad 4 & & & \searrow & 2{,}981719017 \\
& \searrow & 3{,}617302273 & \nearrow & & & \searrow & 2{,}696444745 & \nearrow & \\
& & & \searrow & 1 & & & & \searrow & 0{,}745429754
\end{array}
$$

$$\tag{66}$$

if U_2 is the stable subspace, and

$$
\begin{array}{ccccc}
& & & \nearrow & 16 \\
& \nearrow & 4{,}306671441 & & \\
4 & & & \searrow & 4 \\
& \searrow & 3{,}553066674 & \nearrow & \\
& & & \searrow & 1
\end{array}
$$

$$\tag{67}$$

if the stable subspace is U_3. If we compare (46) with the right hand side of (65) then we will see that the volatility of the modifie price process is much lower and, once again, the tail behavior is more stable. The same effect is reflecte in (66) and (67), though it is more moderated because both U_2 and U_3 contain the price in one period $(8; 8; 2; 2)$, which provokes that the risk neutral probabilities of "up" and "down" in (46) remain $1/3$ and $2/3$ between dates 0 and 1. If we compare (66) and (67), the volatility variation in (67) is more noticeable between dates 1 and 2. This is because U_3 contains the fina pay-off $(16; 4; 4; 1)$, and therefore one should not expect significan variations in its price-process.

Overall, (65), (66) and (67) are examples of DS free (and arbitrage free) pricing processes and they reflec the major finding of this paper. The lack of compatibility is a caveat that can be overcome once one fixe the stable subspace and the desired MPR. These two ingredients will be respected, and then the price modificatio will be "as small as possible". Both ingredients have a clear influenc on the fina price process modification and their selection is an important decision, which could be maid by calibration to market. The modifie process will reflec lower volatilities and will moderate the tail behavior of the SDF. $\qquad\square$

Example 7 (Black and Scholes model) *Let us end this section with a toy example involving the Black and Scholes model. Thus, consider Example 1 and a non-ambiguous complete model such that every square-integrable random payoff at $T = 1/4$ (three months) may by replicated by a self-financin portfolio composed of a riskless asset and a risky one satisfying the usual stochastic differential equation*

$$\frac{dS}{S} = 0{,}04dt + 0{,}08dz, \tag{68}$$

where dz is a Standard Brownian Motion. The risk measure will be the $CVaR$ with the confidenc level 70%. Bearing in mind (29), it is clear that every element in Δ will be essentially bounded. Since the SDF of this model has a log-normal distribution, and consequently it is not essentially bounded,[10] Problem (12) is not feasible and Case 4 in Remark 1 applies, i.e., there are DS (see also Remark 4 and (54)). We have selected a market price of risk equal to $0{,}1$ and have applied Proposition 16 in order to obtain $y^{(\beta}$. Once again, the major differences between $y^{(\beta}$ and y_Π are in the tails of $y^{(\beta}$, as illustrated in Table 1 below, where some values at $T = 1/4$ of both $y^{(\beta}$ and y_Π have been simulated. Similarly, Monte Carlo simulation has been used in order to generate realized paths of the underlying asset under both (68) and the price modification implied by $y^{(\beta}$.

---

[10]Recall that the relationship between the SDF at T (y_Π) and the underlying asset at T (S_T) is given by

$$Log(y_\Pi) = -\frac{r}{\sigma^2} Log(S_T/S_0) + \frac{Tr\left(r - \frac{\sigma^2}{2}\right)}{2\sigma^2},$$

S_0 denoting initial price, r denoting drift and σ denoting volatility ($r = 0{,}04$ and $\sigma = 0{,}08$ in (68)).

Table 2 *contains a simulated path for* 12 *quarters (three years).*

$Table\_2$

$Table\_1$		$Initial\_S$	$Modified\_S$
		1	1
$Initial\_SDF$	$Modified\_SDF$	1,058199144	1,022511198
0,505303277	0,909090909	1,014242609	1,04552915
0,603291783	0,909090909	1,028688069	1,069065263
0,720282238	0,909090909	1,058199144	1,058199144
0,859959505	0,909090909	1,088556834	1,086567375
1,026723012	0,965723012	1,84955982	1,111027309
1,225825331	1,164825331	1,119785428	1,104356022
1,463537608	1,212121212	1,121832846	1,129216399
1,747347095	1,212121212	1, 104060723	1,091133384
2,08619297	1,212121212	1,12190991	1,115696103
		1,128950081	1,048003855
		1,123631111	1,071595677

As in Example 6*, one can observe that the underlying asset moderates its tail behavior when the existence of* DS *is overcome.* ☐

6.   Conclusion

The existence of DS with unbounded GSR or MPR is a major caveat affecting the available arbitrage-free Brownian-motion-linked models for pricing and hedging derivatives. This short-coming is provoked by the pricing models themselves, since modification of the risk measure does not prevent the DS existence. Moreover, the risk measure may be replaced by some deviations and the drawback is not overcome.

The DS existence is neither consistent with the intuition nor consistent with the financia equilibrium. The DS existence should provoke additional trading modifying the pricing model itself and leading to "more correct prices".

The empirical evidence reveals that the real GSR of a DS never equals infinit , though it is fre-

quently high. The natural question is how to modify the Brownian-motion-linked pricing model so as to reach a finit but large enough (consistent with the empirical evidence) GSR, and this has been the focus of this paper. It has been shown that every model can be modifie in such a manner that the new SDF satisfie the two requirements above, $i.e.$, absence of DS and suitable optimal GSR. This is important for several reasons; If the existent models predict an unrealistic price evolution for DS, then they may also make mistakes when pricing and hedging some derivatives, provoking errors that may affect practitioners, supervisors and researchers. Secondly, the lack of DS is much more coherent from a theoretical viewpoint and, compatible with equilibrium. Finally, the major discrepancies between the initial model and the modifie one will affect the tails of their SDF, which seems to be also consistent with many previous empirical finding in asset pricing. It is important to point out that the purposes of this paper were theoretical. The modifie pricing model depended on the desired MPR and the proposed stable subspace. In practice, these ingredients, and the resulting price processes, should be calibrated to market.

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