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AGGREGATE PRODUCTION FUNCTIONS: DOES FIXED CAPITAL MATTER?

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1. Introduction

In a series of papers some years ago,[1] I examined the question of the existence of aggregate production functions. I did so in the context of a model in which each firm's technology was embodied in its capital stock so that capital (of several types) was not physically homogeneous but rather specific to firms while labour and output (each of which could be of several types) were assigned to firms so as to make the entire system produce efficiently. In such a model, the efficient production frontier is readily seen to depend on the amount of each capital type which each firm has and on the total amounts of each labour type and each output type for the system as a whole. The aggregation question was then that of the possibility of simplifying by aggregating over capital (for firms and types) or over labour types or over output types.

The necessary and sufficient conditions for such aggregation turned out to be very strong. Restricting attention to constant returns, they can be roughly summarized as follows: First, an aggregate of the desired kind must exist at the level of each firm. Second,[2] if capital is to be aggregated, firm production functions can differ by at most capital-augmenting technical changes, although the capital to be augmented can already be an otherwise unrestricted aggregate at the firm level. If labour or output is to be aggregated, on the other hand, the corresponding firm-level aggregates must all be the same, but the firm's production functions are not otherwise restricted.[3] In effect, to take output aggregation as an example, this means that every firm must produce the same market basket of outputs as every other firm, differing only as to scale, with the composition of the common basket depending on prices.

These conditions are surely very strong and unlikely to be fulfilled in practice. One question which arises, however, is the extent to which such strength really has to do with the assumption that capital is immobile and technology embodied in it. Surely many of the results depend on that assumption but despite the impetus given to the analysis by the "Cambridge-Cambridge debate",[4] it is not plain that all of them do. Indeed, as we shall see in the present paper, some of the results have at least as much to do with the fact that aggregation over individual productive units (firms) is involved as with the assumption that capital is fixed.

The present paper investigates this question and examines the aggregation issues which arise when all factors are mobile. Aside from the interest in this question which attaches to an investigation of just how much fixity of capital matters, there are some reasons for considering it on its own grounds. First, it may be considered that capital is mobile in the long run. This may be thought of (in the longest run) as the fungibility of capital funds;[5] in some shorter time period there are circumstances in which individual types of capital goods, not interchangeable among themselves, are nevertheless moved about among productive units to secure efficient production. We can think of such cases as brought about by efficient second-hand markets or by a central-planning authority. At a less grand level, we can think of a single firm managing several productive processes and efficiently allocating capital goods as well as other factors to them. The analysis below exhibits the conditions under which the resulting efficient production surface permits of aggregation.

The results obtained are aesthetically rather pleasing. Aside from the obvious fact that mobility of capital goods guarantees the existence of an aggregate over firms for any <u>single</u> capital type (and thus eliminates the need for any other conditions such as capital-augmentation), they are essentially as follows (for constant returns). The efficient production surface for two firms permits aggregation over some group of variables (outputs or factors) if and only if: (1) such an aggregate exists at the level of each firm separately; and (2) <u>either</u> the firm level aggregates are the same in both firms <u>or</u> the two firms' production functions differ by at most an "aggregate-augmenting" technical change.[6] Conditions for aggregation when many firms are involved can be built up from these results.

We thus see the role which fixity of capital plays (beyond preventing the aggregation over firms of a single capital type without special conditions). The mere fact that aggregation over firms is involved, whether or not capital is fixed, restricts aggregation possibilities to the two special cases just described. The effect of assuming capital is fixed is merely to go from a situation in which either special case suffices for aggregation for any group of factors or of outputs to a situation in which one special case applies to aggregation over all fixed factors and the other applies to aggregation over groups of variable factors or of outputs.[7] To put it differently, moving from a fixed to a mobile capital model does not really introduce new cases which permit aggregation; it merely makes the union of previously known cases applicable.

2. The Model

Firms are indexed by v=1, ...,n. Each firm has a production function given by:

$$(2.1)\qquad y(v) = f^{v}(X(v),\ L(v)).$$

This will be interpreted in more than one way. In the simplest interpretation, $y(v)$ is a single output, homogeneous across firms, $X(v)$ is an r-vector of factors to be aggregated and $L(v)$ an s-vector of other factors. $L(v)$ and $X(v)$ are allocated to firms so as to maximize $y = \Sigma y(v)$, subject to the constraints

$$(2.2)\qquad \Sigma L(v) = L;\ \Sigma X(v) = X.$$

Calling the maximized value of output, y^*, this makes y^* a function of L and X, say $G(X,L)$. The question to be analyzed is that of when aggregation over the r factors in X are possible; i.e., when can we write:

$$(2.3)\qquad y^* = G(X,L) = H(\psi(X),L)$$

for some scalar-valued function, $\psi(\cdot)$.

On this interpretation, we can think of X as a vector of (mobile) capital of r different types and $\psi(X)$ as an aggregate capital index. Alternatively, we can think of X as made up of labour of all different types with $\psi(X)$ a total labour index (and L, despite the mnemonic abbreviation) as capital. Next, X can be some, but not all types of labour with L all remaining factors and $\psi(X)$ an aggregate such as "skilled labour" and so on.

Further, it is easy to see that the analysis is not in fact restricted to only a single output. For some purposes, where firms produce different outputs, we can think of $y(v)$ as dollars worth of output at fixed prices so that total revenue is maximized. This locates points on the production possibility frontier. Where firms produce more than one output, we can interpret some elements of L as being outputs alternative to those involved in y.

A second class of interpretations involve output aggregation. Here,

$f^v(\cdot,\cdot)$ must be interpreted as a factor-requirements function, rather than as a production function, with $y(v)$ the amount of the factor demanded given the amount of the other factors, $L(v)$, and the outputs to be produced, $X(v)$. Outputs and other factors are now assigned to firms to minimize total use of y. Aggregation over X now becomes output aggregation. If some outputs are counted in L rather than in X, such aggregation is over certain categories of output and so forth.

To fix ideas, I shall discuss the proofs and give the theorems in terms of aggregation over a set of factors, with (2.1) a production function, remarking on the other interpretations from time to time. In addition, to simplify matters, I shall take $r=2$ and $s=1$ so that (2.1) becomes

$$y(v) = f^v(X_1(v), X_2(v), L(v)) \tag{2.4}$$

and aggregation is over the two elements of X. Greater generality in this regard only makes the proofs notationally more complex without changing them in any substantive way. I discuss such generalization below.

I assume each of the $f^v(\cdot,\cdot,\cdot)$ twice continuously differentiable; subscripts denote differentiation in the obvious way.
The $f^v(\cdot,\cdot,\cdot)$ are assumed non-decreasing in their arguments and every factor is assumed productive in at least one use. For most of the paper I also assume constant returns, commenting on generalizations after the main results. The Hessians of the $f^v(\cdot,\cdot,\cdot)$ are assumed negative semi-definite with rank $(r+s-1)$ which is 2 in the case of 3 factors.

Now, it is possible for (2.3) to hold and aggregation over X to be possible without its being the case that aggregation is possible over X within each firm.[8] This is because changes in L, even if they altered the marginal rate of substitution between $X_1(v)$ and $X_2(v)$ within each firm with

$X_1(v)$ and $X_2(v)$ fixed, might just happen to leave that marginal rate of substituion unaffected in the system as a whole through their effects on the reassignment of the $X_i(v)$. An example is given below. However, it is uninteresting to have an aggregate which can be formed over the entire set of firms unless it can be formed over every subset, since the disappearance of one or more firms would destroy the possibility of aggregation. Hence, throughout this paper I use the following strong definition of aggregation:

Definition 2.1: An aggregate over X will be said to exist if and only if an expression in the form (2.3) exists both for the entire set of firms and for every proper subset.

In other words, aggregation must be possible over every subset of firms. It follows that we must assume the equivalent of (2.3) to get anywhere at all, and I now do so.

Assumption 2.1: Each firm's production function can be written in the form:

$$(2.5) \qquad f^v(X(v), L(v)) = F^v(\phi^v(X(v)), L(v))$$

where $\phi^v(\cdot)$ is scalar valued.

Remark 2.1: Since we have assumed constant returns, both $F^v(\cdot,\cdot)$ and $\phi^v(\cdot)$ can be taken to be homogeneous of degree one in their respective arguments.

Of course the restriction in Assumption 2.1 is already very strong.

3. Aggregation With Two Firms

The heart of the analysis will be performed for two firms (n=2). As will be apparent later the results for n firms follow readily.

Now, there are certain cases which permit aggregation but are basically uninteresting. The first of these, additive separability of each $F^v(\cdot,\cdot)$ in $\psi^v(\cdot)$ and $L(v)$, has already been ruled out for constant returns by our assumption as to the rank of the Hessian of $f^v(\cdot,\cdot)$. Another case remains, however. It may be that efficient factor allocation sometimes requires that all of some factor be allocated to one firm with none allocated to the other. In any region in which this is true for both X_1 and X_2, with all of both X_1 and X_2 allocated to the same firm, it is plain that aggregation is trivial. Such an extreme case can safely be ignored, however, and I shall assume it does not occur. The less extreme cases where not all factors are allocated to both firms remain of interest, however.

One related remark before proceeding. As indicated in the introduction, aggregation will turn out to be possible under either one of two conditions, the first in which the $F^v(\cdot,\cdot)$ can be taken to be the same and the second in which the $\phi^v(\cdot)$ can be taken to be the same. It is possible for one of these conditions to hold for some values of the variables and the other to hold for different values. I shall be precise about this in the proofs, but it is inconvenient to have to remember it in the discussion, so I generally speak of the $F^v(\cdot,\cdot)$ the same or the $\phi^v(\cdot)$ the same as though a given condition held everywhere.

I now proceed to the results. The basic theorem on aggregation to be used is well-known theorem of Leontief (1947a, 1947b) which states that aggregation is possible if and only if the marginal rates of substitution among variables in the aggregate are independent of variables left out of it. The "regions" mentioned in the analysis are in the space of total X_1, X_2 and L. "Aggregation" always means aggregation over X.

<u>Lemma 3.1:</u> In an open region in which all of X_1 and all of L are assigned to firm 1 and all of X_2 assigned to firm 2, aggregation is not possible.

<u>Proof:</u> In this case, changing the amount of L affects the marginal product of X_1 but not that of X_2.

<u>Lemma 3.2</u>: In an open region in which all of either X_1 or X_2 is allocated to a single firm, aggregation is possible if and only if the $F^v(\cdot,\cdot)$ can be taken to be the same, v=1, 2.

<u>Proof</u>: If both the $F^v(\cdot,\cdot)$ can be taken to be the same, then L will certainly be allocated to both firms so as to make the ratio $L(v)/\phi^v(X(v))$ the same in both. Further, by Lemma 3.1, if L is <u>not</u> allocated to both firms, aggregation is impossible. Hence we may as well proceed by assuming that it is so allocated.

Now suppose that all of X_1 is allocated to firm 1 everywhere in the region in question. Then, as L changes, the fact that X_1 is "really" mobile plays no role and we might as well consider it as fixed capital. If X_2 is also fixed (totally allocated to firm 2), then the desired result follows from the theorem for fixed capital aggregation (Theorem 3.2 of Fisher (1965), p. 268) which states that the only differences between the $F^v(\cdot,\cdot)$ must be capital-augmenting. We can take the values of the capital-augmenting parameter into the definition of the $\phi^v(\cdot)$. If X_2 is allocated to both firms, the same result follows from the similar theorem on aggregation over fixed and movable capital goods (Theorem 4.1 of Fisher (1968b), p. 421).

The opposite case, in which the X_i are allocated to both firms but L is not is handled by:

<u>Lemma 3.3</u>: In any open region in which the X_i are assigned to both firms but L is assigned to only one firm, aggregation is possible if and only if the $\phi^v(\cdot)$ can be taken to be the same.

<u>Proof</u>: In such a region, L might just as well be immobile. Aggregation over X is now aggregation over mobile factors in the presence of immobile ones. The desired result now follows from Theorem 5.1 of Fisher (1968a), p. 397.[9]

All of this cannot be very surprising. It amounts to observing that in open regions in which some factor is allocated to only one firm, then that factor might as well be treated as immobile. Aggregation conditions then reduce to those already known for the fixed capital case. Thus the cases so far considered do not really exploit the mobility of all factors. We now begin to study the case in which that mobility can be exploited -- the case in which X_1, X_2 and L are assigned to both firms.

Lemma 3.4: Consider any open region in which all three factors are assigned to both firms. If the $F^v(\cdot,\cdot)$ cannot be taken to be the same, then addition of L to the system as a whole requires some reallocation of X_1 and X_2.

Proof: Suppose not. Then we can treat $X_1(v)$ and $X_2(v)$ as fixed (v=1, 2) when considering the effects of changing L. Since marginal products of all three factors must be the same in both uses both before and after the change, it must be the case that:

$$(3.1) \qquad F^1_{LL} \frac{\partial L(1)}{\partial L} = F^2_{LL} \frac{\partial L(2)}{\partial L}$$

and

$$(3.2) \qquad F^1_{\phi L} \phi^1_1 \frac{\partial L(1)}{\partial L} = F^2_{\phi L} \phi^2_1 \frac{\partial L(2)}{\partial L} ,$$

denoting differentiation by subscripts in the obvious way. This implies:

$$(3.3) \qquad \frac{F^1_{\phi L}\phi^1_1}{F^1_{LL}} = \frac{F^2_{\phi L}\phi^2_1}{F^2_{LL}} .$$

Since the $F^v(.,.)$ are homogeneous of degree one, their first partials are homogeneous of degree zero, and Euler's theorem applied to the marginal products of L shows that (3.3) is equivalent to:

$$(3.4) \qquad \frac{L(1)\phi^1_1}{\phi^1} = \frac{L(2)\phi^2_1}{\phi^2} .$$

Since $F^1_\phi \phi^1_1 = F^2_\phi \phi^2_1$, and $F^1_L = F^2_L$, this implies

(3.5) $$F^1_\phi \cdot \frac{\phi^1}{L(1)} + F^1_L = F^2_\phi \frac{\phi^2}{L(2)} + F^2_L$$

which, by another application of Euler's theorem gives

(3.6) $$\frac{F^1}{L(1)} = \frac{F^2}{L(2)} ,$$

so that it must be the case that L has its <u>average</u> product equalized over firms whenever all marginal products are equalized. We can express this by saying that there exists a function $Q(\cdot,\cdot,\cdot)$, independent of v such that, for v=1, 2,

(3.7) $$\frac{F^v}{L(v)} = Q(F^v_\phi \phi^v_1, F^v_\phi \phi^v_2, F^v_L).$$

However, F^v_L is in in one-to-one correspondence with the "quasi-factor ratio", $\phi^v/L(v)$ and this determines average product. Thus the first two arguments of $Q(.,.,.)$ are superfluous and there exists a function $g(.)$, independent of v such that:

(3.8) $$\frac{F^v}{L(v)} = g(F^v_L)$$

so that L has its average product equalized whenever its marginal product is (in the open region in question). This, however, is known to be equivalent to the $F^v(.,.)$ differing by at most a ϕ^v-augmenting technical change.[10] Since such a change can be absorbed into the definition of the $\phi^v(.)$, it follows that the $F^v(.,.)$ can be taken to be the same, which is a contradiction, and the Lemma is proved.

<u>Lemma 3.5</u>: In any open region in which all three factors are assigned to each firm, aggregation is possible only if either (a) the $F^v(.,.)$ can be taken to be the same or (b) the $\phi^v(.)$ can be taken to be the same.

Proof: Consider changing L with X_1 and X_2 (the system totals) constant. If the $F^v(.,.)$ cannot be taken to be the same, this will require reallocation of at least one of the X_i. If the $\phi^v(.)$ cannot be taken to be the same, there must be some point at which the equality of the marginal rate of substitution between X_1 and X_2 in both firms requires that the ratios $X_1(1)/X_2(1)$ and $X_1(2)/X_2(2)$ not be identical. Then reallocation of at least one of the X_i must alter at least one of these ratios. Such alteration will change the marginal rate of substitution between X_1 and X_2 for the system as a whole (as well as within each firm). Thus that marginal rate of substitution is not invariant to changes in total L, and, by Leontief's theorem, no aggregate exists, completing the proof.

The fact that the conditions of Lemma 3.5 are sufficient as well as necessary is developed as part of the principal result of this section for which we are now ready.

Theorem 3.1: An aggregate over two firms exists if and only if, for any open region in the space of the $X_1(v)$, $X_2(v)$, and $L(v)$[11], at least one of (a) and (b) below holds:

(a) The $F^v(.,.)$ can be taken to be the same.

(b) The $\phi^v(.)$ can be taken to be the same.

Proof: Necessity. This follows directly from Lemmas 3.1, 3.2, 3.3 and 3.5.

Sufficiency. Much of the proof of sufficiency can be derived from these Lemmas as well, but a more self-contained proof is more instructive, needed for the case of all three factors assigned to both firms, and desirable for later purposes.

First suppose that the $F^v(\cdot,\cdot)$ can be taken to be the same.[12] Then optimal allocation of L requires that the quasi-factor ratio, $\phi^v/L(v)$ be the same for both firms. Define

$$(3.9) \qquad J \equiv \phi^1(X(1)) + \phi^2(X(2)) .$$

Then whenever L is optimally allocated

$$(3.10) \qquad \phi^1/L(1) = \phi^2/L(2) = J/L .$$

let $\lambda = L(1)/L = \phi^1/J$. Total output, when optimized, will be given by (omitting the superscripts on the $F^v(.,.)$ since they are both the same):

$$(3.11) \qquad \begin{aligned} y^* &= F(\phi^1, L(1)) + F(\phi^2, L(2)) \\ &= F(\lambda J, \lambda L) + F((1-\lambda)J, (1-\lambda)L) \\ &= F(J,L) \end{aligned}$$

by constant returns. It now follows that X_1 and X_2 must be allocated to the two firms to maximize J as defined in (3.9), whence the value of J depends only on X_1 and X_2 and not on L and F(J,L) is seen from (3.11) to be the required aggregate production function.

Now suppose instead that the $\phi^v(\cdot)$ can be taken to be the same.[13] Optimal allocation of X_1 and X_2 then requires that they be assigned in the same ratio to each firm. In that case, however, the ratios $X_1(v)/X_2(v)$ must be the same as the over-all ratio X_1/X_2 for v=1, 2. Since the marginal rate of substitution between X_1 and X_2 is the same within each firm and depends only on the ratio $X_1(v)/X_2(v)$, that marginal rate of substitution in the system as a whole depends only on X_1/X_2 and not on L. The fact that X_1 and X_2 can be aggregated now follows from Leontief's theorem and the proof is complete.

Note that if either the $F^v(.,.)$ or the $\phi^v(.)$ can be taken to be the same, certain of the cases treated in Lemmas 3.1-3.3 cannot arise.

4. More Than Two Firms

Suppose now that there are many firms ($n \geq 2$).

Corollary 4.1: An aggregate over n firms exists if and only if, for any open region in the space of $X_1(v)$, $X_2(v)$ and $L(v)$ at least one of (a) and (b) below holds:

(a) All the $F^v(.,.)$ can be taken to be the same.

(b) All the $\phi^v(\cdot)$ can be taken to be the same.

Proof: Sufficiency. The proofs given for sufficiency in Theorem 3.1 do not require n=2.

Necessity. By the strong definition of aggregation (Definition 2.1, above), aggregation requires the existence of an aggregate over every pair of firms. Theorem 3.1, therefore, shows that every pair of firms must have either the same $F^v(.,.)$ or the same $\phi^v(.)$ if an aggregate exists. Now suppose that an aggregate exists but that not all firms have $F^v(.,.)$ which can be taken to be the same. Let A be the set of all firms whose $F^v(.,.)$ can be taken to be the same as $F^1(.,.)$. Let B be the set of all remaining firms. Then neither A nor B are empty. However, the existence of an aggregate requires that every firm in A and every firm in B have $\phi^v(\cdot)$ which can be taken to be the same. Evidently, then all the $\phi^v(\cdot)$ can be taken to be the same.

Note that appeal to the strong definition of aggregation is required here. Otherwise we could have the following situation. Suppose n=3 and $F^1(.,.)$ and $F^2(.,\cdot)$ can be taken to be the same but $F^3(.,.)$ is essentially different. Using the construction in the sufficiency proof of Theorem 3.1 which led to (3.11), aggregate over firms one and two to form the aggregate production function $F^1(\tilde{J},\tilde{L})$ where $\tilde{L} \equiv L(1) + L(2)$ and

$$\tilde{J} = \text{Max} \{\phi^1(X(1)) + \phi^2(X(2))\} \tag{4.1}$$

subject to X(1) + X(2) fixed. If it now turns out that $\tilde{J}(\cdot)$ and $\phi^3(\cdot)$ can be taken to be the same, then the third firm can be combined with the aggregate of the first two.

The trouble with this is that unless $\phi^1(\cdot)$ and $\phi^2(\cdot)$ can be taken to be the same, it will not in general be true that they can be taken to be the same as $\tilde{J}(\cdot)$ or $\phi^3(\cdot)$. So if the first firm were not present no aggregate would exist.

Thus, were we to drop the strong definition of aggregation (but retain Assumption 2.1 as to aggregation within each firm), we would permit a somewhat wider class of cases along the lines just described. This widening seems of no practical importance, however.

5. Generalizations

I now briefly consider two generalizations, the first, to more than three factors and the second, to non-constant returns technologies.

The case in which there are more than three factors, with one exception, presents no substantive changes. For example, the proof of Lemma 3.4 would involve matrices in (3.3)[14] but be otherwise unchanged.

The single exception involves the generalization of the cases treated in Lemmas 3.2 and 3.3 above. Suppose that L has more than one element and that one but not all factors in L are assigned to one firm. Suppose further that at least one X_i is not assigned to both firms. Aggregation of X then becomes isomorphic to the problem of capital aggregation when not all capital types are to be included in the aggregate. Closed-form results are not known for this case, although necessary and sufficient conditions are.[15] These conditions are extremely restrictive. The flavor of Theorem 4.1 is preserved even with this exception by observing that at best the assumption that all factors are mobile permits aggregation under various conditions which would hold without such an assumption but does not do more than to permit the union of such conditions.

Further, for the exception just noted to apply, it must be the case that full mobility fails to come into play. I shall ignore this minor complication in the remainder of the discussion.

The case of non-constant return technologies has the property that, in general, no aggregate will exist whether or not the $F^v(\cdot,\cdot)$ or the $\phi^v(\cdot)$ can be taken to be the same.[16] It is easy to see, however, that the necessity as well as the sufficiency results obtained above apply if the $F^v(\cdot,\cdot)$ can be taken to be constant returns in ϕ^v and $L(v)$ and the $\phi^v(\cdot)$ are merely homothetic. They do not require that <u>both</u> the $F^v(\cdot,\cdot)$ <u>and</u> the $\phi^v(\cdot)$ can be taken to be constant returns at the same time.[17] A complete closed-form characterization of the non-constant-returns cases permitting aggregation is not easy to come by.

6. Interpretations

As indicated in the introduction, these results can be interpreted in various ways in terms of factor or output aggregation. I shall give some remarks along these lines for the simplest cases.

In terms of factor aggregation, the existence of aggregates at the firm level (Assumption 2.1) can be thought of as stating that each firm can be regarded as if it had a two-stage production process (although such a description is only a parable and not necessarily a literal one). In the first stage, the factors to be aggregated (the $X_i(v)$) are combined together to produce an intermediate output (the $\phi^v(X(v))$). That intermediate output is then combined with the remaining factors (the $L(v)$) to produce final output. Aggregation of X in the system as a whole can be done if and only if firms are either all alike as regards the first stage of production <u>or</u> all alike as regards the second stage. If they are all alike as regards the first stage (the $\phi^v(\cdot)$ all the same), then the fact that L is mobile plays no role; this is the condition for aggregation of mobile factors when the remainder are fixed.[18] If they are all alike as regards the second stage (the $F^v(\cdot,\cdot)$ all the same), then the fact that the X_i are mobile plays no role; this is the condition for aggregation of fixed factors when the remainder are mobile.[19]

This is not to say that mobility does not help.[20] Apart from the fact that it permits instant aggregation over firms of any single factor, it aids in other ways. If the X_i were fixed, taking the $\phi^v(\cdot)$ as the same would not permit aggregation; if L were fixed, taking the $F^v(\cdot,\cdot)$ as the same would not help.

A different insight into the results can be obtained by considering the case of output aggregation. Here, Assumption 2.1 is the assumption that outputs can be aggregated within each firm with the firm thought of as producing a composite good (the $\phi^v(\cdot)$) which is then split up into individual outputs. The condition that the $\phi^v(\cdot)$ all be the same is the condition that each firm, faced with the same relative output prices, produce the same market basket with individual outputs in the same ratio as every other firm. (What the ratios are can depend on prices.) Firms can then differ in the scale of their composite market basket and in the way in which it is produced, but not in its composition.

The other condition -- that the $F^v(\cdot,\cdot)$ can all be taken to be the same can be thought of as requiring that the production functions with which the composite goods (the $\phi^v(\cdot)$) are produced differ only by Hicks-neutral differences.[21] In this case, however, the make-up of the composite good is unrestricted.

Output aggregation requires, however, either that all composite goods be the same or that the technologies for producing them be essentially the same. One implication of this concerns the case of complete specialization. Suppose that every firm produces a single good, different from that produced by any other firm. Then the $\phi^{v}(\cdot)$ cannot all be the same and aggregation requires only Hicks-neutral differences in technology. With only such differences, however, the production possibility frontier for the entire system will consist only of flats; relative output prices will be fixed (for all outputs actually produced) and it is hardly surprising that output aggregation is possible.

It is interesting to note, however, that even this restrictive case becomes admissible only because all factors are assumed mobile. With fixed capital, output aggregation is possible only with all firms producing the same composite market basket.[22]

When we leave complete specialization in outputs we obtain a somewhat similar, but perhaps less intuitive result. Aggregation of outputs will be possible even if each firm does not produce the same composite bundle as every other, but only if the composite bundles produced are produced with the same technology. In effect, the relative prices of the composite bundles must be fixed.

In general, then, the assumption that all factors are mobile does aid aggregation somewhat. First (a point not the focus of this paper), it permits instantly the aggregation of any single factor across firms. So far as aggregation involving more than one factor is concerned, the conditions are seen still to be very restrictive. Aggregation is possible only under circumstances which, in some sense, would have allowed it anyway. The only gain is that now either the condition which would have permitted aggregation of fixed factors (the $F^{V}(\cdot,\cdot)$ the same) or the condition which would have permitted aggregation of movable factors or outputs (the $\phi^{V}(\cdot)$ the same) will do. Aggregation can thus take place in somewhat wider circumstances; however, in a sense, no new cases are introduced. It thus turns out that aggregation across firms plays quite as big a role in the difficulties of aggregate production functions as does the fixity of capital.

NOTES

1. For present purposes, the important papers in the series are Fisher (1965), (1968a), and (1968b). Fisher (1969) presents a non-technical survey. (I am not, of course, the only one to work in this area. See Fisher (1969) for other references.)

2. Some of the statements which follow assume away the Nataf case in which all firms' production functions are additively separable. See Fisher (1969) and Nataf (1948).

3. There are similar conditions -- and one additional very strong one in the case of capital -- for the construction of partial rather than full aggregates but these will not directly concern us here. See Fisher (1965), (1968a), 1968b), (1969) and (1981).

4. Essentially Robinson (1953-54) and Solow (1955-56).

5. See Brown and Chang (1976) for an analysis which can be so interpreted.

6. This means, in effect, that they will not differ at all, since such changes can be absorbed into the definition of the aggregate.

7. As already observed, the assumption of fixed capital also produces stringent conditions if only some capital types are to be aggregated.

8. In the case in which X is capital and capital is fixed this cannot happen; where L is capital and fixed it may. See Fisher (1965) and (1968a).

9. That theorem is proved for the case in which each firm has a non-zero amount of capital (here L). However, careful reading of the proof of Lemma 3.1 (Fisher (1968a), pp. 394-95) shows that this is immaterial.

10. See Theorems 3.1' and 3.2 of Fisher (1965), p. 268. Note that the fact that X_2 is assigned to both firms is not used in the proof. This corresponds to the fact that the same line of proof as that just given applies

to the case of one of the X's fixed. (See Fisher (1968b), pp. 421-23.) That case was treated as part of Lemma 3.2, above.

11. Note that this is not the same as the open regions used in the Lemmas which were in the space of the factor <u>totals.</u> Of course, it turns out to make no difference.

12. The following proof is based on Solow's proof of the sufficiency of capital-augmentation for the aggregation of fixed capital. (Solow (1964), pp. 104-105. See also Fisher (1969), pp. 559-60.)

13. The following sufficiency proof follows that given in Fisher (1968a), pp. 406-407 for a different theorem.

14. And an appeal to Theorems 7.1' and 7.2 of Fisher (1965), pp. 282-83 instead of Theorem 3.1' and 3.2, p. 268.

15. See Fisher (1965), pp. 274-77, Fisher (1968b), pp. 423-24, Fisher (1969), pp. 561-62, 568-69, and Fisher (1981). Incidentally, the existence of this case shows that the results given in Fisher (1968a) pp. 401-405, that under constant returns absence of specialization is necessary for aggregation of mobile factors in the presence of fixed capital is limited to the case in which <u>all</u> mobile factors are to be included in the aggregate. (The remarks following Theorem 9.1, p. 407, overlook this possibility.) This is not unimportant since, along lines observed in the introduction, output aggregation when capital is fixed <u>and</u> there is more than one mobile factor can be cast into this form. Other cases also arise. The cases permitting aggregation in such circumstances are extremely special, however.

16. Cf. Fisher (1965), p. 270.

17. Cf. Fisher (1965), pp. 270-72, Fisher (1968a), pp. 399-401, and Fisher (1968b) pp. 424-26.

18. See Fisher (1968a).

19. See Fisher (1965). Cf. Fisher (1968b).

20. Cf. Fisher (1968b), pp. 423-24 and Fisher (1969), pp. 568-69.

21. Scalar multiplication of ϕ^v in (2.5) is a Hicks-neutral shift if the $f^v(\cdot,\cdot)$ are interpreted as factor-requirements functions.

22. See Fisher (1968a).

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