# Exchangeable Equilibria

by

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Submitted to the Department of Electrical Engineering & Computer Science
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## Abstract

The main contribution of this thesis is a new solution concept for symmetric games (of complete information in strategic form), the exchangeable equilibrium. This is an intermediate notion between symmetric Nash and symmetric correlated equilibrium. While a variety of weaker solution concepts than correlated equilibrium and a variety of refinements of Nash equilibrium are known, there is little previous work on "interpolating" between Nash and correlated equilibrium.

Several game-theoretic interpretations suggest that exchangeable equilibria are natural objects to study. Moreover, these show that the notion of symmetric correlated equilibrium is too weak and exchangeable equilibrium is a more natural analog of correlated equilibrium for symmetric games.

The geometric properties of exchangeable equilibria are a mix of those of Nash and correlated equilibria. The set of exchangeable equilibria is convex, compact, and semi-algebraic, but not necessarily a polytope. A variety of examples illustrate how it relates to the Nash and correlated equilibria.

The same ideas which lead to the notion of exchangeable equilibria can be used to construct tighter convex relaxations of the symmetric Nash equilibria as well as convex relaxations of the set of all Nash equilibria in asymmetric games. These have similar mathematical properties to the exchangeable equilibria.

An example game reveals an algebraic obstruction to computing exact exchangeable equilibria, but these can be approximated to any degree of accuracy in polynomial time. On the other hand, optimizing a linear function over the exchangeable equilibria is NP-hard. There are practical linear and semidefinite programming heuristics for both problems.

A secondary contribution of this thesis is the computation of extreme points of the set of correlated equilibria in a simple family of games. These examples illustrate that in finite games there can be factorially many more extreme correlated equilibria than extreme Nash equilibria, so enumerating extreme correlated equilibria is not an effective method for enumerating extreme Nash equilibria. In the case of games with a continuum of strategies and polynomial utilities, the examples illustrate that while the set of Nash equilibria has a known finite-dimensional description in terms of moments, the set of correlated equilibria admits no such finite-dimensional characterization.

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*Dedicated to Marianne Cavanaugh,
who once said she would be slightly
disappointed if I never wrote this.*

*Thank you for everything.*

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When writing this thesis I have followed the conventional mathematical style of emphasizing results and proofs thereof rather than the processes by which these arise. Several excellent free software packages have enabled me to compute examples, form conjectures, search for counterexamples, and visualize solutions much more easily and quickly than I ever could have on my own. In particular I would like to credit the authors of the equilibrium solver Gambit [48], the semidefinite program solver SeDuMi [72], the sum-of-squares toolbox SOSTOOLS [59], and the linear matrix inequality parser YALMIP [47].

I will always look back fondly on the summer I spent studying ergodic theory in the SMALL program at Williams College. For the first several years of graduate school I occasionally looked for ways to mix ergodic theory into my game theory research, but the only connection I could find was that I had been drawn to both fields because I liked their names. Eventually an application popped up when I was not looking for it and this forms the basis for Chapter 8. I am grateful to my SMALL advisor Cesar Silva for introducing me to research in mathematics, teaching me ergodic theory, and for showing me the simple proof of Corollary 8.12, which connects ergodicity with extremality and drives the main result of that chapter.

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# Chapter 1

# Introduction

A game is a situation in which multiple agents interact, each led by his own preferences. Game theory seeks to understand such situations, either *describing* how players will play or *prescribing* how they should play. To this end a variety of solution concepts have been proposed to explain how games are played. These all make some assumptions about the players, how rational they are, and how they are connected.

As one would expect, stronger assumptions lead to stronger predictions, but such improvements are tempered by a decrease in the domain of applicability. Since the assumptions may be false or difficult to verify, such conclusions may also be less convincing. On the other hand, weaker solution concepts can lose their meaning in the opposite way, sometimes failing to narrow down the space of possible outcomes at all.

The variety of solution concepts which have been proposed and the abundance of examples of failures of both of these types suggest that such a tradeoff is unavoidable. There is no one "best" solution concept. To understand a game we will often need to employ several to understand what different assumptions about players mean for the outcome and why.

In this thesis we propose a new solution concept in the context of symmetric games, games in which the players are interchangeable. While it would be unreasonable to believe that a game played in real life is exactly symmetric in the sense that the players truly have identical preferences about parallel outcomes, it is a common simplifying assumption which in many cases does not appear far from reality and so it is a reasonable model to study.

For example, we may wish to model a game played by many over the internet, such as bidding for an item in an online auction. Such a situation is a game of incomplete information, because players do not in general know each other's valuations. Nonetheless, it is natural to assume a prior probability distribution over such valuations which is symmetric. If we imagine each player choosing a plan of action as a function of his valuation "before he learns his valuation," then the

| $(u_1, u_2)$ | $W$ | $M$ |
|---|---|---|
| $W$ | (4,4) | (1,5) |
| $M$ | (5,1) | (0,0) |

**Table 1.1.** Chicken. Player 1 chooses rows and player 2 chooses columns.

players do truly face a symmetric situation. In other words, the induced complete information game is symmetric.

Using this type of reasoning we can apply ideas from symmetric games to asymmetric games. Each player imagines himself behind a *veil of ignorance*, unaware of which role he will take in the game. He then imagines taking all roles (or equivalently, a uniform average of the roles) in a symmetrized game. This idea of players "putting themselves in each other's shoes" incorporates a notion of fairness into the analysis of asymmetric games. Rawls explores the philosophical implications of taking this idea as the basis for a theory of justice [60], though in a way which violates some fundamental game-theoretic principles [7, 35].

We take the viewpoint of a Bayesian who has yet to observe the outcome of the game. As shown by Aumann [2], if we believe the players will act rationally (maximizing their utilities given their information) and this is common knowledge, then this restricts our choice of prior over outcomes. In particular the prior must be a correlated equilibrium. Conversely, without further assumptions any correlated equilibrium would do.

In this thesis we will focus on symmetric games of complete information. Given the symmetry, it is natural to focus on symmetric equilibria: those distributions which do not change if the players are relabeled. If statistical independence of actions is to be expected, the result is a symmetric Nash equilibrium, or in other words, a correlated equilibrium in which the players' actions are independent and identically distributed (i.i.d.). More generally, without independence we should expect a symmetric correlated equilibrium.

The main message of this thesis is that unlike the asymmetric case, not all symmetric correlated equilibria are equally reasonable in a symmetric game. We define a subset of these called the exchangeable equilibria and argue that symmetric correlated equilibria which are not exchangeable are unnatural in several senses.

This is most easily illustrated with an example, so consider the game "Chicken" shown in Table 1.1. This is a battle of nerves in which two players are riding their bikes towards each other. Each can choose to be either "Wimpy," and veer off to the side at the last moment, or "Macho," and hold his course. If both players play $M$ they crash, which is unambiguously the worst possible outcome. If both play $W$ then they merely walk away feeling slightly ridiculous. If the players choose different strategies then the Macho one looks good in front of the assembled spectators, the Wimpy one looks bad, and their utilities reflect this.

As pictured here this is a symmetric game: swapping the roles of the players does not change the utilities. This means that both players feel exactly the same about being in parallel situations. The amount the row player would be embarrassed by playing $W$ when the column player plays $M$ is exactly the same as the amount the column player would be embarrassed if the opposite were to happen. Similarly, they both feel precisely the same about the prospect of crashing.

When the situation leads to such a symmetric model, we can also consider a somewhat stronger assumption: that the players are in fact independent copies of a single decision-making agent, so each would pursue the same actions if given the same information. In other words, we could assume that any difference in action is explained purely by a difference in information, rather than by a difference in interpretation or personal preferences (symmetric utilities mean that with respect to the outcome of the game, at least, no such difference in preferences exists). This means assuming that the players base their actions on rolls of the same types of dice and measurements of the same environmental variables, but the resulting actions may differ based on random events such as measurement noise or the inherent unpredictability of a die roll.

If we make such an additional assumption, then conditioned on the state of the world the players' actions will be i.i.d. We define exchangeable equilibria to be those correlated equilibria which are i.i.d. conditioned on some auxiliary variables. Thus the set of exchangeable equilibria contains the symmetric Nash equilibria and is contained in the symmetric correlated equilibria. Any symmetric correlated equilibrium which is not exchangeable cannot be implemented without implicitly breaking symmetry in some way.

In the Chicken example there are two asymmetric Nash equilibria: one player chooses $M$ and the other $W$. Such a situation is stable because neither player wishes to deviate from it if he believes his opponent will not deviate. Viewed as probability distributions over outcomes in the game, we can write these equilibria as $2 \times 2$ matrices:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

If some device, such as a traffic light, randomly chooses between these two distributions with equal probability, the resulting distribution over outcomes is the symmetric correlated equilibrium

$$\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

The players each interpret the internal state of the traffic light in opposite ways. In fact this distribution over outcomes cannot be realized without some symmetry-breaking in the players' interpretation of environmental information. To see this note that distributions over outcomes which are i.i.d. are of the form

$$\begin{bmatrix} 1-p \\ p \end{bmatrix} \begin{bmatrix} 1-p \\ p \end{bmatrix}^T = \begin{bmatrix} (1-p)^2 & p(1-p) \\ p(1-p) & p^2 \end{bmatrix}$$

for some $0 \leq p \leq 1$. The probability that both players choose the same action is $(1-p)^2 + p^2 \geq \frac{1}{2}$. Thus if the players' actions were conditionally i.i.d. on any auxiliary parameter then even unconditionally (by linearity of expectation) both players would choose the same strategy with probability at least $\frac{1}{2}$. This means the symmetric correlated equilibrium above is not exchangeable.

In fact the unique exchangeable equilibrium of Chicken places equal probability on all outcomes:

$$\begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}.$$

In particular it is independent, so it is a symmetric Nash equilibrium. Thus in this case taking the symmetry assumption to its logical conclusion leads to a unique prediction about the outcome of the game, but if we only go halfway then there are other possible outcomes. If we wish to allow symmetry-breaking such outcomes may be reasonable, but otherwise they are not. Viewed another way, the only way for symmetric rational players to completely avoid crashes in Chicken is with a traffic light.

## 1.1 Overview

In this section we outline the body of the thesis, pointing out the contributions made by each chapter in turn.

Chapter 2 covers the background material used frequently in the bulk of the thesis. The concepts and results introduced here are all either known or straightforward extensions of known results. Some background material which is only locally relevant is introduced later.

The main contribution of this thesis is the notion of exchangeable equilibrium defined above. In Chapter 3 we introduce these in their most basic form, in the context of symmetric games. We study the geometry of the set of exchangeable equilibria, proving its basic properties and computing many examples.

The set of exchangeable equilibria includes the symmetric Nash equilibria and is included in the symmetric correlated equilibria. These inclusions can be strict; in fact symmetric correlated equilibria can yield higher social welfare than exchangeable equilibria, which can yield higher social welfare than symmetric Nash equilibria. A symmetric game with an asymmetric Nash equilibrium always has symmetric correlated equilibria which are not exchangeable (under a mild condition on the symmetries of the game). In the case of $2 \times 2$ symmetric bimatrix games the exchangeable equilibria are the convex hull of the symmetric Nash equilibria. For games with more strategies or players these sets can be distinct.

The set of exchangeable equilibria retains many but not all elementary properties of correlated equilibria: it is convex, compact, and semialgebraic (defined by polynomial inequalities), but not polyhedral in general. The existence of exchangeable equilibria can be proven by elementary means, using an extension of Hart and Schmeidler's argument for correlated equilibria [38]. In games with "enough" symmetry, symmetric Nash equilibria are always extreme points of the set of exchangeable equilibria; the corresponding statement with correlated equilibria in place of exchangeable equilibria can fail when there are more than two players [14, 25, 52].

We also give an alternative construction of the set of exchangeable equilibria of a symmetric bimatrix game as a convex relaxation of the set of symmetric Nash equilibria defined in terms of quadratic inequalities. In this case exchangeable equilibria are correlated equilibria which are completely positive when viewed as matrices. To complement this derivation we also give a new characterization of the convex hull of symmetric Nash equilibria in terms of completely positive matrices.

With an understanding of the definitions and geometry in place, Chapter 4 covers game-theoretic interpretations of exchangeable equilibria. We give four ways of viewing exchangeable equilibria of symmetric games. These show that among correlated equilibria, the exchangeable equilibria are the most symmetric, assume the least knowledge of opponents, are the most robust to players' knowledge of the number of opponents, and can be implemented using simple correlation schemes. Such results imply that among correlated equilibria, exchangeable equilibria should be focal (likely to be chosen because they "stand out" in some way). We take these interpretations as evidence that exchangeable equilibria are natural game-theoretic objects to study.

One of these interpretations suggests a way to design tighter convex relaxations of Nash equilibria, which we study in Chapter 5. We call these order $k$ exchangeable equilibria ($k = 1$ reduces to the usual exchangeable equilibria) and show that in games with "enough" symmetry these converge to mixtures of symmetric Nash equilibria as $k$ goes to infinity. Geometrically these have similar properties to exchangeable equilibria, but we do not prove their existence directly for $k > 1$. Constructing such a proof is an open problem; combined with the other results it would yield a new proof of the existence of Nash equilibria.

We use a different modification of the same interpretation to motivate the notion of exchangeable equilibria in asymmetric games in Chapter 6. For simplicity we restrict attention to bimatrix games. The set of asymmetric exchangeable equilibria is convex, compact, and semialgebraic. It is related to but distinct from the symmetric exchangeable equilibria of the symmetrization of the game. There are natural maps from Nash equilibria to symmetric Nash equilibria of the symmetrization to symmetric exchangeable equilibria of the symmetrization to asymmetric exchangeable equilibria of the game to correlated equilibria of the game. In particular the map from symmetric exchangeable equilibria of the symmetrization into asymmetric exchangeable equilibria shows that existence of asymmetric exchangeable equilibria follows automatically from the symmetric theory.

In Chapter 7 we turn to computational questions about exchangeable equilibria. An example in Chapter 3 shows that a game can have a unique exchangeable equilibrium, the probabilities of which are irrational numbers, so computing $\epsilon$-exchangeable equilibria is the best we can hope for. Papadimitriou and Roughgarden [55] turned Hart and Schmeidler's existence proof into an efficient algorithm for computing correlated equilibria (intended for use on games with polynomially many players – with a constant number of players the problem is easily solved by linear programming), so it is natural to try to compute exchangeable equilibria by modifying the algorithm in the same way we modified the existence proof. The results of [55] imply that this modification computes a rational exchangeable equilibrium, a contradiction.

This exposes a previously unknown error in the arithmetic precision analysis of [55]. We give a simple fix for this which computes $\epsilon$-correlated and $\epsilon$-exchangeable equilibria in polynomial time. This algorithm can also compute asymmetric exchangeable equilibria of bimatrix games. It is not known whether an order $k$ exchangeable equilibrium can be computed in polynomial time for any fixed $k > 1$. We show that the problem of optimizing an arbitrary linear functional over exchangeable equilibria is NP-hard.

We then present a variety of linear and semidefinite programming methods for approximating exchangeable equilibria which are useful in practice, particularly for small games where they are exact, but which do not give a priori performance guarantees for larger games. These semidefinite programming methods have been used to compute most of the examples in this thesis.

We move away from exchangeable equilibria in Chapter 8 to discuss extreme points of the set of correlated equilibria, which can be viewed as another type of structured correlated equilibria. It is known that in bimatrix games the extreme Nash equilibria are among the extreme correlated equilibria [25]. We show that there can be factorially many more of the latter, so computing all of these is not an efficient way to compute all extreme Nash equilibria.

We show further than in polynomial games there can be extreme correlated equilibria which are not finitely supported measures. This shows that certain natural attempts to prove the existence of finitely supported correlated equilibria in polynomial games without going through Nash equilibria must fail. It also shows that the set of correlated equilibria does not admit a finite-dimensional representation of the type enjoyed by the Nash equilibria of a polynomial game. This explains past difficulties finding provably efficient algorithms for computing or approximating correlated equilibria in polynomial games [68, 70].

Finally in Chapter 9 we summarize the results of the thesis and in particular draw connections between the work on exchangeable equilibria and extreme correlated equilibria. We close by discussing related open problems.

## 1.2 Previous work

In this section we give context for the results of this thesis. Broadly, these results split into two loosely related parts: the theory of exchangeable equilibria, which forms the bulk of the thesis, and the structure of extreme correlated equilibria, discussed in Chapter 8. We review literature related to these two parts in turn.

### 1.2.1 Leading towards exchangeable equilibria

As we will see, the idea of exchangeable equilibrium arises naturally when one combines the notion of exchangeable random variables with the correlated equilibrium solution concept. Both of these are large areas of study in their own right and we make no attempt at an exhaustive survey of the literature on either. However, several papers have led toward the idea of exchangeable equilibrium in some way or seem to be particularly related to what is accomplished by exchangeable equilibria and we review those here.

One such example is Hart and Schmeidler's elementary proof of existence of correlated equilibria [38]. This was nearly contemporaneous with a similar but independent proof of the same theorem by Nau and McCardle [54]. Papadimitriou and Roughgarden suggested a clever application of the ellipsoid algorithm to turn these ideas into an efficient algorithm for computing correlated equilibria of large games [55]. We will see in Chapter 7 that this algorithm has a technical flaw, but that a repaired version can compute approximate correlated and exchangeable equilibria.

The arguments in [38], [54], and [55] share the somewhat mysterious quality that they work with non-equilibrium product distributions at an intermediate stage, but in the end the product structure vanishes entirely and the result is a correlated equilibrium with no obvious extra structure. Considering what extra structure might be obtained from such an argument, in particular if we require the game to be symmetric and try to make the argument respect the symmetry as much as possible, is one route to our notion of exchangeable equilibrium.

Another example is Brandenburger's survey of epistemic game theory [8]. He mentions that the standard assumption of probabilistic independence of different players' strategies in noncooperative game theory is somewhat suspect and perhaps unnatural. In a footnote he offers the concept of exchangeability as an example of a more natural way to capture our ignorance of the distinctions between random variables. However, this is not explored further.

The notion of exchangeable equilibrium bears at least an outward resemblance to the work of Hillas, Kohlberg, and Pratt on an outside observer's assessment of the outcome of a game [40]. They consider an observer watching a given $n$-player game being played repeatedly. Each time the game is played by a new set of players disjoint from the set of all previous players, so it makes sense to view these interactions as exchangeable. These players do not have access to the history of play, though the observer does.

Hillas, Kohlberg, and Pratt characterize correlated equilibria of the original game by considering when in the extended game the observer can offer advice to some player allowing him to increase his expected payoff. They argue that the players should have a better understanding of the game than the observer, and prove that the limiting distribution of play is a correlated equilibrium if and only if the observer cannot offer such helpful advice. Furthermore, they give a stronger exchangeability-type condition under which play will be a Nash equilibrium.

The paper [40] does not consider solution concepts between Nash and correlated equilibrium. However, its heavy use of exchangeability suggests an ideological kinship with the present work. It seems likely that the concept of exchangeable equilibrium could be reinterpreted in that setup, though we do not do so here.

One interpretation of exchangeable equilibria is that they are those correlated equilibria which extend to symmetric correlated equilibria of games with an arbitrarily large number of identical interactions (Section 4.3). That is to say, they are robust to the number of players: the players could imagine playing these equilibria with any large, even unknown, number of players and could not profitably deviate even if they knew the number of players. This is in contrast to the paper of Myerson on games with many players in which the number of players is modeled probabilistically [51]. That work is in a Bayesian game setting and so not directly comparable to ours, but the distinction between robust equilibria and those which are sensitive to a given probabilistic model is worth making. In a given situation, one assumption or the other may be more natural.

Another interpretation of exchangeable equilibria is that they are those correlated equilibria in which the correlating device takes a specific form: the players choose strategies conditionally i.i.d. on some hidden parameters (Section 4.1). In general it is an interesting to problem to study which equilibria can arise from a particular type of correlating device. Of course the most well-studied class of correlating device makes players choose their strategies independently and corresponds to the mixed Nash equilibria. But there is also Sorin's notion of *distribution equilibria*, correlated equilibria in which each player gets the same payoff conditional on all outcomes [67]. These are in general incomparable to exchangeable equilibria as shown in Example 3.20. Another work of this type is Du's classification of the correlated equilibria which can arise when the players correlate on their hierarchy of beliefs about the play of the game [22].

Exchangeable equilibria have a natural characterization in terms of complete positivity, as does the convex hull of the symmetric Nash equilibria (Section 3.4). These characterizations allow these sets to be computed explicitly for small games or approximated efficiently for larger games using associated semidefinite programs (Section 7.2). With the theory of exchangeable equilibria in place, these computational results are immediate from the known semidefinite relaxations for the completely positive matrices developed by Parrilo [57]. Another semidefinite relaxation has been developed specifically for Nash equilibria of finite games and more general min-max problems by Laraki and Lasserre [45].

### 1.2.2 Literature related to extreme correlated equilibria

The geometry of Nash and correlated equilibria has also been studied extensively. Therefore we again only mention work below if it is directly connected to ours and we do not attempt to be exhaustive.

In Chapter 8 we study extreme points of the set of correlated equilibria of a family of example games which includes both finite games and games with continuous strategy spaces and polynomial utility functions. This builds on previous work which falls roughly into three categories. The first establishes the connection between extreme Nash and correlated equilibria. Second is the study of separable games, a common generalization of finite and polynomial games which serves as an abstract unifying framework to make and test conjectures about such games. Much of this work focuses on bounding the support, or number of strategies played with positive probability, of equilibria. As such we place other results on this topic in this category as well. The third is work on correlated equilibria in infinite games, which require somewhat more care to define and work with than Nash equilibria in infinite games or correlated equilibria in finite games. We discuss each of these categories in turn.

The main result in the first category is that for two-player finite games, extreme Nash equilibria (viewed as product distributions) are a subset of the extreme correlated equilibria. Cripps [14] and Evangelista and Raghavan [25] proved this independently. This result shows that it makes sense to compare the number of extreme Nash and correlated equilibria. It also raises the natural question of whether all extreme Nash equilibria could be enumerated efficiently (say in polynomial time in the size of the output) by enumerating the extreme correlated equilibria.

In a similar vein, Nau et al. [53] show that for non-degenerate finite games with any number of players, the Nash equilibria lie on the boundary of the correlated equilibrium polytope. With three or more players, the Nash equilibria need not be extreme correlated equilibria. For example the three-player poker game analyzed by Nash in [52] has rational payoffs, hence rational extreme correlated equilibria, but its unique Nash equilibrium uses irrational probabilities.

The second category of previous work covers separable games, which are a class of games including polynomial games and finite games which share many properties of finite games. These were first studied during the 1950's by Dresher, Karlin, and Shapley in papers such as [21], [20], and [44], which were later combined in Karlin's book [43]. Their work focuses on the zero-sum case, which contains some of the key ideas for the nonzero-sum case. In particular, they show how to replace the infinite-dimensional mixed strategy spaces (sets of probability distributions over compact metric spaces) with finite-dimensional moment spaces. Carathéodory's theorem [5] then applies to show that finitely-supported Nash equilibria exist.

There are many similarities between separable games and finite games whose payoff matrices satisfy low-rank conditions. Lipton et al. [46] consider two-player finite games and provide bounds on the cardinality of the support of extreme Nash equilibrium strategies in terms of the ranks of the payoff matrices. The main technical tool here is again Carathéodory's theorem.

Germano and Lugosi show that in finite games with three or more players there exist correlated equilibria with smaller support than one might expect for Nash equilibria [29]. The proof is geometrical; it essentially views correlated equilibria as living in a subspace of low codimension and it too uses Carathéodory's theorem.

The bounds on the support of Nash and correlated equilibria in finite and separable games of the previous three paragraphs are all synthesized in [68]; the portion on Nash equilibria has appeared in [69]. The general idea is that simple payoffs (low-rank matrices, low-degree polynomials, etc.) lead to simple Nash equilibria (small support), and those in turn lead to simple correlated equilibria (small support again).

To produce upper bounds on the minimal support of correlated equilibria which depend only on the rank of the payoff matrices and not on the size of the strategy sets, [68] does not bound the support of all extreme correlated equilibria, but rather only those whose support is contained inside a Nash equilibrium of small support, which must exist. Similar results hold for polynomial games with, for example, degree used in place of rank (the notions of degree and rank are generalized in [68] and [69]).

This work left open the question of whether all extreme correlated equilibria have support size which can be bounded in terms of the rank of the payoff matrices, independently of the size of the strategy sets. We show that this is not the case, because our examples have payoffs which are of rank 1 and extreme correlated equilibria of arbitrarily large, even infinite (when the strategy spaces are) support.

The third category of previous work concerns correlated equilibria without finite support, which have been defined and studied by several authors. An important example of this line of research is the paper by Hart and Schmeidler [38]. The definition of correlated equilibria presented in [38] is convenient for proving some theoretical results (they focus on existence) but not usually for computation.

Equivalent characterizations of correlated equilibria in continuous games which are more suitable for computation are developed in [70]. One of these forms the basis for the analysis in Section 8.4. Other such characterizations lead to algorithms for approximating correlated equilibria of continuous games [70].

# Chapter 2

# Background

This chapter covers a variety of standard material required repeatedly throughout the remaining chapters. Additional background is given in later chapters when it is only needed locally.

No claim is made to the originality of anything in this section except the exposition. We define several new terms: *internal* and *external correlated equilibrium*, *good response*, *good set*, *player-transitive*, *player trivial*, *strategy-trivial*, *completely positive tensor*, and *doubly nonnegative tensor*. For some of these the corresponding theory may not have been developed in detail before, but these are natural extensions or applications of standard concepts.

We work with families of probability distributions throughout, so we first fix some notation. For a topological space $T$ we will write $\Delta(T)$ to denote the set of regular Borel probability measures on $T$. Often $T$ will be finite and given the discrete topology by default. If $T$ is a product it will be given the product topology. The mass of a singleton $t \in T$ under a probability measure $\pi \in \Delta(T)$ will be denoted $\pi(t) := \pi(\{t\})$ for simplicity. The distribution which assigns unit mass to $t \in T$ will be denoted $\delta_t \in \Delta(T)$. For finite $T$ we will view elements of the simplex $\Delta(T)$ as column vectors and elements of $\Delta(T^2) := \Delta(T \times T)$ as matrices. To avoid trivialities in our discussion we will often tacitly assume that $|T| \geq 2$.

The symbol $\diamondsuit$ will signal the end of an example throughout, just as $\square$ signals the end of a proof.

## 2.1 Game theory

This section is divided into three parts. In the first part we lay out the basic definitions of finite games and equilibria. The second part reviews Hart and Schmeidler's proof of the existence of correlated equilibria [38]. The third part covers symmetries of games.

### 2.1.1 Games and equilibria

Throughout we will consider only games of complete information in strategic (also called normal) form. That is to say, each player will have a set of possible strategies among which he may choose, perhaps randomly. All players choose their actions simultaneously and then each is awarded a payoff based on the choices of all players. Each player seeks to maximize his payoff in the presence of the other players' actions. All the data defining the game is common knowledge: everyone knows it, everyone knows that everyone knows it, ad infinitum. This is one of the most classical types of game, and the methods of this thesis could certainly be extended to more modern or realistic models. We restrict to this simple case to avoid distractions from the key ideas.

We will extend some of the definitions of this section to certain infinite games in Chapter 8. Until then, all games are finite. More formally:

**Definition 2.1.** *A **(finite) game** has a finite set $P$ of $n \geq 2$ players, each with a finite set $C_i$ of $m_i := |C_i| \geq 2$ **strategies** (also called **pure strategies**) and a **utility** or **payoff function** $u_i : C \to \mathbb{R}$, where $C = \prod C_i$. A game is **zero-sum** if it has two players, called the **maximizer** (denoted $M$) and the **minimizer** (denoted $m$), and satisfies $u_M + u_m \equiv 0$.*

We will also use $m := \max_i m_i$, especially when the $m_i$ are all equal, with the distinction between the identity of the minimizer in a zero-sum game clear from context. For elements of $C_i$ we use Roman letters subscripted with the player's identity, such as $s_i$ and $t_i$. We will typically use the unsubscripted letter $s$ to denote a **strategy profile**, a choice of strategy for each player. For a choice of a strategy for all players except $i$ we use the symbol $s_{-i}$. This allows the abuse of notation (ubiquitous in game theory) in which we write $(t_i, s_{-i})$ for the strategy profile $(s_1, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_n)$ to avoid a proliferation of ellipses.

**Definition 2.2.** *A **mixed strategy** for player $i$ in a game $\Gamma$ is a probability distribution over his pure strategy set $C_i$, and the set of mixed strategies for player $i$ is $\Delta(C_i)$. The set of **mixed strategy profiles** (also called **independent** or **product distributions**) will be denoted $\Delta^\Pi(\Gamma) := \prod_i \Delta(C_i)$.*

For independent distributions it is important that we write $\Delta^\Pi(\Gamma)$ rather than $\Delta^\Pi(C)$, because $\Gamma$ specifies how $C$ is to be thought of as a product. For example, the set $S \times S \times S$ could be viewed as a product of three copies of $S$, or a product of $S$ with $S \times S$, and these lead to different notions of an independent distribution – one is a product of three terms and one is a product of two terms. This distinction will be particularly important when we define powers of games in Section 5.2.

To make the notation fit together we will write $\Delta(\Gamma)$ for $\Delta(C)$. We may then view $\Delta^\Pi(\Gamma)$ as the (nonconvex) subset of $\Delta(\Gamma)$ consisting of product distributions or as a convex subset of $\mathbb{R}^{\sqcup_i C_i}$. The former view will be natural when we define exchangeable equilibria, which live in $\Delta(\Gamma)$, as convex combinations of product distributions. The latter will be useful when looking for product distributions which are fixed by a group action (see the proof of Lemma 3.15); such fixed distributions are easy to find with a convex setup (Proposition 2.22). Which of these views we are using will be clear from context if not explicitly specified.

As usual we extend the domain of $u_i$ from $C$ to $\Delta(\Gamma)$ by linearity, defining $u_i(\pi) = \sum_{s \in C} u_i(s)\pi(s)$, the expected value of $u_i$ under the probability distribution $\pi$. Having done so we can define equilibria. These are distributions over strategy profiles which are self-enforcing in the sense that if a player believes the other players will play according to the distribution, it is in his own best interest to do so as well. The case of product distributions yields the famous Nash equilibria [52]. Without assuming a product structure one obtains Aumann's notion of correlated equilibria. Both are defined below and will be used throughout.

The term *equilibria* is used because these distributions are defined by a fundamental stability property. The use of equilibria in analyzing games is justified in a number of ways. One of the most important is Aumann's proof that if players commonly believe in a probabilistic model describing outcomes of a game, then this distribution must be a correlated equilibrium [2]. This result holds under the standard assumptions on noncooperative games: the game being played is common knowledge, and the players are rational (utility-maximizing) and act independently.

It is natural to expect that under any dynamic procedure for adjusting the players' strategy choices based on self-interest, such distributions would be stable (attracting or not). There is a wealth of literature on such dynamics, when they reach an equilibrium, and which equilibria can be reached, e.g. [36, 37, 65, 76]. Despite the failure of some such procedures to converge and other philosophical objections, equilibria provide a good starting point for understanding games and continue to be widely used when analyzing games in practice.

Even in games where equilibria do not seem to represent reasonable outcomes, this failure often offers some insight. After computing equilibria of a given game we may find that we should not expect real players to act in a perfectly rational, utility-maximizing manner. They may choose to sacrifice some amount of utility in favor of computational complexity or ease of implementation, or they might believe there is some chance their opponents will do this. In this thesis we will set aside most such philosophical issues (except those related to symmetry of games and solutions) and take standard equilibrium notions as given.

**Definition 2.3.** *A **Nash equilibrium** is an n-tuple* $(\rho_1, \ldots, \rho_n) \in \Delta^{\Pi}(\Gamma) = \prod_i \Delta(C_i)$ *of mixed strategies, one for each player, such that* $u_i(s_i, \rho_{-i}) \leq u_i(\rho_i, \rho_{-i})$

*for all strategies* $s_i \in C_i$ *and all players* $i$. *The set of Nash equilibria of a game* $\Gamma$ *is denoted* $\mathrm{NE}(\Gamma)$.

Equivalently:

**Proposition 2.4.** *A Nash equilibrium is an* $n$*-tuple* $(\rho_1, \ldots, \rho_n) \in \Delta^{\Pi}(\Gamma)$ *such that only best responses are played with positive probability:*

$$\rho_i(s_i) > 0 \Rightarrow s_i \in \arg\max_{t_i \in C_i} u_i(t_i, \rho_{-i}).$$

Aumann generalized this equilibrium notion for situations in which there may be some correlation between players' strategy choices, either due to common observations of (noisy) environmental variables or an explicit mediator advising all interested parties [1]. In either case we assume each player knows his own strategy and so is free to consider deviations which depend on this. Such an assumption leads to the following definition (only requiring this condition for constant functions $\zeta_i$ yields the notion of **weak correlated equilibrium**).

**Definition 2.5.** *A* ***correlated equilibrium*** *is a joint distribution* $\pi \in \Delta(\Gamma)$ *such that* $\sum_{s \in C} [u_i(\zeta_i(s_i), s_{-i}) - u_i(s)]\, \pi(s) \leq 0$ *for all functions* $\zeta_i : C_i \to C_i$ *and all players* $i$. *The set of correlated equilibria of a game* $\Gamma$ *is denoted* $\mathrm{CE}(\Gamma)$.

Nash equilibria correspond exactly to the correlated equilibria which are product distributions, so viewing $\Delta^{\Pi}(\Gamma)$ as a subset of $\Delta(\Gamma)$ we can write $\mathrm{NE}(\Gamma) = \mathrm{CE}(\Gamma) \cap \Delta^{\Pi}(\Gamma)$. We introduce the existence theorems for correlated and Nash equilibria in Sections 2.1.2 and 2.1.3.

The correlated equilibrium conditions are expressed by $\sum_{i=1}^{n} m_i^{m_i}$ linear inequalities. This number can be reduced to $\sum_{i=1}^{n} m_i(m_i - 1)$ in the standard way:

**Proposition 2.6.** *A joint distribution* $\pi \in \Delta(\Gamma)$ *is a correlated equilibrium if and only if* $\sum_{s_{-i} \in C_{-i}} [u_i(t_i, s_{-i}) - u_i(s)]\, \pi(s) \leq 0$ *for all strategies* $s_i \neq t_i \in C_i$ *and all players* $i$.

This proposition can be rephrased as follows. Suppose $(X_1, \ldots, X_n)$ is a random vector taking values in $C$. We think of $X_i$ as a (random) strategy recommended to player $i$. Given this information, player $i$ can form his conditional beliefs $\mathrm{Prob}(X_{-i} \mid X_i)$ about the recommendations to the other players given his own recommendation. That is to say, $\mathrm{Prob}(X_{-i} \mid X_i)$ is a random variable taking values in $\Delta(C_{-i})$ which is a function of $X_i$. One can then define the event

$$\{\text{pure strategy } X_i \text{ is a best response to distribution } \mathrm{Prob}(X_{-i} \mid X_i) \text{ for all } i\}.$$

The distribution of $(X_1, \ldots, X_n)$ is a correlated equilibrium if and only if this event happens almost surely. More succinctly:

**Proposition 2.7.** *Let $(X_1, \ldots, X_n)$ be a random vector taking values in $C$ distributed according to $\pi \in \Delta(\Gamma)$. Then $\pi$ is a correlated equilibrium if and only if $X_i$ is a best response to* $\mathrm{Prob}(X_{-i} \mid X_i)$ *almost surely for all $i$.*

Sometimes we consider correlated equilibria in a more general sense, in which players may have access to information besides their actions. The general framework for doing so is illustrated in Figure 2.1. There is some underlying (random) state of the world which is not assumed to be known to the players. The information available to each player is an arbitrary function of this state and some private random noise; the state and all the noise signals are assumed independent. This random noise could for example be measurement noise, or the outcome of some private coin tosses on which a player will base his action – anything which we wish to explicitly assume the other players cannot access.

For mathematical simplicity we typically think of the state of the world, the noises, and the players' information as all taking values in some finite sets. If this is not the case we must make some measurability restrictions. We speak informally and avoid assigning symbols to everything involved to prevent a pointless explosion of notation.

Each player chooses a function $f_i$ mapping his information to actions. We refer to all the data together – the state distribution, the noise distributions, the maps from these to information, and the $f_i$ – as a **correlation scheme.** We assume these data are known to the players: only the realizations of the random quantities are hidden. If no player can improve his expected utility by unilaterally deviating to a different function $f_i'$, we refer to the correlation scheme as an **external correlated equilibrium.** When we wish to emphasize the distinction we will refer the content of Definition 2.5 as an **internal correlated equilibrium.** These two notions were studied by Aumann in [1] and [2], respectively[1], and are closely related:

**Proposition 2.8.** *The distribution of actions in an external correlated equilibrium is an internal correlated equilibrium and every internal correlated equilibrium arises in this way.*

*Proof.* Given an external correlated equilibrium, no player can gain by deviating from $f_i$. In particular no player can gain by deviating to $\zeta_i \circ f_i$ for any $\zeta_i : C_i \to C_i$. Therefore if we push the $f_i$ back into the unobserved part of the model in Figure 2.1, merging it into the noise-adding stage, the result is an external correlated equilibrium in which all players choose the identity function from their

[1]Aumann refers to both notions simply as *correlated equilibrium.* We know of no standard terminology for differentiating the two, and so introduce the terms *internal* and *external* here.

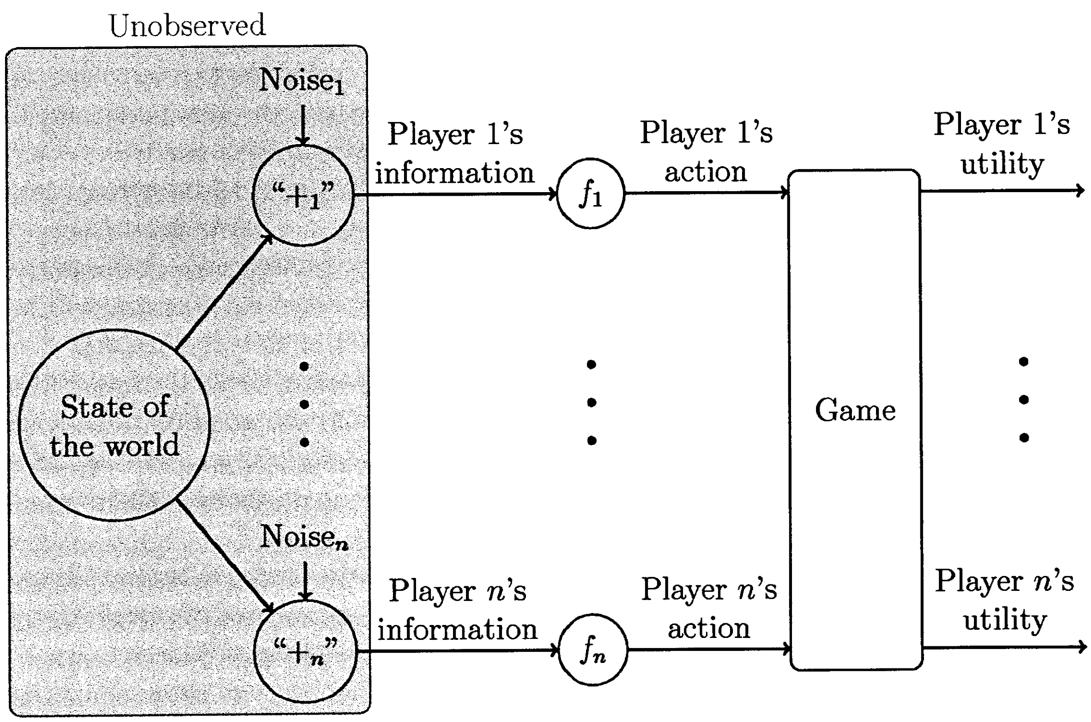


**Figure 2.1.** In a correlation scheme, each player receives noisy information about the state of the world (the "$+_i$" indicates the state is being combined with the noise somehow, not necessarily additively) and chooses his action in the game as a function $f_i$ of this information. If no player can improve his utility by playing a different function of his information, we call all this data (the functions and the information structure together) an external correlated equilibrium.

information to their action. This coincides with the definition of what it means for the distribution over information / actions to be an internal correlated equilibrium.

Conversely, given a $\pi \in \Delta(C)$ we can design a correlation scheme in which the state of the world is a random strategy profile distributed according to $\pi$, each player's information is equal to his personal component of this strategy profile (so the "noise adding" step just strips away the other players' choices of strategy – no additional randomness is needed), and each $f_i$ is the identity on $C_i$. By the definitions this is an external correlated equilibrium if and only if $\pi$ is an internal correlated equilibrium. □

For a given correlation scheme we say that a player **knows the state of the world** if the state of the world is a function of (measurable with respect to) his information. This means that player knows all relevant information which he can know in theory, but not the outcomes of the other players' private environment measurements or coin tosses. We have built some redundancy into the model in the sense that we could have chosen to push all the random noise into the state of the world, making each player's information a deterministic function of the state. The resulting model would allow for less flexibility, in the sense that players knowing the state of the world would mean there could be no private coin tosses. Compare the following statements:

**Proposition 2.9.** *In an external correlated equilibrium, if each player knows the state of the world then the outcome conditioned on the state is a Nash equilibrium almost surely, and every Nash equilibrium arises in this way.*

**Proposition 2.10.** *In an external correlated equilibrium, if each player knows the state of the world and all the noise signals are constant (or equivalently are considered part of the state) then the outcome conditioned on the state is a* pure *Nash equilibrium almost surely, and every pure Nash equilibrium arises in this way.*

The proofs of these propositions amount to little more than repeating the definitions. The main idea is that if the players commonly know the state, then conditioned on this knowledge their play is independent and each is best replying to his opponent. That is to say, the outcome is conditionally a Nash equilibrium, which must be pure if the players cannot privately randomize.

We complete our survey of basic concepts in game theory with a brief discussion of zero-sum games, the first class of games to be studied in detail. The most important result about zero-sum games is that they admit a *value*, but to define this we need to introduce the Minimax Theorem. We will also use this theorem in the following section to prove the existence of correlated equilibria, and similarly in Chapter 3 for exchangeable equilibria.

**Minimax Theorem.** *Let $U$ and $V$ be finite-dimensional vector spaces with compact convex subsets $K \subset U$ and $L \subset V$. Let $\Phi : U \times V \to \mathbb{R}$ be a bilinear map. Then*

$$\sup_{x \in K} \inf_{y \in L} \Phi(x,y) = \inf_{y \in L} \sup_{x \in K} \Phi(x,y),$$

*and the optima are attained.*

The standard modern proof uses the separating hyperplane theorem [5].

**Definition 2.11.** *Given a zero-sum game $\Gamma$, we can apply the Minimax Theorem with $K = \Delta(C_M)$, $L = \Delta(C_m)$, and $\Phi = u_M$. The common value of these two optimization problems is called the* ***value*** *of the game and denoted $v(\Gamma)$. Maximizers on the left hand side are called* ***maximin strategies*** *and the set of such is denoted $\mathrm{Mm}(\Gamma) \subseteq \Delta(C_M)$. Minimizers on the right are called* ***minimax strategies*** *and the set of these is denoted $\mathrm{mM}(\Gamma) \subseteq \Delta(C_m)$.*

We now introduce the notion of a *good reply* in a zero-sum game. This is not a standard definition, but it will simplify the statements of several arguments below. The name is meant to be evocative of the term *best reply*: while a best reply is one which maximizes one's payoff, a good reply is merely one which returns a "good" payoff: at least the value of the game.

**Definition 2.12.** *In a zero-sum game $\Gamma$, we say that a strategy $\sigma \in \Delta(C_M)$ for the maximizer is a* ***good reply to $\theta \in \Delta(C_m)$*** *if $u_M(\sigma, \theta) \geq v(\Gamma)$. We say that a set $\Sigma \subseteq \Delta(C_M)$ of strategies is* ***good against the set*** *$\Theta \subseteq \Delta(C_m)$ if for all $\theta \in \Theta$ there is a $\sigma \in \Sigma$ which is a good reply to $\theta$. If $\Sigma$ is good against $\Delta(C_m)$ we say that $\Sigma$ is* ***good.***[2]

The main result about good sets is:

**Proposition 2.13.** *If $\Gamma$ is a zero-sum game and $\Sigma \subseteq \Delta(C_M)$ is good, then $\Gamma$ has a maximin strategy in $\overline{\mathrm{conv}}(\Sigma)$, i.e., $\overline{\mathrm{conv}}(\Sigma) \cap \mathrm{Mm}(\Gamma) \neq \emptyset$.*

*Proof.* Apply the Minimax Theorem with $K = \overline{\mathrm{conv}}(\Sigma)$ and $L = \Delta(C_m)$. $\square$

It is worth noting that in general a good set need not include a maximin strategy. For example, in any zero-sum game the set $C_M \subsetneq \Delta(C_M)$ is a good set, but some zero-sum games such as matching pennies only have mixed maximin strategies, i.e. $C_M \cap \mathrm{Mm}(\Gamma) = \emptyset$.

The notion of payoff equivalence is a standard way to turn structural information about a game into structural information about equilibria. The definition is chosen to lead directly to the proposition which follows.

[2] *Good sets* are similar in spirit to Voorneveld's *prep sets* [74], but tailored to zero-sum games.

**Definition 2.14.** *Two mixed strategies $\sigma_i, \tau_i \in \Delta(C_i)$ are said to be* ***payoff equivalent*** *if $u_j(\sigma_i, s_{-i}) = u_j(\tau_i, s_{-i})$ for all $s_{-i} \in C_{-i}$ and all players $j$.*

**Proposition 2.15.** *If $\sigma_i$ is payoff equivalent to $\tau_i$ for all $i$, then $(\sigma_1, \dots, \sigma_n)$ is a Nash equilibrium if and only if $(\tau_1, \dots, \tau_n)$ is a Nash equilibrium.*

## 2.1.2 The Hart-Schmeidler argument

In this section we recall the structure of Hart and Schmeidler's proof of the existence of correlated equilibria based on the Minimax Theorem [38]. The goal of this is to frame their argument in a way which will allow us to extend it, redoing as little as possible of the work they have done. We will use a similar argument to prove the existence of exchangeable equilibria (Theorem 3.16).

Hart and Schmeidler's argument begins by associating with a game $\Gamma$ a new zero-sum game $\Gamma^0$ and interpreting correlated equilibria of $\Gamma$ as maximin strategies of this new game. In $\Gamma^0$ the maximizer plays the roles of all the players in $\Gamma$ simultaneously and the minimizer tries to find a profitable unilateral deviation from the strategy profile selected by the maximizer.

**Definition 2.16.** *Given any game $\Gamma$, define a two-player zero-sum game $\Gamma^0$ with $C^0_M := C$, $C^0_m := \bigsqcup_i C_i \times C_i$, and utilities*

$$u^0_M(s, (r_i, t_i)) = -u^0_m(s, (r_i, t_i)) := \begin{cases} u_i(s) - u_i(t_i, s_{-i}) & \text{if } r_i = s_i, \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 2.17.** *Let $\Gamma$ be any game. For any player $i$ in $\Gamma$, $r_i \in C_i$, and $s \in C$ we have $u^0_M(s, (r_i, r_i)) = 0$, so we can bound the value of $\Gamma^0$ by $v(\Gamma^0) \leq 0$. A mixed strategy $\sigma \in \Delta(C^0_M) = \Delta(C)$ for the maximizer in $\Gamma^0$ satisfies $u^0_M(\sigma, (r_i, t_i)) \geq 0$ for all $(r_i, t_i) \in C^0_m$ if and only if $\sigma \in \mathrm{CE}(\Gamma)$. Therefore, if $v(\Gamma^0) = 0$ then $\mathrm{Mm}(\Gamma^0) = \mathrm{CE}(\Gamma)$.*

*Proof.* Immediate from the definitions. □

To prove $v(\Gamma^0) = 0$, and hence the existence of correlated equilibria (Theorem 2.20), we must show that for any $y \in \Delta(C^0_m)$ there is a $\pi \in \Delta(C^0_M)$ such that $u_M(\pi, y) \geq 0$. Hart and Schmeidler actually show that there exists such a $\pi$ with some extra structure, which we summarize in Lemma 2.19. We will exploit this extra structure below to prove Lemma 3.15, a stronger statement in a similar spirit. This in turn allows us to prove the existence of exchangeable equilibria (Theorem 3.16).

Given a $y = (y_1, \dots, y_n) \in \Delta(C^0_m)$, $y_i \in \mathbb{R}^{C_i \times C_i}$, a good reply $\pi$ can be constructed in terms of certain auxiliary games $\gamma(y_i)$. For our purposes it is more important to understand the statement of Lemma 2.19 than to remember the details of this construction. Besides this lemma the only property of $\gamma(y_i)$ we will need is that its definition is independent of how elements of $C_i$ are labeled (Proposition 3.14).

**Definition 2.18.** *For any player $i$ in $\Gamma$ and any nonnegative $y_i \in \mathbb{R}^{C_i \times C_i}$, define the zero-sum game $\gamma(y_i)$ with strategy sets $C_M = C_m := C_i$ and utilities*

$$u_M^{\gamma(y_i)}(s_i, t_i) = -u_m^{\gamma(y_i)}(s_i, t_i) := \begin{cases} \sum_{r_i \neq s_i} y_i^{s_i, r_i} & \text{if } s_i = t_i, \\ -y_i^{s_i, t_i} & \text{otherwise.} \end{cases}$$

**Lemma 2.19** ([38]). *Fix a game $\Gamma$ and consider $\Gamma^0$. If $y \in \Delta(C_m^0)$, then any strategy $\pi \in \mathrm{Mm}(\gamma(y_1)) \times \cdots \times \mathrm{Mm}(\gamma(y_n)) \subset \Delta(C_M^0)$ satisfies $u_M^0(\pi, y) = 0$. In particular $v(\Gamma^0) = 0$, $\pi$ is good against $y$, and $\Delta^\Pi(\Gamma)$ is good.*

*Proof.* First we show $v(\gamma(y_i)) = 0$ for $y_i \geq 0$. If the minimizer plays $\tau_i \in \Delta(C_i)$,

$$u_M^{\gamma(y_i)}(s_i, \tau_i) = \sum_{r_i \neq s_i} [\tau_i(s_i) - \tau_i(r_i)] y_i^{s_i, r_i}$$

is nonnegative when $s_i$ maximizes $\tau_i(s_i)$. Thus choosing $\tau_i$ uniform is a minimax strategy with zero payoff. This strategy is fully supported, so by the Minimax Theorem any $\pi_i \in \mathrm{Mm}(\gamma(y_i))$ satisfies $u_M^{\gamma(y_i)}(\pi_i, t_i) = 0$ for $t_i \in C_i$.

For $\pi \in \Delta^\Pi(\Gamma) \subset \Delta(C_M^0)$ and $y \in \Delta(C_m^0)$,

$$\begin{aligned} u_M^0(\pi, y) &= \sum_i \sum_{s_{-i} \in C_{-i}} \pi_{-i}(s_{-i}) \sum_{s_i, t_i \in C_i} \pi_i(s_i) \left[u_i(s) - u_i(t_i, s_{-i})\right] y_i^{s_i, t_i} \\ &= \sum_i \sum_{s_{-i} \in C_{-i}} \pi_{-i}(s_{-i}) \sum_{s_i, t_i \in C_i} u_i(t_i, s_{-i}) \left[\pi_i(t_i) y_i^{t_i, s_i} - \pi_i(s_i) y_i^{s_i, t_i}\right] \\ &= \sum_i \sum_{s_{-i} \in C_{-i}} \pi_{-i}(s_{-i}) \sum_{t_i \in C_i} u_i(t_i, s_{-i}) \sum_{s_i \neq t_i} \left[\pi_i(t_i) y_i^{t_i, s_i} - \pi_i(s_i) y_i^{s_i, t_i}\right] \\ &= \sum_i \sum_{s_{-i} \in C_{-i}} \pi_{-i}(s_{-i}) \sum_{t_i \in C_i} u_i(t_i, s_{-i}) u_M^{\gamma(y_i)}(\pi_i, t_i), \end{aligned}$$

where the first equality follows by swapping the roles of $s_i$ and $t_i$ in one of the summands. Taking $\pi_i \in \mathrm{Mm}(\gamma(y_i))$ for $i = 1, \ldots, n$ makes every summand zero. $\square$

**Theorem 2.20** ([38]). *For any game $\Gamma$, the value $v(\Gamma^0) = 0$, so $\mathrm{Mm}(\Gamma^0) = \mathrm{CE}(\Gamma)$ and a correlated equilibrium of $\Gamma$ exists.*

*Proof.* Combining Lemma 2.19 and Proposition 2.17, we get $\mathrm{Mm}(\Gamma^0) = \mathrm{CE}(\Gamma)$. Applying Proposition 2.13 to $\Gamma^0$ with $\Sigma = \Delta^\Pi(\Gamma)$ gives existence. □

This proof merits two remarks. First of all, since $\overline{\mathrm{conv}}(\Delta^\Pi(\Gamma)) = \Delta(\Gamma)$, Proposition 2.13 does not yield any benefit in this case over directly applying the Minimax Theorem to $\Gamma^0$. Rather, we have used Proposition 2.13 to illustrate our proof strategy for Theorem 3.16, in which we use a stronger version of Lemma 2.19 to choose $\Sigma$ with $\overline{\mathrm{conv}}(\Sigma) \subsetneq \Delta(\Gamma)$.

Second, note that in this case we know that there is a maximin strategy of $\Gamma^0$ in the good set $\Delta^\Pi(\Gamma)$: this is just the statement of Nash's Theorem (see the following section). However, we cannot conclude this directly from the fact that $\Delta^\Pi(\Gamma)$ is a good set because of the remark after Proposition 2.13.

### 2.1.3 Groups acting on games

The concept of a symmetry of a game extends back at least to Nash's paper [52]. Although we use the language of group theory to discuss symmetries, it is worth noting that we do not use any but the most basic theorems from group theory (e.g., the fact that for any $h$ in a group $G$, the maps $g \mapsto gh$ and $g \mapsto hg$ are bijections from $G$ to $G$).

All groups will be finite throughout. In any group $e$ will denote the identity element. The subgroup generated by group elements $g_1, \dots, g_n$ will be denoted $\langle g_1, \dots, g_n \rangle$. For $n \in \mathbb{N}$ we will write $\mathbb{Z}_n$ for the additive group of integers mod $n$ and $S_n$ for the symmetric group on $n$ letters. We will use cycle notation to express permutations. For example $\sigma = (1\ 2\ 3)(4\ 5)(6)$ is shorthand for

$$\sigma(1) = 2,\ \sigma(2) = 3,\ \sigma(3) = 1,\ \sigma(4) = 5,\ \sigma(5) = 4,\ \text{and } \sigma(6) = 6.$$

**Definition 2.21.** *A* ***left action of the group*** $G$ ***on the set*** $X$ *is a map* $\cdot : G \times X \to X$ *written with infix notation which satisfies the identity condition* $e \cdot x = x$ *and the associativity condition* $g \cdot (h \cdot x) = (gh) \cdot x$. *A* ***right action of*** $G$ ***on*** $X$ *is a map* $\cdot : X \times G \to X$ *such that* $x \cdot e = x$ *and* $(x \cdot g) \cdot h = x \cdot (gh)$.

*We say that an action is* ***linear*** *if it extends to an action on an ambient vector space* $V$ *containing* $X$ *and the map* $x \mapsto x \cdot g$ *on* $V$ *is linear for all* $g \in G$. *An* $x \in X$ *is* $G$***-invariant*** *if* $x \cdot g = x$ *for all* $g \in G$. *The set of* $G$*-invariant elements is denoted* $X_G$.

**Proposition 2.22.** *If* $G$ *acts linearly on the convex set* $X$ *then there is a map* $\mathrm{ave}_G : X \to X_G$ *given by* $\mathrm{ave}_G(x) = \frac{1}{|G|}\sum_{g \in G} x \cdot g$. *In particular if* $X$ *is nonempty then* $X_G$ *is nonempty.*

*Proof.* For any $x \in X$, $\mathrm{ave}_G(x)$ is a convex combination of elements $x \cdot g \in X$, hence $\mathrm{ave}_G(x) \in X$. For any $h \in G$ we have

$$\begin{aligned}\mathrm{ave}_G(x) \cdot h &= \left[\frac{1}{|G|}\sum_{g\in G} x \cdot g\right] \cdot h = \frac{1}{|G|}\sum_{g\in G}(x \cdot g) \cdot h = \frac{1}{|G|}\sum_{g\in G} x \cdot (gh) \\ &= \frac{1}{|G|}\sum_{g\in G} x \cdot g = \mathrm{ave}_G(x),\end{aligned}$$

where we have used linearity, the definition of a group action, and bijectivity of $g \mapsto gh$. ☐

**Example 2.23.** Proposition 2.22 can fail without the implicit assumption that $G$ is finite. For example let $\mathbb{Z}$ act on the convex set $\Delta(\mathbb{Z})$ by translation. An element of the fixed point set $\Delta_{\mathbb{Z}}(\mathbb{Z})$ would be a $p : \mathbb{Z} \to [0, \infty)$ such that $p_i = p_0$ for all $i$ and $\sum_{i\in\mathbb{Z}} p_i = \infty \cdot p_0 = 1$. No such $p$ exists. ◇

A left action of $G$ on $X$ induces right actions on many function spaces defined on $X$. For example $\mathbb{R}^X$ is the space of functions $X \to \mathbb{R}$. For $y \in \mathbb{R}^X$ we can define $y \cdot g \in \mathbb{R}^X$ by $(y \cdot g)(x) = y(g \cdot x)$. The condition that this is a right action of $G$ on $\mathbb{R}^X$ follows immediately from the fact that we began with a left action of $G$ on $X$. For finite $X$ (the case of most interest to us), the same argument shows that $G$ acts on $\Delta(X)$ on the right.

**Definition 2.24.** *We say that* ***a group $G$ acts on the game*** $\Gamma$ *if the following conditions hold. The group $G$ acts on the left on the set of players $P$ and $\bigsqcup_i C_i$, making $g \cdot s_i \in C_{g\cdot i}$ for $s_i \in C_i$. Such actions automatically induce a left action of $G$ on $C = \prod_i C_i$ defined by $(g \cdot s)_{g\cdot i} = g \cdot s_i$. We require that the utilities be invariant under the induced action on the right: $u_{g\cdot i} \cdot g = u_i$, i.e., $u_{g\cdot i}(g \cdot s) = u_i(s)$ for all $i \in P$, $s \in C$, and $g \in G$. We say that $G$ is a* ***symmetry group of*** $\Gamma$, *call elements of $G$* ***symmetries of*** $\Gamma$, *and call $\Gamma$ a* ***symmetric game*** *when $G$ can be inferred from context.*

Note that an action of $G$ on a game can be fully specified by its action on $\bigsqcup_i C_i$ or on $C$. One way to do this is to choose $G$ to be a subgroup of the symmetric group on $\bigsqcup_i C_i$ or $C$ satisfying the above properties.

We will generally view the symmetry group associated to a game as commonly known by the players, just like the rest of the structure of the game (see Section 4.2 for further discussion of this). Reasoning by symmetry, the players can thus view player $i$'s choice between strategies $s_i$ and $t_i$ as the same as player $g \cdot i$'s choice between $g \cdot s_i$ and $g \cdot t_i$ for any $g \in G$. In this way we can transport any assessment of player $i$'s action to player $g \cdot i$ and vice versa. For simplicity we will concern ourselves mostly with the case when the set of players is homogeneous in the sense that any two players can be compared in this fashion. This is captured by the definition of *player-transitivity* below.

Some games have symmetries $g$ with $g \cdot i = i$ and $g \cdot s_i \neq s_i$ for some $i \in P$ and $s_i \in C_i$. Admitting such symmetries corresponds to considering player $i$'s strategies $s_i$ and $g \cdot s_i$ as indistinguishable, so he can never favor one of the other. A game with such symmetries can always be reduced to one without by grouping together strategies thus identified into meta-strategies in which an element of the subset is chosen uniformly at random. The result is a game whose symmetry group is *strategy-trivial*:

**Definition 2.25.** *The* ***stabilizer subgroup of player*** $i$ *is*

$$G_i := \{g \in G \mid g \cdot i = i\},$$

*and acts on* $C_i$ *on the left. We say that the action of* $G$ *is* ***player trivial*** *if* $G_i = G$ *for all* $i$*, or in other words if* $g \cdot i = i$ *for all* $g$ *and* $i$*. The action of* $G$ *is* ***player-transitive*** *if for all* $i, j \in P$ *there exists* $g \in G$ *such that* $g \cdot i = j$*. We call the action of* $G$ ***strategy-trivial*** *if* $g \cdot s_i = s_i$ *for all* $g \in G_i$ *(but not necessarily all* $g \in G$*) and* $s_i \in C_i$*.*

If $g, h \in G$ satisfy $g \cdot i = h \cdot i = j$, then $h^{-1}g \in G_i$. Strategy-triviality means $g \cdot s_i = h \cdot s_i$ for all $s_i \in C_i$, so $G$ identifies $C_i$ with $C_j$ in a canonical way; a similar computation shows that all such identifications are compatible. Player-transitivity means that all the strategy spaces are thus identified. Therefore in a game with a player-transitive strategy-trivial action we often think of all players as having the same strategy set, in which case $G$ acts by permuting the players but leaving the labels of the strategies fixed.

**Definition 2.26.** *A* ***standard symmetric game*** *is a game which has a player-transitive and strategy-trivial symmetry group and which satisfies the additional conditions that* $C_i = C_j$ *for all* $i$ *and* $j$ *and symmetries act by permuting the roles of the players:* $g \cdot (s_{g \cdot 1}, \ldots, s_{g \cdot n}) = (s_1, \ldots, s_n)$*. We will say such a game has a* ***standard symmetry group****.*

We illustrate the notion of group actions on a game using four examples.

**Example 2.27.** Let $\Gamma$ be any game and $G$ any group. Define $g \cdot s = s$ for all $g \in G$ and $s \in C$. This defines a player-trivial and strategy-trivial action of $G$ on $\Gamma$ called the trivial action. ◊

**Example 2.28.** A two-player finite game is often called a **bimatrix game** because it can be described by two matrices $A$ and $B$, such that if player one plays

| $(u_1, u_2)$ | $H_2$ | $T_2$ |
|---|---|---|
| $H_1$ | $(1, -1)$ | $(-1, 1)$ |
| $T_1$ | $(-1, 1)$ | $(1, -1)$ |

**Table 2.1.** Matching pennies. Player 1 chooses rows and player 2 chooses columns.

strategy $i$ and player two plays strategy $j$ then their payoffs are $A_{ij}$ and $B_{ij}$, respectively. If these matrices are square and $B = A^T$ then we call the game a **symmetric bimatrix game**. One example is the game of chicken (Table 1.1), which has $A = \left[\begin{smallmatrix} 4 & 1 \\ 5 & 0 \end{smallmatrix}\right] = B^T$.

To put this in the context of group actions defined above, let each player's strategy set be $C_1 = C_2 = \{1, \ldots, m\}$ indexing the rows and columns of $A$ and $B$. Define $g \cdot (i,j) = (j,i)$ for $(i,j) \in C$, so $g \cdot (g \cdot (i,j)) = (i,j)$. The assumption $B = A^T$ is exactly the utility compatibility condition saying that this specifies an action of $G = \{e, g\} \cong S_2$ on this game. Of course, depending on the structure of $A$ and $B$ there may be other nontrivial symmetries as well. The element $g$ swaps the players but not the strategy labels, so the action of $G$ is standard. $\diamondsuit$

**Example 2.29.** Note that the condition that a bimatrix game be symmetric is *not* that $A = A^T$ and $B = B^T$. Indeed, such a game need not have any nontrivial symmetries. For example, consider the game defined by $A = \left[\begin{smallmatrix} 0 & 2 \\ 2 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. The unique Nash equilibrium of this game is for player 1 to play the mixed strategy $p = \left[\, \frac{1}{4} \; \frac{3}{4} \,\right]$ and player 2 to play $q = \left[\, \frac{1}{3} \; \frac{2}{3} \,\right]$. Since the equilibrium is unique, any symmetry of the game must induce a corresponding symmetry of the equilibrium by Nash's Theorem (below). But the four entries of $p$ and $q$ are all distinct, so the only symmetry of this game is the trivial one. $\diamondsuit$

**Example 2.30.** Consider the game of matching pennies, whose utilities are shown in Table 2.1. The labels $H$ and $T$ stand for heads and tails, respectively, and the subscripts indicate the identities of the players for notational purposes. This a bimatrix game, but it is not a symmetric bimatrix game in the sense of Example 2.28.

Nonetheless this game does have symmetries. The easiest to see is the map $\sigma$ which interchanges the roles of heads and tails. Letting $g$ be the permutation of $\bigsqcup_i C_i$ given in cycle notation as $g = (H_1\ T_1)(H_2\ T_2)$, we define $g \cdot s_i = g(s_i)$. Another symmetry is the permutation $h = (H_1\ H_2\ T_1\ T_2)$. These satisfy $g^2 = e$ and $h^2 = g$, so $G = \langle h \rangle \cong \mathbb{Z}_4$. Note that $g$ acts on $P$ as the identity whereas $h$ swaps the players, so $G$ acts player-transitively, whereas $\langle g \rangle \cong S_2$ acts player-trivially. Neither acts strategy-trivially. In fact there is no way to relabel the strategies to make this a standard symmetric game. $\diamondsuit$

**Example 2.31.** Now we consider an example of an $n$-player game with symmetries. Throughout this example all arithmetic will be done mod $n$. We take the set of players to be $P := \mathbb{Z}_n$ and all strategy sets to be $C_i := \mathbb{Z}_n$. Define

$$u_i(s_1, \dots, s_n) = \begin{cases} 1, & \text{when } s_i = s_{i-1} + 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then we can define a symmetry $g$ by $g(s_i) = s_i + 1$, which increments each player's strategy by one mod $n$, but fixes the identities of the players. Clearly $g$ is a permutation of order $n$.

We can define another symmetry $h$ which maps a strategy for player $i$ to the same numbered strategy for player $i+1$. That is to say, $h$ acts on $C$ by cyclically permuting its arguments. Again, $h$ is a permutation of order $n$. Note that $g$ and $h$ commute, so together they generate a symmetry group $G \cong \mathbb{Z}_n \times \mathbb{Z}_n$. Both $\langle h \rangle \cong \mathbb{Z}_n$ and $G$ act player-transitively, whereas $\langle g \rangle \cong \mathbb{Z}_n$ acts player-trivially. Only $\langle h \rangle$ acts strategy-trivially. If $n$ is composite and factors as $n = kl$ for $k, l > 1$ then $\langle h^k \rangle \cong \mathbb{Z}_l$ acts on $\Gamma$ but neither player-transitively nor player-trivially. $\diamondsuit$

The left actions in the definition of a group action on a game induce linear right actions on function spaces such as $\Delta(\Gamma) \subsetneq \mathbb{R}^C$ and $\Delta^\Pi(\Gamma) \subsetneq \mathbb{R}^{\sqcup_i C_i}$. The inclusion map $\mathbb{R}^{\sqcup_i C_i} \to \mathbb{R}^C$ is $G$-equivariant (commutes with the action of $G$), so with regard to this action it does not matter whether we choose to view $\Delta^\Pi(\Gamma)$ as a subset of $\mathbb{R}^{\sqcup_i C_i}$ or of $\mathbb{R}^C$.

Because of the utility compatibility conditions of a group action on a game, the actions on $\Delta(\Gamma)$ and $\Delta^\Pi(\Gamma)$ restrict to actions on the sets $\mathrm{CE}(\Gamma)$ and $\mathrm{NE}(\Gamma)$, respectively. This allows us to define the $G$-invariant subsets $\Delta_G(\Gamma)$, $\Delta^\Pi_G(\Gamma)$, $\mathrm{CE}_G(\Gamma)$, and $\mathrm{NE}_G(\Gamma)$. The action of the stabilizer subgroup $G_i$ on $C_i$ allows us to define the $G$-invariant subset $\Delta_{G_i}(C_i)$.

**Definition 2.32.** *When the symmetry group is understood from context, elements of* $\mathrm{CE}_G(\Gamma)$ *and* $\mathrm{NE}_G(\Gamma)$ *are called* ***symmetric correlated*** *and* ***Nash equilibria****, respectively.*

The main theorem about symmetric games is:

**Nash's Theorem** ([52]). *A symmetric game has a symmetric Nash equilibrium.*

Compare this with the following trivial improvement on Theorem 2.20.

**Proposition 2.33.** *A symmetric game has a symmetric correlated equilibrium.*

*Proof.* Apply Proposition 2.22 to $\mathrm{CE}(\Gamma)$, which is nonempty by Theorem 2.20. $\square$

A priori we might not expect correlated equilibria with a greater degree of symmetry than predicted by Proposition 2.33 to exist. But viewing $G$-invariant Nash equilibria as correlated equilibria, we see that we can often guarantee much more. Suppose we have an $n$-player game which has identical strategy sets for all players and which is symmetric under cyclic permutations of the players, such as the game in Example 2.31. Then Proposition 2.33 yields a correlated equilibrium $\pi$ which is invariant under cyclic permutations of the players, but need not be invariant under other permutations. On the other hand the Nash equilibrium $\rho = (\rho_1, \dots, \rho_n)$ given by Nash's Theorem satisfies $\rho_1 = \cdots = \rho_n$ so the corresponding product distribution $\pi(s_1, \dots, s_n) = \rho_1(s_1) \cdots \rho_1(s_n)$ is a correlated equilibrium which is invariant under arbitrary permutations of the players. Exchangeable equilibria will be, among other things, correlated equilibria with all of the extra symmetry of symmetric Nash equilibria.

## 2.2 Exchangeability

Throughout the thesis we will deal with collections of random variables which are *exchangeable*, meaning they are in some sense indistinguishable. Such is the case when the measurements they represent are unlabeled or labeled in a way which does not relate to their outcome. We will usually assume these are given to us in some arbitrary order. The defining property of exchangeability, then, is that the joint distribution does not depend on this order.

One example is a sequence of i.i.d. fair coin flips, or indeed any i.i.d. sequence. However, this does not exhaust the possibilities. Another example is the case in which $X_i = X_1$ almost surely for all $i$. This is exchangeable regardless of the distribution on $X_1$, but is only independent in the trivial case when $X_1$ is deterministic.

We begin our formal discussion of exchangeability with finite collections of random variables. The infinite case will follow. Our random variables will take values in a set $T$ with $2 \leq |T| < \infty$ to avoid trivialities on one end and irrelevant complexities on the other.

**Definition 2.34.** *For finite $n$, a probability distribution $\pi \in \Delta(T^n)$ is said to be **$n$-exchangeable** if it is invariant under permutations of the indices, or in other words, if it represents a sequence of random variables $(X_1, \dots, X_n)$ which is equal in distribution to $(X_{\sigma(1)}, \dots, X_{\sigma(n)})$ for all permutations $\sigma$ on $\{1, \dots, n\}$. The set of $n$-exchangeable distributions is denoted $\Delta_{S_n}(T^n)$.*

There is a map $\Delta(T^n) \to \Delta(T^m)$ for any $m \leq n$ given by marginalization onto the first $m$ factors. If the distribution of $(X_1, \dots, X_n)$ is invariant under permutations then clearly so is the distribution of $(X_1, \dots, X_m)$, so this restricts to a natural map $\mu_{n\to m} : \Delta_{S_n}(T^n) \to \Delta_{S_m}(T^m)$. The composition condition $\mu_{m\to l} \circ \mu_{n\to m} = \mu_{n\to l}$ for $l \leq m \leq n$ is immediate.

The image of the map $\mu_{n\to m}$ for $n > m$ is the set of $m$-exchangeable distributions which can be extended to $n$-exchangeable distributions. Why is this important? If we are interested in a sequence of $m$ coin flips, can we not just say the distribution should be an element of $\Delta_{S_m}(T^m)$ and be done? The problem is that even if we are only planning to flip the coin $m$ times or we only care about the outcome of $m$ flips, we generally believe that it would be physically possible to flip the coin an arbitrary number of additional times and obtain results consistent with the first $m$ flips. Our probabilistic model should reflect this.

**Definition 2.35.** *Distributions in the image of $\mu_{n\to m}$ are called $n$-**extendable**.*

**Example 2.36.** Let $m := |T|$. Suppose $X_1$ and $X_2$ have a 2-exchangeable distribution. We will look at one consequence of $n$-extendability: its implications for $\mathrm{Prob}(X_1 = X_2)$.

First suppose the distribution of $X_1$ and $X_2$ is $(m+1)$-extendable and let $X_1, X_2, \ldots, X_{m+1}$ be an $(m+1)$-exchangeable sequence extending these. Since these random variables all take values in a set of size $m$, in any realization the pigeonhole principle implies there will be two which are equal. By exchangeability $\mathrm{Prob}(X_1 = X_2) = \mathrm{Prob}(X_i = X_j)$ for any $i \neq j$. Since there are $\binom{m+1}{2}$ such pairs, the union bound gives $\mathrm{Prob}(X_1 = X_2) \geq \binom{m+1}{2}^{-1}$.

If we merely assume $m$-exchangeability we do not get any such lower bound. To see this note that the distribution in which $X_1, \ldots, X_m$ are assigned distinct values in $T$ in each of the $m!$ ways with equal probability is $m$-exchangeable and yet $\mathrm{Prob}(X_1 = X_2) = 0$. Similar reasoning shows that the bound above can be achieved.

As $n$ increases past $m+1$ we obtain larger lower bounds on $\mathrm{Prob}(X_1 = X_2)$ from the assumption of $n$-extendability. Instead of a combinatorial argument, we will derive a bound analytically in the $n \to \infty$ case below (Example 2.42) using De Finetti's Theorem. ◊

**Example 2.37.** To get a feel for $\mu_{n\to m}$ we will write it down and analyze it in the simplest case, when $T = \{0,1\}$; for a deeper and more general analysis of such maps see [18]. A distribution $\pi \in \Delta_{S_n}(T^n)$ is characterized by the probabilities $\pi(x_1, \ldots, x_n)$ for $x_i \in T$. By definition of an $n$-exchangeable distribution, this probability does not depend on the order of the $x_i$, so in particular there are some $p_0, \ldots, p_n \geq 0$ such that $\pi(x_1, \ldots, x_n) = p_k$ whenever $\sum_{i=1}^n x_i = k$. Since $\pi$ is a probability mass function, $\sum_{k=0}^n \binom{n}{k} p_k = 1$. Conversely, any $p_1, \ldots, p_n \geq 0$ satisfying this equation define a distribution $\pi \in \Delta_{S_n}(T^n)$.

In particular, for any $0 \leq k \leq n$ there is a distribution $\pi^{n,k} \in \Delta_{S_n}(T^n)$ under which $\sum_{i=1}^{n} X_i = k$ almost surely, i.e., $p_k = \binom{n}{k}^{-1}$ and the other $p_i$ are zero. The set $\Delta_{S_n}(T^n)$ is a simplex with extreme points $\pi^{n,0}, \ldots, \pi^{n,n}$. The map $\mu_{n\to n-1}$ which drops $X_n$ is linear and satisfies

$$\mu_{n\to n-1}(\pi^{n,k}) = \frac{k}{n}\pi^{n-1,k-1} + \frac{n-k}{n}\pi^{n-1,k}.$$

With this map in hand we address the question of when an arbitrary $m$-exchangeable distribution is $n$-extendable. Note that $\pi^{n-1,l}$ is in the image of $\mu_{n\to n-1}$ if and only if $l = 0$ or $l = n-1$, so $\mu_{n\to n-1}$ is onto if and only if $n = 2$. For more general $1 \leq m < n$ this shows that $\mu_{n\to m}$ is not onto unless $m = 1$. For any $\pi \in \Delta_{S_1}(T^1) = \Delta(T)$ the distribution $\pi^{\otimes n}$ of $n$ i.i.d. copies of $\pi$ satisfies $\mu_{n\to 1}(\pi^{\otimes n}) = \pi$, so $\mu_{n\to 1}$ is onto. The result is that an arbitrary $m$-exchangeable distribution cannot necessarily be extended to an $n$-exchangeable distribution, except in the trivial case $m = 1$. $\diamondsuit$

A similar argument shows that the same statements about surjectivity of $\mu_{\cdot\to\cdot}$ are true for general finite $T$; we will not prove this fact because it is merely motivational and our results do not logically depend on it. Summing up, we have a sequence of sets and maps

$$\cdots \xrightarrow{\mu_{4\to 3}} \Delta_{S_3}(T^3) \xrightarrow{\mu_{3\to 2}} \Delta_{S_2}(T^2) \xrightarrow{\mu_{2\to 1}} \Delta_{S_1}(T^1) = \Delta(T),$$

where the maps defined by composition of the ones shown are omitted and only the rightmost map is onto. Another way of looking at these maps is the following result whose converse is obvious. It states that if all the $m$-variable marginals of some distribution $\psi \in \Delta(T^n)$ equal the distribution $\pi \in \Delta(T^m)$ (in which case $\pi$ must be symmetric) and we are only interested in $\psi$ through its effect on $\pi$, we may take $\psi$ itself to be symmetric.

**Proposition 2.38.** *Let $m \leq n$. If all $\frac{n!}{(n-m)!}$ of the $m$-variable marginals of a distribution in $\Delta(T^n)$ equal $\pi \in \Delta(S^m)$, then $\pi$ is $n$-extendable.*

*Proof.* Let $K \subseteq \Delta(T^n)$ be the set of distributions all whose marginals equal $\pi$, so $K$ is nonempty by assumption. Since marginalization is linear, $K$ is convex. The linear action of the symmetric group $S_n$ on $\Delta(T^n)$ restricts to an action on $K$, so by Proposition 2.22 there is a distribution in $K_{S_n} := K \cap \Delta_{S_n}(T^n)$ whose marginal is $\pi$. $\square$

We are interested in $m$-exchangeable distributions which are $n$-extendable for all $n \geq m$ to model sequences such as coin flips where there is no natural limit on the number of instances. In a sense we would like to take the intersection "$\bigcap \Delta_{S_n}(T^n)$," but this expression is meaningless (or $\emptyset$, if you prefer) as written because these sets do not all live in the same place. The correct notion for what we would like to work with is the *inverse* (also called *projective*) *limit*

$$\varprojlim \Delta_{S_n}(T^n) := \left\{(\pi^1, \pi^2, \ldots) \mid \mu_{n\to m}(\pi^n) = \pi^m\right\} \subset \prod_{n=1}^{\infty} \Delta_{S_n}(T^n).$$

By the Kolmogorov Consistency Theorem (Theorem 12.1.2 in [23], for example) any element of this limit defines a unique probability measure on $T^\infty$ and this probability measure is invariant under arbitrary permutations of finitely many indices. Conversely any probability measure on $T^\infty$ invariant under such permutations marginalizes to give the corresponding element of the limit. We can therefore identify this inverse limit with $\Delta_{S_\infty}(T^\infty)$, defined as follows.

**Definition 2.39.** *Let $S_\infty$ denote the group of permutations of $\mathbb{N}$ which fix all but finitely many elements. We write $\Delta_{S_\infty}(T^\infty)$ to denote the* ***exchangeable distributions****, the subset of $\Delta(T^\infty)$ which is invariant under permutations of finitely many indices, or in other words joint distributions of sequences of random variables $(X_1, X_2, \ldots)$ which are invariant under permutations of finitely many indices.*

Again there is a map $\Delta(T^\infty) \to \Delta(T^m)$ marginalizing onto the first $m$ factors, which restricts to a map $\mu_{\infty\to m} : \Delta_{S_\infty}(T^\infty) \to \Delta_{S_m}(T^m)$. We also refer to the distributions in $\mu_{\infty\to m}(\Delta_{S_\infty}(T^\infty)$, those which extend to an exchangeable distribution, as **exchangeable**. Under the identification $\Delta_{S_\infty}(T^\infty) \cong \varprojlim \Delta_{S_n}(T^n)$, we have $\mu_{\infty\to m} : (\pi^1, \pi^2, \ldots) \mapsto \pi^m$. The compatibility conditions $\mu_{m\to l} \circ \mu_{n\to m} = \mu_{n\to l}$ mean that the limit $\varprojlim \Delta_{S_n}(T^n)$ acts like an intersection "$\bigcap \Delta_{S_n}(T^n)$" in the following sense. Rather than being about the sets of exchangeable distributions, this statement and proof work mutatis mutandis for any inverse system of nonempty compact Hausdorff spaces.

**Proposition 2.40.** *A distribution in $\Delta_{S_m}(T^m)$ is exchangeable if and only if it is $n$-extendable for all $n \geq m$. More specifically, for any finite $m$ we have the nesting*

$$\Delta_{S_m}(T^m) \supseteq \mu_{m+1\to m}(\Delta_{S_{m+1}}(T^{m+1})) \supseteq \mu_{m+2\to m}(\Delta_{S_{m+2}}(T^{m+2})) \supseteq \cdots,$$

*and image of the marginalization map is*

$$\bigcap_{n=m}^{\infty} \mu_{n\to m}\left(\Delta_{S_n}(T^n)\right) = \mu_{\infty\to m}\left(\Delta_{S_m}(T^\infty)\right).$$

*Proof.* Nesting comes from the compatibility of the marginalization maps. As for the equation, any $\pi^m$ in the right hand side extends to a $\pi = (\pi^1, \pi^2, \ldots) \in \varprojlim \Delta_{S_n}(T^n) \cong \Delta_{S_\infty}(T^\infty)$. By definition $\mu_{n\to m}(\pi^n) = \pi^m$ for all $n \geq m$, so $\pi^m$ is in the left hand side. For the converse, suppose $\pi^m$ is in the left hand side. For $n \geq m$, let $A_n$ be the inverse image of the point $\{\pi^m\}$ under the map $\mu_{n\to m}$. By assumption the $A_n$ are nonempty compact Hausdorff and they form an inverse system under the restrictions of the marginalization maps: the compatibility conditions still hold. The inverse limit of a system of nonempty compact Hausdorff spaces is nonempty (and compact Hausdorff); this generalizes the statement the a nested intersection of nonempty compact Hausdorff spaces is nonempty and can be proven in the same way [24]. Therefore there are $\pi^{m+1}, \pi^{m+2}, \ldots$ such that $\mu_{n\to l}(\pi^n) = \pi^l$ for all $n \geq l \geq m$. Defining $\pi^k = \mu_{m\to k}(\pi^m)$ for all $k < m$ we obtain an element $(\pi^1, \pi^2, \ldots) \in \varprojlim \Delta_{S_n}(T^n)$, so $\pi^m$ is in the right hand side. $\square$

Thus the polyhedral sets $\mu_{n\to m}(\Delta_{S_n}(T^n))$ approximate $\mu_{\infty\to m}(\Delta_{S_\infty}(T^\infty))$. This proposition also yields infinitary versions of some results about $n$-exchangeable distributions. For example, combining with Proposition 2.38 yields:

**Corollary 2.41.** *If all of the m-variable marginals of a distribution in $\Delta(T^\infty)$ equal $\pi \in \Delta(T^m)$, then $\pi \in \mu_{\infty\to m}(\Delta_{S_\infty}(T^\infty))$.*

While $\Delta_{S_\infty}(T^\infty)$ is more concrete than the limit representation, it still does not give a way to generate all exchangeable distributions. We now introduce De Finetti's Theorem, which supplies this. Note that for any $\pi \in \Delta(T)$, the distribution $\pi^{\otimes\infty}$ of countably many random variables each i.i.d. according to $\pi$ is in $\Delta_{S_\infty}(T^\infty)$ (it is evidently exchangeable and is represented by $(\pi, \pi^{\otimes 2}, \pi^{\otimes 3}, \ldots)$ in the inverse limit). Furthermore, the sets $\Delta_{S_n}(T^n)$ for $1 \leq n \leq \infty$ are convex, so any random mixture of such i.i.d. distributions is also in $\Delta_{S_\infty}(T^\infty)$. That is to say, there is a natural weakly continuous linear map $\Delta(\Delta(T)) \to \Delta_{S_\infty}(T^\infty)$ sending $\delta_\pi$ to $\pi^{\otimes\infty}$.

**De Finetti's Theorem.** *The map $\Delta(\Delta(T)) \to \Delta_{S_\infty}(T^\infty)$ is a bijection.*

For a proof see e.g. [64]. The upshot is that we can think of any exchangeable distribution as arising from a two-stage process. First, a random element $\Theta \in \Delta(T)$ is secretly selected according to some distribution in $\Delta(\Delta(T))$. Second, the sequence of random variables $X_1, X_2, \ldots$ is constructed to be i.i.d. according to $\Theta$.

If we are trying to build exchangeable distributions, this limits the class of constructions we have to consider. If we are trying to understand a given exchangeable distribution, we can think of it having been constructed this way whether it was or was not[3].

The inverse map $\Delta_{S_\infty}(T^\infty) \to \Delta(\Delta(T))$ sends the distribution of the exchangeable sequence $X_1, X_2, \ldots$ to the distribution of the limiting empirical distributions $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$. This entire discussion can be generalized to certain classes of infinite $T$, but we will not need the details.

**Example 2.42** (continues Example 2.36). Suppose the distribution of $X_1$ and $X_2$, which take values in a set $T$ of cardinality $m$, is exchangeable (i.e., $\infty$-extendable), the natural limiting case of the original example. By De Finetti's Theorem these are i.i.d. conditioned on some parameter.

Suppose for now that they are i.i.d. with distribution $\pi \in \Delta(S) \subsetneq [0,1]^m$. Then $\mathrm{Prob}(X_1 = X_2) = \|\pi\|_2^2$. Let $e \in \mathbb{R}^T$ denote the all ones vector. By the Cauchy-Schwarz inequality,

$$1 = (\pi \bullet e)^2 \le \|\pi\|_2^2 \, \|e\|_2^2 = \mathrm{Prob}(X_1 = X_2)m,$$

so $\mathrm{Prob}(X_1 = X_2) \ge m^{-1}$ whenever $X_1$ and $X_2$ are i.i.d. By linearity of expectation this bound also holds when they are conditionally i.i.d., so whenever they are exchangeable. This improves the lower bound $\binom{m+1}{2}^{-1}$ proven from $(m+1)$-extendability above by combinatorial means. Since the Cauchy-Schwarz inequality is tight exactly when applied to parallel vectors, this new bound is tight if and only if $X_1$ and $X_2$ are i.i.d. uniform over $T$. $\diamondsuit$

## 2.3 Tensors

If we imagine trying to write the joint distribution of random variables $X_1, \ldots, X_n$ all with finite range, ignoring the constraints imposed by paper, the natural thing to do is to write an $n$-dimensional array with entry $(i_1, \ldots, i_n)$ indicating the probability $\mathrm{Prob}(X_1 = i_1, \ldots, X_n = i_n)$. Such an object is called a **tensor of type**[4] $n$. If $n = 1$ it is a vector and if $n = 2$ it is a matrix. We will need very little theory of tensors in what follows; mostly it is the notation which will be convenient.

To be slightly more formal, let $V_1, \ldots, V_n$ be finite-dimensional real vector spaces with each $V_i$ having a distinguished basis $e_i^1, \ldots, e_i^{d_i}$. Then the **tensor product** $V_1 \otimes \cdots \otimes V_n$ is the real vector space of dimension $d_1 \cdots d_n$ with distinguished basis $e_1^{j_1} \otimes \cdots \otimes e_n^{j_n}$ for $1 \le j_i \le d_i$. An element of such a tensor product is a linear combination of such distinguished basis elements and is called a $d_1 \times \cdots \times d_n$ **tensor**.

[3]The classic example of an exchangeable distribution which does not obviously arise in this way is the Pólya urn model [58].

[4]Often the word "order" is used instead, but this would cause confusion later.

It is called a **nonnegative tensor** if it is entrywise nonnegative (always with respect to the distinguished basis), a **normalized tensor** if its entries sum to one, and a **probability tensor** if it is both. Such definitions automatically define **nonnegative vectors**, **probability matrices**, and so on. Tensor products can be defined without reference to bases (and much more generally), but we will always be interested in probability tensors and related notions where a distinguished basis is available, so we will not need a more general definition.

We can extend the range of the tensor product $\otimes$ multilinearly from sequences of basis elements to all of $V_1 \times \cdots \times V_n$. If $v_i = \sum_{j=1}^{d_i} c_i^{j_i} e_i^{j_i}$ for $c_i^{j_i} \in \mathbb{R}$ then

$$v_1 \otimes \cdots \otimes v_n := \sum c_1^{j_1} \cdots c_n^{j_n} e_1^{j_1} \otimes \cdots \otimes e_n^{j_n}$$

where the product of the $c_i^{j_i}$ is taken in $\mathbb{R}$ and the sum is over all $j$ with $1 \leq j_i \leq d_i$. Such a tensor is said to be **simple** or **rank one**. If the $v_i$ are all probability vectors then $v_1 \otimes \cdots \otimes v_n$ is a probability tensor: it is exactly the joint distribution of $n$ independent random variables with the given distributions.

If $v$ and $w$ are viewed as column vectors then the rank 1 matrix $vw^T$ can be identified with $v \otimes w$ in a natural way. If $\dim V = m \geq 2$ and $\dim W = n \geq 2$ then $V \otimes W$ is the space of $m \times n$ matrices, so not all elements of $V \otimes W$ are of the form $v \otimes w$: only rank 1 matrices are of this form.

Often we will be interested in sequences of random variables $X_1, \ldots, X_n$ all taking values in some fixed finite set. If the distribution of $X_1$ is a probability vector in $V$, then the joint distribution of these random variables is naturally an element of the $n^{\text{th}}$ **tensor power** $V^{\otimes n} := V \otimes \cdots \otimes V$. We define the tensor power of a vector $v$ to be $v^{\otimes n} := v \otimes \cdots \otimes v$. If $v$ is a probability vector then this is the joint distribution of $n$ random variables i.i.d. according to $v$. When performing the tensor power operation we will not distinguish between row and column vectors, using whichever is more legible.

Within the tensor power $V^{\otimes n}$ we can define the space $\mathrm{Sym}^n(V)$ of **symmetric tensors** to be those which are invariant under arbitrary permutations of the indices. For type 2 tensors this reduces to the definition of a symmetric matrix. For $v, w \in V$ some example symmetric tensors in $\mathrm{Sym}^3(V)$ are $v^{\otimes 3}$ and $v \otimes v \otimes w + v \otimes w \otimes v + w \otimes v \otimes v$. Symmetric probability tensors of type $n$ are exactly the same as joint distributions of $n$-exchangeable sequences of random variables $X_1, \ldots, X_n$ with finite range. The tensor powers of vectors are exactly the symmetric simple tensors (although symmetric simple tensors may also be written in other ways, e.g. $(2v) \otimes (2v) = (4v) \otimes v$).

We will often need to write down tensors of type 3, which we will do by writing them as matrices and separating the "pages" by a vertical bar. These pages should be thought of as being stacked like pages of a book rather than being adjacent to each other. For example if $v = \begin{bmatrix}1 & 1\end{bmatrix}$ and $w = \begin{bmatrix}1 & 0\end{bmatrix}$ then the examples from the previous paragraph are

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|cc} 3 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right].$$

## 2.4 Complete positivity

For the purposes of this section, $T$ will be a set of finite cardinality $m$ which we identify with the standard basis vectors in $\mathbb{R}^m$. If we view the distributions in $\Delta(T^n)$ as $m \times \cdots \times m$ tensors of type $n$, those which are exchangeable (can be extended to distributions in $\Delta_{S_\infty}(T^\infty)$) are exactly those which are completely positive, as defined below. After establishing this link with exchangeability, we will examine the geometry of the sets of completely positive tensors.

Completely positive tensors are a direct generalization of completely positive matrices (studied in e.g. [4]) to tensors of higher type. They should not be confused with the completely positive maps used in quantum mechanics.

**Definition 2.43.** *The set* $\mathrm{CP}^n_m$ *of* ***completely positive tensors*** *is defined to be the convex hull of the set* $\{v^{\otimes n} \mid v \in \mathbb{R}^m_{\geq 0}\}$ *of nonnegative symmetric simple tensors, or equivalently the conic hull or set of finite sums of elements of this set.*

The set of nonnegative symmetric simple tensors is positively homogeneous: $c(v^{\otimes n}) = (\sqrt[n]{c}\, v)^{\otimes n}$ for all $c \geq 0$. This gives the equivalence of the definitions.

**Example 2.44.** Fix $n \in \mathbb{N}$ and define matrices $M, B \in \mathbb{R}^{n \times n}_{\geq 0}$ by

$$M_{ij} = \#\{\text{common factors of } i \text{ and } j\} \qquad \text{and} \qquad B_{ij} = \begin{cases} 1 & \text{if } j \text{ divides } i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $M$ is completely positive, because $M = BB^T$:

$$(BB^T)_{ij} = \sum_{k=1}^{n} B_{ik}B_{jk} = \#\{k \mid k \text{ divides } i \text{ and } j\} = M_{ij}. \qquad \diamondsuit$$

**Proposition 2.45.** *The set of normalized completely positive tensors is*

$$\mathrm{conv}\{v^{\otimes n} \mid v \in \Delta(T)\}.$$

*Proof.* One direction is immediate. For the other, let $W = \sum_{i=1}^k v_i^{\otimes n}$ for some $v_i \in \mathbb{R}^m_{\geq 0}$ and $k \in \mathbb{N}$. Summing the entries of $W$ we get $1 = \sum_{i=1}^k \|v_i\|_1^n$. Let $\lambda_i = \|v_i\|_1^n$, so these are nonnegative and sum to one. Then $\frac{v_i}{\|v_i\|_1} \in \Delta(T)$ and

$$W = \sum_{i=1}^{k} \lambda_i \left(\frac{v_i}{\|v_i\|_1}\right)^{\otimes n}.$$

□

**Proposition 2.46.** *The image $\mu_{\infty\to n}(\Delta_{S_\infty}(T^\infty))$ of the marginalization map is the set of normalized completely positive tensors.*

*Proof.* By De Finetti's Theorem the image is $\{\int v^{\otimes n} d\mu(v) \mid \mu \in \Delta(\Delta(T))\}$. The conclusion follows by applying the following lemma to $X := \Delta(T)$ and $f(v) := v^{\otimes n}$ and then applying Proposition 2.45. □

**Lemma 2.47.** *Let $f : X \to V$ be a continuous map from a compact Hausdorff space into a finite-dimensional vector space. Then $\{\int f\, d\mu \mid \mu \in \Delta(X)\} = \operatorname{conv}\{f(x) \mid x \in X\}$.*

*Proof.* The author has given a topological proof (Theorem 2.8 in [69]). Here we instead follow Karlin's convex analytic proof (Theorem 3.1.1 in [43]).

Let the two sets mentioned be $Y$ and $Z$, respectively. If $z \in Z$ then $z = \sum_{i=1}^k \lambda_i f(x_i)$ for distinct $x_i \in X$ and $\lambda$ a probability vector. Defining the probability measure $\mu := \sum_{i=1}^k \lambda_i \delta_{x_i} \in \Delta(X)$ which assigns mass $\lambda_i$ to $x_i$, we get $z = \int f\, d\mu \in Y$.

If $z \notin Z$ then by compactness of $Z$ there exists separating hyperplane, an affine functional $g : V \to \mathbb{R}$ such that $g(z) = -1$ and $g(f(x)) \geq 0$ for all $x \in X$ [5]. For any $\mu \in \Delta(X)$, linearity of integration gives

$$g\left(\int f\, d\mu\right) = \int g(f(x))\, d\mu(x) \geq \int 0\, d\mu(x) = 0,$$

so $z \notin Y$. □

By combining Proposition 2.46 with Proposition 2.40 and removing the normalization conditions, we can construct polyhedral outer approximations to $\mathrm{CP}_m^n$. Later we will see how to construct better approximations in terms of positive semidefinite matrices.

**Proposition 2.48.** *The set $\mathrm{CP}_m^n$ is a closed convex cone. It is also semialgebraic (describable by finite Boolean combinations of polynomial inequalities) and decidable in the sense that a Turing machine can determine in finite time whether a given tensor in $(\mathbb{Q}^m)^{\otimes n}$ is completely positive.*

*Proof.* It is a convex cone by definition. To see that it is closed, suppose that $W_1, W_2, \ldots \in \mathrm{CP}_m^n$ and $W_k \to W_\infty \in (\mathbb{R}^m)^{\otimes n}$. If $W_\infty = 0$ then we are done, so we may assume $W_k \neq 0$ for all $1 \leq k \leq \infty$. Then $\|W_k\|_1 \to \|W_\infty\|_1 > 0$. Let $\hat{W}_k = \frac{W_k}{\|W_k\|_1}$. Then $\hat{W}_k$ is a normalized completely positive tensor for $k < \infty$.

By Proposition 2.45 the set of these is the convex hull of a compact set, hence compact. Therefore the limit $\hat{W}_\infty$ is also completely positive, so $W_\infty$ is as well.

To prove semialgebraicity, we begin with the definition that any completely positive tensor can be written as a sum of tensor powers of nonnegative vectors. These all live in the vector space of symmetric $m \times \cdots \times m$ tensors which has dimension $\binom{n+m-1}{m-1}$. By Carathéodory's theorem on convex cones [5] and homogeneity of the set of tensor powers of nonnegative vectors, any completely positive tensor can be written as a sum of at most $\binom{n+m-1}{m-1}$ such tensors. That is to say

$$W \in CP^n_m \Leftrightarrow \exists v_1, v_2, \ldots, v_{\binom{n+m-1}{m-1}} \in \mathbb{R}^m_{\geq 0} \text{ s.t. } W = \sum_{i=1}^{\binom{n+m-1}{m-1}} v_i^{\otimes n},$$

so membership in $CP^n_m$ is expressible by a first-order formula over the reals. By quantifier elimination this means $\mathrm{CP}^n_m$ is semialgebraic and decidable [3]. □

Quantifier elimination procedures are extremely computationally intensive (to the extent that running them is not even feasible on modern computers for tensors with more than a handful of entries), but suffice to prove decidability. We will see in Section 2.5.4 that faster methods are available for the cases when $m = 2$ or $n = 2$ and $m \leq 4$, but in general no faster algorithms are known.

The definition of $\mathrm{CP}^n_m$ is parsimonious in the sense that we have written it as the convex cone generated by its extreme rays. In particular there are infinitely many of these, so $\mathrm{CP}^n_m$ is not polyhedral.

**Proposition 2.49.** *If $n \geq 2$ then $\{v^{\otimes n} \mid v \in \mathbb{R}^m_{\geq 0}, v \neq 0\}$ is the set of extreme rays of $\mathrm{CP}^n_m$.*

*Proof.* Since these tensors generate $\mathrm{CP}^n_m$, only they can be extreme; we will show that they are. To do so we must show that if $v^{\otimes n} = P + Q$ where $P, Q \in \mathrm{CP}^n_m$ then one of $P$ and $Q$ is a nonnegative multiple of $v^{\otimes n}$ (in which case the other is as well). By the definition of $\mathrm{CP}^n_m$ and induction it suffices to prove that if $v^{\otimes n} = \sum_{i=1}^k u_i^{\otimes n}$ for nonzero $u_i \geq 0$ and some $k$, then one of the $u_i$ is a scalar multiple of $v$. The nonzero and nonnegativity assumptions on $v$ and the $u_i$ automatically imply that this scalar must be positive.

We first treat the case when $n$ is even. Let $\bullet$ denote the Euclidean inner product: the sum of the products of corresponding components of the arguments, be they vectors in $\mathbb{R}^m$ or tensors in $(\mathbb{R}^m)^{\otimes n}$. Then $x^{\otimes n} \bullet y^{\otimes n} = (x \bullet y)^n$. For any $x \in \mathbb{R}^m$ with $v \bullet x = 0$, we have

$$0 = (v \bullet x)^n = v^{\otimes n} \bullet x^{\otimes n} = \sum_{i=1}^k u_i^{\otimes n} \bullet x^{\otimes n} = \sum_{i=1}^k (u_i \bullet x)^n.$$

Since $n$ is even, the summands on the right must all be zero: $u_i \bullet x = 0$. Since this holds for all $x \in \mathbb{R}^m$ with $v \bullet x = 0$, each $u_i$ must be a scalar multiple of $v$.

Now we consider the case when $n \geq 2$ is odd, so $n-1 \geq 2$ as well. Since $v$ is nonzero there is some $j \leq m$ such that $v_j > 0$; let $(u_i)_j$ denote the $j^{\text{th}}$ entry of $u_i$. Looking at the $j^{\text{th}}$ "slice" along one dimension of the supposed decomposition $v^{\otimes n} = \sum_{i=1}^k u_i^{\otimes n}$ we obtain

$$\left(v_j^{\frac{1}{n-1}} \cdot v\right)^{\otimes(n-1)} = v_j \cdot v^{\otimes(n-1)} = \sum_{i=1}^k (u_i)_j \cdot u_i^{\otimes(n-1)} = \sum_{i=1}^k \left((u_i)_j^{\frac{1}{n-1}} \cdot u_i\right)^{\otimes(n-1)}.$$

Since $n-1 \geq 2$ is even, the left hand side is an extreme ray of $\mathrm{CP}_m^{n-1}$. The decomposition on the right then implies that for some $i$ with $(u_i)_j > 0$, $(u_i)_j^{\frac{1}{n-1}} \cdot u_i$ is a scalar multiple of $v_j^{\frac{1}{n-1}} \cdot v$. In particular, $u_i$ is a multiple of $v$. □

The definition of complete positivity makes it easy to give examples of completely positive tensors. The following is a simple criterion to rule out some tensors as not completely positive.

**Proposition 2.50.** *If a tensor $W = \sum_{j \in T^n} c_j e_1^{j_1} \otimes \cdots \otimes e_n^{j_n}$ is completely positive and $c_{i,\ldots,i} = 0$ then $c_j = 0$ whenever one of the coordinates of $j$ is $i$.*

*Proof.* Write $W = \sum_k v_k^{\otimes n}$ with $v_k \in \mathbb{R}^m_{\geq 0}$. Then $c_{i,\ldots,i}$ is the sum of the $n^{\text{th}}$ powers of the $i^{\text{th}}$ coordinates of the $v_k$, so these coordinates are all zero. If $j \in T^n$ has a coordinate equal to $i$ then $c_j$ is a sum of products of the coordinates of the $v_k$, and each of these products includes the $i^{\text{th}}$ coordinate of at least one of the $v_k$, so is zero. □

## 2.5 Semidefinite relaxations

### 2.5.1 Conic programming

Conic programming is a widely-applicable framework for optimization problems. A **conic program** is specified by a linear map $L : V \to W$ between a pair of finite-dimensional vector spaces, a closed convex cone $K \subseteq V$, a linear objective function $f : V \to \mathbb{R}$ and a right-hand side vector $y \in W$:

$$\begin{aligned} \underset{x \in V}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & L(x) = y, \\ & x \in K. \end{aligned}$$

This **primal** comes associated with a **dual** in an entirely syntactic way:

$$\begin{aligned} &\underset{g \in W^*}{\text{maximize}} && g(y) \\ &\text{subject to} && f - L^*(g) \in K^*, \end{aligned}$$

where $L^* : W^* \to V^*$ is the transpose of $L$, defined in terms of the dual vector spaces, and $K^* := \{h \in V^* \mid h(x) \geq 0 \text{ for all } x \in K\}$ is the dual cone. Direct algebraic manipulation shows that if $x$ and $g$ are feasible solutions of the primal and dual, respectively, then $f(x) \geq g(y)$, so the optimal value of the primal is always at least that of the dual. Under favorable conditions such as feasibility of an interior point of each cone (many others are known [5]) these values will in fact be equal and attained at some feasible points.

While these programs look different, both amount to optimizing an affine function over the intersection of an affine space with a closed convex cone. The apparent difference is because the affine space is defined in the primal in terms of relations and in the dual in terms of generators. Basic linear algebra allows us to switch between such representations, so mathematically there is no difference and we will refer to both as conic programs.

## 2.5.2 Linear, semidefinite, and completely positive programming

Families of conic programs are usually named according to the choice of cone $K$. For instance, if $K = \mathbb{R}^n_{\geq 0}$ is the nonnegative orthant then (identifying $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$ using the standard inner product) $K = K^*$ and these problems reduce to

$$\begin{aligned} &\underset{x \in \mathbb{R}^n}{\text{minimize}} && f^T x \\ &\text{subject to} && Lx = y, \\ &&& x \geq 0, \end{aligned} \qquad \text{and} \qquad \begin{aligned} &\underset{g \in \mathbb{R}^m}{\text{maximize}} && y^T g \\ &\text{subject to} && L^T g \leq f, \end{aligned}$$

two standard forms of **linear programs (LPs)**.

For another example let $V$ be the space of symmetric $n \times n$ matrices and $K$ the cone of positive semidefinite matrices (membership in which is denoted $X \succeq 0$). Again we can identify $V$ with $V^*$ using the standard inner product and we obtain $K = K^*$: such cones are called self-dual. The primal and dual problems are now called **semidefinite programs (SDPs)**:

$$\begin{aligned} &\underset{X = X^T \in \mathbb{R}^{n \times n}}{\text{minimize}} && f(X) \\ &\text{subject to} && L(X) = y, \\ &&& X \succeq 0, \end{aligned} \qquad \text{and} \qquad \begin{aligned} &\underset{g \in W^*}{\text{maximize}} && g(y) \\ &\text{subject to} && f - L^*(g) \succeq 0. \end{aligned}$$

If we wish to have multiple symmetric matrices with positive semidefiniteness constraints as variables, we can do so by thinking of these as blocks on the diagonal of a larger matrix. We can use the equality constraints to ensure that the entries not in these blocks are all zero and recall that a block diagonal matrix is positive semidefinite if and only if all the blocks are. Thus the problem can still be cast as a semidefinite program.

An extreme case of this would be to have $n$ positive semidefinite "blocks" all of size $1 \times 1$, which is the same as $n$ nonnegative variables. Therefore linear programs are a special case of semidefinite programs, and in particular we can use linear inequality constraints when we write down semidefinite programs.

Linear and semidefinite programming are well-studied problems which can be solved[5] efficiently both in theory and in practice. Many important practical problems can be modeled exactly or approximately as linear or semidefinite programs.

Another class of conic programs are the **completely positive programs**, where $K$ is the (not self-dual) cone of $n \times n$ completely positive matrices. These can encode a wide class of problems, but unfortunately they are not so easy to solve. For example, let $A \bullet B := \mathrm{Tr}(AB)$ denote the standard inner product on symmetric $n \times n$ matrices, $E$ the matrix of all ones, and suppose we could solve

$$\begin{aligned} \underset{X=X^T \in \mathbb{R}^{n\times n}}{\text{minimize}} \quad & A \bullet X \\ \text{subject to} \quad & E \bullet X = 1, \\ & X \text{ completely positive.} \end{aligned} \tag{2.1}$$

The optimal value would be negative if and only if there were an $x \geq 0$ such that $x^T A x < 0$. The set $\{A \in \mathbb{R}^{n \times n} \mid A = A^T \text{ and } x^T A x \geq 0 \text{ for all } x \geq 0\}$ is called the cone of **copositive matrices**. So solving this optimization problem tells us whether $A$ is copositive, a problem which is known to be co-NP-complete [50]. Therefore solving general completely positive programs is NP-hard.

Often it will happen that we can write down a set and observe that it is the collection of feasible points of a particular conic program or a projection thereof (e.g. we are only interested in a subset of the variables in the conic program), but

[5]There are two subtleties here in the case of semidefinite programs. First, there are polynomially-sized semidefinite programs whose optimal solutions require exponentially many bits to write down (in binary, say), so we require explicit inner and outer bounds on the solution set to ensure we can even write down the answer in polynomial time. Second, the optimal solution may well be irrational and of high algebraic degree, so in practice we must settle for approximate solutions. Given fixed inner and outer bounds on the feasible set, a solution within $\epsilon > 0$ of optimal can be computed in polynomial time in the size of the instance and $\log \frac{1}{\epsilon}$ using the ellipsoid method.

we will have no particular objective function in mind to optimize. In this case we will nonetheless say that the set **is** or **is described by** the conic program. For example, the set of correlated equilibria of a game is a linear program in this sense. We can find a correlated equilibrium by adding to these an arbitrary objective function and solving the linear program, we can plot projections of the set of correlated equilibria by solving a family of linear programs with varying objective, and generally speaking we can understand much about sets of correlated equilibria via the theory of linear programming.

The set of exchangeable equilibria of a game, the main topic of this thesis, is a completely positive program in this sense. As such we do not expect that we will be able to test membership or optimize over the set of these in an efficient way, and it is important to be able to approximate this set in a more tractable way. To do so we apply the standard technique of approximating the cone involved (in this case the completely positive tensors) by a simpler cone in such a way that the resulting problem can easily be transformed into a semidefinite program. This is the content of Section 2.5.4, which builds on the following.

### 2.5.3 Polynomial nonnegativity and sums of squares

For this section $x = (x_1, \dots, x_m)$ will be a vector of indeterminates and $\mathbb{R}[x]^n$ will denote the vector space of homogeneous polynomials in $m$ variables of total degree $n$ with coefficients in $\mathbb{R}$ (along with the zero polynomial). Many problems can be reduced to the question of whether a polynomial in this set takes nonnegative values for all $x \in \mathbb{R}^m$. We define the set of such **nonnegative (homogeneous) polynomials** as $\Psi_{m,2n} := \{f \in \mathbb{R}[x]^{2n} \mid f(x) \geq 0 \text{ for all } x \in \mathbb{R}^m\}$. Note that by homogeneity nonzero polynomials can only be nonnegative if their degree is even. This set is a closed convex cone, but testing membership in it is NP-hard: as we will soon see, testing copositivity reduces to polynomial nonnegativity.

A simple sufficient condition for a polynomial to be nonnegative is that it be a sum of squares of polynomials:

**Definition 2.51.** *We say that a polynomial $f \in \mathbb{R}[x]^{2n}$ is a* ***sum of squares*** *if there exist $g_1, \dots, g_k \in \mathbb{R}[x]^n$ such that $f \equiv \sum_{i=1}^k g_i^2$. The set of such polynomials is denoted $\Sigma_{m,2n}$.*

At any $x \in \mathbb{R}^m$, we have $f(x) = \sum_{i=1}^k [g_i(x)]^2 \geq 0$ since squares in $\mathbb{R}$ are nonnegative, so $\Sigma_{m,2n} \subseteq \Psi_{m,2n}$. The set $\Sigma_{m,2n}$ is also a convex cone, but membership in it is easier to test: it can be reduced to a semidefinite program.

Fix $m$ and $n$ and let $\mu(x)$ be a vector of length $l = \dim \mathbb{R}[x]^n = \binom{n+m-1}{m-1}$ listing all monomials in $\mathbb{R}[x]^n$. Then a polynomial $g \in \mathbb{R}[x]^n$ is the same as an inner product $\mu(x)^T z$ for some $z \in \mathbb{R}^l$. The square of this polynomial is $g^2 =$

$\mu(x)^T zz^T \mu(x)$. A sum of squares in $\Sigma_{m,2n}$ therefore takes the form $\mu(x)^T Z\mu(x)$ where $Z = \sum_{i=1}^k z_i z_i^T$. Such a $Z$ is positive semidefinite, and conversely any positive semidefinite $Z$ can be written as $Z = \sum_{i=1}^k z_i z_i^T$ for some $k$ and some $z_i \in \mathbb{R}^l$.

Note that $\mathcal{H} : Z \mapsto \mu(x)^T Z\mu(x)$ is a linear map $\mathbb{R}^{l\times l} \to \mathbb{R}[x]^{2n}$. We have shown that $\Sigma_{m,2n} = \{\mathcal{H}(Z) \mid Z \succeq 0\}$. Suppose we would like to constrain the decision variables in a conic program so that a polynomial $f \in \mathbb{R}[x]^{2n}$ whose coefficients are affine in the decision variables (perhaps constant, perhaps not) is a sum of squares. Then we can introduce the new symmetric matrix variable $Z \succeq 0$ and write $f = \mathcal{H}(Z)$, a collection of linear equality constraints on the coefficients.

That is to say, we can express sum of squares conditions on polynomials in a semidefinite program. In this way we can easily test whether a polynomial is a sum of squares, optimize over families of sums of squares, and much more [56]. We say that the sums of squares are an **(inner) semidefinite relaxation**[6] or **approximation** of the nonnegative polynomials.

How close is $\Sigma_{m,2n}$ to $\Psi_{m,2n}$? It is a classic result of Hilbert that these sets are equal (and the relaxation is said to be **exact**) in precisely three cases: (1) $m = 2$, (2) $2n = 2$, or (3) $m = 3$ and $2n = 4$ (see Reznick [61] for a modern treatment). The third case is difficult to prove and the second is essentially the fact that symmetric positive semidefinite matrices admit square roots. The first is a simple exercise: dehomogenization yields a univariate polynomial which factors over $\mathbb{C}$ as a product of linear terms. These linear terms group into complex conjugate pairs, which multiply to yield a sum of squares, and real roots, which must each have even multiplicity by the nonnegativity assumption. The product of two sums of squares is again a sum of squares, and homogenizing completes the proof.

While Hilbert showed that these were the only cases where equality holds, his counterexamples in the other cases were not particularly explicit. Motzkin has given the simple example $M(x, y, z) = x^4y^2 + x^2y^4 - 3x^2y^2z^2 + z^6$ of a polynomial of degree 6 in three variables which is not a sum of squares. The fact that $M$ has no odd powers of any variables will be exploited in the next section.

If in a particular case $\Sigma_{m,2n}$ is not a good enough approximation to $\Psi_{m,2n}$, there are tighter approximations available. For instance, if $f$ is a sum of squares then so is $(\sum x_i^2)^r f$ for $r \in \mathbb{N}$, but the converse is not necessarily true. Therefore we obtain a better approximation by asking that $(\sum x_i^2)^r f$ be a sum of squares for some fixed $r$, a condition which we can again write as a semidefinite program. As $r$ increases we obtain a nested sequence of inner approximations to $\Psi_{m,2n}$ which converge in the sense that the closure of their union is $\Psi_{m,2n}$ [61].

[6] The term "relaxation" is used for both inner and outer approximations even though logically it makes more sense for outer approximations.

### 2.5.4 Double nonnegativity

We can use this master example of relaxing $\Psi_{m,2n}$ to $\Sigma_{m,2n}$ to construct SDP relaxations of other cones by expressing them in terms of $\Psi_{m,2n}$ and replacing that with $\Sigma_{m,2n}$, or one of the tighter relaxations if we wish. In this section we will illustrate this technique by deriving an outer SDP relaxation for $\mathrm{CP}^n_m$ called the cone $\mathrm{DNN}^n_m$ of *doubly nonnegative tensors.* The name comes from the well-studied matrix case $n = 2$.

It will be clear from the final result that $\mathrm{DNN}^n_m$ is an SDP relaxation of $\mathrm{CP}^n_m$, so we could avoid going through the derivation by pulling it out of thin air and checking that it contains $\mathrm{CP}^n_m$. The point of deriving it is to illustrate that this is a general systematic procedure which we could use to derive tighter relaxations as well.

Fix $m, n \geq 2$ and endow the space of $\mathrm{Sym}^n(\mathbb{R}^m)$ of symmetric tensors with the Euclidean inner product $A \bullet B = \sum_{j \in \{1,\ldots,m\}^n} A_j B_j$. Using this we can identify this vector space with its dual and view $\mathrm{CP}^n_m$ and its dual cone

$$(\mathrm{CP}^n_m)^* = \{A \in \mathrm{Sym}^n(\mathbb{R}^m) \mid A \bullet B \geq 0 \text{ for all } B \in \mathrm{CP}^n_m\},$$

the set of **copositive tensors**, as subsets of $\mathrm{Sym}^n(\mathbb{R}^m)$. We can simplify this definition slightly by recalling that $\mathrm{CP}^n_m$ is the collection of sums of nonnegative symmetric simple tensors. A tensor $A$ has a nonnegative inner product with all such sums if and only if it has a nonnegative inner product with all such simple tensors. Furthermore

$$A \bullet x^{\otimes n} = \sum_{j \in \{1,\ldots,m\}^n} A_j x_{j_1} \cdots x_{j_n},$$

so

$$(\mathrm{CP}^n_m)^* = \left\{ A \in \mathrm{Sym}^n(\mathbb{R}^m) \,\middle|\, \sum_{j \in \{1,\ldots,m\}^n} A_j x_{j_1} \cdots x_{j_n} \geq 0 \text{ for all } x \geq 0 \right\}.$$

The copositive tensors are those which make a certain polynomial in variables $x_1, \ldots, x_m$ nonnegative whenever the variables are nonnegative. Since the nonnegative reals are exactly the squares, a symmetric tensor $A$ is copositive if and only if

$$A \bullet \left[x_1^2, \ldots, x_m^2\right]^{\otimes n} := \sum_{j \in \{1,\ldots,m\}^n} A_j x_{j_1}^2 \cdots x_{j_n}^2 \geq 0$$

for all $x \in \mathbb{R}^m$. In other words, if we define a linear map $L : \mathrm{Sym}^n(\mathbb{R}^m) \to \mathbb{R}[x_1, \ldots, x_m]^{2n}$ by the left hand side of this inequality, then $(\mathrm{CP}^n_m)^*$ is the inverse image of $\Psi_{m,2n}$ under $L$. Replacing the nonnegative polynomials with sums of squares gives us an inner SDP relaxation $L^{-1}(\Sigma_{m,2n}) \subseteq L^{-1}(\Psi_{m,2n}) = (\mathrm{CP}^n_m)^*$.

Taking the dual of convex cones reverses inclusions and every closed convex cone is its own double dual [5]. Therefore we get an outer relaxation for the completely positive tensors: $\mathrm{CP}^n_m = (\mathrm{CP}^n_m)^{**} \subseteq (L^{-1}(\Sigma_{m,2n}))^*$. The name for the relaxation comes from the $n = 2$ case, Example 2.54 below.

**Definition 2.52.** *The set of* ***doubly nonnegative tensors*** *is*

$$\mathrm{DNN}^n_m := (L^{-1}(\Sigma_{m,2n}))^*.$$

We now use the technique of symmetry reduction to give a more explicit characterization of $\mathrm{DNN}^n_m$. As in the previous section, we can define $\mu(x)$ to be a vector of all monomials in $\mathbb{R}[x_1, \ldots, x_m]^n$. We refer to $\alpha = (\alpha_1, \ldots, \alpha_m)$ as the **multi-degree** of the monomial $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$. We will say that two such multi-degrees $\alpha$ and $\beta$ (or the corresponding monomials) **have the same parity** if $\alpha_i = \beta_i \mod 2$ for all $i$. We will assume that the monomials are listed in $\mu(x)$ in an order such that all monomials having the same parity are in a continuous segment. There will be at most $2^{m-1}$ such blocks: $n = \sum_i \alpha_i$ means that the parities are not independent.

Suppose $L(A) \in \Sigma_{m,2n}$. Then we can write $L(A) = \mu(x)^T Z \mu(x)$ for some $Z \succeq 0$. Observe that for any $A$, the polynomial $L(A)$ only has monomials in which all exponents are of even degree in all the variables. Therefore we in fact have

$$L(A) = \mu(\pm x_1, \ldots, \pm x_m)^T Z \mu(\pm x_1, \ldots, \pm x_m)$$

for any choice of signs (same on both sides). For any such sign choice there is a diagonal matrix $D$ with $\pm 1$ on the diagonal such that $\mu(\pm x_1, \ldots, \pm x_m) = D\mu(x_1, \ldots, x_m)$, so $L(A) = \mu(x)^T D^T Z D \mu(x)$ and $D^T Z D \succeq 0$ (conjugation preserves semidefiniteness).

This gives an action of the finite group $(S_2)^m$ (the product of $m$ copies of $S_2$) on the convex set $\mathcal{Z} := \{Z \succeq 0 \mid L(A) = \mu(x)^T Z \mu(x)\}$. Viewing $Z$ as a matrix indexed by the monomials in $\mu(x)$, the effect of flipping the sign of $x_i$ is to flip the sign of $Z_{\alpha,\beta}$ if and only if $\alpha_i \neq \beta_i \mod 2$.

Since we have a finite group acting linearly on a convex set $\mathcal{Z}$, Proposition 2.22 gives a $Z \in \mathcal{Z}$ which is fixed by this action. Such a $Z$ has $Z_{\alpha,\beta} = 0$ whenever $\alpha$ and $\beta$ do not have the same parity. The way the monomials in $\mu(x)$ were ordered, the means that $Z$ is block diagonal with blocks corresponding to the segments of $\mu(x)$. From now on when we say a matrix is block diagonal, this is what we will mean. Letting $\mathcal{H}(Z) = \mu(x)^T Z \mu(x)$ as in the previous section, we have shown that

$$L^{-1}(\Sigma_{m,2n}) = \{A \in \mathrm{Sym}^n(\mathbb{R}^m) \mid \exists \text{ block diagonal } Z \succeq 0 \text{ with } \mathcal{H}(Z) = L(A)\}.$$

To an index $j \in \{1, \ldots, m\}^n$ we can associate a multi-degree $\alpha = (\alpha_1, \ldots, \alpha_m)$ by $\alpha_i = \#\{k : j_k = i\}$, the number of times $i$ appears listed in $j$. Doing so generates all multi-degrees of monomials in $\mu(x)$. The symmetry of a tensor $A \in \mathrm{Sym}^n(\mathbb{R}^m)$ means that two indices $j, j'$ corresponding to the same $\alpha$ must satsify $A_j = A_{j'}$. The map $L$ sends a symmetric tensor $A$ to the polynomial for which the coefficient of $x_1^{2\alpha_1} \cdots x_m^{2\alpha_m}$ is a fixed multiple of $A_j$ for any index $j$ corresponding to multi-degree $\alpha$. The multiplier is the number of indices corresponding to this multi-degree.

Therefore $L$ is injective and has image equal to the subspace $V$ of $\mathbb{R}[x]^{2n}$ for which all powers of all variables are even. If we view this space as its range, $L$ is invertible. We will denote its inverse by $R : V \to \mathrm{Sym}^n(\mathbb{R}^m)$ since $L^{-1}$ has been reserved for the inverse image. Let $W$ denote the space of block diagonal matrices $Z$. The restriction $\mathcal{H}|_W$ maps $W$ to $V$, so

$$L^{-1}(\Sigma_{m,2n}) = \{R(\mathcal{H}|_W(Z)) \mid Z \succeq 0 \text{ is block diagonal}\}.$$

Then

$$\begin{aligned}
\mathrm{DNN}_m^n &= (L^{-1}(\Sigma_{m,2n}))^* = \{B \in \mathrm{Sym}^n(\mathbb{R}^m) \mid A \bullet B \geq 0 \text{ for all } A \in L^{-1}(\Sigma_{m,2n})\} \\
&= \{B \in \mathrm{Sym}^n(\mathbb{R}^m) \mid R(\mathcal{H}|_W(Z)) \bullet B \geq 0 \text{ for all block diagonal } Z \succeq 0\} \\
&= \{B \in \mathrm{Sym}^n(\mathbb{R}^m) \mid Z \bullet \mathcal{H}|_W^*(R^*(B)) \geq 0 \text{ for all block diagonal } Z \succeq 0\},
\end{aligned}$$

where $\mathcal{H}|_W^*$ and $R^*$ are adjoints and $\bullet$ is the standard inner product. By definition $\mathcal{H}|_W^*(R^*(B))$ is in $W$, i.e. it is block diagonal. The tensor $B$ is doubly nonnegative if and only if each block of $\mathcal{H}|_W^*(R^*(B))$ makes a nonnegative inner product with all positive semidefinite matrices. Since the cone of positive semidefinite matrices is self-dual, this happens if and only if each block of $\mathcal{H}|_W^*(R^*(B))$ is itself positive semidefinite, which in turn happens if and only if $\mathcal{H}|_W^*(R^*(B))$ is positive semidefinite. If we define $T = \mathcal{H}|_W^* \circ R^*$ we can write out $T$ explicitly in coordinates (being careful about which inner products were used to define the adjoints), which proves the following.

**Theorem 2.53.** *Fix $m$ and $n$ and define the map $T$ from $\mathrm{Sym}^n(\mathbb{R}^m)$ to block diagonal matrices indexed by pairs of multi-degrees of monomials in $\mathbb{R}[x_1, \ldots, x_m]^n$ by*

$$[T(B)]_{\alpha\beta} = \begin{cases} B_j, & j \text{ has associated multi-degree } \frac{\alpha+\beta}{2} \in \mathbb{Z}_{\geq 0}^m, \\ 0, & \alpha \text{ and } \beta \text{ have different parity so } \frac{\alpha+\beta}{2} \notin \mathbb{Z}_{\geq 0}^m. \end{cases}$$

*Then*

$$\begin{aligned}
\mathrm{DNN}_m^n &= \{B \in \mathrm{Sym}^n(\mathbb{R}^m) \mid T(B) \succeq 0\} \\
&= \{B \in \mathrm{Sym}^n(\mathbb{R}^m) \mid \text{each block of } T(B) \text{ is positive semidefinite}\}.
\end{aligned}$$

If $\alpha$ is the multi-degree associated with index $j$, then $[T(B)]_{\alpha\alpha} = B_j$. In particular positive semidefiniteness of $T(B)$ means that the diagonal entries are nonnegative, so $B$ is a nonnegative tensor. This is a good sanity check, because nonnegativity is a baseline requirement we would expect from a convex relaxation of $\mathrm{CP}^n_m$, all of whose elements are nonnegative. It is straightforward to show directly that for $x \in \mathbb{R}^m_{\geq 0}$ we have $T(x^{\otimes n}) \succeq 0$, so by convexity $\mathrm{CP}^n_m \subseteq \mathrm{DNN}^n_m$ as expected. We now look at several values of $m$ and $n$ explicitly, focusing on the question of exactness: when does $\mathrm{CP}^n_m = \mathrm{DNN}^n_m$?

**Example 2.54.** The case $n = 2$ is that of symmetric matrices. Categorizing the quadratic monomials in $m$ variables according to the parity of their multi-degrees, we get one size $m$ group $x_1^2, \ldots, x_m^2$ of all the squares and then $\binom{m}{2}$ size one groups each consisting of $x_i x_j$ for some $i \neq j$. Therefore for a matrix $B$ the matrix $T(B) \in \mathbb{R}^{\binom{m+1}{2} \times \binom{m+1}{2}}$ has its upper left $m \times m$ block equal to $B$, while its remaining diagonal entries list the off-diagonal entries of $B$ and the other elements are zero. Thus $B$ is doubly nonnegative if and only if $B \succeq 0$ and the off-diagonal entries of $B$ are nonnegative. Since $B \succeq 0$ implies that the diagonal entries of $B$ are nonnegative, this agrees with the standard:

**Definition 2.55.** *A matrix is* ***doubly nonnegative*** *if it is symmetric, element-wise nonnegative, and positive semidefinite.*

This case is the reason for the name *doubly nonnegative.* Exactness of the relaxation has been well-studied in the matrix case, where it is known that $\mathrm{CP}^2_m = \mathrm{DNN}^2_m$ if and only if $m \leq 4$. For a discussion of the history of this result and a geometric proof, see [33]. The case $m = 2$ is easy to show by hand, the others are successively more difficult, and there is an explicit counterexample known in the case $m = 5$. For our purposes, the utility of this result is that it provides a way to decide complete positivity easily for $4 \times 4$ and smaller matrices. $\diamondsuit$

**Example 2.56.** Next we consider the case $m = 2$ of $2 \times \cdots \times 2$ symmetric tensors of type $n$. We formed the relaxation $\mathrm{DNN}^n_m$ of $\mathrm{CP}^n_m$ by relaxing $\Psi_{m,2n}$ to $\Sigma_{m,2n}$, but in the case $m = 2$ we have also proven $\Psi_{2,2n} = \Sigma_{2,2n}$, so we have not actually relaxed anything: double nonnegativity and complete positivity are equivalent in this case.

A tensor in $\mathrm{Sym}^n(\mathbb{R}^2)$ is indexed by $j \in \{0,1\}^n$ and two entries are forced to be equal by symmetry if and only if their indices have the same number of ones. Thus a tensor $B$ can be described by $n+1$ numbers $b_0, \ldots, b_n$, where $b_i$ is the value of all $B_j$ such that $\sum j_k = i$. The bivariate monomials of degree $n$ are of the form $x_1^i x_2^{n-i}$ and so split into two categories by parity of $i$. The number of monomials of each parity in turn depends on the parity of $n$, so the conditions split into two cases. If $n = 2l + 1$ is odd, then $B$ is doubly nonnegative / completely positive if and only if

$$\begin{bmatrix} b_0 & b_1 & \cdots & b_l \\ b_1 & b_2 & \cdots & b_{l+1} \\ \vdots & \vdots & \ddots & \vdots \\ b_l & b_{l+1} & \cdots & b_{2l} \end{bmatrix} \succeq 0 \quad \text{and} \quad \begin{bmatrix} b_1 & b_2 & \cdots & b_{l+1} \\ b_2 & b_3 & \cdots & b_{l+2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{l+1} & b_{l+2} & \cdots & b_{2l+1} \end{bmatrix} \succeq 0.$$

If $n = 2l$ is even, then $B$ is doubly nonnegative / completely positive if and only if

$$\begin{bmatrix} b_0 & b_1 & \cdots & b_l \\ b_1 & b_2 & \cdots & b_{l+1} \\ \vdots & \vdots & \ddots & \vdots \\ b_l & b_{l+1} & \cdots & b_{2l} \end{bmatrix} \succeq 0 \quad \text{and} \quad \begin{bmatrix} b_1 & b_2 & \cdots & b_l \\ b_2 & b_3 & \cdots & b_{l+1} \\ \vdots & \vdots & \ddots & \vdots \\ b_l & b_{l+1} & \cdots & b_{2l-1} \end{bmatrix} \succeq 0.$$

◇

**Example 2.57.** A $3 \times 3 \times 3$ symmetric tensor $B$ has 27 entries but the symmetry requirement leaves only 10 free parameters. We can view such a tensor as three $3 \times 3$ matrices $B_1, B_2, B_3$ stacked above each other. The triviariate monomials of degree 3 split into four groups based on parity: $x_1 \cdot (x_1^2, x_2^2, x_3^2)$, $x_2 \cdot (x_1^2, x_2^2, x_3^2)$, $x_3 \cdot (x_1^2, x_2^2, x_3^2)$, and the singleton $x_1x_2x_3$. Therefore $T(B)$ is block diagonal with three $3 \times 3$ blocks $B_1$, $B_2$, and $B_3$ and one $1 \times 1$ block $B_{123}$ (the only entry of $B$ not on the diagonal of any of the matrices $B_i$). Therefore $B \in \mathrm{DNN}_3^3$ if and only if the three "slices" $B_i$ are positive semidefinite and $B_{123} \geq 0$.

Define the tensor

$$A = \left[\begin{array}{ccc|ccc|ccc} 0 & 1/3 & 0 & 1/3 & 1/3 & -1/2 & 0 & -1/2 & 0 \\ 1/3 & 1/3 & -1/2 & 1/3 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & -1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \in \mathrm{Sym}^3(\mathbb{R}^3).$$

Then $L(A) = M$, the Motzkin polynomial in $\Psi_{3,6} \setminus \Sigma_{3,6}$. That means $A \in (\mathrm{CP}_3^3)^* \setminus L^{-1}(\Sigma_{3,6})$. Dualizing, we get $\mathrm{CP}_3^3 \subsetneq \mathrm{DNN}_3^3$: complete positivity and double nonnegativity disagree for $3 \times 3 \times 3$ tensors. By solving the semidefinite program with constraints $B \in \mathrm{DNN}_3^3$ and $A \bullet B = -6$ (any negative number would do) we can find an explicit tensor

$$B = \left[\begin{array}{ccc|ccc|ccc} 639 & 30 & 80 & 30 & 30 & 32 & 80 & 32 & 40 \\ 30 & 30 & 32 & 30 & 639 & 80 & 32 & 80 & 40 \\ 80 & 32 & 40 & 32 & 80 & 40 & 40 & 40 & 30 \end{array}\right] \in \mathrm{DNN}_3^3 \setminus \mathrm{CP}_3^3. \qquad ◇$$

Since the image of $L$ is the space of polynomials in $\mathbb{R}[x_1, \ldots, x_m]^{2n}$ all of whose terms have even degree in all the variables, $L^{-1}(\Sigma_{m,2n}) \subsetneq L^{-1}(\Psi_{m,2n})$ if and only if there is a $p \in \Psi_{m,2n} \setminus \Sigma_{m,2n}$ whose terms have even degree in all the variables. The Motzkin polynomial fulfills this role for $m = n = 3$. We can use this to derive such a $p$ whenever $m, n \geq 3$.

A polynomial $p \in \mathbb{R}[x_1, \ldots, x_m]^{2n}$ is nonnegative if and only it is nonnegative when viewed as a polynomial in $\mathbb{R}[x_1, \ldots, x_{m+1}]^{2n}$. By Newton polytope considerations, such a $p$ is in $\Sigma_{m,2n}$ if and only if it is in $\Sigma_{m+1,2n}$. Adding a variable in this way does not change the fact that all terms of $p$ have even degree in all variables. Therefore $L^{-1}(\Sigma_{m,2n}) \subsetneq L^{-1}(\Psi_{m,2n})$ implies $L^{-1}(\Sigma_{m+1,2n}) \subsetneq L^{-1}(\Psi_{m+1,2n})$. The same statements apply to $p$ and $x_1^2 \cdot p(x_1, \ldots, x_m) \in \mathbb{R}[x_1, \ldots, x_m]^{2n+2}$, so $L^{-1}(\Sigma_{m,2n}) \subsetneq L^{-1}(\Psi_{m,2n})$ implies $L^{-1}(\Sigma_{m,2n+2}) \subsetneq L^{-1}(\Psi_{m,2n+2})$.

Combining these two procedures and repeatedly applying them to the Motzkin polynomial, we get $L^{-1}(\Sigma_{m,2n}) \subsetneq L^{-1}(\Psi_{m,2n})$ for all $m, n \geq 3$. Taking dual cones, we therefore have $\mathrm{CP}_m^n \subsetneq \mathrm{DNN}_m^n$ for all $m, n \geq 3$. We summarize this and the results proven or cited above as:

**Theorem 2.58.** *For all $m$ and $n$, $\mathrm{CP}_m^n \subseteq \mathrm{DNN}_m^n$. Equality holds if and only if $m = 2$ or $n = 2$ and $m \leq 4$.*

# Chapter 3

# Symmetric Exchangeable Equilibria

In this chapter we define exchangeable equilibria of symmetric games and establish their basic properties. For now we will view these purely formally. We defer discussion of interpretations of exchangeable equilibria to the following chapter.

We first define a class of generalized exchangeable distributions, which are closely connected with exchangeable distributions in the usual sense when the symmetry group $G$ of the game is standard (acts by permuting players transitively but does not permute strategies of any individual player). We then define exchangeable equilibria to be correlated equilibria which are exchangeable in this general sense and examine the geometric content of this definition. Next we present a series of examples to illustrate the theory.

In the final section we restrict to the symmetric bimatrix case and show how symmetric correlated equilibria, exchangeable equilibria, and the convex hull of symmetric Nash equilibria all arise naturally as successively stronger convex relaxations of a well-known system of quadratic inequalities defining symmetric Nash equilibria. The exchangeable equilibria and the convex hull of symmetric Nash equilibria are both characterized in terms of completely positive matrices. A reader preferring a more concrete approach may prefer to read the final section first as motivation for the general case; indeed, that order better reflects the sequence in which the concepts were originally developed.

## 3.1 Generalized exchangeable distributions

When $\Gamma$ is a standard symmetric game with $G$ as its symmetry group, $\Delta_G^\Pi(\Gamma)$ is the set of i.i.d. distributions in $n$ random variables taking values in $C_1$. By De Finetti's Theorem and Proposition 2.46, the distributions in $\operatorname{conv}(\Delta_G^\Pi(\Gamma))$ are exactly those which can be extended to exchangeable distributions in infinitely many random variables. The following definition lacks this elegant interpretation when $G$ is not standard, but will allow us to develop the theory of exchangeable equilibria with minimal assumptions on the action of $G$.

**Definition 3.1.** *Viewing $\Delta_G^\Pi(\Gamma)$ as a nonconvex subset of the convex set $\Delta_G(\Gamma)$, the set of* ***(generalized) exchangeable*** *probability distributions is*

$$\Delta_G^X(\Gamma) := \operatorname{conv}\left(\Delta_G^\Pi(\Gamma)\right) \subseteq \Delta_G(\Gamma).$$

Often the word *generalized* will be clear from context and will be omitted. Note that the symbol $\Delta_G^X$ does *not* refer to the $G$-invariant elements of some set $\Delta^X$; rather, it is an abuse of notation chosen to fit with the symbols $\Delta_G$ and $\Delta_G^\Pi$. To get a feel for these sets, we will look at them in the context of some examples.

**Example 3.2** (continues Example 2.27). When $G$ acts trivially we can ignore it entirely. Not all distributions are independent so $\Delta_G^\Pi(\Gamma) \subsetneq \Delta_G(\Gamma) = \Delta(\Gamma)$, but $\Delta_G^X(\Gamma) = \Delta_G(\Gamma)$. By definition one inclusion is automatic. To prove the reverse note that for any $s \in C$, $\delta_s = \delta_{s_1} \cdots \delta_{s_n} \in \Delta^\Pi(\Gamma) = \Delta_G^\Pi(\Gamma)$. But for any $\pi \in \Delta(\Gamma)$ we can write $\pi = \sum_{s \in C} \pi(s)\delta_s$, and such a convex combination of the $\delta_s$ is in $\Delta_G^X(\Gamma)$ by definition. $\diamondsuit$

**Example 3.3** (continues Example 2.28). A symmetric bimatrix game is standard, so $\Delta_G^X$ is the set of normalized completely positive matrices (Proposition 2.45). $\diamondsuit$

**Example 3.4** (continues Example 2.30). In matching pennies the map on $C$ induced by the symmetry $h$ is the cyclic permutation

$$((H_1, H_2)\ (T_1, H_2)\ (T_1, T_2)\ (H_1, T_2)).$$

In particular, a $G$-invariant probability distribution must assign equal probability to all four outcomes in $C$. There is only one such distribution and it is independent, so $\Delta_G^\Pi(\Gamma) = \Delta_G^X(\Gamma) = \Delta_G(\Gamma)$. $\diamondsuit$

**Example 3.5** (continues Example 2.31). Recall that in this game there are $n$ players and the $C_i$ are the same for all $i$. The group $G$ permutes the players cyclically. Therefore the elements of $\Delta_G^\Pi(\Gamma)$ are invariant under arbitrary permutations of the players, hence so are the elements of $\Delta_G^X(\Gamma)$. (The converse statement is false; that is to say, there are probability distributions over $C$ which are invariant under arbitrary permutations of the players but are not in $\Delta_G^X(\Gamma)$.) On the other hand, an element of $\Delta_G(\Gamma)$ need only be invariant under cyclic permutations of the players. $\diamondsuit$

The basic properties of $\Delta_G^X(\Gamma)$ are summarized in the following propositions.

**Proposition 3.6.** *The set $\Delta_G^X(\Gamma)$ is a compact convex semialgebraic set with*

$$\Delta_G^\Pi(\Gamma) \subseteq \Delta_G^X(\Gamma) \subseteq \Delta_G(\Gamma).$$

*If the action of $G$ is player-transitive then the set of extreme points of $\Delta_G^X(\Gamma)$ is $\Delta_G^\Pi(\Gamma)$. If $G$ is standard then $\Delta_G^X(\Gamma)$ is the set of normalized completely positive tensors in $\mathrm{CP}_m^n$.*

*Proof.* The inclusions are immediate. The set $\Delta^{\Pi}(\Gamma)$ viewed as a set of $n$-tuples, being a product of simplices, is compact. To view $\Delta^{\Pi}(\Gamma)$ as a set of product distributions in $\Delta(\Gamma)$ we take the image of the set of tuples under the continuous map $(\sigma_1, \ldots, \sigma_n) \mapsto \sigma_1 \otimes \cdots \otimes \sigma_n$, so this is also compact. The action of $G$ on this set is continuous, thus the fixed point subset $\Delta_G^{\Pi}(\Gamma)$ is closed, hence compact. The convex hull of a compact set is compact [5]. Semialgebraicity follows from Carathéodory's theorem and quantifier elimination just as in Proposition 2.48.

By definition an element of $\Delta_G^X(\Gamma)$ can be written as a convex combination of elements of $\Delta_G^{\Pi}(\Gamma)$, so only elements of the latter set can be extreme. If the action of $G$ is player-transitive, the elements of $\Delta_G^{\Pi}(\Gamma)$ are nonnegative symmetric simple tensors. Therefore they are extreme rays of $\mathrm{CP}_m^n$ by Proposition 2.49. But $\Delta_G^X(\Gamma)$ is a convex subset of $\mathrm{CP}_m^n$ all of whose elements are normalized. In particular this means $\Delta_G^X(\Gamma)$ cannot contain two elements which are scalar multiples of each other, so a point in $\Delta_G^{\Pi}(\Gamma)$ which is an extreme ray of $\mathrm{CP}_m^n$ is an extreme point of $\Delta_G^X(\Gamma)$.

The final claim is by definition of $\Delta_G^{\Pi}(\Gamma)$ for a standard game and Proposition 2.45. □

Example 3.2 shows how the above can fail without the assumption on $G$. In that example $\Delta_G^X(\Gamma) = \Delta_G(\Gamma)$ only has finitely many extreme points.

**Proposition 3.7.** *The linear extension of the inclusion map $\Delta_G^{\Pi}(\Gamma) \to \Delta_G(\Gamma)$ is weakly continuous and maps $\Delta(\Delta_G^{\Pi}(\Gamma))$ onto $\Delta_G^X(\Gamma)$.*

*Proof.* Apply Lemma 2.47 with $X = \Delta_G^{\Pi}(\Gamma)$ and $f : \Delta_G^{\Pi} \to \Delta_G(\Gamma)$ the inclusion map. □

## 3.2 Definition and properties

We are now ready to define exchangeable equilibria.

**Definition 3.8.** *The set of **(symmetric) exchangeable equilibria** of a game $\Gamma$ with symmetry group $G$ is*

$$\mathrm{XE}_G(\Gamma) := \mathrm{CE}(\Gamma) \cap \Delta_G^X(\Gamma).$$

Note that similarly to $\Delta_G^X$, the symbol $\mathrm{XE}_G(\Gamma)$ is chosen to match $\mathrm{NE}_G(\Gamma)$ and $\mathrm{CE}_G(\Gamma)$; it does *not* denote $G$-invariant elements of some set $\mathrm{XE}(\Gamma)$ of "asymmetric exchangeable equilibria." We leave such an asymmetric notion undefined until Chapter 6. Until then we are free to drop the word *symmetric* in our discussion.

For now we observe that if the action of $G$ is trivial, Example 3.2 above shows that $\Delta_G^X(\Gamma) = \Delta(\Gamma)$, so $\mathrm{XE}_G(\Gamma) = \mathrm{CE}(\Gamma)$. In particular this shows that we would not get a novel concept if we tried to define the asymmetric exchangeable equilibria as $\mathrm{XE}_G(\Gamma)$ for a trivial action of $G$. To get the correct notion we will need some additional insight from studying symmetric exchangeable equilibria in detail.

Intersecting everything in Proposition 3.6 with $\mathrm{CE}(\Gamma)$ yields the following.

**Proposition 3.9.** *The set* $\mathrm{XE}_G(\Gamma)$ *is a compact convex semialgebraic set with* $\mathrm{conv}(\mathrm{NE}_G(\Gamma)) \subseteq \mathrm{XE}_G(\Gamma) \subseteq \mathrm{CE}_G(\Gamma)$. *If the action of $G$ is player-transitive then elements of* $\mathrm{NE}_G(\Gamma)$ *are (among the) extreme points of* $\mathrm{XE}_G(\Gamma)$.

In the following section we will see examples in which all of these containments are strict. There is one case in which the first containment cannot be strict:

**Theorem 3.10.** *If $\Gamma$ is a symmetric bimatrix game with $m = 2$ strategies per player then* $\mathrm{conv}(\mathrm{NE}_{S_2}(\Gamma)) = \mathrm{XE}_{S_2}(\Gamma)$.

*Proof.* Let $A = \left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right]$ be the payoff matrix for the row player, $b := x - z$ and $c := w - y$. An exchangeable equilibrium $W = \left[\begin{smallmatrix} p & q \\ q & r \end{smallmatrix}\right]$ must satisfy the correlated equilibrium constraints and complete positivity, which is the same as double nonnegativity as shown in Example 2.56. Altogether the conditions are:

$$\begin{aligned} pr &\geq q^2 && \text{(semidefiniteness)},\\ p,q,r &\geq 0 && \text{(nonnegativity)},\\ bp &\geq cq && \text{(incentive constraint \#1)},\\ cr &\geq bq && \text{(incentive constraint \#2), and}\\ p+2q+r &= 1 && \text{(normalization)}. \end{aligned}$$

Let us show that any extreme point $W$ of $\mathrm{XE}_{S_2}(\Gamma)$ is in $\mathrm{conv}(\mathrm{NE}_{S_2}(\Gamma))$. If $pr = q^2$ then $\mathrm{rank}(W) = 1$, so it is a Nash equilibrium. If $q = 0$ then we can write $W = p\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] + r\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ as a convex combination of symmetric pure Nash equilibria.

The remaining case is that semidefiniteness and nonnegativity are not tight. Extremality requires that the remaining three linear conditions on $p$, $q$, and $r$ be tight and linearly independent. Multiplying the tight incentive constraints yields $(bc)pr = (bc)q^2$. But $pr > q^2$, so $b = 0$ or $c = 0$. Then $bp = cq$ and $p, q > 0$ give $b = c = 0$. The incentive constraints are trivial, hence linearly dependent. □

By Nash's Theorem symmetric games always admit symmetric Nash equilibria. Many symmetric games also have asymmetric Nash equilibria. In the strategy-trivial case, such games also admit symmetric correlated equilibria which are not exchangeable, so the second containment in Proposition 3.9 is strict. In Examples 3.18 and 3.19 below we will see some ways in which the assumptions of this theorem cannot be weakened.

**Theorem 3.11.** *If a game* $\Gamma$ *with strategy-trivial symmetry group* $G$ *has* $\mathrm{NE}_G(\Gamma) \subsetneq \mathrm{NE}(\Gamma)$ *then* $\mathrm{XE}_G(\Gamma) \subsetneq \mathrm{CE}_G(\Gamma)$. *In particular, if* $\pi \in \mathrm{NE}(\Gamma) \setminus \mathrm{NE}_G(\Gamma)$ *then* $\mathrm{ave}_G(\pi) \in \mathrm{CE}_G(\Gamma) \setminus \mathrm{XE}_G(\Gamma)$.

*Proof.* Let $\pi = x_1 \otimes \cdots \otimes x_n \in \mathrm{NE}(\Gamma)$ and $\psi = \mathrm{ave}_G(\pi)$. The action of $G$ maps $\mathrm{CE}(\Gamma)$ to itself, so $\psi \in \mathrm{CE}_G(\Gamma)$. We will prove the contrapositive of the theorem statement in the form: if $\psi \in \Delta_G^X(\Gamma)$ then $\pi \in \Delta_G^\Pi(\Gamma)$. We first do this in the case that $G$ is standard (i.e. also player-transitive) and then generalize. Standardness means $\Delta_G^X(\Gamma) \subseteq \Delta_{S_n}(C_1^n)$, so $\psi$ is $S_n$-invariant and completely positive.

Standardness also implies that the action of $G$ factors through the map $G \to S_n$ induced by the action of $G$ on the players. Modding out by the kernel of this action, we can take $G$ to be a subgroup of $S_n$ without changing any of the equilibrium sets in the theorem statement. For any $\sigma \in S_n$, the distribution $\psi = \psi \cdot \sigma$ is the average of $\pi$ over the right coset $G\sigma$. Averaging over all cosets we obtain $\psi = \mathrm{ave}_{S_n}(\pi)$.

We now show that for any $r \in \mathbb{R}^m$, $r \bullet x_1 = \cdots = r \bullet x_n$. For $\alpha \in \mathbb{R}$, define $r_\alpha := r - \alpha e$. Define a linear functional $L_{r_\alpha}$ on $(\mathbb{R}^m)^{\otimes n}$ by $L_{r_\alpha}(W) := W \bullet (r_\alpha \otimes r_\alpha \otimes e^{\otimes(n-2)})$. Then for any $v \in \mathbb{R}^m_{\geq 0}$, we have $L_{r_\alpha}(v^{\otimes n}) = (r_\alpha \bullet v)^2 (e \bullet v)^{n-2} \geq 0$. By linearity, $L_{r_\alpha}$ is nonnegative on $\mathrm{CP}_m^n$.

By assumption on $\psi$:

$$\begin{aligned}
0 \leq L_{r_\alpha}(\psi) &= L_{r_\alpha}\left(\frac{1}{n!}\sum_{\sigma \in S_n} \pi \cdot \sigma^{-1}\right) = \frac{1}{n!}\sum_{\sigma \in S_n} L_{r_\alpha}(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}) \\
&= \frac{1}{n!}\sum_{\sigma \in S_n} (r_\alpha \bullet x_{\sigma(1)})(r_\alpha \bullet x_{\sigma(2)})(e \bullet x_{\sigma(3)}) \cdots (e \bullet x_{\sigma(n)}) \\
&= \frac{1}{n!}\sum_{\sigma \in S_n} (r_\alpha \bullet x_{\sigma(1)})(r_\alpha \bullet x_{\sigma(2)}) = \frac{(n-2)!}{n!}\sum_{i=1}^{n}\sum_{j \neq i}(r_\alpha \bullet x_i)(r_\alpha \bullet x_j) \\
&= \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j \neq i}(r \bullet x_i - \alpha)(r \bullet x_j - \alpha).
\end{aligned}$$

Let $p_i := r \bullet x_i$. We have shown $\sum_{i=1}^n \sum_{j \neq i}(p_i - \alpha)(p_j - \alpha) \geq 0$ for all $\alpha \in \mathbb{R}$. Clearly this is the case if all the $p_i$ are equal. We now show that this is the only case in which this inequality can hold for all $\alpha$. Let $a = n(n-1)$, $b = -2(n-1)\sum_i p_i$ and $c = (\sum_i p_i)^2 - \sum_i p_i^2$ be the coefficients of this quadratic polynomial in $\alpha$. Then $a\alpha^2 + b\alpha + c$ is nonnegative for all $\alpha$, so its minimum value $\frac{4ac - b^2}{4a}$ is also nonnegative. Rearranging terms in the inequality $b^2 \leq 4ac$ we obtain

$$n \sum_i p_i^2 \leq \left(\sum_i p_i\right)^2.$$

The Cauchy-Schwarz inequality applied to $p = [p_1\ p_2\ \cdots\ p_n]$ and $e$ is the reverse. But the Cauchy-Schwarz inequality is only tight when applied to parallel vectors, so $p$ is a multiple of $e$. That is to say, $r \bullet x_1 = \cdots = r \bullet x_n$. Since $r \in \mathbb{R}^m$ was arbitrary, $x_1 = \cdots = x_n$ and $\pi \in \Delta_G^\Pi(\Gamma)$. This completes the standard case.

In general the action of a group on a game partitions the set of players into multiple orbits; the player-transitive case is when there is exactly one orbit. In each orbit, the cardinalities of all players' strategy sets are equal, because each of these is mapped bijectively onto each other one by the action of some group element. When the action is strategy-trivial, these bijections are unique and compatible, so the strategy sets of all players in an orbit are identified in a canonical way and we can take them to be identical.

Reordering the players so players in an orbit are numbered consecutively, elements of $\Delta_G^\Pi(\Gamma)$ are of the form $v_1^{\otimes l_1} \otimes \cdots \otimes v_k^{\otimes l_k}$, where $k$ is the number of orbits, the $l_k$ are their sizes, and $\sum_k l_k = n$. Marginalizing out the strategies of all players except those in a given orbit $j$ (which we may assume to contain more than one player), the above argument for the standard case shows that if $\psi \in \Delta_G^X(\Gamma)$ then the mixed strategies $x_i$ of players in orbit $j$ are all equal. Repeating this argument for all orbits, we obtain $\pi \in \Delta_G^\Pi(\Gamma)$. □

Compare the final statement of Proposition 3.9 with the similar statement of Proposition 8.6, first proven in [14] and [25]. This second result states that for generic two-player games, which have finitely many Nash equilibria (see e.g. [32]), elements of NE($\Gamma$) are among the extreme points of CE($\Gamma$). Without the two-player or genericity assumptions this result can fail. In contrast Proposition 3.9 shows that the corresponding statement about $\mathrm{NE}_G(\Gamma)$ and $\mathrm{XE}_G(\Gamma)$ holds without assumptions about the number of players or genericity of the game, as long as the action of $G$ is player-transitive. This is one sense in which the set of exchangeable equilibria can be thought of as closer to the set of (symmetric) Nash equilibria than the set of correlated equilibria is.

We now give a direct proof (i.e. without applying Nash's theorem) that an exchangeable equilibrium exists along the lines of the correlated equilibrium existence proof in Section 2.1.2. We again consider the zero-sum game $\Gamma^0$ and prove that a certain set is good in this game (Lemma 3.15). The difference is that the action of $G$ yields a smaller good set, $\Delta_G^\Pi(\Gamma)$. To prove this lemma we need the following three symmetry results.

**Proposition 3.12.** *If $G$ acts on $\Gamma$ then $G$ acts player-trivially on $\Gamma^0$ by*

$$g \cdot (s, (r_i, t_i)) := (g \cdot s, (g \cdot r_i, g \cdot t_i)).$$

*Proof.* The action thus defined on the strategy sets of $\Gamma^0$, which is obviously player-trivial, respects the utilities. This is a matter of applying the definitions:

$$\begin{aligned} u_M^0(g \cdot (s, (r_i, t_i))) &= u_M^0(g \cdot s, (g \cdot r_i, g \cdot t_i)) \\ &= \begin{cases} u_{g \cdot i}(g \cdot s) - u_{g \cdot i}(g \cdot t_i, g \cdot s_{-i}) & \text{if } g \cdot r_i = g \cdot s_i, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} u_i(s) - u_i(t_i, s_{-i}) & \text{if } r_i = s_i, \\ 0 & \text{otherwise} \end{cases} \\ &= u_M^0(s, (r_i, t_i)) \end{aligned}$$

for all values of the parameters. For $u_m^0$ multiply by $-1$. □

**Proposition 3.13.** *If $G$ acts player-trivially on a zero-sum game, then a set $\Sigma \subseteq \Delta_G(C_M)$ is good if and only if it is good against $\Delta_G(C_m)$.*

*Proof.* For all $g \in G$, $\sigma \in \Delta_G(C_M)$, and $\theta \in \Delta(C_m)$ we have $u_M(\sigma, \theta \cdot g) = u_M(\sigma \cdot g, \theta \cdot g) = u_M(\sigma, \theta)$, so $u_M(\sigma, \theta) = u_M(\sigma, \mathrm{ave}_G(\theta))$. □

**Proposition 3.14.** *The map $\mathrm{Mm}(\gamma(\cdot))$ is natural in the sense that if $\sigma : C_i \to C_j$ is a bijection and $y_i = y_j \circ (\sigma, \sigma)$, then composition with $\sigma$ maps $\mathrm{Mm}(\gamma(y_j))$ to $\mathrm{Mm}(\gamma(y_i))$.*

*Proof.* For all $s_i, t_i \in C_i$ we have

$$\begin{aligned} u_M^{\gamma(y_j)}(\sigma(s_i), \sigma(t_i)) &= \begin{cases} \sum_{r_j \neq \sigma(t_i)} y_j^{\sigma(s_i), r_j} & \text{if } \sigma(s_i) = \sigma(t_i), \\ -y_j^{\sigma(s_i), \sigma(t_i)} & \text{otherwise} \end{cases} \\ &= \begin{cases} \sum_{r_i \neq t_i} y_j^{\sigma(s_i), \sigma(r_i)} & \text{if } s_i = t_i, \\ -y_i^{s_i, t_i} & \text{otherwise} \end{cases} \\ &= \begin{cases} \sum_{r_i \neq t_i} y_i^{s_i, r_i} & \text{if } s_i = t_i, \\ -y_i^{s_i, t_i} & \text{otherwise} \end{cases} \\ &= u_M^{\gamma(y_i)}(s_i, t_i), \end{aligned}$$

so $\gamma(y_i)$ and $\gamma(y_j)$ are the same game up to a relabeling of the strategy sets. Thus maximin strategies of $\gamma(y_j)$, appropriately relabeled, give maximin strategies of $\gamma(y_i)$ by the same argument which shows that if a group $G$ acts on a game $\Gamma$ then it acts on $\mathrm{NE}(\Gamma)$. □

**Lemma 3.15.** *If $G$ acts on the game $\Gamma$ then the set $\Delta_G^{\Pi}(\Gamma)$ is good in the zero-sum game $\Gamma^0$ of Definition 2.16.*

*Proof.* By Proposition 3.12 and Proposition 3.13, it suffices to consider only $y \in \Delta_G(C_m'^0)$, and show that there is a $\pi \in \Delta_G^\Pi(\Gamma)$ which is good against $y$. Lemma 2.19 states that any $\pi \in Z(y) := \mathrm{Mm}(\gamma(y_1)) \times \cdots \times \mathrm{Mm}(\gamma(y_n)) \subset \Delta^\Pi(\Gamma)$ is good against $y$.

By Proposition 3.14 the action of $G$ on $\Delta^\Pi(\Gamma)$ restricts to a linear action of $G$ on $Z(y)$ since $y \in \Delta_G(C_m'^0)$. Viewing $Z(y)$ as a convex subset of $\mathbb{R}^{\sqcup_i C_i}$, Proposition 2.22 shows the $G$-invariant subset $Z_G(y) \subseteq \Delta_G^\Pi(\Gamma)$ is nonempty, so $\Delta_G^\Pi(\Gamma)$ is good. □

**Theorem 3.16.** *A game with symmetry group $G$ has an exchangeable equilibrium.*

*Proof.* By Theorem 2.20, $\mathrm{Mm}(\Gamma^0) = \mathrm{CE}(\Gamma)$. Lemma 3.15 shows we can apply Proposition 2.13 to $\Gamma^0$ with $\Sigma = \Delta_G^\Pi(\Gamma)$, proving that $\mathrm{Mm}(\Gamma^0) \cap \Delta_G^X(\Gamma) = \mathrm{XE}_G(\Gamma)$ is nonempty. □

It is worth contrasting this with the proof that symmetric correlated equilibria exist (Proposition 2.33). Both involve averaging arguments to produce symmetric solutions. The difference is that in the proof of Proposition 2.33 the averaging occurs within the set $\Delta(\Gamma)$, whereas in the case of Theorem 3.16 (in particular Lemma 3.15), the averaging occurs within $\Delta^\Pi(\Gamma)$, viewed as a convex subset of $\mathbb{R}^{\sqcup_i C_i}$. By averaging within this smaller set, we guarantee that the resulting correlated equilibrium has the additional structure such as symmetries discussed at the end of Section 2.1.3.

The latter averaging argument requires a bit more care. In particular, Proposition 2.33 is an immediate corollary of Theorem 2.20 on the existence of correlated equilibria. On the other hand, to prove Theorem 3.16 we have to "lift the hood" on Theorem 2.20 and use Lemma 2.19 on good sets. By doing so we exhibit a correlated equilibrium which we can prove lies in $\Delta_G^X(\Gamma)$ instead of just $\Delta_G(\Gamma)$.

A nonempty semialgebraic set defined in terms of polynomials with rational coefficients contains a point with algebraic coordinates (by induction on the projection theorem for semialgebraic sets, Theorem 2.76 in [3]), so any symmetric game with rational utilities has an algebraic exchangeable equilibrium. In the case of symmetric bimatrix games we can sharpen this by invoking Nash's theorem.

**Theorem 3.17.** *A symmetric bimatrix game $\Gamma$ with rational utilities admits a rational exchangeable equilibrium: $\mathrm{XE}_{S_2}(\Gamma) \cap \mathbb{Q}^{m \times m} \neq \emptyset$.*

*Proof.* By Nash's theorem a symmetric Nash equilibrium $(x, x) \in \mathrm{NE}_{S_2}(\Gamma)$ exists. Let $D_i \subseteq C_i$ be the set of strategies to which $x$ assigns positive probability. Then the conditions $y \in \Delta(D_1) \subseteq \Delta(C_1)$ and $s_1$ is a best response to $y$ for all $s_1 \in D_1$ are a finite number of linear inequalities on $y$. The coefficients are differences of utilities, so rational, and the constraints are feasible as the choice $y = x$ shows.

A feasible $y$ is a best response to itself, so the pair $(y, y) \in \mathrm{NE}_{S_2}(\Gamma)$. Any extreme point of the feasible polytope is rational, so in particular a rational Nash equilibrium exists and this is a rational exchangeable equilibrium. □

## 3.3 Examples

All the examples presented in this section are standard symmetric games sized so that complete positivity coincides with double nonnegativity; this is an important part of what enables us to do the computations. Many are games of **identical interest**, i.e., games in which every player receives the same utility in all outcomes. It is somewhat surprising that these games exhibit any noteworthy behavior, because identical interest games are often thought of as somewhat trivial: they always have pure Nash equilibria, for example, and are often viewed as single-player decision problems in disguise. However, the pure Nash equilibria need not be symmetric, so when restricting attention to symmetric equilibria identical interest games seem to be a less trivial class.

**Example 3.18** ($2 \times 2$ game with $\mathrm{conv}(\mathrm{NE}_{S_2}) = \mathrm{XE}_{S_2} = \mathrm{CE}_{S_2}$). Consider the coordination game with utility matrices $A_1 = B_1 = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$. Parametrize symmetric probability matrices as $\left[\begin{smallmatrix}p & q\\ q & r\end{smallmatrix}\right]$ with $p, q, r \geq 0$ and $p + 2q + r = 1$. Then the correlated equilibrium conditions are exactly $p \geq q$ and $r \geq q$. Note that together these imply that $pr \geq q^2$, so any correlated equilibrium is automatically positive semidefinite (the principal minors are nonnegative), hence completely positive by Theorem 2.58 and so exchangeable. Therefore the set of symmetric exchangeable equilibria is the same as the set of symmetric correlated equilibria and a simple computation shows that these sets are equal to

$$\mathrm{CE}_{S_2} = \mathrm{XE}_{S_2} = \mathrm{conv}\left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0.25 & 0.25\\ 0.25 & 0.25\end{bmatrix}\right\} = \mathrm{conv}(\mathrm{NE}_{S_2}).$$

This game also admits asymmetric correlated equilibria such as $\begin{bmatrix}1/3 & 1/3\\ 0 & 1/3\end{bmatrix}$, which shows that we cannot replace Nash equilibria with correlated equilibria in the statement of Theorem 3.11. ◇

**Example 3.19** ($2 \times 2$ game with $\mathrm{conv}(\mathrm{NE}_{S_2}) = \mathrm{XE}_{S_2} \subsetneq \mathrm{CE}_{S_2}$). Compare the game in Example 3.18 with the anti-coordination game having utility matrices $A_2 = B_2 = \left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$. Parametrizing probability matrices in the same way, the correlated equilibrium conditions are now reversed: $q \geq p$ and $q \geq r$. If either of these were strict we would have $q^2 > pr$ and the probability matrix would have negative determinant, so it could not correspond to an exchangeable equilibrium. Thus the only exchangeable equilibrium is the one with $p = q = r = \frac{1}{4}$. This is also the only symmetric Nash equilibrium.

| $(u_1 = u_2)$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | 2 | 2 | 0 |
| $b$ | 2 | 1 | 2 |
| $c$ | 0 | 2 | 2 |

**Table 3.1.** A game for which the convex hull of the symmetric Nash equilibria is strictly contained in the set of exchangeable equilibria which is itself strictly contained in the set of symmetric correlated equilibria.

These two games are in some sense isomorphic as bimatrix games but not as symmetric bimatrix games, and this accounts for the fact that their sets of symmetric exchangeable (and Nash) equilibria are not isomorphic. In the following chapter we will see that this difference corresponds to the fact that when these games are played by many players simultaneously, there is a symmetric (with respect to the players) way to break symmetry between the two pure strategies in the case of the coordination game but not in the case of the anticoordination game.

This game can also be viewed as having a larger symmetry group $S_2 \times S_2$, where one factor swaps the players as usual and the other fixes the players but swaps the strategies of both. This symmetry group is not strategy-trivial and $\Delta_{S_2 \times S_2}(\Gamma)$ is a singleton containing only the uniform distribution. Therefore $\mathrm{CE}_{S_2 \times S_2}(\Gamma) = \mathrm{XE}_{S_2 \times S_2}(\Gamma)$ even though $\mathrm{NE}_{S_2 \times S_2}(\Gamma) \subsetneq \mathrm{NE}(\Gamma)$, so without strategy-triviality the conclusion of Theorem 3.11 can be false. ◇

**Example 3.20** ($3 \times 3$ game with $\mathrm{conv}(\mathrm{NE}_{S_2}) \subsetneq \mathrm{XE}_{S_2} \subsetneq \mathrm{CE}_{S_2}$, exchangeable equilibria incomparable to Sorin's distribution equilibria). Consider the game in Table 3.1. First, we show strict containment in $\mathrm{XE}_{S_2} \subsetneq \mathrm{CE}_{S_2}$. Second, we show strict containment in $\mathrm{conv}(\mathrm{NE}_{S_2}) \subsetneq \mathrm{XE}_{S_2}$. Third, we observe that $\mathrm{XE}_{S_2}$ is not polyhedral. These results are summarized in Figure 3.1. Finally we show that in general no containment holds in either direction between exchangeable equilibria and Sorin's notion of distribution equilibria [67].

The matrix

$$W^1 = \frac{1}{4}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is a correlated equilibrium because each player gets his maximum payoff with probability one. By Proposition 2.50 it is not completely positive, so it is a correlated equilibrium which is not exchangeable. In fact we can say more: $W^1$ is not 3-extendable. Suppose $X_1, X_2, X_3$ were random variables with a 3-exchangeable distribution taking values in $\{a, b, c\}$ and having $W^1$ as the marginal distribution

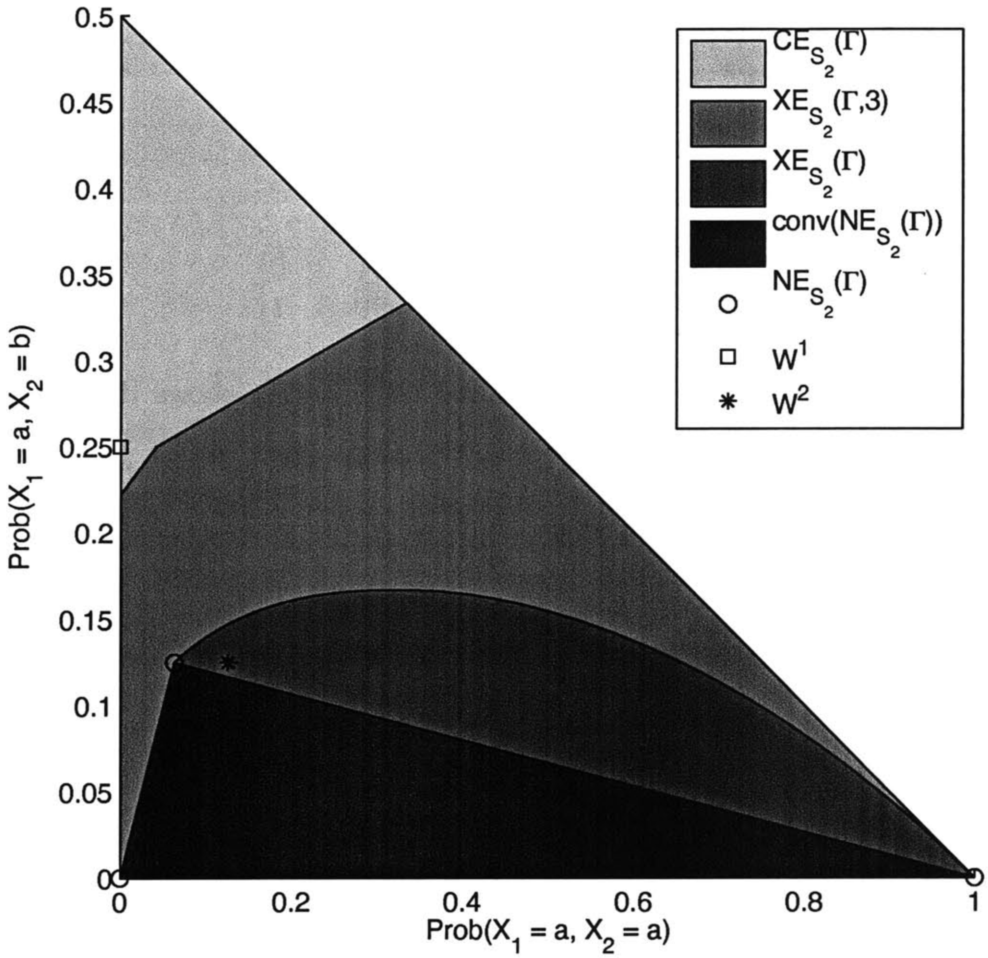


**Figure 3.1.** Comparison of equilibrium sets for the game in Table 3.1. These sets are naturally sets of symmetric $3 \times 3$ matrices, but we have chosen a projection into two dimensions which highlights the separation between the sets. The set $\mathrm{XE}_{S_2}(\Gamma, 3)$, discussed in Section 4.2, denotes the correlated equilibria which are marginals of distributions in $\Delta_{S_3}(C_1^3)$. The set of exchangeable equilibria is not polyhedral despite the fact that the other sets are.

of $X_1$ and $X_2$. This would imply

$$\begin{aligned}\frac{1}{4} &= \operatorname{Prob}(X_1 = a)\\ &\leq \operatorname{Prob}\left((X_1 = a, X_2 \neq b) \vee (X_1 = a, X_3 \neq b) \vee (X_2 = b, X_3 = b)\right)\\ &\leq \operatorname{Prob}(X_1 = a, X_2 \neq b) + \operatorname{Prob}(X_1 = a, X_3 \neq b) + \operatorname{Prob}(X_2 = b, X_3 = b)\\ &= 2W_{11}^1 + 2W_{13}^1 + W_{22}^1 = 0,\end{aligned}$$

a contradiction, so no such 3-exchangeable distribution exists.

One can also verify that

$$W^2 = \frac{1}{8}\begin{bmatrix}1 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 1\end{bmatrix} = \frac{1}{8}\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$$

is a correlated equilibrium, and the exhibited factorization shows that $W^2$ is completely positive, hence exchangeable. Suppose for a contradiction that $W^2$ were a convex combination of Nash equilibria. Then at least one of the Nash equilibria in the convex combination would have to assign positive probability to the strategy profile $(b, b)$. Suppose player 2 did not play $c$ with positive probability in such a Nash equilibrium. Given this information, player 1 prefers $a$ to $b$, so player 1 cannot choose $b$ with positive probability in such an equilibrium, a contradiction. Symmetric arguments show that each player must play all his strategies with positive probability. Therefore this Nash equilibrium has full support. But $W^2$ has entries which are zero, hence this Nash equilibrium cannot be included in an expression of $W^2$ as a convex combination of Nash equilibria. Thus $W^2$ is not a convex combination of Nash equilibria.

This argument also shows that the only symmetric Nash equilibrium which assigns positive probability to $b$ is the one with full support, which a simple computation shows to be $\begin{bmatrix}\frac{1}{4} & \frac{1}{2} & \frac{1}{4}\end{bmatrix}$. The only other symmetric Nash equilibria are $\begin{bmatrix}1 & 0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0 & 1\end{bmatrix}$. There can be no symmetric Nash equilibrium which assigns zero probability to $b$ but positive probability to both $a$ and $c$. Such an equilibrium would have to assign equal probability to $a$ and $c$, but $b$ is the unique best response to such a mixture.

The set of correlated equilibria of a game is always polyhedral. The set of Nash equilibria is generically finite, so its convex hull is polyhedral for generic games, and for this game in particular. It is visually evident, and can be proven algebraically, that the projection of the set of exchangeable equilibria pictured in Figure 3.1 is not polyhedral. In fact it is an algebraic curve of degree 11 which factors into three linear components, a quadratic component, and a degree six component over $\mathbb{Q}$. Two of the linear components are easily visible (the bottom and left of the convex hull of the symmetric Nash equilibria), and the third corresponds to the maximum $y$ value, attained along the line segment joining $(\frac{11}{36}, \frac{1}{6})$ to $(\frac{1}{3}, \frac{1}{6})$. The quadratic component corresponds to the curved portion of the boundary to the right of this maximum and is defined by the vanishing of $x^2 + 2xy + 4y^2 - x$. The degree six component is the curved portion of the boundary to the left of the maximum.

To the author's knowledge, beside exchangeable equilibria (and the generalizations in Chapters 5 and 6) Sorin's notion of distribution equilibria is the only solution concept known which lies between the Nash and correlated equilibria.

**Definition 3.21** ([67]). *A **distribution equilibrium** is a correlated equilibrium such each player's expected utility conditioned on his recommendation is a constant (potentially different for each player) almost surely.*

Since at a Nash equilibrium players are indifferent between the strategies they play with positive probability, any Nash equilibrium viewed as a product distribution is a distribution equilibrium. Similarly we may define symmetric distribution equilibria to be distribution equilibria which are symmetric, so symmetric Nash equilibria are symmetric distribution equilibria.

Under $W^1$ the expected utility of either player conditioned on his recommendation is always 2, so this is a symmetric distribution equilibrium which is not exchangeable. On the other hand under $W^2$ the expected utility conditioned on receiving recommendation $a$ is 2 and conditioned on recommendation $b$ is $\frac{3}{2}$. Therefore $W^2$ is an exchangeable equilibrium which is not a distribution equilibrium, and these two intermediate equilibrium notions are incomparable. $\diamondsuit$

**Example 3.22** (Exchangeable equilibrium with higher payoff than symmetric Nash equilibria). Let $\Gamma$ be the symmetric bimatrix game with payoffs

$$A = B^T := \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The expected utility of both players under a symmetric distribution $W \in \Delta_{S_2}(\Gamma)$ is $W \bullet A = W \bullet B = W \bullet \frac{A+B}{2}$. The largest value in $\frac{A+B}{2}$ is $3/2$ and occurs in entries $(1,2)$ and $(2,1)$ only. The unique symmetric distribution placing all mass on these entries is

$$W^1 = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so the only $W$ with $W \bullet A \geq \frac{3}{2}$ is $W = W^1$. One can verify that $W^1$ is a correlated equilibrium but it is not exchangeable by Proposition 2.50 because it has zero

| $(u_1 = u_2)$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | 0 | 3 | 2 |
| $b$ | 3 | 0 | 2 |
| $c$ | 2 | 2 | 3 |

**Table 3.2.** A game with correlated equilibria supported on $\{a,b\} \times \{a,b\}$ but no exchangeable equilibria supported on this set.

diagonal, so $W^1 \in \mathrm{CE}_{S_2}(\Gamma) \setminus \mathrm{XE}_{S_2}(\Gamma)$. Furthermore $W^1$ achieves a higher utility than any exchangeable equilibrium.

The distribution

$$W^2 := \frac{5}{7}\begin{bmatrix}1/8\\7/8\\0\end{bmatrix}\begin{bmatrix}1/8\\7/8\\0\end{bmatrix}^T + \frac{2}{7}\begin{bmatrix}1/8\\0\\7/8\end{bmatrix}\begin{bmatrix}1/8\\0\\7/8\end{bmatrix}^T = \frac{1}{64}\begin{bmatrix}1 & 5 & 2\\5 & 35 & 0\\2 & 0 & 14\end{bmatrix}$$

is exchangeable and it is also a correlated equilibrium, so it is an exchangeable equilibrium yielding expected utility $W \bullet A = W \bullet B = \frac{17}{16}$ to both players.

Suppose for a contradiction that there were a symmetric Nash equilibrium $x$ with $x^T Ax > 1$. Since $x$ is a probability vector, $[Ax]_1, [Ax]_3 \leq 1$. But $x^T Ax$ is a convex combination of $[Ax]_1$, $[Ax]_2$, and $[Ax]_3$, so $[Ax]_2 > 1$. Therefore strategy 2 is the unique best response to $x$. Only best responses are played with positive probability in a Nash equilibrium, so $x = \begin{bmatrix}0 & 1 & 0\end{bmatrix}$. But then the expected utility $x^T Ax = 1$, a contradiction. Thus there is no symmetric Nash equilibrium which yields utility greater than 1. In particular $W^2 \in \mathrm{XE}_{S_2}(\Gamma) \setminus \mathrm{conv}(\mathrm{NE}_{S_2}(\Gamma))$ and $W^2$ achieves a higher utility than any symmetric Nash equilibrium.

Indeed, Gambit [48] computes that there are two symmetric pure Nash equilibria $\begin{bmatrix}0 & 1 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0 & 1\end{bmatrix}$ as well as one symmetric mixed Nash equilibrium $\begin{bmatrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{2}\end{bmatrix}$. Both pure equilibria yield utility 1 and the mixed equilibrium yields utility $\frac{3}{4}$. $\diamond$

**Example 3.23** (Supports of exchangeable equilibria). A natural question is whether the existence of symmetric correlated equilibria in a symmetric bimatrix game with a given support (set of pairs assigned positive probability) implies the existence of exchangeable equilibria with this support. One can trivially construct counterexamples, such as the symmetric correlated equilibrium $\begin{bmatrix}0 & 1/2\\1/2 & 0\end{bmatrix}$ in Example 3.19.

A weaker question would be whether the existence of a correlated equilibrium with support within some principal submatrix guarantees the existence of an exchangeable equilibrium with support within that principal submatrix. Again, the answer is no, as the following example shows.

Consider the symmetric game of identical utilities in Table 3.2. For $0 \le \epsilon \le \frac{1}{6}$ the probability matrix

$$\begin{bmatrix} \epsilon & \frac{1}{2}-\epsilon & 0 \\ \frac{1}{2}-\epsilon & \epsilon & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a correlated equilibrium of this game. Now suppose $p$, $q$, and $r$ were such that

$$\begin{bmatrix} p & q & 0 \\ q & r & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

were a completely positive matrix with entries summing to one. Then this matrix would be positive semidefinite, implying $p+r \ge 2q$. In other words, at most half of the probability would be assigned to outcomes with a payoff of three and the rest of the probability would be assigned to outcomes with a payoff of zero. Therefore the expected payoff of such a distribution would be at most $\frac{3}{2}$, and it would be in either player's best interest to deviate unilaterally from his recommendation and play strategy $c$. Thus such a matrix could not be a correlated equilibrium, so there do not exist any symmetric exchangeable equilibria with this support. $\diamondsuit$

**Example 3.24** (Exchangeable equilibria need not optimize utility over correlated equilibria). It was shown in Aumann's original paper on correlated equilibria [1] that correlated equilibria can achieve utilities outside the convex hull of those achievable by Nash equilibria. The same can happen with exchangeable equilibria in place of Nash equilibria. Consider the symmetric bimatrix game with utilities

$$A = \begin{bmatrix} 0 & 60 & 30 & 40 \\ 30 & 0 & 60 & 40 \\ 60 & 30 & 0 & 40 \\ 40 & 40 & 40 & 41 \end{bmatrix}$$

and $B = A^T$, as considered in [25, 49, 54] and originally due to Aumann. Let $W$ be a symmetric correlated equilibrium of this game. Then we must have $W \bullet A \ge 40$ or else the row player could unilaterally improve by switching to the bottom strategy. One can verify that

$$W^1 = \frac{1}{27}\begin{bmatrix} 1 & 4 & 4 & 0 \\ 4 & 1 & 4 & 0 \\ 4 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is a correlated equilibrium which achieves the minimum utility $W^1 \bullet A = 40$.

On the other hand by symmetry of $W$ one has

$$W \bullet A = \frac{1}{2} W \bullet A + \frac{1}{2} W^T \bullet A = \frac{1}{2} W \bullet A + \frac{1}{2} W \bullet A^T = W \bullet \frac{A+B}{2} \leq 45,$$

since $\frac{A+B}{2} \leq 45$ elementwise. One can verify that

$$W^2 = \frac{1}{6} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is a correlated equilibrium achieving the maximum utility $W^2 \bullet A = 45$.

We will show that this game has a unique exchangeable equilibrium, which has utility $W \bullet A = 41$. To see this suppose $W$ is an exchangeable equilibrium. Then $W \succeq 0$, so

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \bullet W \geq 0,$$

and thus

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \bullet W \geq \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \bullet W.$$

That is to say, at least a third of the mass placed in the upper left $3 \times 3$ block of $W$ is on the diagonal of that block. The entries of $\frac{A+B}{2}$ corresponding to the diagonal of this block are zero and the entries corresponding to the rest of this block are 45. Therefore the expected utility of the players conditioned on their joint recommendation being in this $3 \times 3$ block is at most $\frac{1}{3}(0) + \frac{2}{3}(45) = 30$.

Define $S$ and $T$ to be the events that the row player and column player, respectively, receive a recommendation within their first three strategies. Suppose for a contradiction that $W$ is an exchangeable equilibrium which places positive probability on $S$. Since $W \succeq 0$, the first three diagonal entries of $W$ cannot all be zero. In particular, the probability of $T$ given $S$ is positive. Letting $(i, j)$ be distributed according to $W$, we can then compute

$$\begin{aligned} \mathbb{E}[A_{ij} \mid S] &= \mathbb{E}[A_{ij} \mid S \wedge T] \operatorname{Prob}(T \mid S) + \mathbb{E}[A_{ij} \mid S \wedge \neg T] \operatorname{Prob}(\neg T \mid S) \\ &\leq 30 \operatorname{Prob}(T \mid S) + 40 \operatorname{Prob}(\neg T \mid S) < 40, \end{aligned}$$

so the row player can improve by playing the bottom row independent of his recommendation, contradicting the assumption that $W$ is a correlated equilibrium.

Therefore $W$ places zero probability on $S$ and by symmetry also on $T$. That is to say, $W_{44} = 1$, so $W$ is a symmetric pure strategy Nash equilibrium with utility $A_{44} = 41$. In particular no correlated equilibrium which maximizes or minimizes utility is an exchangeable equilibrium. ◇

**Example 3.25** (Exchangeable equilibria need not optimize trace over correlated equilibria). We have seen in Examples 2.36 and 2.42 and Proposition 2.46 that exchangeability of a distribution in $\Delta_{S_2}(C_1^2)$ gives a lower bound on the trace of the corresponding matrix. For this reason it is natural to ask whether exchangeable equilibria optimize trace over correlated equilibria. To see that they do not, we consider the symmetric bimatrix game with utilities

$$A = B^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Let $L : \mathbb{R}^{4\times4} \to \mathbb{R}^{4\times4}$ be the map which circularly shifts the columns of a matrix right by one and the rows down by one. Since $A$ is a circulant matrix (invariant under $L$), the set of correlated equilibria is invariant under $L$. So is the set of $4 \times 4$ completely positive matrices and the trace function, so among the set of symmetric correlated equilibria of maximal trace there is one which is circulant and similarly for symmetric exchangeable equilibria. The same argument works for minimal trace.

The general form of a $4 \times 4$ symmetric circulant probability matrix is

$$W = \begin{bmatrix} a & b & c & b \\ b & a & b & c \\ c & b & a & b \\ b & c & b & a \end{bmatrix},$$

where $a, b, c \geq 0$ and $4a + 8b + 4c = 1$. The correlated equilibrium condition says exactly that $b \geq a$ and $b \geq c$. Therefore the maximal trace of a symmetric correlated equilibrium is achieved by setting $a = b = \frac{1}{12}$ and $c = 0$, yielding a trace of $\frac{1}{3}$. The minimal trace of zero is achieved by setting $b = \frac{1}{8}$ and $a = c = 0$.

Now we impose the additional condition that $W \succeq 0$, so $W$ is an exchangeable equilibrium. The positivity of $2 \times 2$ principal minors gives $a \geq b$, and the reverse inequality comes from the correlated equilibrium conditions, so $a = b$. Multiplying $W$ on the left by $v = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$ gives $vW = (c-b)v$, so $c-b$ is an eigenvalue of $W$ and we must have $c \geq b$. The correlated equilibrium conditions again give the reverse inequality, so $a = b = c = \frac{1}{16}$ and we see that any symmetric exchangeable equilibrium must have trace $\frac{1}{4}$, and so cannot optimize the trace over correlated equilibria. ◇

**Example 3.26** ($2\times 2\times 2$ game with $\mathrm{conv}(\mathrm{NE}_{S_3}) \subsetneq \mathrm{XE}_{S_3} \subsetneq \mathrm{CE}_{S_3}$). Consider the three-player, two-strategy game of identical interest with utilities

$$u(s_1, s_2, s_3) = \begin{cases} 1, & \text{when } s_1 = s_2 = s_3, \\ 0, & \text{otherwise,} \end{cases}$$

written in tensor form as

$$u = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

This coordination game has three symmetric Nash equilibria:

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \text{ and } \frac{1}{8}\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right].$$

An element of $\Delta_{S_3}(\Gamma)$ can be written as

$$\left[\begin{array}{cc|cc} w & x & x & y \\ x & y & y & z \end{array}\right];$$

to be in $\mathrm{conv}(\mathrm{NE}_{S_3}(\Gamma))$ such a tensor must satisfy $x = y$ because this linear constraint is satisfied by all the symmetric Nash equilibria.

The correlated equilibrium conditions are $w \geq y$ and $z \geq x$. Let $\lambda \in [0,1]$. The exchangeable distribution

$$\begin{aligned} \pi_\lambda &:= \lambda \left[\tfrac{2}{3}\ \ \tfrac{1}{3}\right]^{\otimes 3} + (1-\lambda)\left[\tfrac{1}{3}\ \ \tfrac{2}{3}\right]^{\otimes 3} = \frac{\lambda}{27}\left[\begin{array}{cc|cc} 8 & 4 & 4 & 2 \\ 4 & 2 & 2 & 1 \end{array}\right] + \frac{1-\lambda}{27}\left[\begin{array}{cc|cc} 1 & 2 & 2 & 4 \\ 2 & 4 & 4 & 8 \end{array}\right] \\ &= \frac{1}{27}\left[\begin{array}{cc|cc} 1+7\lambda & 2+2\lambda & 2+2\lambda & 4-2\lambda \\ 2+2\lambda & 4-2\lambda & 4-2\lambda & 8-7\lambda \end{array}\right] \end{aligned}$$

is a correlated equilibrium if and only if $1+7\lambda \geq 4-2\lambda$ and $8-7\lambda \geq 2+2\lambda$. Therefore $\pi_\lambda \in \mathrm{XE}_{S_3}(\Gamma)$ if and only if $\lambda \in \left[\frac{1}{3}, \frac{2}{3}\right]$. This distribution only satisfies $x = y$ only when $\lambda = \frac{1}{2}$, so $\pi_\lambda \in \mathrm{XE}_{S_3}(\Gamma) \setminus \mathrm{conv}(\mathrm{NE}_{S_3}(\Gamma)$ for $\lambda \in \left[\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{2}{3}\right]$. When $\lambda = \frac{1}{2}$ we have

$$\begin{aligned} \pi_{\frac{1}{2}} &= \frac{1}{54}\left[\begin{array}{cc|cc} 9 & 6 & 6 & 6 \\ 6 & 6 & 6 & 9 \end{array}\right] \\ &= \frac{1}{18}\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] + \frac{1}{18}\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] + \frac{8}{9}\cdot\frac{1}{8}\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right], \end{aligned}$$

so $\pi_{\frac{1}{2}} \in \mathrm{conv}(\mathrm{NE}_{S_3}(\Gamma))$.

On the other hand

$$\frac{1}{5}\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \in \mathrm{CE}_{S_3}(\Gamma) \setminus \mathrm{XE}_{S_3}(\Gamma)$$

because $\det\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right] = -1$. Therefore we have $\mathrm{conv}(\mathrm{NE}_{S_3}(\Gamma)) \subsetneq \mathrm{XE}_{S_3}(\Gamma) \subsetneq \mathrm{CE}_{S_3}(\Gamma)$. A similar argument yields the corresponding containment for the $n$-player coordination game for all $n \geq 3$.

Continuing with the three-player coordination game, let us calculate the extreme points of $\mathrm{XE}_{S_3}(\Gamma)$. Since complete positivity and double nonnegativity are the same for $2 \times 2 \times 2$ tensors, the conditions for a point to be in $\mathrm{XE}_{S_3}(\Gamma)$ are

$$\begin{aligned} w &\geq y \geq 0, \\ z &\geq x \geq 0, \\ wy &\geq x^2, \\ xz &\geq y^2, \end{aligned} \tag{3.1}$$

and normalization $w + 3x + 3y + z = 1$. We will drop the normalization condition, calculate extreme rays of the resulting closed convex cone, and normalize at the end.

Any such extreme ray satisfies $x = y$, $x > y$, or $x < y$. In the first case one can check that such an extreme ray is one of the three symmetric Nash equilibria given above. The second case reduces to the third by applying the symmetry of the game which swaps the two strategies, thereby swapping $x \leftrightarrow y$ and $w \leftrightarrow z$. Thus it suffices to compute extreme rays with $y > x$; until otherwise specified we will assume $y > x$.

We can decompose the equilibrium tensor as

$$\left[\begin{array}{cc|cc} w & x & x & y \\ x & y & y & z \end{array}\right] = \left[\begin{array}{cc|cc} w-y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|cc} y & x & x & y \\ x & y & y & z \end{array}\right],$$

where both summands satisfy the constraints (3.1). Extremality means that one of these summands is a nonnegative multiple of the other. Since $y > x \geq 0$, the first summand must be zero, $w = y$, and the constraint $wy \geq x^2$ is automatic. Therefore we want to compute extreme rays of the system

$$\begin{aligned} y &> x, \\ z &\geq x \geq 0, \\ xz &\geq y^2. \end{aligned}$$

Since $y^2 > 0$ the bottom constraint gives $x, y, z > 0$. Furthermore if $x = z$ then $x^2 \geq y^2$, contradicting $y > x > 0$, so $z > x$. For extremality we need *some* constraint to be tight – the set where no constraints are tight is open and so has no extreme rays – and the only candidate now is $xz \geq y^2$. Thus our extreme ray must satisfy

$$\begin{aligned} y, z > x > 0, \\ xz = y^2. \end{aligned}$$

That is to say, it is of the form

$$\left[\begin{array}{cc|cc} y & x & x & y \\ x & y & y & \frac{y^2}{x} \end{array}\right] \tag{3.2}$$

for $y > x > 0$. Conversely, all such tensors are extreme. To see this, decompose such a tensor into two nonzero tensors

$$\left[\begin{array}{cc|cc} w_1 & x_1 & x_1 & y_1 \\ x_1 & y_1 & y_1 & z_1 \end{array}\right] + \left[\begin{array}{cc|cc} w_2 & x_2 & x_2 & y_2 \\ x_2 & y_2 & y_2 & z_2 \end{array}\right],$$

both of which are exchangeable equilibria. Since $\left[\begin{smallmatrix} x & y \\ y & \frac{y^2}{x} \end{smallmatrix}\right]$ has zero determinant, it is an extreme ray of the cone of positive semidefinite matrices. Therefore there are $\lambda_1, \lambda_2 \geq 0$ such that $x_i = \lambda_i x$, $y_i = \lambda_i y$, and $z_i = \lambda_i \frac{y^2}{x}$. The correlated equilibrium conditions give $w_i \geq y_i$, but we have $w_1 + w_2 = y = y_1 + y_2$, so these must be tight. Therefore $w_i = y_i = \lambda_i y$, which means both tensors are nonnegative multiples of (3.2), so that tensor is extreme.

Adding the normalization condition back in, we get that such an extreme exchangeable equilibrium satisfies $3x^2 + 4xy + y^2 = x$, the equation for an ellipse. Since $w = y$ and $z = 1 - 3x - 4y$ by normalization, these points $(w, x, y, z)$ do in fact lie on an ellipse in the plane $[w = y, w + 3x + 3y + z = 1]$ in $\mathbb{R}^4$. They form an arc of the ellipse cut off by the strict inequalities $y > x > 0$. A priori the endpoints where one of these becomes tight need not correspond to extreme exchangeable equilibria, but in this case the endpoints are actually the symmetric Nash equilibria

$$\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \text{ and } \frac{1}{8}\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right],$$

which we know to be extreme by Proposition 3.9.

Symmetry gives another such ellipse for $x > y$. The result is that the set of exchangeable equilibria of this game is formed as follows. We begin with the three symmetric Nash equilibria, which correspond to points spanning the plane $[x = y, w{+}3x{+}3y{+}x = 1]$ in $\mathbb{R}^4$. We draw two copies of an elliptical arc lying in the independent subspaces $[w = y, w{+}3x{+}3y{+}z = 1]$ and $[z = x, w{+}3x{+}3y{+}z = 1]$ connecting the two symmetric pure Nash equilibria to the mixed one. The set of

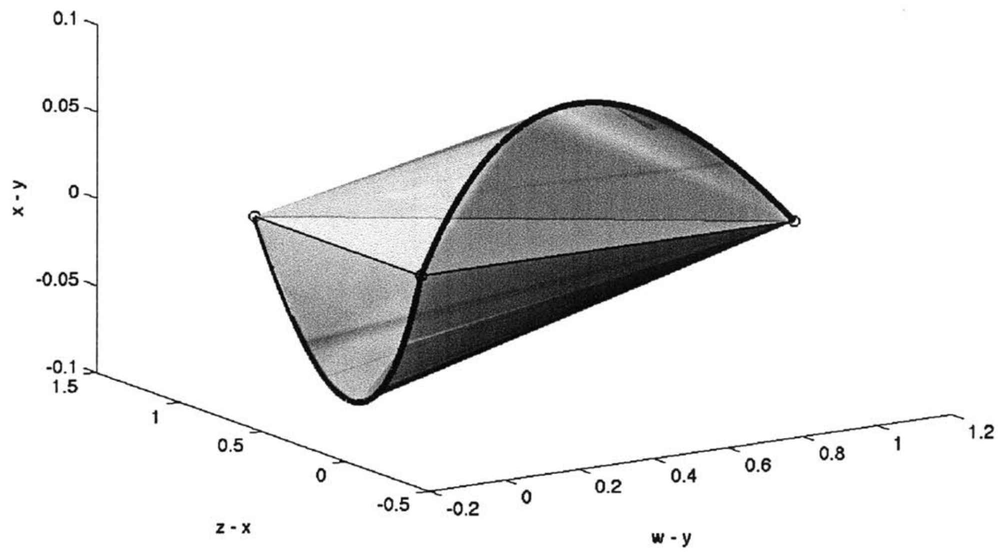


**Figure 3.2.** Exchangeable equilibria of the game in Example 3.26. This set is formed by beginning with the three symmetric Nash equilibria (circles), joining the two pure ones to the mixed ones by arcs from an ellipse (bold curves), and taking the convex hull. The set of symmetric correlated equilibria is polyhedral and so strictly contains the exchangeable equilibria.

exchangeable equilibria is the convex hull of the resulting set, which is a subset of a reducible algebraic curve. This is shown in Figure 3.2. ◇

**Example 3.27** ($2\times2\times2$ game with unique irrational exchangeable equilibrium). Let $\Gamma$ be the three player symmetric game of identical utilities with common strategy space $C_i = \{0,1\}$ and common utility function

$$u(s_1,s_2,s_3) = \begin{cases} 0 & \text{when } s_1+s_2+s_3 = 3, \\ s_1+s_2+s_3 & \text{else,} \end{cases}$$

which we may alternatively write as the $2\times2\times2$ symmetric tensor

$$u = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & 0 \end{array}\right].$$

The exchangeable equilibria are the tensors of the form

$$\left[\begin{array}{cc|cc} w & x & x & y \\ x & y & y & z \end{array}\right] \tag{3.3}$$

subject to $w,x,y,z \geq 0$ (nonnegativity), $w+3x+3y+z=1$ (normalization), and the conditions:

$$2y - 2x \geq w, \tag{3.4}$$
$$x \geq 2z - 2y, \tag{3.5}$$
$$wy \geq x^2, \tag{3.6}$$
$$xz \geq y^2. \tag{3.7}$$

If any of the variables $w,x,y,z$ were zero then these conditions would imply they were all zero, contradicting normalization. So $w,x,y,z > 0$ and some algebra yields:

Multiply (3.4) by $y$ and apply (3.6): $$2y^2 - 2xy \geq wy \geq x^2 \tag{3.8}$$

Multiply (3.5) by $x$ and apply (3.7): $$x^2 \geq 2xz - 2xy \geq 2y^2 - 2xy \tag{3.9}$$

Equality throughout (3.8) and (3.9): $$x^2 = 2xz - 2xy = 2y^2 - 2xy = wy \tag{3.10}$$

Quadratic formula, $x,y>0$: $$x = (\sqrt{3}-1)y \tag{3.11}$$

Substitute (3.11) into (3.10): $$z = \frac{x^2+2xy}{2x} = \frac{x+2y}{2} = \frac{\sqrt{3}+1}{2}y$$

Substitute (3.11) into (3.10): $$w = \frac{x^2}{y} = \left(\sqrt{3}-1\right)^2 y.$$

Normalization gives a unique solution. Let $p = 1 - \frac{1}{\sqrt{3}}$, so $\frac{p}{1-p} = \sqrt{3} - 1$ and $\frac{1-p}{p} = \frac{\sqrt{3}+1}{2}$. Then the unique exchangeable equilibrium is given by $w = p^3$, $x = p^2(1-p)$, $y = p(1-p)^2$, and $z = (1-p)^3$, or in tensor form:

$$\operatorname{conv}(\mathrm{NE}_{S_3}(\Gamma)) = \mathrm{XE}_{S_3}(\Gamma) = \left\{ \begin{bmatrix} p \\ 1-p \end{bmatrix}^{\otimes 3} \right\}.$$

In particular $p, w, x, y, z \in \mathbb{Q}\left[\sqrt{3}\right] \setminus \mathbb{Q}$ are irrational, so $\Gamma$ has no exchangeable equilibrium with rational numbers for probabilities. $\diamondsuit$

## 3.4 Convex relaxations of Nash equilibria

For this section only we restrict attention to symmetric bimatrix games. We write down the standard complementarity conditions defining (symmetric) Nash equilibria, and consider three ways of formally relaxing or weakening these conditions to yield convex sets. We identify these relaxations as the symmetric correlated equilibria, the symmetric exchangeable equilibria, and the convex hull of the symmetric Nash equilibria. Both the exchangeable equilibria and the convex hull of symmetric Nash equilibria are characterized in terms of completely positive matrices. The results are summarized in Table 3.3.

While the set $\mathrm{XE}_{S_2}$ is defined in terms of $\mathrm{CP}^2_m$ for an $m \times m$ game, we characterize $\operatorname{conv}(\mathrm{NE}_{S_2})$ in terms of $\mathrm{CP}^2_{2m}$. Thus we obtain an easily-checkable positive semidefiniteness condition for a matrix to be in $\operatorname{conv}(\mathrm{NE}_{S_2})$ from Theorem 2.58 only in the case $2m \leq 4$, i.e., $2 \times 2$ games. To demonstrate this we give another proof of Theorem 3.10.

We now recall the well-known complementarity characterization of symmetric Nash equilibria (see e.g. Von Stengel's survey [73], which treats the asymmetric case in a similar way). Let $A \in \mathbb{R}^{m \times m}$ denote the utility matrix of player 1, defined by $A_{ij} = u_1(i, j)$, and define the auxiliary matrix

$$P := \begin{bmatrix} e & -A \\ 0 & I \end{bmatrix} \in \mathbb{R}^{2m \times (m+1)},$$

where $e \in \mathbb{R}^m$ is a column vector of all ones and $0$ and $I$ are the zero vector and identity matrix of the appropriate sizes.

**Proposition 3.28.** *The vector $w \in \mathbb{R}^m$ is a symmetric Nash equilibrium if and only if $\sum w_i = 1$ and there exists $v \in \mathbb{R}$ such that*

$$z := P \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} ve - Aw \\ w \end{bmatrix} \in \mathbb{R}^{2m}$$

*is elementwise nonnegative and complementary in the sense that* $z_i z_{i+m} = 0$ *for* $1 \leq i \leq m$.

Rewriting this characterization in terms of a matrix $W \in \mathbb{R}^{m\times m}$, a vector $\gamma \in \mathbb{R}^{1\times m}$, and a scalar $\alpha$ (thinking of $W = ww^T$, $\gamma = vw^T$, and $\alpha = v^2$), we get the following characterization of the matrices $W \in \mathrm{NE}_{S_2}(\Gamma)$.

**Proposition 3.29.** *The matrix* $W \in \Delta_{S_2}(C_1^2)$ *is in* $\mathrm{NE}_{S_2}(\Gamma)$ *if and only if there exist* $\gamma \in \mathbb{R}^{1\times m}$ *and* $\alpha \in \mathbb{R}$ *such that the symmetric matrix*

$$Z := P\begin{bmatrix}\alpha & \gamma \\ \gamma^T & W\end{bmatrix}P^T := \begin{bmatrix}\alpha ee^T - A\gamma^T e^T - e\gamma A^T + AWA^T & e\gamma - AW \\ \gamma^T e^T - WA^T & W\end{bmatrix}$$

*satisfies:*

- *$Z$ has rank* 1,
- *$Z \geq 0$ elementwise, and*
- $Z_{i,i+m} = 0$ *for* $1 \leq i \leq m$.

There are a number of straightforward ways to produce convex relaxations of the set $\mathrm{NE}_{S_2}(\Gamma)$ using the characterization in Proposition 3.29. We will analyze three of these, which lead in turn to $\mathrm{CE}_{S_2}$, $\mathrm{XE}_{S_2}$, and $\mathrm{conv}(\mathrm{NE}_{S_2})$. For the first we drop the rank 1 condition and recover the set of symmetric correlated equilibria.

**Proposition 3.30.** *The following are equivalent for* $W \in \Delta_{S_2}(C_1^2)$*:*

1. $W \in \mathrm{CE}_{S_2}(\Gamma)$,
2. *$W$ satisfies all conditions of Proposition 3.29 except the rank* 1 *condition, and*
3. *there exists* $\gamma \in \mathbb{R}^{1\times m}$ *such that* $e\gamma \geq AW$ *with equality on the diagonal.*

*Proof.* Note that $\alpha$ only appears in the upper left block of $Z$, so by making $\alpha$ large we can make the entries of this block nonnegative. Thus we may drop the variable $\alpha$ and ignore the upper left block of $Z$. The lower right block of $Z$ is just $W$, the nonnegativity of which is implicit in the assumption $W \in \Delta_{S_2}(C_1^2)$.

Thus the final two constraints written in Proposition 3.29 reduce in this case to the condition that the matrix $e\gamma - AW$ be elementwise nonnegative with zeros on the diagonal. By matrix multiplication and symmetry of $W$ we have $(AW)_{ij} = \sum_{k=1}^m A_{ik}W_{kj} = \sum_{k=1}^m A_{ik}W_{jk}$. Thus the condition on $e\gamma - AW$ is equivalent to the condition that $\gamma_j = \sum_{k=1}^m A_{jk}W_{jk}$ and $\sum_{k=1}^m A_{ik}W_{jk} \leq \gamma_j$ for all $i$ and $j$. In other words, we can drop $\gamma$ and get the equivalent condition that $\sum_{k=1}^m [A_{jk} - A_{ik}]W_{jk} \geq 0$ for all $i, j$. □

By restricting $W$ to be a convex combination of matrices of the form $ww^T$ for $w \geq 0$ we get a second, tighter relaxation.

**Corollary 3.31.** *The set* $\mathrm{XE}_{S_2}(\Gamma)$ *of (symmetric) exchangeable equilibria is obtained by relaxing the rank* 1 *condition in Proposition 3.29 to complete positivity of* $W$.

For the third and final relaxation of this section, note that whenever $W$ satisfies the conditions of Proposition 3.29 it is a symmetric Nash equilibrium so we can take $v = w^T A w$ and $Y := \left[\begin{smallmatrix} \alpha & \gamma \\ \gamma^T & W \end{smallmatrix}\right] = \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]^T$ to get $Z := PYP^T$ to be elementwise nonnegative. Since it is also rank one, $Z$ will be completely positive. The lower right block of $Z$ is $W$, so complete positivity of $Z$ implies complete positivity of $W$. Thus we can obtain an even tighter relaxation than exchangeable equilibria by replacing the rank 1 condition by complete positivity of $Z$. In fact, this is the tightest convex relaxation.

**Theorem 3.32.** *The set* $\mathrm{conv}(\mathrm{NE}_{S_2}(\Gamma))$ *is obtained by relaxing the rank* 1 *condition in Proposition 3.29 to complete positivity of* $Z$.

The main technical tool in the proof of this theorem is:

**Lemma 3.33.** *Define*

$$K = \mathrm{cone}\left\{yy^T | Py \geq 0\right\}$$

*and*

$$L = \left\{Y | PYP^T \text{ is completely positive}\right\}.$$

*If $P$ has full column rank, then $K = L$.*

*Proof.* For any $yy^T$ in the generating set of $K$ we have $Py \geq 0$ so $Pyy^TP^T$ is completely positive, hence $yy^T \in L$. Since $L$ is a convex cone, $K \subseteq L$.

Conversely, let $Y \in L$. Then we can write $PYP^T = F^TF$ for some rectangular elementwise nonnegative matrix $F$. For any real vector $x$ with $P^Tx = 0$ we have

$$\|Fx\|^2 = x^TF^TFx = x^TPYP^Tx = 0^TY0 = 0,$$

so $Fx = 0$. That is to say the null space of $P^T$ is contained in the null space of $F$, so the row space of $F$ is contained in the row space of $P^T$.

In particular there is a matrix $B$ such that $F = BP^T$, so $PYP^T = F^TF = PB^TBP^T$. Since $P$ has full column rank there exists a matrix $Q$ such that $QP = I$, the identity matrix. Then $Y = QPYP^TQ^T = QPB^TBP^TQ^T = B^TB$. If $y_1, \ldots, y_k$ are the columns of $B^T$, then $Py_i$ is the $i^{\text{th}}$ column of $F^T$, so $Py_i \geq 0$. Therefore $Y = \sum_i y_iy_i^T \in K$. $\square$

*Proof of Theorem 3.32.* Let $Y = \left[\begin{smallmatrix} \alpha & \gamma \\ \gamma^T & W \end{smallmatrix}\right]$ be any matrix satisfying the constraints. Then $PYP^T$ is completely positive by assumption and $P$ has full column rank, so Lemma 3.33 implies that $Y = \sum y_i y_i^T$ for some $y_i$ such that $Py_i \geq 0$. Let $Y_i = y_i y_i^T$. Then $PY_iP^T$ is completely positive for each $i$, hence elementwise nonnegative for each $i$. Thus for an entry of $PYP^T = \sum_i PY_iP^T$ to be zero, the corresponding entry of $PY_iP^T$ must be zero for all $i$. Therefore $PY_iP^T$ satisfies the complementarity condition.

Define

$$\lambda_i := \begin{bmatrix} 0 \\ e \end{bmatrix}^T Y_i \begin{bmatrix} 0 \\ e \end{bmatrix} = \left\| y_i^T \begin{bmatrix} 0 \\ e \end{bmatrix} \right\|^2.$$

Then $\lambda_i \geq 0$ and since $W$ is normalized, $\sum_i \lambda_i = 1$. Reindex all the quantities so that $\lambda_i > 0$ for $i = 1, \ldots, l$ and $\lambda_i = 0$ for $i = l+1, \ldots, k$. For $i = 1, \ldots, l$ define $\hat{y}_i = \frac{y_i}{\sqrt{\lambda_i}}$ and $\hat{Y}_i = \hat{y}_i \hat{y}_i^T$. Write $\hat{y}_i = \begin{bmatrix} u_i \\ w_i \end{bmatrix}$. Then the components of $w_i$ sum to one, and $W_i = w_i w_i^T$ is a symmetric Nash equilibrium.

Now consider $y_i$ for $i = l+1, \ldots, k$. By definition of $P$ the bottom block of $y_i$ is the same as that of $Py_i$, hence nonnegative. Since $\lambda_i = 0$ these elements sum to zero and must all be zero. Therefore

$$W = \sum_{i=1}^{l} \lambda_i w_i w_i^T,$$

where $\lambda_i \geq 0$ and $\sum_{i=1}^{l} \lambda_i = 1$, so $W$ is a convex combination of symmetric Nash equilibria. $\square$

It is tempting to try to use this characterization to compute a distribution in $\operatorname{conv}(\mathrm{NE}_{S_2})$ by finding such a $Z$ with an explicit nonnegative factorization proving complete positivity. In fact, the above argument shows that we can easily find a symmetric Nash equilibrium from this factorization. So computing such a distribution along with a factorization is as hard as computing Nash equilibria (see [10, 11, 13, 15, 16, 30, 63]). Nonetheless, we can use this characterization for approximate computation in which no such factorization is computed as described in Section 7.1.7, as well as to give another proof that exchangeable equilibria are the convex hull of symmetric Nash equilibria for $2 \times 2$ symmetric bimatrix games (Theorem 3.10).

*Proof #2 of Theorem 3.10.* Let $W \in \mathrm{XE}_{S_2}(\Gamma) \subset \mathbb{R}^{2\times 2}$. Clearly $W$ is not the zero matrix. If $W$ has rank 1 then $W$ is a correlated equilibrium which is a symmetric product distribution, so $W \in \mathrm{NE}_{S_2}(\Gamma)$. Otherwise $W$ has rank 2, so it is invertible and completely positive, hence positive definite.

| Equilibrium type (†) | Condition on $W$ (*) | Condition on $Z$ (**) |
|---|---|---|
| $\mathrm{NE}_{S_2}$ | $\mathrm{rank}(W) = 1$ | $Z \geq 0$ |
| $\mathrm{NE}_{S_2}$ | | $\mathrm{rank}(Z) = 1, Z \geq 0$ |
| $\mathrm{conv}(\mathrm{NE}_{S_2})$ | | $Z$ completely positive |
| $\mathrm{XE}_{S_2}$ | $W$ completely positive | $Z \geq 0$ |
| $\mathrm{CE}_{S_2}$ | | $Z \geq 0$ |

**Table 3.3.** Each row characterizes a type of equilibrium by a condition on $W$ and a condition on $Z$ using a statement of the form (3.12).

Since $W \in \mathrm{CE}_{S_2}(\Gamma)$, Proposition 3.30 implies that we can find a $\gamma \in \mathbb{R}^{1\times m}$ and an $\alpha \in \mathbb{R}$ satisfying all conditions of Proposition 3.29 (except the rank condition on $W$). Note that we can freely increase $\alpha$ without violating any of these conditions. If we take $\alpha > \gamma W^{-1}\gamma^T$, then the Schur complement condition implies that the matrix $\left[\begin{smallmatrix} \alpha & \gamma \\ \gamma^T & W \end{smallmatrix}\right]$ is positive definite, hence the matrix $Z = P\left[\begin{smallmatrix} \alpha & \gamma \\ \gamma^T & W \end{smallmatrix}\right]P^T$ is positive semidefinite. But by assumption $Z \in \mathbb{R}^{4\times 4}$ is also elementwise nonnegative, so it is completely positive by Theorem 2.58. Theorem 3.32 then implies $W \in \mathrm{conv}(\mathrm{NE}_{S_2}(\Gamma))$. □

We summarize the results of this section in Table 3.3. Each row of the table corresponds to a statement of the form:

$$W \text{ is a(n) } (\dagger) \iff W \text{ is symmetric, normalized, and satisfies } (*) \text{ and } \exists \alpha, \gamma \text{ such that } Z \text{ satisfies complementarity and } (**), \tag{3.12}$$

Note that $W$ is a principal submatrix of $Z$, so each of the statements about $Z$ in column (**) of the table implies the corresponding statement about $W$. Therefore conditions are only listed in column (*) if they are not automatically implied by the conditions in column (**).

Chapter 4

# Interpretations of Symmetric Exchangeable Equilibria

Several different game-theoretic setups give rise to the notion of exchangeable equilibria. Of course these are all related (some more obviously than others), but each has something different to add to the overall picture. Taken together, the fact that these interpretations each give rise to the same solution concept is evidence that exchangeable equilibria are natural objects worthy of study.

Throughout this chapter we restrict attention to the motivating case of a standard symmetric game. This means that all players have the same strategy set, the symmetric product distributions are the i.i.d. distributions, and the generalized exchangeable distributions are the mixtures of these. By Proposition 2.46 these are exactly the distributions of $n$ random variables which can be extended to exchangeable distributions (in the classical sense) of an infinite sequence of random variables.

## 4.1 Hidden variable interpretation

A basic tenet of decision theory is that preferences are specific to each individual, and so interpersonal comparison of utilities is meaningless. In practice this belief is often dropped in favor of other simplifying assumptions which can be justified within the setting at hand. One important example of this phenomenon is zero-sum bimatrix games. The zero-sum condition says not only that the players' preferences over outcomes are opposite, but also that their preferences over lotteries over outcomes are opposite. This is a very strong condition indeed, and while it may never be true exactly, it is often a reasonable approximation to model situations in which there seems to be no opportunity for cooperation.

Similarly, symmetric games are an idealization in which we take the roles and preferences of the players to be identical. Even when this assumption is not directly applicable, we can make it so by passing to the symmetrization as discussed in Section 5.5.2 below. This corresponds to placing the players behind a *veil of ignorance* [60] where they "forget their roles" and imagine themselves confronting the situations faced by all players in the game. The resulting game is symmetric by construction.

When symmetry is a natural approximation from the start or is imposed by symmetrization, it seems reasonable to take this idea to its logical conclusion and treat the players as independent instances of an identical decision-making agent, or clones, as it were. Doing so, we should expect all players would make the same choice when confronted with the same situation. They interpret their observations of the environment in the same way. If the setup is such that we have reason to believe that the players will choose their strategies statistically independently, then our solution concept of choice should be symmetric Nash equilibrium. Here we explicitly use the assumption that the players are identical; in particular they have not been provided with a way to break the symmetry so as to choose an asymmetric Nash equilibrium.

This raises a natural question: what is the appropriate solution concept to use if statistical independence of actions is not a reasonable assumption and some correlation may be expected? For a general game the answer would be correlated equilibrium[1], so the obvious answer in the symmetric setting is symmetric correlated equilibrium.

It is our goal to argue that this obvious answer is wrong. Or it might be more fair to say that it is half wrong. It is correct insofar as the symmetry of the situation rules out any correlated equilibria which are not symmetric. However, we claim that symmetry also rules out symmetric correlated equilibria which are not exchangeable.

To see this we return to the notion of correlation schemes and external correlated equilibria illustrated in Figure 2.1. We begin with the tautological statement that each player bases his action on the state of the world. Perhaps he plays the same action at all states, perhaps not. Insofar as he does not have full knowledge of the state of the world his action must be based on some noisy observations of the world. Since the players are identical they all make the same measurements and coin flips and they all interpret them in the same way, but the outcomes of these random events may differ. In this way and only this way may the players' actions differ. More formally:

[1]See Aumann's argument in [2]. The key assumption forming the basis of that argument is that the players have common prior beliefs. While this is debatable in some contexts, it is certainly true in the symmetric setting considered here. Put another way, the philosophical or epistemic argument for correlated equilibrium is stronger in symmetric games than in general games.

**Definition 4.1.** *We say that a correlation scheme is* **symmetric** *if all players have the same noise distribution, mapping from state and noise to information, and mapping $f_i$ from information to action. We say that an external correlated equilibrium is* **symmetric** *if it arises from a a symmetric correlation scheme.*

For example, the players may wish to base their actions on tomorrow's weather. This is an aspect of the underlying state of the world, albeit one which is not known precisely to the players (it is a **hidden variable**), so they can only condition their actions on their estimates of the weather. Perfect correlation may occur if these estimates are obtained from a common weather report. Weaker correlation may arise if both players make their own forecasts in terms of independent measurements of atmospheric data.

Under a symmetric correlation scheme, the players' noise distributions are i.i.d., so their information, and hence actions, are i.i.d. conditioned on the state. We immediately obtain the expected generalization of Proposition 2.9:

**Proposition 4.2.** *In a symmetric external correlated equilibrium, if each player knows the state of the world then the outcome conditioned on the state is a symmetric Nash equilibrium almost surely, and every symmetric Nash equilibrium arises in this way.*

On the other hand the generalization of Proposition 2.8 looks a bit different:

**Proposition 4.3.** *The distribution of actions in a symmetric external correlated equilibrium is an exchangeable equilibrium and every exchangeable equilibrium arises in this way.*

*Proof.* By Proposition 2.8 this distribution is a correlated equilibrium. It is also i.i.d. conditioned on the state, so it is exchangeable.

Conversely, let $\pi$ be any exchangeable equilibrium. Then we can write $\pi = \sum_{i=1}^{k} \lambda_i x_i^{\otimes n}$ for some $x_i \in \Delta(C_1)$ and some probability vector $\lambda$. Construct a symmetric correlation scheme with $k$ states of the world chosen according to the distribution $\lambda$. In state $i$, choose each player's information i.i.d. according to $x_i$. Let the mappings from information to action all be the identity map. Then the distribution over actions is $\pi$, and since this is in particular an internal correlated equilibrium, the resulting correlation scheme is a symmetric external correlated equilibrium. $\square$

We might say that any internal correlated equilibrium which is not symmetric explicitly breaks symmetry, whereas one which is not exchangeable implicitly breaks symmetry since it cannot be implemented by a symmetric correlation scheme. It is natural to focus on the exchangeable equilibria, which do not break symmetry at all.

| $(u_{\text{Row}}, u_{\text{Column}})$ | My name is Alice | My name is Bob |
|---|---|---|
| My name is Alice | (0,0) | (1,1) |
| My name is Bob | (1,1) | (0,0) |

**Table 4.1.** Anti-coordinating based on player identity.

## 4.2 Unknown opponent interpretation

As in the previous section on the hidden variable interpretation, here we again focus on games in which the symmetry includes not only the action spaces and payoffs, but also the roles of the players in a larger sense. To illustrate this idea, consider the anti-coordination game of Example 3.19 with its strategies named as in Table 4.1. We will first give two examples of choices of players without this symmetry and then an example with this symmetry. Note that because of the symmetry of the payoffs, we do not have to select who will be the row player and who will be the column player. In particular this information is not available to help the players coordinate on an equilibrium.

Suppose first that this game were played by two people named Alice and Bob whose names were common knowledge. It is reasonable to predict that the players would have no trouble coordinating on the pure strategy Nash equilibrium in which each player correctly identifies his own name. Suppose secondly that this game were played by two friends named Alice. It may still be possible for them to coordinate based on, say, the common knowledge that one of them has the middle name Roberta and the other does not.

Suppose finally that two arbitrary Alices are selected from the record-breaking crowd at this year's Conference of Game Theorists Named Alice. They are sequestered in separate rooms and asked to choose the action they would play in this game against an opponent chosen from the same conference who has been given the same information. It is common knowledge that both players are attending the conference, but no further information is given. As Bayesian observers, how should we expect them to play? The standard Bayesian way to capture our ignorance of any distinctions between the possible players is to say that the distribution over outcomes should be symmetric. Based on the conference title we can assume common knowledge of rationality, so by Aumann's result [2] we should expect the Alices to play a correlated equilibrium. In particular we should not expect them to do something foolish like both playing "My name is Alice" with probability one.

What else can we say about the outcome? The best symmetric correlated equilibrium in terms of payoff is $W = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$. Is this a reasonable solution?

We claim that $W$ would not be played. Suppose for a contradiction that $W$ were the expected distribution of outcomes and consider asking a third Alice (recall the record attendance) what strategy she would play in the game. The chosen strategies $A_1$, $A_2$, and $A_3$ of the three Alices would all be random variables, and their joint distribution would be symmetric under arbitrary permutations of the Alices. In particular all the pairwise distributions of two of these random variables would be given by $W$. Consulting $W$, we see that the events $E_1 = \{A_2 = A_3\}$, $E_2 = \{A_1 = A_3\}$, and $E_3 = \{A_1 = A_2\}$ would each occur with probability zero, therefore so would their union $E = E_1 \cup E_2 \cup E_3$. But there are only two strategies in the game, so in any realization at least two of the $A_i$ must be equal by the pigeonhole principle. That is to say, $E$ must occur with probability one regardless of the distribution of the $A_i$, a contradiction.

On the other hand, the mixed strategy Nash equilibrium in which each player randomizes over her actions with equal probability does not suffer from this difficulty: one can create a sequence of i.i.d. Bernoulli(1/2) random variables for any number of Alices. Such a sequence has a distribution which is exchangeable and has the desired mixed equilibrium as its marginal distribution corresponding to any choice of two players.

Strictly speaking we have only argued that the observed correlated equilibrium should be $N$-extendable, where $N$ is the number of Alices at the conference. However, with $N$ assumed to be large, it is more natural to consider equilibria which could be extended to an arbitrary number of attendees. In particular this ensures that we select only the equilibria which are robust to the exact size of the pool and the players' knowledge thereof.

Being a bit more formal, we consider situations in which the players are drawn from a pool of $N$ interchangeable agents ($n \leq N \leq \infty$). We interpret interchangeability as meaning that all $N$ agents choose a hypothetical action for the game, regardless of whether they are selected to play, and the joint distribution $\psi$ of these actions is $N$-exchangeable. Whichever $n$ agents are chosen to play the game, we assume they choose their actions in their best interest, in the sense that their joint distribution is a correlated equilibrium. If this is true for one selection of $n$ agents it is true for any other by symmetry: the distribution of actions of any $n$ agents is $\mu_{N\to n}(\psi)$.

**Definition 4.4.** *For a standard symmetric game $\Gamma$ and $n \leq N \leq \infty$ we define the set of $N$-**exchangeable equilibria** to be the inverse image*

$$\mathrm{XE}_G(\Gamma, N) := \mu_{N\to n}^{-1}(\mathrm{CE}(\Gamma)).$$

Up to marginalization, $\infty$-exchangeable equilibria are the same as exchangeable equilibria. Proposition 2.40 immediately yields the following result, which says that $N$-exchangeable equilibria approximate exchangeable equilibria for large $N$.

**Proposition 4.5.** *For a standard symmetric game the images of the sets of $N$-exchangeable equilibria under the marginalization maps are nested*

$$\cdots \xrightarrow{\mu_{n+3\to n+2}} \mathrm{XE}_G(\Gamma, n+2) \xrightarrow{\mu_{n+2\to n+1}} \mathrm{XE}_G(\Gamma, n+1) \xrightarrow{\mu_{n+1\to n}} \mathrm{XE}_G(\Gamma, n)$$

*and*

$$\bigcap_{N=n}^{\infty} \mu_{N\to n}(\mathrm{XE}_G(\Gamma, N)) = \mu_{\infty\to n}(\mathrm{XE}_G(\Gamma, \infty)) = \mathrm{XE}_G(\Gamma).$$

Therefore we may view exchangeable equilibria as exactly those correlated equilibria arising when players are selected from a large pool of interchangeable agents and their identities are unknown to each other. It makes sense to say that the notion of exchangeable equilibria eliminates those correlated equilibria which are "unreasonable" on the grounds of symmetry arguments such as the one made above. Of course, in settings such as the Alice and Bob example where these kinds of symmetry arguments do not apply, we may expect other equilibria.

Finally, note that by Proposition 2.38 we can weaken the interchangeability assumption somewhat. As long as all $n$-variable marginals of any joint distribution of $N$ random variables are equal, we can view these as marginals of an $N$-exchangeable distribution. Thus we can avoid explicitly assuming the joint distribution is $N$-exchangeable.

## 4.3 Many player interpretation

We have interpreted $N$-exchangeable equilibria as lifts of equilibria of the $n$-player game $\Gamma$. We can also interpret these directly as symmetric equilibria of an extended $N$-player game for $n \leq N < \infty$ (for technical reasons we avoid directly analyzing games with infinitely many players). To do so suppose that instead of picking $n$ players out of a pool of $N$ in an arbitrary fashion, we pick the $n$ players uniformly at random (without replacement). The set

$$Q = \{\text{ordered } n\text{-tuples of distinct elements of } \{1, \ldots, N\}\},$$

specifies the possible assignments of players in the pool to roles in $\Gamma$ and has cardinality $|Q| = \frac{N!}{(N-n)!}$. If a player is chosen, he gets whatever utility he obtains playing $\Gamma$; otherwise he gets zero. This defines a new game $\Gamma^{(N)}$ with strategy sets $C_i^{(N)} = C_1$ and utilities given by averaging over the possible selections of players:

$$u_i^{(N)}(s_1, \ldots, s_N) := \frac{(N-n)!}{N!} \sum_{j=1}^{n} \sum_{\substack{p \in Q: \\ p_j = i}} u_j(s_{p_1}, \ldots, s_{p_n}).$$

Scaling these utilities by $\frac{N!}{(N-n)!}$ we obtain a game in which all $n$-tuples in $Q$ play the game simultaneously, choosing the same strategy in each instance. Since scaling utilities does not change the equilibria, this gives us (essentially) another interpretation of $\Gamma^{(N)}$.

**Example 4.6.** We look again at the anti-coordination game $\Gamma$ whose utilities are shown in Table 4.1 (also studied in Example 3.19). The story is different from the one in the previous section, so we simply refer to the strategies as '$A$' and '$B$'. The utilities of the $N$-player game $\Gamma^{(N)}$ are given by

$$u_i^{(N)}(s_1,\ldots,s_N) = \frac{2}{N(N-1)}\#\{j : s_j \neq s_i\},$$

so each player wants to choose whichever strategy is chosen by fewer of his opponents. This is a version of "The Minority Game" introduced by Challet and Zhang [9]. One interpretation is that the $N$ players are choosing between two equally good restaurants, $A$ and $B$, so each player wants to eat at the less crowded restaurant. $\diamondsuit$

The game $\Gamma^{(N)}$ is symmetric under arbitrary permutations of the $N$ players, so it is natural to focus on the set of symmetric correlated equilibria $\mathrm{CE}_{S_N}\left(\Gamma^{(N)}\right)$, which is contained in $\Delta_{S_N}\left(C_1^N\right)$. These are exactly the $N$-exchangeable equilibria of $\Gamma$.

**Proposition 4.7.** *Let $\Gamma$ be a game with standard symmetry group $G$. Then $\Gamma^{(N)}$ has symmetry group $S_N$ and $\mathrm{CE}_{S_N}\left(\Gamma^{(N)}\right) = \mathrm{XE}_G(\Gamma, N)$.*

*Proof.* We have already made the observation about the symmetry group, so we begin by rewriting the utilities of $\Gamma^{(N)}$. For any $j$ we can find a $g \in G$ such that $g \cdot j = 1$. For $p \in Q$ with $p_j = i$,

$$u_j(s_{p_1},\ldots,s_{p_n}) = u_1(s_i, s_{p_{g^{-1}\cdot 2}},\ldots,s_{p_{g^{-1}\cdot n}}).$$

As $p_{-j}$ ranges over all $(n-1)$-tuples of distinct elements of $\{1,\ldots,N\}\setminus\{i\}$ so does $(p_{g^{-1}\cdot 2},\ldots,p_{g^{-1}\cdot n})$. Therefore

$$\sum_{\substack{p\in Q:\\ p_j=i}} u_j(s_{p_1},\ldots,s_{p_n}) = \sum_{\substack{p\in Q:\\ p_1=i}} u_1(s_i, s_{p_2},\ldots,s_{p_n})$$

is independent of $j$. Substituting this back in, we obtain

$$u_i^{(N)}(s_1,\ldots,s_N) = \frac{n(N-n)!}{N!}\sum_{\substack{p\in Q:\\ p_1=i}} u_1(s_i, s_{p_2},\ldots,s_{p_n}).$$

Let $\psi \in \Delta_{S_N}\left(C_1^N\right)$ and $\pi = \mu_{N\to n}(\psi)$. For any $\zeta : C_1 \to C_1$ we have

$$\begin{aligned}
&\sum_{s_1,\dots,s_N} u_i^{(N)}(s_1,\dots,\zeta(s_i),\dots,s_N)\psi(s_1,\dots,s_N)\\
&\qquad = \sum_{s_1,\dots,s_N} \frac{n(N-n)!}{N!} \sum_{\substack{p\in Q:\\ p_1=i}} u_1(\zeta(s_i), s_{p_2},\dots,s_{p_n})\psi(s_1,\dots,s_N)\\
&\qquad = \frac{n(N-n)!}{N!} \sum_{\substack{p\in Q:\\ p_1=i}} \sum_{s_1,\dots,s_N} u_1(\zeta(s_i), s_{p_2},\dots,s_{p_n})\psi(s_1,\dots,s_N)\\
&\qquad = \frac{n(N-n)!}{N!} \sum_{\substack{p\in Q:\\ p_1=i}} \sum_{s_1,\dots,s_n} u_1(\zeta(s_1), s_2,\dots,s_n)\pi(s_1,\dots,s_n)\\
&\qquad = \frac{n(N-n)!}{N!}\frac{(N-1)!}{(N-n)!} \sum_{s_1,\dots,s_n} u_1(\zeta(s_1), s_2,\dots,s_n)\pi(s_1,\dots,s_n)\\
&\qquad = \frac{n}{N} \sum_{s_1,\dots,s_n} u_1(\zeta(s_1), s_2,\dots,s_n)\pi(s_1,\dots,s_n).
\end{aligned}$$

Therefore $\zeta$ will improve a player's payoff in $\Gamma^{(N)}$ under $\psi$ if and only if it will do so in $\Gamma$ under $\pi$ (by symmetry and player-transitivity it suffices to look at deviations $\zeta$ for player 1 in $\Gamma$), so $\psi \in \mathrm{CE}_{S_N}\left(\Gamma^{(N)}\right)$ if and only if $\pi \in \mathrm{CE}(\Gamma)$. $\square$

Thus exchangeable equilibria correspond to symmetric correlated equilibria of $N$-player extensions of the game for arbitrary $N$. Given that the original game was symmetric, it is reasonable to expect that equilibria with a high degree of symmetry would be focal (apt to be chosen over other equilibria because they attract attention by being "better" in some way). This is another reason we might expect players to play an exchangeable equilibrium. This proposition also provides another route to existence of exchangeable equilibria.

*Proof #2 of Theorem 3.16 (standard case only).* In Proposition 2.33 we used the existence of correlated equilibria and an averaging argument to obtain the existence of symmetric correlated equilibria. By Proposition 4.7 this means $\mathrm{XE}_G(\Gamma, N) \neq \emptyset$ for all $n \leq N < \infty$ These sets $\mu_{N\to n}(\mathrm{XE}_G(\Gamma, N))$ are polytopes and so compact. They are also nested so their intersection is nonempty, and equal to $\mathrm{XE}_G(\Gamma)$ by Proposition 4.5. $\square$

**Example 4.8** (continues Example 4.6). Which equilibria should we expect to be played in The Minority Game? There are an abundance of pure equilibria of $\Gamma^{(N)}$; these are exactly the strategy profiles in which $\lfloor \frac{N}{2} \rfloor$ players choose one restaurant and $\lceil \frac{N}{2} \rceil$ choose the other. These all have the disadvantage of not being symmetric in the players. While such equilibria can be justified with an evolutionary model [9], they do not make as much sense in a symmetric one-shot game.

We can symmetrize by constructing a correlated equilibrium which picks one of these pure Nash equilibria uniformly at random. This yields a $\pi^N \in \mathrm{XE}_G(\Gamma, N)$. One can show that if $N$ is odd then $\pi = \pi^{N+1}$ is the unique $\pi \in \Delta_{S_{N+1}}(C_1^{N+1})$ such that $\mu_{N+1\to N}(\pi) = \pi^N$ (the extra player in $\Gamma^{(N+1)}$ goes to the less populated restaurant). On the other hand, if $N$ is even then there is no $\pi \in \Delta_{S_{N+1}}(C_1^{N+1})$ such that $\mu_{N+1\to N}(\pi) = \pi^N$. This means $\pi^N$ is not $(N+2)$-extendable for any $N$.

In particular, there is no symmetric way to extend any of the $\pi^L$ to a correlated equilibrium of all the games $\Gamma^{(N)}$ simultaneously. Therefore expecting the players to actually play one of the $\pi^N$ means assuming that the value of $N \pm 1$ (depending on the parity of $N$) is common knowledge. This assumption does not seem realistic in this model.

It is more realistic to look for a solution which is robust to the value of $N$. This could be accomplished in a number of ways, such as by fixing some value $N$ which is much greater than the actual expected number of players and using $\pi^N$, or by assuming the players have a probability distribution over possible values of $N$ [51]. Exchangeable equilibria are those which are the most robust to the value of $N$; namely, they are correlated equilibria of $\Gamma^{(N)}$ for all $N$.

The unique exchangeable equilibrium of the anti-coordination game in Table 4.1 is the mixed Nash equilibrium (proven in Example 3.19 above). That is to say, the only Bayesian rational strategy of the players in The Minority Game which is symmetric and makes sense regardless of the number of players is for everyone to choose a restaurant uniformly at random. $\diamondsuit$

Finally, the fact that the exchangeable equilibria correspond to symmetric correlated equilibria of some $N$-player extension for all $N$ is robust to how exactly we formulate $\Gamma^{(N)}$. For example, the above formulation corresponds to every ordered list of $n$ out of the $N$ players playing a copy of $\Gamma$. We could alter this to by choosing only a subset of these interactions. An exchangeable equilibrium extends simultaneously to a symmetric correlated equilibrium of all these games. Furthermore, we could allow the players in these extensions to consider different deviations for different copies of $\Gamma$, in which case the exchangeable equilibria would be exactly the correlated equilibria in which the players choose the same action against all opponents. The proofs of both of these facts are similar to the proof of Proposition 4.7.

## 4.4 Sealed envelope implementation

For $n \leq N \leq \infty$ we can assign another interpretation to $\mathrm{XE}_G(\Gamma, N)$ in terms of which correlated equilibria can be implemented by a certain simple type of correlation scheme. Suppose $\Gamma$ is to be played by $n$ players who have access to a common pile of $N$ indistinguishable sealed envelopes, each of which contains a slip of paper on which an element of $C_1$ is written. Each player is allowed to choose one envelope and base his choice of action in $\Gamma$ on its contents, which he examines privately. We assume that there is some mechanism in place preventing multiple players from choosing the same envelope (e.g. reducing the utilities of such players below the minimum utility in $\Gamma$).

The fact that the envelopes are indistinguishable means that the distribution $\psi$ over their contents should be $N$-exchangeable. Then $\psi$ is in $\mathrm{XE}_G(\Gamma, N)$ if and only if it is a Nash equilibrium of the larger game including the envelope selection process for both players to choose distinct envelopes in an arbitrary fashion and then play the strategy written inside the envelope they choose. In particular, indistinguishability means they cannot exploit the knowledge of which envelope the other player will choose in any way.

A natural question at this point is when this behavior is still in equilibrium if the players are allowed to observe the contents of $k > 1$ envelopes. This leads to the notion of order $k$ exchangeable equilibria, the subject of Chapter 5.

Chapter 5

# Higher Order Exchangeable Equilibria

We motivate higher order exchangeable equilibria in the context of standard symmetric games via the sealed envelope implementation of Section 4.4. We then give the definition in the general case. We examine when these converge to convex combinations of Nash equilibria as the order grows. Finally we show that existence of higher order exchangeable equilibria is equivalent to Nash's Theorem on existence of Nash equilibria.

## 5.1 Refinement of the sealed envelope implementation

Fix a game $\Gamma$ with standard symmetry group $G$. An $N$-extendable correlated equilibrium can be viewed as the distribution of the contents of $N$ indistinguishable envelopes each containing an element of $C_1$ so that it is a Nash equilibrium of the game with envelope selection for the $n$ players to each pick a different envelope and play its contents.

Now fix $k \in \mathbb{N}$. We can refine this definition by allowing each player to take up to $k$ of the envelopes (assuming $N$ is large enough) and base his play of $\Gamma$ on the contents of all of these. If it is a Nash equilibrium for him to take only one and play its contents then we call the distribution of envelopes an **order $k$ $N$-exchangeable equilibrium**. The case $k = 1$ reduces to the previously defined notion of $N$-exchangeable equilibrium.

**Definition 5.1.** *The set of **order $k$ $N$-exchangeable equilibria** of a game $\Gamma$ with player-transitive strategy-trivial symmetry group $G$ is written $\mathrm{XE}_G^k(\Gamma, N)$.*

The order $k$ $N$-exchangeable equilibrium concept is most natural when $N \geq nk$, so each player can simultaneously choose $k$ envelopes. However, since we stipulate that no player can gain by choosing more than one envelope, the definition makes sense as long as there is one envelope for everyone and an additional $k-1$ envelopes for one player to contemplate choosing. In other words, we only require $N \geq n+k-1$.

For $N \geq n+k-1$, to have $\psi \in \mathrm{XE}_G^k(\Gamma, N)$ means therefore that $\psi \in \Delta_{S_N}(C_1^N)$ and $\mu_{N \to n+k-1}(\psi)$ satisfies a certain incentive condition. This condition is that if $X_1, \ldots, X_N$ are distributed according to $\psi$, then conditioned on $X_{n+1}, \ldots, X_{n+k-1}$, the distribution of $X_1, \ldots, X_n$ is a correlated equilibrium of $\Gamma$ almost surely. Since the correlated equilibrium constraints are homogeneous, this condition is just a finite number of linear constraints on $\mu_{N \to n+k-1}(\psi)$, hence on $\psi$.

Note that if $\psi$ is i.i.d. then looking at multiple envelopes cannot convey any extra information, so $\psi \in \mathrm{XE}_G^k(\Gamma, N)$ if and only if $\mu_{N \to n}(\psi) \in \mathrm{NE}_G(\Gamma)$. Thus the symmetric Nash equilibria are naturally included in the order $k$ $N$-exchangeable equilibria for all $k \in \mathbb{N}$ and $N \geq n+k-1$.

**Example 5.2** (continues Example 3.20). We revisit the $3 \times 3$ symmetric bimatrix game $\Gamma$ of identical utilities

$$A = A^T = B = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix},$$

for which we showed $\mathrm{conv}(\mathrm{NE}_{S_2}(\Gamma)) \subsetneq \mathrm{XE}_{S_2}(\Gamma) \subsetneq \mathrm{CE}_{S_2}(\Gamma)$. We have seen that

$$W^2 := \frac{1}{2} \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}^T + \frac{1}{2} \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}^T \in \mathrm{XE}_{S_2}(\Gamma) \setminus \mathrm{conv}(\mathrm{NE}_{S_2}(\Gamma)).$$

One way to extend this distribution to an exchangeable distribution of random variables $X_1, X_2, \ldots$ is to flip a fair coin to choose either the first or third strategy, then flip this coin infinitely many times to determine whether each $X_i$ should equal either the second strategy or this previously chosen strategy.

The resulting distribution is in $\mathrm{XE}_{S_2}(\Gamma, \infty)$ by definition, but it is not in $\mathrm{XE}_{S_2}^2(\Gamma, \infty)$. To see this, note that there is a positive probability that a player who opens two envelopes will see the first and second strategies as his recommendations. In this case he knows that the original coin flip selected the first strategy. Conditioned on this information, his opponent's recommendation is one of the first two strategies with equal probability. The unique best response to this information is to play the first strategy. This gives a better payoff than following the recommendation of the envelope which suggested the second strategy, so this distribution is not in $\mathrm{XE}_{S_2}^2(\Gamma, \infty)$.

Now we compute $\mu_{\infty \to 3}(\mathrm{XE}_{S_2}^2(\Gamma, \infty))$ for this game. We will show that this set is the convex hull of the images of the three symmetric Nash equilibria of $\Gamma$. We look at the marginal onto three copies because $n + k - 1 = 2 + 2 - 1 = 3$, so this is the marginal on which the incentive constraints are naturally placed. Choosing $N = \infty$ allows us to use results about complete positivity. We can parametrize such distributions as symmetric tensors:

$$W = \left[\begin{array}{ccc|ccc|ccc} a & b & c & b & d & e & c & e & f \\ b & d & e & d & g & h & e & h & i \\ c & e & f & e & h & i & f & i & j \end{array}\right].$$

The conditional correlated equilibrium conditions are

$$\begin{array}{ccc} b \geq 2c, & d \geq 2e, & e \geq 2f, \\ a \geq c, & b \geq e, & c \geq f, \\ 2e \geq d, & 2h \geq g, & 2i \geq h, \\ 2b \geq d, & 2d \geq g, & 2e \geq h, \\ e \geq 2c, & h \geq 2e, & i \geq 2f, \\ f \geq c, & i \geq e, & j \geq f. \end{array}$$

In particular $c = f$ and $d = 2e = h$:

$$W = \left[\begin{array}{ccc|ccc|ccc} a & b & c & b & 2e & e & c & e & c \\ b & 2e & e & 2e & g & 2e & e & 2e & i \\ c & e & c & e & 2e & i & c & i & j \end{array}\right].$$

Since $W$ is the marginal of an exchangeable distribution, it is completely positive and so doubly nonnegative. In particular, by Example 2.57 the slices of $W$ are all positive semidefinite and $e \geq 0$. One of the $2 \times 2$ minor conditions for semidefiniteness yields $0 \leq 2ec - e^2 = e(2c - e)$. Therefore we either have $e = 0$ or $e \leq 2c$. But $c \geq 0$ so $e \leq 2c$ either way. The reverse inequality is one of the equilibrium conditions, so $e = 2c$:

$$W = \left[\begin{array}{ccc|ccc|ccc} a & b & c & b & 4c & 2c & c & 2c & c \\ b & 4c & 2c & 4c & g & 4c & 2c & 4c & i \\ c & 2c & c & 2c & 4c & i & c & i & j \end{array}\right].$$

The determinant of the first slice of $W$ is

$$a(4c^2 - 4c^2) - b(bc - 2c^2) + c(2bc - 4c^2) = 0 - bc(b - 2c) + 2c^2(b - 2c) = -c(b - 2c)^2.$$

Since this slice is positive semidefinite this determinant must be nonnegative, meaning $c = 0$ or $b = 2c$. If $c = 0$ then positive semidefiniteness of the upper left

$2 \times 2$ block would imply $b^2 \leq 4ac = 0$, so $b = 0 = 2c$ in this case as well. Therefore $b = 2c$ regardless. Applying the same reasoning to the last slice gives $i = 2c$:

$$W = \left[\begin{array}{ccc|ccc|ccc} a & 2c & c & 2c & 4c & 2c & c & 2c & c \\ 2c & 4c & 2c & 4c & g & 4c & 2c & 4c & 2c \\ c & 2c & c & 2c & 4c & 2c & c & 2c & j \end{array}\right].$$

From the equilibrium conditions we have $g \leq 2d = 4e = 8c$. If $g < 8c$ then in particular $c > 0$ and the upper left $2 \times 2$ minor of the middle slice of $W$ would be $2cg - 16c^2 < 16c^2 - 16c^2 = 0$, contradicting positive semidefiniteness. Thus $g = 8c$:

$$W = \left[\begin{array}{ccc|ccc|ccc} a & 2c & c & 2c & 4c & 2c & c & 2c & c \\ 2c & 4c & 2c & 4c & 8c & 4c & 2c & 4c & 2c \\ c & 2c & c & 2c & 4c & 2c & c & 2c & j \end{array}\right].$$

The only remaining correlated equilibrium conditions which are not automatically satisfied with equality when $W$ is of this form are $a, j \geq c \geq 0$ and the normalization condition $a + 62c + j = 1$. Any such $W$ can be written

$$W = (a - c) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^{\otimes 3} + 64c \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}^{\otimes 3} + (j - c) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^{\otimes 3},$$

where $a - c \geq 0$, $64c \geq 0$, $j - c \geq 0$, and $a - c + 64c + j - c = a + 62c + j = 1$. Conversely, these normalized nonnegative symmetric simple tensors are just the symmetric Nash equilibria of $\Gamma$ (extended to i.i.d. distributions on three random variables), so any such convex combination really is the marginal of an order 2 $\infty$-exchangeable equilibrium. Summarizing:

$$\begin{aligned} \mu_{\infty \to 3}(\mathrm{XE}^2_{S_2}(\Gamma, \infty)) &= \operatorname{conv} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^{\otimes 3}, \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}^{\otimes 3}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^{\otimes 3} \right\} \\ &= \operatorname{conv} \left( \mu_{3 \to 2}^{-1} \left( \mathrm{NE}_{S_2}(\Gamma) \right) \right). \end{aligned}$$ ♢

It is not always the case that the order 2 $\infty$-exchangeable equilibria are the same as the convex combinations of symmetric Nash equilibria.

**Example 5.3** (continues Example 3.26). Consider the symmetric three player coordination game with identical utilities

$$u = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

In this game a player should choose whichever strategy he believes is more likely for both of his opponents to choose.

Fix $p \in \left(\frac{1}{2}, 1\right)$. Suppose we have an infinite sequence of envelopes containing elements of the strategy space $C_1 = \{0, 1\}$ chosen according to the distribution $\frac{1}{2}\left[\begin{smallmatrix} p \\ 1-p \end{smallmatrix}\right]^{\otimes\infty} + \frac{1}{2}\left[\begin{smallmatrix} 1-p \\ p \end{smallmatrix}\right]^{\otimes\infty}$. That is to say, we flip a fair coin to choose which strategy to favor and conditioned on that choose the envelope contents i.i.d. with a $p$-biased coin.

If a player observes the contents of two of these envelopes and his observations match, then believes it more likely than not that the initial coin flip biased the contents towards this observation. If his observations do not match then he assigns equal posterior probability to both biases. No matter what, he always believes both of his opponents are at least as likely to play the strategy from his first envelope as they are the other strategy. Therefore this distribution is an order 2 $\infty$-exchangeable equilibrium.

Suppose a player observes the contents of three envelopes to be $0, 1, 1$. Then he believes the coin is more likely to be biased towards strategy 1, so his opponents are more likely to see a 1 than a 0. Therefore it is not in his best interest to play 0, so this distribution is not an order 3 $\infty$-exchangeable equilibrium.

The incentive conditions for an order 3 $\infty$-exchangeable equilibrium concern the distribution of $3 + 3 - 1 = 5$ envelopes. Let us see which distributions are in $\mu_{\infty\to 5}(\mathrm{XE}^3_{S_3}(\Gamma, \infty))$. Such a distribution is specified by 6 numbers $p_0, \ldots, p_5$, where $p_i$ is the probability that a given sequence in $\{0, 1\}^5$ with exactly $i$ ones will appear. Modulo normalization, if a player observes the last two envelopes his beliefs about the distribution of the remaining three will be:

$$\left[\begin{array}{cc|cc} p_0 & p_1 & p_1 & p_2 \\ p_1 & p_2 & p_2 & p_3 \end{array}\right], \left[\begin{array}{cc|cc} p_1 & p_2 & p_2 & p_3 \\ p_2 & p_3 & p_3 & p_4 \end{array}\right], \text{ or } \left[\begin{array}{cc|cc} p_2 & p_3 & p_3 & p_4 \\ p_3 & p_4 & p_4 & p_5 \end{array}\right],$$

depending on the observed contents. To obtain an order 3 $\infty$-exchangeable equilibrium these must each satisfy the correlated equilibrium conditions:

$$\begin{array}{lll} p_0 \geq p_2, & p_1 \geq p_3, & p_2 \geq p_4, \\ p_3 \geq p_1, & p_4 \geq p_2, & p_5 \geq p_3, \end{array}$$

or more concisely, $p_5 \geq p_1 = p_3$ and $p_0 \geq p_2 = p_4$. The marginals of exchangeable distributions are exactly the completely positive distributions. By Example 2.56 complete positivity and double nonnegativity agree and can be written (substituting in $p_3 = p_1$ and $p_4 = p_2$)

$$\begin{bmatrix} p_0 & p_1 & p_2 \\ p_1 & p_2 & p_1 \\ p_2 & p_1 & p_2 \end{bmatrix} \succeq 0 \qquad \text{and} \qquad \begin{bmatrix} p_1 & p_2 & p_1 \\ p_2 & p_1 & p_2 \\ p_1 & p_2 & p_5 \end{bmatrix} \succeq 0.$$

The lower right $2 \times 2$ minor of the left matrix and the upper left $2 \times 2$ minor of the right matrix give $p_2^2 \geq p_1^2$ and $p_1^2 \geq p_2^2$. Since $p_1, p_2 \geq 0$ we must have $p_1 = p_2$.

The set of distributions in $\Delta_{S_5}(C_1^5)$ with $p_0, p_5 \geq p_1 = p_2 = p_3 = p_4$ is exactly

$$\begin{aligned} \mu_{\infty\to 5}(\mathrm{XE}^3_{S_3}(\Gamma,\infty)) &= \operatorname{conv}\left\{ \begin{bmatrix}1\\0\end{bmatrix}^{\otimes 5}, \begin{bmatrix}1/2\\1/2\end{bmatrix}^{\otimes 5}, \begin{bmatrix}0\\1\end{bmatrix}^{\otimes 5} \right\} \\ &= \operatorname{conv}\left(\mu^{-1}_{5\to 3}(\mathrm{NE}_{S_3}(\Gamma))\right). \end{aligned} \qquad \diamondsuit$$

The relationships between the sets $\mathrm{XE}^k_G(\Gamma, N)$ are summarized in Figure 5.1. It is clear from the definitions that order $k+1$ $N$-exchangeable equilibria are automatically order $k$ $N$-exchangeable equilibria. Similarly, marginalization maps order $k$ $(N+1)$-exchangeable equilibria to order $k$ $N$-exchangeable equilibria. Since the inverse limit of $\Delta_{S_N}(C_1^N)$ is $\Delta_{S_\infty}(C_1^\infty)$, the inverse limit of $\mathrm{XE}^k_G(\Gamma, N)$ for fixed $k$ is $\mathrm{XE}^k_G(\Gamma, \infty)$.

It it natural to extend $\mathrm{XE}^k_G(\Gamma, N)$ to the $k = \infty$ case when $N = \infty$ by defining $\mathrm{XE}^\infty_G(\Gamma,\infty) := \bigcap_{k=1}^\infty \mathrm{XE}^k_G(\Gamma,\infty)$. This corresponds to players being allowed to examine any finite number of envelopes. In the examples even for fairly small fixed $k$, the order $k$ $\infty$-exchangeable equilibria were the same as mixtures of Nash equilibria. Why is this?

De Finetti's theorem states that conditioned on observing the contents of many envelopes (say $k$), the remaining envelopes are approximately i.i.d. with distribution approximately equal to the observed empirical distribution. The order $k$ incentive conditions imply that all the observations are best responses to the beliefs, so the [$n^{\text{th}}$ tensor power of the] empirical distribution is close to being a symmetric Nash equilibrium. The more envelopes observed, the closer to a Nash equilibrium the distribution is likely to be.

This argument suggests that $\mathrm{XE}^\infty_G(\Gamma,\infty)$ is exactly the set of distributions in which a random Nash equilibrium is selected and conditioned on that the envelope contents are generated i.i.d. according to this Nash equilibrium. In the remainder of the chapter we will set this statement in a more formal and general context and prove it.

## 5.2 Powers of games

To define order $k$ exchangeable equilibria for games with arbitrary symmetry group $G$ we will need two notions of a power of a game $\Gamma$. These are larger games in which multiple copies of $\Gamma$ are played simultaneously[1]. We need to develop

[1] These definitions can be generalized to define two possible products of games, so that the powers we define reduce to $k$-fold products of a game with itself. If we remove all mention of

$$\begin{tikzcd}
 & & & & \Delta(\mathrm{NE}_{S_2}) \arrow[d, equal] \\
 & & & & \mathrm{XE}^{\infty}_{S_2}(\infty) \arrow[d] \\
 & & & \iddots & \vdots \arrow[d] \\
 & & \mathrm{XE}^3_{S_2}(4) \arrow[d] & \cdots \arrow[l] & \mathrm{XE}^3_{S_2}(\infty) \arrow[l] \arrow[d] \\
 & \mathrm{XE}^2_{S_2}(3) \arrow[d] & \mathrm{XE}^2_{S_2}(4) \arrow[l] \arrow[d] & \cdots \arrow[l] & \mathrm{XE}^2_{S_2}(\infty) \arrow[l] \arrow[d] \\
 \mathrm{XE}^1_{S_2}(2) \arrow[d, equal] & \mathrm{XE}^1_{S_2}(3) \arrow[l] \arrow[d, equal] & \mathrm{XE}^1_{S_2}(4) \arrow[l] \arrow[d, equal] & \cdots \arrow[l] & \mathrm{XE}^1_{S_2}(\infty) \arrow[l] \arrow[d, equal] \\
 \mathrm{CE}_{S_2} \arrow[d] & \mathrm{XE}_{S_2}(3) \arrow[l] \arrow[d] & \mathrm{XE}_{S_2}(4) \arrow[l] \arrow[d] & \cdots \arrow[l] & \mathrm{XE}_{S_2} \arrow[l] \arrow[d] \\
 \Delta_{S_2}(C_1^2) & \Delta_{S_3}(C_1^3) \arrow[l] & \Delta_{S_4}(C_1^4) \arrow[l] & \cdots \arrow[l] & \Delta_{S_\infty}(C_1^\infty) \arrow[l]
\end{tikzcd}$$

**Figure 5.1.** Commutative diagram expressing relations between the sets of order $k$ $N$-exchangeable equilibria for a symmetric bimatrix game. The vertical maps are inclusions and the horizontal maps are marginalizations. Ellipses imply equality in the (inverse) limit. Dependence on the game $\Gamma$ has been suppressed to unclutter the diagram.

the theory of these a bit before making the connection with the previous section in Proposition 5.11. Throughout we will take as fixed a game $\Gamma$ with symmetry group $G$ and a number $k \in \mathbb{N}$.

**Definition 5.4.** *The $k^{th}$* ***power of*** *$\Gamma$, denoted $\Pi^k\Gamma$, is a game in which $k$ independent copies of $\Gamma$ are played simultaneously. More specifically, $\Pi^k\Gamma$ has $kn$ players labeled by pairs $i, j$, $1 \leq i \leq n$, $1 \leq j \leq m$, strategy spaces $\Pi^k C_{ij} := C_i$ for all $i, j$ with generic element $s_i^j$, and utilities $\Pi^k u_{ij}(s_1^1, \ldots, s_n^k) := u_i(s_1^j, s_2^j, \ldots, s_n^j)$.*

*The* ***contracted $k^{th}$ power of*** *$\Gamma$, denoted $\Xi^k\Gamma$, is a game in which $k$ copies of $\Gamma$ are played simultaneously, but all by the same set of players. Specifically, $\Xi^k\Gamma$ has $n$ players, strategy spaces $\Xi^k C_i := C_i^k$ with generic element $(s_i^1, \ldots, s_i^k)$ for all $i$, and utilities $\Xi^k u_i(s_1^1, \ldots, s_n^k) := \sum_j u_i(s_1^j, s_2^j, \ldots, s_n^j)$.*

A schematic illustration of these powers is shown in Figure 5.2.

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symmetry groups (so in particular exchangeable distributions and exchangeable equilibria are no longer defined) all the statements we make about Nash and correlated equilibria of these powers extend to corresponding statements about products with identical proofs. We will not need this level of generality, however, so to avoid complicating notation we focus on powers.

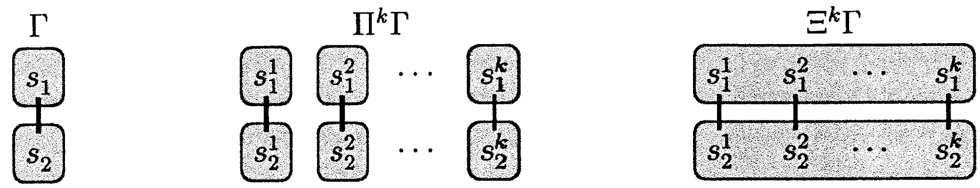

**Figure 5.2.** Representing a 2-player game $\Gamma$ with players choosing strategies $s_1$ and $s_2$ as drawn on the left, the powers $\Pi^k\Gamma$ and $\Xi^k\Gamma$ are formed as shown. A shaded box represents actions controlled by a single player, whose utility is given by the sum over all interactions.

**Proposition 5.5.** *Let $\Gamma$ be a game with symmetry group $G$ and fix $k \in \mathbb{N}$. Then both powers $\Pi^k\Gamma$ and $\Xi^k\Gamma$ are games with symmetry group $G \times S_k$ and they satisfy:*

- $\Delta_{G\times S_k}\left(\Pi^k\Gamma\right) = \Delta_{G\times S_k}\left(\Xi^k\Gamma\right)$
- $\Delta^{\Pi}_{G\times S_k}\left(\Pi^k\Gamma\right) \subsetneq \Delta^{\Pi}_{G\times S_k}\left(\Xi^k\Gamma\right)$
- $\Delta^{X}_{G\times S_k}\left(\Pi^k\Gamma\right) \subseteq \Delta^{X}_{G\times S_k}\left(\Xi^k\Gamma\right)$.

*Furthermore, distributions in $\Delta^{X}_{G\times S_k}\left(\Pi^k\Gamma\right)$ are invariant under the larger group $\langle S_k^n, G^k\rangle$, where the powers denote group products. That is to say, they are invariant under separate permutations applied to the copies of player 1, the copies of player 2, etc., and also under separate elements of $G$ applied to the first copy, the second copy, etc.*

*Proof.* Both powers are invariant under arbitrary permutations of the copies and under symmetries in $G$ applied to all of the copies simultaneously. In fact in the case of $\Pi^k\Gamma$ we can apply a different symmetry in $G$ to each copy independently so that $\Pi^k\Gamma$ is invariant under the larger group $G \wr S_k$ (the wreath product of $G$ and $S_k$), but we will not need this fact.

Since $G \times S_k$ acts on $\Pi^k C$ and $\Xi^k C$ in the same way, we get the first equality. The game $\Pi^k\Gamma$ has more players than $\Xi^k\Gamma$, so $\Delta^{\Pi}_{G\times S_k}\left(\Pi^k\Gamma\right)$ has stronger independence conditions than $\Delta^{\Pi}_{G\times S_k}\left(\Xi^k\Gamma\right)$, yielding the strict containment. Taking convex hulls gives the third containment.

Elements of $\Delta^{\Pi}_{G\times S_k}\left(\Pi^k\Gamma\right)$ are of the form $x^{\otimes k}$ where $x = x_1 \otimes \cdots \otimes x_n \in \Delta^{\Pi}_{G}(\Gamma)$. Such distributions are invariant under all the mentioned symmetries, so any convex combination of these is as well. $\square$

Since $\Delta_{G\times S_k}(\Pi^k\Gamma) = \Delta_{G\times S_k}(\Xi^k\Gamma)$, we can compare the conditions for a distribution to be a correlated equilibrium of $\Pi^k\Gamma$ or $\Xi^k\Gamma$. Many of the relationships between these and the equilibria of $\Gamma$ which we prove below are summarized in Figure 5.3.

**Proposition 5.6.** *Let $(X_1^1, \dots, X_n^k)$ be a random vector taking values in $C^k$ distributed according to $\pi \in \Delta_{G\times S_k}\left(\Pi^k\Gamma\right) = \Delta_{G\times S_k}\left(\Xi^k\Gamma\right)$. Then*

- *$\pi$ is a correlated equilibrium of $\Pi^k\Gamma$ if and only if $X_i^j$ is a best response (in $\Gamma$) to $\mathrm{Prob}(X_{-i}^j \mid X_i^j)$ almost surely for all $i$ and $j$, and*

- *$\pi$ is a correlated equilibrium of $\Xi^k\Gamma$ if and only if $X_i^j$ is a best response (in $\Gamma$) to $\mathrm{Prob}(X_{-i}^j \mid X_i^1, \dots, X_i^k)$ almost surely for all $i$ and $j$.*

*Proof.* By Proposition 2.7, $\pi$ is a correlated equilibrium of $\Pi^k\Gamma$ if and only if $X_i^j$ is a best response in $\Pi^k\Gamma$ to $\mathrm{Prob}(X^1, \dots, X^{j-1}, X_{-i}^j, X^{j+1}, \dots, X^k \mid X_i^j)$ almost surely for all $i$ and $j$. But the utility of player $ij$ in $\Pi^k\Gamma$ is $u_i(X_1^j, \dots, X_n^j)$, so player $ij$ can ignore $X_r^l$ whenever $l \neq j$.

Similarly $\pi$ is a correlated equilibrium of $\Xi^k\Gamma$ if and only if $(X_i^1, \dots, X_i^k)$ is a best response in $\Xi^k\Gamma$ to $\mathrm{Prob}(X_{-i}^1, \dots, X_{-i}^m \mid X_i^1, \dots, X_i^k)$ almost surely for all $i$. The utility of player $i$ in $\Xi^k\Gamma$ is $\sum_j u_i(X_1^j, X_2^j, \dots, X_n^j)$ and no $X_i^j$ appears in more than one term of this sum. Thus the sum is maximized when each term is maximized independently. That is to say $\pi \in \mathrm{CE}_{G\times S_k}(\Xi^k\Gamma)$ if and only if $X_i^j$ is a best response in $\Gamma$ to $\mathrm{Prob}(X_{-i}^j \mid X_i^1, \dots, X_i^k)$ almost surely for all $i$ and $j$. $\square$

This characterization allows us to prove the following containments between equilibrium sets. One can construct examples showing that in general none of the inclusions in this proposition can be reversed. In particular, no containment holds between $\mathrm{XE}_{G\times S_k}\left(\Pi^k\Gamma\right)$ and $\mathrm{XE}_{G\times S_k}\left(\Xi^k\Gamma\right)$ in either direction. This is connected to the fact that the inclusion between the sets of correlated equilibria of $\Pi^k\Gamma$ and $\Xi^k\Gamma$ goes in the opposite direction from the inclusion between the sets of Nash equilibria.

**Proposition 5.7.** *The symmetric equilibria of $\Pi^k\Gamma$ and $\Xi^k\Gamma$ satisfy*

$$\mathrm{NE}_{G\times S_k}\left(\Pi^k\Gamma\right) \subseteq \mathrm{NE}_{G\times S_k}\left(\Xi^k\Gamma\right) \subseteq \mathrm{CE}_{G\times S_k}\left(\Xi^k\Gamma\right) \subseteq \mathrm{CE}_{G\times S_k}\left(\Pi^k\Gamma\right).$$

*Proof.* We use Proposition 5.6. If $X_i^j$ are distributed according to $\pi \in \Delta_{G\times S_k}^{\Pi}(\Pi^k\Gamma)$ then the $X_i^j$ are all independent, so the conditional distributions in Proposition 5.6 are equal to the corresponding unconditional distributions and both conditions are equivalent. This proves the first containment. The second containment follows because Nash equilibria are always correlated equilibria. For the third containment, suppose $X_i^j$ is a best response to $\mathrm{Prob}(X_{-i}^j \mid X_i^1, \dots, X_i^k)$ almost surely. Summing over possible values of $X_i^{-j}$ we get that $X_i^j$ is a best response to $\mathrm{Prob}(X_{-i}^j \mid X_i^j)$ almost surely. $\square$

These sets are closely related to the equilibria of $\Gamma$. We will express these relationships in terms of the marginalization map from $\Delta_{G\times S_k}(\Pi^k\Gamma)$ to $\Delta_G(\Gamma)$ which sends the distribution of the random vector $(X_1^1,\dots,X_n^k)$ to the distribution of $(X_1^1,\dots,X_n^1)$. We denote this marginalization map by $\mu_{kn\to n}$, although strictly speaking this is an abuse of our established notation because (in the case when $G$ is trivial, say) distributions in $\Delta_{G\times S_k}(\Pi^k\Gamma)$ need not be $kn$-exchangeable, so it matters which variables we choose to marginalize.

This marginalization map has two natural right inverses. First, the power map $\mathrm{pow}^k$ which sends $\pi$ to $\pi^{\otimes k}$. Second, the diagonal map $\mathrm{diag}^k$ which sends the distribution of random vector $(X_1,\dots,X_n)$ to the distribution of random vector $(X_1^1,\dots,X_n^k)$ where $X_i^j = X_i$ almost surely for all $i,j$.

The following proposition records for completeness how these maps interact with equilibria. We only use a few of these facts in what follows.

**Proposition 5.8.** *A distribution $\psi\in\Delta_{G\times S_k}(\Pi^k\Gamma)$ is a correlated equilibrium of $\Pi^k\Gamma$ if and only if $\mu_{kn\to n}(\psi)$ is a correlated equilibrium of $\Gamma$. A distribution $\pi\in\Delta_G(\Gamma)$ is a correlated equilibrium of $\Gamma$ if and only if $\mathrm{diag}^k(\pi)$ is a correlated equilibrium of $\Xi^k\Gamma$ if and only if $\mathrm{pow}^k(\pi)$ is a correlated equilibrium of $\Xi^k\Gamma$. Marginalization maps $\mathrm{CE}_{G\times S_k}(\Pi^k\Gamma)$ and $\mathrm{CE}_{G\times S_k}(\Xi^k\Gamma)$ onto $\mathrm{CE}_G(\Gamma)$.*

*A distribution $\pi\in\Delta_G^\Pi(\Gamma)$ is a Nash equilibrium of $\Gamma$ if and only if $\mathrm{pow}^k(\pi)$ is a Nash equilibrium of $\Pi^k\Gamma$. This map $\mathrm{NE}_G(\Gamma)\to\mathrm{NE}_{G\times S_k}(\Pi^k\Gamma)$ is onto, as is the marginalization map from $\mathrm{NE}_{G\times S_k}(\Xi^k\Gamma)$ to $\mathrm{NE}_G(\Gamma)$.*

*Proof.* Immediate from the definitions and Propositions 5.6 and 5.7. □

**Proposition 5.9.** *Marginalization maps $\mathrm{XE}_{G\times S_k}(\Pi^k\Gamma)$ and $\mathrm{XE}_{G\times S_k}(\Xi^k\Gamma)$ onto $\mathrm{XE}_G(\Gamma)$.*

*Proof.* By Propositions 5.7 and 5.8 any correlated equilibrium of either power maps to a correlated equilibrium of $\Gamma$. Product distributions for either power marginalize to product distributions for $\Gamma$, so by linearity so do exchangeable distributions. Therefore exchangeable equilibria of either power map into exchangeable equilibria of $\Gamma$.

To see that this map is onto we use a different argument for each power. Let $\pi=\sum_j\lambda^j x_1^j\otimes\cdots\otimes x_n^j$ be a decomposition of a $\pi\in\mathrm{XE}_G(\Gamma)$ as a finite convex combination where $x_1^j\otimes\cdots\otimes x_n^j$ are in $\Delta_G^\Pi(\Gamma)$. Then

$$\mathrm{diag}^k(\pi)=\sum_j\lambda^j\,\mathrm{diag}^k(x_1^j)\otimes\cdots\otimes\mathrm{diag}^k(x_n^j)\in\Delta^X_{G\times S_k}(\Xi^k\Gamma)$$

and by Proposition 5.8 this is also a correlated equilibrium of $\Xi^k\Gamma$, so it is an exchangeable equilibrium which marginalizes to $\pi$. On the other hand

$$\sum_j \lambda^j \operatorname{pow}^k(x_1^j \otimes \cdots \otimes x_n^j) = \sum \lambda^j \operatorname{pow}^k(x_1^j) \otimes \cdots \otimes \operatorname{pow}^k(x_n^j) \in \Delta^X_{G\times S_k}(\Pi^k\Gamma)$$

and this must also be a correlated equilibrium of $\Pi^k\Gamma$ by Propositions 5.7 and 5.8 since it marginalizes to $\pi$. □

Note that we lifted $\pi$ to an exchangeable equilibrium of $\Xi^k\Gamma$ which is a correlated equilibrium of $\Pi^k\Gamma$ but not necessarily an exchangeable distribution for $\Pi^k\Gamma$, so not an exchangeable equilibrium of $\Pi^k\Gamma$. Conversely, the lift of $\pi$ to an exchangeable equilibrium of $\Pi^k\Gamma$ is an exchangeable distribution of $\Xi^k\Gamma$ but not necessarily a correlated equilibrium thereof, so not an exchangeable equilibrium of $\Xi^k\Gamma$. This raises the natural question of whether there always exists a lift which is simultaneously an exchangeable equilibrium of both powers. We will see in the coming sections that there need not be.

For any $1 \le p < k$ a similar abuse of notation gives a marginalization map

$$\mu_{kn\to pn} : \Delta_{G\times S_k}(\Pi^k\Gamma) \to \Delta_{G\times S_p}(\Pi^p\Gamma).$$

This map respects the structure of all the sets of distributions mentioned in Proposition 5.5, i.e., it restricts to maps $\Delta^\Pi_{G\times S_k}(\Pi^k\Gamma) \to \Delta^\Pi_{G\times S_p}(\Pi^p\Gamma)$, $\Delta^X_{G\times S_k}(\Xi^k\Gamma) \to \Delta^X_{G\times S_p}(\Xi^p\Gamma)$, etc. By Proposition 5.6 it also respects the equilibrium structure of these games in the sense that $\mathrm{XE}_{G\times S_k}(\Pi^k\Gamma)$ maps into $\mathrm{XE}_{G\times S_p}(\Pi^p\Gamma)$, $\mathrm{NE}_{G\times S_k}(\Xi^k\Gamma)$ maps into $\mathrm{NE}_{G\times S_p}(\Xi^p\Gamma)$, etc.

We have shown that if $p = 1$ then all of these maps are onto. In particular, $\mu_{kn\to n}(\mathrm{XE}_{G\times S_k}(\Pi^k\Gamma)) = \mathrm{XE}_G(\Gamma) = \mu_{kn\to n}(\mathrm{XE}_{G\times S_k}(\Xi^k\Gamma))$, so neither of these marginalized sets of exchangeable equilibria need approach the convex hull of the Nash equilibria of $\Gamma$ as $k$ gets large. Alone neither power's set of exchangeable equilibria provides a tighter convex relaxation for the symmetric Nash equilibria of $\Gamma$ than $\mathrm{XE}_G(\Gamma)$ itself does. We will see that taking their intersection fixes this.

## 5.3 Order $k$ exchangeable equilibria

**Definition 5.10.** *The set of **order $k$ exchangeable equilibria of** $\Gamma$ is*

$$\mathrm{XE}^k_G(\Gamma) := \mathrm{XE}_{G\times S_k}\left(\Pi^k\Gamma\right) \cap \mathrm{XE}_{G\times S_k}\left(\Xi^k\Gamma\right),$$

*or equivalently by Propositions 5.5 and 5.7,*

$$\mathrm{XE}^k_G(\Gamma) := \Delta^X_{G\times S_k}\left(\Pi^k\Gamma\right) \cap \mathrm{CE}_{G\times S_k}\left(\Xi^k\Gamma\right).$$

$$\begin{tikzcd}
\mathrm{CE}_{G\times S_k}(\Pi^k\Gamma) & & \mathrm{CE}_{G\times S_k}(\Xi^k\Gamma) \arrow[ll, hook'] & & \mathrm{CE}_G(\Gamma) \\
\mathrm{XE}_{G\times S_k}(\Pi^k\Gamma) \arrow[u, hook] & & \mathrm{XE}_{G\times S_k}(\Xi^k\Gamma) \arrow[u, hook] & {} \arrow[r, two heads, "\mu_{kn\to n}"] & \mathrm{XE}_G(\Gamma) \arrow[u, hook] \\
& \mathrm{XE}^k_G(\Gamma) \arrow[ul, hook] \arrow[ur, hook] & & & \mu_{kn\to n}(\mathrm{XE}^k_G(\Gamma)) \arrow[u, hook] \\
\mathrm{NE}_{G\times S_k}(\Pi^k\Gamma) \arrow[ur, hook] \arrow[rr, hook] & & \mathrm{NE}_{G\times S_k}(\Xi^k\Gamma) \arrow[uu, hook] & & \mathrm{NE}_G(\Gamma) \arrow[u, hook]
\end{tikzcd}$$

**Figure 5.3.** At the left is a summary of the containments between equilibrium sets of the powers $\Pi^k\Gamma$ and $\Xi^k\Gamma$ proven in Section 5.2. An arrow $A \hookrightarrow B$ indicates $A \subseteq B$. Marginalization maps each of these onto the set at the same height on the right.

The relationship between $\mathrm{XE}^k_G(\Gamma)$ and the sets of equilibria of the $k^{\text{th}}$ powers is summarized in Figure 5.3.

These order $k$ exchangeable equilibria are related to order $k$ $\infty$-exchangeable equilibria, as defined in Section 5.1, just as exchangeable equilibria are related to $\infty$-exchangeable equilibria by Proposition 4.5:

**Proposition 5.11.** *In a standard symmetric game,*

$$\mathrm{XE}^k_G(\Gamma) = \mu_{\infty\to nk}(\mathrm{XE}^k_G(\Gamma,\infty)).$$

*Proof.* The $k^{th}$ power $\Pi^k\Gamma$ is a standard symmetric game with symmetry group $G \times S_k$. The set $\Delta^X_{G\times S_k}(\Pi^k\Gamma) = \mu_{\infty\to nk}(\Delta_{S_\infty}(C_1^\infty))$. Let $X_1, X_2, \ldots$ be an exchangeable sequence of random variables taking values in $C_1$. By Proposition 5.6, the marginal distribution of $X_1, \ldots, X_{nk}$ is a correlated equilibrium of $\Xi^k\Gamma$ if and only if, conditioned on $k-1$ of these random variables, any other $n$ of them form a correlated equilibrium of $\Gamma$. $\square$

We do not give a direct proof of the existence of order $k$ exchangeable equilibria in the style of the existence proofs for exchangeable equilibria. Rather, we observe that existence follows immediately from Nash's theorem. In the following two sections we will show that this existence result is in fact equivalent to Nash's Theorem, in the sense that we can use it to prove Nash's Theorem. Therefore a hypothetical direct proof of Proposition 5.12 would yield a new proof of Nash's Theorem.

**Proposition 5.12.** *A game with symmetry group $G$ has an order $k$ exchangeable equilibrium for all $k \in \mathbb{N}$.*

*Proof.* By Propositions 5.5 and 5.7 we have

$$\mathrm{NE}_{G\times S_k}(\Pi^k\Gamma) \subseteq \mathrm{XE}_{G\times S_k}(\Pi^k\Gamma) \cap \mathrm{XE}_{G\times S_k}(\Xi^k\Gamma),$$

so by Nash's Theorem this intersection is nonempty. □

For $1 \leq p < k$, the marginalization map sends $\mathrm{XE}_{G\times S_k}(\Pi^k\Gamma)$ into $\mathrm{XE}_{G\times S_p}(\Pi^p\Gamma)$ and $\mathrm{XE}_{G\times S_k}(\Xi^k\Gamma)$ into $\mathrm{XE}_{G\times S_p}(\Xi^p\Gamma)$. Therefore it sends $\mathrm{XE}_G^k(\Gamma)$ into $\mathrm{XE}_G^p(\Gamma)$. Projecting the order $k$ exchangeable equilibria into $\Delta_G(\Gamma)$ for all $k \in \mathbb{N}$ we obtain

$$\begin{aligned}\mathrm{NE}_G(\Gamma) \subseteq \mathrm{conv}(\mathrm{NE}_G(\Gamma)) \subseteq \cdots \subseteq \mu_{3n\to n}(\mathrm{XE}_G^3(\Gamma)) \subseteq \mu_{2n\to n}(\mathrm{XE}_G^2(\Gamma))\\ \subseteq \mathrm{XE}_G(\Gamma) \subseteq \mathrm{CE}_G(\Gamma).\end{aligned} \tag{5.1}$$

This raises a natural question: does $\bigcap_{k=1}^{\infty} \mu_{kn\to n}(\mathrm{XE}_G^k(\Gamma)) = \mathrm{conv}(\mathrm{NE}_G(\Gamma))$? We will take up this question in the following sections.

## 5.4 Order $\infty$ exchangeable equilibria

As $k$ varies the sets of higher order exchangeable equilibria live in different spaces but the marginalization maps form a sequence

$$\cdots \to \mathrm{XE}_G^4(\Gamma) \to \mathrm{XE}_G^3(\Gamma) \to \mathrm{XE}_G^2(\Gamma) \to \mathrm{XE}_G(\Gamma),$$

so the natural way to take an "intersection" of these is the inverse limit.

**Definition 5.13.** *The set of* ***order $\infty$ exchangeable equilibria*** *is*

$$\mathrm{XE}_G^{\infty}(\Gamma) := \varprojlim \mathrm{XE}_G^k(\Gamma).$$

By the Kolmogorov consistency theorem elements of this inverse limit can be viewed as distributions of arrays of random variables $X_i^j$, $1 \leq i \leq n$, $1 \leq j < \infty$. The marginal distribution in which we only look at the variables with $j \leq k$ is in $\mathrm{XE}_G^k(\Gamma)$. Conversely if this is true for all $k$ then the distribution of the $X_i^j$ is an order $\infty$ exchangeable equilibrium.

**Proposition 5.14.** *A game with symmetry group $G$ has an order $\infty$ exchangeable equilibrium.*

*Proof.* This follows from Proposition 5.12 and the fact that the inverse limit of a sequence of nonempty compact Hausdorff spaces is nonempty. □

The power map $\mathrm{pow}^k$ sends $\Delta^\Pi_G(\Gamma) \to \Delta^\Pi_{G\times S_k}(\Pi^k\Gamma)$. Extending by linearity we get a map $\Delta(\Delta^\Pi_G(\Gamma)) \to \Delta^X_{G\times S_k}(\Pi^k\Gamma)$ and these maps are compatible with the marginalization maps in the sense that the diagram

$$\begin{tikzcd}
& \Delta(\Delta^\Pi_G(\Gamma)) \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\cdots \arrow[r] & \Delta^X_{G\times S_3}(\Pi^3\Gamma) \arrow[r] & \Delta^X_{G\times S_2}(\Pi^2\Gamma) \arrow[r] & \Delta^X_G(\Gamma)
\end{tikzcd}$$

commutes. Using De Finetti's Theorem (and an argument similar to the one proving Proposition 6.6 below) one can show that indeed $\Delta(\Delta^\Pi_G(\Gamma)) = \varprojlim \Delta^X_{G\times S_k}(\Pi^k\Gamma)$. Since $\mathrm{XE}^k_G(\Gamma) \subseteq \Delta^X_{G\times S_k}(\Pi^k\Gamma)$, we can pass to inverse limits and view $\mathrm{XE}^\infty_G(\Gamma)$ as a subset of $\Delta(\Delta^\Pi_G(\Gamma))$. In particular it is the subset consisting of those $\pi \in \Delta(\Delta^\Pi_G(\Gamma))$ which map into $\mathrm{XE}^k_G(\Gamma)$ for all $k$. In the next section we compute this subset explicitly when $G$ is player-transitive.

**Proposition 5.15.** *Viewing* $\mathrm{XE}^\infty_G(\Gamma)$ *as a subset of* $\Delta(\Delta^\Pi_G(\Gamma))$*, we have*

$$\Delta(\mathrm{NE}_G(\Gamma)) \subseteq \mathrm{XE}^\infty_G(\Gamma).$$

*Proof.* Let $\pi \in \Delta(\mathrm{NE}_G(\Gamma))$. The map $\Delta(\Delta^\Pi_G(\Gamma)) \to \Delta^X_{G\times S_k}(\Pi^k\Gamma)$ sends $\pi$ to a mixture of Nash equilibria of $\Pi^k\Gamma$ by Propositions 5.7 and 5.8. In particular it maps $\pi$ into $\mathrm{XE}^k_G(\Gamma)$ for all $k$, so $\pi \in \mathrm{XE}^\infty_G(\Gamma)$. □

## 5.5 Nash equilibria from higher order exchangeable equilibria

We have seen that Nash's theorem leads to the existence of order $k$ exchangeable equilibria for finite $k$ which in turn leads to the existence of order $\infty$ exchangeable equilibria. In this section we complete the loop, going from order $\infty$ exchangeable equilibria to Nash equilibria.

### 5.5.1 The player-transitive case

**Theorem 5.16.** *If* $G$ *acts player-transitively on* $\Gamma$*, then* $\Delta(\mathrm{NE}_G(\Gamma)) = \mathrm{XE}^\infty_G(\Gamma)$.

*Proof.* One inclusion is Proposition 5.15. For the converse let $\mathcal{R}$ be a random variable taking values in $\Delta^\Pi_G(\Gamma)$ distributed according to $\pi \in \mathrm{XE}^\infty_G(\Gamma)$. Let $X^j_i$,

$1 \leq i \leq n$, $1 \leq j < \infty$, be random variables taking values in $C_i$ with distribution $\mathcal{R}_i$ which are conditionally independent given $\mathcal{R}$. We must show that $\mathcal{R}_i$ is a best response to $\mathcal{R}_{-i}$ almost surely. We will do this by approximating $\mathcal{R}_i$ and $\mathcal{R}_{-i}$ in terms of the $X_i^j$.

For each $k \in \mathbb{N}$ the distribution of the finite collection of random variables $X_i^j$ with $j \leq k$ is an order $k$ exchangeable equilibrium, so Proposition 5.6 states that for any $1 \leq j \leq k$ the strategy $X_i^j$ is a best response to the random conditional distribution $\text{Prob}(X_{-i}^j \mid X_i^1, \ldots, X_i^k)$ almost surely.

By the last part of Proposition 5.5, $\text{Prob}(X_{-i}^j \mid X_i^1, \ldots, X_i^k) \equiv \text{Prob}(X_{-i}^1 \mid X_i^1, \ldots, X_i^k)$ for all $i$, $j$, and $k$. We define $\mathcal{P}_i^k$ to be this common random conditional distribution. Let $\mathcal{Y}_i^j$ be the random variable taking values in $\Delta(C_i)$ which is the empirical distribution of $X_i^1, \ldots, X_i^j$. Then $\mathcal{Y}_i^j$ is a mixture of best responses to $\mathcal{P}_i^k$ whenever $j \leq k$, hence $\mathcal{Y}_i^j$ is itself a best response. We will show that $\mathcal{Y}_i^j$ and $\mathcal{P}_i^k$ converge to $\mathcal{R}_i$ and $\mathcal{R}_{-i}$, respectively, as $j$ and $k$ go to infinity.

Let $\Sigma_i$ be the completion of the $\sigma$-algebra generated by $X_i^1, X_i^2, \ldots$ and define $\mathcal{P}_i^\infty := \text{Prob}(X_{-i}^1 \mid \Sigma_i)$. Then $\mathcal{P}_i^k \to \mathcal{P}_i^\infty$ almost surely as $k$ goes to infinity (Theorem 10.5.1 in [23]). Therefore $\mathcal{Y}_i^j$ is a best response to $\mathcal{P}_i^\infty$ for all $j$ almost surely. By the strong law of large numbers, $\mathcal{Y}_i^j$ converges almost surely to $\mathcal{R}_i$ as $j$ goes to infinity, so $\mathcal{R}_i$ is a best response to $\mathcal{P}_i^\infty$ almost surely. Furthermore, $\mathcal{R}_i$ is measurable with respect to $\Sigma_i$ because the $\mathcal{Y}_i^j$ are.

The $X_i^j$ are conditionally independent given $\mathcal{R}$, so we have $\mathcal{P}_i^\infty = \mathbb{E}(\text{Prob}(X_{-i}^1 \mid \mathcal{R}) \mid \Sigma_i)$. Since $G$ acts player-transitively, for any player $j$ we have $\mathcal{R}_j = \mathcal{R}_i \cdot g$ for some $g \in G$, hence $\mathcal{R}_j$ is measurable with respect to $\Sigma_i$ and so is $\mathcal{R}$. In particular $\text{Prob}(X_{-i}^1 \mid \mathcal{R})$ is measurable with respect to $\Sigma_i$ and we obtain

$$\mathcal{P}_i^\infty = \mathbb{E}(\text{Prob}(X_{-i}^1 \mid \mathcal{R}) \mid \Sigma_i) = \text{Prob}(X_{-i}^1 \mid \mathcal{R}) = \mathcal{R}_{-i}.$$

This shows that $\mathcal{R}_i$ is a best response to $\mathcal{R}_{-i}$ almost surely for all $i$, so $\mathcal{R} \in \text{NE}_G(\Gamma)$ almost surely and $\pi \in \Delta(\text{NE}_G(\Gamma))$. $\square$

Therefore if $G$ is player transitive we get convergence in (5.1). If $G$ is the trivial group then any correlated equilibrium $\pi \in \text{CE}(\Gamma)$ lifts to $\sum_{s \in C} \pi(s)\delta_{\delta_s} \in \text{XE}_G^\infty(\Gamma)$, so the image of $\text{XE}_G^\infty(\Gamma)$ in $\Delta(\Gamma)$ is $\text{CE}(\Gamma)$ whereas the image of $\Delta(\text{NE}_G(\Gamma))$ is $\text{conv}(\text{NE}(\Gamma))$. These sets are different for some games (e.g., chicken), so the above theorem can fail without the player-transitivity assumption.

We can recover Nash's theorem in the player-transitive case from the above theorem, but we stress that this does not give a new proof of Nash's theorem; rather, it is a proof that the existence of Nash equilibria is equivalent to the existence of higher order exchangeable equilibria.

**Nash's Theorem** (player-transitive case). *A game with player-transitive symmetry group $G$ has a $G$-invariant Nash equilibrium.*

*Proof.* Combine Proposition 5.14 with Theorem 5.16, noting that $\Delta(\emptyset) = \emptyset$. □

### 5.5.2 Arbitrary symmetry groups

In this section we show how to embed an arbitrary game $\Gamma$ with symmetry group $G$ in a game $\Gamma^{\mathrm{Sym}}$ with a player-transitive symmetry group, preserving the existence of $G$-invariant Nash equilibria. This allows us to drop the player-transitivity assumption from the previous section, completing the cycle of implications:

$$\begin{array}{ccc} \mathrm{NE}_G(\Gamma) \neq \emptyset & \Longrightarrow & \mathrm{XE}^k_G(\Gamma) \neq \emptyset \\ \Big\Uparrow & & \Big\Downarrow \\ \mathrm{NE}_G(\Gamma) \neq \emptyset & \Longleftarrow & \mathrm{XE}^\infty_G(\Gamma) \neq \emptyset \\ \text{(player transitive } G\text{)} & & \end{array}$$

There are a variety of ways to symmetrize games. The one we have chosen is a natural $n$-player generalization of von Neumann's tensor-sum symmetrization discussed in [28]. The idea is that each of the $n$ players in $\Gamma^{\mathrm{Sym}}$ plays all the roles of the players in $\Gamma$ simultaneously. The players in $\Gamma^{\mathrm{Sym}}$ play $n!$ copies of $\Gamma$, one for each assignment of players in $\Gamma^{\mathrm{Sym}}$ to roles in $\Gamma$. A player's utility in $\Gamma^{\mathrm{Sym}}$ is the sum of his utilities over the copies.

Viewing the game $\Gamma$ as the actual situation faced by a society, considering $\Gamma^{\mathrm{Sym}}$ amounts to each player forgetting his true role and imagining all the roles he could have taken in the society. Behind this so-called *veil of ignorance* [60], the players face a symmetric situation.

**Definition 5.17.** *Given an $n$-player game $\Gamma$ with strategy sets $C_i$ and utilities $u_i$ we define its **symmetrization** $\Gamma^{\mathrm{Sym}}$ to be the $n$-player game with strategy sets $C_i^{\mathrm{Sym}} := C$ (with typical strategy $s^i = (s^i_1, \dots, s^i_n)$) and utilities*

$$u_i^{\mathrm{Sym}}(s) := \sum_{\tau \in S_n} u_{\tau(i)}\big(d(\tau \star s)\big),$$

*where $s = (s^1, \dots, s^n) \in C^{\mathrm{Sym}} = C^n$, $\star : S_n \times C^{\mathrm{Sym}} \to C^{\mathrm{Sym}}$ is defined by $(\tau \star s)^k := s^{\tau^{-1}(k)}$, and $d : C^{\mathrm{Sym}} \to C$ is defined by $[d(s)]_k := s^k_k$.*

We now show that $\Gamma^{\mathrm{Sym}}$ is a game with player-transitive symmetry group. We will use $\star$ to denote the action on $\Gamma^{\mathrm{Sym}}$ to distinguish it from the action $\cdot$ on $\Gamma$.

**Proposition 5.18.** *If $\Gamma$ is a game with symmetry group $G$ then $\Gamma^{\mathrm{Sym}}$ is a game with player-transitive symmetry group $G \times S_n$, where $\sigma \in S_n$ acts by $\star$ as defined above and $g \in G$ acts by*

$$g \star (s^1, \dots, s^n) \mapsto (g \cdot s^1, \dots, g \cdot s^n).$$

*Proof.* Note that $\star$ defines an action of $G$ on $C^{\mathrm{Sym}}$. Also, for $\sigma, \tau \in S_n$ we have

$$\big(\tau \star (\sigma \star s)\big)^k = (\sigma \star s)^{\tau^{-1}(k)} = s^{\sigma^{-1}(\tau^{-1}(k))} = s^{(\tau\sigma)^{-1}(k)} = \big((\tau\sigma) \star s\big)^k,$$

so $\star$ is an action of $S_n$ on $C^{\mathrm{Sym}}$ as well. These actions commute, so together they define an action $\star$ of $G \times S_n$ on $C^{\mathrm{Sym}}$. Note that the induced actions on players are $g \star i = i$ and $\sigma \star i = \sigma(i)$.

To show that this is an action of $G \times S_n$ on $\Gamma^{\mathrm{Sym}}$ it suffices to show that the utilities of $\Gamma^{\mathrm{Sym}}$ are invariant under the action of any $\sigma \in S_n$ and any $g \in G$. To see the former, let $\sigma \in S_n$. Then we have

$$\begin{aligned} u^{\mathrm{Sym}}_{\sigma\star i}(\sigma \star s) &= \sum_{\tau \in S_n} u_{\tau(\sigma(i))}\Big(d\big(\tau \star (\sigma \star s)\big)\Big) = \sum_{\tau \in S_n} u_{(\tau\sigma)(i)}\Big(d\big((\tau\sigma) \star s\big)\Big) \\ &= \sum_{\tau \in S_n} u_{\tau(i)}\big(d(\tau \star s)\big) = u^{\mathrm{Sym}}_i(s), \end{aligned}$$

where we have used in the penultimate equation the fact that $S_n$ is a group, so the map $\tau \mapsto \tau\sigma$ is a bijection. To see the latter, let $g \in G$ and let $\gamma \in S_n$ be the permutation induced by $g$ on the set of players in $\Gamma$. Then we have $d(g \star s) = g \cdot d(\gamma^{-1} \star s)$, so

$$\begin{aligned} u^{\mathrm{Sym}}_{g\star i}(g \star s) &= \sum_{\tau \in S_n} u_{\tau(i)}\Big(d\big(\tau \star (g \star s)\big)\Big) = \sum_{\tau \in S_n} u_{\tau(i)}\Big(d\big(g \star (\tau \star s)\big)\Big) \\ &= \sum_{\tau \in S_n} u_{\tau(i)}\Big(g \cdot d\big(\gamma^{-1} \star (\tau \star s)\big)\Big) = \sum_{\tau \in S_n} u_{(\gamma^{-1}\tau)(i)}\Big(d\big((\gamma^{-1}\tau) \star s\big)\Big) \\ &= \sum_{\tau \in S_n} u_{\tau(i)}\big(d(\tau \star s)\big) = u^{\mathrm{Sym}}_i(s), \end{aligned}$$

where the fourth equation follows because $g$ is a symmetry of $\Gamma$. Clearly $S_n$ acts transitively on the set of players. □

**Nash's Theorem.** *A game with symmetry group $G$ has a $G$-invariant Nash equilibrium.*

*Proof.* Let $\Gamma$ be a game with symmetry group $G$. Then $\Gamma^{\mathrm{Sym}}$ is a game with player-transitive symmetry group $G \times S_n$ by Proposition 5.18, so it has a $(G \times S_n)$-symmetric Nash equilibrium by the player-transitive version of Nash's Theorem.

By definition of the action of $G \times S_n$ on $\Gamma^{\text{Sym}}$, this Nash equilibrium is of the form $(\rho, \dots, \rho)$, with $\rho \in \Delta_G(\Gamma)$. Notice that for each player $i$, each utility $u_k^{\text{Sym}}(s^1, \dots, s^n)$ is a sum of functions which only depend on $s_j^i$ for a single value of $j$. Thus $\rho$ is payoff equivalent to the product of its marginals $\rho_1 \times \cdots \times \rho_n \in \Delta_G^{\Pi}(\Gamma)$. Therefore we can take the Nash equilibrium $(\rho, \dots, \rho)$ to be such that $\rho \in \Delta_G^{\Pi}(\Gamma)$ by Proposition 2.15.

It remains to verify that $\rho \in \text{NE}_G(\Gamma)$. For any $s^i \in C$ we can compute

$$\begin{aligned} u_i^{\text{Sym}}(\rho, \dots, \rho, s^i, \rho, \dots, \rho) &= \sum_{\tau \in S_n} u_{\tau(i)} \left(\rho_1, \dots, \rho_{\tau(i)-1}, s^i_{\tau(i)}, \rho_{\tau(i)+1}, \dots, \rho_n\right) \\ &= (n-1)! \sum_{j=1}^{n} u_j(\rho_1, \dots, \rho_{j-1}, s_j^i, \rho_{j+1}, \dots, \rho_n). \end{aligned}$$

For each value of $j$ we can vary the $s_j^i$ component of $s^i$ independently and it is a best response for player $i$ to play $\rho$ in $\Gamma^{\text{Sym}}$ if the rest of the players play $\rho$, so we must have

$$u_j(\rho_1, \dots, \rho_{j-1}, s_j, \rho_{j+1}, \dots, \rho_n) \leq u_j(\rho)$$

for all players $j$ and all $s_j \in C_j$, i.e., $\rho \in \text{NE}_G(\Gamma)$. □

Chapter 6

# Asymmetric Exchangeable Equilibria

In this chapter we take up the question of what an exchangeable equilibrium of a game which is not (necessarily) symmetric should be. Throughout we will use the term **asymmetric** as shorthand to mean "not necessarily symmetric," so for example asymmetric games will include symmetric games and asymmetric exchangeable equilibria will include symmetric exchangeable equilibria.

To simplify notation, we restrict attention to a bimatrix game $\Gamma$ throughout. The row player chooses a strategy in $C_1 = \{1, \ldots, r\}$ and the column player chooses a strategy in $C_2 = \{1, \ldots, c\}$. The matrices $A$ and $B$ denote the utilities of the row and column players, respectively.

Before we begin let us recall what an exchangeable equilibrium of $\Gamma$ should *not* be. As shown in Example 3.2, $\text{conv}\{xy^T \mid x \in \mathbb{R}^r_{\geq 0}, y \in \mathbb{R}^c_{\geq 0}\} = \mathbb{R}^{r \times c}_{\geq 0}$. Therefore $\text{CE}(\Gamma) \cap \text{conv}\{xy^T \mid x, y \geq 0\} = \text{CE}(\Gamma)$ does not yield an interesting definition of asymmetric exchangeable equilibrium.

## 6.1 Partial exchangeability

We will generalize the notion of exchangeable equilibrium to bimatrix games by analogy with the sealed envelope implementation in the symmetric case. There we had an infinite number of indistinguishable envelopes from which the players could choose, each containing an element from their common strategy space. Now it is natural to have two pools of envelopes, one containing elements of $C_1$ and the other containing elements of $C_2$. The two types of envelopes will be distinguishable (perhaps each has $C_1$ or $C_2$ written on the outside), but the infinitely many instances of each will not be distinguishable from each other. What does exchangeability mean in this context?

We model the contents of the envelopes as an infinite sequence of random variables $R_1, S_1, R_2, S_2, \ldots$ with the $R_i$ taking values in $C_1$ and the $S_i$ taking values in $C_2$. Indistinguishability means that we can tell from looking at an envelope if its contents are an $R_i$ or an $S_i$, but not the value of $i$. In particular $R_1$

has no more connection with $S_1$ than it does with $S_{37}$; the index $i$ is meaningless. That is to say, of the following two definitions the first more closely captures what we mean by exchangeability of two pools of random variables; at times we will also need the second, weaker definition. These both fall under the general heading of **partially exchangeable** random variables [19] (this subject is much broader and the terms below are not standard, just convenient for the present discussion).

**Definition 6.1.** *We say that a sequence $R_1, S_1, R_2, S_2, \ldots$ of two types of random variables is **individually exchangeable** if the distribution is invariant under permuting finitely many $R_i$ or finitely many $S_i$. That is to say, whenever $g : \mathbb{N} \to \mathbb{N}$ is bijective and fixes all but finitely many numbers then the distributions of*

$$\begin{aligned} &R_1, S_1, R_2, S_2, \ldots, \\ &R_{g(1)}, S_1, R_{g(2)}, S_2, \ldots, \\ &R_1, S_{g(1)}, R_2, S_{g(2)} \ldots \end{aligned}$$

*are all the same. If the distribution of $R_{g(1)}, S_{g(1)}, R_{g(2)}, S_{g(2)}, \ldots$ is the same as that of $R_1, S_1, R_2, S_2, \ldots$ for all such $g$ then we say the sequence is **simultaneously exchangeable**.*

**Example 6.2.** If the sequence $R_1, S_1, R_2, S_2, \ldots$ is simultaneously exchangeable then the subsequence $R_1, S_2, R_3, S_4, \ldots$ is individually exchangeable. Indeed, permuting the odd-numbered pairs in the first sequence permutes the $R_{2i-1}$ in the second while fixing the $S_{2i}$, and permuting the even-numbered pairs does the opposite. ♢

**Example 6.3.** An example of an individually exchangeable sequence is a sequence of independent random variables with all the $R_i$ sharing a distribution and all the $S_i$ sharing a distribution, which may be different from that of the $R_i$. In other words, the pairs $(R_i, S_i)$ are i.i.d. according to some distribution in $\Delta^{\Pi}(\Gamma)$. For simultaneously exchangeable random variables an example is when the pairs $(R_i, S_i)$ are i.i.d. according to some distribution in $\Delta(\Gamma)$. The sets of both types of exchangeable distributions are convex (invariance under a given permutation is a linear condition), so mixtures of such distributions give examples without independence. ♢

Applying the definition twice, observe that if $R_1, S_1, R_2, S_2, \ldots$ is individually exchangeable we may permute the $R_i$ and $S_i$ in two different ways without affecting the distribution. In particular we may choose to permute them in the same way, so the sequence is simultaneously exchangeable. Simultaneous exchangeability is the same as exchangeability of the sequence of pairs $(R_1, S_1), (R_2, S_2), \ldots$ in the usual sense of Section 2.2. This in turn implies that the sequences $R_1, R_2, \ldots$ and $S_1, S_2, \ldots$ are both exchangeable. The converse statements are false.

**Example 6.4.** A simultaneously exchangeable sequence need not be individually exchangeable. Suppose $C_1 = C_2 = \{1,2\}$ and the $R_i$ are i.i.d. uniform with $S_i \equiv R_i$ for all $i$. Then the pairs $(R_1, S_1), (R_2, S_2), \ldots$ are i.i.d. so this sequence is simultaneously exchangeable. Note that in a individually exchangeable sequence, the joint distribution of $R_1$ and $S_1$ is the same as the joint distribution of $R_1$ and $S_2$. However in this case the joint distribution of $R_1$ and $S_1$ is $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$, while the joint distribution of $R_1$ and $S_2$ is $\begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$. Therefore this sequence is not individually exchangeable. $\diamondsuit$

**Example 6.5.** The sequences $R_1, R_2, \ldots$ and $S_1, S_2, \ldots$ can both be exchangeable without $R_1, S_1, R_2, S_2, \ldots$ being simultaneously exchangeable. For example take the $R_i$ to be i.i.d. and $S_i \equiv R_{i+1}$. Then the joint distribution of $R_1, S_1, R_2, S_2$ is not the same as that of $R_2, S_2, R_1, S_1$ because $S_1 = R_2$ almost surely but $S_2 \neq R_1$ with positive probability. Therefore the sequence $R_1, S_1, R_2, S_2, \ldots$ is not simultaneously exchangeable. $\diamondsuit$

Let us see what De Finetti's theorem tells us about these two new notions of exchangeability. First suppose that we have a simultaneously exchangeable sequence. We can view this as arising by first choosing a random distribution $P$ in $\Delta(\Gamma)$ according to some distribution in $\Delta(\Delta(\Gamma))$. We then choose pairs $(R_i, S_i)$ i.i.d. according to $P$. The empirical distribution of $(R_1, S_1), \ldots, (R_n, S_n)$ must converge almost surely to $P$ as $n$ goes to infinity by the strong law of large numbers.

Now suppose the sequence is individually exchangeable. If we swap $R_1$ and $R_2$, the empirical distributions will change by at most $\mathcal{O}\left(\frac{1}{n}\right)$, so almost surely their limit $P$ will not change. Therefore conditioned on $P$ we have that $(R_1, S_1)$ and $(R_2, S_2)$ are i.i.d. according to $P$, but also $(R_2, S_1)$ and $(R_1, S_2)$ are i.i.d. according to $P$. In particular, conditioned on $P$, $R_1$ and $S_1$ are independent. That is to say, $P \in \Delta^\Pi(\Gamma)$ almost surely. In summary, we have proven that the two types of exchangeable random variables in Definition 6.1 are always mixtures of the independent distributions in Example 6.3:

**Proposition 6.6.** *Just as the set of exchangeable distributions taking values in $C_1$ can be naturally identified with $\Delta(\Delta(C_1))$, the set of simultaneously exchangeable distributions can be identified with $\Delta(\Delta(\Gamma))$ and the set of individually exchangeable distributions can be identified with $\Delta(\Delta^\Pi(\Gamma))$.*

We used the corresponding result for exchangeable distributions, De Finetti's Theorem, to show that the marginal distribution of $n$ variables in an exchangeable sequence takes the form $\sum_k \lambda_k x_k^{\otimes n}$ for a probability vector $\lambda$ and $x_k \in \Delta(C_1)$ (Propositions 2.45 and 2.46). The same proof works in this context.

**Proposition 6.7.** *The joint distribution of $R_1, S_1, \ldots, R_n, S_n$ of a truncated individually exchangeable sequence or simultaneously exchangeable sequence takes the form*

$$\sum_k \lambda_k \, (x_k \otimes y_k)^{\otimes n} \qquad \text{or} \qquad \sum_k \lambda_k \, W_k^{\otimes n},$$

*respectively, for a probability vector $\lambda$, $x_k \in \Delta(C_1)$, $y_k \in \Delta(C_2)$, and $W_k \in \Delta(\Gamma)$. Conversely any such distribution is the truncation of an exchangeable sequence of the corresponding type.*

As Example 6.2 might suggest, if we are only interested in the marginals corresponding terms in the sequence with different indices then the conditions we obtain are the same for simultaneously and individually exchangeable sequences. The following instance of this principle will be important in the next section.

**Proposition 6.8.** *Let $R_1, S_1, R_2, S_2, \ldots$ be a* simultaneously *exchangeable sequence (so in particular it could be individually exchangeable) with marginal distributions $W \in \Delta(\Gamma)$ of $R_1$ and $S_2$, $X \in \Delta_{S_2}(C_1^2)$ of $R_1$ and $R_2$, and $Y \in \Delta_{S_2}(C_2^2)$ of $S_1$ and $S_2$. Then*

$$\begin{aligned} &\begin{bmatrix} X & W \\ W^T & Y \end{bmatrix} \text{ is completely positive,} \\ &\begin{bmatrix} X & W \\ W^T & Y \end{bmatrix} \begin{bmatrix} e \\ -e \end{bmatrix} = 0, \text{ and} \\ \begin{bmatrix} e \\ e \end{bmatrix}^T &\begin{bmatrix} X & W \\ W^T & Y \end{bmatrix} \begin{bmatrix} e \\ e \end{bmatrix} = 4. \end{aligned} \tag{6.1}$$

*Conversely, any $X$, $W$, and $Y$ for which the matrix $\left[\begin{smallmatrix} X & W \\ W^T & Y \end{smallmatrix}\right]$ has these properties arise as the marginals of an* individually *exchangeable sequence.*

*Proof.* First suppose $R_1, S_1, R_2, S_2, \ldots$ is simultaneously exchangeable. By Proposition 6.7 the joint distribution of $R_1$, $S_1$, $R_2$, and $S_2$ is of the form $\sum_k \lambda_k \, W_k \otimes W_k$. Let $x_k = W_k e$ and $y_k = W_k^T e$ be the marginals of $W_k$. Then $X = \sum_k \lambda_k \, x_k \otimes x_k$, $Y = \sum_k \lambda_k \, y_k \otimes y_k$, and $W = \sum_k \lambda_k \, x_k \otimes y_k$, so altogether

$$\begin{bmatrix} X & W \\ W^T & Y \end{bmatrix} = \sum_k \lambda_k \begin{bmatrix} x_k \\ y_k \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}^T$$

is a completely positive matrix. Since the marginal distributions of $X$, $W$, and $Y$ agree this matrix also has $\left[\begin{smallmatrix} e \\ -e \end{smallmatrix}\right]$ in its kernel. The matrix contains four probability distributions, so its elements sum to 4.

Conversely suppose that $\left[\begin{smallmatrix} X & W \\ W^T & Y \end{smallmatrix}\right]$ satisfies the conditions. By the usual scaling technique, we can write

$$\begin{bmatrix} X & W \\ W^T & Y \end{bmatrix} = \sum_k \lambda_k b_k b_k^T,$$

where each $b_k \in \mathbb{R}^{r+c}$ is a nonnegative column vector whose components sum to 2. This scaling makes the elements of $b_k b_k^T$ sum to 4, so $\lambda$ is a probability vector. Dropping any zero terms, we may assume without loss of generality that all components of $\lambda$ are strictly positive. Then

$$\begin{aligned} 0 = \begin{bmatrix} e \\ -e \end{bmatrix}^T 0 = \begin{bmatrix} e \\ -e \end{bmatrix}^T \begin{bmatrix} X & W \\ W^T & Y \end{bmatrix} \begin{bmatrix} e \\ -e \end{bmatrix} = \sum_k \lambda_k \begin{bmatrix} e \\ -e \end{bmatrix}^T b_k b_k^T \begin{bmatrix} e \\ -e \end{bmatrix} \\ = \sum_k \lambda_k \left\| b_k^T \begin{bmatrix} e \\ -e \end{bmatrix} \right\|^2, \end{aligned}$$

so $b_k^T \left[\begin{smallmatrix} e \\ -e \end{smallmatrix}\right] = 0$ for all $k$. Therefore we can write $b_k = \left[\begin{smallmatrix} x_k \\ y_k \end{smallmatrix}\right]$ for some nonnegative vectors $x_k \in \mathbb{R}^r$ and $y_k \in \mathbb{R}^c$ with $x_k^T e = y_k^T e$ and $x_k^T e + y_k^T e = 2$. That is to say, $x_k \in \Delta(C_1)$ and $y_k \in \Delta(C_2)$. If we choose a random $k$ according to the distribution $\lambda$ and conditioned on that choose $R_i$ and $S_i$ all independently with distributions $x_k$ and $y_k$, respectively, the result is an individually exchangeable sequence $R_1, S_1, R_2, S_2, \ldots$ whose appropriate marginals are $X$, $W$, and $Y$. □

**Example 6.9.** If we relax complete positivity in (6.1) to elementwise nonnegativity, the resulting conditions are necessary for there to exist four random variables $R_1$, $S_1$, $R_2$, and $S_2$ whose appropriate marginals are $X$, $W$, and $Y$. In the absence of complete positivity these conditions are not sufficient. The matrix

$$\begin{bmatrix} X & W \\ W^T & Y \end{bmatrix} := \left[\begin{array}{cc|cc} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ \hline 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \end{array}\right]$$

cannot correspond to any such distribution on these four random variables, because this would mean $S_2 = R_1 = R_2 = S_1 \neq S_2$ almost surely, a contradiction. ◇

## 6.2 Defining asymmetric exchangeable equilibria

For a natural extension of the sealed envelope implementation of symmetric exchangeable equilibria to the asymmetric case, let us suppose there is an individually exchangeable sequence of random variables $R_1, S_1, R_2, S_2, \dots$ with the $R_i$ taking values in $C_1$ and the $S_i$ taking values in $C_2$. Each player is allowed to choose one envelope and base his play on its contents. To define asymmetric exchangeable equilibria, we will ask that it be a Nash equilibrium of this extended game for player $i$ to choose one of the envelopes containing a recommendation from $C_i$ and play its contents. We may call the distribution of $R_1, S_1, R_2, S_2, \dots$ an "$\infty$-exchangeable equilibrium". As in the symmetric case such an infinite-dimensional object is useful conceptually for interpretations, but for concreteness and computational purposes we define exchangeable equilibrium as the finitely-describable counterpart of this.

If player 1 chooses $R_i$ and player 2 chooses $S_j$ then the joint distribution $W \in \Delta(\Gamma)$ of their recommendations does not depend on $i$ and $j$ by the individual exchangeability assumption. For concreteness we will suppose player 1 chooses $R_1$ and player 2 chooses $S_2$. We consider only unilateral deviations: if player 1 decides to instead take envelope $S_1$ then the joint distribution of recommendations will be $Y \in \Delta_{S_2}(C_2^2)$; if player 2 switches to $R_2$ the joint distribution of recommendations will be $X \in \Delta_{S_2}(C_1^2)$. The matrices $X$, $W$, and $Y$ satisfy (6.1) by Proposition 6.8, from which we see that the sequence $R_1, S_1, R_2, S_2, \dots$ could just as well have been taken to be simultaneously exchangeable if we added the extra constraint that the players' choices of envelopes could not have the same index.

All the incentive constraints for an exchangeable equilibrium can be defined in terms of $X$, $W$, and $Y$. We want it to be in both players' best interests to choose the envelope with their label and play its contents, so they cannot improve by deviating to any function of its contents or any function of the contents of the alternative envelope they could have chosen. In symbols the conditions are:

$$\begin{aligned}
\sum_{ij} u_1(f(i), j) W_{ij} &\leq \sum_{ij} u_1(i, j) W_{ij} \text{ for all } f : C_1 \to C_1, \\
\sum_{j_1 j_2} u_1(f(j_1), j_2) Y_{ij} &\leq \sum_{ij} u_1(i, j) W_{ij} \text{ for all } f : C_2 \to C_1, \\
\sum_{ij} u_2(i, f(j)) W_{ij} &\leq \sum_{ij} u_2(i, j) W_{ij} \text{ for all } f : C_2 \to C_2, \\
\sum_{i_1 i_2} u_2(i_1, f(i_2)) X_{ij} &\leq \sum_{ij} u_2(i, j) W_{ij} \text{ for all } f : C_1 \to C_2,
\end{aligned} \tag{6.2}$$

where indices $i$ run over $C_1$ and indices $j$ run over $C_2$. Note that two of these are exactly the conditions for $W$ to be a correlated equilibrium. The other two relate $W$ to $X$ and $Y$. Just as the original Definition 2.5 of correlated equilibrium had exponentially many inequalities and we replaced these with polynomially many in Proposition 2.6, we will see in the next section that the conditions (6.2) can be expressed with polynomially many inequalities.

Putting these things together:

**Definition 6.10.** *The triple $(X, W, Y)$ is an* ***(asymmetric) exchangeable equilibrium*** *of the bimatrix game $\Gamma$ if it satisfies* (6.1) *and* (6.2). *The set of such is denoted* $\mathrm{XE}_-(\Gamma)$.

The dash subscript on $\mathrm{XE}_-(\Gamma)$ serves two purposes: to emphasize that no symmetries of $\Gamma$ have been used in the construction and to make it clear that the symmetric exchangeable equilibria are not the same as the symmetric elements of the set of asymmetric exchangeable equilibria (Example 6.14 below). This definition has many properties in common with the symmetric notion (cf. Proposition 3.9):

**Proposition 6.11.** *The set of exchangeable equilibria of a bimatrix game is convex, compact, and semialgebraic. The map $(X, W, Y) \mapsto W$ sends exchangeable equilibria to correlated equilibria. The map $(x, y) \mapsto (xx^T, xy^T, yy^T)$ sends Nash equilibria to exchangeable equilibria. The images of Nash equilibria are (among the) extreme points of* $\mathrm{XE}_-(\Gamma)$.

*Proof.* The first two sentences follow from the discussion above. If $(x, y)$ is a Nash equilibrium then $\left[\begin{smallmatrix} X & W \\ W^T & Y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^T$ is completely positive and satisfies the other conditions of (6.1) because $x$ and $y$ are probability vectors. The correlated equilibrium conditions in (6.2) follow as usual from the definition of Nash equilibrium. The other incentive constraints hold automatically because $X$ and $Y$ are independent distributions, so the players cannot take advantage of these alternative signals in any way. As for extremality, $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^T$ is an extreme point of the larger set of $(r + c) \times (r + c)$ completely positive matrices with entries summing to 4 (Proposition 2.49), and so must also be extreme in $\mathrm{XE}_-(\Gamma)$. $\square$

**Example 6.12.** Correlated equilibria can achieve higher social welfare $W \bullet (A+B)$ than asymmetric exchangeable equilibria. Let $\Gamma$ be the symmetric bimatrix game "Chicken" with utilities

$$A = B^T = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}.$$

We first compute the unique correlated equilibrium which maximizes social welfare. Since the game is symmetric the set of correlated equilibria is invariant under swapping players. The social welfare is also invariant under this action, so the maximum social welfare is achieved by a symmetric correlated equilibrium $W = \left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right]$. The correlated equilibrium incentive constraints are $b \geq a, c$. The social welfare is $8a + 12b$ and normalization requires $a + 2b + c = 1$. We can always move mass from $c$ to $b$ while remaining a correlated equilibrium and increasing social welfare, so an optimal symmetric correlated equilibrium has $c = 0$. The social welfare is then $2a+6$, which is maximized by making $a$ as large as possible: $a = b = \frac{1}{3}$.

Thus $W^* = \left[\begin{smallmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{smallmatrix}\right]$ is a correlated equilibrium of maximum social welfare and the only such among symmetric correlated equilibria. Any not necessarily symmetric correlated equilibrium $\hat{W}$ of maximum social welfare has $\frac{1}{2}\hat{W} + \frac{1}{2}\hat{W}^T = W^*$. In other words it is of the form $\hat{W} = \left[\begin{smallmatrix} 1/3 & 1/3+p \\ 1/3-p & 0 \end{smallmatrix}\right]$ for some $p \in [-1/3, 1/3]$. The only $p$ such that $\hat{W}$ is a correlated equilibrium is $p = 0$. Therefore $W^*$ is the unique correlated equilibrium with maximum social welfare.

If there were an exchangeable equilibrium $(X, W, Y)$ achieving this maximum social welfare, $W$ would in particular be a correlated equilibrium, making $W = W^*$. By symmetry $(Y, W^{*T}, X)$ would also be an exchangeable equilibrium achieving the same social welfare, so by averaging there would be an asymmetric exchangeable equilibrium of the form $(X, W^*, X)$. We will show that there is no $X = \left[\begin{smallmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{smallmatrix}\right]$ such that $(X, W^*, X)$ is an exchangeable equilibrium.

The marginalization conditions give

$$\begin{aligned} X_{11} + X_{12} &= W^*_{11} + W^*_{12} = \frac{2}{3} \\ X_{12} + X_{22} &= W^*_{21} + W^*_{22} = \frac{1}{3}. \end{aligned}$$

Putting these together we get $X_{11} = X_{22} + \frac{1}{3}$. Suppose for a contradiction that $X_{22} > X_{12}$. If the column player were to choose an envelope from his opponent's pile and play the opposite of its contents, his expected utility would be

$$\begin{aligned} \sum_{i_1 i_2} u_2(i_1, f(i_2))X_{ij} &= 4X_{12} + X_{22} + 5X_{11} \\ &> 5X_{12} + 5X_{11} = 5(X_{12} + X_{11}) = \frac{10}{3} = \sum_{ij} u_2(i,j)W^*_{ij}, \end{aligned}$$

contradicting the incentive conditions. Therefore $X_{22} \leq X_{12}$.

Since $X$ is positive semidefinite, $X_{22} = 0$ would mean $X_{12} = 0$, contradicting the fact that these numbers sum to $\frac{1}{3}$. So $X_{22} > 0$. The lower right $3 \times 3$ minor of $\left[\begin{smallmatrix} X & W^* \\ W^* & X \end{smallmatrix}\right]$ must be nonnegative and is equal to $X_{22}(X_{11}X_{22} - X_{12}^2) - \frac{1}{9}X_{22}$. Dividing by $X_{22}$ we obtain

$$\frac{1}{9} \leq X_{11}X_{22} - X_{12}^2 = \left(X_{22} + \frac{1}{3}\right)X_{22} - X_{12}^2 = X_{22}^2 + \frac{1}{3}X_{22} - X_{12}^2 \leq \frac{1}{3}X_{22} \leq \frac{1}{3}X_{12},$$

so $X_{12} \geq X_{22} \geq \frac{1}{3}$. This contradicts $X_{12} + X_{22} = \frac{1}{3}$.

Therefore there is no such asymmetric exchangeable equilibrium achieving the maximum social welfare achieved by correlated equilibria. In fact MATLAB computes that in this case the maximum social welfare over asymmetric exchangeable equilibria is achieved by the two pure Nash equilibria. $\diamondsuit$

We now look at connections with the notion of symmetric exchangeable equilibria.

**Proposition 6.13.** *Let* $\Gamma$ *be a symmetric bimatrix game. Then* $W \mapsto (W, W, W)$ *maps* $\mathrm{XE}_{S_2}(\Gamma)$ *into* $\mathrm{XE}_{-}(\Gamma)$. *Conversely if* $(W, W, W) \in \mathrm{XE}_{-}(\Gamma)$ *then* $W \in \mathrm{XE}_{S_2}(\Gamma)$.

*Proof.* If $W$ is completely positive we can write $W = \sum_k w_k w_k^T$, in which case

$$\begin{bmatrix} W & W \\ W & W \end{bmatrix} = \sum_k \lambda_k \begin{bmatrix} w_k \\ w_k \end{bmatrix} \begin{bmatrix} w_k \\ w_k \end{bmatrix}^T$$

is also completely positive and satisfies the other constraints of (6.1). This corresponds to the situation when $R_1, S_1, R_2, S_2, \ldots$ is exchangeable in the usual sense. In particular no player can gain by choosing a different envelope because the joint distribution between his envelope and his opponents will be the same regardless. Since $W$ is a correlated equilibrium, all the conditions of (6.2) are met.

For the converse note that if $(W, W, W) \in \mathrm{XE}_{-}(\Gamma)$ then $W$ is a correlated equilibrium by (6.2). Also $\left[\begin{smallmatrix} W & W \\ W & W \end{smallmatrix}\right]$ is completely positive and the corresponding factorization thereof restricts to a factorization of its upper left block $W$, so it too is completely positive. $\square$

**Example 6.14.** If $\Gamma$ is a symmetric bimatrix game, it can happen that $W$ is symmetric and $(X, W, Y) \in \mathrm{XE}_{-}(\Gamma)$, but $W \notin \mathrm{XE}_{S_2}(\Gamma)$. For example, let $\Gamma$ be the anti-coordination game in Example 3.19. Let $e_1$ and $e_2$ denote the standard unit column vectors in $\mathbb{R}^2$. Then $(e_1, e_2)$ and $(e_2, e_1)$ are Nash equilibria of $\Gamma$, so

$$(e_1 e_1^T, e_1 e_2^T, e_2 e_2^T), (e_2 e_2^T, e_2 e_1^T, e_1 e_1^T) \in \mathrm{XE}_{-}(\Gamma)$$

by Proposition 6.11. Since $\mathrm{XE}_{-}(\Gamma)$ is convex, the average of these two is an exchangeable equilibrium with $W = \left[\begin{smallmatrix} 0 & 1/2 \\ 1/2 & 0 \end{smallmatrix}\right]$, which is not completely positive, so not a symmetric exchangeable equilibrium. $\diamondsuit$

Another connection with symmetric exchangeable equilibria comes from passing from the asymmetric bimatrix game $\Gamma$ to its symmetrization $\Gamma^{\mathrm{Sym}}$. Recall from Section 5.5.2 that $\Gamma^{\mathrm{Sym}}$ is a symmetric bimatrix game in which two copies of $\Gamma$ are played simultaneously, with each player in $\Gamma^{\mathrm{Sym}}$ taking different roles in the two copies. Therefore each player's strategy space is $C_i^{\mathrm{Sym}} = C$. A symmetric exchangeable equilibrium of $\Gamma^{\mathrm{Sym}}$ is therefore a joint distribution over random variables $R_1, S_1, R_2, S_2$ which factors as $\sum_k \lambda_k W_k \otimes W_k$, and hence by Proposition 6.7 a truncation of a simultaneously exchangeable sequence $R_1, S_1, R_2, S_2, \ldots$. Player $i$ receives the pair $(R_i, S_i)$ and the incentive condition is that using all this information he cannot do better than playing $R_i$ and $S_i$ in the appropriate copies of $\Gamma$.

These are stronger incentive constraints than those imposed in asymmetric exchangeable equilibria: If $R_1$ is a best response against $S_2$ even given the extra information $S_1$, then $R_1$ is still a best response without this extra information, and it is also at least as good as any response formulated in terms of $S_1$ only without $R_1$. On the other hand the exchangeability constraints are weaker; only simultaneous exchangeability rather than individual. However by Proposition 6.8 the relevant marginals still satisfy the conditions (6.1), and therefore they are an asymmetric exchangeable equilibrium. In other words, we can trade the stronger incentive constraints for the stronger exchangeability property. This proves:

**Proposition 6.15.** *For distributions $\pi \in \Delta(C^2)$ over random variables $R_1$, $S_1$, $R_2$, and $S_2$, the triple marginalization map*

$$\pi \mapsto (\mu_{R_1R_2}(\pi), \mu_{R_1S_2}(\pi), \mu_{S_1S_2}(\pi))$$

*maps* $\mathrm{XE}_{S_2}(\Gamma^{\mathrm{Sym}})$ *into* $\mathrm{XE}_{-}(\Gamma)$.

Combining this with Theorem 3.16 we get:

**Theorem 6.16.** *A bimatrix game has an asymmetric exchangeable equilibrium.*

**Example 6.17.** Not all asymmetric exchangeable equilibria arise in this way from symmetric exchangeable equilibria of the symmetrization. Consider the bimatrix game $\Gamma$ shown in Table 6.1. The matrix

$$\begin{aligned}
\begin{bmatrix} X & W \\ W^T & Y \end{bmatrix} &:= \frac{1}{55}\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 30 & 5 & 20 & 15 & 0 \\ 0 & 5 & 15 & 10 & 0 & 10 \\ \hline 0 & 20 & 10 & 24 & 6 & 0 \\ 0 & 15 & 0 & 6 & 9 & 0 \\ 0 & 0 & 10 & 0 & 0 & 10 \end{array}\right] \\
&= \frac{1}{11}\left[\begin{array}{c} 0 \\ 1 \\ 1 \\ \hline 2 \\ 0 \\ 0 \end{array}\right]\left[\begin{array}{c} 0 \\ 1 \\ 1 \\ \hline 2 \\ 0 \\ 0 \end{array}\right]^T + \frac{1}{55}\left[\begin{array}{c} 0 \\ 5 \\ 0 \\ \hline 2 \\ 3 \\ 0 \end{array}\right]\left[\begin{array}{c} 0 \\ 5 \\ 0 \\ \hline 2 \\ 3 \\ 0 \end{array}\right]^T + \frac{2}{11}\left[\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \\ 1 \end{array}\right]\left[\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \\ 1 \end{array}\right]^T
\end{aligned} \tag{6.3}$$

is completely positive and one can check that it satisfies the incentive conditions so $(X, W, Y) \in \mathrm{XE}_{-}(\Gamma)$.

Let

$$\pi := \frac{1}{55}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 5 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix}0\\1\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}^T + \frac{1}{55}\begin{bmatrix}0\\2\\0\\0\\3\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\2\\0\\0\\3\\0\\0\\0\\0\end{bmatrix}^T + \frac{2}{11}\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}^T \in \Delta^X_{S_2}(\Gamma^{\mathrm{Sym}}),$$

where the rows and columns are indexed by strategies in $\Gamma^{\mathrm{Sym}}$ as in Table 6.1. Then $(X, W, Y)$ is the image of $\pi$ under the triple marginalization map of Proposition 6.15. Note that $\pi$ is not a correlated equilibrium of $\Gamma^{\mathrm{Sym}}$: if a player is told to play $cf$, he believes his opponent will play $cf$, in which case he can do strictly better by deviating to $af$.

In fact $\pi$ is the only such lift of $(X, W, Y)$ to $\Delta^X_{S_2}(\Gamma^{\mathrm{Sym}})$. Let $\pi'$ be any lift. The entries of $X$, $W$, and $Y$ are sums of certain entries of $\pi'$, so zeros in $(X, W, Y)$ force the corresponding summands to be zero. This forces $\pi'$ to have zeros everywhere that $\pi$ does except at $\pi'_{bd,ce} = \pi'_{ce,bd}$. Since $\pi'$ must be completely positive and the diagonal entry $\pi'_{ce,ce} = 0$, Proposition 2.50 then yields $\pi'_{bd,ce} = \pi'_{ce,bd} = 0$ as well. With these zeros in place, the marginalization conditions fix the remaining elements so $\pi' = \pi$.

This means any exchangeable lift of $(X, W, Y)$ is not a correlated equilibrium, so the asymmetric exchangeable equilibrium $(X, W, Y)$ of $\Gamma$ does not lift to a symmetric exchangeable equilibrium of $\Gamma^{\mathrm{Sym}}$. Approximations of the sets of equilibrium payoffs illustrating this are shown in Figure 6.1. $\diamondsuit$

$\Gamma$:

| $(u_1, u_2)$ | $d$ | $e$ | $f$ |
|---|---|---|---|
| $a$ | $(1,1)$ | $(6,8)$ | $(9,5)$ |
| $b$ | $(2,4)$ | $(9,5)$ | $(3,2)$ |
| $c$ | $(3,5)$ | $(5,0)$ | $(8,9)$ |

$\Gamma^{\text{Sym}}$:

| $(u_1, u_2)$ | $ad$ | $bd$ | $cd$ | $ae$ | $be$ | $ce$ | $af$ | $bf$ | $cf$ |
|---|---|---|---|---|---|---|---|---|---|
| $ad$ | $(2,2)$ | $(5,3)$ | $(6,4)$ | $(7,9)$ | $(10,10)$ | $(11,11)$ | $(10,6)$ | $(13,7)$ | $(14,8)$ |
| $bd$ | $(3,5)$ | $(6,6)$ | $(7,7)$ | $(10,6)$ | $(13,7)$ | $(14,8)$ | $(4,3)$ | $(7,4)$ | $(8,5)$ |
| $cd$ | $(4,6)$ | $(7,7)$ | $(8,8)$ | $(6,1)$ | $(9,2)$ | $(10,3)$ | $(9,10)$ | $(12,11)$ | $(13,12)$ |
| $ae$ | $(9,7)$ | $(6,10)$ | $(1,6)$ | $(14,14)$ | $(11,17)$ | $(6,13)$ | $(17,11)$ | $(14,14)$ | $(9,10)$ |
| $be$ | $(10,10)$ | $(7,13)$ | $(2,9)$ | $(17,11)$ | $(14,14)$ | $(9,10)$ | $(11,8)$ | $(8,11)$ | $(3,7)$ |
| $ce$ | $(11,11)$ | $(8,14)$ | $(3,10)$ | $(13,6)$ | $(10,9)$ | $(5,5)$ | $(16,15)$ | $(13,18)$ | $(8,14)$ |
| $af$ | $(6,10)$ | $(3,4)$ | $(10,9)$ | $(11,17)$ | $(8,11)$ | $(15,16)$ | $(14,14)$ | $(11,8)$ | $(18,13)$ |
| $bf$ | $(7,13)$ | $(4,7)$ | $(11,12)$ | $(14,14)$ | $(11,8)$ | $(18,13)$ | $(8,11)$ | $(5,5)$ | $(12,10)$ |
| $cf$ | $(8,14)$ | $(5,8)$ | $(12,13)$ | $(10,9)$ | $(7,3)$ | $(14,8)$ | $(13,18)$ | $(10,12)$ | $(17,17)$ |

**Table 6.1.** Top: the game $\Gamma$. Bottom: the symmetrization $\Gamma^{\text{Sym}}$. The game $\Gamma$ has asymmetric exchangeable equilibria not lifting to symmetric exchangeable equilibria of $\Gamma^{\text{Sym}}$.

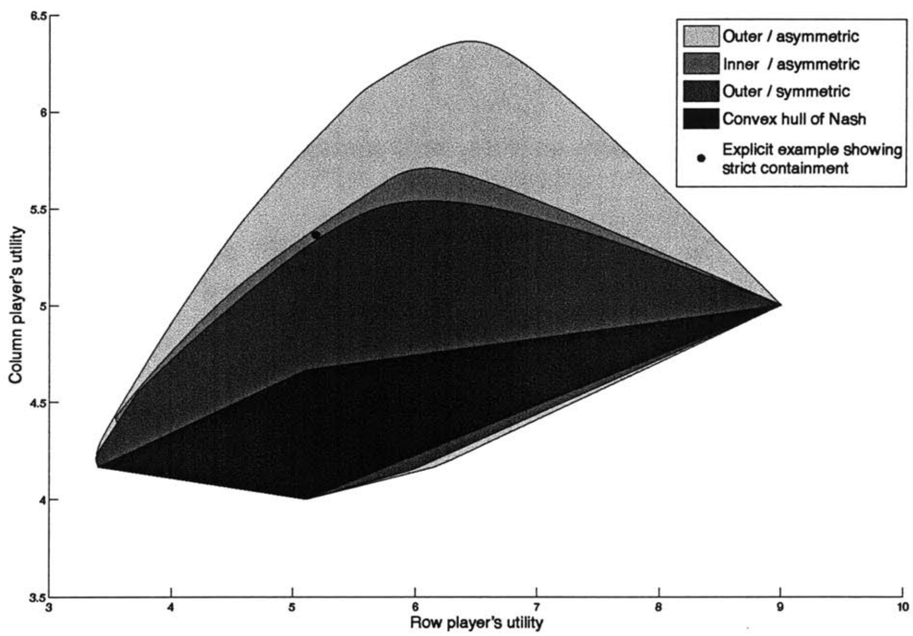


**Figure 6.1.** Inner and outer approximations to the sets of utilities achievable by exchangeable equilibria of the games $\Gamma$ and $\Gamma^{\mathrm{Sym}}$ shown in Table 6.1. The point singled out corresponds to the asymmetric exchangeable equilibrium $(X, W, Y)$ shown not to lift to a symmetric exchangeable equilibrium of the symmerization. Note how this point lies inside the inner approximation to $\mathrm{XE}_{-}(\Gamma)$ but outside the outer approximation to $\mathrm{XE}_{S_2}(\Gamma^{\mathrm{Sym}})$. This gives a computational proof of the lack of a lift.

## 6.3 Convex relaxations of Nash equilibria

Just as in the symmetric case (Section 3.4), the exchangeable equilibrium conditions can be derived mechanically as convex relaxations of Nash equilibria. We again begin with the complementarity characterization of Nash equilibria, now for general bimatrix games. Define the auxiliary matrix

$$P := \begin{bmatrix} e & 0 & 0 & -A \\ 0 & e & -B^T & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \in \mathbb{R}^{2(r+c)\times(2+r+c)},$$

where $e$, 0, and $I$ denote column vectors, zero matrices, and identity matrices of appropriate sizes.

**Proposition 6.18.** *The pair $(x, y)$ of vectors $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^c$ is a Nash equilibrium if and only if these vectors are normalized and there exist $u, v \in \mathbb{R}$ such that*

$$z := P \begin{bmatrix} u \\ v \\ x \\ y \end{bmatrix} = \begin{bmatrix} ue - Ay \\ ve - B^T x \\ x \\ y \end{bmatrix} \in \mathbb{R}^{2(r+c)}$$

*is elementwise nonnegative and complementary in the sense that $z_i z_{i+r+c} = 0$ for $1 \le i \le r + c$.*

Let

$$h := \begin{bmatrix} u \\ v \\ x \\ y \end{bmatrix} \in \mathbb{R}^{2+r+c}$$

and

$$f := \begin{bmatrix} 0 \\ 0 \\ e \\ -e \end{bmatrix} \in \mathbb{R}^{2+r+c},$$

with blocks of equal size. Note that $f^T h = e^T x - e^T y = 1 - 1 = 0$. This is an extra linear constraint which we did not have in the symmetric case, but which we will use in the relaxations. If we express this proposition in terms of $H = hh^T$ we get:

**Proposition 6.19.** *The matrix $W \in \Delta(\Gamma)$ is a Nash equilibrium if and only if the other entries of the symmetric matrix*

$$H := \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \epsilon & \zeta & \eta \\ \gamma^T & \zeta^T & X & W \\ \delta^T & \eta^T & W^T & Y \end{bmatrix}$$

*can be filled in so that $Hf = 0$ and $Z := PHP^T$ (see Figure 6.2) satisfies:*

- *$Z$ has rank 1,*
- *$Z \geq 0$ elementwise, and*
- *$Z_{i,i+r+c} = 0$ for $1 \leq i \leq r + c$.*

Having already covered the symmetric game versions of the relaxations, the easiest to extend to this context is the generalization of Theorem 3.32:

**Theorem 6.20.** *The set* $\operatorname{conv}(\mathrm{NE}(\Gamma))$ *is obtained by relaxing the rank* 1 *condition in Proposition 6.19 to complete positivity of $Z$.*

We omit the proof, which is almost identical to that of Theorem 3.32. The only change is that now once we show that $H = \sum h_i h_i^T$ and we wish to show that $H_i = h_i h_i^T$, when suitably scaled, satisfies the conditions of Proposition 6.19, we must also show that $h_i^T f = 0$. But this follows immediately from

$$0 = f^T 0 = f^T H f = \sum_i (h_i^T f)^2.$$

Before approaching the other two relaxations, let us unpack the conditions in Proposition 6.19 a bit, starting with the meaning of $X$ and $Y$. To fit into $H$, the matrix $X$ must be $r \times r$ and symmetric; similarly $Y$ must be $c \times c$ and symmetric. The lower right quadrant of $Z$ (a $2 \times 2$ block of blocks) is equal to $\left[\begin{smallmatrix} X & W \\ W^T & Y \end{smallmatrix}\right]$, so nonnegativity of this means $X$ and $Y$ must be elementwise nonnegative. The condition $Hf = 0$ in particular means that $Xe = We$ and $e^T W = e^T Y$. That is to say, $X \in \Delta_{S_2}(C_1^2)$ and $Y \in \Delta_{S_2}(C_2^2)$ are probability distributions and furthermore, their marginals agree with the two marginals of the (asymmetric) probability matrix $W$.

The other two relaxations we consider will consist of dropping the rank constraint entirely and replacing it with the condition that $\left[\begin{smallmatrix} X & W \\ W^T & Y \end{smallmatrix}\right]$ be completely positive. In both cases we can make the entries in the upper left quadrant of $Z$ (see Figure 6.2) nonnegative by making $\alpha$, $\beta$, and $\epsilon$ large enough. No other

$$A, B, W \in \mathbb{R}^{r\times c}, \qquad \alpha, \beta, \epsilon \in \mathbb{R}, \qquad \gamma, \zeta \in \mathbb{R}^{1\times r}, \qquad \delta, \eta \in \mathbb{R}^{1\times c}, \qquad X \in \mathbb{R}^{r\times r}, \qquad Y \in \mathbb{R}^{c\times c},$$

$$H := \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \epsilon & \zeta & \eta \\ \gamma^T & \zeta^T & X & W \\ \delta^T & \eta^T & W^T & Y \end{bmatrix} \in \mathbb{R}^{(2+r+c)\times(2+r+c)}, \qquad f := \begin{bmatrix} 0 \\ 0 \\ e \\ -e \end{bmatrix} \in \mathbb{R}^{2+r+c}, \qquad Hf = \begin{bmatrix} \gamma e - \delta e \\ \zeta e - \eta e \\ Xe - We \\ W^T e - Ye \end{bmatrix},$$

$$P := \begin{bmatrix} e & 0 & 0 & -A \\ 0 & e & -B^T & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \in \mathbb{R}^{2(r+c)\times(2+r+c)},$$

$$Z := PHP^T = \begin{bmatrix} \alpha ee^T - A\delta^T e^T - e\delta A^T + AYA^T & \beta ee^T - A\eta^T e^T - e\gamma B + AW^T B & e\gamma - AW^T & e\delta - AY \\ \beta ee^T - B^T\gamma^T e^T - e\eta A^T + B^T WA^T & \epsilon ee^T - B^T\zeta^T e^T - e\zeta B + B^T XB & e\zeta - B^T X & e\eta - B^T W \\ \gamma^T e^T - WA^T & \zeta^T e^T - XB & X & W \\ \delta^T e^T - YA^T & \eta^T e^T - W^T B & W^T & Y \end{bmatrix}.$$

**Figure 6.2.** Matrices for relaxations of Nash equilibria of bimatrix games.

constraints are placed on these variables, so we can remove them entirely along with any consideration of the upper left quadrant of $Z$. The remaining constraints we have to understand are:

$$\begin{aligned}
&e\gamma \geq AW^T \text{ with equality on the diagonal,}\\
&e\eta \geq B^T W \text{ with equality on the diagonal,}\\
&e\delta \geq AY,\\
&e\zeta \geq B^T X,\\
&\gamma e = \delta e,\\
&\zeta e = \eta e,
\end{aligned}$$

where the first four come from the nonnegativity and complementarity constraints on $Z$ and the last two come from $Hf = 0$.

The same analysis as in the proof of Proposition 3.30 (which shows that in the symmetric case dropping the rank condition yields symmetric correlated equilibria) shows that the existence of $\gamma$ and $\eta$ satisfying the first two constraints listed above are exactly the conditions for $W$ to be a correlated equilibrium of $\Gamma$.

These conditions also imply $\gamma e = \operatorname{Tr}(AW^T) = \sum_{ij} A_{ij}W_{ij}$ is the expected utility for the row player under $W$, so the condition $\gamma e = \delta e$ reduces to $\sum_j \delta_j = \sum_{ij} A_{ij}W_{ij}$ and we can eliminate $\gamma$ once we constrain $W$ to be a correlated equilibrium. The constraint $e\delta \geq AY$ means that $\delta_j \geq [AY]_{ij}$ for all $1 \leq i \leq r$ and $1 \leq j \leq c$, or equivalently, $\delta_j \geq \max_i [AY]_{ij}$ for all $j$. These constraints can simultaneously be satisfied for all $j$ within the budget $\sum_j \delta_j = \sum_{ij} A_{ij}W_{ij}$ if and only if

$$\sum_j \max_i [AY]_{ij} \leq \sum_{ij} A_{ij}W_{ij}.$$

This in turn is equivalent to the condition that

$$\sum_{j_1 j_2} u_1(f(j_1), j_2) Y_{j_1 j_2} \leq \sum_{ij} u_1(i,j) W_{ij}$$

for all functions $f : C_2 \to C_1$. This is one of the constraints from (6.2).

Rewriting the constraints in this way allows us to eliminate $\delta$. We can similarly eliminate $\zeta$ and $\eta$ and replace them with the condition that

$$\sum_{i_1 i_2} u_2(i_1, f(i_2)) X_{i_1,i_2} \leq \sum_{ij} u_2(i,j) W_{ij}$$

for all $f : C_1 \to C_2$, the last of the conditions in (6.2). Going backwards, the exponentially many incentive constraints (6.2) can be rewritten using only polynomially many linear inequalities by introducing the auxiliary variables $\delta$ and $\zeta$ and constrainting these to satisfy the elementwise nonnegativity and complementarity constraints in Proposition 6.19. We have shown:

**Proposition 6.21.** *The matrix* $\left[\begin{smallmatrix} X & W \\ W^T & Y \end{smallmatrix}\right]$ *is completely positive and there exist values of the Greek variables satisfying all the conditions of Proposition 6.19 except the rank condition if and only if* $(X, W, Y)$ *is an asymmetric exchangeable equilibrium.*

This is indeed a relaxation because $Z$ being elementwise nonnegative and having rank one automatically implies that $\left[\begin{smallmatrix} X & W \\ W^T & Y \end{smallmatrix}\right]$ is completely positive. As expected removing the rank condition entirely yields the correlated equilibria, analogously to Proposition 3.30 in the symmetric case.

**Proposition 6.22.** *The matrix* $W \in \Delta(\Gamma)$ *is a correlated equilibrium of* $\Gamma$ *if and only if there exist values for the remaining variables satisfying all the conditions of Proposition 6.19 except the rank condition.*

*Proof.* We have dealt with the Greek variables already. For any $W$, we can set $X = Wee^TW^T$ and $Y = W^Tee^TW$. This corresponds to distributing $R_1$ and $S_2$ according to $W$ and independently distributing $R_2$ and $S_1$ according to $W$. In particular this means that $S_1$ is never informative about $S_2$ and $R_2$ is never informative about $R_1$, so it is in neither player's best interest to choose these – the associated conditions are satisfied automatically. The only conditions which remain are those which say that $W$ is a correlated equilibrium. $\square$

Chapter 7

# Computation of Symmetric Exchangeable Equilibria

This chapter is divided into two sections. In the first we address questions of theoretical computational complexity. We show that, appropriately formulated, the problem of computing or approximating a single exchangeable equilibrium of a game can be solved in polynomial time. We focus on symmetric exchangeable equilibria, but note that we can use the same algorithm to approximate asymmetric exchangeable equilibria of bimatrix games efficiently by computing symmetric exchangeable equilibria of the symmetrization (this only yields a polynomial blowup) and applying the natural approximate version of Proposition 6.15. The computational complexity of the corresponding problem for higher order exchangeable equilibria is not known. We close the first section by showing that optimizing a linear functional over (approximate) exchangeable equilibria is $NP$-hard, even in the symmetric bimatrix case.

In the second section we discuss linear and semidefinite relaxations as practical approaches to these problems. These relaxations can be solved efficiently in practice but do not yield a priori guarantees about approximation error. Unlike the provably efficient methods, these extend immediately to higher order exchangeable equilibria.

## 7.1 Computational complexity

In their seminal paper [55], Papadimitriou and Roughgarden show how to convert the Hart-Schmeidler argument from an existence proof for correlated equilibria into an efficient algorithm for computing correlated equilibria in large games (polynomially many players, exponentially many strategy profiles – for smaller games any linear programming algorithm is sufficient). Since the Hart-Schmeidler argument can be modified to preserve symmetry and prove the existence of exchangeable equilibria (Theorem 3.16) it is natural to ask whether Papadimitriou and Roughgarden's "Ellipsoid Against Hope" algorithm can be similarly modified to compute exchangeable equilibria.

Mathematically the same modification seems to apply (being ellipsoid-based, these algorithms are entirely theoretical and impractical, so it is difficult to actually implement it). However, there is a problem. The ellipsoid method runs for finitely many steps and uses rational[1] arithmetic, so produces rational output. On the other hand a rational exchangeable equilibrium need not exist, as in Example 3.27.

In this section we review the Ellipsoid Against Hope algorithm, note how it can be symmetrized to compute exchangeable equilibria, and then address this paradox by isolating a flaw in the treatment of numerical precision issues in [55]. By introducing explicit bounds on the variables, we show how to resolve the problem and efficiently compute approximate correlated and exchangeable equilibria to an arbitrary degree of accuracy. In the case of exchangeable equilibria, this is the best we could hope for (at least in rational numbers) because of Example 3.27. Finally, we show that optimizing a linear functional over the (approximate) exchangeable equilibria is NP-hard.

### 7.1.1 Background

**Definitions from [55]** As much as possible we will build on the notation of [55]; we offer a brief summary here and refer the reader to that paper for more details. The primary exceptions are our use of $\delta$ for rounding resolution where Papadimitriou and Roughgarden use $\epsilon$ and our use of $\|U\|_\infty$ in place of $u$. In this chapter we reserve $\epsilon$ for talking about $\epsilon$-correlated equilibria, defined below, and $u$ is reserved for utilities throughout the thesis. For this chapter we will assume further that utilities are integer-valued.

Define $m := \max_i |C_i|$ and let $N := |C| \le m^n$ be the number of strategy profiles. Since $N$ is exponential in $n$ we must avoid writing down utility functions in full as $n$ grows. To do so we consider **succinct games of polynomial type with the polynomial expectation property** as defined in [55], or just **succinct games** for short.

The condition that a probability distribution $x$ is a correlated equilibrium can be expressed with one homogeneous linear constraint for each player $i$ and pair of strategies $s_i, t_i \in C_i$ which says it is not profitable to play $t_i$ when $s_i$ is recommended. This yields a total of $M \le nm^2$ linear inequalities in $x$. We write these as $Ux \ge 0$, where $U \in \mathbb{R}^{M \times N}$ is a matrix defined in terms of the utilities. The magnitude $\|U\|_\infty$ of the largest entry of $U$ is at most twice the absolute value of the largest utility of the game.

[1]We use "rational" in the algebraic sense in this chapter, rather than the game- or decision-theoretic sense. A vector, matrix, or tensor is rational if its components are.

A correlated equilibrium is a tensor with exponentially many components, so we do not expect to be able to write it out in full. We are interested in games with a **polynomial correlated equilibrium scheme**, i.e. an efficient way of computing and expressing a correlated equilibrium so we can sample from it repeatedly in randomized polynomial time.

The type of polynomial correlated equilibrium scheme we consider consists of computing a convex combination of polynomially many product distributions (with all numbers rational and of polynomial description length) which is a correlated equilibrium. Note that a product distribution over the strategy spaces can be expressed using $\sum_{i=1}^{n} |C_i| \leq mn$ numbers. We can sample from such a joint distribution without computing all $N$ probabilities explicitly using a two stage process. Viewing the weights in the convex combination as defining a probability distribution over these product distributions, we randomly choose a product distribution. Then we sample from the chosen product by sampling from each component of the product independently.

**Further definitions** An **$\epsilon$-correlated equilibrium** is a weakening of a correlated equilibrium in which no player can expect to improve his utility by more than $\epsilon$ by deviating from his recommendation. A probability distribution $x$ is an $\epsilon$-correlated equilibrium if and only if

$$\epsilon \geq \max_i \sum_{s_i \in C_i} \max_{t_i \in C_i} (-[Ux]_{s_i,t_i}).$$

The strategy $t_i$ at which the inner maximum is achieved for each $s_i$ defines a function $t_i : C_i \to C_i$. The best player $i$ can do in response to recommendation $s_i$ is to play $t_i(s_i)$, in which case his expected utility gain is the sum above.

When discussing computation of $\epsilon$-correlated equilibria, "the input" will refer to the description of a (succinct) game instance as well as $\epsilon$ written in binary. To say something can be computed in polynomial time then means that the run time is polynomial jointly in the description length of the game and $\log \frac{1}{\epsilon}$. Relaxing correlated equilibrium to $\epsilon$-correlated equilibrium and adopting this new notion of input, we obtain the notion of a **polynomial $\epsilon$-correlated equilibrium scheme**.

An **$\epsilon$-exchangeable equilibrium** of a symmetric game with symmetry group $G$ is an $\epsilon$-correlated equilibrium which is exchangeable. To talk about computing such things, we must specify how the symmetric game is to be represented as input. Each element of $G$ acts by permuting the elements of $\bigsqcup C_i$ and such a permutation can be written in a number of bits polynomial in the size of the game. One of our main cases of interest is when $G = S_n$ acts by permuting the players, but this group has $n!$ elements so requiring them all to be listed as part of the input is unreasonable. Rather, we will assume the input contains a list of generators of the group $G$ identified with specific permutations of $\bigsqcup C_i$. This solves the problem: $S_n$ is generated by polynomially many transpositions.

We will assume the game is specified succinctly as usual, and satisfies the polynomial expectation property. We will assume further that the utilities are *promised* to be invariant under the given generators, and therefore under $G$. With this representation for symmetric games, we define a **polynomial $\epsilon$-exchangeable equilibrium scheme** in the natural way.

## 7.1.2 The Ellipsoid Against Hope algorithm

To set up for the algorithm we rewrite Hart and Schmeidler's minimax proof of existence of correlated equilibria, reviewed in Section 2.1.2, in the language of linear programming, as done in Section 3.1 of [55].

First, we write linear constraints expressing that $x$, a vector of variables indexed by strategy profiles (actually a tensor which we view as being "flattened" into one long vector), is a correlated equilibrium of the input game. The constraints come in three types: incentive constraints which say that no player can improve by deviating from his recommendation, nonnegativity constraints on the variables, and a normalization constraint making $x$ a probability measure. All the constraints except normalization are homogeneous (have zero constant term).

Second, we remove the normalization constraint and replace it with an objective of maximizing the sum of the elements of $x$, forming the primal linear program

$$\begin{aligned} \underset{x \in \mathbb{R}^C}{\text{maximize}} \quad & \sum_{s \in \prod C_i} x_s \\ \text{subject to} \quad & Ux \geq 0 \\ & x \geq 0. \end{aligned} \tag{$P$}$$

The zero vector is always feasible and any nonzero feasible solution can be normalized to give a correlated equilibrium. Therefore $(P)$ is unbounded if and only if a correlated equilibrium exists, and finding one amounts to computing a nonzero feasible solution.

Third, we dualize $(P)$ to produce the linear program[2]

$$\begin{aligned} \underset{y \in \mathbb{R}^{\sqcup_i C_i \times C_i}}{\text{minimize}} \quad & 0 \\ \text{subject to} \quad & U^T y \leq -e \\ & y \geq 0, \end{aligned} \tag{$D$}$$

[2]Strictly speaking this is what one gets by replacing all variables in the usual dual (see [6]) with their negatives. We will use this convention for all duals in the present chapter to agree with the notation of [55].

with the goal of showing $(D)$ is infeasible. By strong linear programming duality this would suffice to prove $(P)$ unbounded. To prove $(D)$ infeasible we must show that any given $y \geq 0$ violates $U^T y \leq -e$.

Rather than exhibiting a particular violated constraint, we give a convex combination of these constraints which is not satisfied. The Hart-Schmeidler argument shows that for any $y \geq 0$ one can find an explicit probability distribution $x$ over outcomes of the game (viewed for now as a row vector) such that $xU^T y = 0$. Furthermore, $x$ can be taken to be a product distribution, i.e., $x_s = x^1_{s_1} \cdots x^n_{s_n}$. The vector $y$ of dual variables splits into a vector $y^i$ for each player $i$ dualizing that player's incentive constraints. The factor $x^i$ of the product can be any steady-state distribution of a certain Markov chain defined only in terms of $y^i$, so we can take $x^i$ to be rational in $y^i$. We refer to such a choice $x(y)$ as a **Hart-Schmeidler separation oracle.**

To turn this into an algorithm, Papadimitriou and Roughgarden employ the ellipsoid method, going through the above existence proof in reverse. They run the ellipsoid algorithm on $(D)$ using a Hart-Schmeidler separation oracle ($x$ can be computed efficiently by solving a linear program of polynomial size), which necessarily concludes that the problem is infeasible. Such a run of the ellipsoid algorithm produces a sequence of cuts of the form $xU^T y \leq -1$ for various values of $x$. These cuts may be taken directly from the oracle, or may be rounded first to avoid exponential blowup of the number of bits required after polynomially many iterations[3]. (In fact the $-1$ on the right hand side must be relaxed slightly when rounding, but this will not be important here.)

Let $L$ be the number of iterations for which the ellipsoid algorithm runs on $(D)$. Stacking the row vectors $x$ corresponding to these $L$ cuts into a matrix $X \in \mathbb{R}^{L \times N}$, we can write a modified dual

$$\begin{aligned} \underset{y \in \mathbb{R}^{\sqcup C_i \times C_i}}{\text{minimize}} \quad & 0 \\ \text{subject to} \quad & XU^T y \leq -e \\ & y \geq 0, \end{aligned} \tag{$D'$}$$

The claim is that $(D')$ is infeasible (we will see below why this is incorrect), so

[3]In a response to a preprint of this chapter Jiang and Leyton-Brown [41] have shown how to efficiently go from such an $x$ to an explicit violated inequality of the linear program, avoiding such rounding issues. They can thus compute an exact correlated equilibrium of a large game efficiently, but their procedure breaks symmetry so cannot be used for our main goal of computing exchangeable equilibria.

the modified primal

$$\begin{array}{ll} \underset{\alpha\in\mathbb{R}^L}{\text{maximize}} & \sum_{k=1}^{L} \alpha_k \\ \text{subject to} & UX^T\alpha \geq 0 \\ & \alpha \geq 0 \end{array} \tag{$P'$}$$

with variables $\alpha_k$ is unbounded. This implies that a correlated equilibrium can then be computed by solving $(P')$ for any nonzero solution, say by the ellipsoid algorithm again.

### 7.1.3 Paradox

**Proposition 7.1.** *Given a symmetric game as input, the Ellipsoid Against Hope algorithm claims to compute a rational exchangeable equilibrium.*

*Proof.* Suppose an iteration of the ellipsoid algorithm on the dual $(D)$ corresponding to a symmetric $n$-player game begins with a symmetric ellipsoid, i.e., one invariant under the induced action of $G$ on the dual variables. The center point of the ellipsoid is also symmetric. Therefore the same method for extending the Hart-Schmeidler argument to prove existence of exchangeable equilibria in Theorem 3.16 applies here and we may take $x$ to be a symmetric product distribution[4].

If we use a symmetric cut on a symmetric ellipsoid, symmetry will not be broken and the ellipsoid for the next iteration will also be symmetric. Thus if we start with a symmetric ellipsoid, we can take the cut $x$ to be a symmetric product distribution at each iteration. The algorithm then outputs a convex combination of symmetric product distributions. The ellipsoid algorithm for linear programming uses rational arithmetic and a Hart-Schmeidler oracle gives rational output on rational input, so the end result is rational. □

This contradicts Example 3.27, which shows that such an equilibrium need not exist.

### 7.1.4 Resolution

In this section we pinpoint the issue with the analysis of the Ellipsoid Against Hope algorithm in [55]. While the contradiction derived above depends on the parameters of the algorithm being chosen in a symmetric fashion, in this section we do not need such assumptions.

[4]Jiang and Leyton-Brown's [41] derandomization procedure breaks this symmetry, thereby avoiding Proposition 7.1 and the associated algebraic obstruction of Example 3.27.

To see why the algorithm fails, we must recall some details of the ellipsoid algorithm for determining infeasibility of a system of linear inequalities. The input is specified by giving coefficients and right hand sides, assumed integral. Perturbing the problem in the standard way, we can assume it is either infeasible or strictly feasible.

Using a bound in terms of problem data, we can find an ellipsoid $E$ large enough that if there were a feasible solution there would be one in $E$. We can also compute a small volume $v$, so that if the problem were feasible then the feasible set would have volume at least $v$ inside $E$.

The algorithm then runs, shrinking the volume of the ellipsoid at each iteration. Assuming the problem is infeasible, the ellipsoid will eventually have volume less than $v$. At this point we know that the problem was infeasible, so the algorithm can terminate.

If we run this algorithm on the dual problem $(D)$, which we know is infeasible, the algorithm will of course tell us this. The problem lies in claiming that the modified dual $(D')$ will also be infeasible.

The rationale given in [55] for this statement is that the run of the ellipsoid algorithm on $(D)$ is a valid run of the ellipsoid algorithm on $(D')$. This is true insofar as the cuts made at each iteration are valid for $(D')$. But the initial ellipsoid and volume $v$ need not be valid choices for $(D')$. So this run of the ellipsoid algorithm tells us that the intersection of $E$ and the feasible set of $(D')$ has small volume. But we cannot conclude from this that $(D')$ is infeasible. Depending on the size of the coefficients in $(D')$ (after clearing denominators to make them integer) such a conclusion may require a much larger starting ellipsoid $E$ and a much smaller final volume $v$ .

One can try to patch up this problem by starting with a bigger ellipsoid $E$ and smaller volume $v$ before running the ellipsoid algorithm on $(D)$, hoping that the these parameters will be good enough to ensure that the run of the ellipsoid algorithm is valid for $(D')$ as well. But the corresponding run of the ellipsoid algorithm may produce even larger coefficients in $(D')$.

To try to get around this, one can also round the constraints of $(D')$ so that the coefficients do not get so large as to invalidate the conclusion of the ellipsoid algorithm. But the larger the starting ellipsoid, the more delicate the rounding must be to try to avoid making the problem feasible, as shown below. It turns out that there is no combination of initial ellipsoid and rounding scheme which can ensure that $(D')$ will be infeasible. Thus the modified primal linear program $(P')$ need not be unbounded, so we cannot necessarily compute a correlated equilibrium by solving it.

For example, suppose we try to tweak the parameters of Section 3.3 in [55] ("The Issue of Arithmetic Precision"). Recall that $\|U\|_\infty$ denotes the magnitude of the largest coefficient in $(D)$ (all coefficients are assumed integer), and $n, M, N \geq 2$ are other parameters depending on the dimensions of the problem. At each iteration we round the elements of the $x_i$ for each cut to multiples of some[5] $\delta = \frac{1}{K}$, $K \in \mathbb{N}$.

Let $R$ be the radius of the ball with which we start the ellipsoid algorithm. To be able to certify infeasibility of $(D)$, we must take $R$ to be large; in [55] the bound $R \geq \|U\|_\infty^N$ is used. With the unspecified $R$ in place of $\|U\|_\infty^N$ the analysis in [55] states[6] that to avoid introducing errors when rounding, we should take $\delta \leq (4nMNR\|U\|_\infty)^{-1}$, so

$$\delta R \leq (4nMN\|U\|_\infty)^{-1} < 1. \tag{7.1}$$

The size of the largest coefficient in $(D')$ will still be at most $\|U\|_\infty$, being convex combinations of the coefficients in $(D)$, but these coefficients are no longer integer.

The bound on $R$ needed to ensure that the ellipsoid algorithm correctly determines infeasibility assumes that the coefficients are integer. The coefficients of $(D')$ have common denominator $\delta^{-n} = K^n$, so at least if we clear denominators in the naïve fashion the largest coefficient of the equivalent integer version of $(D')$ will be of order $\|U\|_\infty \delta^{-n}$. Using the same type of bound on $R$ as before to ensure that the ellipsoid algorithm works correctly on $(D')$, we obtain $R \geq (\|U\|_\infty \delta^{-n})^N$, so

$$\begin{aligned}\delta R &\geq \|U\|_\infty^N \delta^{1-nN} \geq \|U\|_\infty^N (4nMNR\|U\|_\infty)^{nN-1} \\ &\geq \|U\|_\infty^N (4nMN\|U\|_\infty^{N+1})^{nN-1} \gg 1,\end{aligned} \tag{7.2}$$

contradicting (7.1). Thus we cannot choose $\delta$ and $R$ so the run of the ellipsoid algorithm on $(D)$ guarantees infeasibility of $(D')$.

**Example 7.2** (continues Example 3.27). Let us see how this problem manifests itself when the Ellipsoid Against Hope algorithm is applied to the game in Example 3.27. As shown in the proof of Proposition 7.1 we can take the $x$ for each cut to be a rational symmetric product distribution. We will show that regardless of the sequence of rational symmetric cuts, the modified dual $(D')$ is feasible.

It suffices to show that there exists a symmetric feasible solution, i.e., one with $y^1 = y^2 = y^3$. Since each player has two strategies, each $y^i$ has four components, one representing each possible choice of deviation from a strategy for player $i$ to another strategy for player $i$. The two components representing trivial deviations (from a strategy to itself) have coefficient zero in all dual constraints, so we can drop these two dual variables as irrelevant.

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[5] We use $\delta$ where [55] uses $\epsilon$ to avoid confusion with the $\epsilon$ in "$\epsilon$-correlated equilibrium".

[6] The error analysis in [55] seems to include an extra factor of 2 so that the 4 in the expression for $\delta$ could safely be replaced with a 2, but that will not make a significant difference here.

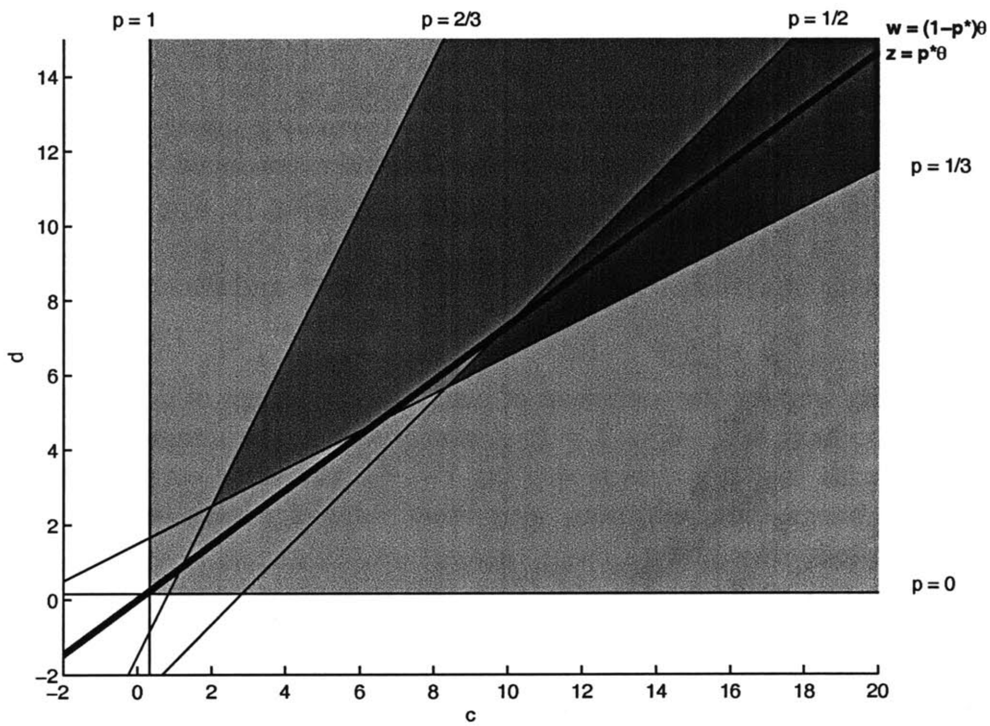


**Figure 7.1.** The feasible region of the modified dual $(D')$ as more cuts are added. The three successively darker regions correspond to the sets of constraints $p \in \{0, 1\}$, $p \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, and $p \in \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$, respectively. As more cuts are added, the feasible region shrinks off to infinity in the northeast direction. However, any finite number of cuts with $p$ values chosen from $[0, 1] \setminus \{1 - \frac{1}{\sqrt{3}}\}$ are jointly feasible, containing all points sufficiently far along on the bold ray.

This leaves two variables for the symmetrized dual:

$$\begin{array}{rl} \text{(constraint for strategy profile with } s_1+s_2+s_3=0) & 1 \le 3c \\ \text{(constraint for strategy profiles with } s_1+s_2+s_3=1) & 1 \le 2c-d \\ \text{(constraint for strategy profiles with } s_1+s_2+s_3=2) & 1 \le -2c-2d \\ \text{(constraint for strategy profile with } s_1+s_2+s_3=3) & 1 \le 6d, \end{array} \tag{SymD}$$

where $c$ represents the likelihood of a player to deviate from strategy 0 to strategy 1 and $d$ represents the reverse. Note that the first and last constraint imply that $c, d > 0$, while the third implies $c + d < 0$, so (SymD) is infeasible as expected.

Now let us compute the modified dual constraint produced by taking a convex combination of these constraints weighted according to a symmetric product distribution of the form $\left[\begin{smallmatrix} p \\ 1-p \end{smallmatrix}\right]^{\otimes 3}$ for $0 \le p \le 1$. Multiplying the constraint corresponding to $s_1 + s_2 + s_3 = k$ by $\binom{3}{k} p^k (1-p)^{3-k}$ and summing over $k$ yields

$$1 \le -3p(3p^2 - 6p + 2)c + 3(1-p)(3p^2 - 6p + 2)d.$$

The only way for the coefficient of $c$ and the coefficient of $d$ to simultaneously vanish is to have $3p^2 - 6p + 2 = 0$. Unsurprisingly, the unique solution to this equation with $0 \le p \le 1$ is $p = p^* := 1 - \frac{1}{\sqrt{3}}$, the value corresponding to the unique exchangeable equilibrium. Any other value of $p$ leads to a nondegenerate linear inequality in $c$ and $d$.

Let $c = (1 - p^*)\theta$ and $d = p^*\theta$. Then this inequality becomes

$$1 \le 3\big[(1-p)p^* - p(1-p^*)\big](3p^2 - 6p + 2)\theta.$$

The bracketed term has the same sign as $3p^2 - 6p + 2$ within the interval $[0, 1]$, so the coefficient of $\theta$ is positive for all $p \in [0,1] \setminus \{p^*\}$. We can simultaneously satisfy any finite number of such inequalities arising from values $p \in [0,1] \setminus \{p^*\}$ by making $\theta$ large enough. In particular, as long as we use cuts for which $x$ is a rational symmetric product distribution, the linear program $(D')$ will be feasible. For an example see Figure 7.1. ◇

In the rounded version of $(D')$ discussed in Section 3.3 of [55], the weaker constant term $\frac{-e}{2}$ is used in $(D')$ instead of $-e$ when $X$ is rounded. In the present section we considered arbitrary rational $X$, so we can view these as coming from some rounding scheme if we wish. This means that the conclusion that $(D')$ is feasible also applies to the rounded version with weaker constant term.

### 7.1.5 Approximate Ellipsoid Against Hope

We can use an algorithm similar to the Ellipsoid Against Hope algorithm to compute an $\epsilon$-correlated equilibrium of a succinctly-representable game in time jointly polynomial in the description length of the game and $\log \frac{1}{\epsilon}$. The corrected version of Theorem 4.1 from [55] (expanded to include exchangeable equilibria) is:

**Theorem 7.3.** *If $\Gamma$ is a succinct game of polynomial type and has the polynomial expectation property, then it has a polynomial $\epsilon$-correlated equilibrium scheme. If $\Gamma$ is symmetric then it has a polynomial $\epsilon$-exchangeable equilibrium scheme.*

The goal of this section is to prove this theorem. The idea is as follows. While we cannot be sure that $(D')$ is infeasible, we can guarantee that all feasible solutions are large. This means that $(P')$ is almost unbounded, in the sense that there is a normalized solution which only violates the constraints by a small amount. This is an $\epsilon$-correlated equilibrium.

**Algorithm** Assume $\epsilon > 0$ is specified by giving an integer $R = \frac{1}{\epsilon}$ in binary as part of the input. Form the bounded dual

$$\begin{aligned} \underset{y \in \mathbb{R}^{\sqcup C_i \times C_i}}{\text{minimize}} \quad & 0 \\ \text{subject to} \quad & U^T y \leq -e \\ & Re \geq y \geq 0, \end{aligned} \tag{$D_R$}$$

and run the ellipsoid algorithm on it, starting from the ball centered at zero of radius[7] $R\sqrt{N}$, which contains the box $[0, R]^N$.

At iteration $k$, let $y_k$ be the center of the current ellipsoid and choose the cut as follows. If $y_k \notin [0, R]^N$, choose one of the violated box constraints as a cut. If $y_k \in [0, R]^N$, let $x_k$ be a cut from a Hart-Schmeidler oracle (symmetric if the game is), so $x_k U^T y_k = 0$. Round each of the terms $x_k^i$ of the product $x_k$ to a probability distribution $\tilde{x}_k^i$ whose components are multiples of

$$\delta := (2nMNR \left\|U\right\|_\infty)^{-1}$$

and form the distribution $\tilde{x}_k$ which is the product of the $\tilde{x}_k^i$ (rounding consistently preserves symmetry). Use $\tilde{x}_k U^T y \leq -1$ as the cut.

Run the ellipsoid algorithm until the ellipsoid has volume at most

$$v := N^{-N} (N \left\|U\right\|_\infty \delta^{-n})^{-N^2(N+1)}.$$

After $L$ (to be specified below) iterations, stack the $L$ row vectors $\tilde{x}_k$ into one large matrix $X$. There is a probability vector $\alpha$ such that $X^T\alpha$ is an $\epsilon$-correlated

[7]This square root is not a problem – the ellipsoid algorithm only needs the squared radius $R^2N$ as input.

equilibrium and one can be computed in polynomial time by solving the linear program

$$\begin{array}{lll}
\underset{\alpha\in\mathbb{R}^L,\ \xi,\ \xi_{\cdot,\cdot}\in\mathbb{R}}{\text{minimize}} & \xi & \\
\text{subject to} & \left[UX^T\alpha\right]_{s_it_i} \geq -\xi_{i,s_i}, & \text{players } i \in P, \text{ strategies } s_i, t_i \in C_i, \\
& \sum_{s_i\in C_i} \xi_{i,s_i} \leq \xi, & \text{players } i \in P, \\
& \sum_{k=1}^{L} \alpha_k = 1, & \\
& \alpha \geq 0. &
\end{array} \qquad (P^*)$$

**Correctness** There are two parts. First we must show that the above cuts are a legal run of the ellipsoid algorithm on $(D_R)$. There is clearly no problem with the box cuts, so it remains to show that the rounded Hart-Schmeidler cuts are valid. Such a cut $\tilde{x}_k U^T y \leq -1$ is a convex combination of the constraints of $(D_R)$, so it is a valid inequality. To see that it is violated at $y_k$, note that $\|\tilde{x}_k - x_k\|_\infty \leq n\delta$ by the triangle inequality, so

$$\begin{aligned}
\left|\tilde{x}_k U^T y_k\right| &= \left|(\tilde{x}_k - x_k)U^T y_k\right| \leq \|\tilde{x}_k - x_k\|_\infty MN \|U\|_\infty \|y_k\|_\infty \\
&\leq \delta n M N R \|U\|_\infty = \frac{1}{2},
\end{aligned}$$

and the cut is valid.

Second, we must show that solving $(P^*)$ produces an $\epsilon$-correlated equilibrium. The fact that a feasible solution with value $\xi$ is a $\xi$-correlated equilibrium follows from the definitions. It remains to show that $(P^*)$ has a feasible solution with objective value at most $\epsilon$. If we form $(D')$ as in Section 7.1.2 from the run of the ellipsoid algorithm and this problem happens to be infeasible, then the arguments of that section show that $(P^*)$ has optimal value zero and we are done. We will assume for the remainder of this section that $(D')$ is feasible.

Form the bounded modified dual

$$\begin{array}{ll}
\underset{y\in\mathbb{R}^{\cup C_i\times C_i}}{\text{minimize}} & 0 \\
\text{subject to} & XU^T y \leq -e \\
& Re \geq y \geq 0,
\end{array} \qquad (D'_R)$$

The above run of the ellipsoid algorithm is also valid for $(D'_R)$ – in this context the idea behind the Ellipsoid Against Hope algorithm is correct. The upper bounds on the $y$ variables ensure that the initial ellipsoid is valid and all the cuts used are actually constraints of this problem. The ellipsoid algorithm certifies that the volume of the feasible set of $(D'_R)$ is at most $v$.

Clearing the denominator $\delta^{-n}$ in the constraints $XU^Ty \leq -e$ yields equivalent constraints with integer coefficients, all bounded in absolute value by $\|U\|_\infty \delta^{-n} \geq R$. By Lemma 8.4 in [6], if the feasible set of $(D'_R)$ were full-dimensional it would have volume greater than $v$, contradicting the analysis above. Therefore we conclude that the feasible set is not full-dimensional.

Fix $\rho > 1$. Perturbing the constraints of $(D'_R)$, we get an infeasible problem

$$\begin{array}{ll} \underset{y \in \mathbb{R}^{\sqcup C_i \times C_i}}{\text{minimize}} & 0 \\ \text{subject to} & XU^Ty \leq -\rho e \\ & Re \geq y \geq 0, \end{array}$$

The dual of this problem,

$$\begin{array}{ll} \underset{\alpha \in \mathbb{R}^L,\ \beta \in \mathbb{R}^M}{\text{maximize}} & \rho \sum_{k=1}^{L} \alpha_k - R \sum_{j=1}^{M} \beta_j \\ \text{subject to} & UX^T\alpha + \beta \geq 0 \\ & \alpha, \beta \geq 0, \end{array}$$

is feasible $(\alpha, \beta = 0)$, hence unbounded by strong linear programming duality. Thus there exist $\alpha$ and $\beta$ satisfying

$$\begin{aligned} \rho \sum_{k=1}^{L} \alpha_k &> R \sum_{j=1}^{M} \beta_j \\ UX^T\alpha + \beta &\geq 0 \\ \alpha, \beta &\geq 0. \end{aligned}$$

The first and last constraints give $\sum \alpha_k > 0$. Normalizing, we get $\alpha, \beta$ such that

$$\begin{aligned} \sum_{j=1}^{M} \beta_j &< \frac{\rho}{R} \\ \sum_{k=1}^{L} \alpha_k &= 1 \\ UX^T\alpha + \beta &\geq 0 \\ \alpha, \beta &\geq 0. \end{aligned}$$

Letting $\xi_{i,s_i} = \max_{t_i \in C_i} \beta^i_{s_i,t_i}$ and $\xi = \max_i \sum_{s_i \in C_i} \xi_{i,s_i}$ we get a feasible solution of $(P^*)$ with $\xi$ being a sum of some of the elements of $\beta$. Thus $\xi < \frac{\rho}{R} \leq \rho\epsilon$. Since $\rho > 1$ was arbitrary, an optimal solution of $(P^*)$ will have value at most $\epsilon$.

## 7.1.6 Run time

First we check that the number of iterations $L$ of the ellipsoid algorithm is polynomial in the input. For concreteness, we use the fixed-precision version of the ellipsoid algorithm presented in Theorem 3.2.1 of [34]. The volume of the initial ellipsoid is $\mathrm{vol}(E_1) \leq (2R\sqrt{N})^N$ and we run until the volume is $\mathrm{vol}(E_L) \leq v = N^{-N}(N\|U\|_\infty \delta^{-n})^{-N^2(N+1)}$. By Lemma 3.2.10 in [34], we can take $\log \mathrm{vol}(E_{i+1}) \leq \log \mathrm{vol}(E_i) - \frac{1}{5N}$. Thus it suffices to take

$$L = 5N\left\lceil N\log\frac{2\sqrt{N}}{\epsilon} + N\log N + N^2(N+1)\log(N\|U\|_\infty) + N^2(N+1)n\log\frac{2nMN\|U\|_\infty}{\epsilon}\right\rceil,$$

which is polynomial in the input size.

Having worked around the error in [55], the rest of the analysis from that paper goes through in this new setting. In particular, the assumption that a succinct game has polynomial type and the polynomial expectation property implies that each element of the matrix product $UX^T$, and in particular each coefficient of the cuts made in the ellipsoid algorithm, can all be computed in polynomial time in the size of the input.

The ellipsoid method therefore runs in polynomial time on $(D_R)$. From the history of this computation we can form the linear program $(P^*)$ in polynomial time. We use the ellipsoid algorithm again to solve this in polynomial time, computing an $\epsilon$-correlated equilibrium which is a polynomial sum of products. If the game is symmetric all these products can be taken to be symmetric, since the algorithm starts with a symmetric ellipsoid. This proves Theorem 7.3.

## 7.1.7 Hardness of optimizing over exchangeable equilibria

Suppose we would like to minimize a linear function (to some desired degree of accuracy) over the set $\mathrm{XE}_G(\Gamma)$ of exchangeable equilibria of a game or perhaps over the $\epsilon$-exchangeable equilibria for a given epsilon. The Approximate Ellipsoid Against Hope algorithm no longer applies directly: it can identify a subset of $\Delta_G^X(\Gamma)$ guaranteed to contain a single $\epsilon$-exchangeable equilibrium, but this set need not contain all the $\epsilon$-exchangeable equilibria.

It stands to reason that the Approximate Ellipsoid Against Hope algorithm, which is based on linear programming and duality, could be modified to optimize a linear function over the $\epsilon$-exchangeable equilibria in polynomial time. It turns out this is false, at least assuming $P \neq NP$, because this problem is NP-hard.

To formalize this we must view the task as a decision problem, so we can ask whether the minimum of the linear function (specified by its rational coefficients) is at least some rational number $q$.

Let $\theta$ denote the zero game with two players having $m$ strategies each. This is a symmetric bimatrix game. Any distribution in $\Delta(\Gamma)$ is an $\epsilon$-correlated equilibrium for any $\epsilon$, so the set of $\epsilon$-exchangeable equilibria is the set of normalized $m \times m$ completely positive matrices. Suppose we are given a matrix $A \in \mathbb{Q}^{m\times m}$ and we would like to optimize the linear function $A \bullet X$ over $X$ in this set of equilibria. This is exactly the optimization problem (2.1), so if $q = 0$ we are asking whether $A$ is a copositive matrix, a co-NP-complete problem [50].

This is a special case of the problem of optimizing over $\epsilon$-exchangeable equilibria, so that problem is NP-hard.

## 7.2 Linear and semidefinite relaxations

Although the Approximate Ellipsoid Against Hope algorithm finds $\epsilon$-exchangeable equilibria efficiently in theory, its use of the ellipsoid method makes running it impractical. While the ellipsoid method runs in polynomial time, the exponent and constants are large enough that it is prohibitively slow in practice. Furthermore, the numerical issues, discussed in some detail above, make it necessary to keep track of many bits of precision at each iteration to ensure convergence. The inability to use standard fixed or floating point arithmetic makes this method even slower. If we are willing to give up guaranteed bounds on the error $\epsilon$, there are more practical heuristics available.

The Approximate Ellipsoid Against Hope algorithm works by computing a polynomially-sized representation of a (polyhedral) subset of $\Delta_G^X(\Gamma)$ which contains an $\epsilon$-correlated equilibrium by construction. We can skip this step and instead choose any subset of $\Delta_G^X(\Gamma)$ with a small description in terms of linear or semidefinite constraints. Finding the $\epsilon$-correlated equilibrium in this set with minimum $\epsilon$ is then a linear or semidefinite program which can be solved efficiently (in theory and practice) by interior point methods.

We have seen this technique in action in many of the examples above which were chosen to be small enough so that complete positivity coincides with double nonnegativity and the set of exchangeable equilibria is exactly describable by a semidefinite program. For larger games the situation is not so straightforward, but we can still provide inner approximations to the completely positive matrices.

Let us see how this might work in the case of symmetric bimatrix games. A known polyhedral inner approximation to the completely positive matrices is the set of nonnegative **diagonally dominant** matrices. These are the matrices for which the diagonal element in each row is at least as large as the sum of all other elements. It is easy to show that the extreme rays of the diagonally dominant matrices are of the form $v \otimes v$ where $v$ is a vector with at most two 1 entries and zeros elsewhere. These are all nonnegative symmetric rank one matrices, so conic combinations of these are completely positive as claimed.

We can form tighter polyhedral approximations by merely choosing any finite set of vectors $v$ which are in some sense "evenly spaced" throughout $\Delta(C_1)$. Conic combinations of tensor powers of these are automatically completely positive, and the denser the set of vectors is chosen, the tighter the relaxation.

There is a semidefinite generalization of this approach, which is most easily described for matrices. If $X = \sum x_k x_k^T$, $x_k \in \mathbb{R}^l_{\geq 0}$ is a completely positive matrix and $P \in \mathbb{R}^{m \times l}$ is an elementwise nonnegative matrix, the conjugate $PXP^T = \sum (Px_k)(Px_k)^T \in \mathbb{R}^{m \times m}$ is also completely positive. Choosing several $X$ and $P$ matrices in this way we also get that sum $\sum_i P_i X_i P_i^T$ is completely positive whenever the $X_i$ are and the $P_i$ are elementwise nonnegative. If we take $l = 1$ then the $X_i$ are scalars and constraining these to be nonnegative yields exactly the polyhedral approximation above.

If the $X_i$ are $4 \times 4$ or smaller their complete positivity is equivalent to double nonnegativity and the set of matrices of the form $\sum_i P_i X_i P_i^T$ is an inner semidefinite approximation to $\mathrm{CP}^2_m$. For example, if we let $e_1, \ldots, e_m$ denote the unit column vectors in $\mathbb{R}^m$ and we let the $P_i$ vary over all $\binom{m}{4}$ ways of choosing four of these and concatenating them into an $m \times 4$ matrix (the order does not matter), constraining $X_i \geq 0$ and $X_i \succeq 0$, then we get a semidefinite description of:

$$\begin{aligned}&\left\{\sum_i P_i X_i P_i^T \;\middle|\; X_i \geq 0, X_i \succeq 0\right\}\\ &\qquad = \operatorname{conv}\left\{vv^T \mid v \in \mathbb{R}^m_{\geq 0} \text{ has at most four nonzero entries}\right\} \subseteq \mathrm{CP}^2_m,\end{aligned}$$

where the containment is strict if and only if $m > 4$ by Proposition 2.49 on extreme rays of the set of completely positive matrices.

These methods can be used just as well to approximate higher order exchangeable equilibria as well as asymmetric exchangeable equilibria, and in some cases will find exact equilibria. Example 6.17 of an asymmetric exchangeable equilibrium which does not lift to a symmetric exchangeable equilibrium of the symmetrization was found using this inner approximation of $\mathrm{CP}^2_6$ to give an inner approximation to $\mathrm{XE}_-(\Gamma)$ and comparing it to the outer approximation of $\mathrm{XE}_{S_2}(\Gamma^{\mathrm{Sym}})$ given by relaxing $\mathrm{CP}^2_9$ to $\mathrm{DNN}^2_9$ (see Figure 6.1). Note that none of the factors in the completely positive factorization (6.3) has more than four nonzero entries.

We can also give outer approximations of the set of exchangeable equilibria by replacing completely positive tensors with an outer approximation, such as doubly nonnegative tensors. Recall that these were constructed in terms of the sum of squares relaxation of positive polynomials, so we can use tighter relaxations of positive polynomials to construct tighter relaxations of completely positive tensors if necessary. Such relaxations in general are not exact, but we can easily minimize linear functionals over these to get lower bounds on the NP-hard problem of minimizing linear functionals over the exchangeable equilibria, and these techniques again apply equally to higher order and asymmetric exchangeable equilibria.

We can theoretically use the inner approximations to completely positive tensors to give upper bounds for the minimum of a linear functional over the exchangeable equilibria, but the corresponding inner approximation of the exchangeable equilibria may be empty, yielding a trivial bound of $+\infty$. In such cases we will still be able to give upper bounds to the minimum over $\epsilon$-exchangeable equilibria, for sufficiently large $\epsilon$.

Chapter 8

# Structure of Extreme Correlated Equilibria

In finite games the set of correlated equilibria is a compact convex polytope, and therefore seemingly much simpler than the set of Nash equilibria, which can be essentially any algebraic variety [17]. Even in the simple case of two-player finite games, the set of Nash equilibria is a union of finitely many polytopes: seemingly more complicated than the set of correlated equilibria. The relationship between these two sets is that all the extreme points of the polytopes making up the Nash equilibria, viewed as product distributions, are extreme points of the correlated equilibrium polytope. This result, discovered independently by Evangelista and Raghavan [25] and Cripps [14], is reviewed in Section 8.1.1.

In this chapter[1] we construct two-player zero-sum games in which the set of correlated equilibria has many more extreme points than the set of Nash equilibria has. This behavior does not seem to be pathological in any way: it occurs in very simple finite games and the simplest of infinite games. We take this as evidence that this complexity is likely to be quite common.

A *polynomial game* is one with compact intervals for strategy spaces and polynomial utility functions. Polynomial games are one of the simplest classes of games with infinite strategy sets, and formally similar to finite games in many ways [20, 21, 43, 44, 68, 69]. One of the most important connections is that the set of Nash equilibria of a polynomial game, which is a priori an infinite-dimensional object, can be given a finite-dimensional description in terms of fixed points of a certain self-map defined on compact, convex, but nonpolyhedral subsets of $\mathbb{R}^k$. Geometrically this looks like the finite case, though there the sets are simplices.

Given this representation it is natural to ask whether the set of correlated

[1] A version of this chapter has been published as [71]. It is logically independent of Chapters 3 through 7 on exchangeable equilibria. In fact its development precedes that of exchangeable equilibria. Thematically this material is closer to the author's master's thesis [68], but it was not completed in time for inclusion therein. Nonetheless there are connections to exchangeable equilibria under the surface. These are discussed in the final chapter.

equilibria of a polynomial game also admits a finite-dimensional characterization. The finite-dimensional description of Nash equilibria is in terms of moments so it is natural to consider whether a similar characterization of correlated equilibria in terms of joint moments exists. In this chapter we show that in a very general sense such a characterization does not exist. We do so by exhibiting an example polynomial game with extreme points which are not finitely supported; a convex set of measures characterized by finitely many conditions on its (generalized) moments cannot have such extreme points.

**Outline** The remainder of this chapter is organized as follows. We cover background material in Section 8.1. Section 8.2 introduces the examples to be studied. The two types of examples are closely related – the finite game examples are just restrictions of the strategy spaces in the infinite game example to fixed finite sets. This allows us to analyze both examples on equal footing. In Section 8.3 we define and compute the extreme Nash equilibria of these examples, counting them in the finite game example. Then we define and analyze the extreme correlated equilibria in Section 8.4, comparing the results with those about Nash equilibria as we go.

## 8.1 Background

First we fix notation. When $T$ is a topological space, $\Delta^*(T)$ will denote the set of regular finite Borel measures on $T$. In particular $\Delta(T)$ is the set of measures in $\Delta^*(T)$ with unit mass. If $T$ is finite it will be given the discrete topology by default so $\Delta(T)$ is a simplex and $\Delta^*(T)$ is an orthant in $\mathbb{R}^T$.

### 8.1.1 Extreme equilibria in finite games

In this section we present shorter, less computational proofs of the main results in [39], [14], and [25] on the structure of the Nash and correlated equilibrium sets of two-player finite games. In [39] Heuer and Millham define maximal Nash sets and give a characterization which leads to the notion of extreme Nash equilibria. Evangelista and Raghavan [25] and Cripps [14] independently show the same result relating these to extreme points of the correlated equilibrium polytope.

**Definition 8.1.** *Let $\Gamma$ by a bimatrix game. A product set $E_1 \times E_2 \subseteq \mathrm{NE}(\Gamma)$ with $E_i \subseteq \Delta(C_i)$ is called a **Nash set**. A **maximal Nash set** is one which is maximal with respect to inclusion.*

**Example 8.2.** If $(\sigma, \tau) \in \mathrm{NE}(\Gamma)$ then $\{(\sigma, \tau)\}$ is a Nash set but need not be maximal. If this is the unique Nash equilibrium then this Nash set is maximal. $\diamondsuit$

**Example 8.3.** If $\Gamma$ is a zero-sum game, then $\mathrm{NE}(\Gamma) = \mathrm{Mm}(\Gamma) \times \mathrm{mM}(\Gamma)$, the product of the sets of maximin and minimax strategies, so $\mathrm{NE}(\Gamma)$ is the unique maximal Nash set. $\diamondsuit$

**Proposition 8.4** ([39]). *Let $E_1 \times E_2$ be a maximal Nash set of a bimatrix game. Then the $E_i$ are convex polytopes defined by a set of inequalities of the form* (8.1) *below with $C_i = B_i \cup Z_i$.*

*Proof.* Let $B_i \subseteq C_i$ be the set of pure strategies which are best responses to all strategies in $E_{-i}$. Let $Z_i \subseteq C_i$ be the set of pure strategies which are not chosen with positive probability under any mixed strategy in $E_i$. Only best responses are chosen with positive probability in a Nash equilibrium, so $C_i = B_i \cup Z_i$.

Define a set $F_i \subseteq \Delta(C_i)$ to be the collection of $\tau_i$ satisfying the linear constraints

$$\begin{aligned}
\sum_{s_i \in C_i} [u_{-i}(s_{-i}, s_i) - u_{-i}(t_{-i}, s_i)]\tau_i(s_i) &= 0 \quad \text{for } s_{-i}, t_{-i} \in B_{-i}, \\
\sum_{s_i \in C_i} [u_{-i}(s_{-i}, s_i) - u_{-i}(t_{-i}, s_i)]\tau_i(s_i) &\geq 0 \quad \text{for } s_{-i} \in B_{-i}, t_{-i} \in C_{-i}, \\
\tau_i(s_i) &= 0 \quad \text{for } s_i \in Z_i, \\
\tau_i(s_i) &\geq 0 \quad \text{for } s_i \in C_i, \\
\sum_{s_i \in C_i} \tau_i(s_i) &= 1.
\end{aligned} \tag{8.1}$$

Then $F_1 \times F_2 \supseteq E_1 \times E_2$ is a Nash set, so $E_i = F_i$. $\square$

**Definition 8.5.** *A pair $(\sigma_1, \sigma_2)$ is an **extreme Nash equilibrium** if there is a maximal Nash set $E_1 \times E_2$ with $\sigma_i$ an extreme point of $E_i$ for $i = 1, 2$. A $\pi \in \Delta(\Gamma)$ is an **extreme correlated equilibrium** if it is an extreme point of* $\mathrm{CE}(\Gamma)$.

**Proposition 8.6** ([14, 25]). *If $(\sigma_1, \sigma_2)$ is an extreme Nash equilibrium of a bimatrix game then $\sigma_1 \times \sigma_2$ is an extreme correlated equilibrium.*

*Proof.* Let $E_1 \times E_2$ be a maximal Nash set with $\sigma_i$ an extreme point of $E_i$ for $i = 1, 2$. We can assume the $E_i$ are defined by the conditions (8.1) for some $B_i$ and $Z_i$. An extreme point of a polytope defined by linear equations and inequalities is a point where certain inequalities are tight these inequalities are not all tight at any other point. Therefore we can enlarge the sets $B_i$ and $Z_i$ to $B_i^*$ and $Z_i^*$, making $\sigma_1$ and $\sigma_2$ the unique pair satisfying the homogeneous conditions

$$\begin{aligned}
\sum_{s_i \in C_i} [u_{-i}(s_{-i}, s_i) - u_{-i}(t_{-i}, s_i)]\tau_i(s_i) &= 0 \quad \text{for } s_{-i}, t_{-i} \in B^*_{-i}, \\
\sum_{s_i \in C_i} [u_{-i}(s_{-i}, s_i) - u_{-i}(t_{-i}, s_i)]\tau_i(s_i) &\geq 0 \quad \text{for } s_{-i} \in B^*_{-i}, t_{-i} \in C_{-i}, \\
\tau_i(s_i) &= 0 \quad \text{for } s_i \in Z^*_i, \\
\tau_i(s_i) &\geq 0 \quad \text{for } s_i \in C_i,
\end{aligned} \tag{8.2}$$

which are also normalized $\sum \sigma_i(s_i) = 1$. Dropping normalization, any $\tau_i$ satisfying (8.2) is of the form $\tau_i = \lambda_i \sigma_i$ for some $\lambda_i \geq 0$.

The following constraints are the correlated equilibrium conditions with some extra inequalities made tight:

$$\begin{aligned}
\sum_{s_i \in C_i} [u_{-i}(s_{-i}, s_i) - u_{-i}(s_{-i}, t_i)]\pi(s_i, s_{-i}) &= 0 \quad \text{for } s_{-i}, t_{-i} \in B^*_{-i} \\
\sum_{s_i \in C_i} [u_{-i}(s_{-i}, s_i) - u_{-i}(s_{-i}, t_i)]\pi(s_i, s_{-i}) &\geq 0 \quad \text{for } s_{-i}, t_{-i} \in C_{-i} \\
\pi(s) &= 0 \quad \text{if } s_1 \in Z^*_1 \text{ or } s_2 \in Z^*_2, \\
\pi(s) &\geq 0 \quad \text{for all } s \in C, \\
\sum_{s \in C} \pi(s) &= 1.
\end{aligned} \tag{8.3}$$

We will show that (8.3) defines a unique $\pi$. For each $s_{-i} \in C_{-i}$, $\pi(\cdot, s_{-i})$ satisifes (8.2), so there is a $\lambda_{-i}(s_{-i}) \geq 0$ such that $\pi(s_i, s_{-i}) = \lambda_{-i}(s_{-i})\sigma_i(s_i)$ for all $s \in C$. Summing over all $s \in C$ we get $\sum_{s_i \in C_i} \lambda_i(s_i) = \sum_{s \in C} \pi(s) = 1$, so $\lambda_i \in \Delta(C_i)$ for $i = 1, 2$. We have $\pi = \sigma_1 \times \lambda_2 = \lambda_1 \times \sigma_2$, so $\sigma_i = \lambda_i$ for each $i$ and any distribution satisfying (8.3) must equal $\sigma_1 \times \sigma_2$. This distribution does in fact satisfy these conditions, so it is the unique correlated equilibrium at which these particular constraints are tight. Therefore it is extreme. $\square$

The final example in Nash's paper [52] is a three player game with rational utilities and a unique Nash equilibrium. This equilibrium uses irrational probabilities, so viewed as a product distribution at least one of the probabilities is irrational. On the other hand the set of correlated equilibria is a polytope defined by linear inequalities with rational coefficients, so all its extreme points are rational and in particular not equal to the Nash equilibrium.

Therefore this result does not generalize in the obvious way to finite games with more than two players. A weaker statement which does generalize is given in [53], but we do not use this result in this work.

## 8.1.2 Ergodic theory

To prove that certain correlated equilibria of a polynomial game are extreme (Proposition 8.26) we will need some ideas from ergodic theory. The first is the standard definition of compatibility between a measure and a transformation on a space. The second expresses one notion of what it means for a transformation to "mix up" a space – in this case that the space cannot be partitioned into two sets of positive measure which do not interact under the transformation. Using these we state the main ergodic theorem and a corollary which is key to the proof of Proposition 8.26.

**Definition 8.7.** *Given a measure $\mu \in \Delta^*(S)$ on a space $S$, a measurable function $g : S \to S$ is called* ***($\mu$-)measure preserving*** *if $\mu(g^{-1}(A)) = \mu(A)$ for all measurable $A \subseteq S$. Note that if $g$ is invertible (in the measure theoretic sense that an almost everywhere inverse exists), then this is equivalent to the condition that $\mu(g(A)) = \mu(A)$ for all $A$.*

**Definition 8.8.** *Given a measure $\mu \in \Delta^*(S)$, a $\mu$-measure preserving transformation $g$ is called* ***ergodic*** *if $\mu(A \triangle g^{-1}(A)) = 0$ implies $\mu(A) = 0$ or $\mu(S \setminus A) = 0$, where $A \triangle B$ denotes the symmetric difference $(A \setminus B) \cup (B \setminus A)$.*

**Example 8.9.** Fix a finite set $S$ and a function $g : S \to S$. Let $\mu$ be counting measure on $S$. Then $g$ is measure preserving if and only if it is a permutation. In this case a set $T$ satisfies $\mu(g^{-1}(T) \triangle T) = 0$ if and only if $g^{-1}(T) = T$ if and only if $T$ is a union of cycles of $g$. Therefore $g$ is ergodic if and only if it consists of a single cycle. ◊

**Example 8.10.** Fix $\alpha \in \mathbb{R}$. Let $S = [0, 1)$ and let $\mu$ be Lebesgue measure on $S$. Define $g : S \to S$ by $g(x) = (x + \alpha) \mod 1 = (x + \alpha) - \lfloor x + \alpha \rfloor$. Then $g$ is $\mu$-measure preserving because Lebesgue measure is translation invariant. It can be shown that $g$ is ergodic if and only if $\alpha$ is irrational. For a proof and more examples, see [66]. ◊

The following is one of the core results of ergodic theory. We will only use it to prove the corollary which follows, so it need not be read in detail. The proof can be found in any text on ergodic theory, e.g. [66].

**Theorem 8.11** (Birkhoff's ergodic theorem). *Fix a probability measure $\mu$ and a $\mu$-measure preserving transformation $g$. Then for any $f \in \mathcal{L}^1(\mu)$:*

- $\tilde{f}(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(g^k(x))$ *exists $\mu$-almost everywhere,*
- $\tilde{f} \in \mathcal{L}^1(\mu)$,
- $\int \tilde{f}\, d\mu = \int f\, d\mu$,

- $\tilde{f}(g(x)) = \tilde{f}(x)$ *$\mu$-almost everywhere, and*
- *if $g$ is ergodic then $\tilde{f}(x) = \int f\, d\mu$ $\mu$-almost everywhere.*

**Corollary 8.12.** *Suppose $\mu$ and $\nu$ are probability measures such that $\nu$ is absolutely continuous with respect to $\mu$. If a transformation $g$ preserves both $\mu$ and $\nu$ and $g$ is ergodic with respect to $\mu$, then $\nu = \mu$.*

*Proof.* Fix any measurable set $A$. Let $f$ be the indicator function for $A$, i.e. the function equal to unity on $A$ and zero elsewhere. Applying Birkhoff's ergodic theorem to $f$ and $\mu$ yields $\tilde{f}(x) = \mu(A)$ $\mu$-almost everywhere. Since $\nu$ is absolutely continuous with respect to $\mu$, $\tilde{f}(x) = \mu(A)$ $\nu$-almost everywhere also. If we now apply Birkhoff's ergodic theorem to $\nu$ we get:

$$\nu(A) = \int f\, d\nu = \int \tilde{f}\, d\nu = \int \mu(A)\, d\nu = \mu(A). \qquad \square$$

## 8.2 Description of the examples

We focus on two related examples, one with finite strategy sets and one with infinite strategy sets. We develop these in parallel by analyzing arbitrary games satisfying the following condition. The condition does not have any game theoretic content; it was merely chosen for simplicity and the results to which it leads.

**Assumption 8.13.** *The game is a zero-sum strategic form game with two players, called $X$ and $Y$. The strategy sets $C_X$ and $C_Y$ are compact subsets of $J := [-1, 1]$, each of which contains at least one positive element and at least one negative element. Player $X$ chooses a strategy $x \in C_X$ and player $Y$ chooses $y \in C_Y$. The utility functions[2] are $u_X(x, y) = xy = -u_Y(x, y)$.*

**Example 8.14.** Fix an integer $l > 0$. Let $C_X$ and $C_Y$ each have $2l$ elements, $l$ of which are positive and $l$ of which are negative. If we take $l = 1$ and $C_X = C_Y = \{-1, 1\}$ then we recover the matching pennies game shown in Table 2.1 (up to relabeling). $\diamond$

**Example 8.15.** Let $C_X = C_Y = [-1, 1]$. Then the game is essentially the mixed extension of matching pennies. That is to say, suppose two players play matching pennies and choose their strategies independently, playing 1 with probabilities

[2] By inspection of the utilities we can see that for any $C_X$ and $C_Y$, the rank of this game in the sense of [69] is $(1, 1)$ (and in fact also in the stronger sense of Theorem 3.3 of that paper). The notion of the rank of a game is related to the rank of the payoff matrices and will not play a significant role in this thesis; we merely wish to note that under this definition of complexity of payoffs, the games we consider are uniformly simple.

$p \in [0,1]$ and $q \in [0,1]$. Define the utilities for the mixed extension to be the expected utilities under this random choice of strategies. Letting $x = 2p - 1$ and $y = 2q - 1$, the utility to the first player is $xy$ and the utility to the second player is $-xy$. Therefore this example is the mixed extension of matching pennies, up to an affine scaling of the strategies. $\diamondsuit$

Usually one looks at pure equilibria of the mixed extension of a game; these are exactly the mixed equilibria of the original game. We will instead be looking at mixed Nash equilibria and correlated equilibria of the mixed extension itself, a game with a continuum of actions. The relationship between correlated equilibria of the mixed extension and those of the original game is much more complicated than the corresponding relationship for mixed Nash equilibria. This drives the results of this chapter.

## 8.3 Extreme Nash equilibria

We now characterize and count the extreme points of the sets of Nash equilibria in games satisfying Assumption 8.13. Since the games are zero-sum, the set of Nash equilibria can be viewed as a Cartesian product of two (weak*) compact convex sets, the sets of maximin and minimax strategies [31]. The Krein-Milman theorem [62] completely characterizes such sets by their extreme points, explaining our focus on extreme points throughout.

The Nash equilibria of games satisfying Assumption 8.13 take the following particularly simple form.

**Proposition 8.16.** *A pair $(\sigma,\tau) \in \Delta(C_X) \times \Delta(C_Y)$ is a Nash equilibrium of a game satisfying Assumption 8.13 if and only if $\int x\, d\sigma(x) = \int y\, d\tau(y) = 0$.*

*Proof.* If $\int x\, d\sigma(x) = 0$ then $u_Y(\sigma, y) = 0$ for all $y \in C_Y$, so any $\tau \in \Delta(C_Y)$ is a best response to $\sigma$. If $\int y\, d\tau(y) = 0$ as well then $\sigma$ is also a best response to $\tau$, so $(\sigma,\tau)$ is a Nash equilibrium.

Suppose for a contradiction that there exists a Nash equilibrium $(\sigma,\tau)$ such that $\int x\, d\sigma(x) > 0$; the other cases are similar. Player $y$ must play a best response, so $\int y\, d\tau(y) < 0$, which is possible by assumption. Player $x$ plays a best response to that, so $\int x\, d\sigma(x) < 0$, a contradiction. $\square$

For games which are zero-sum but not necessarily finite, we extend Definition 8.5 as follows.

**Definition 8.17.** *A Nash equilibrium $(\sigma,\tau)$ of a zero-sum game is called* ***extreme*** *if $\sigma$ and $\tau$ are extreme points of the maximin and minimax sets, respectively.*

Applying Proposition 8.16 to this definition, we can characterize the extreme Nash equilibria of games satisfying Assumption 8.13.

**Proposition 8.18.** *Consider a game satisfying Assumption 8.13. A pair* $(\sigma, \tau) \in \Delta(C_X) \times \Delta(C_Y)$ *is an extreme Nash equilibrium if and only if* $\sigma$ *and* $\tau$ *are each either* $\delta_0$ *or of the form* $\alpha\delta_a + \beta\delta_b$ *where* $a < 0$, $b, \alpha, \beta > 0$, $\alpha + \beta = 1$, *and* $\alpha a + \beta b = 0$.

*Proof.* By Proposition 8.16 we must show that these distributions are the extreme points of the set of probability distributions having zero mean. Since $\delta_0$ is an extreme point of the set of probability distributions, it must be an extreme point of the subset which has zero mean. To see that $\alpha\delta_a + \beta\delta_b$ is also an extreme point, suppose we could write it as a convex combination of two other probability distributions with zero mean. The condition that both be positive measures implies that both must be of the form $\alpha'\delta_a + \beta'\delta_b$. But $\alpha$ and $\beta$ as specified above are the unique coefficients which make this be a probability measure with zero mean. Therefore $\alpha' = \alpha$ and $\beta' = \beta$, so $\alpha\delta_a + \beta\delta_b$ cannot be written as a nontrivial convex combination of probability distributions with zero mean, i.e., it is an extreme point.

Suppose $\sigma$ were an extreme point which was not of one of these types. Then $\sigma$ could not be supported on one or two points, so either $[0, 1]$ or $[-1, 0)$ could be partitioned into two sets of positive measure. We will only treat the first case; the second is similar. Let $[0, 1] = A \cup B$ where $A \cap B = \emptyset$ and $\sigma(A), \sigma(B) > 0$. Since $\sigma$ has zero mean we must have $\sigma([-1, 0)) > 0$ as well.

For a set $D$ we define the restriction measure $\sigma|_D$ by $\sigma|_D(C) := \sigma(D \cap C)$ for all measurable $C$. Then $\sigma = \sigma|_A + \sigma|_B + \sigma|_{[-1,0)}$. Let $a = \int_A x\, d\sigma(x)$, $b = \int_B x\, d\sigma(x)$, and $c = \int_{[-1,0)} x\, d\sigma(x)$. Since $\sigma([-1, 0)) > 0$ and $x$ is less than zero everywhere on $[-1, 0)$, we must have $c < 0$ and similarly $a, b \geq 0$. By assumption $a + b + c = 0$. Therefore we can write:

$$\sigma = \left(\sigma|_A + \frac{a}{|c|}\sigma|_{[-1,0)}\right) + \left(\sigma|_B + \frac{b}{|c|}\sigma|_{[-1,0)}\right)$$

Being an extreme point of the set of probability measures with zero mean, $\sigma$ must be an extreme ray of the set of positive measures with first moment equal to zero. But this means that we cannot write $\sigma = \sigma_1 + \sigma_2$ where the $\sigma_i$ are positive measures with zero first moment unless $\sigma_i$ is a multiple of $\sigma$. Neither of the measures in parentheses above is a multiple of $\sigma$, a contradiction. □

We illustrate this proposition on both examples introduced in Section 8.2.

**Example 8.19** (continues Example 8.14). In this case neither $C_X$ nor $C_Y$ contains zero, so the only extreme Nash equilibria are those in which $\sigma$ and $\tau$ are of the form $\alpha\delta_a + \beta\delta_b$ for $a < 0$ and $b > 0$. For any choice of $a$ and $b$ simple algebra gives unique $\alpha$ and $\beta$ satisfying the conditions of Proposition 8.18. There are $l$ possible choices for each of $a$ and $b$ for each of the two players, so there are $l^4$ extreme Nash equilibria. ◇

**Example 8.20** (continues Example 8.15). Since $C_X = C_Y = [-1,1]$, there are infinitely many extreme Nash equilibria in this case. However, they are all finitely supported and the size of the support of each player's strategy is always either one or two. Furthermore the condition that $(\sigma, \tau)$ be a Nash equilibrium is equivalent to both having zero mean. This illustrates the general facts that in games with polynomial utility functions the Nash equilibrium conditions only involve finitely many moments of $\sigma$ and $\tau$ (in this case, only the mean) and the extreme Nash equilibria (when defined, say for zero-sum games) have uniformly bounded support [43]. ◇

## 8.4 Extreme correlated equilibria

In this section we will show that in bimatrix games, the number of extreme correlated equilibria can be much larger than the number of extreme Nash equilibria. It is meaningful to compare these because all extreme Nash equilibria of a bimatrix game, viewed as product distributions, are automatically extreme correlated equilibria (Proposition 8.6).

In the case of polynomial games we will show that there can be extreme correlated equilibria with arbitrarily large finite support and with infinite support. This implies that the set of correlated equilibria cannot be characterized in terms of finitely many joint moments.

**Roadmap** The analysis proceeds in several steps which will be technical at times, so we start with an outline of what follows.

- We begin by defining correlated equilibria in games satisfying Assumption 8.13 using a characterization from [70].
- Proposition 8.23 shows how this characterization can be simplified under our choice of utility functions.
- We use this simpler characterization to construct a family of finitely supported extreme correlated equilibria in Proposition 8.24.
- Then we note that all extreme correlated equilibria of the games in Examples 8.14 and 8.19 are of this form, so this allows us to count the extreme correlated equilibria and determine their asymptotic rate of growth as the number of pure strategies grows.

- Using the ergodic theory introduced in Section 8.1.2 we construct in Proposition 8.26 a large family of extreme correlated equilibria without finite support for the game in Examples 8.15 and 8.20.

- Finally we show that if a set can be represented by finitely many moments then all its extreme points have uniformly bounded finite support. This shows that the set of correlated equilibria of the game in Examples 8.15 and 8.20 cannot be represented by finitely many moments and completes the analysis.

Having completed the roadmap, we are ready to begin. Defining correlated equilibria in games with infinite strategy sets such as polynomial games requires a bit more care than defining these in finite games. The standard definition as in e.g. [38] is to modify Definition 2.5 by only requiring the incentive condition hold for measurable functions $\zeta_i$. This definition is not well suited for computation, so we use the following equivalent characterization.

**Proposition 8.21** ([70]). *A probability distribution $\mu \in \Delta(C_X \times C_Y)$ is a correlated equilibrium of a game satisfying Assumption 8.13 if and only if*

$$\int_{A\times J} (xy - x'y)\, d\mu(x,y) \geq 0 \quad \textit{and} \quad \int_{J\times A} (xy - xy')\, d\mu(x,y) \leq 0$$

*for all $x' \in C_X$, $y' \in C_Y$, and measurable $A \subseteq J$.*

*Proof.* When $\mu$ is finitely supported this is clearly equivalent to Definition 2.5. The general case is part (1) of Corollary 2.14 in [70] with the present utilities substituted in. □

Note that these conditions are homogeneous (that is, invariant under positive scaling) in $\mu$. The only condition on $\mu$ that is not homogeneous is the probability measure condition $\mu(J \times J) = 1$. We will often ignore this condition to avoid having to normalize every expression, referring to a measure $\mu \in \Delta^*(C_X \times C_Y)$ satisfying the conditions of the proposition as a correlated equilibrium.

**Definition 8.22.** *When we need to distinguish these notions, we will refer to a measure $\mu \in \Delta^*(C_X \times C_Y)$ satisfying the conditions of Proposition 8.21 as a* ***homogeneous correlated equilibrium*** *and a measure $\mu \in \Delta(C_X \times C_Y)$ satisfying the conditions as a* ***proper correlated equilibrium****. In the context of homogeneous correlated equilibria the term* ***extreme*** *will refer to extreme rays; for proper correlated equilibria it will refer to extreme points.*

When $\mu \neq 0$ is a homogenous correlated equilibrium, $\frac{1}{\mu(J\times J)}\mu$ is a proper correlated equilibrium. The set of homogenous correlated equilibria is a convex cone. The extreme rays of this cone are exactly those measures which are positive multiples of the extreme points of the set of proper correlated equilibria.

The following proposition characterizes correlated equilibria of games satisfying Assumption 8.13 and is analogous to Proposition 8.16 for Nash equilibria. Note how the Nash equilibrium measures were characterized in terms of their moments but the correlated equilibria are not. Whereas the Nash equilibria are pairs of mixed strategies with zero mean for each player, condition (3) of this proposition says that the correlated equilibria are joint distributions such that regardless of each player's own recommendation, the conditional mean of his opponent's recommended strategy is zero.

**Proposition 8.23.** *For a game satisfying Assumption 8.13 and a measure $\mu \in \Delta^*(C_X \times C_Y)$ such that $xy \neq 0$ $\mu$-a.e., the following are equivalent:*

1. *$\mu$ is a correlated equilibrium;*

2. $$\kappa_x(A) := \int_{A\times J} xy\, d\mu(x,y) \quad \textit{and} \quad \kappa_y(A) := \int_{J\times A} xy\, d\mu(x,y)$$
   *are both the zero measure, i.e., equal zero for all measurable $A \subseteq J$;*

3. $$\lambda_x(A) := \int_{A\times J} y\, d\mu(x,y) \quad \textit{and} \quad \lambda_y(A) := \int_{J\times A} x\, d\mu(x,y)$$
   *are both the zero measure.*

*Proof.* ($1 \Rightarrow 2$) We will consider only $\kappa_x$; $\kappa_y$ is similar. The conditions of Proposition 8.21 with $A = J$ imply that

$$x' \int_{J\times J} y\, d\mu(x,y) \le \int_{J\times J} xy\, d\mu(x,y) \le y' \int_{J\times J} x\, d\mu(x,y)$$

for all $x' \in C_X, y' \in C_Y$. By assumption it is possible to choose $x'$ and $y'$ either positive or negative, so $\int_{J\times J} xy\, d\mu(x,y) = 0$. A similar argument with any $A$ implies that $\int_{A\times J} xy\, d\mu(x,y) \ge 0$. Therefore we have

$$0 = \int_{J\times J} xy\, d\mu(x,y) = \int_{A\times J} xy\, d\mu(x,y) + \int_{(J\setminus A)\times J} xy\, d\mu(x,y) \ge 0 + 0 = 0$$

for all $A$, so the inequality must be tight and we get $\int_{A\times J} xy\, d\mu(x,y) = 0$ for all $A$.

($2 \Leftrightarrow 3$) By definition $d\kappa_x = x\, d\lambda_x$ and by assumption $\lambda_x(0) = 0$. If one of these measures is zero then so is the other, and respectively with $y$ in place of $x$.

($2$ & $3 \Rightarrow 1$) The integrals in Proposition 8.21 vanish. □

**Proposition 8.24.** *Fix a game satisfying Assumption 8.13. Let $k > 0$ be even and $x_1, \ldots, x_{2k}$ and $y_1, \ldots, y_{2k}$ be such that:*

1. *$x_i \in C_X$ and $y_i \in C_Y$ are all nonzero;*
2. *the sequence $x_1, x_3, \ldots, x_{2k-1}$ has distinct elements and alternates in sign;*
3. *the sequence $y_1, y_3, \ldots, y_{2k-1}$ has distinct elements and alternates in sign;*
4. *$x_{2i} = x_{2i-1}$ and $y_{2i} = y_{2i+1}$ for all $i$ when subscripts are interpreted mod $2k$.*

*Then $\mu = \sum_{i=1}^{2k} \frac{1}{|x_i y_i|}\delta_{(x_i,y_i)}$ is an extreme correlated equilibrium.*

*Proof.* To show that $\mu$ is a correlated equilibrium define $d\kappa(x,y) = xy\,d\mu(x,y)$. Then $\kappa = \sum_{i=1}^{2k} \operatorname{sign}(x_i)\operatorname{sign}(y_i)\delta_{(x_i,y_i)}$. Defining the projection $\kappa_x$ as in Proposition 8.23, we have

$$\begin{aligned}\kappa_x &= \sum_{i=1}^{2k} \operatorname{sign}(x_i)\operatorname{sign}(y_i)\delta_{x_i} = \sum_{i=1}^{k} \operatorname{sign}(x_{2i})\left(\operatorname{sign}(y_{2i}) + \operatorname{sign}(y_{2i-1})\right)\delta_{x_{2i}} \\ &= \sum_{i=1}^{k} \operatorname{sign}(x_{2i})(0)\delta_{x_{2i}} = 0,\end{aligned}$$

because $x_{2i} = x_{2i-1}$ and $y_{2i}$ differs in sign from $y_{2i-1}$ by assumption. The same argument shows that $\kappa_y = 0$, so $\mu$ is a correlated equilibrium.

To see that $\mu$ is extreme, suppose $\mu = \mu' + \mu''$ where $\mu'$ and $\mu''$ are correlated equilibria. Clearly $\mu' = \sum_{i=1}^{2k} \alpha_i \delta_{(x_i,y_i)}$ for some $\alpha_i \geq 0$. Define $d\kappa' = xy\,d\mu'(x,y)$, so $\kappa' = \sum_{i=1}^{2k} \alpha_i x_i y_i \delta_{(x_i,y_i)}$. By assumption

$$\kappa'_x = \sum_{i=1}^{k} x_{2i}\left(\alpha_{2i-1}y_{2i-1} + \alpha_{2i}y_{2i}\right)\delta_{x_{2i}}$$

is the zero measure. Since the $x_{2i}$ are distinct and nonzero we must have $\alpha_{2i-1}y_{2i-1} + \alpha_{2i}y_{2i} = 0$ for all $i$. Similarly since $\kappa'_y = 0$ we have $\alpha_{2i+1}x_{2i+1} + \alpha_{2i}x_{2i} = 0$ for all $i$ (with subscripts interpreted mod $2k$).

The $x_i$ and $y_i$ are all nonzero, so fixing one $\alpha_i$ fixes all the others by these equations. That is to say, these equations have a unique solution up to multiplication by a scalar, so $\mu'$ is a positive scalar multiple of $\mu$. But the splitting $\mu = \mu' + \mu''$ was arbitrary, so $\mu$ is extreme. □

An argument along the lines of the proof of Proposition 8.24 shows that any finitely supported correlated equilibrium $\mu$ whose support does not contain any

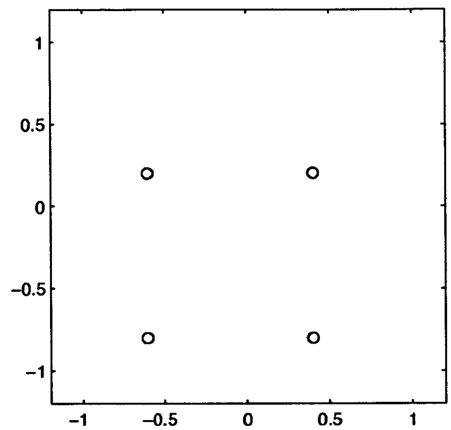


**Figure 8.1.** The support of an extreme correlated equilibrium. In the notation of Proposition 8.24, $k = 2$, $x_1 = 0.4$, $x_3 = -0.6$, $y_1 = 0.2$, and $y_3 = -0.8$.

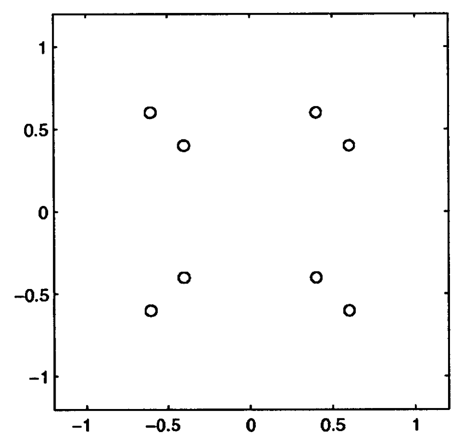


**Figure 8.2.** The support of another extreme correlated equilibrium. In the notation of Proposition 8.24, $k = 4$, $x_1 = 0.4$, $x_3 = -0.4$, $x_5 = 0.6$, $x_7 = -0.6$, $y_1 = 0.6$, $y_3 = -0.4$, $y_5 = 0.4$, and $y_7 = -0.6$.

points with $x = 0$ or $y = 0$ can be written as $\mu = \mu' + \mu''$ where $\mu' \neq 0$ is a correlated equilibrium and $\mu'' \neq 0$ is a correlated equilibrium of the form studied in Proposition 8.24. Therefore a finitely supported $\mu$ cannot be extreme unless it is of this form.

**Example 8.25** (continues Example 8.19). For some examples of the supports of extreme correlated equilibria of games of this type, see Figures 8.1 and 8.2.

To count the number of extreme correlated equilibria of this game we must count the number of essentially different sequences of $x_i$ and $y_i$ of the type mentioned in Proposition 8.24. Fix $k$ and let $k = 2r$ where $1 \leq r \leq l$. Note that cyclically shifting the sequences of $x_i$'s and $y_i$'s by two does not change $\mu$, nor does reversing the sequence. Therefore we can assume without loss of generality that $x_1, y_1 > 0$. We then have $l$ possible choices for $x_1, y_1, x_3$, and $y_3$, $l-1$ possible choices for $x_5, x_7, y_5$, and $y_7$, etc., for a total of $\left(\frac{l!}{(l-r)!}\right)^4$ possible choices of the $x_i$ and $y_i$. These will always be essentially different (i.e., give rise to different $\mu$) unless we cyclically permute the sequences of $x_i$ and $y_i$ by some multiple of four, in which case the resulting sequence is essentially the same. The number of such cyclic permutations is $r$. Therefore the total number of extreme correlated equilibria is

$$e(n) = \sum_{r=1}^{n} \frac{1}{r} \left(\frac{l!}{(l-r)!}\right)^4.$$

We will see that $e(l) = \Theta\left(\frac{1}{l}(l!)^4\right)$. That is to say, $e(l)$ is asymptotically upper and lower bounded by a constant times $\frac{1}{l}(l!)^4$. The expression $\frac{1}{l}(l!)^4$ is just the final term in the summation for $e(l)$, so the lower bound is clear. Define

$$f(l) := \frac{e(l)}{\frac{1}{l}(l!)^4} = \sum_{r=0}^{l-1} \frac{l}{l-r} \cdot \frac{1}{(r!)^4},$$

where the second equality is by reindexing. Then $f(l) \geq 1$ for all $l$. We will now show that $f(l)$ is also bounded above. Intuitively this is not surprising since the terms in the summation for $f(l)$ die off extremely quickly as $s$ grows.

For all $1 \leq r < l-1$ we have that the ratio of term $r+1$ in the summation to term $r$ is:

$$\frac{\frac{l}{l-r-1} \cdot \frac{1}{((r+1)!)^4}}{\frac{l}{l-r} \cdot \frac{1}{(r!)^4}} = \frac{l-r}{l-r-1} \cdot \frac{1}{(r+1)^4} \leq \frac{1}{8},$$

so for $l > 1$ we can bound the sum by a geometric series:

$$f(l) - 1 = \sum_{r=1}^{l-1} \frac{l}{l-r} \cdot \frac{1}{(r!)^4} \leq \frac{l}{l-1} \sum_{r=0}^{\infty} \frac{1}{8^r} = \frac{8l}{7(l-1)} \leq \frac{16}{7}.$$

Therefore $1 \leq f(l) \leq \frac{23}{7}$ for all $l$, so $e(l) = \Theta\left(\frac{1}{l}(l!)^4\right)$ as claimed. Comparing this to the results of the previous section in which we saw that the number of extreme Nash equilibria of this game is $l^4$, we see that in this case there is a super-exponential separation between the number of extreme Nash and the number of extreme correlated equilibria. This implies, for example, that computing all extreme correlated equilibria is not an efficient method for computing all extreme Nash equilibria, even though all extreme Nash equilibria are extreme correlated equilibria and recognizing whether an extreme correlated equilibrium is an extreme Nash equilibrium is easy. There are simply too many extreme correlated equilibria. ◊

Next we prove a more abstract version of Proposition 8.24 which includes certain extreme points which are not finitely supported. To do so we use the ergodic theory concepts introduced in Section 8.1.2.

**Proposition 8.26.** *Fix measures $\nu_1, \nu_2, \nu_3$, and $\nu_4 \in \Delta^*((0,1])$ and maps $f_i : (0,1] \to (0,1]$ such that $\nu_{i+1} = \nu_i \circ f_i^{-1}$ (interpreting subscripts mod 4). The portion of the measure $\mu$ in the $i^{th}$ quadrant of $J \times J$ will be constructed in terms of $f_i$ and $\nu_i$. Define $j_i : (0,1] \to J \times J$ by $j_1(x) = (x, f_1(x))$, $j_2(x) = (-f_2(x), x)$, $j_3(x) = (-x, -f_3(x))$, and $j_4(x) = (f_4(x), -x)$. Let $|\kappa| = \sum_{i=1}^4 \nu_i \circ j_i^{-1}$. If Assumption 8.13 is satisfied,* $\operatorname{supp}|\kappa| \subseteq C_X \times C_Y$*, and $\frac{1}{|xy|} \in \mathcal{L}^1(|\kappa|)$ then $d\mu = \frac{1}{|xy|}\, d|\kappa|$ is a correlated equilibrium.*

*By assumption $f_4 \circ f_3 \circ f_2 \circ f_1 : (0,1] \to (0,1]$ is $\nu_1$-measure preserving. If it is also ergodic with respect to $\nu_1$, then $\mu$ is extreme.*

*Proof.* First we must show that $\mu$ is a correlated equilibrium. It is a finite measure by the assumption $\frac{1}{|xy|} \in \mathcal{L}^1(|\kappa|)$ and $xy \neq 0$ $\mu$-a.e. by definition. Define $g : J \times J \to J \times J$ as follows.

$$g(x,y) = \begin{cases} j_1(x) & \text{if } x > 0, y < 0 \\ j_2(y) & \text{if } x > 0, y > 0 \\ j_3(-x) & \text{if } x < 0, y > 0 \\ j_4(-y) & \text{if } x < 0, y < 0 \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

The function $g$ is $|\kappa|$-measure preserving. To see this fix any measurable set $B \subseteq (0,1] \times (0,1]$. Let $A = j_1^{-1}(B)$. Then $|\kappa|(B) = |\kappa|(A \times (0,1]) = \nu_1(A)$ by definition of $|\kappa|$. But $g^{-1}(B) = g^{-1}(A \times (0,1]) = A \times [-1,0)$, so

$$\begin{aligned} |\kappa|(g^{-1}(B)) &= |\kappa|(A \times [-1,0)) = \nu_4(j_4^{-1}(A \times [-1,0))) = \nu_4(f_4^{-1}(A)) \\ &= \nu_1(A) = |\kappa|(B). \end{aligned}$$

Therefore $g$ is measure preserving for subsets of $(0,1] \times (0,1]$. The arguments for the other quadrants are similar and since $g$ maps each quadrant into a different quadrant, $g$ is measure preserving on its entire domain.

Define the signed measure $\kappa$ by $d\kappa = xy\, d\mu = \operatorname{sign}(x)\operatorname{sign}(y)\, d|\kappa|$. We have seen that $|\kappa|(A \times (0,1]) = |\kappa|(A \times [-1,0))$, so $\kappa(A \times (0,1]) = -\kappa(A \times [-1,0))$. Since $\kappa(A \times \{0\}) = 0$, we have $\kappa(A \times J) = 0$, or using the terminology of Proposition 8.23, $\kappa_x(A) = 0$. A similar argument implies $\kappa_x(A) = 0$ if $A \subseteq [-1,0)$.

Clearly $\kappa_x(0) = 0$ by definition of $\kappa_x$, so $\kappa_x$ is the zero measure. In the same way we can show that $\kappa_y$ is the zero measure, so $\mu$ is a correlated equilibrium by Proposition 8.23.

Now we will show via several steps that $\mu$ is extreme. Write $\mu = \mu_1 + \mu_2$ where the $\mu_i$ are nonzero correlated equilibria. Since these are all positive measures, the $\mu_i$ are absolutely continuous with respect to $\mu$. Define $d|\kappa_i| = |xy|\, d\mu_i$ and $d\kappa_i = xy\, d\mu_i$.

Next we show that $g$ is $|\kappa_i|$-measure preserving. We will demonstrate this fact for $B \subseteq (0,1] \times (0,1]$. As above, we define $A = j_1^{-1}(B)$. Then $|\kappa_i|(B) = |\kappa_i|(A \times (0,1])$ since $(A \times (0,1]) \triangle B$ has $|\kappa|$ measure zero and $|\kappa_i|$ is absolutely continuous with respect to $|\kappa|$. Furthermore, $|\kappa_i|(g^{-1}(B)) = |\kappa_i|(A \times [-1,0))$. But $\mu_i$ is a correlated equilibrium so $\kappa_i(A \times (0,1]) = -\kappa_i(A \times [-1,0))$. Hence $|\kappa_i|(g^{-1}(B)) = |\kappa_i|(A \times [-1,0)) = |\kappa_i|(A \times (0,1]) = |\kappa_i|(B)$. Again, the proof is the same for $B$ contained in other quadrants, so $g$ is $|\kappa_i|$-measure preserving.

For the second-to-last step we prove that $g$ is ergodic with respect to $|\kappa|$. Suppose $B \subseteq J \times J$ is such that $|\kappa|(g^{-1}(B) \triangle B) = 0$. Let $Q_i$ be the intersection of $B$ with the $i^{\text{th}}$ quadrant. Then $|\kappa|(g^{-1}(Q_{i+1}) \triangle Q_i) = 0$, so $|\kappa|(g^{-4}(Q_1) \triangle Q_1) = 0$. Let $A = j_1^{-1}(Q_1)$. Then $|\kappa|(g^{-4}(Q_1) \triangle Q_1) = \nu_1((f_4 \circ f_3 \circ f_2 \circ f_1)^{-1}(A) \triangle A) = 0$. By assumption the map $f_4 \circ f_3 \circ f_2 \circ f_1$ is ergodic, so $\nu_1(A) = 0$ or $\nu_1(A) = \nu_1((0,1]) = |\kappa|((0,1] \times (0,1])$. Therefore $|\kappa|(Q_1) = \nu_1(A) = 0$ or $|\kappa|(Q_1) = |\kappa|((0,1] \times (0,1])$. In either case since $g$ is $|\kappa|$-measure preserving we get $|\kappa|(Q_i) = |\kappa|(Q_1)$ for all $i$. Therefore $|\kappa|(B) = 0$ or $|\kappa|(B) = |\kappa|(J \times J)$, so $g$ is ergodic with respect to $|\kappa|$.

Normalizing $|\kappa|$ and $|\kappa_i|$ to be probability measures, we can apply Corollary 8.12 to obtain $|\kappa_i| = \frac{|\kappa_i|(J \times J)}{|\kappa|(J \times J)} |\kappa|$. By definition the set on which $|xy|$ is zero has $\mu$ measure zero. Therefore

$$d\mu_i = \frac{1}{|xy|}\, d|\kappa_i| = \frac{|\kappa_i|(J \times J)}{|\kappa|(J \times J)} \frac{1}{|xy|}\, d|\kappa| = \frac{|\kappa_i|(J \times J)}{|\kappa|(J \times J)}\, d\mu,$$

so $\mu_i = \frac{|\kappa_i|(J \times J)}{|\kappa|(J \times J)} \mu$ and $\mu$ is extreme. □

Above we have constructed $\mu$ and $g$ so that $g$ maps the quadrants counterclockwise – quadrant 1 to quadrant 2, etc. However, the same argument would go through if $g$ mapped the quadrants clockwise.

To view Proposition 8.24 as a special case of Proposition 8.26, let each $\nu_i$ be a uniform probability measure over a finite subset of $(0,1]$. The function $g$ is defined by $g(x_i, y_i) = (x_{i+1}, y_{i+1})$ and the $f_i$ are defined to be compatible with this. The map $f_4 \circ f_3 \circ f_2 \circ f_1$ is a permutation on the support of $\nu_1$, which is precisely the positive values of $x_i$. By construction this permutation consists of a single cycle, hence it is ergodic.

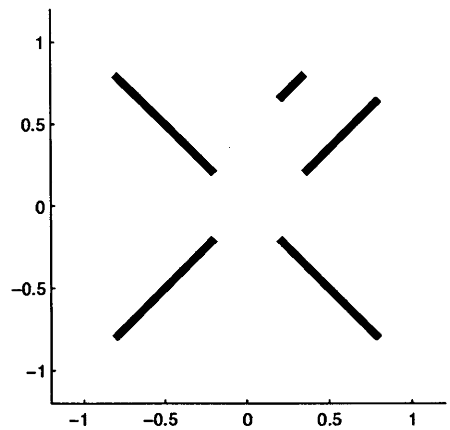


**Figure 8.3.** The support set of an extreme correlated equilibrium which is not finitely supported. Extremality of this equilibrium depends sensitively on the choices of endpoints for the line segments. In this case there are segments connecting: $(0.2, -0.2)$ to $(0.8, -0.8)$; $(-0.2, -0.2)$ to $(-0.8, -0.8)$; $(-0.2, 0.2)$ to $(-0.8, 0.8)$; $\left(0.2, 0.2 + \frac{1}{\sqrt{5}}\right)$ to $\left(0.8 - \frac{1}{\sqrt{5}}, 0.8\right)$; and $\left(0.8 - \frac{1}{\sqrt{5}}, 0.2\right)$ to $\left(0.8, 0.2 + \frac{1}{\sqrt{5}}\right)$.

**Example 8.27** (continues Example 8.20). We can combine Example 8.10 and Proposition 8.26 to exhibit extreme points of the set of correlated equilibria for this game which are not finitely supported. Let $0 < a < b < 1$. Let $\nu_i$ be Lebesgue measure on $[a, b)$ for all $i$. Fix $\alpha$ such that $\frac{\alpha}{b-a}$ is irrational. Define $f_1 : [a, b) \to [a, b)$ by $f(x) = (x - a + \alpha \mod (b - a)) + a$. This is just an affinely scaled version of Example 8.10 so $f_1$ is $\nu_i$-measure preserving and ergodic. Define $f_1$ on $(0, 1] \setminus [a, b)$ arbitrarily, because that is a set of measure zero. Let $f_2, f_3, f_4 : (0, 1] \to (0, 1]$ be the identity. These data satisfy all the assumptions of Proposition 8.26. In particular, since $0 < a < b < 1$, $xy$ is bounded away from zero on the support of $|\kappa|$. Therefore $\frac{1}{|xy|} \in \mathcal{L}^1(|\kappa|)$. Since $\nu_i$ is not finitely supported, $\mu$ is an extreme correlated equilibrium which is not finitely supported. The support of $\mu$ is shown in Figure 8.3 with parameters $a = 0.2$, $b = 0.8$, and $\alpha = \frac{1}{\sqrt{5}}$. ◇

**Definition 8.28.** *Given a compact Hausdorff space $K$ we say that a set of measures $\mathcal{M} \subseteq \Delta^*(K)$ is **describable by moments** if there exists an integer $d$, bounded Borel measurable maps $g_1, \ldots, g_d : K \to \mathbb{R}$, and a set $M \subseteq \mathbb{R}^d$ such that a measure $\mu$ is in $\mathcal{M}$ if and only if $\left(\int g_1\, d\mu, \ldots, \int g_d\, d\mu\right) \in M$.*

The results of [43] show that the maximin and minimax strategy sets of a two-player zero-sum polynomial game can always be described by moments.

Introducing a similar notion for $n$-tuples of measures, the set of Nash equilibria can always be described by moments in any polynomial game [69]. However, combining this example with the following proposition we see that the set of correlated equilibria of a polynomial game cannot in general be described by moments.

This is important because the finite-dimensional representation in terms of moments is the primary tool for computing and characterizing Nash equilibria of polynomial games. One is therefore naturally drawn to try to find such a representation for the set of correlated equilibria. The example and this proposition show that no such representation exists in general.

**Proposition 8.29.** *Let $\mathcal{M} \subseteq \Delta^*(K)$ be a set of measures describable by moments. Then all extreme points of $\mathcal{M}$ have finite support and this support is uniformly bounded by $d$, where $d$ is the integer associated with the description of $\mathcal{M}$ by moments.*

*Proof.* Let $g_1, \ldots, g_d : K \to \mathbb{R}$ be the maps describing $\mathcal{M}$. Suppose there exists a measure $\mu \in \mathcal{M}$ which is extreme and supported on more than $d$ points, so we can partition the domain of $\mu$ into $d+1$ sets $B_1, \ldots, B_{d+1}$ of positive measure. For $c = (c_1, \ldots, c_{d+1}) \in \mathbb{R}^{d+1}_{\geq 0}$, define $\mu_c = \sum_{i=1}^{d+1} c_i \mu|_{B_i}$. The map $c \mapsto \mu_c$ is injective. Define

$$K = \left\{ c \in \mathbb{R}^{d+1}_{\geq 0} \;\middle|\; \int g_i \, d\mu_c = \int g_i \, d\mu \text{ for } i = 1, \ldots, d \right\},$$

so $(1, 1, \ldots, 1) \in K$. Linearity of integration implies that the nonempty set $K$ is the intersection of an affine space of dimension at least one with the positive orthant. By Carathéodory's theorem (or equivalently the statement that a feasible linear program has a basic feasible solution), the extreme points of $K$ each have at most $d$ nonzero entries [5]. Thus $(1, 1, \ldots, 1)$ is not an extreme point of $K$, so we can write $(1, 1, \ldots, 1) = \lambda c + (1 - \lambda) c'$ for $0 < \lambda < 1$ and $(1, 1, \ldots, 1) \neq c, c' \in K$. Therefore $\mu = \mu_{(1,1\ldots,1)} = \lambda \mu_c + (1 - \lambda) \mu_{c'}$ is not extreme. $\square$

# Chapter 9

# Future Directions

The motivation for the work presented in this thesis has been to push the concept of correlated equilibrium and its associated machinery as far as possible. We have focused on correlated equilibria which possess extra structure, and when such equilibria can be proven to exist or even be computed using techniques designed for correlated equilibria. A simple source of such structures is results about games of a particular type having Nash equilibria of a special form.

The main motivating example of this thesis has been symmetric games, which have symmetric Nash equilibria. We have observed that these have extra structure, in the sense that even if we ignore their product form, they lie in the convex set $\Delta_G^X(\Gamma)$ which is smaller than the convex set $\Delta_G(\Gamma)$ in which the symmetric correlated equilibria lie. Indeed $\Delta_G^X(\Gamma)$ may even have additional symmetries which $\Delta_G(\Gamma)$ does not have, and so be lower-dimensional than $\Delta_G(\Gamma)$. An example not based on symmetry is polynomial games, discussed in Chapter 8. Nash equilibria in polynomial games are tuples of probability distributions over $[-1, 1]$, but such games admit Nash equilibria with finite support, i.e., in which only finitely many strategies are played with positive probability. These are in particular correlated equilibria with finite support.

The existence of Nash equilibria of these special forms is easy to prove using the same fixed point techniques traditionally used to prove the existence of Nash equilibria. Forgetting about the product structure, we can instead try to prove weaker statements about existence of the corresponding types of correlated equilibria, using convex methods such as the Minimax Theorem (or equivalently separating hyperplanes or linear programming duality).

This is more than just a mathematical exercise. When such proofs do work, as in the case of exchangeable equilibria, they can lead to efficient algorithms for computing the corresponding equilibria, and such algorithms are generally believed not to exist for Nash equilibria. Furthermore, they make stronger predictions about the play of the game than correlated equilibria under weaker assumptions on the players' behavior than is required for Nash equilibria.

In other cases, such as higher order exchangeable equilibria of symmetric games and finitely supported correlated equilibria of polynomial games, such proofs remain elusive. Of course, it is difficult to draw positive conclusions from such a negative statement – perhaps this reflects nothing more than a limitation of the author's creativity. However, there is a common thread between these two difficult cases which suggests that something deeper may be involved.

## 9.1 Open questions

We close with a discussion of this and other future research directions suggested by the results of this thesis.

### 9.1.1 Higher order exchangeable equilibria

As shown in Chapter 5, the existence of higher order exchangeable equilibria implies Nash's theorem in full generality. So a new, perhaps convexity-based, proof of the existence of higher order exchangeable equilibria would lead to a new proof of Nash's theorem. This would likely also yield algorithms for computing order $k$ exchangeable equilibria efficiently for fixed $k$, and so extend both the existence and computational results for exchangeable equilibria proven in this thesis.

Since the resulting proof of Nash's theorem would use a limiting argument in $k$, this would not necessarily contradict the evidence that computing Nash equilibria is hard [10, 11, 13, 15, 16, 30, 63]. On a related note, all questions about the rate of convergence of [say, projections of] $\mathrm{XE}^k_G(\Gamma)$ to $\Delta(\mathrm{NE}_G(\Gamma))$ remain open, including the question of whether this convergence always or at least generically happens at finite $k$. It is plausible that genericity may be relevant because separation of the Nash equilibria (and so finiteness thereof) may play a role.

### 9.1.2 Finitely-supported correlated equilibria in polynomial games

To our knowledge no direct proof of the existence of finitely supported correlated equilibria in polynomial games is known. One can prove the existence of a Nash equilibrium whose support cardinality is bounded a priori in terms of the degrees of the utility functions, then derive bounds for correlated equilibria from this, but we know of no proof for correlated equilibria which does not go through the existence of Nash equilibria. In Chapter 8 we showed that attempts to prove such results by showing that any extreme correlated equilibrium has bounded or even finite support must fail.

Again, convexity-based proofs of these true statements could yield provably efficient algorithms for computing or approximating such equilibria. Several heuristics are known which seem to converge quickly in practice [70], but none is proven to run in polynomial time. This is a somewhat surprising state of affairs since in most ways polynomial games seem to behave like finite games [68], for which correlated equilibria can be computed easily.

### 9.1.3 The correlated equilibrium conundrum

The proof that every polynomial game has a finitely supported Nash equilibrium proceeds by identifying two mixed strategies in $\Delta([-1,1])$ if they are payoff equivalent, i.e., yield the same payoff regardless of the actions of the opponents. One then observes that the space of equivalence classes is finite-dimensional and each contains a finitely supported distribution.

Attempting to replicate this result for correlated equilibria fails; as we have seen the set of correlated equilibria of a polynomial game is not so easily reduced to a finite-dimensional object. The reason is that for the purposes of correlated equilibria, the distinction between two payoff equivalent strategies cannot be ignored.

**Example 9.1.** Let $\Gamma$ be the following symmetric bimatrix game:

$$A = B^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Let player $i$'s strategy set be $C_i = \{a_i, b_i, c_i\}$. The first two columns of $A$ are equal, as are the first two rows, so $a_1$ is payoff equivalent to $b_1$. Symmetrically, $a_2$ is payoff equivalent to $b_2$. Define two distributions over outcomes of $\Gamma$:

$$W^1 = \frac{1}{4}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad W^2 = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

One can verify that $W^1$ is a correlated equilibrium, and $W^2$ is obtained from $W^1$ by moving some mass from $(a_1, c_2)$ and $(c_1, a_2)$ to $(b_1, c_2)$ and $(c_1, b_2)$, which are profiles of payoff equivalent strategies. Nonetheless, $W^2$ is not a correlated equilibrium.

Why does this happen? The reason is that when we consider correlated strategies, we are interested not only in what utility our recommendations give us, but also in what information they give. While we may not care from a payoff perspective whether strategy $a_i$ or $b_i$ has been suggested, the two possibilities may allow us to make completely different inferences about the behavior of our opponent. It is this information which may be payoff-relevant. If player 1 receives the recommendation $b_1$ under $W^2$ then he knows with probability one that player 2 has received the recommendation $c_2$, hence it is in the interest of player 1 to deviate to $c_1$.

This corresponds to the fact that in Hart and Schmeidler's associated zero-sum game $\Gamma^0$ we have

$$u_M^0((a_1,a_2),(a_1,c_1)) = 1 \neq 0 = u_M^0((b_1,b_2),(a_1,c_1)),$$

so $(a_1,a_2)$ and $(b_1,b_2)$ are not payoff equivalent in $\Gamma^0$. $\diamondsuit$

This same issue blocks a natural attempt to prove the existence of higher order exchangeable equilibria. In $\Xi^k\Gamma$ the additive separability of the utilities means that any mixed strategy in $\Delta(C_i^k)$ for player $i$ is payoff equivalent to the product of its marginals. Applying this transformation to the strategies of each player maps $\Delta^{\Pi}_{G\times S_k}(\Xi^k\Gamma)$ to $\Delta^{\Pi}_{G\times S_k}(\Pi^k\Gamma)$. If this transformation changed strategies for the maximizer in $(\Xi^k\Gamma)^0$ to payoff equivalent strategies, then we could apply it within the proof of Theorem 3.16 to get correlated equilibria of $\Xi^k\Gamma$ in $\Delta^{X}_{G\times S_k}(\Pi^k\Gamma)$, i.e., order $k$ exchangeable equilibria of $\Gamma$. But as we have seen moving mass between payoff equivalent strategies in $\Gamma$ can "break" correlated equilibria, and this proof attempt fails.

Perhaps there is some subclass of correlated equilibria bigger than the Nash equilibria but without this fragility – maybe correlated equilibria arising from a particular type of correlation scheme. An appropriate definition and constructive proof of existence of such equilibria could resolve the questions raised in the previous subsections. On the other hand the nonexistence of such a class would point to a strange fundamental lack of robustness in game-theoretic correlation.

### 9.1.4 Rational exchangeable equilibria

Another example of a statement which follows easily from Nash's Theorem for which we know of no other proof is the existence of rational exchangeable equilibria in bimatrix games. The symmetric case is Theorem 3.17 and a similar proof works in the asymmetric case. Therefore there is no algebraic obstruction to computing such equilibria exactly, but we do not know whether this can be done in polynomial time.

The Approximate Ellipsoid Against Hope algorithm can compute completely positive matrices which are almost correlated equilibria, and by rounding could compute correlated equilibria which are almost completely positive. It is not clear how to modify it to do both simultaneously.

Related questions for multiplayer games such as the complexity of determining whether a game admits a rational exchangeable equilibrium or computing one if it does are similarly open. Is there an obstruction to the existence of a rational exchangeable equilibrium which can be seen from the utilities directly, without resorting to computing all the exchangeable equilibria explicitly?

### 9.1.5 Symmetric identical interest games

An identical interest game is one in which all players receive the same utility at every outcome. Finding Nash equilibria of such games is trivial; any maximum of the utility function will do. But when the game is also symmetric, it is not clear that this will help find symmetric Nash equilibria. Many of the examples in this thesis are symmetric identical interest games, and illustrate that the structure of the symmetric equilibria is less trivial than that of the asymmetric equilibria. Can symmetric Nash equilibria of symmetric identical interest games be computed more efficiently than Nash equilibria in general, or is this problem also PPAD-complete?

On a related note, a computer search of 10,000 randomly generated $4 \times 4$ symmetric identical interest games did not find any in which the (generically unique) exchangeable equilibrium of maximum expected utility was not in fact a symmetric Nash equilibrium. Is the natural conjecture true, and does it generalize? For larger games, computational experiments would be difficult due to the difficulty of optimizing over exchangeable equilibria. Can exchangeable equilibria nonetheless help compute symmetric Nash equilibria of identical interest games?

### 9.1.6 Further computational questions

As shown in Section 7.1.7 it is NP-hard to optimize an arbitrary linear functional over exchangeable equilibria. What about optimizing utility (equivalent to social welfare by symmetry)?

Generally speaking ellipsoid-based algorithms are good for proving that a problem is solvable in polynomial time but are not effective in practice. Is there a more practical polynomial time algorithm for approximating exchangeable equilibria than the Approximate Ellipsoid Against Hope algorithm?

Is there some sort of "rounding" technique to go from exchangeable equilibria to $\epsilon$-Nash equilibria? Complexity results such as [12] suggest that the resulting $\epsilon$ would be strictly positive, but even for fixed small positive $\epsilon$ the complexity of computing such equilibria is open.

Are there generally applicable sufficient conditions for the uniqueness of exchangeable equilibrium which can be checked directly from the utilities? These would immediately imply uniqueness and polynomial-time computability of the symmetric Nash equilibrium.

### 9.1.7 Applications of exchangeable equilibria

Which symmetric games arise in practice and what does the notion of exchangeable equilibria say about these? It would be particularly interesting to find natural classes of games for which we know $\mathrm{XE}_G(\Gamma) \subsetneq \mathrm{CE}_G(\Gamma)$ (either by Theorem 3.11 or some other means) or $\mathrm{XE}_G(\Gamma) = \mathrm{conv}(\mathrm{NE}_G(\Gamma))$. One possible approach would be to extend the notion of exchangeable equilibria to games of incomplete information. Many games such as auctions can be viewed as symmetric if analyzed from the perspective of players who have not yet learned their values / types.

However, the definition of exchangeable equilibrium rests on the definition of correlated equilibrium, and it is not clear what this should be in games of incomplete information [26,27]. Or rather, there are a variety of possibilities depending on which information structures one wishes to allow. One must address questions like whether the correlating device should be correlated with the players' types, etc. Different kinds of correlated equilibrium will lead to different notions of exchangeable equilibrium and it remains to be seen what predictions these will make and to what extent they share the properties of the complete information case covered in this thesis.

### 9.1.8 Structuralist game theory

The motivation for much of the work in this thesis has been to put together a picture of game theory analogous to the picture mathematicians have of their more traditional subjects, such as algebraic topology. That is to say, the goal is to understand games through structure-preserving maps and their effects on equilibria. Though much of the work in game theory is mathematically sophisticated, this viewpoint is not currently[1] widespread, or at least not overtly discussed.

There appears to be an immediate roadblock for this approach: a category of games would need a notion of a map between games, but it is not obvious what that should mean. There is a reasonable definition of isomorphism, but these alone do not give an interesting category. Attempts have been made to add other morphisms [75], but so far these seem forced and have limited applicability.

**Functorial transformations** Nonetheless, the approach is not dead in the water. There are various transformations one can apply which "should" be analogous to functors, if only games formed a category. The behavior of these can tell us a lot.

For example, given a game $\Gamma$ one can form its mixed extension $\Gamma^\Delta$, whose pure strategies are the mixed strategies of $\Gamma$ and whose utilities are extended

[1]This perspective is implicit in the early work on game theory by pure mathematicians, such as Dresher, Gale, Shapley, Gross, Karlin, etc., before game theory became a major field in its own right. This view seems to have faded as game theory was subsumed by economics.

by linearity. If $\Gamma$ is a finite game then $\Gamma^{\Delta}$ has simplices for strategy spaces and multilinear utility functions. In Chapter 8 we analyzed the mixed extension of matching pennies.

The behavior of Nash equilibria under the transformation $(-)^{\Delta}$ is straightforward. By definition the pure Nash equilibria of $\Gamma^{\Delta}$ are the mixed Nash equilibria of $\Gamma$. The mixed Nash equilibria of $\Gamma^{\Delta}$ are related to those of $\Gamma$ in a simple way. The game $\Gamma^{\Delta}$ has a unique Nash equilibrium if and only if the unique Nash equilibrium of $\Gamma$ is in pure strategies.

The behavior of correlated equilibria under $(-)^{\Delta}$ is subtler. A canonical map sends correlated equilibria of $\Gamma^{\Delta}$ to correlated equilibria of $\Gamma$ and this is onto; every correlated equilibrium of $\Gamma$ lifts to a correlated equilibrium of $\Gamma^{\Delta}$ in a simple way. But this map also sends many distributions which are not correlated equilibria of $\Gamma^{\Delta}$ to correlated equilibria of $\Gamma$. Even in simple cases, such as when $\Gamma$ is "Matching Pennies," the set of correlated equilibria of $\Gamma^{\Delta}$ can be extremely complex as discussed in Chapter 8 above.

Another "functor" takes a standard symmetric game $\Gamma$ and produces the symmetric $N$-player game $\Gamma^{(N)}$ introduced in Section 4.3. The behavior of symmetric Nash equilibria is easy to understand: $x^{\otimes n}$ is a symmetric Nash equilibrium of $\Gamma$ if and only if $x^{\otimes N}$ is one of the $N$-player extension $\Gamma^{(N)}$.

Again a natural map sends symmetric correlated equilibria of $\Gamma^{(N)}$ to symmetric correlated equilibria of $\Gamma$. This time, however, the map need not be onto; some symmetric correlated equilibria of $\Gamma$ may not lift to $\Gamma^{(N)}$. Those which do lift for all $N$ are exactly the exchangeable equilibria.

A variety of transformations have been introduced in the literature, such as symmetrizations which form symmetric games out of arbitrary games and products which combine multiple games into one. Typically the effects of such operations on the Nash equilibria are well-understood by design, but the effects on other solution concepts, in particular correlated equilibria, less so. It would be fruitful to study such examples in detail.

At a deeper level, is it possible to understand the entire collection of such "functorial" transformations? Doing so would require fleshing out the intuitive idea of naturality possessed by the examples above. One approach would be to solve the problem of turning the collection of games into an honest category, rather than merely an analogy.

**Formal solution concepts** Another approach would consider transformations which respect strategic structure in the sense that they take games for which all solution concepts agree to games for which all solution concepts agree. Formalizing this would require a precise definition of solution concept. Beyond being a map from games to objects of some type (a **formal solution concept**, perhaps), solution

$$\begin{tikzcd}
\Gamma \arrow[dr] \arrow[drr] \arrow[drrr] \arrow[ddd, "T"'] & & & \\
& BR(\Gamma) \arrow[r] \arrow[d, "f_T"] & CE(\Gamma) \arrow[r] \arrow[d, dotted, leftrightarrow, "??"] & NE(\Gamma) \arrow[d, dotted, leftrightarrow, "?"] \\
& BR(T(\Gamma)) \arrow[r] & CE(T(\Gamma)) \arrow[r] & NE(T(\Gamma)) \\
T(\Gamma) \arrow[ur] \arrow[urr] \arrow[urrr] & & &
\end{tikzcd}$$

**Figure 9.1.** Schematic commutative diagram showing the effect of a transformation $T$, such as the mixed extension $(-)^{\Delta}$ or the $N$-player extension $(-)^{(N)}$ on solutions of a game $\Gamma$. The effect on the best response correspondence ($BR$) is predictable, and the effect on Nash equilibria ($NE$) is typically easy to understand. The effect on correlated equilibria ($CE$) tends to be the most interesting part. Note how Nash equilibria are a function of (subordinate to) correlated equilibria, which are in turn a function of the best response correspondence.

concepts are usually an "I know it when I see it" kind of thing.

To see that such a definition may be possible, define a formal solution concept $Q$ to be **subordinate** to $P$ if $Q$ factors through $P$. For example, Nash equilibria are subordinate to correlated equilibria: the Nash equilibria are the independent correlated equilibria. If a transformation $T$ and formal solution concept $P$ are such that $P \circ T = f_T \circ P$ for some $f_T$, we might say $T$ **determines** $P$, because $f_T$ shows how $P$ varies when $T$ is applied. When $P(\Gamma) = P(\Gamma')$ we have $Q(T(\Gamma)) = Q(T(\Gamma'))$.

If all "reasonable" solution concepts were subordinate to a fixed "master" solution concept $P$, we could take $P$-determining as the definition for "respecting strategic structure." One $P$ (not usually considered a solution concept in its own right) to which the most common solution concepts are all subordinate is the best response correspondence $BR$, which associates with a game the set of strategies for each player which are best responses to each probability distribution over opponents' strategies.

For example, Nash equilibria are the fixed points of the best response correspondence, hence subordinate to $BR$. Iterated elimination of strictly dominated strategies is subordinate to $BR$ because it is equivalent to correlated rationalizability, which is more obviously subordinate to $BR$. Both $(-)^{\Delta}$ and $(-)^{(N)}$ determine $BR$. For a schematic picture of these ideas see Figure 9.1.

For a non-example, the transformation which multiplies all utilities by $-1$ does not determine $BR$. We do not view a player acting to minimize his utility as rational when his preference is to maximize. In other words the backwards variants of standard solution concepts in which we replace maximization by minimization are not subordinate to $BR$.

**Desired outcomes** Far from just being "abstract nonsense," such work could have important consequences for our understanding of computational game theory. In particular, the PPAD-completeness proofs [10, 11, 15, 16] which give the state-of-the-art understanding of the complexity of computing Nash equilibria are based on reductions which do not seem to be natural in the senses discussed above. Is this a necessary feature of such reductions? Or is there a game-theoretic way to understand the reduction from $n$ player games to 4 player games (say)? An answer to this question, particularly a negative one, requires a concrete notion of naturality for transformations between games, and the strength of such an answer depends on the strength of this notion of naturality.

Another problem where these ideas could have an impact is finding classes of games for which equilibria can be computed efficiently. For example Kannan and Theobald's notion of the rank of a bimatrix game [42] defines classes of games for which we can compute equilibria, but these classes are not natural in even the most basic sense: they are not closed under positive affine transformations. Closing under such operations gives more natural classes, but it becomes hard to test membership in these classes. A better understanding of naturality may show how to expand these classes in a way which is natural, makes is easy to test membership, and does not increase the complexity of computing equilibria. Alternatively it may turn out that in this setting naturality and computation are fundamentally at odds.

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# Notation

| | |
|---|---|
| $\square$ | end of a proof |
| $\diamondsuit$ | end of an example |
| $:=$ | is defined to be equal to |
| $\mathbb{N}$ | natural numbers $\{1, 2, \ldots\}$ |
| $\mathbb{Z}$ | integers $\{0, \pm 1, \pm 2, \ldots\}$ |
| $\mathbb{Q}$ | rational numbers |
| $\mathbb{R}$ | real numbers |
| $\mathbb{R}_{\geq 0}$ | nonnegative real numbers |
| $\mathbb{C}$ | complex numbers |
| $\lVert - \rVert_p$ | $\ell_p$ norm |
| $\bullet$ | Euclidean inner product |
| $\overline{X}$ | closure of $X$ |
| $\mathrm{conv}(X)$ | convex hull of $X$ |
| $\overline{\mathrm{conv}}(X)$ | closure of the convex hull of $X$ |
| $\mathrm{cone}(X)$ | conic hull of $X$ |
| $\Delta(T)$ | set of regular Borel probability measures on $T$ |
| $\delta_t$ | Dirac measure / point mass at $t$ |
| $\mathrm{Prob}(Y \mid X)$ | conditional probability distribution of $Y$ given $X$ |
| $\Gamma$ | a game |
| $P$ | set of players |
| $n$ | number of players |
| $C_i$ | pure strategy set of player $i$ |
| $m_i$ | number of strategies for player $i$ |
| $m$ | maximum number of strategies for any player |
| $C$ | set of pure strategy profiles / outcomes |
| $s_i$ | strategy of player $i$ |
| $s_{-i}$ | profile of strategies for all players except $i$ |
| $u_i$ | utility / payoff function for player $i$ |
| $\Delta(\Gamma)$ | distributions over outcomes |

| | |
|---|---|
| $\Delta^{\Pi}(\Gamma)$ | product distributions over outcomes |
| $\mathrm{NE}(\Gamma)$ | Nash equilibria of $\Gamma$ |
| $\mathrm{CE}(\Gamma)$ | correlated equilibria of $\Gamma$ |
| $\mathrm{Mm}(\Gamma)$ | maximin strategies of the zero-sum game $\Gamma$ |
| $\mathrm{mM}(\Gamma)$ | minimax strategies of the zero-sum game $\Gamma$ |
| $\Gamma^0$ | Hart and Schmeidler's zero-sum game associated to $\Gamma$ |
| $\gamma(y_i)$ | Hart and Schmeidler's auxiliary game associated to $y_i$ |
| $G$ | a group |
| $\langle g_1, \ldots, g_k \rangle$ | subgroup generated by $g_1, \ldots, g_k$ |
| $S_n$ | group of permutations of $n$ letters |
| $\mathbb{Z}_n$ | group of integers mod $n$ |
| $\cdot$ | a group action |
| $(-)_G$ | $G$-invariant elements of $G$ action on $(-)$ |
| $\mu_{n \to m}$ | marginalization $\Delta_{S_n}(T^n) \to \Delta_{S_m}(T^m)$ |
| $\otimes$ | tensor product |
| $(-)^{\otimes n}$ | $n^{\text{th}}$ tensor power |
| $\mathrm{Sym}^n(-)$ | symmetric tensors in $(-)^{\otimes n}$ |
| $\mathrm{CP}_m^n$ | completely positive tensors in $(\mathbb{R}^m)^{\otimes n}$ |
| $\exists$ | there exists |
| $V^*$ | dual of vector space $V$ |
| $K^*$ | dual of cone $K$ |
| $\mathbb{R}[x_1, \ldots, x_m]^n$ | homogeneous polynomials of degree $n$ in $x_1, \ldots, x_m$ |
| $\Psi_{m,2n}$ | nonnegative polynomials in $\mathbb{R}[x_1, \ldots, x_m]^{2n}$ |
| $\Sigma_{m,2n}$ | sums of squares in $\mathbb{R}[x_1, \ldots, x_m]^{2n}$ |
| $\mathrm{DNN}_m^n$ | double nonnegative tensors in $(\mathbb{R}^m)^{\otimes n}$ |
| $\Delta_G^X(\Gamma)$ | generalized exchangeable distributions |
| $\mathrm{XE}_G(\Gamma)$ | (symmetric) exchangeable equilibria of $\Gamma$ |
| $e$ | all ones column vector |
| $I$ | identity matrix |
| $\mathrm{XE}_G(\Gamma, N)$ | $N$-exchangeable equilibria of $\Gamma$ |
| $\Gamma^{(N)}$ | $N$-player extension of $\Gamma$ |
| $\mathrm{XE}_G^k(\Gamma, N)$ | order $k$ $N$-exchangeable equilibria of $\Gamma$ |
| $\Pi^k\Gamma$ | $k^{\text{th}}$ power of $\Gamma$ |
| $\Xi^k\Gamma$ | contracted $k^{\text{th}}$ power of $\Gamma$ |
| $\mathrm{XE}_G^k(\Gamma)$ | order $k$ exchangeable equilibria of $\Gamma$ |
| $\Gamma^{\mathrm{Sym}}$ | symmetrization of $\Gamma$ |
| $r$ | number of strategies of the row player in a bimatrix game |
| $c$ | number of strategies of the column player in a bimatrix game |
| $\mathrm{XE}_-(\Gamma)$ | asymmetric exchangeable equilibria of bimatrix game $\Gamma$ |

| | |
|---|---|
| $N$ | number of strategy profiles |
| $M$ | number of correlated equilibrium constraints |
| $U$ | matrix of correlated equilibrium constraints |
| $\|U\|_\infty$ | largest value in correlated equilibrium constraints |
| $\epsilon$ | error tolerance |
| $\delta$ | quantization for rounding |
| $R$ | radius of bounding ball for ellipsoid algorithm |
| $v$ | stopping volume for ellipsoid algorithm |
| $\Delta^*(T)$ | set of finite Borel measures on $T$ |
| $J$ | interval $[-1, 1]$ |