A Partial Order Approach to Decentralized Control

by

Parikshit Shah

B.Tech., Indian Institute of Technology, Bombay (2003)
S.M., Stanford University (2005)

Submitted to the Department of Electrical Engineering and Computer
Science
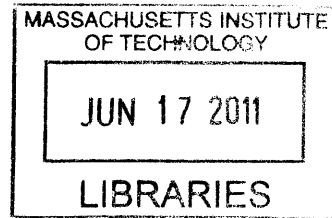in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2011

Author ....
Department of Electrical Engineering and Computer Science
April 29, 2011

Certified by ....
Pablo A. Parrilo
Professor
Thesis Supervisor

Accepted by...
Leslie A. Kolodziejski
Chairman, Department Committee on Graduate Students

A Partial Order Approach to Decentralized Control

by

Parikshit Shah

Submitted to the Department of Electrical Engineering and Computer Science
on April 29, 2011, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Abstract

In this thesis we consider the problem of decentralized control of linear systems. We employ the theory of partially ordered sets (posets) to model and analyze a class of decentralized control problems. Posets have attractive combinatorial and algebraic properties; the combinatorial structure enables us to model a rich class of communication structures in systems, and the algebraic structure allows us to reparametrize optimal control problems to convex problems.

Building on this approach, we develop a state-space solution to the problem of designing $\mathcal{H}_2$-optimal controllers. Our solution is based on the exploitation of a key separability property of the problem that enables an efficient computation of the optimal controller by solving a small number of uncoupled standard Riccati equations. Our approach gives important insight into the structure of optimal controllers, such as controller degree bounds that depend on the structure of the poset. A novel element in our state-space characterization of the controller is a pair of transfer functions, that belong to the incidence algebra of the poset, are inverses of each other, and are intimately related to estimation of the state along the different paths in the poset.

We then view the control design problem from an architectural viewpoint. We propose a natural architecture for poset-causal controllers. In the process, we establish interesting connections between concepts from order theory such as Möbius inversion and control-theoretic concepts such as state estimation, innovation, and separability principles. Finally, we prove that the $\mathcal{H}_2$-optimal controller in fact posseses the proposed controller structure, thereby proving the optimality of the architecture.

Thesis Supervisor: Pablo A. Parrilo
Title: Professor

Acknowledgments

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Decentralized versus Centralized Control

The advent of large-scale engineering systems has created a paradigm shift in the way systems are analyzed, designed, and built. Consider some typical examples of such large scale systems: the internet, vehicle platoons, satellite arrays, smart power grids, and sensor networks to name a few. These systems are diverse in their detailed physical behavior, but when viewed through the lens of systems theory exhibit an important common feature. From a systems-theoretic perspective, all these examples consist of an interconnection of a large number of *subsystems*, each of which individually is benign and amenable to standard analysis and design techniques. Indeed, these individual subsystems can all be analyzed via well-understood *centralized* systems theory. However, understanding the systems theory of these interconnected systems is a far more complex task.

This complexity creates interesting tension between theory and practice. Since many large scale systems require a decision-making/control layer (which requires information about the system), the notion of information exchange between subsystems plays a critical role. In practice, the systems engineer would like to keep information exchange to a minimum. Exchanging information between subsystems involves building communica-

tion channels which is often expensive. Even in cases where it may not be prohibitively expensive (for example wireless links), the *network* aspect of the problem may create significant challenges (such as interference). Establishing all-to-all communication (or alternatively a global centralized control center), even if feasible from a physical standpoint, presents the daunting challenge of collecting and processing all the information in one centralized location. Thus, the system designer has every incentive to keep communication among subsystems at a minimum. On the other hand, the theoretical underpinnings for such communication-constrained systems need to be better understood.

Thus, on the one end of the spectrum is the fully centralized systems and control theory, a classical and well understood area. On the other extreme of the spectrum are large-scale systems that demand decentralization, but where little theoretical understanding, and consequently knowledge of sound design principles, is available. This gulf between theory and practice compels us to develop a theory of decentralized control. Some of the critical questions that must be addressed include:

- "What kinds of communication constraints between subsystems should be enforced?"

- "Given a communication architecture, what is the optimal controller? How does the designer *compute* it?"

- "What are the broad architectural principles involved in such controller design?"

These are the motivating questions driving our work [47, 44, 48, 49] in this thesis. Indeed, these three questions serve as a broad outline of this thesis. We study these questions in a *linear systems* setup and devote a chapter to each of the above questions.

1.2 Information Flow in Systems

As emphasized earlier, a critical feature of large scale systems is the ability of different subsystems to communicate with each other. This communication may more abstractly be

16

viewed as a *flow of information*. Interestingly, the flow of information between subsystems is an important source of *complexity*. Understanding how to design controllers for systems with arbitrary communication is not always an easy task.

In a seminal paper, Witsenhausen [62] presented the now famous Witsenhausen counterexample. This example consisted of a seemingly simple two-step decision making problem with linear dynamics, quadratic cost function and Gaussian noise (a so-called "LQG" problem). He showed that optimal controllers for this simple problem were nonlinear, a surprising result at the time. In later work Mitter and Sahai [32] proved that in fact linear controllers could be "arbitrarily suboptimal" for this problem. At the heart of the complexity in this example is a tension between communication and control, or more broadly, the flow of information. In subsequent work [63, 37] the question of when optimal controllers are linear was addressed. Fundamental to this question, unsurprisingly, is the type of information flow, namely classical versus nonclassical information flow in systems.

From a computational standpoint, Papadimitriou and Tsitsiklis [36] showed that a problem related to the Witsenhausen counterexample was NP-hard. Indeed, several classes of decentralized control problems are now understood to be computationally intractable [36, 35, 54, 10, 9]. On the positive side, several authors have shown classes of problems to be computationally *tractable* [44, 47, 25, 5, 18, 58, 38]. The important distinction that seems to separate the tractable classes of problems from the intractable ones is the *flow of information* among subsystems. Developing the right language and set of tools to describe information flow and characterize tractable information flows is vital. In this thesis, we present *partially ordered sets* (posets) as a crucial tool that enables us to (a) describe information flows of a specific type, and (b) characterize a large class of tractable decentralized control problems.

1.2.1 Poset-Causal Information Flows

We mention two important notions that motivate this poset-based information structure. The first notion is that of *acyclic* information flow. In many areas of engineering and computer science problems have a natural underlying graph structure. In a large number of cases, it is known that "tree-like" or "acyclic" graph structures are more tractable than general graphs. We mention statistical inference on graphical models as an important motivating example [27, 2]. Another important notion in the context of decentralized decision-making is the notion of *transitivity* of information flow, which we clarify below.

As a motivating example consider (abstractly) a system consisting of three subsystems as shown in Fig.1-1. Each subsystem has a decision-maker, that makes certain choices based on the information available to him. The arrows indicate the flow of information. Subsystem 1's decision is a function of the information (formally captured by the notion of a state) available only at subsystem 1. Subsystem 1 communicates its state information to subsystem 2, so that subsystem 2's decision is a function of the state information of subsystem 2 as well as subsystem 1. Similarly, subsystem 2 communicates its state information to subsystem 3 so that the decision at subsystem 3 is a function of the state information of subsystems 2 and 3. (A subtle but significant clarification needs to be made regarding whether subsystem 3 has access to information about subsystem 1, which we discuss momentarily). In this sense the diagram in Fig. 1-1 describes a decision-making scenario where the available information at different subsystems is hierarchical in nature. It is convenient to view the available state information as an "information flow" with subsystems communicating local state information to other subsystems. In our setup, communication among subsystems is one-directional, comprising an *acyclic* information flow.

Another important feature in this setup, is whether or not subsystem 3 is allowed to access information about subsystem 1 or not. Suppose that subsystem 3 is not allowed to see this information. Since subsystem 2 is allowed to communicate its own information (in particular its own decisions) to subsystem 3, this may encourage subsystem 2 to engage

Figure 1-1: An example of a decentralized system.

in complicated protocols whereby it chooses decisions which are seemingly suboptimal, but via which it can *signal* information about subsystem 1 to subsystem 3, thereby making the decision *globally* optimal. It turns out [25, 22] that understanding such phenomena can be enormously complicated, and it is first important to understand the "no-signalling-incentive" situations. To do away with this complication, we assume that subsystem 3 has access to all the information about subsystem 1 that subsystem 2 has. Viewed more generally, this is a transitivity property about the information flow. As we will see, posets provide a natural abstraction to describe and generalize such acyclic, transitive information flows. The high-level problem described in this chapter will be made precise in later chapters. In Chapter 3 (Section 3.1) the notion of poset-causal systems is introduced. The notions of information flow, hierarchical information and distributed decision making are all formalized in a control-theoretic setup.

1.3 Computational Considerations

While understanding classes of tractable information flows is an important conceptual task, developing efficient algorithms for computing optimal controllers within these classes is equally critical. More classical treatments of control theory have developed an algebraic approach for solving control problems via the Youla parametrization [57, 20]. In the context of decentralized control, Youla-based approaches have been well-studied as well [44].

These approaches have the attractive feature that they are able to reduce optimal control problems to convex optimization problems in the Youla parameter. However these techniques have drawbacks.

The main problem with Youla based techniques is that this parametrization is infinite-dimensional. There is no way to *a priori* bound the degree of the controller being designed. Moreover, even if the degree were bounded, one is still left with the problem of optimizing over the locations of the poles and the residues. Techniques do exist whereby one designs a sequence of controllers which approach optimality, but the degrees of the controllers in the sequence are not necessarily well-behaved. To compound these issues, these methods suffer from numerical instability.

In the centralized setting, researchers have found ways around these issues in the case of both $\mathcal{H}^2$ and $\mathcal{H}^\infty$ performance metric. This involves the construction of a state space solution to the problem. State space solutions have several nice features: (a) There is an optimal centralized controller of bounded degree (the degree needs to be at most the degree of the plant) (b) the approach provides insight into the structure of the optimal controller (c) efficient algorithms exist to construct the state space solutions, and (d) an extensive body of theoretical work exists that addresses many different aspects of these problems. While there is a considerable body of work dealing with state-space solutions for centralized control problems [16, 21], in this thesis we present novel state-space solutions for decentralized control problems.

1.4   Main Contributions

The preceding discussion regarding decentralization, tractability, information flows and computation sets the agenda for our thesis. Broadly, there are two themes in this thesis. The first theme is the development of the notion of poset-causality as a notion of information flow in systems. The second theme is related to computational and architectural

issues related to design of decentralized control of such poset-causal systems. The main contributions in this thesis are the following:

1. We introduce the notion of a partially ordered sets (posets) as a means of modeling causality-like communication constraints between subsystems in a decentralized control setting.

2. We exploit algebraic properties of posets to show that optimal control problems over poset-causal systems can be convexified.

3. We show that a number of seemingly disparate examples studied in the decentralized control literature are specific instances of this poset-causality paradigm, so that posets in fact form a unifying theme in decentralized control.

4. We consider the problem of designing $\mathcal{H}_2$ optimal decentralized controllers for poset-causal systems using state-space techniques. We show a certain crucial separability property of the problem under consideration. This separability makes it possible to decompose the decentralized control problem over posets into a collection of standard centralized control problems.

5. We give an explicit state-space solution procedure. To construct the solution, one needs to solve standard Riccati equations (corresponding to the different sub-problems). Using the solutions of these Riccati equations, one constructs certain block matrices and provides a state-space realization of the controller.

6. We provide bounds on the degree of the optimal controller in terms of a parameter $\sigma_\mathcal{P}$ that depends only on the order-theoretic structure of the poset.

7. We describe the structural form of the optimal controller. We introduce a novel pair of transfer functions (Φ, Γ) which are inverses of each other, and which capture the prediction structure in the optimal controller. We call Φ the *propagation filter*, it corresponds to propagation of certain signals upstream. We call Γ the *differential*

21

filter, it corresponds to computation of differential improvement in the prediction of the state at different subsystems.

8. We state a new and intuitive decomposition of the structure of the optimal controller into certain local control laws.

9. We then address the question: "What is a reasonable architecture of controllers for poset-causal systems? What should be the role of controller states, and what computations should be involved in the controller?" We propose a controller architecture that involves natural concepts from order theory and control theory as building blocks. In the process we establish a new and significant connection between Möbius inversion on posets (a concept with deep connections with many diverse areas) and decentralized control. As a consequence, we gain further understanding into the structure of the optimal controller, the roles of the filters Φ and Γ (mentioned in item 7) in terms of state prediction and Möbius inversion.

10. We show that a natural coordinate transformation of the state variables yields a novel *separation principle* within this architecture.

11. We show that the optimal $\mathcal{H}_2$ controller (with state-feedback) has precisely the proposed architecture.

1.5   Related Work

This thesis connects two different mathematical themes, namely partially ordered sets, and decentralized control. Both these themes (individually) have a rich literature, and in this section we cite some pertinent pieces of work in these two areas.

The first theme, namely posets, are very well studied objects in combinatorics. The associated notions of incidence algebras, Möbius inversion and Galois connections were studied in generality by Rota [39] in a combinatorics setting. Since then, order-theoretic

concepts have been used in engineering and computer science; we mention a few specific works below. Cousot and Cousot used these ideas to develop tools for formal verification of computer programs in their seminal paper [14]. In control theory too, these ideas have been used by some authors in the past, albeit in somewhat different settings. Ho and Chu used posets to study team theory problems [25]. They were interested in sequential decision making problems where agents must make decisions at different time steps. They study the form of optimal decision-makers when the problems have poset structure. Mullans and Elliot [34] use posets to generalize the notion of time and causality, and study evolution of systems on locally finite posets. Del Vecchio and Murray [56] have used ideas from lattice and order theory to construct estimators for discrete states in hybrid systems. Poset-causal systems are also related to the class of systems studied more classically in the context of *hierarchical systems* [31, 19], where abstract notions of hierarchical organization of large-scale systems were introduced and their merits were argued for.

The second theme, namely that of decentralized control also has a rich literature dating back to the 1970s, we mention the classical survey of Sandell et. al [45], the work of Wang and Davison [61], and the books by Gündes and Desoer [24] and Šiljak [55]. In more recent work, Blondel and Tsitsiklis [9] have shown that in certain instances, decentralized control problems are computationally intractable, in particular they show that the problem of finding bounded-norm, block-diagonal stabilizing controllers in the presence of output-feedback is NP-hard. In other recent work, several authors including Voulgaris [58, 59, 60], Bamieh and Voulgaris [5] and Fan et. al [18] presented cases where decentralized control problems are computationally tractable. Lall and Rotkowitz generalize these ideas in a framework of a property called quadratic invariance [44], we discuss connections to their work later. In past work [47], Shah and Parrilo have shown that posets provide a unifying umbrella to describe these tractable examples under an appealing theoretical framework. Related also is the literature on *team theory* which may be viewed as distributed decision-making over a finite time horizon. Important contributions were made in the classical work

23

of Radner [37] and Ho and Chu [25]. More recently team theory problems have also been studied by Gattami and Bernhardson [22] and Rantzer [38].

While the above-mentioned literature deals with understanding tractable classes of control problems, finding computationally efficient algorithms for the same is equally crucial. Without a doubt, a major advance in this area has been the advent of state-space techniques. In the context of centralized control, we mention the influential work of Doyle et. al [16]. In the context of decentralized control, state-space methodologies have been proposed in [28, 46, 64, 52, 44]. Our state-space solution procedure is perhaps the closest in spirit to the work of Rotkowitz and Lall [44] and Swigart and Lall [52], but significantly more computationally efficient and insightful than [44], and applicable to a much more general class of problems than studied in [52]. More detailed comparisons to these works will be made in later chapters.

1.6   Organization of Thesis

This thesis is organized as follows. In Chapter 2 we introduce some of the necessary background, including concepts from order theory and control theory that will be used throughout the thesis. In Chapter 3, we introduce the notion of poset-causal systems. We establish connections between poset-causality, convexity and the Youla parametrization. We also demonstrate that many examples studied in the decentralized control literature may be unified in the poset causality framework. In Chapter 4 we consider the problem of computing the $\mathcal{H}_2$-optimal poset-causal controller for a poset-causal system. By exploiting certain separability properties of the problem we develop a state-space solution for the optimal control problem. We also describe the structure of the optimal controller. In Chapter 5, we study the control design problem over posets from a broader architectural viewpoint. We describe an intuitive controller architecture and establish some important related properties such as optimality and separation principles. In Chapter 6 we conclude the thesis and

outline directions of future research.

Chapter 2

Background

In this chapter we establish the necessary background that will be used throughout this thesis. Broadly, this thesis draws ideas from two areas: order theory and control theory. In the first section we establish the order theoretic background including concepts such as partially ordered sets, incidence algebras and related algebraic properties. In the second section we establish the necessary control theoretic background.

2.1   Order Theoretic Background

In this section we introduce some of the order-theoretic preliminaries that will play a central role in this thesis.

2.1.1   Partially Ordered Sets and Incidence Algebras

Definition 2.1. *A partially ordered set (or* poset*) $\mathcal{P} = (P, \preceq)$ consists of a set P along with a binary relation $\preceq$ which has the following properties:*

1. $a \preceq a$ (reflexivity),

2. $a \preceq b$ and $b \preceq a$ implies $a = b$ (antisymmetry),

3. $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ (transitivity).

We will sometimes use the notation $a \prec b$ to denote the strict order relation $a \preceq b$ but $a \neq b$.

Posets may be finite or infinite, depending upon the cardinality of the underlying set P. In this thesis, we will have occasion to deal with both finite and infinite posets. When we talk about finite-dimensional Linear Time-Invariant (LTI) systems, we will model decentralization constraints with finite posets. When we talk about systems with time-delays and spatially invariant (distributed parameter) systems, the underlying state space is infinite dimensional and we will then use infinite posets to model decentralization constraints.

It is possible to represent a poset graphically via a *Hasse diagram* by representing the transitive reduction of the poset as a graph (i.e. by drawing only the minimal order relations graphically, an upward arrow representing the relation $\preceq$, with the remaining order relations being implied by transitivity).

Example 2.1. *An example of a poset with three elements (i.e., $P = \{a, b, c\}$) with order relations $a \preceq b$ and $a \preceq c$ is shown in Figure 2-1. In the diagram (known as a Hasse diagram), an up arrow indicates the order relation $\preceq$.*
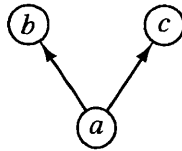
Figure 2-1: A poset on the set $\{a, b, c\}$.

Definition 2.2. *Let $\mathcal{P} = (P, \preceq)$ be a poset. Let $\mathbb{Q}$ be a field. The set of all functions $f : P \times P \to \mathbb{Q}$ with the property that $f(x, y) = 0$ if $y \not\preceq x$ is called the* incidence algebra *of $\mathcal{P}$ over $\mathbb{Q}$. It is denoted by $I(\mathcal{P})$.* *

---

*Standard definitions of the incidence algebra use an opposite convention, namely $f(x, y) = 0$ if $x \not\preceq y$. Thus, the matrix representation of the incidence algebra is typically a transposal of matrix representations that appear here. For example, while the incidence algebra of a chain is the set of lower-triangular matrices

28

The ring will usually be clear from the context (most often it will be either the field of rational proper transfer functions or the reals). When the set P is finite, the set of functions in the incidence algebra may be thought of as *matrices* with a specific sparsity pattern given by the order relations of the poset.

Definition 2.3. *Let $\mathcal{P}$ be a poset. The function $\zeta \in I(\mathcal{P})$ defined by*

$$\zeta(x,y) = \begin{cases} 0, & \text{if } y \not\leq x \\ 1, & \text{otherwise} \end{cases}$$

is called the zeta-function *of $\mathcal{P}$.*

Clearly, the zeta-function of the poset is an element of the incidence algebra.

Example 2.2. *The matrix representation of the zeta function for the poset from Example 1 is as follows:*

$$\zeta = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The incidence algebra is the set of all matrices in $\mathbb{Q}^{3 \times 3}$ which have the same sparsity pattern as its zeta function.

Given two functions $f, g \in I(\mathcal{P})$, their sum $f+g$ and scalar multiplication cf are defined as usual. For finite posets, the product $h = f \cdot g$ is defined as follows:

$$h(x,y) = \sum_{z \in P} f(x,z)g(z,y). \tag{2.1}$$

(The definition can be appropriately extended to infinite posets by replacing summation by integration, we will do so when dealing with spatially distributed systems.) As mentioned

---

in this thesis, in standard treatments it would appear as upper-triangular matrices. We reverse the convention so that in a control theoretic setting one may interpret such matrices representing *poset-causal* maps. This reversal of convention entails transposition of other standard objects like the zeta and the Möbius functions.

above, we will frequently think of the functions in the incidence algebra of a poset as square matrices (of appropriate dimensions) inheriting a sparsity pattern dictated by the poset. The above definition of function multiplication is made so that it is consistent with standard matrix multiplication.

Theorem 2.1. *Let $\mathcal{P}$ be a finite poset. Under the usual definition of addition and multiplication as defined in (1) the incidence algebra is an associative algebra (i.e. it is closed under addition, scalar multiplication and function multiplication).*

Proof. Closure under addition and scalar multiplication is obvious. Let $f, g \in \mathcal{I}(\mathcal{P})$, and consider elements x, y such that $y \not\leq x$. If $y \not\leq x$, there cannot exist a z such that $y \leq z \leq x$. Hence, in the sum (5.4), either $f(x, z) = 0$ or $g(z, y) = 0$ for every z, and thus $h(x, y) = 0$. □

A standard corollary of this theorem is the following [50, Theorem 1.2.3].

Corollary 2.1. *Let $\mathcal{P}$ be a finite poset and let $A \in \mathcal{I}(\mathcal{P})$ be an invertible matrix. Then $A^{-1} \in \mathcal{I}(\mathcal{P})$.*

Often we will abuse notation and think of incidence algebras at the block matrix level. To element $i \in P$ we associate m_i rows and n_i (consecutive) columns of the matrix. Then if $j \not\leq i$ we set the (i, j) *block matrix* of size $m_i \times n_j$ to be zero. Thus we may think of rectangular matrices (which are square at the block level) as being in the incidence algebra.

Example 2.3. *The following block matrix may be viewed as being in the incidence algebra of the poset shown in Fig. 2-1. In this matrix, $m_1 = 2, n_1 = 1, m_2 = 2, n_2 = 2, m_3 = 1, n_3 = 1$.*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

This notion can be made more rigorous using the notion of a *quoset* as described below. In only one section of this thesis will we need to think formally of quosets and their associated algebras. For the most part, thinking of incidence algebras at the block level will suffice.

2.1.2 Quosets and Structural Matrix Algebras

It is possible to define a more general notion of a partial order in the absence of anti-symmetry. Indeed, one can equip the set P with an equivalence relation, and impose a order relation on the *quotient set* modulo the equivalence relation. The resulting object is called a quotient poset or *quoset*, (sometimes called a *preorder* in the literature). There is a corresponding algebraic object, analogous to the incidence algebra, called the *structural matrix algebra* [3].

Definition 2.4. *A quoset $Q = (Q, \precsim)$ is a set Q with a binary relation $\precsim$ such that $\precsim$ is reflexive and transitive.*

Thus it is possible for distinct elements i, j to satisfy $i \precsim j$ and $j \precsim i$ (we will call such elements equivalent and denote this by $i \simeq j$).

The analogue of an incidence algebra generalized to quosets is the following:

Definition 2.5. *Let $\mathbb{Q}$ be a field and $Q = (Q, \precsim)$ be a quoset. Let the* structural matrix algebra M *be the set of functions $f : Q \times Q \to \mathbb{Q}$ with the property that $f(i, j) = 0$ if $j \not\precsim i$ for all i, j.*

We leave it as an easy exercise to the reader to verify that M is an associative algebra. Figure 2-2 shows an example of a quoset and the sparsity pattern of the associated structural matrix algebra.

31

$$\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & 0 & 0 & * \end{bmatrix}$$
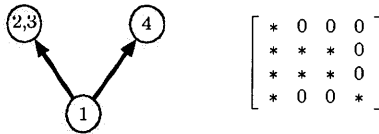
Figure 2-2: A quoset and the sparsity pattern of its associated structural matrix algebra. Elements with a '*' indicate possible nonzero elements.

2.1.3 Galois Connections

In some situations, we will be dealing with two different posets whose order relations are closely related to one another via a notion of "similarity". (As an example if poset were a sub-poset of the other, one would like to say that they are "similar"). One natural way of modeling such a situation is using the notion of *Galois connections*.

Definition 2.6. *Let* $\mathcal{P} = (P, \leq)$ *and* $Q = (Q, \sqsubseteq)$ *be finite posets. A pair of maps* (ϕ, ψ) *where* $\phi : P \to Q$ *and* $\psi : Q \to P$ *is said to form a* Galois connection *if it satisfies the following property:*

$$q \sqsubseteq \phi(p) \Leftrightarrow \psi(q) \leq p \text{ for all } p \in \mathcal{P} \text{ and } q \in Q.$$

Indeed, if $\mathcal{P}$ and Q are isomorphic (as posets), the isomorphism and its inverse constitutes a Galois connection. More generally, if $\mathcal{P}$ and Q are related by a Galois connection then the posets $(\phi(P), \sqsubseteq)$ and $(\psi(Q), \leq)$ are isomorphic [17].

Example 2.4. *Figure 2-3 shows two posets* $\mathcal{P}$ *and* Q *related by a Galois connection. Note that poset* $\mathcal{P}$ *is isomorphic to the subposet of* Q *with the elements* $\{1, 3\}$.

Example 2.5. *We present another example of a pair of posets* $(\mathcal{P}, Q)$ *related by a Galois connection. The blue arrows indicate the maps* ϕ *and* ψ.

32
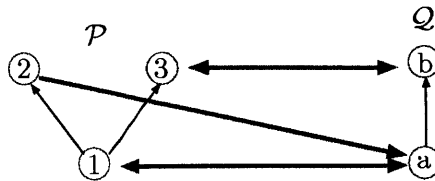
Figure 2-3: A pair of posets related by a Galois connection.
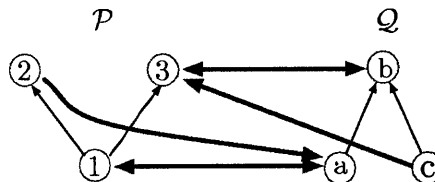
Figure 2-4: Another pair of posets related by a Galois connection.

2.2 Control Theoretic Background

In this section we describe the basic control theoretic setup we will be considering in this thesis. For most of this thesis we consider finite-dimensional linear time-invariant systems (except for a brief period when we consider systems with time-delays and distributed-parameter systems). In what follows, we remind the reader representations of such systems in terms of frequency domain representations and state-space descriptions.

2.2.1 Finite Dimensional Systems

Let us begin by considering finite-dimensional linear time invariant systems. It is often convenient to take an *input-output* view of the system in terms of the following block diagram (Fig. 2-5). We will not particularly emphasize the continuous or discrete time
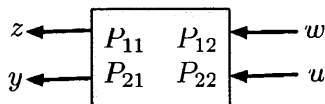
Figure 2-5: A standard input-output LTI system.

cases as all our results will apply equally well to both the settings. In Fig. 2-5, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the plant output, $w \in \mathbb{R}^{n_w}$ is the exogenous input, $z \in \mathbb{R}^{n_z}$ is the system output. We will be interested in representing our systems via transfer function matrices in the standard way [66] as

$$P(\omega) = \begin{bmatrix} P_{11}(\omega) & P_{12}(\omega) \\ P_{21}(\omega) & P_{22}(\omega) \end{bmatrix}, \tag{2.2}$$

where $P(\omega) \in \mathbb{C}^{(n_w+n_u)\times(n_z+n_y)}$ is the overall system transfer function. We will assume throughout that the resulting system is controllable and detectable. In this thesis, often the *plant* P_{22} will play a special role, so we will abbreviate notation and define $G := P_{22}$. We assume that G is strictly proper, so that stabilization of P is equivalent to stabilization of G.

While dealing with finite dimensional LTI systems the signal and operator spaces will be the standard ones [44]. In some sections we will be dealing with systems with time-delays, in these cases the systems are no longer finite-dimensional, and the relevant spaces will need to be appropriately extended (see [42, 43, 44]). We denote $\mathcal{R}_p^{m\times n}$ to be the set of rational-proper transfer matrices of dimension $m \times n$. We denote the set of stable proper transfer matrices by $\mathcal{RH}_\infty$. The entries of P can be shown to be rational proper transfer functions, i.e. $P \in \mathcal{R}_p^{(n_w+n_u)\times(n_z+n_y)}$.

The fundamental question in control theory is that of controller design. In terms of the input-output view under consideration (see Fig. 2-5), the problem may be viewed as designing a transfer function matrix $K \in \mathcal{R}_p^{n_y\times n_u}$ with certain desirable properties (to be formalized below). Note that once a suitable K has been chosen, one *interconnects* G with K in the feedback loop as shown in Fig. 2-6

This interconnection *induces* a map from w to z which may be represented by a transfer function as:

$$f(P, K) := P_{11} + P_{12}K(I - GK)^{-1}P_{21}.$$
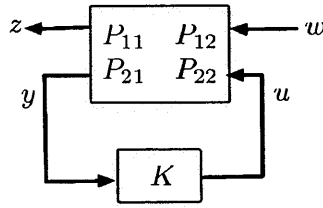
Figure 2-6: Interconnection of the system P with the controller K.

In this thesis we will explore the question of choosing appropriate "optimal" controllers. However, a far more basic requirement of a controller K is that it be *stabilizing*. Informally, this means that upon interconnection, the controller ensures that bounded energy disturbances w only produce bounded energy signals within the closed-loop (this property is known as *internal stability*). We refer the reader to [66] for a formal introduction of this basic concept. There is a subtle distinction between stability and internal stability. We will always require controllers to be internally stabilizing, but nevertheless informally refer to them as being stabilizing.

A formal statement of the classical optimal controller-synthesis problem is the following:

$$\begin{array}{ll} \underset{K}{\text{minimize}} & \| f(P,K) \| \\ \\ \text{subject to} & K \text{ stabilizes } P \end{array} \tag{2.3}$$

Very generally, $\| \cdot \|$ represents any norm on $\mathcal{R}_p^{n_z \times n_w}$, chosen to appropriately capture the performance of the closed-loop system. At that level of generality, the problem of computing optimal solutions may be challenging. Two canonical and well-studied norms in control theory are the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$ norms, we will defer their formal treatment to later chapters.

The well-studied classical problem stated in (2.3) requires only that the controller be stabilizing. In this thesis, we will additionally require that the controller also be *decentralized*. These decentralization constraints on the controller will manifest themselves in

the form of certain structural constraints on K. Indeed the decentralized control problem considered in this thesis is of the form:

$$\underset{K}{\text{minimize}} \quad \| f(P, K) \|$$

$$\text{subject to} \quad K \text{ stabilizes } P \qquad (2.4)$$

$$K \in S,$$

where S represents a class of structural constraints. In our thesis, S represents some subspace of the space of controllers (typically a subspace of *sparse* transfer function matrices with a fixed sparsity pattern). It may be noted that for general P and S there is no known technique for solving problem (2.4). Indeed, the reader may recall from the Chapter 1 that certain variants of the problem are known to be NP-hard [9].

In this thesis, we will consider structures S that arise from posets. Specifically, S will correspond to the subspace $\mathcal{I}(\mathcal{P})$, the incidence algebra of a fixed poset $\mathcal{P}$ (we will also briefly consider other types of structures S that arise from Galois connections). These types of constraints will have natural interpretations in terms of the flow of information in the system.

2.2.2 Youla Parametrization

Problem (2.3) as presented is a nonconvex problem in K. The nonconvexity arises as a consequence of the *linear-fractional* nature of the objective function in the controller variable K (recall that norms are convex functions). The constraints K stabilizing and $K \in S$ (where S is a subspace) are both convex (since the set of stabilizing controllers forms a subspace [11, pg. 154]). If the subspace constraint $K \in S$ were absent (the so-called "centralized" problem), several techniques exist for solving the problem (2.3) [20]. One approach towards a solution to the problem is via an explicit parameterization of all stabilizing controllers for the problem (2.3). It is desirable to have the closed-loop transfer function be an

affine function in the parameter, so that the problem becomes convex. There are different approaches to perform the parametrization, for example the Youla parametrization [44] and the so-called R-parametrization [11].

Let $\mathcal{H}_{stab}$ denote the set of *all* stable closed loop transfer matrices achieved by controllers that internally stabilize the plant, i.e.

$$\mathcal{H}_{stab} = \left\{ P_{11} + P_{12}K(I - GK)^{-1}P_{21} \mid K \text{ stabilizes } P \right\}. \tag{2.5}$$

Let

$$h_G : \mathcal{R}^{n_y \times n_u} \to \mathcal{R}^{n_y \times n_u}$$
$$K \mapsto K(I - GK)^{-1}. \tag{2.6}$$

Let us define $R := h_G(K)$. Then it is well-known [11] that $\mathcal{H}_{stab}$ can be parameterized in terms of R via

$$\mathcal{H}_{stab} = \{ P_{11} + P_{12}RP_{21} \mid RG \in \mathcal{RH}_\infty, R \in \mathcal{RH}_\infty, I + GR \in \mathcal{RH}_\infty, (I + GR)G \in \mathcal{RH}_\infty \}. \tag{2.7}$$

If G is stable, (2.7) reduces to a simpler parameterization:

$$\mathcal{H}_{stab} = \{ P_{11} + P_{12}RP_{21} \mid R \in \mathcal{RH}_\infty \}. \tag{2.8}$$

Under reasonably mild conditions (namely well-posedness of the interconnection between P and K [66]), the map h_G is invertible. Hence, given R, the controller K may be uniquely recovered by

$$K = h_G^{-1}(R) = (I + RG)^{-1}R. \tag{2.9}$$

An interesting feature of the map h_G is that for certain classes of information structures S the map is *structure-preserving*. Indeed when the structure under consideration is an in-

cidence algebra, the maps h_G and h_G^{-1} preserve the incidence algebra structure as described below:

Lemma 2.1. *Assume $G \in \mathcal{I}(\mathcal{P})$. Then $h_G(K) \in \mathcal{I}(\mathcal{P})$ if and only if $K \in \mathcal{I}(\mathcal{P})$.*

Proof. Follows from the definition of h_G, Theorem 2.1 and Corollary 2.1.   □

2.2.3 Quadratic Invariance

While the Youla parametrization enables one to convexify centralized control problems of the form (2.3), it is natural to ask under what conditions one can also expect an exact convex reformulation of the more challenging subspace-constrained problem (2.4). Motivated by this question, Rotkowitz and Lall present a property known as *quadratic invariance* in their paper [44], defined below:

Definition 2.7. *A plant G and a subspace (of controllers) S is defined to be quadratically invariant if for every $K \in S$, $KGK \in S$.*

If the plant G and the constraint S in problem (2.4) possess quadratic invariance then the Youla parametrization allows an exact convex reformulation of (2.4).

More formally, let $h(K, G) := h_G(K)$ (defined in (2.5)), and K_{nom} be a nominal stable and stabilizing controller. Then Lall and Rotkowitz [44] show that the set of all feasible controllers is given by:

$$C_{stab} = \{K_{nom} - h(h(K_{nom}, G), Q) \mid Q \in S, Q \text{ stable}\}.$$

Then the decentralized control problem (2.4) has the exact reformulation:

$$\begin{aligned}
\underset{K}{\text{minimize}} \quad & \| P_{11} + P_{12}QP_{21} \| \\
\text{subject to} \quad & Q \text{ stable} \\
& Q \in S.
\end{aligned} \qquad (2.10)$$

38

If Q^* is the optimal solution of the above problem then the optimal controller is given by $K^* = K_{nom} - h(h(K_{nom}, G), Q^*)$. We will explore the connections between quadratic invariance and our poset-based approach in detail in later chapters. The appealing property of (2.10) is that it is an exact convex reformulation of the nonconvex problem (2.4).

2.2.4 State Space Realizations

While a frequency domain transfer function representation of the form (2.2) is a natural way to describe an LTI system, it is often very useful to present the system via a *state-space realization* [66]. We consider the following state-space description of the system shown in Fig. 2-5 in discrete time:

$$
\begin{aligned}
x[t+1] &= Ax[t] + B_1w[t] + B_2u[t] \\
z[t] &= C_1x[t] + D_{11}w[t] + D_{12}u[t] \\
y[t] &= C_2x[t] + D_{21}w[t] + D_{22}u[t].
\end{aligned}
\tag{2.11}
$$

(Note that we describe the discrete time case here for convenience and note that continuous time descriptions are analogous in nature). A convenient (and standard [66]) notation we will often use to compactly represent (2.11) is:

$$
\begin{bmatrix} z \\ y \end{bmatrix} =
\left[\begin{array}{c|cc}
A & B_1 & B_2 \\
\hline
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{array} \right]
\begin{bmatrix} w \\ y \end{bmatrix}
\tag{2.12}
$$

In this thesis we will often encounter situations where the plant (i.e. the map from u to y) of the state space system (2.11) is in the incidence algebra $\mathcal{I}(\mathcal{P})$ for some poset $\mathcal{P}$. This will be ensured by assuming that A, B_2, C_2 and D_{22} are in the (block) incidence algebra $\mathcal{I}(\mathcal{P})$. This assumption will have a natural interpretation in terms of the flow of information in the system that will be made precise at the appropriate juncture.

39

2.2.5 Systems with Time Delays

In Chapter 3, we will encounter a class of systems where the dynamics in the plant and controller suffer from propagation delays. The class of time-delayed systems under consideration will be Linear and Time-Invariant (LTI), however these systems are not finite dimensional and hence state-space descriptions are inconvenient.

By suitably extending the class of signal spaces and operator spaces [43], it is possible to define and work with transfer function representations of such systems. Indeed, the optimization problem (2.3), the Youla parametrization and convex reformulation results described in Section 2.2.2 all extend naturally to this setup.

Another useful representation of such systems is via the *impulse response*. Given a time-delayed system G with input u and output y, the impulse response $\psi(t)$ is defined to be the output $y(t)$ obtained under a unit impulse input $\delta(t)$. For a general input,

$$y(t) = \int_{t \in \mathbb{R}} \psi(t - \tau) u(\tau) d\tau,$$

and the above is often represented using the standard convolution operation $y(t) = \psi(t) * u(t)$. (This definition is stated for the situation where the input u and output y are single-dimensional, but may be generalized in a natural way to the multi-dimensional setting).

2.2.6 Spatially Invariant Systems

In this section, we briefly introduce the notion of spatially invariant systems. The formal presentation of the subject is quite detailed and technical, and for the purposes of this thesis somewhat tangential (since in the context of spatially distributed systems we wish to mostly emphasize the connection between communication constraints and posets). For further details, we encourage the reader to see [30, 7, 15, 8, 4, 6] and the references therein.

Spatially invariant systems are a class of distributed parameter infinite-dimensional systems that evolve along the spatio-temporal coordinates $(x, t) \in \mathcal{X} \times \mathcal{T}$ where $\mathcal{X}$ is the spatial

40

domain and $\mathcal{T}$ is the temporal domain (or time). We assume that $\mathcal{X} = \mathbb{R}^n$ and the temporal domain $\mathcal{T}$ may be assumed to be $\mathbb{R}_{\geq 0}$. (In the fully general case, one can assume that these domains have the structure of a locally compact abelian group, but for the sake of simplicity, and to fix ideas we make these choices.) In this thesis, we will study the class of systems that are linear and spatially and temporally invariant, i.e spatio-temporal systems that are invariant under translations along the spatial and temporal coordinates (we will assume the temporal invariance implicitly, and call such systems spatially invariant systems). Just as LTI systems are characterized by impulse responses $h(t)$ (such a description being possible due to time-invariance), spatially invariant systems cab be completely described by a spatio-temporal impulse response $\psi(x, t)$.

Much like finite-dimensional LTI systems and time-delayed systems described above, spatially invariant systems also have natural state-space and frequency domain descriptions (though both spatial and temporal frequency variables are now present). Indeed, such systems may be thought to posses transfer functions (which depend on both the temporal frequency variable s and spatial frequency variable λ). As mentioned above, the precise treatment of these concepts is well-studied and fairly technical. In this thesis, we intend to emphasize the connections with partially ordered sets, and for those purposes the somewhat informal treatment described here will suffice.

Example 2.6. *The canonical example to keep in mind while thinking of such spatially invariant systems is those described by linear PDEs, such as the wave equation:*

$$\partial_t^2 \psi(x, t) = c^2 \partial_x^2 \psi(x, t) + u(x, t). \tag{2.13}$$

41

This has a state-space description:

$$\partial_t \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ c^2 \partial_x^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u$$

$$\psi = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.$$

This also has a compact transfer function description:

$$G(\lambda, s) = \frac{1}{s^2 + c^2 \lambda^2}.$$

Chapter 3

Poset-Causal Systems, Convexity, and the Youla Parameterization

In this chapter we begin our study with a class of systems known as poset-causal systems. These systems are composed of collection of subsystems such that the interconnection between them has a certain generalized spatial causality structure. This chapter revolves around two themes involving poset-causal systems. The first, and major theme of this chapter involves unification of a broad class of (previously studied, as well as new) class of *decentralization structures* into a common theoretical framework. Variations of this theme occur throughout this chapter in different contexts: we consider finite-dimensional systems, systems with delays and distributed parameter systems. In all these settings we show that many previously studied decentralization structures possess a single unifying concept, namely poset-causality.

The second theme in this chapter is convexity, and the role of the Youla parametrization in facilitating convexity. It is well-known in optimization that convexity is an important and fundamental property enabling computational tractability. Thus, in order to solve optimal control problems involving decentralization tractably, it is reasonable to demand convexity. Here again, the algebraic properties of partially ordered sets play a fundamental role that

enable convex reformulations. We study this, and connections to related properties that enable convexity such as quadratic invariance, in this chapter. The message we wish to convey to the reader through the chapter is that many classes of decentralized control problems that are considered to be tractable have an underlying poset structure. The algebraic properties that this structure brings with it ensures computational tractability.

In this chapter we first consider finite-dimensional linear time invariant systems with communication constraints, both within the plant and the controller. In first part of the chapter we are interested in the setting where the system is composed of several interacting subsystems. This subsystem approach enables us to partition the overall transfer function into several local transfer functions. The communication constraints among the subsystems manifest themselves via sparsity constraints in the plant and controller transfer functions. These sparsity constraints can often be modeled quite naturally using posets. In this chapter we consider several specific classes of such communication constraints that arise naturally in many decentralized decision-making problems.

We then consider systems with time-delays. Decentralized systems composed of interacting subsystems with communication delays have been studied in the past by Rotkowitz and Lall [44]. We show that these results may be naturally interpreted using posets.

Many practical control problems are also naturally *spatially distributed*, i.e. the overall subsystem is composed of many subsystems, each of which is at a different spatial location. Spatially distributed systems (and the related notion of distributed parameter systems) have also been extensively studied (see [30, 7, 15, 8]). It is natural to study decentralized control in the spatially distributed setting since many spatially distributed systems are also large scale and lumped in the sense that the controller may interface with the system at only a relatively small number of spatial coordinates and thus may face natural communication constraints. The problem of decentralized control of spatially distributed systems becomes considerably simpler when the system has a property known as *spatial invariance*. Intuitively, this means that the overall system is not only time-invariant but also invariant under

spatial translations. Such systems have been studied in some detail by Bamieh et. al. [4, 6].

We also study the problem of decentralized control of spatially invariant distributed systems based on the partial order framework developed in [47, 48]. We show that this framework allows one to study several interesting classes of decentralized problems. To study communication structures for spatially invariant systems, it is sufficient to study the *spatio-temporal impulse response*, which constitutes the impulse response in the joint spatial and temporal domain (denoted by $h(x, t)$, where x is the spatial domain and t is the temporal domain) when the system reacts to an impulse at the origin $(x, t) = (0, 0)$. The communication structure of the system determines the *support* of $h(x, t)$. We show that modeling the communication structure via posets allows us to generalize the results of Bamieh et. al. [4, 6]. Similar results were simultaneously and independently developed by Rotkowitz [41].

The main contributions of this chapter may be summarized as follows:

- We introduce the notion of a partially ordered set (poset) as a means of modeling causality-like communication constraints between subsystems in a decentralized control setting.

- We exploit algebraic properties of the problem to show that the set of controllers that satisfy the sparsity constraints can be parameterized explicitly.

- We generalize the poset based model from the setting where plant and controller have same communication constraints to the setting where they may have different constraints.

- We study systems with time delays. It had been shown in a previous chapter [42], that subject to certain conditions on the delays between subsystems (namely triangle inequality), the resulting problem was quadratically invariant (and thus amenable to convex optimization). We show that there is a natural poset associated with systems with time delays with this subadditivity property, and that the computational

45

tractability is simply an algebraic consequence of this underlying poset.

- We introduce a poset-based framework to study decentralized control of spatially distributed systems. This model naturally extends the results on time delayed systems mentioned in the previous item. We generalize some previously known results of [6] regarding funnel-causal systems.

- We study the relationship between posets and quadratic invariance. We show that quadratic invariance can be naturally interpreted as a transitivity property, and that under certain natural settings, poset structures and quadratic invariance are *exactly equivalent*. We introduce the notion of a quoset, which is a poset modulo an equivalence relation. We show that under similar but somewhat more general conditions, quadratic invariance is equivalent to *quosets*.

This chapter is organized as follows. In Section 3.1 we introduce the class of finite-dimensional LTI poset-causal systems and establish how optimal decentralized controllers may be computed by convex optimization using the Youla parametrization. In Section 3.3 we extend these results beyond poset-causal systems to a more general class using the notion of Galois connections. In Section 3.4 we unify known results related to decentralized systems with time-delays into a poset-causal framework. In Section 3.5, we unify and generalize known results about distributed spatially invariant systems using the poset framework. In Section 3.6, we discuss the connections between quadratic invariance and poset-causality.

3.1 Control of Poset Causal Systems

We begin introducing poset-causality in the context of finite-dimensional LTI systems. The interconnection between subsystems in these systems will obey certain causal relationships, which will be formalized using posets. We will call such decentralized systems

poset causal systems. These are a (reasonably large) class of structured decentralized systems that have a one-directional or causal flow of information. We will study the task of designing controllers that mirror these structural constraints. We first begin with some examples of communication structures that can be modeled using posets.

3.1.1 Examples of Communication Structures Arising from Posets

In this section we study some examples of posets. Several classes of communication structures have been studied in the decentralized control literature [59], [60], we show how these classes can be unified in a poset framework. The intuition behind modeling communication among subsystems via posets is as follows. We say that subsystems i and j satisfy $i \preceq j$ if an input at subsystem i affects the output at subsystem j. It means that subsystem j is more *information-rich*. We will formalize this notion in the next section.

Independent subsystems

The trivial poset on the set $\{1, 2, \ldots, n\}$ where there are no partial order relations between any of the elements (i.e. all the elements are independent of each other) corresponds to the case where the subsystems exchange no communication whatsoever (all subsystems have access to only their own information, thus K and G are diagonal). The corresponding incidence algebra for this poset is the set of diagonal matrices. It is readily seen that this is just the case where one is required to stabilize n independent plants using independent controllers, a problem that reduces to a classical control problem.

At the other extreme is the case where the poset is *totally ordered*. This is the case of nested control [59], which we study below.

Nested systems

This is a class of systems where the transfer functions have a block-triangular structure. Nested systems have been analyzed by Voulgaris [60], [59]. Such structures arise in cases
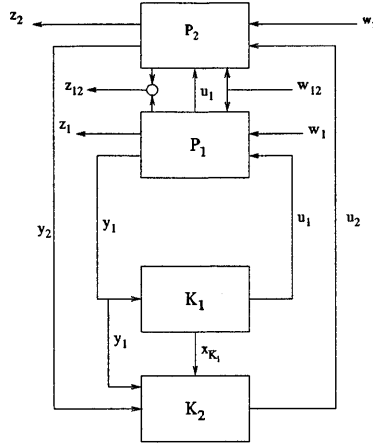
Figure 3-1: A system with nested communication constraints.

where there are several subsystems are arranged hierarchichally, such as when each subsystem is contained within a subsequent subsystem so that the arrangement forms a nest. There is one-way communication among the subsystems (say from the inside to the outside).

For simplicity, consider a system with just two subsystems P_1 and P_2. The internal subsystem P_1 can communicate information to the outer subsystem P_2 (but not vice-versa). The task at hand is to design a controller that obeys this same nested-communication architecture. The following is the set of plant outputs, control inputs, exogenous outputs and exogenous inputs respectively:

$$y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad z := \begin{bmatrix} z_1 \\ z_{12} \\ z_2 \end{bmatrix} \quad w := \begin{bmatrix} w_1 \\ w_{12} \\ w_2 \end{bmatrix}.$$

The sparsity pattern generated by the communication constraints for the controller and the plant are as follows:

$$G = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \quad K = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}.$$

Figure 3-1 depicts such a nested system in a block diagram. It is easy to see that G

48

and K are matrices in the incidence algebra generated by the poset over $\{1, 2\}$ with $1 \preceq 2$. This is consistent with the intuition that since subsystem 2 has access to information from subsystem 1 (input 1 can affect output 2) to make decisions (K_{21} is allowed to be non-zero), subsystem 2 is more information-rich. Voulgaris [59] showed that for such nested systems, the optimal control problem can be reduced to a convex problem in the Youla parameter. In the next section we will see that this result follows as a special case of a more general result that is true for all poset-causal systems.

Other examples

The example cited in the above subsection shows that nested systems are just special cases of those arising from posets. Clearly, many other communication structures can be modeled as posets. Some such examples include multi-chains, lattices and transitive closures of directed acyclic graphs. A few such examples are shown in Fig. 3-2.
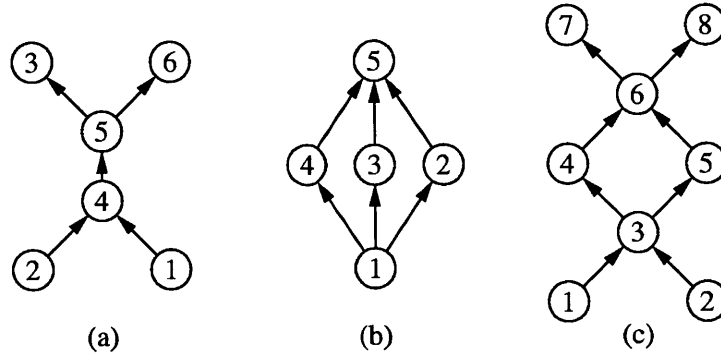
Figure 3-2: Examples of other poset communication structures: (a) A multi-chain (b) A lattice (c) A directed acyclic graph.

3.2 Systems with Same Plant and Controller Communication Constraints

We now consider systems that are composed by interconnecting several subsystems. Each subsystem is assumed to be linear, time-invariant, and finite-dimensional. For concrete-

ness, we consider systems evolving in discrete time, though we emphasize that all results presented in this thesis extend naturally to the continous time case. We consider an input-output framework where each subsystem is represented as a transfer function matrix G. We view $G \in \mathcal{R}_p^{n \times n}$ as a system that is composed of n subsystems. Subsystem i consists of input i and output i (the transfer function between which is G_{ii}). In addition, input i can also affect another subsystem (say subsystem j) in which case $G_{ij} \neq 0$. As in [44], we would like to consider communication constraints between the subsystems being modeled as sparsity constraints on the matrix G. To this end we define some terminology.

Suppose we have a collection of subsystems that are interconnected in a way that is consistent with the partial order structure of a poset $\mathcal{P} = (\{1, \ldots, n\}, \preceq)$. The partial order represents the communication structure in the plant as follows:

Definition 3.1. *The plant $G \in \mathcal{R}_p^{n \times n}$ is said to be $\mathcal{P}$-poset-causal if whenever $j \npreceq i$, an input at subsystem j does not affect subsystem i (i.e. $G_{ij} = 0$).*

This definition formalizes the notion of subsystem level causality in the plant, i.e. that $j \preceq i$ implies that i is in the cone of influence of j since $G_{ij} \neq 0$. In this section we are interested in the case where the controller K mirrors the communication constraints of the plant, i.e. if $j \npreceq i$ then $K_{ij} = 0$ (i.e $K \in \mathcal{I}(\mathcal{P})$. This formalizes a notion of information richness, since if $j \preceq i$ then the controller for i has access to more information than the controller for j (K_{ij} is allowed to be nonzero, whereas K_{ji} is forced to be zero).

Example 3.1. *Consider the poset with six elements as shown Figure 3-2(a). An input at subsystem 4 can only affect (the outputs of) subsystems 3, 5 and 6. In other words, subsystems 3, 5 and 6 are in the cone of influence of subsystem 4. In the language of posets, this is stated as $4 \preceq 3$ and $4 \preceq 5$ and $4 \preceq 6$. Thus, the language of posets enables us to model such causal relationships between subsystems in decentralized systems. Variations of this theme will recur in this chapter in later sections will generalize this notion to other types of constraints.*

50

We denote the set of all stabilizing controllers that lie in the incidence algebra by $C_{stab}(\mathcal{P})$. Let the set of all achievable closed loop transfer functions that are stabilized by $K \in C_{stab}(\mathcal{P})$ be denoted by $\mathcal{H}_{stab}(\mathcal{P})$. Recall that we are interested in solving the optimal control problem:

$$\underset{K}{\text{minimize}} \quad \| f(P, K) \|$$

$$\text{subject to} \quad K \text{ stabilizes } P \qquad\qquad (3.1)$$

$$K \in \mathcal{I}(\mathcal{P}),$$

To convexify (3.1) it will be necessary to reparameterize $C_{stab}(\mathcal{P})$ and $\mathcal{H}_{stab}(\mathcal{P})$. Our approach will be as follows. First we will construct an explicit stabilizing controller in $C_{stab}(\mathcal{P})$. Using this controller in the feedback loop, we reduce the problem to the case where the plant is stable and then use equation (2.8) to parameterize the set of all closed loop maps. Before we do so, we make an important remark and state a related assumption.

Stabilization

Remark Suppose we have a plant $G \in \mathcal{I}(\mathcal{P})$ with G_{ij} unstable for some $i \neq j$. The task of internally stabilizing the plant G with a controller $K \in \mathcal{I}(\mathcal{P})$ is *impossible*. This is because G_{ij} does not have a feedback path. To illustrate this consider an example with two subsystems forming a nest (i.e. the block-triangular case we saw in the preceding subsection). Suppose

$$G = \begin{bmatrix} 0 & G_{12} \\ 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix},$$

where K is some stabilizing controller in the incidence algebra. By Theorem 2.1 and Corollary 2.1 it is easy to see that $R = K(I - GK)^{-1}$ is also in the incidence algebra (i.e. it is

lower triangular). However, if this were the case, one can readily check that

$$(I + GR)G = \begin{bmatrix} 0 & G_{12} \\ 0 & 0 \end{bmatrix}.$$

It is impossible for this to be stable, thus the controller cannot be internally stabilizing, yielding a contradiction. As mentioned already, the problem here is that G_{12} has no feedback path around it to internally stabilize it. This fact is illustrated in Fig. 3-3.
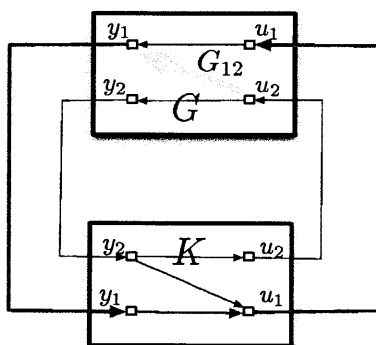
Figure 3-3: Feedback interconnection showing absence of feedback path around G_{21}

This is true in general, and we will state this formally. First we recall that the configuration shown in Figure 3-4 involving an unstable G is internally unstable *for all* K because the signal z is unbounded when the input u is non-zero.
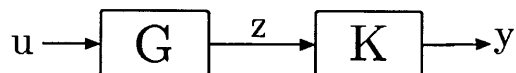
Figure 3-4: Open loop interconnection cannot be internally stable.

Lemma 3.1. *Let G be a poset-causal system. If for some i, j (distinct indices) G_{ij} is unstable, then the plant is not internally stabilizable by a $\mathcal{P}$-poset-causal controller $K \in \mathcal{I}(\mathcal{P})$.*

Proof. We show that in a feedback interconnection of G and K where both are $\mathcal{P}$- poset-causal, G_{ij} has no feedback path around it. Let $i \neq j$ and suppose (for the sake of contra-

diction) that there is a feedback element K_{lm} around G_{ij}. First, note that since $G_{ij} \neq 0$ (it is in fact unstable, so must be non-zero), we have $j \leq i$. Since the output of subsystem i in the plant is connected only to the input of controller i, and the output of controller i is connected only to the input of subsytem i in the plant, it must be the case that $l = j$ and $m = i$. Hence we require K_{ji} to be nonzero. Since K is in the incidence algebra and $K_{ji} \neq 0$ we have $i \leq j$. We thus established that $i \leq j$ and $j \leq i$, so that by antisymmetry (Definition 2.1), $i = j$, a contradiction. $\qquad\square$

The preceding argument demonstrates that the task of stabilizing a poset-causal plant with a poset-causal controller is feasible only when the off-diagonal entries G_{ij} are stable. We will see that the G_{ij} being stable for all $i \neq j$ alongwith G_{ii} being stabilizable for all i is also a sufficient condition for stabilizability of the overall system. For technical reasons that will become apparent, we will assume something stronger, namely that the G_{ii} are stabilizable by a stable controller. We next show that when these conditions hold, a stabilizing controller in the incidence algebra may be explicitly constructed. We formalize this result.

Lemma 3.2. *Let G be a poset-causal plant. Then G is internally stabilizable by a controller $K \in \mathcal{I}(\mathcal{P})$ if and only if:*

- *G_{ij} is stable for $i \neq j$.*

- *G_{ii} is stabilizable for all $i \in P$.*

Furthermore, G is internally stabilizable by a stable controller $K \in \mathcal{I}(\mathcal{P})$ if G_{ii} is stabilizable by a stable controller for all $i \in P$.

Proof. Since $G \in \mathcal{I}(\mathcal{P})$, without loss of generality, we assume that G is lower triangular. (It is always possible to put it in triangular form by constructing a linear extension of the poset i.e. extending the partial order to a total order which is consistent [1, Prop 1.4]). Let us use the notation $R = K(I - GK)^{-1}$ so that $R_{ii} = K_{ii}(I - G_{ii}K_{ii})^{-1}$.

Note that by Lemma 3.1, G_{ij} (for $i \neq j$) being stable are necessary conditions. Since $K(I - GK)^{-1}$ is lower-triangular, its diagonal entries are simply $R_{ii} = K_{ii}(I - G_{ii}K_{ii})^{-1}$. Corresponding expressions hold for the diagonal entries of RG, $(I + GR)G$, and $I + GR$. By (2.7) these diagonal entries are stable if and only if K_{ii} can internally stabilize G_{ii}. Hence the conditions stated in the statement of the lemma are necessary.

To see sufficiency, let us pick a controller K which is diagonal and with diagonal entries K_{ii} such that it internally stabilizes the G_{ii}. It can be easily verified that the off-diagonal entries of these matrices are stable because they are sums and products of stable entries (recall that by assumption the G_{ij} are stable for $i \neq j$). Hence all four of the transfer functions are stable, and by (2.7) we have a stable closed loop. Lastly, if the G_{ii} are stabilizable by a stable controller, a choice of a diagonal stable K such that K_{ii} is stable and stabilizes G_{ii} for all $i \in P$. This controller internally stabilizes G. $\qquad\square$

Henceforth, we assume the following:

- G_{ij} are stable for $i \neq j$

- G_{ii} is stabilizable by a stable K_{ii} for all $i \in P$.

Parametrization of all stabilizing controllers in the incidence algebra

An important step in reparametrizing the nonconvex problem (3.1) is understanding the class of feasible controllers. In the centralized control problem, the Youla parameterization provides a complete characterization of the class of achievable stable closed loops and the set of feasible controllers. We now describe how to extend such a parametrization for the poset-causal case.

Given a poset-causal plant $G \in \mathcal{I}(\mathcal{P})$ we provide a characterization of the set of poset-causal controllers $K \in \mathcal{I}(\mathcal{P})$ which internally stabilize the plant, along with the set of achievable stable closed loops.

Let us define $R = h_G(K)$ as defined in (2.6). We begin by noting that from Lemma 2.1, $K \in I(\mathcal{P})$ if and only if $R \in I(\mathcal{P})$. Note that from (2.7), if we treat $R = h_G(K)$, as a parameter the set of achievable stable closed loops by controllers $K \in I(\mathcal{P})$ is given by

$$\mathcal{H}_{stab} = \{P_{11} + P_{12}RP_{21} \mid R \in \mathcal{RH}_\infty \cap I(\mathcal{P}), RG \in \mathcal{RH}_\infty, I + GR \in \mathcal{RH}_\infty, (I + GR)G \in \mathcal{RH}_\infty\}.$$

This is a complete parametrization of the set of achievable closed loops. It is an appealing construction because the set of closed-loops is an *affine* function of the parameter R. One important drawback of the above is that it is not a *free* parametrization. The parameter R is constrained to have certain stabilization properties indicated by the last three inclusion requirements in the preceding formula. Indeed, for practical as well as theoretical reasons, it would be more appealing to express the set of closed loops as a free parametrization. This can indeed be done by using the idea of *pre-stabilization*, which we describe next.

We first note that by Lemma 3.2, it is straightforward to choose a nominal controller K_{nom} which is stable, diagonal (and hence trivially in the incidence algebra) and also stabilizing (one only needs to stabilize the diagonal elements separately). We use K_{nom} in the closed loop to stabilize the plant, so that the problem is reduced to the case of a stable plant. Now we treat the system with K_{nom} in the closed loop as the "new plant", which is already stable. Let

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{G} \end{bmatrix},$$

where $\tilde{P}$ is the closed loop map obtained by interconnection of K_{nom} with G. It is well-known (see for example [44, Theorem 17]) that the set of all achievable stable closed-loops is given by

$$\mathcal{H}_{stab}(\mathcal{P}) = \left\{ \tilde{P}_{11} + \tilde{P}_{12}R\tilde{P}_{21} \mid R \in \mathcal{RH}_\infty \cap I(\mathcal{P}) \right\}.$$

Finally, the set of all stabilizing controllers in the incidence algebra is

$$C_{stab}(\mathcal{P}) = \left\{ (I + R\tilde{G})^{-1} R \mid R \in \mathcal{RH}_{\infty} \cap \mathcal{I}(\mathcal{P}) \right\}.$$

Using this parameterization, one can reduce the optimal control problem (3.1) to the convex problem:

$$\text{minimize}_{R} \quad \| \tilde{P}_{11} + \tilde{P}_{12} R \tilde{P}_{21} \|$$

$$\text{subject to} \quad R \in \mathcal{RH}_{\infty}$$

$$R \in \mathcal{I}(\mathcal{P}),$$

Remark Rotkowitz and Lall have studied a property known as *quadratic invariance* (see Definition 2.7) in the context of Youla domain convexification. They show that this algebraic condition describes a large class of problems which are amenable to convex reparametrization. In subsequent work [29] it is also shown that in a certain sense this is the largest class of problems which is amenable to convex reparametrization in the Youla domain. We remark that a poset-causal plant G and the subspace of poset causal controllers (i.e. $S = \mathcal{I}(\mathcal{P})$) is quadratically invariant for the following reason. Since G is poset-causal, $G \in \mathcal{I}(\mathcal{P})$ and the information constraint is also $K \in \mathcal{I}(\mathcal{P})$. By Theorem 2.1, $\mathcal{I}(\mathcal{P})$ is an algebra of matrices, hence $KGK \in \mathcal{I}(\mathcal{P})$.

We emphasize that poset-causal systems form a subclass of quadratically invariant problems. While the attendant convexity guarantees are thus unsurprising, they form a large interesting subclass with intuitive combinatorial structure and rich algebraic structure. This additional structure, present in poset-causal systems but absent in a general quadratically invariant setup, can be exploited to obtain much stronger results as will become apparent later in the thesis.

As an example of this, note that in our approach (unlike in a quadratically invariant setup) we do not need to assume *a priori* knowledge of a nominal stable controller $K \in \mathcal{I}(\mathcal{P})$ to obtain a convex reformulation. Rather, as described above, we can explicitly

construct such a nominal controller (assuming that the subsystems (i.e. diagonal elements) themselves have a stable controller). That enables us to obtain a free parametrization that does not have any constraints apart from $R \in \mathcal{RH}_\infty \cap \mathcal{I}(\mathcal{P})$.

This distinction of being able to explicitly produce nominal stabilizing and stable controllers is important. The task of producing a nominal stable, stabilizing and decentralized controller in general settings may potentially be as hard as the optimization problem itself (testing feasibility may be as hard as optimization). In our case, the poset structure allows us to explicitly produce a stabilizing controller.

3.3 Systems with different plant and controller communication constraints

3.3.1 Modeling via Galois Connections

In this section we examine a more general setting where the controller is not necessarily required to mirror the communication constraints of the plant. In the previous section, we viewed the system as a collection of interconnected subsystems with the poset describing an information hierarchy on the subsystems. We alter this view here to deal with different information structure in the plant and the controller while retaining a partial order point of view. Instead of having a partial order on subsystems, we now impose a partial order on the set of inputs and a (possibly different) partial order on the set of outputs. The communication constraints between the inputs and outputs in the plant and the controller are given by a pair of maps between the two posets. We show that if the pair of maps have the special property of being a *Galois connection* [39, 23] the controller synthesis problem (subject to the communication constraints) is amenable to convex optimization.

The main idea behind using Galois connections to describe poset-like decentralization constraints is as follows. Suppose one has a plant with a set of inputs and a set of outputs.

Often, cases may arise where some inputs are more influential than other inputs (in that the effect of an impulse can be observed at more output ports). This hierarchy among the inputs in terms of their "cone of influence" defines a natural partial ordering on the set of inputs. The task at hand is to design a certain decentralized controller where the inputs to the controller (which are the outputs of the plant) also obey certain "similar" causality constraints. We show that it is possible to solve such decentralized control problems when the notion of similarity being used is the notion of a Galois connection.

We first remind the reader that a Galois connection formally captures the notion of similarity between posets (see Definition 2.6). Let $G \in \mathcal{R}_p^{n_y \times n_u}$ be a system with n_u inputs and n_y outputs. Let $\mathcal{P} = (\{1, \ldots, n_u\}, \preceq)$ and $Q = (\{1, \ldots, n_y\}, \sqsubseteq)$ be posets on the index sets of the inputs and outputs respectively.

Definition 3.2. *(1) We say that the plant G is* communication-constrained *by ϕ if whenever $\phi(j) \not\sqsubseteq i$ input j cannot communicate with output i (i.e. $G_{ij} = 0$).*

(2) We say that the controller K is communication-constrained *by ψ if whenever $\psi(j) \not\preceq i$, j cannot communicate with input i (i.e. $K_{ij} = 0$).*

We will call a plant $G \in S(\phi)$ and a controller specification $K \in S(\psi)$ a *Galois-causal* problem.

The set of all controllers that are communication-constrained by ψ is a subspace. This subspace is denoted by $S(\psi)$. Similarly, the set of all plants that are constrained by ϕ are denoted by $S(\phi)$. These definitions can be interpreted as follows. Let $\uparrow i = \{k \in Q : k \geq i\}$. Then given an input at j in the plant, $\uparrow \phi(j)$ is exactly those outputs in the cone of influence of input j.

Remark These definitions generalize the notion of an incidence algebra to the case when we have two posets. For instance if the two posets are the same (i.e. $\mathcal{P} = Q$) and we choose the Galois connection to be the identity map (i.e. $\phi = \psi = id$), then it can be easily verified that $S(\phi) = S(\psi) = \mathcal{I}(\mathcal{P})$.

58

Theorem 3.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be posets on the index sets of the inputs and outputs respectively. Let $G \in \mathcal{R}_p^{n_y \times n_u} \in S(\phi)$ and $G \in \mathcal{R}_p^{n_u \times n_y} \in S(\psi)$. Then $KGK \in S(\psi)$ and $GKG \in S(\phi)$.*

Proof. We prove $KGK \in S(\psi)$. Suppose $G \in S(\phi)$ and $K \in S(\psi)$. Assume $\psi(j) \not\preceq i$ (so $K_{ij} = 0$), and assume for the sake of contradiction that $(KGK)_{ij} \neq 0$. Note that $(KGK)_{ij} = \sum_k \sum_l K_{ik} G_{kl} K_{lj}$. If $\psi(k) \not\preceq i$ then $K_{ik} = 0$. Similarly if $\psi(j) \not\preceq l$ then $K_{lj} = 0$ and if $\phi(l) \not\sqsubseteq k$ then $G_{kl} = 0$. Hence, nonzero terms in the above summation may only occur for indices that satisfy $\psi(k) \preceq i$, $\phi(l) \sqsubseteq k$, and $\psi(j) \preceq l$. Using Definition 2.6, since (ϕ, ψ) form a Galois connection $\psi(k) \preceq i \Leftrightarrow k \sqsubseteq \phi(i)$, and $\psi(j) \preceq l \Leftrightarrow j \sqsubseteq \phi(l)$ Hence, $j \sqsubseteq \phi(l) \sqsubseteq k \sqsubseteq \phi(i)$. These three conditions imply that $j \sqsubseteq \phi(i) \Leftrightarrow \psi(j) \preceq i$, contradicting our initial assumption and thereby proving that $(KGK)_{ij} = 0$.

We now prove that $G \in S(\phi) \Rightarrow GKG \in S(\phi)$. Suppose $\phi(j) \not\sqsubseteq i$ (so that $G_{ij} = 0$). Then $(GKG)_{ij} = \sum_{kl} G_{ik} K_{kl} G_{lj}$. If $G_{ik} \neq 0$ then $\phi(k) \sqsubseteq i$, $K_{kl} \neq 0$ then $\psi(l) \preceq k$, and if $G_{lj} \neq 0$ then $\phi(j) \sqsubseteq l$. Now $\psi(l) \preceq k \Leftrightarrow l \sqsubseteq \phi(k)$. Hence $\phi(j) \sqsubseteq l \sqsubseteq \phi(k) \sqsubseteq i$ so that $\phi(j) \sqsubseteq i$, a contradiction.

$\square$
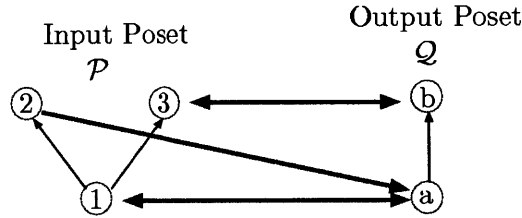
Figure 3-5: Posets $\mathcal{P}$ and $\mathcal{Q}$ with a pair of maps that form a Galois connection.

Remark A consequence of Theorem 3.1 is that problems with such communication constraints are quadratically invariant [44] (see Definition 2.7), and can hence be recast as convex problems in the Youla parameter. The emphasis here is on the fact that this enables us to extend the class of decentralization structures that can be captured under the poset umbrella.

Example 3.2. *Consider a system with three inputs ($P = \{1, 2, 3\}$) and two outputs ($Q = \{a, b\}$).* *The input and output spaces are endowed with a partial order structure and a pair of maps* *(ϕ, ψ) that form a Galois connection (see Fig. 3-5).*

$$\phi(1) = a \quad \psi(a) = 1$$
$$\phi(2) = a \quad \psi(b) = 3$$
$$\phi(3) = b.$$

This order-theoretic structure results in communication constraints on the plant G and controller K as shown in (3.2) below. The constraints in the plant G have a certain sparsity structure that is dictated by the posets $\mathcal{P}$ and Q and the map ϕ. When appropriately interpreted, these constraints naturally generalize the notion of causality and "cone of influence" type interpretation that we encountered in the incidence algebra case. In the plant each input j has associated to it a set of outputs in its "cone of influence", which is exactly the set $\uparrow \phi(j)$. For example, for the input 1, its cone of influence is $\uparrow \phi(1) = \{a, b\}$. Similarly the cone of influence of input 3 is $\uparrow \phi(3) = \{b\}$. A similar interpretation holds for the controller: each input to the controller i has a "cone of influence" which is precisely $\uparrow \psi(i)$.

$$
G = \begin{bmatrix} * & * & 0 \\ * & * & * \end{bmatrix} \quad
K = \begin{bmatrix} * & 0 \\ * & 0 \\ * & * \end{bmatrix} \tag{3.2}
$$

It can be easily verified that the problem with these sparsity constraints is quadratically invariant. The quadratic invariance of the problem depends only on the interconnection between the inputs and outputs in the controller and plant. The emphasis here is that when such constraints are modeled using order-theoretic considerations as explained above, quadratic invariance and the attendant convexity guarantees follow.

Example 3.3. *As another example of sparsity patterns governed by Galois connections consider the posets shown in Fig. 3-7. The maps (ϕ, ψ) are given by:*
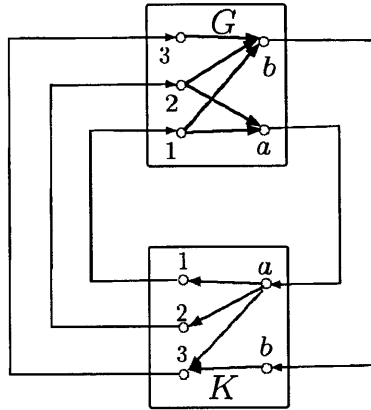
Figure 3-6: Communication constraints within G and K resulting from the poset and Galois connection shown in Figure 3-5.
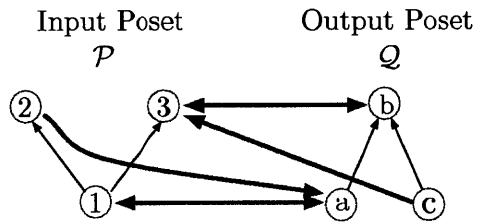
Figure 3-7: Posets $\mathcal{P}$ and $\mathcal{Q}$ with a pair of maps that form a Galois connection.

$$\phi(1) = a \quad \psi(a) = 1$$
$$\phi(2) = a \quad \psi(b) = 3$$
$$\phi(3) = b \quad \psi(c) = 3.$$

The sparsity patterns of G and K associated to this poset pair $(\mathcal{P}, \mathcal{Q})$ with the Galois connection maps (ϕ, ψ) is

$$G = \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & 0 & 0 \end{bmatrix} \quad K = \begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & * & * \end{bmatrix}.$$

It is straightforward to verify that this pair of sparsity patterns is quadratically invariant.

3.4 Systems with Time Delays

As mentioned earlier, one of the goals of this chapter is to show that many examples of decentralized control problems studied in the literature can be modeled via posets. In this section we provide another example of this involving certain structured time-delayed systems. It was shown by Rotkowitz and Lall [44] that systems involving time-delays which obey the triangle inequality may be studied in a quadratic invariance setup. In this section we show that such systems have a natural underlying poset structure, and its associated incidence algebra structure implies the convexity guarantees. The emphasis here is on the construction of the underlying poset and not the resulting quadratic invariance. We hope to convince the reader through these and other examples of the fundamental role that posets seem to play in much of the current theory on decentralized control.

In this section we consider LTI systems with time delays. Given a decentralized plant with communication delays between the different subsystems, we consider the task of designing controllers for the subsystems which interact according to a similar delay structure.

It has been known [42, 43] that such communication structures are amenable to convex reparametrization due to their quadratic invariance. In this section we show that posets arise naturally in this setup, that they describe the communication constraints in an intuitive way, and that the partial order structure results in convexity.

Consider a system with n subsystems (let $\mathcal{N} = \{1, \ldots, n\}$). Let the system be described by the transfer function matrix G where $G_{ij}(\omega)$ describes the frequency response between input of system j and output of system i. An equivalent way to describe the plant is to specify the impulse responses $g_{ij}(t)$. Define the delay between the subsystems i and j (denoted by D_{ij}) as follows (see Figure 3-12):

$$D_{ij} = \sup \left\{ \tau : g_{ij}(t) = 0 \text{ for all } t \leq \tau \right\}.$$

Note that since all systems are assumed to be causal, the delays D_{ij} are nonnegative.
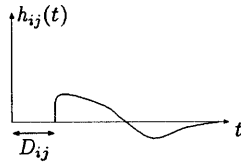
Figure 3-8: Impulse response $h_{ij}(t)$ along with the associated delay D_{ij}.

We define a relation $\preceq$ on $\mathcal{N} \times \mathbb{R}$ as follows.

Definition 3.3. *We say that $(j, t_1) \preceq (i, t_2)$ if*

$$t_2 - t_1 \geq D_{ij}.$$

Since the systems we are dealing with are time invariant, what this condition means intuitively is that $(j, t_1) \preceq (i, t_2)$ if system i at time t_2 is in the cone of influence of an impulse applied at system j at time t_1. We show next that if the delays satisfy a triangle inequality then the relation $\preceq$ described in Definition 3.3 is a partial order relation.

63

Proposition 3.1. *Suppose $D_{ii} = 0$ (i.e. effect of input on output within same subsystems is without delay), $D_{ij} > 0$ (there is nonzero delay between distinct subsystems) and the D_{ij} satisfy*

$$D_{ij} + D_{jk} \geq D_{ik}, \tag{3.3}$$

for all i, j, k distinct. Then $\leq$ in Definition 3.3 is a partial order relation.

Proof. Since $D_{ii} = 0$, by definition $(i, t_1) \leq (i, t_1)$. If $(i, t_1) \leq (j, t_2)$ and $(j, t_2) \leq (i, t_1)$) then $t_1 - t_2 \geq 0$ and $t_2 - t_1 \geq 0$ (since delays are nonnegative), thus by definition $t_1 = t_2$. Since $D_{ij} > 0$ for $i \neq j$ it must be the case that $i = j$ giving anti-symmetry. If $(i, t_1) \leq (j, t_2)$ and $(j, t_2) \leq (k, t_3)$, we have $t_1 \leq t_2 \leq t_3$. Further, $t_2 - t_1 \geq D_{ji}$ and $t_3 - t_2 \geq D_{kj}$. By (3.3), $t_3 - t_1 \geq D_{ji} + D_{kj} \geq D_{ki}$ and hence $(i, t_1) \leq (k, t_3)$, verifying transitivity. $\square$

Note that this triangle inequality structure on the delays is *exactly* the condition that appears in [42]. What is interesting here is that these delays actually arise from a natural poset structure, as we have just pointed out (the poset is determined purely by the delays, the actual functional form of the impulse response does not matter). Furthermore, the set of impulse responses $g_{ij}(t)$ which satisfy this delay structure actually forms an algebra of functions under convolution, as the next proposition shows.

Definition 3.4. *Let $\Psi = \{D_{ij}\}_{1 \leq i, j \leq n}$ be a given set of delays. Let $\mathcal{I}_\Psi$ denote the set of (matrix) impulse responses $G(t)$ with the property that $g_{ij}(t) = 0$ if $(j, 0) \not\leq (i, t)$.*

Intuitively $g_{ij}(t) = 0$ means that the effect of an impulse at time $t = 0$ at subsystem j has not reached the output of subsystem i at time t. Thus $\mathcal{I}_\Psi$ is precisely the set of systems which obeys the delay structure prescribed by Ψ. Given a set of impulse responses $F = \{f_{ij}(t)\}$ and $G = \{g_{ij}(t)\}$ define $F * G$ to be the matrix of impulse responses with $(F * G)_{ij}(t) = \sum_{k=1}^n f_{ik} * g_{kj}(t)$.

Proposition 3.2. *Given a set of delays Ψ which satisfy the conditions of Proposition 3.1. If $F = \left[f_{ij}(t)\right]_{1 \leq i, j \leq n}, G = \left[g_{ij}(t)\right]_{1 \leq i, j \leq n}$ such that $F, G \in \mathcal{I}_\Psi$, then $F * G \in \mathcal{I}_\Psi$.*

Proof. Suppose $(j, 0) \not\leq (i, t)$. It suffices to show that $(F * G)_{ij}(t) = 0$. Now,

$$(F * G)_{ij}(t) = \sum_{k=1}^{n} \int_{\mathbb{R}_+} f_{ik}(t - \tau) g_{kj}(\tau) d\tau.$$

If $(F * G)_{ij}(t) \neq 0$ then there must be some k, τ such that $g_{kj}(\tau) \neq 0$ and $f_{ik}(t - \tau) \neq 0$. This in turn means that $\tau \geq D_{kj}$ and $t - \tau \geq D_{ik}$. Thus $(j, 0) \leq (k, \tau)$ and $(k, \tau) \leq (i, t)$. By transitivity, $(j, 0) \leq (i, t)$, contrary to our assumption. □

Since the impulse responses form a convolutional algebra, the transfer function matrices $F(\omega)$ and $G(\omega)$ form a multiplicative algebra and are thus quadratically invariant. This allows us to conclude the following proposition.

Proposition 3.3. *Consider a set of delay constraints* Ψ *such that they satisfy the triangle inequality (3.3). Given a plant* $G \in \mathcal{I}_\Psi$ *with same delay constraints, the set of controllers* $K \in \mathcal{I}_\Psi$ *is quadratically invariant with respect to* G.

Remark This recovers another well-known result known in the decentralized control literature [42, 43] i.e. if the plant and controller have the same delay structure Ψ (with the triangle inequality), then designing optimal controllers with this delay structure is amenable to convex reparametrization. We emphasize again the interesting connections between convexity and underlying poset structure in this class of problems.

3.5 Spatially Invariant Systems

It is possible to extend the results of the preceding section on time-delayed systems to a class of infinite dimensional systems that are spatially distributed [48]. While these results were proposed in [48] by Shah and Parrilo, similar results were independently and simultaneously developed by Rotkowitz et. al. in [41]. These results generalized in multiple directions the previous results of Bamieh and Voulgaris [6]. In the spirit of this chapter of unifying past results into a poset framework, we briefly review our results in this section.

Recall that spatially invariant systems are a class of spatially distributed dynamical systems (sometimes called distributed parameter systems) that evolve in a spatio-temporal domain. Such systems have been extensively studied and analysed [15]. A detailed treatment of the subject is fairly involved and technical. More pertinent to our discussion is the study of decentralized structures in the context of spatially distributed systems, an interesting and well-studied topic [6, 4, 33]. In this section, we aim to establish connections to such decentralization structures arising naturally in this context to the notion of poset-causality. We will see that, once again, known classes of tractable problems have an underlying poset structure.

Recall from Chapter 2 the notion of spatially invariant systems that evolve along spatial coordinates ($x \in \mathcal{X}$) as well as temporal coordinates ($t \in \mathcal{T}$). Much like temporal invariance, we say that a system is spatio-temporally invariant if the effect of an impulse at spatial coordinate x_1 at time t_1 at another location x_2 at time t_2 depends only on $x_2 - x_1$ and $t_2 - t_1$. Such systems may be specified by their *spatio-temporal impulse response* $h(x, t)$. This function describes the response of the system at location (x, t) under the influence of an impulse at $(0, 0)$. Given a system $h(x, t)$ one defines the *support function* $f(x)$ as follows:

$$f(x) = \sup \{\tau : h(x, t) = 0 \text{ for all } t \leq \tau\}. \tag{3.4}$$

The support function evaluated at x describes the *delay* involved in the effect of an impulse at the origin to reach x. Note that this support function provides the natural generalization of delay between subsystems that we encountered in section 3.4. For example, if the system under consideration were described by the wave equation, then the support function would be exactly the light cone centered at the origin (see Fig. 3-9).

66
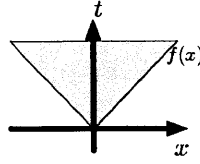
Figure 3-9: Light cone of a wave generated at the origin.

Partial Order Formulation

For concreteness we fix $\mathcal{X} = \mathbb{R}^n$ to be the domain of the spatial variable x and $\mathcal{T} = \mathbb{R}_{\geq 0}$ to be the domain of the temporal variable t. Let $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be the support function. We define a partial order on the tuple (x, t) as follows:

Definition 3.5. *The relation* $(x_1, t_1) \leq (x_2, t_2)$ *holds if*

1. $t_1 \leq t_2$ *(in the standard ordering on* $\mathbb{R}$*),*

2. $f(x_2 - x_1) \leq t_2 - t_1$ *(in the standard ordering on* $\mathbb{R}$*).*

Proposition 3.4. *Suppose the support function* $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *satisfies the following properties:*

1. $f(0) = 0$,

2. $f(x) > 0$ *for* $x \neq 0$,

3. $f(x_1 + x_2) \leq f(x_1) + f(x_2)$ $\qquad\qquad\qquad\qquad$ (3.5)

    for all $x_1, x_2 \in \mathbb{R}^n$ *(subadditivity).*

Then the relation $\leq$ *in Definition 3.5 is a partial order relation.*

Proof. We need to verify the three defining properties of partial order relations, namely reflexivity, anti-symmetry, and transitivity.

The relation $(x_1, t_1) \leq (x_1, t_1)$ holds trivially, condition 1 in Definition 3.5 is satisfied trivially, and condition 2 is satisfied because $f(0) = 0$.

Suppose $(x_1, t_1) \leq (x_2, t_2)$ and $(x_2, t_2) \leq (x_1, t_1)$. Then, by Definition 3.5 condition 1,

clearly $t_1 = t_2$. Thus $f(x_2 - x_1) \leq 0$, and since $f(x) \geq 0$ with equality only at the origin, it must be the case that $x_1 = x_2$. Thus $(x_1, t_1) = (x_2, t_2)$.

Finally, suppose that $(x_1, t_1) \leq (x_2, t_2)$ and $(x_2, t_2) \leq (x_3, t_3)$. Then $f(x_3 - x_2) \leq t_3 - t_2$ and $f(x_2 - x_1) \leq t_2 - t_1$. By subadditivity,

$$f(x_3 - x_1) \leq f(x_3 - x_2) + f(x_2 - x_1) \leq t_3 - t_1.$$

Hence, transitivity holds. □

Note that subadditivity of the support function provides the natural generalization of the triangle inequality of delays between subsystems that we saw in Section 3.4. Once a partial order is defined on the space, one can think of the space as a poset $\mathcal{P} = (P, \preceq)$ (in our case the set $P = \mathbb{R}^n \times \mathbb{R}_{\geq 0}$). By defining a multiplication rule on functions of the form $h : P \times P \to \mathbb{R}$ one can define the incidence algebra associated with the poset.

Rather than considering all functions of the form $h : P \times P \to \mathbb{R}$, which are of the form $h((x_1, t_1), (x_2, t_2))$ we restrict our attention to those functions which are *spatially and temporally invariant*, i.e. the value of the function depends only on $x_1 - x_2$ and $t_1 - t_2$. More precisely, these functions are of the form $h((x_1, t_1), (x_2, t_2)) = h(x_1 - x_2, t_1 - t_2)$.

Definition 3.6. *Let f be a support function satisfying (3.5) and $\mathcal{P} = (\mathbb{R}^n \times \mathbb{R}_{\geq 0}, \preceq)$ where the relation $\preceq$ is as defined in Definition 3.5. The set of functions $h : P \times P \to \mathbb{R}$ with the property that :*

1. $h((x_1, t_1), (x_2, t_2)) = h(x_1 - x_2, t_1 - t_2)$

   (called spatial invariance)

2. $h(x_1 - x_2, t_1 - t_2) = 0$ *for* $(x_1, t_1) \not\preceq (x_2, t_2)$

   (called order sparsity).

is called the spatially invariant incidence algebra *with respect to the support function f. It is denoted by $\mathcal{I}_f$.*

Consider a spatially invariant system where $g(\xi, \tau)$ is the spatio-temporal impulse re-

sponse (i.e. the response of the system at (ξ, τ) under an impulse at $x = 0, t = 0$).Due to spatial and temporal invariance, the impulse response of the system at (ξ, τ) under impulse at an arbitrary (x, t) will be simply $g(\xi - x, \tau - t)$. (For invariant systems, it is enough to specify the impulse response at the origin). Setting $g((\xi, \tau), (x, t)) = g(x - \xi, t - \tau)$, one can view g as being an element of the spatially invariant incidence algebra.

We now justify the reason for calling the object defined in Definition 3.6 an algebra. We show next that one can define a natural multiplication operation on $\mathcal{I}_f$, and that $\mathcal{I}_f$ is closed under this multiplication, justifying its description as an "algebra". We first define the multiplication operation.

Definition 3.7. *Let* $h_1(x_1 - x_2, t_1 - t_2), h_2(x_1 - x_2, t_1 - t_2) \in \mathcal{I}_f$ *be two functions in the spatially invariant incidence algebra. Then,*

$$
\begin{aligned}
h_3&((x_1, t_1), (x_2, t_2)) \\
&\triangleq h_1(x_1 - x_2, t_1 - t_2) \star h_2(x_1 - x_2, t_1 - t_2) \qquad (3.6) \\
&\triangleq \int_{\mathbb{R}_{\geq 0}} \int_G h_1(x_1 - x, t_1 - t) h_2(x - x_2, t - t_2) dx dt.
\end{aligned}
$$

We now show the closure property of the incidence algebra.

Proposition 3.5. *Let* $h_1, h_2 \in \mathcal{I}_f$ *be two functions in the spatially invariant incidence algebra. Then the following statements are true:*

(a) $h_1 + h_2 \in \mathcal{I}_f$,

(b) *For every scalar* $c, c \cdot h_1 \in \mathcal{I}_f$,

(c) $h_1 \star h_2 \in \mathcal{I}_f$.

Proof. Note that to prove membership in $\mathcal{I}_f$ one needs to prove two things, namely spatial invariance and order sparsity. Verifying spatial invariance and order sparsity of (a) and (b) are trivial exercises. We next verify the invariance property of $h_1 \star h_2$.

Let $h_3 = h_1 \star h_2$. Then,

$$h_3((x_1 + \xi, t_1 + \tau), (x_2 + \xi, t_2 + \tau))$$

$$= h_1((x_1 + \xi) - (x_2 + \xi), (t_1 + \tau) - (t_2 + \tau)) \star h_2((x_1 + \xi) - (x_2 + \xi), (t_1 + \tau) - (t_2 + \tau))$$

$$= h_1(x_1 - x_2, t_1 - t_2) \star h_2(x_1 - x_2, t_1 - t_2)$$

$$= h_3((x_1, t_1), (x_2, t_2)).$$

Hence h_3 is also spatially (and temporally) invariant. Having justified the invariance, from now on we will write $h_3(x_1 - x_2, t_1 - t_2)$.

To prove order sparsity, suppose h_1 and h_2 are poset causal with respect to a support function f. Consider $h_3(x_1 - x_2, t_1 - t_2)$ such that $(x_1, t_1) \not\preceq (x_2, t_2)$. Then by Definition 3.6, $h_1(x_1 - x_2, t_1 - t_2) = 0$ and $h_2(x_1 - x_2, t_1 - t_2) = 0$. Furthermore, there cannot exist a (ξ, τ) with both $(x_1, t_1) \preceq (\xi, \tau)$ and $(\xi, \tau) \preceq (x_2, t_2)$ simultaneously true (if they were both true, then by associativity if the partial order relation we would have $(x_1, t_1) \preceq (x_2, t_2)$ yielding a contradiction). Thus, in (3.6), each term in the integrand is zero, thus giving $h_3(x_1 - x_2, t_1 - t_2) = 0$. $\qquad\square$

Definition 3.8. *Given a spatially invariant distributed system with impulse response $h(x, t)$, the system is said to be* poset-causal *if the impulse response satisfies order sparsity with respect to a function f satisfying the conditions 1, 2 and 3 of Proposition 3.4.*

Since we defined multiplication in such a way that it is consistent with convolution of impulse response functions, we get the following important theorem as a direct consequence of Proposition 3.5:

Theorem 3.2. *The composition of spatially invariant poset-causal systems is also spatially invariant and poset-causal.*

Much like standard LTI systems, infinite-dimensional spatially invariant systems also have frequency domain representations, Youla parametrization and other standard proper-

70

ties [4, 6]. We mention that the fact that poset-causal spatially invariant systems form a convolutional algebra in the time domain readily implies that they form a multiplicative algebra in the frequency domain. This in turn enables one to exploit the Youla parametrization to express the set of achievable closed loop maps as an affine function in the Youla parameter, thereby raising the possibility of tractable computational methods. We do not dwell on the computational aspects of the problem here. Instead we study the poset aspects and show how conditions that we arrived at using our framework enables us to refine previously known results.

3.5.1 Relation to Funnel Causality

In [6], Bamieh and Voulgaris introduce a specific class of communication constraints for spatially invariant systems. They call such systems *funnel causal* systems. In their paper, the authors show that convolution of funnel causal impulse responses are also funnel causal, and that such systems are thus closed under composition. Finally, the authors show that due to this closure property, the set of all stabilizing funnel causal controllers can be described in the Youla domain in a convex fashion, thus making it amenable to optimization.

Our results generalize these results by Bamieh and Voulgaris. We show in this subsection that their main result regarding closure under composition of funnel-causal systems is *essentially a statement about poset causal systems*. We show that funnel causal systems are a sub-class of poset causal systems, i.e. if a system is funnel causal, one can construct a poset and an associated incidence algebra that contains the impulse response of the given system. In fact funnel causal systems form a *proper subset* of poset causal systems, indeed in the next subsection we will provide examples of poset-causal systems that are not funnel causal.

We will show that Theorem 3.2 completely generalizes the results in [6]. The outline of the argument is as follows. Funnel causal systems are defined in terms of concave support functions (in one dimension), whereas poset-causal systems are defined in terms of

sub-additive support functions as defined in (3.5). We first show that for functions in one dimension, concave functions are subadditive. Thus, if f is concave (thus funnel causal), f is sub additive and by Proposition 3.4 the system is poset causal. Proposition 3.4 shows that such systems have a naturally associated poset and incidence algebra. By Theorem 3.2 poset-causal (thus funnel-causal) systems are closed under composition. In [6], the authors define funnel causal systems and the related notion of propagation functions (which are essentially support functions) in the following way.

Definition 3.9 ([6]). *A scalar valued function $f(x)$ is said to be a propagation function if f is nonnegative, $f(0) = 0$ and such that $\{f(x), x \geq 0\}$ and $\{f(x), x \leq 0\}$ are concave respectively.*

Definition 3.10 ([6]). *A system is said to have the property of funnel causality if its impulse response is such that*

$$h(x, t) = 0 \quad for \quad t < f(x),$$

where $f(x)$ is a propagation function.

This definition essentially imposes a concave, "funnel" shape on the support (propagation) function of the spatio-temporal impulse. We next show that such propagation functions are in fact subadditive. Hence, by Definition 3.5 and Proposition 3.4 they can be endowed with a partial order with respect to the propagation function f.

Proposition 3.6. *If $f : \mathbb{R} \to \mathbb{R}$ is such that $f(0) = 0$, $f(x) > 0$ for $x \neq 0$ and $\{f(x), x \geq 0\}$, $\{f(x), x \leq 0\}$ are concave, then f is subadditive.*

Proof. Let us restrict attention to the case where $x \geq 0$ (the case where $x \leq 0$ is similar). We first show that over this range, the function f is monotonically increasing (it will be

72

decreasing over $x \le 0$). By concavity, for every $\gamma \in (0, 1)$ and $t > 0$ we have

$$f(x + t) \ge \gamma f(x) + (1 - \gamma)f\left(x + \tfrac{1}{1-\gamma}t\right)$$
$$> \gamma f(x),$$

where the last inequality follows from the fact that $f(x) > 0$ for $x \ne 0$. Since this inequality is true for every $\gamma \in (0, 1)$ it must be true that $f(x + t) \ge f(x)$.

Let $a, b \ge 0$ (the case where $a, b \le 0$ is similar, and the case where $a \le 0, b \ge 0$ will be addressed last). We want to show that $f(a + b) \le f(a) + f(b)$. Without loss of generality, assume $a \le b$. Then $b = \gamma a + (1 - \gamma)b$ for some $\gamma \in [0, 1]$ (in fact $\gamma = \frac{a}{b}$). Let L_1 represent the straight line that passes through the points $(a, f(a))$ and $(a+b, f(a+b))$ (see Fig. 3-10). Consider the point $(b, r) \in L_1$ (thus r satisfies $r = \gamma f(a) + (1-\gamma)f(a+b)$). By non-negativity and concavity of f, we have $0 \le r \le f(b)$.
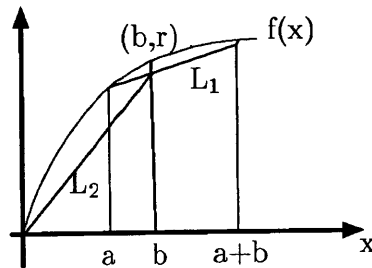
Figure 3-10: Subadditivity of concave support functions.

Thus,

$$f(a + b) - f(b) \le f(a + b) - r$$
$$= \frac{r - f(a)}{b - a}a, \tag{3.7}$$

where the last equality follows from elementary properties of the straight line L_1.

Let L_2 be the straight line between the points $(0, 0)$ and (b, r). Let $(a, t) \in L_2$ be the

point on this straight line at a. Then

$$
\begin{aligned}
t &= \eta(0) + (1 - \eta)r \\
&\leq (1 - \eta)f(b) \\
&\leq f(a),
\end{aligned}
$$

where the last inequality follows from the facts that the point $(a, (1 - \eta)f(b))$ is a point on the straight line L_3 which connects $(0, 0)$ and $(b, f(b))$ and that f is concave. Substituting $t \leq f(a)$ in (3.7), we get

$$
\begin{aligned}
f(a + b) - f(b) &\leq \frac{r - t}{b - a} a \\
&= t \\
&\leq f(a).
\end{aligned}
$$

The second equality follows from the fact that the points $(0, 0), (a, t)$, and (b, r) all lie on the straight line L_2. Thus, for $a, b \geq 0$, $f(a + b) \leq f(a) + f(b)$. As mentioned, the proof for the case when $a, b \leq 0$ is similar.

Now let $a \leq 0, b \geq 0$. Let $a + b \geq 0$ (the other case is similar). Since $a + b \leq a$ and f is increasing, $f(a + b) \leq f(a) \leq f(a) + f(b)$ (recall that the function is nonnegative everywhere). This completes the proof. □

As a corollary, we recover the following result by Bamieh [6, Lemma 1].

Corollary 3.1. *Composition of spatially invariant funnel causal systems is also spatially invariant and funnel causal.*

Proof. By Proposition 3.6, the propagation function in subadditive. By Proposition 3.4, the propagation function f satisfies all the conditions to define a partial order relation on $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$. Hence, if $h(x, t)$ is funnel causal with respect to the propagation function f, it is also poset causal with respect to (the partial order defined by) f. □
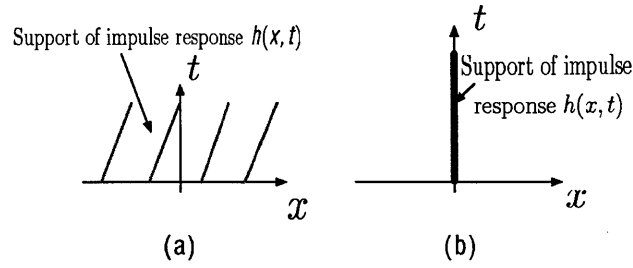
Figure 3-11: Examples of poset causal systems. (a) A centralized causal system. (b) A completely decentralized causal system.

3.5.2 Examples of Poset Causal Systems

In this subsection, we consider some examples of poset-causal systems to show how some interesting communication structures can be modeled via this poset framework.

Example 3.4. *We begin with the trivial example where the support function $f(x) = -\infty$ for all $x \neq 0$ and $f(0) = 0$. Note that subadditivity is trivially satisfied. Then the partial order on $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ is simply $(x_1, t_1) \leq (x_2, t_2)$ if $t_1 \leq t_2$. Hence, the set of poset-causal impulse responses is the set of impulse responses such that $h(x_1 - x_2, t_1 - t_2) = 0$ if $t_1 > t_2$. This is simply the impulse responses for the set of centralized causal systems where information propagates instantaneously. The spatial variables have no communication constraints, all subsystems in space have access to all information about other subsystems upto the present time t.*

On the opposite end of the spectrum, we have the completely decentralized case as we illustrate in Example 3.5 below.

Example 3.5. *Let $f(x) = \infty$ for $x \neq 0$ and $f(0) = 0$. In this case, $(x_1, t_1) \neq (x_2, t_2)$ if*

$$1) \quad t_1 \leq t_2$$

$$2) \quad f(x_2 - x_1) \leq t_2 - t_1.$$

However, if $x_2 \neq x_1$, $f(x_2 - x_1) = \infty$, hence the second condition in Definition 3.5 can never

75

be satisfied *for distinct x_1 and x_2. This corresponds to the case when all subsystems are incomparable with respect to each other. This resulting incidence algebra corresponds to the set of systems that are causal and completely decentralized, i.e. the impulse response $h(x, t)$ (which corresponds to an impulse at $(x, t) = (0, 0)$) has support only on the surface $x = 0, t \geq 0$, and is zero for other values of x.*

Example 3.6. *A class of systems that has been studied in the literature corresponds to the case where the support function $f(x) = c|x|$ where x is understood to be one-dimensional. Such systems have been called* cone causal *systems. Note that $f(0) = 0$, $f(x) > 0$ for $x \neq 0$. Subadditivity of f follows from the triangle inequality (alternatively, from the concavity of $|x|$). Hence, f satisfies all the conditions to prescribe a partial order on $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$. The impulse-response support for functions in this incidence algebra are depicted in Figure 3-9. Such systems draw motivation from the following interpretation. Suppose the system responds to an impulse at the origin. Then $h(x, t)$ is going to be supported on a* light cone *originating at the origin with speed of light equal to c. In other words the system has a constant (but finite) speed of signal propagation. Such examples arise naturally in physical systems, for example linear wave equations as described by (2.13).*

Example 3.7. *Another class of poset causal systems that have been studied in the literature are funnel causal systems. As described in Section 3.5.1 funnel causal systems are subclasses of poset causal systems. For more details and examples, the reader is referred to [6].*

Example 3.8. *The examples described up to this point correspond to systems with a single-dimensional spatial domain ($\mathcal{X} = \mathbb{R}$), we now consider an examples with multi-dimensional spatial domains ($\mathcal{X} = \mathbb{R}^n$). (We remind the reader that the result for funnel-causal systems as presented in [6], closure under convolution only holds for one dimensional systems). The advantage of our approach is that it abstracts the essential property for convolutional closure to hold. This essential property that we identify is subadditivity, which arises naturally for many classes of (multi-dimensional) functions.*

76

A natural class of support functions $f(x)$ are the p-norms $f(\cdot) = \|\cdot\|_p$ for $p \geq 1$. Clearly,

$f(0) = 0$ and $f(x) > 0$ for $x \neq 0$ by definition of a norm. Also, by the triangle inequality

for norms,

$$f(x_1 + x_2) = \|x_1 + x_2\|_p \leq \|x_1\|_p + \|x_2\|_p = f(x_1) + f(x_2).$$

Again, since f satisfies all the properties necessary to impose a poset structure on the

system, the incidence algebra based argument tell us that the corresponding impulses will

be closed under convolution.

It is interesting to note that, in [6], the authors identified concavity of f as being the

property essential to having convolutional closure. In this example, norms in general are

not concave, on the contrary, they are convex, yet we have convolutional closure. This

further strengthens the argument that sub-additivity is a more fundamental property. (In the

one-dimensional case, of course, the p-norms coincide with the absolute value function).

In the next example we further investigate the relationship between funnel causality

and poset causality. As already mentioned, the property at the heart of funnel causality

is concavity, whereas the property at the heart of poset causality is sub-additivity. We

have already shown that in one dimension, concavity implies sub-additivity so that funnel-

causal systems are poset causal. It is natural to wonder whether the converse is true, i.e.

whether all subadditive functions are concave. The p norms on $\mathbb{R}^n$ for $n > 1$ and $p > 1$)

are examples of subadditive functions which are non-concave (in fact, they are convex).

Example 3.9 below is an another example of a sub-additive function on the real line which

is not concave (nor convex).

Example 3.9. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ (see Fig. 3-12 given by*

$$f(x) = \begin{cases} |x| \ \textit{for} \ |x| \leq 1 \\ 2 - x \ \textit{for} \ 1 < x \leq 1 + \epsilon \\ 2 + x \ \textit{for} \ -1 - \epsilon \leq x < -1 \\ 1 - \epsilon \ \textit{for} \ |x| > 1 + \epsilon. \end{cases}$$

77
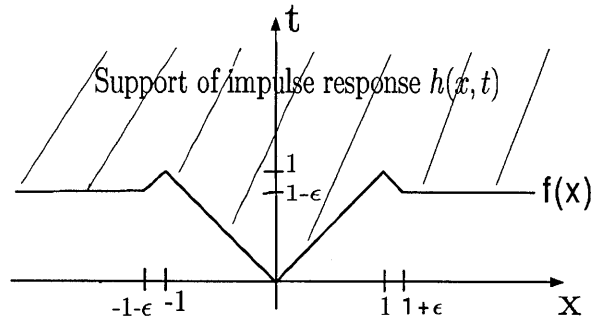
Figure 3-12: A sub-additive non-concave support function.

Here we assume that ϵ is a sufficiently small positive number, say $0 < \epsilon < \frac{1}{4}$. One needs to verify sub-additivity, i.e. $f(a+b) \leq f(a)+f(b)$, it is straightforward to do so by verifying several sub-cases.

3.6 Quadratic Invariance and Poset Structure

We remind the reader that quadratic invariance [44] characterizes the class of problems that can be convexified in the Youla domain as described in Section 2.2.3. In this section we want to study the connection between quadratic invariance and posets. We have seen that poset structure implies that the problem is quadratically invariant. We are now interested in understanding the converse, i.e. "does quadratic invariance imply existence of poset-like structure?" We will see that quadratic invariance, in a certain restricted setting, closely resembles the transitivity property. As argued earlier, posets provide the right language to describe transitive relations. In what follows, we make this connection more concrete. Connections between quadratic invariance and *partially nested structures* as defined in a team-theoretic setting by Ho and Chu [25] have been studied and pointed out by Rotkowitz [40]. The team theoretic problem considers a scenario where there are multiple decision makers who must each make a decision in some order. The paper considers a scenario

where the order in which decisions are made satisfy certain *precedence relations*. (Though this terminology is not used in these papers, these precedence relations are, in fact, partial order relations.) The paper by Ho [25] shows that problems with this precedence structure (called partially nested problems) are amenable to convex optimization, and moreover, that optimal controllers are linear. Rotkowitz shows that existence of these precedence relations is equivalent to quadratic invariance. Our results are similar in spirit, in fact Proposition 3.7 (below) is essentially contained in [40]. However, we provide a finer characterization of quadratic invariance in terms of posets and quosets.

Consider the problem of designing an optimal controller $K \in S$ as described in problem (2.4). In this section we revisit the model where decentralization constraints are viewed as sparsity constraints on the controller. Let $K \in \mathcal{R}^{n_y} \times \mathcal{R}^{n_u}$. Define a subset of indices of K via $\mathcal{J} \subseteq \left\{1, \ldots, n_y\right\} \times \{1, \ldots, n_u\}$. Then the subspace constraint is defined as $K_{ij} = 0$ for all $(i, j) \in \mathcal{J}$. Define

$$\text{Sparsity}(S) = \left\{K \mid K_{ij} = 0 \text{ for } (i, j) \in \mathcal{J} \text{ and } K_{ij} = 1 \text{ for } (i, j) \in \mathcal{J}^c\right\}.$$

Quadratic invariance reduces to the following transitive property in this model [44, Theorem 26]:

Theorem 3.3. *The subspace S is quadratically invariant with respect to a specified plant G if and only if for all $K \in Sparsity(S)$ and all i, j, k, l,*

$$K_{ij}G_{jk}K_{kl}(1 - K_{il}) = 0. \tag{3.8}$$

Remark Let us interpret equation (3.8) in an intuitive way. Let us denote the constraint $K_{ij} \neq 0$ by $i \rightarrow_K j$ (which denotes that there is a path from i to j in the controller) and $G_{jk} \neq 0$ by $j \rightarrow_G k$ (i.e. that there is a path from j to k in the plant). Then the equation (3.8) states that:

$$i \rightarrow_K j, j \rightarrow_G k, k \rightarrow_K l \text{ implies } i \rightarrow_K l. \tag{3.9}$$

The transitive structure becomes more apparent now. What quadratic invariance is saying is that the overall graph of the closed loop (which is comprised of a combination of subgraphs of the plant and the controller) is transitively closed. The condition means that if l is not allowed to communicate to i in the controller then there must exist no path from l to i around the closed loop (because such a path would produce a way for l to communicate to i by going once around the closed loop).

When the graph inside the plant and the controller is identical, quadratic invariance reduces to transitive closure of this (identical) graph. We next show that in this scenario quadratic invariance corresponds to existence of poset structure.

3.6.1 Existence of Posets

Consider a plant G and a decentralized control problem of the form (2.4) with sparsity constraints. Let $\mathcal{J}$ be the index set on which K is required to be sparse so that the constraint set S is described by

$$S = \left\{ K \mid K_{ij} = 0 \text{ for } (i, j) \in \mathcal{J} \right\}.$$

We consider the square case i.e. $n_y = n_u$. Let $\mathcal{N} = \left\{ 1, \ldots, n_y \right\}$. We say that a given decentralized control problem is *plant-controller symmetric* if the given plant also satisfies the sparsity constraints of the controller (i.e. $G \in S$). In this setup, notice that quadratic invariance (3.8) is equivalent to the fact that $\mathcal{J}^c$ is transitively closed, i.e.

$$(i, j) \in \mathcal{J}^c, (j, k) \in \mathcal{J}^c, (k, l) \in \mathcal{J}^c \Rightarrow (i, l) \in \mathcal{J}^c. \tag{3.10}$$

Proposition 3.7. *Consider a plant-controller symmetric control problem. Suppose the following assumptions are true of the index set:*

1. $(i, i) \in \mathcal{J}^c$

2. For distinct i and j we have $(i, j) \in \mathcal{J}^c \Rightarrow (j, i) \in \mathcal{J}$.

3. The problem is quadratically invariant.

Then there exists a poset $\mathcal{P}$ over n_y elements such that S is the incidence algebra of $\mathcal{P}$.

Proof. Since both G and K are $n_y \times n_y$ matrices, it is enough to construct a poset on n_y elements and show that the sparsity pattern of S exactly corresponds to the incidence algebra of this poset. Let us define our candidate for the partial order $\preceq$ as follows: $i \preceq j$ if $(i, j) \in \mathcal{J}^c$. We need to verify that this is indeed a partial order relation.

Since $(i, i) \in \mathcal{J}^c$, we clearly have $i \preceq i$ thus verifying reflexivity. If $i \preceq j$ and $j \preceq i$ then it must be the case that $(i, j) \in \mathcal{J}^c$ and $(j, i) \in \mathcal{J}^c$. However the second assumption in the statement of the proposition excludes the possibility of such i, j being distinct, thus $i = j$ and we have anti-symmetry. Finally, suppose we have $i \preceq j$ and $j \preceq l$ (i.e. $(i, j) \in \mathcal{J}^c$ and $(j, l) \in \mathcal{J}^c$). Choose index k such that $k = j$ and use quadratic invariance to conclude from equation (3.10) that $(i, l) \in \mathcal{J}^c$. Thus $i \preceq l$, verifying transitivity.

The incidence algebra of this poset is the set of elements such that $K_{ij} = 0$ if $i \not\preceq j$, (i.e. $(i, j) \in \mathcal{J}$) which is exactly the definition of S. $\qquad\qquad\square$

3.6.2 Existence of Quosets

We now generalize Proposition 3.7. It turns out that one can in fact relax the second assumption (anti-symmetry). It is possible to have a more general notion of a partial order in the absence of anti-symmetry. In that setting, distinct elements can be equivalent, and the partial order is defined on the quotient set modulo the equivalence. The resulting object is similar to a poset (called a quotient poset or *quoset*, sometimes it is called a *preorder* in the literature). There is a corresponding algebraic object, analogous to the incidence algebra, called the *structural matrix algebra* [3]. We introduced these notions in section 2.1.2.

The analogue of Proposition 3.7 to quosets is the following.

Proposition 3.8. *Consider a plant-controller symmetric control problem. Suppose the following assumptions are true of the index set:*

81

1. $(i, i) \in \mathcal{J}^c$

2. The problem is quadratically invariant.

Then there exists a quoset Q over n_y elements such that S is the structural matrix algebra of Q.

Proof. Again we construct a candidate quoset and verify the associated properties. We say that $i \precsim j$ if $(i, j) \in \mathcal{J}^c$. The verification of the properties are very similar to that of Proposition 3.7. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We have thus seen that the second condition from Proposition 3.7 can be relaxed, and that in the relaxed setting quadratic invariance is equivalent to existence of quoset structure in the problem. What happens when condition (1) is relaxed (i.e. allow constraints $K_{ii} = 0$ for some i)? We answer this in the next proposition.

. **Definition 3.11.** *Given $\mathcal{J}$, we call $\bar{\mathcal{J}} = \mathcal{J} \setminus \bigcup (i, i)$ the reflexive closure of $\mathcal{J}$. This is simply the operation of adding the reflexive relation to the set $\mathcal{J}^c$ which may not a priori satisfy reflexivity.*

We will say that the set $\mathcal{J}^c$ possesses *quoset structure* if the collection of relations $(i, j) \in \mathcal{J}^c$ satisfy the axioms of a quoset, i.e.

1. $(i, i) \in \mathcal{J}^c$

2. $(i, j) \in \mathcal{J}^c$ and $(j, k) \in \mathcal{J}^c$ implies $(i, k) \in \mathcal{J}^c$.

Proposition 3.9. *Suppose we have a plant-controller symmetric control problem with a specified index set (of sparsity constraints) $\mathcal{J}$. (The sparsity constraints are thus $K_{ij} = 0$ for $(i, j) \in \mathcal{J}$.) The problem is quadratically invariant if and only if $(\bar{\mathcal{J}})^c$ has a quoset structure.*

Proof. We first note that taking reflexive closure of a transitively closed set does not affect any of the relations between distinct elements. Define $\mathcal{I} = \mathcal{J}^c$. Define $i \precsim j$ if $(i, j) \in \mathcal{I}$.

Suppose we add the reflexive relations so that $\mathcal{I}' = \mathcal{I} \cup (\bigcup_{i \in N} \{(i, i)\})$. Consider the transitive closure of $\mathcal{I}'$. The only way new relations can be added is by combining transitive relations with the newly added reflexive relations. Thus if $i \precsim j$ and $j \precsim k$, we know that for distinct i, j, k we already have $i \precsim k$. If $j = i$ or $j = k$ we get no new relations. Hence $\mathcal{I}'$ is its own transitive closure.

Suppose the reflexive closure is a quoset. We know that in the closure operation, no new relations between distinct elements were introduced, hence transitivity is unaffected. By (3.10) the problem is quadratically invariant. Conversely, if the problem is quadratically invariant, we know from (3.10) that $\mathcal{I}$ is transitively closed. Thus if we take the reflexive closure, by Proposition 3.8 the resulting set is a quoset. □

3.7 Conclusion

We presented a poset based framework to study decentralized control problems. We showed the connection between partial order structure and several classes of decentralized control problems that have been studying in the past. Indeed, in our view posets provide a language and set of tools to study all these different cases in a unified setting. We also showed the close connection between posets and the algebraic property known as quadratic invariance.

The work in this chapter shows that all these classes of problems have an important algebraic closure property. This property allows optimal control problems to be reformulated into convex ones in the Youla domain, thereby opening the possibility of devising efficient computational procedures. We remind the reader, however, that statement of the control problem as a convex problem in the Youla domain merely hints at this possibility. The reformulated problem, though convex, is nevertheless *infinite-dimensional*. While computational techniques do exist for approximating the solution, they suffer from numerical instability and severe issues related to undesirable growth of controller orders near optimality (see [46] and the references therein). This is one important reason (among others)

to seek state-space solutions, the topic of the next chapter.

Chapter 4

$\mathcal{H}_2$ Optimal Control over Posets

4.1 Introduction

While it is possible to design optimal decentralized controllers for a fairly large class of systems known as quadratically invariant systems in the frequency domain via the Youla parametrization, there are some important drawbacks with such an approach. Typically Youla domain techniques are not computationally efficient, and the degree of optimal controllers synthesized with such techniques is not always well-behaved. In addition to computational efficiency, issues related to numerical stability also arise. Typically, operations at the transfer function level are inherently less stable numerically. Moreover, such approaches typically do not provide insight into the structure of the optimal controller. These drawbacks emphasize the need for *state-space techniques* to synthesize optimal decentralized controllers. State-space techniques are usually computationally efficient, numerically stable, and provide degree bounds for optimal controllers. In our case we will also show that the solution provides important insight into the structure of the controller.

In this chapter we consider the problem of designing $\mathcal{H}_2$ optimal decentralized controllers for poset-causal systems. The control objective is the design of optimal feedback laws that have access to local state information. We emphasize here that different subsys-

tems *do not* have access to the global state, but only the local states of the systems in a sense that will be made precise in the next section. The main contributions in the chapter are as follows:

- We show a certain crucial separability property of the problem under consideration. This result is outlined in Theorem 4.2. This makes it possible to decompose the decentralized control problem over posets into a collection of standard centralized control problems.

- We give an explicit state-space solution procedure in Theorem 4.3. To construct the solution, one needs to solve standard Riccati equations (corresponding to the different sub-problems). Using the solutions of these Riccati equations, one constructs certain block matrices and provides a state-space realization of the controller.

- We provide bounds on the degree of the optimal controller in terms of a parameter $\sigma_\mathcal{P}$ that depends only on the order-theoretic structure of the poset (Corollary 4.2).

- In Theorem 4.4 we briefly describe the structural form of the optimal controller. We introduce a novel pair of transfer functions (Φ, Γ) which are inverses of each other, and which capture the prediction structure in the optimal controller. We call Φ the *propagation filter*, it plays a role in propagating local signals (such as states) upstream based on local information. We call Γ the *differential filter*, it corresponds to computation of differential improvement in the prediction of the state at different subsystems. The discussion related to structural aspects is brief and informal in this chapter. In Chapter 5 we discuss architectural issues formally and in depth.

In an interesting paper by Swigart and Lall [51], the authors consider a state-space approach to the $\mathcal{H}_2$ optimal controller synthesis problem over a particular poset with two nodes corresponding to the nested case (see Section 3.1.1). Their approach is restricted to the finite time horizon setting (although in a subsequent chapter [52], they extend this to

86

the infinite time horizon setting), and uses a particular decomposition of certain optimality conditions. In this nested controller setting, they synthesize optimal controllers and provide insight into the structure of the optimal controller. By using our new separability condition (which is related to their decomposition property, but which we believe to be more fundamental) we significantly generalize those results in this chapter. We provide a solution for *all posets* and for the *infinite time horizon*. In recent work [44], Rotkowitz and Lall proposed a state-space technique to solve $\mathcal{H}_2$ optimal control problems for quadratically invariant systems (which could be used for poset-causal systems). Two important drawbacks of their approach are: (a) one would need to solve significantly larger Riccati equations thereby greatly increasing the required computational effort, and (b) the lack of insight into the form of the optimal controller. Our approach for poset-causal systems is more efficient computationally. Moreover, our approach also provides insight into the structure of the optimal controllers.

The rest of this chapter is organized as follows. In Section 4.2 we introduce the necessary preliminaries regarding the problem. In Section 4.3 we describe our solution strategy. In Section 4.4 we present the main results. We devote Section 4.5 to a discussion of the main results, and their illustration via examples. Section 4.6 contains the proofs of the main results.

4.2 Preliminaries

In this section we introduce some additional concepts from order theory, the control theoretic setup and some notation. Once again $\mathcal{P} = (P, \preceq)$ is a poset, $\mathcal{I}(\mathcal{P})$ is its incidence algebra. Some examples that will be useful to keep in mind in this chapter are shown in Fig. 4-1.

Let $\mathcal{P} = (P, \preceq)$ be a poset and let $p \in P$. We define $\uparrow p = \{q \in P \mid p \preceq q\}$ (we call it the *upstream set* of p). Let $\uparrow\uparrow p = \{q \in P \mid p \preceq q, q \neq p\}$ (called the *strict* upstream set). Simi-
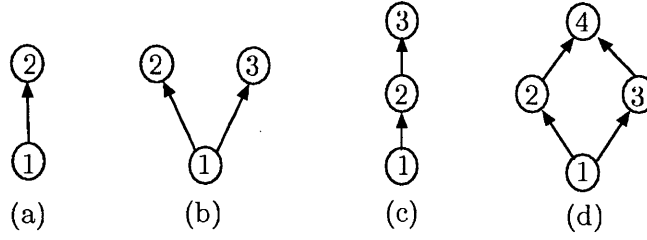
Figure 4-1: Hasse diagrams of some posets.

larly, let $\downarrow p = \{q \in P \mid q \preceq p\}$ (called a *downstream set*), and $\downarrow\downarrow p = \{q \in P \mid q \preceq p, q \neq p\}$. We will often refer to elements $q \in \uparrow\uparrow p$ as *upstream elements* of p and $q \in \downarrow\downarrow p$ as *downstream elements* of p. Define an *interval* $[i, j] = \{p \in P \mid i \preceq p \preceq j\}$. A *chain* is a subset $C \subseteq P$ which is *totally ordered* (i.e. any two elements of C are comparable). A *minimal element* of the poset is an element $p \in P$ such that if $q \preceq p$ for some $q \in P$ then $q = p$. (A maximal element is defined analogously).

In the poset shown in Fig. 4-1(d), $\uparrow 1 = \{1, 2, 3, 4\}$, whereas $\uparrow\uparrow 1 = \{2, 3, 4\}$. Similar $\downarrow 1 = \emptyset, \downarrow 4 = \{1, 2, 3, 4\}$, and $\downarrow\downarrow 4 = \{1, 2, 3\}$. Given $i \preceq j$, let $[i \to j]$ denote the set of all chains from i to j of the form $\{i, i_1\}, \ldots, \{i_k, j\}$ such that $i \preceq i_1 \preceq \cdots \preceq i_k \preceq j$. For example, in the poset in Fig. 4-1(c), $[1 \to 3] = \{\{1, 2, 3\}, \{1, 3\}\}$. A standard corollary of Theorem 2.1 is the following.

Corollary 4.1. *Suppose $A \in I(\mathcal{P})$. Then A is invertible if and only if A_{ii} is invertible for all $i \in P$. Furthermore $A^{-1} \in I(\mathcal{P})$, and the inverse is given by:*

$$[A^{-1}]_{ij} = \begin{cases} A_{ii}^{-1} \sum_{p_{ij} \in [j \to i]} \prod_{\{l,k\} \in p_{ij}} (-A_{lk} A_{kk}^{-1}) & \text{if } i \neq j \\ A_{ii}^{-1} & \text{if } i = j. \end{cases}$$

4.2.1 Control Theoretic Preliminaries

We consider the following state-space system in discrete time:

88

$$x[t+1] = Ax[t] + w[t] + Bu[t]$$

(4.1)

$$z[t] = Cx[t] + Du[t].$$

In this chapter we present the discrete time case only, however, we wish to emphasize that analogous results hold in continuous time in a straightforward manner. In this chapter we consider what we will call *poset-causal systems*. We think of the system matrices (A, B, C, D) to be partitioned into blocks in the following natural way. Let $\mathcal{P} = (P, \leq)$ be a poset with $P = \{1, \ldots, s\}$. We think of this system as being divided into s sub-systems, with sub-system i having some states $x_i[t] \in \mathbb{R}^{n_i}$, and control inputs $u_i[t] \in \mathbb{R}^{m_i}$ for $i \in \{1, \ldots, s\}$. The external output is $z[t] \in \mathbb{R}^p$. The signal $w[t]$ is a disturbance sig-· nal. (To use certain standard state-space factorization results, we assume that $C^T D = 0$ and $D^T D \succ 0$). The states and inputs are partitioned in the natural way such that the sub-systems correspond to elements of the set P with $x[t] = [x_1[t] \,|x_2[t]\,|\ldots|x_s[t]\,]^T$, and $u[t] = [u_1[t]\,|u_2[t]\,|\ldots|u_s[t]\,]^T$. This naturally partitions the matrices A, B, C, D into appropriate blocks so that $A = \left[A_{ij}\right]_{i,j \in P}$, $B = \left[B_{ij}\right]_{i,j \in P}$, $C = \left[C_j\right]_{j \in P}$ (partitioned into columns), $D = \left[D_j\right]_{j \in P}$. (We will throughout deal with matrices at this block-matrix level, so that A_{ij} will unambiguously mean the (i, j) block of the matrix A.) Using these block partitions, one can define the incidence algebra at the block matrix level in the natural way. We denote by $\mathcal{I}_A(\mathcal{P}), \mathcal{I}_B(\mathcal{P})$ the block incidence algebras corresponding to the block partitions of A and B. Often, matrices will have different (but compatible) dimensions and the block structure will be clear from the context. In these cases, we will abuse notation and will drop the subscript and simply write $\mathcal{I}(\mathcal{P})$.

Definition 4.1. *We say that a state-space system is* $\mathcal{P}$*-poset-causal* (or simply poset-causal) *if* $A \in \mathcal{I}_A(\mathcal{P})$ *and* $B \in \mathcal{I}_B(\mathcal{P})$.

Example 4.1. *We use this example to illustrate ideas and concepts throughout this chapter.*

89

Consider the system

$$x[t + 1] = Ax[t] + w[t] + Bu[t]$$

$$z[t] = Cx[t] + Du[t]$$

$$y[t] = x[t],$$

with matrices

$$A = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ -1 & -0.25 & 0 & 0 \\ -1 & 0 & -0.2 & 0 \\ -1 & -1 & -1 & -0.1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad (4.2)$$

$$C = \begin{bmatrix} I^{4\times4} \\ 0^{4\times4} \end{bmatrix} \qquad D = \begin{bmatrix} 0^{4\times4} \\ I^{4\times4} \end{bmatrix}. \qquad (4.3)$$

This system is poset-causal with the underlying poset described in Fig. 4-1(d). Note that in this system, each subsystem has a single input, a single output and a single state. The matrices A and B are in the incidence algebra of the poset.

Recall that the standard notion of causality in systems theory is based crucially on an underlying totally ordered index set (time). Systems (in LTI theory these are described by impulse responses) are said to be *causal* if the support of the impulse response is consistent with the ordering of the index set: an impulse at time zero is only allowed to propagate in the increasing direction with respect to the ordering. This notion of causality can be readily extended to situations where the underlying index set is only *partially ordered*. Indeed this abstract setup has been studied by Mullans and Elliott [34], and an interesting algebraic theory of systems has been developed.

Our notion of poset-causality is very much in the same spirit. We call such systems poset-causal due to the following analogous property among the sub-systems. If an input

90

is applied to sub-system i via u_i at some time t, the effect of the input is seen by the states x_j for all sub-systems $j \in\uparrow i$ (at or after time t). Thus $\uparrow i$ may be seen as the cone of influence of input i. We refer to this causality-like property as *poset-causality*. This notion of causality enforces (in addition to causality with respect to time), causality with respect to the subsystems via a poset. For most of this chapter we will deal with systems that are poset-causal (with respect to some arbitrary but fixed finite poset $\mathcal{P}$). Before we turn to the problem of optimal control we state an important result regarding stabilizability of poset-causal systems of the form (4.1) by poset-causal controllers.

Theorem 4.1. *The poset-causal system* (4.1) *is stabilizable by a poset-causal controller* $K \in \mathcal{I}(\mathcal{P})$ *if and only if the* (A_{ii}, B_{ii}) *are stabilizable for all* $i \in P$.

Proof. See Section 4.6.   □

Note that this result may be viewed as a specialization of the stabilization result in Chapter 3 Lemma 3.2 for state-space systems of the form (4.1) with state feedback. In this chapter, we make the following important assumption about the stabilizability of the sub-systems. By the preceding theorem, this assumption is necessary and sufficient to ensure that the systems under consideration have feasible controllers.

Assumption 1: Given the poset-causal system of the form (4.1), we assume that the sub-systems (A_{ii}, B_{ii}) are stabilizable for all $i \in \{1, \ldots, s\}$.

In the absence of this assumption, there is no poset-causal stabilizing controller to (4.1), and hence the problem of finding an optimal one becomes vacuous. This assumption is necessary and sufficient for the problem to be well-posed. Moreover, in what follows, we will need the solution of certain standard Riccati equations. Assumption 1 ensures that all of these Riccati equations have well-defined stabilizing solutions. This stabilizing property of the Riccati solutions will be useful for proving internal stability of the closed loop system.

The system (4.1) may be viewed as a map from the inputs w, u to outputs z, x via

$$z = P_{11}w + P_{12}u$$

$$x = P_{21}w + P_{22}u$$

where

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} C(zI - A)^{-1} & C(zI - A)^{-1}B + D \\ (zI - A)^{-1} & (zI - A)^{-1}B \end{bmatrix}$$

$$= \left[\begin{array}{c|cc} A & I & B \\ \hline C & 0 & D \\ I & 0 & 0 \end{array} \right].$$

(4.4)

A controller $u = Kx$ induces a map T_{zw} from the disturbance input w to the exogenous output z via

$$T_{zw} = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}.$$

Thus, after the controller is interconnected with the system, the closed-loop map is T_{zw}. The objective function of interest is to minimize the $\mathcal{H}_2$ norm [66] of T_{zw} which we denote by $\|T_{zw}\|$.

Information Constraints on the Controller

Given the system (4.1), we are interested in designing a controller K that meets certain specifications. In traditional control problems, one requires K to be proper, causal and stabilizing. We impose additional constraints on the controller related to decentralization. The decentralization constraint of interest in this chapter is one where the controller mirrors the structure of the plant, and is therefore also in the block incidence algebra $\mathcal{I}_K(\mathcal{P})$ (we will henceforth drop the subscripts and simply refer to the incidence algebra $\mathcal{I}(\mathcal{P})$). This translates into the requirement that input u_i (which corresponds to the input at subsystem i) only has access to states x_j for $j \in \downarrow i$ thereby enforcing poset-causality constraints also on

92

the controller. In this sense the controller has access to *local* states, and we thus refer to it as a *decentralized* state-feedback controller.

Problem Statement

Given the poset-causal system (4.4) with poset $\mathcal{P} = (P, \preceq)$, $|P| = s$, solve the optimization problem:

$$\underset{K}{\text{minimize}} \quad \|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\|^2$$

$$\text{subject to} \quad K \in \mathcal{I}(\mathcal{P}) \tag{4.5}$$

$$K \text{ stabilizing.}$$

The main problem under consideration is to solve the above stated optimal control problem in the controller variable K. The space of solutions of K is the set of all rational proper transfer function matrices that internally stabilize the system (4.1). In the absence of the decentralization constraints $K \in \mathcal{I}(\mathcal{P})$ this is a standard, well-studied control problem that has an efficient finite-dimensional *state-space* solution [66]. The main objective of this chapter is to construct such a solution for the poset-causal case.

Notation

Given a matrix Q, let $Q(j)$ denote the jth column of Q. We denote the ith component of the vector $Q(j)$ to be $Q(j)_i$. For a poset $\mathcal{P}$ with incidence algebra $\mathcal{I}(\mathcal{P})$, we denote the sparsity pattern of the jth column of the matrices in $\mathcal{I}(\mathcal{P})$ by:

$$\mathcal{I}(\mathcal{P})^j := \{v | v_i = 0 \text{ for } j \npreceq i\}.$$

In the above definition v is understood to be a vector composed of $|P|$ blocks, with sparsity being enforced at the block level.

Given the data (A, B, C, D), we will often need to consider sub-matrices or embed a sub-matrix into a full dimensional matrix by zero padding. Some notation for that purpose

we will use is the following:

1. Define $Q^{\uparrow j} = [Q_{ij}]_{i \in \uparrow j}$ (so that it is the jth column shortened to include only the nonzero entries).

2. Also define $A(\uparrow j) = [A(i)]_{i \in \uparrow j}$ so that it is the sub-matrix of A containing all rows and exactly those columns corresponding to the set $\uparrow j$.

3. Define $A(\uparrow j, \uparrow j) = [A_{kl}]_{k,l \in \uparrow j}$ so that it is the $(\uparrow j, \uparrow j)$ sub-matrix of A (containing exactly those rows and columns corresponding to the set $\uparrow j$).

4. Sometimes, given a block $|\uparrow j| \times |\uparrow j|$ matrix we will need to embed it into a block matrix indexed by the original poset (i.e. a $s \times s$ matrix) by padding it with zeroes. Given K (a block $|\uparrow j| \times |\uparrow j|$ matrix) we define:

$$[\hat{K}]_{lm} = \begin{cases} K_{lm} & \text{if } l, m \in \uparrow j \\ 0 & \text{otherwise.} \end{cases}$$

5. $E_i = [\ 0 \ \ \dots \ \ I \ \ \dots \ \ 0\ ]^T$ be the tall block matrix (indexed with the elements of the poset) with an identity in the ith block row.

6. Let $S \subseteq P$. Define $E_S = [E_i]_{i \in S}$. Note that given a block $s \times s$ matrix M, $ME_{\uparrow j} = M(\uparrow j)$ is a matrix containing the columns indexed by $\uparrow j$.

7. Given matrices $A_i, i \in P$, we define the block diagonal matrix:

$$\text{diag}(A_i) = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_s \end{bmatrix}.$$

Recall that every poset $\mathcal{P}$ has a linear extension (i.e. a total order on P which is consistent with the partial order $\leq$). For convenience, we fix such a linear extension of $\mathcal{P}$, and all

indexing of our matrices throughout the chapter will be consistent with this linear extension (so that elements of the incidence algebra are lower triangular).

Example 4.2. *Let $\mathcal{P}$ be the poset shown in Fig. 4-1(d). We continue with Example 4.1 to illustrate notation. (Note that $\uparrow 2 = \{2, 4\}$). As per the notation defined above,*

$$
A^{\uparrow 2} = \begin{bmatrix} -.25 \\ -1 \end{bmatrix} \quad
A(\uparrow 2) = \begin{bmatrix} 0 & 0 \\ -0.25 & 0 \\ 0 & 0 \\ -1 & -0.1 \end{bmatrix} \quad
A(\uparrow 2, \uparrow 2) = \begin{bmatrix} -0.25 & 0 \\ -1 & -0.1 \end{bmatrix}.
$$

$$
\text{Also, if } K(\uparrow 2, \uparrow 2) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \text{ then } \hat{K}(\uparrow 2, \uparrow 2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}.
$$

4.3 Solution Strategy

In this section we first remind the reader of a standard reparametrization of the problem known as the Youla parametrization. Using this reparametrization, we illustrate the main technical idea of this chapter using an example.

4.3.1 Reparametrization

Problem (4.5) as stated has a nonconvex objective function. Typically [44, 47], this is convexified by a bijective change of parameters given by $R := K(I - P_{22}K)^{-1}$. When the sparsity constraints are poset-causal (or quadratically invariant, more generally), this change of parameters preserves the sparsity constraints, and R inherits the sparsity constraints of K. The resulting infinite-dimensional problem is convex in R.

95

For poset-causal systems with state-feedback we will use a slightly different parametrization. Firstly, we note that for poset-causal systems, the matrices A and B are both in the block incidence algebra. As a consequence of (4.4), P_{21} and P_{22} are also in the incidence algebra. This structure, which follows from the closure properties of an incidence algebra, will be extensively used. Since $P_{21}, P_{22} \in I(\mathcal{P})$ the optimization problem (4.5) maybe be reparametrized as follows. Set

$$Q := K(I - P_{22}K)^{-1}P_{21}. \tag{4.6}$$

Note that the map $K \mapsto K(I - P_{22}K)^{-1}P_{21}$ is bijective (provided the inverse exists). Given Q, K can be recovered using

$$K = QP_{21}^{-1}(I + P_{22}QP_{21}^{-1})^{-1}. \tag{4.7}$$

Moreover, since I, P_{21}, P_{22} all lie in the incidence algebra, $K \in I(\mathcal{P})$ if and only if $Q \in I(\mathcal{P})$. Using this reparametrization the optimization problem (4.5) can be recast as:

$$\begin{aligned} \underset{Q}{\text{minimize}} \quad & \|P_{11} + P_{12}Q\|^2 \\ \text{subject to} \quad & Q \in I(\mathcal{P}). \end{aligned} \tag{4.8}$$

Remarks 1. We note that $P_{21} = (zI - A)^{-1}$, and hence (4.7), which involves P_{21}^{-1} may potentially be improper. However, we will *prove* that for the optimal Q in (4.8), this expression is proper and corresponds to a rational controller K^*.

2. For the objective function to be bounded, the optimal Q would have to render $P_{11} + P_{12}Q$ stable. However, one also requires that the overall system is internally stable. We relax this requirement on Q and later show that K^* is nevertheless internally stabilizing. Thus (4.8) is in fact a *relaxation* of (4.5) and thus its optimal value (we call it v_2^*) is no larger than the optimal value of (4.5). We show that the final controller

96

K^* achieves this lower bound v_2^* and is also internally stabilizing.

We would like to emphasize the very important role played by the availability of *full state-feedback*. As a consequence of state-feedback, we have that $P_{21} = (zI - A)^{-1}$. Thus P_{21} is square, invertible (though the inverse is improper), and in the incidence algebra. It is this very important feature of P_{21} that allows us to use this modified parametrization mentioned (4.6) in the preceding paragraph. This parametrization enables us to rewrite the problem in the form (4.8). This form will turn out to be crucial to our main separability result (Theorem 4.2), which enables us to separate the decentralized problem into a set of decoupled centralized problems.

A main step in our solution strategy will be to reduce the optimal control problem to a set of standard centralized control problems, whose solutions may be obtained by solving standard Riccati equations. The key result about centralized $\mathcal{H}_2$ optimal control is as follows.

Lemma 4.1. *Consider a system given by (4.4), along with the following optimal control problem:*

$$\underset{Q}{minimize} \quad \|P_{11} + P_{12}Q\|^2. \tag{4.9}$$

Suppose the pair (A, B) is controllable, $C^T D = 0$, and $D^T D \succ 0$. Then the following Riccati equation has a unique symmetric and positive definite solution:

$$X = C^T C + A^T X A - A^T X B (D^T D + B^T X B)^{-1} B^T X A. \tag{4.10}$$

Let K be obtained from this unique positive definite solution via:

$$K = (D^T D + B^T X B)^{-1} B^T X A. \tag{4.11}$$

Then the optimal solution to (4.9) is given by:

$$Q = -K(zI - (A - BK))^{-1}$$

$$= \left[\begin{array}{c|c} A - BK & I \\ \hline -K & 0 \end{array} \right].$$

(4.12)

(We will often refer to the trio of equations (4.10), (4.11), (4.12) by $(K, Q, P) = \text{Ric}(A, B, C, D)$.)

Proof. The proof is based on standard spectral factorization techniques. A proof may be found in [52, Lemma 5, Lemma 8]. □

4.3.2 Separability of Optimal Control Problem

We next illustrate the main solution strategy via a simple example. Consider the decentralized control problem for the poset in Fig. 4-1(b). Using the reformulation (4.8) the optimal control problem (4.5) may be recast as:

$$\min_{Q} \left\| P_{11} + P_{12} \begin{bmatrix} Q_{11} & 0 & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & 0 & Q_{33} \end{bmatrix} \right\|^2$$

Note that $P_{12}(\uparrow 1) = P_{12}, P_{12}(\uparrow 2) = P_{12}(2)$ (second column of P_{12}), and $P_{12}(\uparrow 3) = P_{12}(3)$. Similarly $Q^{\uparrow 1} = \begin{bmatrix} Q_{11}^T & Q_{21}^T & Q_{31}^T \end{bmatrix}^T, Q^{\uparrow 2} = Q_{22}$, and $Q^{\uparrow 3} = Q_{33}$. Due to the column-wise separability of the $\mathcal{H}_2$ norm, the problem can be recast as:

$$\min_{Q} \left\| P_{11}(1) + P_{12} \begin{bmatrix} Q_{11} \\ Q_{21} \\ Q_{31} \end{bmatrix} \right\|^2 + \| P_{11}(2) + P_{12}(2)Q_{22} \|^2$$
$$+ \| P_{11}(3) + P_{12}(3)Q_{33} \|^2$$

Since the sets of variables appearing in each of the three quadratic terms is different, the problem now may be decoupled into three separate sub-problems, each of which is a standard centralized control problem. For instance, the solution to the second sub-problem can be obtained by noting the realizations of $P_{11}(2)$ and $P_{12}(2)$ and then using (4.12). In this instance,

$$(K, Q^*_{22}, P) = \text{Ric}(A_{22}, B_{22}, C_2, D_2).$$

In a similar way, the entire matrix Q^* can be obtained, and by design $Q^* \in \mathcal{I}(\mathcal{P})$ (and is stabilizing). To obtain the optimal K^*, one can use (4.7). In fact, using (4.7) it is possible to give an explicit state-space formula for K^*, this is the main content of Theorem 4.3 in the next section.

4.4 Main Results

In this section, we present the main results of the chapter. The proofs are available in Section 4.6.

4.4.1 Problem Decomposition and Computational Procedure

Theorem 4.2 (Decomposition Theorem). *Let $\mathcal{P}$ be a poset and $\mathcal{I}(\mathcal{P})$ be its incidence algebra. Consider a poset-causal system given by (4.4). The problem (4.8) is equivalent to the following set of $|P|$ independent decoupled problems:*

$$\underset{Q}{\text{minimize}} \quad \|P_{11}(j) + P_{12}(\uparrow j)Q^{\uparrow j}\|^2 \quad \forall j \in P. \tag{4.13}$$

Theorem 4.2 is essentially the first step towards a state-space solution. The advantage of this equivalent reformulation of the problem is that we now have $s = |P|$ sub-problems, each over a different set of variables (thus the problem is *decomposed*). Moreover, each sub-problem corresponds to a particular *standard centralized* control problem, and thus the

optimal Q in (4.5) can be computed by simply solving each of these sub-problems.

The subproblems described in (4.13) have the following interpretation. Once a controller K, or equivalently Q is chosen a map T_{zw} from the exogenous inputs w to the outputs z is induced. Let us denote by $T_{zw}(1)$ to be the map from the first input z_1 to all the outputs w (this corresponds to the first column of T_{zw}). Similarly, the map from z_i to w for $i \in P$ is given by $T_{zw}(i)$. These subproblems correspond to the computation of the *optimal* maps $T_{zw}^{*(i)}$ for all $i \in P$ from the ith input z_i to the output w. The decomposability of the $\mathcal{H}_2$ norm implies that these maps may be computed separately, and the performance of the overall system is simply the aggregation of these individual maps.

Our next theorem provides an efficient computational technique to obtain the required state-space solution. To obtain the solution, one needs to solve Riccati equations corresponding to the sub-problems we saw in Theorem 4.2. We combine these solutions to form certain simple block matrices, and after simple LFT transformations, one obtains the optimal controller K^*.

Before we state the theorem, we introduce some relevant notation. Let

$$(K(\uparrow j, \uparrow j), Q(j), P(j)) = \text{Ric}\,(A(\uparrow j, \uparrow j), B(\uparrow j, \uparrow j), C(\uparrow j), D(\uparrow j)) \quad \forall j \in P.$$

The matrix

$$A_c^j := A(\uparrow j, \uparrow j) - B(\uparrow j, \uparrow j)K(\uparrow j, \uparrow j) \quad \forall j \in P$$

corresponding to the closed loop state transition matrix will appear often. We introduce two matrices related to the above solution, namely:

$$\mathbf{A} = \text{diag}(A(\uparrow j, \uparrow j) - B(\uparrow j, \uparrow j)K(\uparrow j, \uparrow j))$$

$$\mathbf{K} = \text{diag}(K(\uparrow j, \uparrow j)).$$

100

We will see later on that $\mathbf{A}$ is the closed-loop state transition matrix under a particular indexing of the states. We introduce three matrices related to structure of the poset, namely:

$$\Pi_1 = \begin{bmatrix} E_1 & 0 & \ldots & 0 \\ 0 & E_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & E_1 \end{bmatrix},$$

$$\Pi_2 = \mathrm{diag}\left(\begin{bmatrix} E_2 & \ldots & E_{|\uparrow j|} \end{bmatrix} \right),$$

$$R = \begin{bmatrix} E_{\uparrow 1} & \ldots & E_{\uparrow s} \end{bmatrix}. \tag{4.14}$$

(In Π_1, the jth diagonal block E_1 has $|\uparrow j|$ number of block rows. To be precise, it is a $(\sum_{k \in \uparrow j} n_k) \times n_j$ matrix with the first $n_j \times n_j$ block as the identity and the rest zeroes.) These matrices also have a natural interpretation. In writing the overall states of the closed loop in vector form, we first write the states of subsystem 1 (i.e. x_1) of the plant, then the states of the controller for subsystem 1 (i.e. $q(1)$), then subsystem 2 plant states and controller states, and so on. In this indexing, Π_1 is a *projection operator* that projects onto the coordinates of all the state variables $x_1, \ldots, x_2$. Π_2 is simply the matrix that projects onto the orthogonal complement, i.e. the controller variables $q(1), \ldots, q(s)$. The optimal controller and other related objects can be expressed in terms of the following matrices:

$$A_\Phi = \Pi_2^T \mathbf{A} \Pi_2,$$

$$B_\Phi = \Pi_2^T \mathbf{A} \Pi_1,$$

$$C_\Phi = R\Pi_2, \tag{4.15}$$

$$C_Q = -R\mathbf{K}.$$

We illustrate this notation further by means of a numerical example in Section 4.5.4.

Theorem 4.3 (Computation of Optimal Controller). *Consider the poset-causal system of the form* (4.4), *with* (A_{ii}, B_{ii}) *stabilizable for all* $i \in P$. *Consider the following Riccati*

101

equations:

$$(K(\uparrow j, \uparrow j), Q(j), P(j)) = Ric\,(A(\uparrow j, \uparrow j), B(\uparrow j, \uparrow j), C(\uparrow j), D(\uparrow j)) \quad \forall\, j \in P.$$

Then the optimal solution to the problem (4.5) is given by the controller:

$$K^* = \left[\begin{array}{c|c} A_\Phi - B_\Phi C_\Phi & B_\Phi \\ \hline C_Q(\Pi_2 - \Pi_1 C_\Phi) & C_Q \Pi_1 \end{array} \right]. \qquad (4.16)$$

Moreover, the controller $K^* \in \mathcal{I}(\mathcal{P})$ and is stabilizing.

Recall that n_i denotes the degree of the ith sub-system in (4.1). Let $n_{\max} = \max_i n_i$ be the largest degree of the sub-systems. Let $n(\uparrow\uparrow i) = \sum_{j \in \uparrow\uparrow i} n_j$. Let $\sigma_\mathcal{P} = \sum_{j \in P} |\uparrow\uparrow j|$ (note that this is a purely combinatorial quantity, dependent only on the poset). As we mentioned in the introduction, one of the advantages of state-space techniques is that they provide graceful degree bounds for the optimal controller. As a consequence of Theorem 4.3 we have the following:

Corollary 4.2 (Degree Bounds). *The degree d_{K^*} of the optimal controller is bounded above by*

$$d_{K^*} \leq \sum_{j \in P} n(\uparrow\uparrow j).$$

In particular, $d_{K^*} \leq \sigma_\mathcal{P} n_{\max}$.

4.4.2 Structure of the Optimal Controller

Having established the computational aspects, we now turn to some structural aspects of the optimal controller. Theorem 4.4 sheds some light on the same. We first introduce a pair of very important objects (Φ, Γ), called the *propagation filter* and the *differential filter*,

respectively. Define the block $s \times s$ transfer function matrices (Φ, Γ) via:

$$\Phi = \left[\begin{array}{c|c} A_\Phi & B_\Phi \\ \hline C_\Phi & I \end{array} \right] \qquad \Gamma = \left[\begin{array}{c|c} A_\Phi - B_\Phi C_\Phi & B_\Phi \\ \hline -C_\Phi & I \end{array} \right]. \tag{4.17}$$

Note that both Φ and Γ are invertible (since their "D" matrices are equal to I), and in fact, they are inverses of each other, i.e., $\Gamma \Phi = \Phi \Gamma = I$. We sometimes denote the entries $\Phi_{ij} = \Phi_{i \leftarrow j}$ and similarly $\Gamma_{ij} = \Gamma_{i \leftarrow j}$ to emphasize a certain interpretation of these quantities. We note that $\Phi_{i \leftarrow i} = \Gamma_{i \leftarrow i} = I$ (this can be seen from the fact that the corresponding entries in the "C" matrices of the transfer functions is zero). We show (Lemma 4.4 in the Section 4.6), that $\Phi, \Gamma \in \mathcal{I}(\mathcal{P})$. Moreover, the fact that $\Phi^{-1} = \Gamma$ in conjunction with Corollary 4.1 gives the following expression:

$$\Gamma_{i \leftarrow j} = \sum_{p_{ij} \in [j \to i]} \prod_{\{l,k\} \in p_{ij}} (-\Phi_{l \leftarrow k}). \tag{4.18}$$

We will show that $\Phi_{l \leftarrow k}$, in fact, corresponds to a *specific filter* that propagates local signals upstream. For example, in Fig. 4-1(a), if x_1 is the state at subsystem 1, $\Phi_{21} x_1$ is the prediction of state x_2 at subsystem 1. On the other hand, Γ has an interesting dual interpretation. As one proceeds "upstream" through the poset, more information is available, and consequently the prediction of the global state becomes more accurate. The transfer function Γ plays the role of computing the differential improvement in the prediction of the global state. For this reason, we call it the *differential filter*. Interestingly, it is intimately related to the notion of a *Möbius inversion* on a poset, a generalization of differentiation to posets. We briefly discuss these ideas in the ensuing discussion. Before stating the next theorem, we introduce the transfer function matrix K_Φ, which is defined column-wise via:

$$K_\Phi(j) = \hat{K}(\uparrow j, \uparrow j) \Phi(j).$$

Theorem 4.4 (Structure of Optimal Controller). *The optimal controller* (4.16) *is of the form:*

$$u[t] = -K_\Phi \Gamma x[t]$$

$$= -\sum_{j \in P} \hat{K}(\uparrow j, \uparrow j) \Phi(j)(\Gamma x)_j[t].$$

Remark Let us denote the vector $e(j) = \Phi(j)(\Gamma x)_j$. We will interpret $e(j)$ as the differential improvement in the prediction of the global state x at subsystem j. Denoting $\hat{K}(\uparrow j, \uparrow j)$ by K_j, note that the control law takes the form $u[t] = \sum_{j \in P} K_j e(j)$. This structural form suggests that the controller uses the differential improvement of the global state at the different subsystems as the atoms of local control laws, and that the overall control law is an aggregation of these local control laws.

4.4.3 Interpretation of Φ and Γ

Due to the information constraints in the problem, at subsystem j only states in $\downarrow j$ are available, states of other subsystems are unavailable. A reasonable architecture for the controller would involve *predicting* the unknown states at subsystem j from the available information. We first note that at a particular subsystem it may be possible to compute only a partial prediction of the state. This is illustrated by the following example.

Example 4.3. *Consider the system shown in Fig. 4-2 with dynamics*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}[t+1] = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}[t+1] + \begin{bmatrix} B_{11} & 0 & 0 \\ 0 & B_{22} & 0 \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}[t+1].$$

Note that subsystem 1 has no information about the state of subsystem 2. Moreover, the state x_1 or input u_1 does not affect the dynamics of 2 (their respective dynamics are uncoupled). Hence the only sensible prediction of x_2 at subsystem 1 (which we denote by

$$\begin{array}{c} 3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\[2em] 1 \qquad\qquad 2 \\[1em] \begin{bmatrix} x_1 \\ 0 \\ x_3(1) \end{bmatrix} \qquad \begin{bmatrix} 0 \\ x_2 \\ x_3(2) \end{bmatrix} \end{array}$$
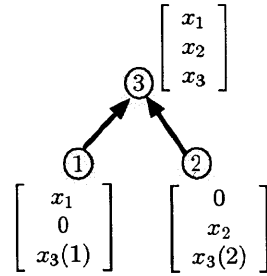
Figure 4-2: Local state information at the different subsystems. The quantities $x_3(1)$ and $x_3(2)$ are partial state predictions of x_3.

$x_2(1)$) is $x_2(1) = 0$. However, both the states x_1, x_2 and inputs u_1, u_2 affect x_3 and u_3. Let us denote $x_3(1)$ to be the prediction of state x_3 at subsystem 1. Since x_2 and u_2 are unknown, the state $x_3(1)$ can at best be a partial prediction of x_3 (i.e. $x_3(1)$ is the prediction of the component of x_3 that is affected by subsystem 1). Similarly $x_3(2)$ is only a partial prediction of x_3. Indeed, one can show that $x_3(1) + x_3(2)$ is a more accurate prediction of the state x_3, and when suitably designed, their sum converges to the true state x_3.

In this chapter we will not discuss how the state predictions are computed, we defer that to Chapter 5. However, we mention that Γ has an interesting related role. At subsystem j the state x_j becomes available for the first time (with respect to the subposet $\downarrow j$). The quantity $(\Gamma x)_j$ measures the *differential improvement* in the knowledge of state x_j, i.e. the difference between the true state x_j and its best prediction from downstream information.

We next examine the role of Φ. Consider a system of the form:

$$q[t + 1] = Hq[t],$$

where $q \in \mathbb{R}^n$. Given q_1 it is possible to compute $q_2, \ldots, q_n$ by *propagation* by noting that

105

$(zI - H)q = 0$. Rewriting these equations, we obtain that

$$-E_{\{2,\dots,n\}}^T HE_1 q_1 + E_{\{2,\dots,n\}}^T (zI - H)E_{\{2,\dots,n\}} \begin{bmatrix} q_2 \\ \vdots \\ q_n \end{bmatrix} = 0$$

to obtain

$$\begin{bmatrix} q_2 \\ \vdots \\ q_n \end{bmatrix} = (zI - H_2)^{-1} H_1 q_1, \tag{4.19}$$

where $H_2 = E_{\{2,\dots,n\}}^T HE_{\{2,\dots,n\}}$ and $H_1 = E_{\{2,\dots,n\}}^T HE_1$. The map $\Phi_H = (zI - H_2)^{-1} H_1$ from q_1 to $q_2, \dots, q_n$ is simply a propagation of the "upstreatm" states based on q_1.

Recall that in the solution procedure, we solved problems of the form

$$(K(\uparrow j, \uparrow j), Q(j), P(j)) = \text{Ric}(A(\uparrow j, \uparrow j), B(\uparrow j, \uparrow j), C(\uparrow j), D(\uparrow j)),$$

where the $K(\uparrow j, \uparrow j)$ are static gains. Suppose that in the closed loop system there are signals $q(j)$ at subsystems $j \in P$ with $q(j) \in \mathbb{R}^{|\uparrow j|}$ that evolve according to the following local relationship:

$$q(j)[t + 1] = (A(\uparrow j, \uparrow j) - B(\uparrow j, \uparrow j)K(\uparrow j, \uparrow j)) q(j)[t]. \tag{4.20}$$

Note that the evolution of the $q(j)$ are mutually decoupled. Let the first component of $q(j)$ be denoted by $q_j(j)$ and the remaining $|\uparrow\uparrow j|$ components be denoted by $q_{\uparrow\uparrow j}(j)$. Then by using the preceding argument, one can then compute $q_{\uparrow\uparrow j}(j)$ based on $q_j(j)$ via propagation, this would simply be given by applying formula (4.19). Rewriting this in state-space form,

we obtain that the required transfer function between $q_j(j)$ and $q_{\uparrow\uparrow j}(j)$ is given by

$$\left[\begin{array}{c|c} A_\Phi(j) & B_\Phi(j) \\ \hline E_{\uparrow\uparrow j} & 0 \end{array}\right].$$

(4.21)

It is easy to see that this is precisely the jth column of Φ (applying the concatenation formula for transfer functions, we can recover the formula for Φ), and thus $\Phi_{i \leftarrow j}$ is simply a computation of $q_{\uparrow\uparrow j}(j)$ based on $q_j(j)$ via propagation. In this sense, Φ plays the role of propagating decoupled local signals.

In Chapter 5, we will establish that the *differential improvements* in the local state predictions obey the above decoupled relationship (4.20) as a consequence of an elegant separation principle. (Thus $q(j)$ will correspond to the differential improvement in the state prediction at subsystem j). As we already mentioned $(\Gamma x)_j$ is the differential improvement in x_j at subsystem j. Since Φ plays the role of propagating decoupled local signals, it follows that $\Phi_{ij}(\Gamma x)_j$ is the differential improvement in the prediction of the state x_i for $i \in \uparrow\uparrow j$ at subsystem j. We illustrate this with an example

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \;\; \textcircled{4}$$
$$\textcircled{3} \;\; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4(3) \end{bmatrix}$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3(2) \\ x_4(2) \end{bmatrix} \;\; \textcircled{2}$$
$$\textcircled{1} \;\; \begin{bmatrix} x_1 \\ x_2(1) \\ x_3(1) \\ x_4(1) \end{bmatrix}$$
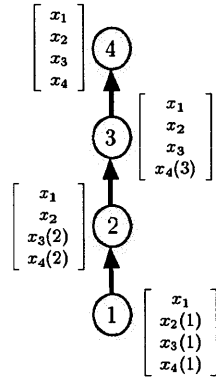
Figure 4-3: Local state information at the different subsystems for a 4-chain.

Example 4.4. *Consider the poset shown in Fig. 4-3. Consider a poset causal system consistent with this poset. The system is composed of four subsystems, let us call them S_1,*

S_2, S_3 and S_4. Let us denote $x_i(j)$ to be the prediction of state x_i at subsystem j. Note that if $i \prec j$ then $x_i(j) = x_i$ since subsystem j has access to all downstream states. Note that the differential improvement in the state x_1 at S_1 is simply x_1, so that $(\Gamma x)_1 = x_1$. Furthermore, at subsystem S_2, the best downstream prediction of x_2 is $x_2(1)$. The prediction at S_2 itself is x_2 (the true state is available here). The differential improvement in the prediction of x_2 at S_2 is given by $(\Gamma x)_2 = x_2 - x_2(1)$. Similarly at subsystems S_3 and S_4 the differential improvements in x_3 and x_4 respectively are $(\Gamma x)_3 = x_3 - x_3(2)$, and $(\Gamma x)_4 = x_4 - x_4(2)$. Furthermore $\Phi_{42}(\Gamma x)_2 = x_4(2) - x_4(1)$ is the differential improvement in the prediction of state x_4 at S_2. A complete list of the differential improvements is shown in Table I. The vector Γx corresponds to the diagonal entries in Table I.

Subsystem/State	S_1	S_2	S_3	S_4
Improvement in x_1	x_1	0	0	0
Improvement in x_2	$x_2(1)$	$x_2 - x_2(1)$	0	0
Improvement in x_3	$x_3(1)$	$x_3(2) - x_3(1)$	$x_3 - x_3(2)$	0
Improvement in x_4	$x_4(1)$	$x_4(2) - x_4(1)$	$x_4(3) - x_4(2)$	$x_4 - x_4(3)$

Table 4.1: Table showing the differential improvement in the different state predictions.

Remark The observant reader will notice that the formulae for the differential improvements bear a remarkable resemblance to finite difference formulae. Indeed Γ is intimately related to the *Möbius inversion formula* on a poset [1], [39] a concept that generalizes the notion of differentiation (more precisely finite differences) to arbitrary posets. The states in the optimal controller q in fact correspond to these generalized finite differences of state predictions of the form $\Phi_{ij}(\Gamma x)_j$ as will be explained in the next subsection. We do not dwell on the deeper relationship to the Möbius inversion and its role in the architecture of the controller in this chapter, and defer that discussion to Chapter 5.

4.4.4 Role of Controller States and the Closed Loop

Before we interpret the prediction structure further, let us first examine the controller states q. Note that $q \in \mathbb{R}^{\sum_{j \in P} n(\uparrow\uparrow j)}$ is a vector with $|P|$ blocks, each block of size $n(\uparrow\uparrow j)$. We can divide the states in the following way: to subsystem 1 we assign the first $|\uparrow\uparrow 1|$ states (we will represent them by $q(1)$. It is a vector corresponding to prediction of states associated to $\uparrow\uparrow 1$. Thus $q_i(1)$ is related to the prediction of the state at the ith subsystem of $\uparrow\uparrow 1$, etc. (more precisely it will correspond to the differential improvement in the prediction of the state). Similarly, to the second subsystem we assign the next $|\uparrow\uparrow 2|$ states and so on. More formally, we let

$$q = [q_j(i)]_{i \in P, j \in \uparrow\uparrow i},$$

so that $q(i)$ corresponds to the states of the controller associated with the ith subsystem. This $q(i)$ is a vector of length $|\uparrow\uparrow i|$ and $q_j(i)$ is the state associated with prediction improvements of x_j using x_i for $j \in \uparrow\uparrow i$.

We interpret the role of the controller states $q_i(j)$. Recall from our previous discussion that the differential improvement in the predictions may be computed using Φ and Γ. At subsystem j, the differential improvement in the prediction of state x_i is given by $\Phi_{ij}(\Gamma x)_j$. Moreover, for $i \prec j$, these differential improvements are zero because the precise state x_i is available at both subsystems i and j. The states of the controller are precisely these differential improvements in the prediction of the state:

$$q_i(j) = \Phi_{ij}(\Gamma x)_j. \tag{4.22}$$

More compactly, $\Phi(i)(\Gamma x)_i$ is the differential improvement in the global state at subsystem i.

Remark If j is a minimal element on the poset, $(\Gamma x)_j = x_i$, and the $\Phi_{ij}(\Gamma x)_j = q_i(j)$

109

correspond to (possibly partial) state predictions.

Indeed the optimal controller only needs to have these differential improvements to construct the global control law.

We emphasize at this juncture that while the formula for the optimal controller stated in Theorem 4.3 and the structural form described in Theorem 4.4 are proved in this chapter, we will formally prove the following two assertions in the next chapter:

- The fact that Γ has the role of computing differential improvements in the prediction of the state,

- That the controller states q in fact correspond precisely to these differential improvements.

Note that when the system (4.4) is connected with the controller (4.16), one obtains the closed-loop dynamics (in the absence of external disturbances):

$$
\begin{bmatrix} x[t+1] \\ q[t+1] \end{bmatrix} = \begin{bmatrix} A + BC_Q\Pi_1 & BC_Q(\Pi_2 - \Pi_1 C_\phi) \\ B_\Phi & A_\Phi - B_\Phi C_\Phi \end{bmatrix} \begin{bmatrix} x[t] \\ q[t] \end{bmatrix}.
\tag{4.23}
$$

By performing a change of coordinates of the state variables with respect to $\Gamma = \begin{bmatrix} \Pi_1 & \Pi_2 - \Pi_1 C_\Phi \end{bmatrix}$, we obtain the following dynamics:

$$
(\Pi_1(x - C_\Phi q) + \Pi_2 q)\,[t+1] = A\,(\Pi_1(x - C_\Phi q) + \Pi_2 q)\,[t].
\tag{4.24}
$$

The reader may easily verify that given the vector $q = [q_i(j)]_{j \in P, i \in \uparrow\uparrow j}$, C_Φ acts on q to produce the vector:

$$
C_\Phi q = \left[\sum_{j \in \downarrow\downarrow i} q_i(j) \right]_{j \in P}.
$$

As mentioned above, $q_i(k)$ represents the differential improvement in the prediction of state x_i at subsystem k. It follows then that from information available about $x_i(k)$ for $k \in \downarrow\downarrow j$,

the best prediction of state the state at i is $\sum_{k \in \downarrow\downarrow j} q_i(k)$. Hence the vector

$$e(i) := \begin{bmatrix} x_i - \sum_{k \in \downarrow\downarrow i} q_i(k) \\ q_{\uparrow\uparrow i}(i) \end{bmatrix}$$

corresponds to the differential improvement in the prediction of the collection of states $x_{\uparrow i}$ at subsystem i. Let us stack all the vectors $e(i)$ for $i \in P$ in a single vector to obtain $e := [e(i)]_{i \in P}$. In this setup, the matrices Π_1 and Π_2 (defined in (4.14)) act very naturally on the vector e:

- *Action of* Π_1: This is a projection onto the first components of the $e(i)$ so that $\Pi_1 e = [x_i - \sum_{k \in \downarrow\downarrow i} q_i(k)]_{i \in P}$. Note that $\Pi_1 e = \Gamma x$, i.e. the vector of differential improvements in the state x_i at subsystem i for all $i \in P$.

- *Action of* Π_2: This is a projection onto the remaining components so that $\Pi_2 e = [q_{\uparrow\uparrow i}(i)]_{i \in P}$.

The optimal control law can also be expressed in terms of the state variables using (4.16):

$$\begin{aligned} u &= C_Q(\Pi_1(x - C_\Phi q) + \Pi_2 q) \\ &= \sum_{i \in P} \hat{K}(\uparrow i, \uparrow i)\Phi(i)(\Gamma x)_i \quad \text{(by Theorem 4.4)}. \end{aligned} \tag{4.25}$$

Note that $\Phi(i)(\Gamma x)_i$ is a vector containing the differential improvement in the prediction of the global state at subsystem i. Each term $\hat{K}(\uparrow i, \uparrow i)\Phi(i)(\Gamma x)_i$ may be viewed as a local control law acting on the local differential improvement in the predicted state. The overall control law has the elegant interpretation of being an aggregation of these local control laws.

Example 4.5. *Let us consider the poset from Fig. 4-1(d), and examine the structure of the controller. (For simplicity, we let $K_j = \hat{K}(\uparrow j, \uparrow j)$, the gains obtained by solving the Riccati*

equations). The control law may be decomposed into local controllers as:

$$u = K_1 \begin{bmatrix} I \\ \Phi_{21} \\ \Phi_{31} \\ \Phi_{41} \end{bmatrix} x_1 + K_2 \begin{bmatrix} 0 \\ I \\ 0 \\ \Phi_{42} \end{bmatrix} (\Gamma x)_2 + K_3 \begin{bmatrix} 0 \\ 0 \\ I \\ \Phi_{43} \end{bmatrix} (\Gamma x)_3 + K_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} (\Gamma x)_4$$

$$= K_1 \begin{bmatrix} x_1 \\ q_2(1) \\ q_3(1) \\ q_4(1) \end{bmatrix} + K_2 \begin{bmatrix} 0 \\ x_2 - q_2(1) \\ 0 \\ q_4(2) \end{bmatrix} + K_3 \begin{bmatrix} 0 \\ 0 \\ x_3 - q_3(1) \\ q_4(3) \end{bmatrix} + K_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ (x_4 - q_4(1)) - q_4(2) - q_4(3) \end{bmatrix}.$$

Each term in the above expression has the natural interpretation of being a local control signal corresponding to differential improvement in predicted states, and the final controller can be viewed as an aggregation of these.

Note that zeros in the above expression imply no improvement on the local state. For example, at subsystem 2 there is no improvement in the predicted value of x_3 because the state x_2 does not affect subsystem 3 due to the poset-causal structure. There is no improvement in the predicted value of state x_3 at subsystem 4 either, because the best available prediction of x_3 from downstream information $\downarrow\downarrow 4$ is x_3 itself. While this interpretation has been stated informally here, it has been made precise in Chapter 5.

4.4.5   A Separation Principle

Note that closed-loop dynamics are given by (4.23). Upon changing coordinates, one obtains (4.24), a block-diagonal realization of the closed-loop dynamics. Writing out these

112

dynamics more explicitly, one obtains:

$$
\begin{bmatrix} \left(x_i - \sum_{j \in \downarrow i} q_i(j)\right)[t+1] \\ q(i)[t+1] \end{bmatrix} = (A(\uparrow i, \uparrow i) - B(\uparrow i, \uparrow i)K(\uparrow i, \uparrow i)) \begin{bmatrix} \left(x_i - \sum_{j \in \downarrow i} q_i(j)\right)[t] \\ q(i)[t] \end{bmatrix}.
$$

$$(4.26)$$

The dynamics for the differential improvement in the state predictions at the different subsystem are thus decoupled and evolve independently. This constitutes an elegant *separation principle*.

4.5 Discussion and Examples

4.5.1 The Nested Case

Consider the poset on two elements $\mathcal{P} = (\{1, 2\}, \preceq)$ with the only order relation being $1 \preceq 2$ (Fig. 4-1(a)). This is the poset corresponding to the communication structure in the "Two-Player Problem" considered in [51]. We show that their results are a specialization of our general results in Section 4.4 restricted to this particular poset.

We begin by noting that from the problem of designing a nested controller can be recast as:

$$
\underset{Q}{\text{minimize}} \quad \left\| P_{11} + P_{12} \begin{bmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{bmatrix} \right\|^2
$$

By Theorem 4.2 this problem can be recast as:

$$
\underset{Q}{\text{minimize}} \quad \left\| P_{11}^1 + P_{12} \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} \right\|^2 + \left\| P_{11}^2 + P_{12}^2 Q_{22} \right\|^2
$$

We wish to compare this to the results obtained in [51]. It is possible to obtain precisely this same decomposition in the finite time horizon where the $\mathcal{H}_2$ norm can be replaced by the Frobenius norm and separability can be used to decompose the problem. For each

of the sub-problems, the corresponding optimality conditions may be written (since they correspond to simple constrained-least squares problems). These optimality conditions correspond exactly to the decomposition of optimality conditions they obtain (the crucial Lemma 3 in their paper). We point out that the decomposition is a simple consequence of the separability of the Frobenius norm.

Let us now examine the structure of the optimal controller via Theorem 4.4. Note that $\uparrow 1 = \{1,2\}$ and $\uparrow 2 = \{2\}$. Based on Theorem 4.3, we are required to solve $(P(1), K) = \mathrm{Ric}(A, B, C, D)$, and $(P(2), J) = \mathrm{Ric}(A_{22}, B_{22}, C_2, D_2)$. Noting that in this example $\Gamma_{2\leftarrow 1} = -\Phi_{2\leftarrow 1}$, a straightforward application of Theorem 4.4 yields the following:

$$u_1[t] = -(K_{11} + K_{12}\Phi_{2\leftarrow 1})x_1[t]$$

$$u_2[t] = -(K_{21} + K_{22}\Phi_{2\leftarrow 1})x_1[t] - J(x_2[t] - \Phi_{21}x_1[t]),$$

which is precisely the structure of the optimal controller given in [51] (though they present the results in a finite-time horizon framework). It is possible to show (as Swigart et. al indeed do in [51]) that $\Phi_{2\leftarrow 1}$ is an predictor of x_2 based on x_1. Thus the controller for u_1 predicts the state of x_2 from x_1, uses the estimate as a surrogate for the actual state, and uses the gain K_{21} in the feedback loop. The controller for u_2 (perhaps somewhat surprisingly) also estimates the state x_2 based on x_1 using $\hat{x}_2 = \Phi_{21}x_1$ (this can be viewed as a "simulation" of the controller for u_1). The prediction error for state 2 is then given by $x_2 - \hat{x}_2 = x_2 - \Phi_{21}x_1$. The control law for u_2 may be rewritten as

$$u_2 = -(K_{21}x_1 + K_{22}\hat{x}_2 + Je_2).$$

Thus this controller uses predictions of x_2 based on x_1 along with prediction errors in the feedback loop. We will see in a later example, that this prediction of states higher up in the poset is prevalent in such poset-causal systems, which results in somewhat larger order

114

controllers.

Analogous to the results in [51], it is possible to derive the results in this chapter for the finite time horizon case (this is a special case corresponding to FIR plants in our discrete-time setup). We do not devote attention to the finite time horizon case in this chapter, but just mention that similar results follow in a straightforward manner.

4.5.2 Discussion Regarding Computational Complexity

Note that the main computational step in the procedure presented in Theorem 4.3 is the solution of the s sub-problems. The jth sub-problem requires the solution of a Riccati equation of size at most $|\uparrow j|n_{max} = O(s)$ (when the degree n_{max} is fixed). Assuming the complexity of solving a Riccati equation using linear algebraic techniques is $O(s^4)$ [13] the complexity of solving s of them is at most $O(s^5)$. We wish to compare this with the only other known state-space technique that works on all poset-causal systems, namely the results of Rotkowitz and Lall [44]. In this paper, they transform the problem to a standard centralized problem using Kronecker products. In the final computational step, one would be required to solve a single large Riccati equation of size $O(s^2)$, resulting in a computational complexity of $O(s^8)$.

4.5.3 Discussion Regarding Degree Bounds

It is insightful to study the asymptotics of the degree bounds in the setting where the sub-systems have fixed degree and the number of sub-systems s grows. As an immediate consequence of the corollary, the degree of the optimal controller (assuming that the degree of the sub-systems n_{max} is fixed) is at most $O(s^2)$ (since $n(\uparrow j) \leq s$). In fact, the asymptotic behaviour of the degree can be sub-quadratic. Consider a poset $(\{1, \ldots, s\}, \preceq)$ with the only order relations being $1 \preceq i$ for all i. Here $|\uparrow 1| = s$, and $|\uparrow i| = 1$ for all $i \neq 1$. Hence, $\sum_j |\uparrow j| - s \leq s$, and thus $d^* \leq s n_{max}$. In this sense, *the degree of the optimal controller is governed by the poset parameter σ_P.*

115

4.5.4 Numerical Example

In this section, we consider a numerical example for the poset shown in Fig. 4-1(d). The system has one state and one input per subsystem, and we synthesize the optimal controller. The data for the is the same as in Example 4.1 with the matrices A, B, C, D as given in (4.2). For this problem the relevant matrices that are used in constructing the controller are:

$$\Pi_1 = \begin{bmatrix} E_1 & E_5 & E_7 & E_9 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\Pi_2 = \begin{bmatrix} E_2 & E_3 & E_4 & E_6 & E_8 \end{bmatrix}.$$

(Recall that E_i is the 9×1 i^{th} unit vector.) Note that $\uparrow 1 = \{1, 2, 3, 4\}, \uparrow 2 = \{2, 4\}, \uparrow 3 = \{3, 4\}, \uparrow 4 = \{4\}$. Accordingly, the Riccati subproblems that we need to solve are given by

$$(K(\uparrow 1, \uparrow 1), Q(1), P(1)) = \text{Ric}(A, B, C, D),$$

$$(K(\uparrow 2, \uparrow 2), Q(2), P(2)) = \text{Ric}\left(\begin{bmatrix} A_{22} & A_{24} \\ A_{42} & A_{44} \end{bmatrix}, \begin{bmatrix} B_{22} & B_{24} \\ B_{42} & B_{44} \end{bmatrix}, \begin{bmatrix} C_2 & C_4 \end{bmatrix}, \begin{bmatrix} D_2 & D_4 \end{bmatrix} \right),$$

$$(K(\uparrow 3, \uparrow 3), Q(3), P(3)) = \text{Ric}\left(\begin{bmatrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{bmatrix}, \begin{bmatrix} B_{33} & B_{34} \\ B_{43} & B_{44} \end{bmatrix}, \begin{bmatrix} C_3 & C_4 \end{bmatrix}, \begin{bmatrix} D_3 & D_4 \end{bmatrix} \right),$$

$$(K(\uparrow 4, \uparrow 4), Q(4), P(4)) = \text{Ric}(A_{44}, B_{44}, C_4, D_4).$$

116

Upon solving these, we obtain

$$
K(\uparrow 1, \uparrow 1) = \begin{bmatrix} -.5340 & -.0230 & -.0139 & .0021 \\ -.2701 & -.2470 & -.1277 & -.0155 \\ -.2710 & -.1205 & -.2289 & -.0155 \\ .0315 & -.3028 & -.3125 & -.0352 \end{bmatrix} \qquad K(\uparrow 2, \uparrow 2) = \begin{bmatrix} -0.2983 & -0.0180 \\ -0.3507 & -0.0407 \end{bmatrix}
$$

$$
K(\uparrow 3, \uparrow 3) = \begin{bmatrix} -.2747 & -.0180 \\ -.3620 & -.0407 \end{bmatrix} \qquad\qquad K(\uparrow 4, \uparrow 4) = -.0501.
$$

From these, it is possible to construct $A(\uparrow j, \uparrow j) - B(\uparrow j, \uparrow j)K(\uparrow j, \uparrow j)$ for $j \in \{1, 2, 3, 4\}$, and from that construct $\mathbf{A} = \mathrm{diag}(A(\uparrow j, \uparrow j) - B(\uparrow j, \uparrow j)K(\uparrow j, \uparrow j))$ given by

$$
\mathbf{A} = \begin{bmatrix}
.0340 & .0230 & .0139 & -.0021 & 0 & 0 & 0 & 0 & 0 \\
-.1959 & .0200 & .1416 & .0134 & 0 & 0 & 0 & 0 & 0 \\
-.1959 & .1435 & .0428 & .0134 & 0 & 0 & 0 & 0 & 0 \\
.0435 & -.3067 & -.3170 & -.0359 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & .0483 & .0180 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -.3510 & -.0412 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .0747 & .0180 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -.3633 & -.0412 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.0499
\end{bmatrix}.
$$

Using (4.15) one readily obtains A_Φ and B_Φ to be

$$
A_\Phi = \begin{bmatrix}
.0200 & .1416 & .0134 & 0 & 0 \\
.1435 & .0428 & .0134 & 0 & 0 \\
-.3067 & -.3170 & -.0359 & 0 & 0 \\
0 & 0 & 0 & -.0412 & 0 \\
0 & 0 & 0 & 0 & -.0412
\end{bmatrix} \qquad B_\Phi = \begin{bmatrix}
-.1959 & 0 & 0 & 0 \\
-.1950 & 0 & 0 & 0 \\
.0435 & 0 & 0 & 0 \\
0 & -.3510 & 0 & 0 \\
0 & 0 & -.3633 & 0
\end{bmatrix}.
$$

Note that given $K(\uparrow 2, \uparrow 2)$ (a 2×2 matrix) one needs to construct $\hat{K}(\uparrow 2, \uparrow 2)$ (a 4×4 matrix) by zero padding. For instance, we have

$$
\hat{K}(\uparrow 2, \uparrow 2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -0.2983 & 0 & -0.0180 \\ 0 & 0 & 0 & 0 \\ 0 & -0.3507 & 0 & -0.0407 \end{bmatrix}.
$$

From these, one constructs C_Q using (4.15) to be

$$
C_Q = \begin{bmatrix} .5340 & .0230 & .0139 & -.0021 & 0 & 0 & 0 & 0 & 0 \\ .2701 & .2470 & .1277 & .0155 & .2983 & .0180 & 0 & 0 & 0 \\ .2710 & .1205 & .2289 & .0155 & 0 & 0 & .2747 & .0180 & 0 \\ -.0315 & .3028 & .3125 & .0352 & .3507 & .0407 & .3620 & .0407 & .0501 \end{bmatrix}.
$$

We use these quantities to obtain the controller K^* using formula (4.16). The controller

$$
K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right],
$$

has the following realization:

$$
A_K = \begin{bmatrix} 0.0200 & 0.1416 & 0.0134 & 0 & 0 \\ 0.1435 & 0.0428 & 0.0134 & 0 & 0 \\ -0.3067 & -0.3170 & -0.0359 & 0 & 0 \\ 0.3510 & 0 & 0 & -0.0412 & 0 \\ 0 & 0.3633 & 0 & 0 & -0.0412 \end{bmatrix}
$$

$$B_K = \begin{bmatrix} -0.1959 & 0 & 0 & 0 \\ -0.1959 & 0 & 0 & 0 \\ 0.0435 & 0 & 0 & 0 \\ 0 & -0.3510 & 0 & 0 \\ 0 & 0 & -0.3633 & 0 \end{bmatrix}$$

$$C_K = \begin{bmatrix} 0.0230 & 0.0139 & -0.0021 & 0 & 0 \\ -0.0513 & 0.1277 & 0.0155 & 0.0181 & 0 \\ 0.1205 & -0.0458 & 0.0155 & 0 & 0.0180 \\ -0.0479 & -0.0495 & -0.0149 & -0.0094 & -0.0094 \end{bmatrix}$$

$$D_K = \begin{bmatrix} 0.5340 & 0 & 0 & 0 \\ 0.2701 & 0.2983 & 0 & 0 \\ 0.2710 & 0 & 0.2747 & 0 \\ -0.03154 & 0.3507 & 0.3620 & 0.0512 \end{bmatrix}.$$

Note that the optimal controller is of degree 5. This matches the bound obtained in Corollary 4.2 exactly. Note also that the matrix D_K is in the incidence algebra (and so is the controller K itself, as can be verified from the transfer function). Finally, this controller can be verified to be stabilizing. Let $h_{\text{open}}, h_{\text{centralized}}, h_{\text{decentralized}}$ be the open loop, optimal centralized closed loop and optimal decentralized closed loop $\mathcal{H}_2$ norms. We obtain the following values:

$$h_{\text{open}} = 4.8620$$

$$h_{\text{centralized}} = 2.2675$$

$$h_{\text{decentralized}} = 2.2892.$$

119

4.6 Proofs of the Main Results

Proof of Theorem 4.1. Note that one direction is trivial. Indeed if the (A_{ii}, B_{ii}) are stabilizable, one can pick a diagonal controller with diagonal elements K_{ii} such that $A_{ii} + B_{ii}K_{ii}$ is stable for all $i \in P$. This constitutes a stabilizing controller.

For the other direction let

$$K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

be a poset-causal controller for the system. We will first show that without loss of generality, we can assume that A_K, B_K, C_K, D_K are block lower triangular (so that K has a realization where all matrices are block lower triangular).

First, note that since $K \in \mathcal{I}(\mathcal{P})$, $D_K \in \mathcal{I}(\mathcal{P})$. Recall, that we assumed throughout that the indices of the matrices in the incidence algebra are labeled so that they are consistent with a linear extension of the poset, so that D_K is lower triangular. Note that the controller K is a block $s \times s$ transfer function matrix which has a realization of the form:

$$K = \left[\begin{array}{c|ccc} A_K & B_K(1) & \dots & B_K(s) \\ \hline C_K(1) & D_K(1,1) & \dots & D_K(1,s) \\ \vdots & \vdots & \ddots & \\ C_K(s) & D_K(s,1) & \dots & D_K(s,s) \end{array}\right]$$

Since the controller $K \in \mathcal{I}(\mathcal{P})$, we have that $K_{sj} = 0$ for all $j \neq s$ (recall that s is the cardinality of the poset). This vector of transfer functions (given by the last column of K with the (s, s) entry deleted) is given by the realization:

$$\bar{K}_s := \left[\begin{array}{c} C_K(1) \\ \vdots \\ C_K(s-1) \end{array}\right](zI - A_K)^{-1}B_K(s) + \left[\begin{array}{c} D_K(1,s) \\ \vdots \\ D_K(s-1,s) \end{array}\right] = 0.$$

Since this transfer function is zero, in addition to $D_K(j, s) = 0$ for all $j = 1, \dots, s - 1$, it

120

must also be the case that the controllable subspace of $(A_K, B_K(s))$ is contained within the unobservable subspace of $\left(\begin{bmatrix} C_K(1)^T & \cdots & C_K(s-1)^T \end{bmatrix}^T, A_K \right)$. By the Kalman decomposition theorem [12, pp. 247], there is a realization of this system of the form:

$$\bar{K}_s = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array} \right],$$

where $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ are of the form:

$$\bar{A} = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{21} & A_{32} & A_{33} \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} 0 \\ 0 \\ B_3 \end{bmatrix} \tag{4.27}$$

$$\bar{C} = \begin{bmatrix} C_1 & 0 & 0 \end{bmatrix}$$

$$\bar{D} = 0.$$

As an aside, we remind the reader that this decomposition has a natural interpretation. For example, the subsystem

$$\left(\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} C_1 & 0 \end{bmatrix} \right)$$

corresponds to the observable subspace, where the system is uncontrollable, etc. (The usual Kalman decomposition as stated in standard control texts is a block 4×4 decomposition of the state-transition matrix. Here we have a smaller block 3×3 decomposition because of the collapse of the subspace where the system is required to be both controllable and observable).

Thus this decomposition allows us to infer the specific block structure (4.27) on the

matrices $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$. As a result of this block structure, there is a realization of the overall controller (A_K, B_K, C_K, D_K), where all the matrices have the block structure

$$
\begin{bmatrix}
M_{1,1} & \cdots & M_{1,s-1} & 0 \\
\vdots & \ddots & & \vdots \\
M_{s-1,1} & \cdots & M_{s-1,s-1} & 0 \\
M_{s,1} & \cdots & M_{s,s-1} & M_{s,s}
\end{bmatrix}.
$$

One can now repeat this argument for the upper $(s-1) \times (s-1)$ sub-matrix of K. By repeating this argument for first $s-1, s-2, \ldots, 1$ we obtain a realization of K where all four matrices are block lower triangular.

Note that given the controller K (henceforth assumed to have a lower triangular realization), the closed loop matrix A_{cl} is given by

$$
A_{cl} = \begin{bmatrix}
A + BD_K & BC_K \\
B_K & A_K
\end{bmatrix}.
$$

By assumption the (open loop) system is poset-causal, hence A and B are block lower triangular. As a result, each of the blocks $A + BD_K, BC_K, B_K, A_K$ are block lower triangular. A straightforward permutation of the rows and columns enables us to put A_{cl} into block lower triangular form where the diagonal blocks of the matrix are given by

$$
\begin{bmatrix}
A_{jj} + B_{jj}D_{K_{jj}} & B_{jj}C_{K_{jj}} \\
B_{K_{jj}} & A_{K_{jj}}
\end{bmatrix}. \tag{4.28}
$$

Note that the eigenvalues of this lower triangular matrix (and thus of A_{cl}, since permutations of rows and columns are spectrum-preserving) are given by the eigenvalues of the diagonal blocks. The matrix A_{cl} is stable if and only if all its eigenvalues are within the unit disk in the complex plane, i.e. the above blocks are stable for each $j \in P$. Note that (4.28) is

obtained as the closed-loop matrix precisely by the interconnection of

$$\left[\begin{array}{c|c} A_{jj} & B_{jj} \\ \hline I & 0 \end{array}\right]$$

with the controller

$$\left[\begin{array}{c|c} A_{K_{jj}} & B_{K_{jj}} \\ \hline C_{K_{jj}} & D_{K_{jj}} \end{array}\right].$$

Hence, (4.28) (and thus the overall closed loop) is stable if and only if (A_{jj}, B_{jj}) are stabilizable for all $j \in P$, and $(A_{K_{jj}}, B_{K_{jj}}, C_{K_{jj}}, D_{K_{jj}})$ are chosen to stabilize the pair. $\qquad\square$

Proof of Theorem 2. If $G = [G_1, \ldots G_k]$ is a matrix with G_i as its columns, then

$$\|G\|_F^2 = \sum_{i=1}^{k} \|G_i\|_F^2,$$

where $\| \cdot \|_F$ denotes the Frobenius norm. This separability property of the Frobenius norm immediately implies the following separability property for the $\mathcal{H}_2$ norm: If $H = [H_1, \ldots H_k]$ is a transfer function matrix with H_i as its columns, then

$$\|H\|^2 = \int_C \|H(z)\|_F^2 dz = \sum_{i=1}^{k} \int_C \|H_i(z)\|_F^2 dz = \sum_{i=1}^{k} \|H_i\|^2,$$

(In the above C denotes the unit circle in the complex plane). The separability property of the $\mathcal{H}_2$ norm can be used to simplify (4.9). Recall that $P_{11}(j)$, $Q(j)$ denote the jth columns of P_{11} and Q respectively. Using the separability we can rewrite (4.9) as

$$\begin{aligned} \underset{Q}{\text{minimize}} \quad & \textstyle\sum_{j\in P} \|P_{11}(j) + P_{12}Q(j)\|^2 \\ \text{subject to} \quad & Q(j) \in \mathcal{I}(\mathcal{P})^j \end{aligned}$$

(4.29)

123

The formulation in (4.29) can be further simplified by noting that for $Q^j \in I(\mathcal{P})^j$,

$$P_{12}Q(j) = P_{12}(\uparrow j)Q^{\uparrow j}. \tag{4.30}$$

The advantage of the representation (4.30) is that, in the right hand side the variable $Q^{\uparrow j}$ is *unconstrained*. Using this we may reformulate (4.29) as:

$$\underset{Q}{\text{minimize}} \quad \sum_{j \in P} \|P_{11}(j) + P_{12}(\uparrow j)Q^{\uparrow j}\|^2 \tag{4.31}$$

Since the variables in the $Q^{\uparrow j}$ are distinct for different j, this problem can be separated into s sub-problems as follows:

$$\underset{Q}{\text{minimize}} \quad \|P_{11}(j) + P_{12}(\uparrow j)Q^{\uparrow j}\|^2$$
$$\text{for all } j \in P. \tag{4.32}$$

$\square$

Note that each sub-problem is a *standard* $\mathcal{H}_2$ optimal centralized control problem, and can be solved using canonical procedures. Once the optimal Q is obtained by solving these sub-problems, the optimal controller may be synthesized using (4.7). The following lemma describes the solutions to the individual sub-problems (4.13) in Theorem 4.2.

Lemma 4.2. *Let* (A, B, C, D) *be as given in* (4.1) *with* A, B *in the block incidence algebra* $I(\mathcal{P})$. *Let*

$$(K(\uparrow j, \uparrow j), Q(j), P(j)) = Ric(A(\uparrow j, \uparrow j), B(\uparrow j, \uparrow j), C(\uparrow j), D(\uparrow j)). \tag{4.33}$$

Then the optimal solution of each sub-problem (4.13) *is given by:*

$$(Q^{\uparrow j})^* = \left[\begin{array}{c|c} A(\uparrow j, \uparrow j) - B(\uparrow j, \uparrow j)K(\uparrow j, \uparrow j) & E_1 \\ \hline -K(\uparrow j, \uparrow j) & 0 \end{array} \right]. \tag{4.34}$$

124

(We remind the reader that in the above E_1 is the block $|\uparrow j| \times 1$ matrix which picks out the first column corresponding of the block $|\uparrow j| \times |\uparrow j|$ matrix before it.)

Proof. Let P_{11}, P_{12} be as described in (4.4). Consider the following optimization problem:

$$\text{minimize}_{\bar{Q}} \quad \|P_{11}(\uparrow j) + P_{12}(\uparrow j)\bar{Q}\|^2. \tag{4.35}$$

We note that the first column of $\bar{Q}$ is precisely $Q^{\uparrow j}$ and the first column of the overall matrix in the objective function is precisely $P_{11}(j) + P_{12}(\uparrow j)Q^{\uparrow j}$. By the separability of the $\mathcal{H}_2$ norm (4.35) may be reformulated as:

$$\text{minimize}_{Q^{\uparrow j}, \bar{Q}'} \quad \|P_{11}(j) + P_{12}(\uparrow j)Q^{\uparrow j}\|^2 + \|P_{11}(\uparrow \uparrow j) + P_{12}(\uparrow j)\bar{Q}'\|^2. \tag{4.36}$$

(Here, $\bar{Q}'$ is the matrix obtained by deleting the first column of $\bar{Q}$). As a result of this decomposition property, the optimal $Q^{\uparrow j}$ can be seen to be the first column of the optimal $\bar{Q}$. Note that the solution to (4.13) can be obtained from (4.10), (4.11), (4.12) by solving

$$(K(\uparrow j, \uparrow j), \bar{Q}^*, P(j)) = \text{Ric}(A(\uparrow j, \uparrow j), B(\uparrow j, \uparrow j), C(\uparrow j), D(\uparrow j)),$$

with

$$\bar{Q}^* = \left[\begin{array}{c|c} A(\uparrow j, \uparrow j) - B(\uparrow j, \uparrow j)K(\uparrow j, \uparrow j) & I \\ \hline -K(\uparrow j, \uparrow j) & 0 \end{array} \right].$$

Since $(Q^{\uparrow j})^*$ is the first column of $\bar{Q}^*$, we obtain the required expression. □

Lemma 4.3. *The optimal solution to* (4.8) *is given by*

$$Q^* = \left[\begin{array}{c|c} \mathbf{A} & \Pi_1 \\ \hline C_Q & 0 \end{array} \right]. \tag{4.37}$$

125

Proof. We note that Lemma 4.2 gives an expression for the individual columns of Q^*.

Using Lemma 4.2 and the LFT formula for column concatenation:

$$\begin{bmatrix} G_1 & G_2 \end{bmatrix} = \left[\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ \hline C_1 & C_2 & D_1 & D_2 \end{array}\right],$$

we obtain the required expression. $\qquad\qquad\square$

Lemma 4.4. *The transfer function matrices* Φ, Γ *and* K_Φ *are in the incidence algebra* $I(\mathcal{P})$.

Proof. Let us define block $s \times 1$ transfer functions as follows:

$$\Phi(j) = \left[\begin{array}{c|c} A_\Phi(j) & B_\Phi(j) \\ \hline E_{\uparrow\uparrow j} & I \end{array}\right] \tag{4.38}$$

$$K_\Phi(j) = \left[\begin{array}{c|c} A_\Phi(j) & B_\Phi(j) \\ \hline -\hat{K}(\uparrow j, \uparrow j)E_{\uparrow\uparrow j} & -\hat{K}(\uparrow j, \uparrow j)E_j \end{array}\right] \tag{4.39}$$

Note that $K_\Phi(j) = \hat{K}(\uparrow j, \uparrow j)\Phi(j)$. Also, note that if i is such that $j \not\le i$ then the ith entry of $\Phi(j)$ is zero since the corresponding row of $E_{\uparrow\uparrow j}$ is zero. By similar reasoning, $K_\Phi(j)$ also has this property. Thus, when we construct the matrices

$$\Phi = \begin{bmatrix} \Phi(1) & \ldots & \Phi(s) \end{bmatrix}$$
$$K_\Phi = \begin{bmatrix} K_\Phi(1) & \ldots & K_\Phi(s) \end{bmatrix}$$

by column concatenation, we see that both $\Phi \in I(\mathcal{P})$ and $K_\Phi \in I_K(\mathcal{P})$. Since $\Gamma = \Phi^{-1}$, we have $\Gamma \in I(\mathcal{P})$. $\qquad\qquad\square$

Lemma 4.5. *The following matrix identities are true:*

$$\Pi_2^T A \Pi_1 = B_\Phi$$

$$(\Pi_1^T + C_\Phi \Pi_2^T) A \Pi_1 = A + B C_Q \Pi_1$$

$$(\Pi_1^T + C_\Phi \Pi_2^T) A (\Pi_2 - \Pi_1 C_\Phi) = B C_Q (\Pi_2 - \Pi_1 C_\Phi)$$

$$\Pi_2^T A (\Pi_2 - \Pi_1 C_\Phi) = A_\Phi - B_\Phi C_\Phi.$$

(4.40)

Proof. Note that the first relation follows directly from the definition, as stated in the third equality in (4.15). Next, we point out that $\Pi_1^T + C_\Phi \Pi_2^T = R$. Hence,

$$(\Pi_1^T + C_\Phi \Pi_2^T) A = RA$$

$$= \left[\begin{array}{ccc} A_c^1 E_{\uparrow 1} & \cdots & A_c^s E_{\uparrow s} \end{array} \right] \Pi_1$$

$$= AR + BC_Q.$$

Since $AR\Pi_1 = A$, we have the second relation.

For the third relation we note again that

$$(\Pi_1^T + C_\Phi \Pi_2^T) A (\Pi_2 - \Pi_1 C_\Phi) = RA(\Pi_2 - \Pi_1 C_\Phi)$$

$$= (AR + BC_Q)(\Pi_2 - \Pi_1 C_\Phi)$$

$$= AR\Pi_2 - (AR\Pi_1)R\Pi_2 + BC_Q(\Pi_2 - \Pi_1 C_\Phi) \text{ (since } C_\Phi = R\Pi_1)$$

$$= AR\Pi_2 - AR\Pi_2 + BC_Q(\Pi_2 - \Pi_1 C_\Phi) \text{ (since } AR\Pi_1 = A)$$

$$= BC_Q(\Pi_2 - \Pi_1 C_\Phi).$$

For the fourth relation, note that

$$\Pi_2^T A (\Pi_2 - \Pi_1 C_\Phi) = \Pi_2^T A \Pi_2 - \Pi_2^T A \Pi_1 C_\Phi$$

$$= A_\Phi - B_\Phi C_\Phi.$$

127

Lemma 4.6. *The matrix* **A** *is stable.*

Proof. Recall that $\mathbf{A} = \mathrm{diag}(A(\uparrow j, \uparrow j) - B(\uparrow j, \uparrow j)K(\uparrow j, \uparrow j))$. Since $A(\uparrow j, \uparrow j)$ and $B(\uparrow j, \uparrow j)$ are lower triangular with A_{kk}, B_{kk}, $k \in \uparrow j$ along the diagonals respectively, we see that the pair $(A(\uparrow j, \uparrow j), B(\uparrow j, \uparrow j))$ is stabilizable by Assumption 1 (simply picking a diagonal K which stabilizes the diagonal terms would suffice to stabilize $(A(\uparrow j, \uparrow j), B(\uparrow j, \uparrow j))$). Hence, there exists a stabilizing solution to $\mathrm{Ric}(A(\uparrow j, \uparrow j), B(\uparrow j, \uparrow j), C(\uparrow j), D(\uparrow j))$ and the corresponding controller $K(\uparrow j, \uparrow j)$ is stabilizing. Thus $A(\uparrow j, \uparrow j) - B(\uparrow j, \uparrow j)K(\uparrow j, \uparrow j))$ is stable, and thus so is **A**. □

Given transfer functions M and K, their feedback interconnection is usually described through a linear fractional transformation of the form $f(M, K) = M_{11} + M_{12}K(I - M_{22}K)^{-1}M_{21}$. State space formulae for this interconnection are standard [66, pp. 179] and will be useful for evaluating several quantities in what follows.

Lemma 4.7. *Given transfer function matrices M and K with realizations*

$$
M = \left[
\begin{array}{c|cc}
A & B_1 & B_2 \\
\hline
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{array}
\right], \quad
K = \left[
\begin{array}{c|c}
A_K & B_K \\
\hline
C_K & D_K
\end{array}
\right],
$$

the Linear Fractional Transformation (LFT) $f(M, K) = M_{11} + M_{12}K(I - M_{22}K)^{-1}M_{21}$ is given by the state-space formula

$$
f(M, K) = \left[
\begin{array}{cc|c}
A + B_2\hat{R}^{-1}D_KC_2 & B_2\hat{R}^{-1}C_K & B_1 + B_2\hat{R}^{-1}D_KD_{21} \\
B_KR^{-1}C_2 & A_K + B_KR^{-1}D_{22}C_K & B_KR^{-1}D_{21} \\
\hline
C_1 + D_{12}D_KR^{-1}C_2 & D_{12}\hat{R}^{-1}C_K & D_{11} + D_{12}D_KR^{-1}D_{21}
\end{array}
\right], \quad (4.41)
$$

where $\hat{R} = I - D_KD_{22}$ and $R = I - D_{22}D_K$.

Proof. The proof is standard, see for example [66, pp. 179] and the references therein. □

Proof of Theorem 4.3. Consider again the optimal control problem (4.5):

$$\underset{K}{\text{minimize}} \quad \|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\|^2$$

$$\text{subject to} \quad K \in \mathcal{I}(\mathcal{P}) \tag{4.42}$$

$$K \text{ stabilizing.}$$

Let v_1^* be the optimal value of (4.42). Consider, on the other hand the optimization problem:

$$\underset{Q}{\text{minimize}} \quad \|P_{11} + P_{12}Q\|^2$$

$$\text{subject to} \quad Q \in \mathcal{I}(\mathcal{P}). \tag{4.43}$$

Let v_2^* be the optimal value of (4.43). Recall that the optimal solution Q^* of (4.43) was obtained in Lemma 4.3 via (4.37). We note that if K^* is an optimal solution to (4.42) then the corresponding $\bar{Q} := K^*(I - P_{22}K^*)^{-1}P_{21}$ is feasible for (4.43). Hence $v_2^* \leq v_1^*$. We will show that the controller in (4.16) is optimal by showing that $\bar{Q} = Q^*$ (so that $v_1^* = v_2^*$). We will also show that $K^* \in \mathcal{I}(\mathcal{P})$ and is internally stabilizing. Since it achieves the lower bound v_2^* and is internally stabilizing, it *must* be optimal.

Given K^* as per (4.16), one can evaluate $\bar{Q} := K^*(I - P_{22}K^*)^{-1}P_{21}$ using (4.41) to obtain:

$$\bar{Q} = \left[\begin{array}{cc|c} A + BC_Q\Pi_1 & BC_Q(\Pi_2 - \Pi_1 C_\Phi) & I \\ B_\Phi & A_\Phi - B_\Phi C_\Phi & 0 \\ \hline C_Q\Pi_1 & C_Q(\Pi_2 - \Pi_1 C_\Phi) & 0 \end{array} \right].$$

Recall that Q^* (4.37) is the optimal solution to (4.8) (which constitutes a lower bound to the problem we are trying to solve). We are trying to show that it is achievable by explicitly producing K^* such that $\bar{Q} := K^*(I - P_{22}K^*)^{-1}P_{21}$ and $\bar{Q} = Q^*$, thereby proving the optimality of K^*.

While Q^* in (4.37) and $\bar{Q}$ obtained above appear different at first glance, their state-

space realizations are actually equivalent modulo a coordinate transformation. Recall that Π_2 is a matrix (composed of standard unit vectors) that spans the orthogonal complement of the column span of Π_1. As a result the matrix $\begin{bmatrix} \Pi_1 & \Pi_2 \end{bmatrix}$ is a permutation matrix. Define the matrices

$$\Lambda := \begin{bmatrix} \Pi_1 & \Pi_2 \end{bmatrix} \begin{bmatrix} I & -C_\Phi \\ 0 & I \end{bmatrix}, \quad \Lambda^{-1} = \begin{bmatrix} I & C_\Phi \\ 0 & I \end{bmatrix} \begin{bmatrix} \Pi_1^T \\ \Pi_2^T \end{bmatrix}.$$

Note that Λ is a square, invertible matrix. Changing state coordinates on Q^* using Λ via:

$$\mathbf{A} \mapsto \Lambda^{-1} \mathbf{A} \Lambda$$

$$\Pi_1 \mapsto \Lambda^{-1} \Pi_1$$

$$C_Q \mapsto C_Q \Lambda$$

along with the relations (4.40) from Lemma 4.5, we see that the transformed realization of Q^* is equal to the realization of $\bar{Q}$, and hence $Q^* = \bar{Q}$.

Using (4.4) for the open loop, (4.16) for the controller and the LFT formula (4.41) to compute the closed loop map, one obtains that the closed-loop state transition matrix is given by

$$\begin{bmatrix} A + BC_Q\Pi_1 & BC_Q(\Pi_2 - \Pi_1 C_\Phi) \\ B_\Phi & A_\Phi - B_\Phi C_\Phi \end{bmatrix} \cong \mathbf{A}.$$

By Lemma 4.6, the closed loop is internally stable.

By the column concatenation formula and (4.38) we have

$$\begin{bmatrix} \Phi \\ K_\Phi \end{bmatrix} = \left[\begin{array}{c|c} A_\Phi & B_\Phi \\ \hline C_\Phi & I \\ C_Q\Pi_2 & C_Q\Pi_1 \end{array} \right]$$

Using the state-space coprime factorization formula [65, pp. 52] it is straightforward to verify that for the expression for K^* in (4.16), $K^* = K_\Phi \Phi^{-1}$. Since by Lemma 4.4 both $\Phi \in \mathcal{I}(\mathcal{P})$ and $K_\Phi \in \mathcal{I}(\mathcal{P})$, we have $K^* \in \mathcal{I}(\mathcal{P})$. □

Proof of Theorem 4.4. In the preceding proof, we established that $K_\Phi(j) = -\hat{K}(\uparrow j, \uparrow j)\Phi(j)$ and that $K^* = K_\Phi \Phi^{-1}$. This directly gives the first expression in the statement of the theorem. The second expression is a simple manipulation of the first expression. □

4.7 Conclusions

In this chapter we provided a state-space solution to the problem of computing an $\mathcal{H}_2$-optimal decentralized controller for a poset-causal system. We introduced a new decomposition technique that enables one to separate the decentralized problem into a set of centralized problems. We gave explicit state-space formulae for the optimal controller and provided degree bounds on the controller. We illustrated our technique with a numerical example. Our approach also enabled us to provide insight into the structure of the optimal controller. We introduced a pair of transfer functions (Φ, Γ) and showed that they were intimately related to the prediction structure. While some architectural aspects of the controller were hinted at, the emphasis in this chapter was much more on the computational aspects of the problem. In the next chapter, we will see a detailed treatment of the architectural issues.

Chapter 5

A General Controller Architecture

5.1 Introduction

While understanding computational aspects of controller synthesis is important, it is equally important to understand structural aspects of controller design. Understanding structural aspects provides important insight into design principles, physical implementation, and also a theoretical perspective that enables the way forward to tackle more complex problems. Indeed, structural properties for optimal poset-causal controllers (that we will study in this chapter) may be used as valuable design principles when dealing with more complicated, non-poset-like communication architectures. With this motivation in mind, we next explore architectural issues for controller design of poset-causal systems.

While the previous chapter was devoted to a state-space solution to the $\mathcal{H}_2$-optimal control problem, in this chapter we are concerned with answering the following question: "What is a reasonable architecture of controllers for poset-causal systems? What should be the role of controller states, and what computations should be involved in the controller?" This chapter focuses on answering this *architectural* question. The main aspects discussed in this chapter are the following:

- A controller architecture that involves natural concepts from order theory and control

theory as building blocks,

- A natural coordinate transformation of the state variables yeilds a novel *separation principle*,

- A proof that the optimal $\mathcal{H}_2$ controller (with state-feedback) studied in Chapter 4 has precisely the proposed controller structure,

The controller structure that we propose in this chapter is of the form $U = \zeta(\mathbf{G} \circ \mu(X))$. Here the matrix X has the interpretation of being a collection of local predictions of the state at different subsystems, and U the local predictions of the inputs. As we will see later, the operators μ and ζ are generalized notions of differentiation and integration on the poset so that $\mu(X)$ may be interpreted as the differential improvement or "correction" in the prediction of the local state. The quantity $\mathbf{G} \circ \mu(X)$ may therefore be interpreted as a local "differential contribution" to the overall control signal. The overall control law then aggregates all these local contributions by "integration" along the poset using ζ.

In Chapter 4, some structural aspects of the optimal controller were hinted at. Specifically, we introduced a pair of transfer functions Φ, Γ, in terms of which the optimal controller was interpreted. In this chapter, we relate these transfer functions to ζ and μ, and make their interpretation explicit.

An outline for this chapter is as follows: In Section 5.2 we introduce the necessary preliminaries for the ensuing discussion. In Section 5.3 we introduce the basic building blocks involved in the controller architecture. In Section 5.4 we propose the architecture, establish the separability principle and explain its optimality property with respect to the $\mathcal{H}_2$ norm.

5.2 Preliminaries

In this chapter we use the standard definitions and notations related to posets introduced in Chapter 2. As usual, a poset is denoted by $\mathcal{P} = (P, \leq)$ and its incidence algebra by $\mathcal{I}(\mathcal{P})$. Typical examples of posets to keep in mind are shown in Fig. 5-1.
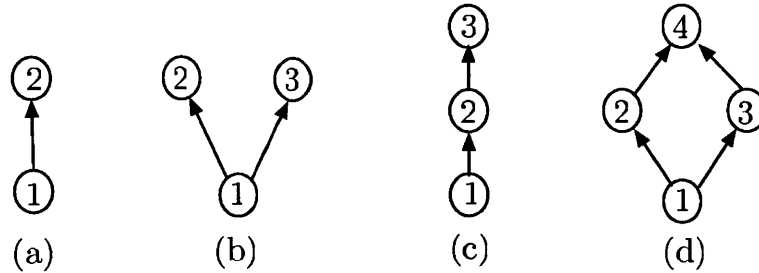
Figure 5-1: Hasse diagrams of some posets.

We again consider the following discrete-time state-space system:

$$x[t + 1] = Ax[t] + w[t] + Bu[t]$$

$$z[t] = Cx[t] + Du[t] \qquad\qquad (5.1)$$

$$y[t] = x[t].$$

In this chapter we present the discrete time case only, however, we wish to emphasize that analogous results hold in continuous time in a straightforward manner. In this chapter we consider what we will call *poset-causal systems*.

As in Chapter 4, we think of this system as being divided into s sub-systems, with sub-system i having some states $x_i[t] \in \mathbb{R}^{n_i}$, and we let $N = \sum_{i \in P} n_i$ be the total degree of the system. The control inputs at the subsystems are $u_i[t] \in \mathbb{R}^{m_i}$ for $i \in \{1, \ldots, s\}$. In this chapter, to simply facilitate convenient notation, we will often assume $n_i = 1$, and $m_i = 1$. We emphasize that this is only done to simplify the presentation, the results hold for arbitrary block sizes n_i and m_i. The external output is $z[t] \in \mathbb{R}^p$. The signal $w[t]$ is

135

a disturbance signal. The states and inputs are partitioned in the natural way such that the sub-systems correspond to elements of the poset $\mathcal{P}$ with $x[t] = [x_1[t] \,|\, x_2[t] \,|\, \ldots \,|\, x_s[t] \,]^T$, and $u[t] = [u_1[t] \,|\, u_2[t] \,|\, \ldots \,|\, u_s[t] \,]^T$. This naturally partitions the matrices A, B, C, D into appropriate blocks so that $A = \left[A_{ij}\right]_{i,j\in P}$, $B = \left[B_{ij}\right]_{i,j\in P}$, $C = \left[C_j\right]_{j\in P}$ (partitioned into columns), $D = \left[D_j\right]_{j\in P}$. (We will throughout deal with matrices at this block-matrix level, so that A_{ij} will unambiguously mean the (i, j) block of the matrix A.) Using these block partitions, one can define the incidence algebra at the block matrix level in the natural way. We denote by $\mathcal{I}_A(\mathcal{P}), \mathcal{I}_B(\mathcal{P})$ the block incidence algebras corresponding to the block partitions of A and B. Often, matrices will have different (but compatible) dimensions and the block structure will be clear from the context. In these cases, we will abuse notation and will drop the subscript and simply write $\mathcal{I}(\mathcal{P})$.

The system (5.1) may be viewed as a map from the inputs w, u to outputs z, x via

$$z = P_{11}w + P_{12}u$$

$$x = P_{21}w + P_{22}u$$

where

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & I & B \\ \hline C & 0 & D \\ I & 0 & 0 \end{array}\right]. \tag{5.2}$$

(We refer the reader to [66] as a reminder of standard LFT notation used above). In this chapter we will assume that $A \in \mathcal{I}(\mathcal{P})$ and $B \in \mathcal{I}(\mathcal{P})$. Indeed, this assumption ensures that the plant $P_{22}(z) = (zI - A)^{-1}B \in \mathcal{I}(\mathcal{P})$.

We remind the reader that we call such systems *poset-causal* due to the following analogous property among the sub-systems. If an input is applied to sub-system i via u_i at some time t, the effect of the input is seen by the states x_j for all sub-systems $j \in\uparrow i$ (at or after time t). Thus $\uparrow i$ may be seen as the cone of influence of input i. We refer to this causality-

136

like property as *poset-causality*. This notion of causality enforces (in addition to causality with respect to time), a causality relation between the subsystems with respect to a poset.

Infomation Constraints in Controller

In this chapter, we will be interested in the design of poset-causal controllers of the form:

$$K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right]. \tag{5.3}$$

We will require that the controller also be poset-causal, i.e. that $K \in \mathcal{I}(\mathcal{P})$. In later sections we will present a general architecture for controllers with this structure with some elegant properties.

The decentralization constraint of interest in this chapter is one where the controller mirrors the structure of the plant, and is therefore also in the block incidence algebra $\mathcal{I}_K(\mathcal{P})$ (we will henceforth drop the subscripts and simply refer to the incidence algebra $\mathcal{I}(\mathcal{P})$). Equivalently, the notion of poset-causality for a controller may be define as follows:

Definition 5.1. *Let $\mathcal{P}$ be a poset. A control law (5.3) is said to be* poset-causal *if u_i uses as input only on x_j for $j \in \downarrow i$ (i.e. downstream information).*

Note that requiring $K \in \mathcal{I}(\mathcal{P})$ is equivalent to enforcing poset-causality constraints on the controller in the sense of Definition 5.1.

Remark Note that the control law for u_i may be possibly *dynamic*, so that the states x_j for $j \in \downarrow i$ may be used to compute predictions of other unknowns states (which we call local states at i). The control input u_i may then depend on the x_j and the predictions that were computed (based on the known states). In fact, the controller may also depend on the local state predictions obtained from subsystem j for $j \in \downarrow i$ (since the information available at j is a subset of the information available at i). We call the collection of all states and state predictions from $\downarrow i$ to be the *downstream* information at i. We will later propose a

137

controller architecture that indeed uses all the downstream information in a very structured way.

5.2.1 Notation

Since we are dealing with poset-causal systems (with respect to the poset $\mathcal{P} = (P, \preceq)$), most vectors and matrices will be naturally indexed with respect to the set P (at the block level). Recall that every poset $\mathcal{P}$ has a linear extension (i.e. a total order on P which is consistent with the partial order $\preceq$). For convenience, we fix such a linear extension of $\mathcal{P}$, and all indexing of our matrices throughout the chapter will be consistent with this linear extension (so that elements of the incidence algebra are lower triangular).

Given a matrix M, M_{ij} will as usual denote the $(i, j)^{th}$ entry. The ith column will be denoted by M^i and the ith row will be denoted by $M^{(i)}$. Given a block $|P| \times |P|$ matrix, we will sometimes need to extract rows and columns corresponding to certain subsets of P. If $S, T \subseteq P$ then $M(S)$ is the sub-matrix containing the columns whose indices belong to S, and $M(S, T)$ is the sub-matrix containing rows and columns indexed by S and T respectively. We will also need to deal with the inverse operation: we will be given an $|S| \times |S|$ matrix K (indexed by some subset $S \subseteq P$) and we will wish to embed it into a $|P| \times |P|$ matrix by zero-padding the locations corresponding to row and column locations in $P \setminus S$. We will denote this embedded matrix by $\hat{K}$. Finally, given a vector q of length $|P|$, the vector $q_{\uparrow i}$ is the sub-vector with components indexed in the set $\uparrow i$ ($q_{\uparrow\uparrow i}$, $q_{\downarrow i}$ etc. are defined similarly).

5.3 Ingredients of the Architecture

The controller architecture that we propose is composed of three main ingredients:

- The notion of *local variables*,

- A notion of a local product, denoted by "$\circ$",

138

- A pair of operators ζ, μ that operate on the local variables in a way that is consistent with the order-theoretic structure of the poset. These operators, called the *zeta function* and the *Möbius* function respectively, are classical objects and play a central role in much of order theory, number theory and combinatorics [39].

5.3.1 Local Variables and Local Products

We begin with the notion of global variables.

Definition 5.2. *We call a function* $z : P \to \mathbb{R}$ *a* global *variable*.

Remark Typical global variables that we encounter will be the overall state x and the input u. When $n_i = 1$ and $m_i = 1$, the state x and input u may be viewed as global variables as defined above. For arbitrary block sizes, the definition of a global variable (and some of the other definitions that follow) must be suitably altered in an obvious way.

Note that the overall system is composed of $s = |P|$ subsystems. One can imagine each subsystem maintaining a copy (more precisely a prediction) of the global variable. We call these predictions local variables.

Definition 5.3. *Let z be a global variable. We call a map* $Z^i : \uparrow i \to \mathbb{R}$ *the local variable at i associated to z.*

Remark We think of Z^i as a vector in $\mathbb{R}^{|P|}$. Note that only the components in $\uparrow i \subseteq P$ matter, and we set the remaining elements to be zero, these elements are not relevant. We collect the local variables (viewed as columns) $Z^i, i \in P$ formally into a single matrix variable (called the *matrix representation*) $Z := \begin{bmatrix} Z^1 \dots Z^s \end{bmatrix}$. We will use the indexing $Z^i = [Z^i_j]_{j \in P}$, so that Z^i_j denotes the local prediction of z_j at subsystem i. We will use this indexing exclusively for local variables so that unambiguously whenever this notation occurs, it refers to a local quantity.

139

Because of the way local variables are constructed, the matrix Z is in the incidence algebra. Only the indices of Z_j^i where $i \leq j$ matter, the remaining elements of Z are formally set to zero and are never to be looked at. Hence, we can identify the space of local variables with the incidence algebra so that we can denote the space of local variables also by $I(\mathcal{P})$. By abuse of terminology, we will refer to Z itself as the local variable. The two local variables we will encounter are X (local state variables) and U (local input variables).

Example 5.1. *We illustrate the concepts of global variables and local variables with an example. Consider the poset shown in Fig. 5-1(d). Then we can define the local variable x and a corresponding (matrix representation of) global variable X as follows:*

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ x_2(1) & x_2 & 0 & 0 \\ x_3(1) & 0 & x_3 & 0 \\ x_4(1) & x_4(2) & x_4(3) & x_4 \end{bmatrix}.$$

We define the following important product:

Definition 5.4. *Let $\mathbf{G} = \{G(1), \ldots, G(s)\}$ be a collection of maps $G(i) : \uparrow i \times \uparrow i \to \mathbb{R}$. To make the multiplication of $G(i)$ with X^i compatible, we view $G(i)$ as matrices of size $|P| \times |P|$ with support on the rows and columns in $\uparrow i$, and zeros elsewhere. Let X be a local variable. We define the local product $\mathbf{G} \circ X$ columnwise via*

$$(\mathbf{G} \circ X)^i \triangleq G(i) X^i \quad \text{for all } i \in P. \tag{5.4}$$

Remark We call the matrices $G(i)$ the *local gains*. Local products give rise to decoupled local relationships in the following natural way. Let X, Y be local variables. If they are related via $Y = \mathbf{G} \circ X$ then the relationship between X and Y is said to be *decoupled*. This is because, by definition,

$$Y^k = G(k) X^k \quad \text{for all } k \in P.$$

140

Thus the maps relating the pairs (X^k, Y^k) are *decoupled* across all $k \in P$ (i.e. Y^k depends only on X^k and not on X^j for any other $j \neq k$). Note that if $Y = G \circ X$, then Y is again a local variable, and in its corresponding matrix representation, only the entries Y^i_j for $i \leq j$ are relevant, the remaining entries are zero.

Example 5.2. *Continuing with Example 5.1, let us define the local gains by*

$$\mathbf{G} = \{G(1), G(2), G(3), G(4)\},$$

where,

$$G(1) = \begin{bmatrix} G_{11}(1) & G_{12}(1) & G_{13}(1) & G_{14}(1) \\ G_{21}(1) & G_{22}(1) & G_{23}(1) & G_{24}(1) \\ G_{31}(1) & G_{32}(1) & G_{33}(1) & G_{34}(1) \\ G_{41}(1) & G_{42}(1) & G_{43}(1) & G_{44}(1) \end{bmatrix} \quad G(2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & G_{22}(2) & 0 & G_{24}(2) \\ 0 & 0 & 0 & 0 \\ 0 & G_{42}(2) & 0 & G_{44}(2) \end{bmatrix}$$

$$G(3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & G_{33}(3) & G_{34}(3) \\ 0 & 0 & G_{43}(3) & G_{44}(3) \end{bmatrix} \quad G(4) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{44}(4) \end{bmatrix}.$$

Then

$$\mathbf{G} \circ X = \begin{bmatrix} G(1) \begin{bmatrix} X_{11} \\ X_{21} \\ X_{31} \\ X_{41} \end{bmatrix} & G(2) \begin{bmatrix} 0 \\ X_{22} \\ 0 \\ X_{42} \end{bmatrix} & G(3) \begin{bmatrix} 0 \\ 0 \\ X_{33} \\ X_{43} \end{bmatrix} & G(4) \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_{44} \end{bmatrix} \end{bmatrix}.$$

Definition 5.5. *Let $M \in \mathbb{R}^{s \times s}$ be a matrix and $X \in I(\mathcal{P})$ a local variable. We define $Y = MX$ to be the usual matrix multiplication of matrices M and X but restricted to entries*

141

$i \leq j$, *i.e.*

$$Y^i_j = \begin{cases} \sum_{k \in P} M_{jk} X^k_i & for \ i \leq j \\ 0 & otherwise. \end{cases}$$

Throughout this chapter products of matrices and local variables of the form MX are *always* viewed as objects in $\mathcal{I}(\mathcal{P})$ so that the output is a local variable with zeroes at indices where $i \not\leq j$. The product XM as a local variable in $\mathcal{I}(\mathcal{P})$ is similarly defined.

5.3.2 The Möbius and Zeta operators

We first remind the reader of two important order-theoretic notions, namely *zeta functions and Möbius functions*. These are both well-known concepts in order theory that generalize discrete integration and finite differences (i.e. discrete differentiation) to posets.

Definition 5.6. *Let* $\mathcal{P} = (P, \preceq)$. *The* zeta matrix ζ *is defined to be the matrix* $\zeta : P \times P \to \mathbb{R}$ *such that* $\zeta(i, j) = 1$ *whenever* $j \preceq i$ *and zeroes elsewhere. The* Möbius operator *is* $\mu := \zeta^{-1}$.

These matrices may be viewed as operators acting on functions on the poset $f : P \to \mathbb{R}$ (the functions being expressed as row vectors). The matrices ζ, μ, which are members of the incidence algebra, act as linear transformations on f in the following way:

$$\zeta : \mathbb{R}^{|P|} \to \mathbb{R}^{|P|} \qquad\qquad \mu : \mathbb{R}^{|P|} \to \mathbb{R}^{|P|}$$
$$f \mapsto f\zeta^T \qquad\qquad f \mapsto f\mu^T.$$

Note that $\zeta(f)$ is also a function on the poset given by

$$(\zeta(f))_i = \sum_{j \leq i} f_j. \tag{5.5}$$

This may be naturally interpreted as a discrete integral of the function f over the poset.

142

The role of the Möbius function is the opposite: it is a generalized finite difference (i.e. a discrete form of differentiation over the poset). If $f : P \to \mathbb{R}$ is a local variable then the function $\mu(f) : P \to \mathbb{R}$ may be computed recursively by:

$$(\mu(f))_i = \begin{cases} f_i \text{ for } i \text{ a minimal element,} \\ f_i - \sum_{j<i} (\mu(f))_j \text{ otherwise.} \end{cases} \quad (5.6)$$

Example 5.3. *Consider the poset in Figure 5-1(c). The zeta function and the Möbius function are given by:*

$$\zeta = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad \mu = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

If $f = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$, *then*

$$\zeta(f) = \begin{bmatrix} f_1 & f_1 + f_2 & f_1 + f_2 + f_3 \end{bmatrix}$$

$$\mu(f) = \begin{bmatrix} f_1 & f_2 - f_1 & f_3 - f_2 \end{bmatrix}.$$
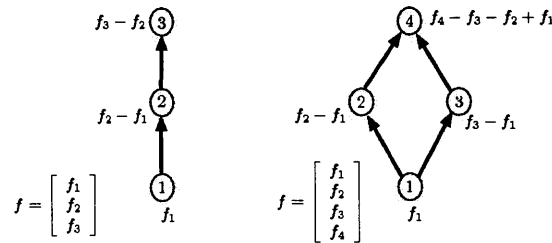

Figure 5-2: Two posets with their Möbius functions. The functions f are functions on the posets, and the values of $\mu(f)$ at element i are indicated next the the relevant elements.

We now define modified versions of the zeta and Möbius operators that extend the

actions of μ and ζ from global variables x to local variables X. Let ζ and μ be matrices as defined in Definition 5.6.

Definition 5.7. *Let* $X = \begin{bmatrix} X^1, \ldots, X^s \end{bmatrix}$ *be a local variable. Define the operators*

$$\mu : I(\mathcal{P}) \to I(\mathcal{P}) \qquad and \qquad \zeta : I(\mathcal{P}) \to I(\mathcal{P})$$

acting via

$$\zeta(X) = X\zeta^T \qquad\qquad \mu(X) = X\mu^T, \qquad\qquad (5.7)$$

where the multiplication is to be interpreted restricted to $I(\mathcal{P})$ (i.e. in the sense of Definition 5.5).

Lemma 5.1. *The operators ζ and μ may be written more explicitly as*

$$\zeta(X)^i_j \triangleq \sum_{k \leq i} X^k_j \qquad\qquad \mu(X)^i_j \triangleq X^i_j - \sum_{k < i} \mu(X)^k_j. \qquad (5.8)$$

Proof. The proofs follow in a straightforward fashion from (5.5) and (5.6). ☐

Note that if $Y = \mu(X)$ then $Y = [Y^i_j]_{i \leq j}$ is also a local variable in $I(\mathcal{P})$. As before, we only define the entries Y^i_j when $i \leq j$ and formally set the remaining entries to zero so that Y has a matrix representation. By abuse of notation, let $\zeta(X) := \begin{bmatrix} \zeta(X)^1, \ldots, \zeta(X)^s \end{bmatrix}$ and $\mu(X) := \begin{bmatrix} \mu(X)^1, \ldots, \mu(X)^s \end{bmatrix}$. The entries corresponding to subsystem j are denoted by $\zeta(X)^j_i$ and $\mu(X)^j_i$. Note that ζ has the natural interpretation of *aggregating or integrating* the local variables X^k for $k \in P$, whereas μ performs the inverse operation of differentiation of the local variables.

Example 5.4. *We illustrate the action of μ acting on a local variable. Consider the local*

144

variable X from Example 5.1. It is easy to verify that

$$\mu(X) = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ x_2(1) & x_2 - x_2(1) & 0 & 0 \\ x_3(1) & 0 & x_3 - x_3(1) & 0 \\ x_4(1) & x_4(2) - x_4(1) & x_4(3) - x_4(1) & x_4 - x_4(3) - x_4(2) + x_4(1) \end{bmatrix}.$$

Lemma 5.2. *The operators (μ, ζ) are inverses of each other so that for all local variables $X \in \mathcal{I}(\mathcal{P})$,*

$$\zeta(\mu(X)) = \mu(\zeta(X)) = X.$$

Proof. We show that for $i \leq j$, $\mu(\zeta(X))^i_j = X^i_j$ (the other proof is similar). We prove this by induction on i. Note that for minimal i this is indeed true and assume (as the induction hypothesis) it is true for all $k < i$. From (5.8),

$$\mu(\zeta(X))^i_j = \zeta(X)^i_j - \sum_{k < i} \mu(\zeta(X))^k_j$$

$$= \zeta(X)^i_j - \sum_{k < i} X^k_j \quad \text{(by induction hypothesis)}$$

$$= X^i_j.$$

$\square$

Since ζ and μ may be interpreted as integration and differentiation operators, the above statement may be viewed as a "poset" version of the fundamental theorem of calculus. We mention one more important property of these operators.

Lemma 5.3. *Let $A, X \in \mathcal{I}(\mathcal{P})$. Then $\mu(AX) = A\mu(X)$, and $\zeta(AX) = A\zeta(X)$.*

Proof. Let Y be the local variable defined as $Y = AX$. From Definition 5.7, $\mu(Y) = Y\mu^T =$

145

$(AX)\mu^T$. Since $\mu(Y)$ is a local variables $\mu(Y)^i_j = 0$ for $j \not\leq i$. Hence, for $j \leq i$ we have

$$
\begin{aligned}
\mu(Y)^i_j &= \sum_{k \in P} Y^i_k \mu^j_k \\
&= \sum_{k \leq i, k \leq j} Y^i_k \mu^j_k \quad \text{(since } Y, \mu \in \mathcal{I}(\mathcal{P})) \\
&= \sum_{k \leq j} Y^i_k \mu^j_k \quad \text{(since } j \leq i) \\
&= \sum_{k \leq j} \sum_{k \leq p \leq i} A^i_p X^p_k \mu^j_k \quad \text{(since } A, X \in \mathcal{I}(\mathcal{P})) \\
&= \sum_{p: p \leq i} A^i_p \sum_{k: k \leq p, k \leq j} X^p_k \mu^j_k \quad \text{(switching order of summation)} \\
&= \sum_{p \leq i} A^i_p \mu(X)^p_j \\
&= [A\mu(X)]^i_j.
\end{aligned}
$$

The proof for ζ is identical. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5.4 Proposed Architecture

5.4.1 Local States and Local Inputs

Having defined local and global variables, we now specialize these concepts to our state-space system (5.1). We will denote x_j to be the true state at subsystem j. We denote $x_j(i)$ to be a prediction of state x_j at subsystem i. Recall the information constraints at subsystem i:

- Information about $\uparrow\uparrow i$: This state information is unavailable, so a (possibly partial) prediction of x_j for $j \in \uparrow\uparrow i$ is formed. We denote this prediction by $x_j(i)$. Computing these predictions is the role of the controller states.

- Information about $\downarrow i$: Complete state information about x_j for $j \in \downarrow i$ is available, so that $x_j(i) = x_j$. Moreover, the predictions from downstream $x_k(j)$ for all $k \in P$ and

$j \le i$ are also available.

- State information about uncomparable elements is not available and irrelevant.

These ideas can be formalized by defining a local (state) variable X such that X^i_j captures the best available information about the state x_j at subsystem i.

Definition 5.8. *Let x, the state of the system (at some time t) be viewed as a global variable. Define the local variable $X \in \mathcal{I}(\mathcal{P})$ to be the local state with respect to x. It satisfies following properties:*

1. $X^i_i = x_i$, the true value of the state,

2. If $i < j$, then X^i_j is a prediction of x_j.

The local state variable may be split as

$$X = X_c + X_p, \tag{5.9}$$

where X_c is a strictly lower triangular matrix with entries $X^i_{c_j} = x_j(i)$ if $i < j$, and $X_p = \mathrm{diag}(x)$ is a diagonal matrix. Note that X_c corresponds to the controller states whose role is to compute the state predictions, and X_p corresponds to the plant states.

Example 5.5. *Consider the poset shown in Fig. 5-1(d). The matrix X shown in Example 5.1 satisfies the conditions for a being a local state variable. The free variables in this example are $x_2(1), x_3(1), x_4(1), x_4(2), x_4(3)$. The plant states are x_1, x_2, x_3, x_4.*

In this way, the local state X will correspond to partial predictions of the state x. We now clarify the notion of a partial prediction with an example.

Example 5.6. *Consider the system shown in Fig. 5-3 with dynamics*

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}[t+1] =
\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}[t+1] +
\begin{bmatrix} B_{11} & 0 & 0 \\ 0 & B_{22} & 0 \\ B_{31} & B_{32} & B_{33} \end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}[t+1].
$$

$$\begin{array}{c}
\text{(3)} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\[2mm]
\text{(1)} \quad\quad \text{(2)} \\[2mm]
\begin{bmatrix} x_1 \\ 0 \\ x_3(1) \end{bmatrix} \quad \begin{bmatrix} 0 \\ x_2 \\ x_3(2) \end{bmatrix}
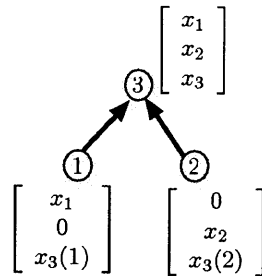\end{array}$$

Figure 5-3: Local state information at the different subsystems. The quantities $x_3(1)$ and $x_3(2)$ are partial state predictions.

Note that subsystem 1 has no information about the state of subsystem 2. Moreover, the state x_1 or input u_1 does not affect the dynamics of 2 (their respective dynamics are uncoupled). Hence the only sensible prediction of x_2 at subsystem 1 is $x_2(1) = 0$ (the situation for $u_2(1)$ is identical). However, both the states x_1, x_2 and inputs u_1, u_2 affect x_3 and u_3. Since x_2 and u_2 are unknown, the state $x_3(1)$ can at best be a partial prediction of x_3 (i.e. $x_3(1)$ is the prediction of the component of x_3 that is affected by subsystem 1). Similarly $x_3(2)$ is only a partial prediction of x_3. Indeed, one can show that $x_3(1) + x_3(2)$ is a more accurate prediction of the state x_3, and when suitably designed, their sum converges to the true state x_3.

State Innovations

We now give a natural interpretation of the operator $\mu(X)$ in terms of the state corrections with the help of an example.

Example 5.7. *Consider the poset shown in Fig. 5-4, and let us inspect the predictions of the state x_6 at the various subsystems. The prediction of x_6 at 1 is $x_6(1)$ and the prediction of x_6 at 2 is $x_6(2)$. The differential improvement in the prediction at subsystem 2 regarding the state x_6 is $x_6(2) - x_6(1)$ (we also refer to this as a "correction"). At subsystems 3 and*

148

$$x_6 - (x_6(2) + x_6(3) + x_6(4))$$
$$+ 2x_6(1)$$

$$x_6(4) - x_6(1)$$

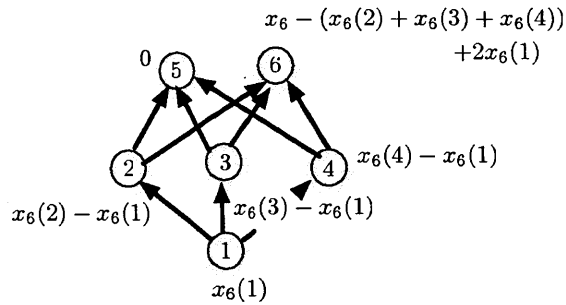$$x_6(2) - x_6(1) \qquad x_6(3) - x_6(1)$$

$$x_6(1)$$

Figure 5-4: Poset showing the differential improvement of the prediction of state x_6 at various subsystems.

4, *the formulae for the corrections are similar. The correction in x_6 at subsystem 5 is zero. These corrections are depicted in Fig. 5-4.*

Local Inputs

Recall that the subsystems need to keep track of not only the states, but also of the inputs at the different subsystems. For example, if i and j are distinct subsystems, a prediction of u_j is needed at i in order to produce a reasonable prediction of x_j. To this end we define the local variables $U = \left[U^1, \ldots, U^s \right]$ such that $U^i_j = u_j(i)$ is a prediction of the input u_j at subsystem i. Moreover, $U^i_i = u_i$ (the true input at subsystem i). These definitions of U^i_j are analogous to those of X^i_j in Definition 5.8. In a natural way μ acts on U to produce $\mu(U)$, which provides information about differential improvement in the prediction of the inputs.

5.4.2 Control Law

We now formally propose the following control law:

$$U = \zeta(\mathbf{G} \circ \mu(X)). \tag{5.10}$$

This control law may be interpreted as follows. The quantity $\mu(X)$ is a differentiation-like operation on the poset that computes local improvements or corrections in the prediction

149

of the global state. The quantity $\mathbf{G} \circ \mu(X)$ is a local correction based on the corrections. Finally, using ζ the local corrections are again aggregated along the poset via "integration". We make the following remarks about this control law.

Remarks   1. Note that a control law of the form (5.10) mimics a centralized controller. Suppose that $G(i) = K$ for all $i \in P$ (so that $\mathbf{G}$ is "constant" across $i \in P$). Then the control law reduces to $U = \zeta(K\mu(X)) = KX$. Suppose all the subsystems have access to the global state so that $X^i = x$. Then $U^i = Kx$, i.e. each subsystem implements a centralized control law.

2. We note that (5.10) specifies U which amounts to specifying the input $U^i_i = u_i$ at subsystem $i \in P$. It also specifies $U^i_j = u_j(i)$ for $i \prec j$ which is the prediction of the input u_j at an upstream subsystem i.

3. The equation (5.10) describes a *general controller architecture* parametrized by the gains $\mathbf{G}$. Specification of a controller is equivalent to specifying the local gain matrices $G(i)$ in $\mathbf{G}$.

4. Since $G(i)$ is supported (i.e. non-zero) only on rows and columns in $\uparrow i$, the controller is feasible with respect to the information constraints and u_i depends only on x_j and $x_k(j)$ for $j \prec i$. Such a matrix may be represented using the notation $G(i) = \hat{F}(i)$, where $F(i) \in \mathbb{R}^{|\uparrow i| \times |\uparrow i|}$.

5. The control law (5.10) may be alternatively written as $U^i = \sum_{k \leq i} G(k)\mu(X)^k$. The control law has the following interpretation. If i is a minimal element of the poset $\mathcal{P}$, then $\mu(X)^i = X^i$, the vector of partial predictions of the state at i. The local control law uses these partial predictions with the gain $G(i)$. If i is a non-minimal element it aggregates all the control laws from $\downarrow\downarrow i$ and adds a *correction term* based on the correction in the global state-prediction $\mu(X)^i$. This correction term is precisely $G(i)\mu(X)^i$.

Example 5.8. *Consider a poset causal system where the underlying poset is shown in Fig 5-1(d). The controller architecture described above is of the form* $U_i = \sum_{k \prec i} G(k)\mu(X)^k$ *(where U^i is a vector containing the predictions of the global input at subsystem i). Noting that $U_i^i = u_i$, we write out the control law explicitly to obtain:*

$$
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = G(1) \begin{bmatrix} x_1 \\ x_2(1) \\ x_3(1) \\ x_4(1) \end{bmatrix} + G(2) \begin{bmatrix} 0 \\ x_2 - x_2(1) \\ 0 \\ x_4(2) - x_4(1) \end{bmatrix} + G(3) \begin{bmatrix} 0 \\ 0 \\ x_3 - x_3(1) \\ x_4(3) - x_4(1) \end{bmatrix} +
$$

$$
G(4) \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 - x_4(2) - x_4(3) + x_4(1) \end{bmatrix}.
$$

5.4.3 State Prediction

Recall that at subsystem i the states x_j for $j \in \uparrow\uparrow i$ are unavailable and must be predicted. Typically, one would predict those states via an observer, however, those states are *unobservable*; only the state x_k for $k \in \downarrow i$ are observable, and in fact directly available. In this situation, rather than using an observer, one constructs a *predictor* to predict the unobservable variables; these correspond to the controller states X_c.

To compute the predictions X_c^i at subsystem i one uses the plant state X_p^i. In addition, one also uses the *downstream* states x_j for $j \prec i$. It is convenient to represent the downstream states as a matrix X_d defined as:

$$
[X_d]_{ij} = \begin{cases} x_i \text{ if } i \prec j \\ 0 \text{ otherwise.} \end{cases} \tag{5.11}
$$

Remark Note that X_d is *not* a local variable, it is to be viewed as a (upper triangular)

151

matrix. In fact, if we define Z via

$$[Z]_{ij} = \begin{cases} 1 \text{ if } i < j \\ 0 \text{ otherwise,} \end{cases} \tag{5.12}$$

then $X_d = X_p Z$.

Example 5.9. *For the poset in Fig. 5-1(d),*

$$X_d = \begin{bmatrix} 0 & x_1 & x_1 & x_1 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The downstream input matrix U_d is similarly defined via $U_d = U_p Z$.

Consider the (poset-causal) system

$$x[t + 1] = Ax[t] + Bu[t]. \tag{5.13}$$

The following dynamics on the local states imitate the above equation using only locally available information (i.e. states x_j and inputs u_j for $j \leq i$) to yield predictions.

$$X^i[t + 1] = A(X_c^i + X_p^i + X_d^i)[t] + B(U^i[t] + U_d^i)[t].$$

This may be compactly written as the following difference equation on $\mathcal{I}(\mathcal{P})$:

$$X[t + 1] = A(X_c + X_p + X_d)[t] + B(U + U_d)[t]. \tag{5.14}$$

This equation makes transparent the dependence of the local state $X = X_c + X_p$ on different terms. The term AX_c corresponds to the dynamical evolution of the predictions, The term

152

AX_p corresponds to the use of the plant state x_i as an input to predict the upstream states states $x_j(i)$. Similarly the term AX_d represents the use of *downstream* state information x_k for $k < i$ in predicting upstream states. The terms involving the true and predicted inputs (involving U_d and $U = U_c + U_p$) have a similar interpretation.

We define μ acting on the matrix X_d via $\mu(X_d) = X_d \mu^T$. We view $\mu(X_d)$ as a *local* variable by restricting attention to entries indexed by $\mu(U_d)^i_j$ for $i \leq j$.

Lemma 5.4. *Let X_d be as defined in (5.11). Then $\mu(X_d) = 0$.*

Proof. Note that since $\mu(X_d) = X_d \mu^T$ (restricted to $\mathcal{I}(\mathcal{P})$), the formula (5.8) holds. Using this formula and the fact that X_d takes value zero on $\mathcal{I}(\mathcal{P})$, the conclusion follows. □

Remark Since U_d is analogously defined, $\mu(U_d) = 0$.

5.4.4 Separation Principle

Applying μ to (5.14) we obtain the following *modified closed-loop dynamics* in the new variables $\mu(X)$:

$$\mu(X)[t + 1] = A\mu(X)[t] + B\mu(U)[t]. \tag{5.15}$$

Let us define $\mathbf{A} + \mathbf{BG} = \{A + BG(1), \ldots, A + BG(s)\}$. From (5.10), and the fact that $\mu(\zeta(Z)) = Z$ we will momentarily see that the modified closed-loop dynamics are:

$$\mu(X)[t + 1] = (\mathbf{A} + \mathbf{BG}) \circ \mu(X)[t]. \tag{5.16}$$

These dynamics describe how the differential improvements in the state evolve. If one picks U such that $\mu(U)$ stabilizes $\mu(X)$, the differential improvements all converge to zero and the state predictions become accurate asymptotically. We show that (5.10) achieves this with an appropriate choice of local gains.

Theorem 5.1. *Let $F(i)$ be chosen such that $A(\uparrow i, \uparrow i) + B(\uparrow i, \uparrow i)F(i)$ is stable for all $i \in P$. Then the control law (5.10) with $G(i) = \hat{F}(i)$ renders (5.15) stable.*

153

Proof. Since $U = \zeta(\mathbf{G} \circ \mu(X))$ it follows that

$$\mu(U) = \mu(\zeta(\mathbf{G} \circ \mu(X)))$$

$$= \mathbf{G} \circ \mu(X).$$

As a consequence, $\mu(U)^i = \hat{F}(i)\mu(X)^i$ for all $i \in P$. Hence the closed-loop dynamics (5.15) become:

$$\mu(X)^i[t+1] = \left(A + B\hat{F}(i)\right)\mu(X)^i[t].$$

Recalling that $\mu(X)$ is a local variable so that $\mu(X)^i$ (viewed as a vector) is non-zero only on $\uparrow i$ it is easy to see that these dynamics are stabilized exactly when $F(i)$ are picked such that $A(\uparrow i, \uparrow i) + B(\uparrow i, \uparrow i)F(i)$ are stable.

$\square$

The dynamics of the different subsystems $\mu(X)^i$ are *decoupled*, so that the gains $G(i)$ may be picked independent of each other. This may be viewed as a *separation principle*. Henceforth, we will assume that the gains $G(i)$ have been picked in this manner. Since the closed loop dynamics of the states $x_i(j)$ are related by a invertible transformation (i.e. $X = \zeta(\mu(X))$) if the modified closed-loop dynamics (5.16) are stabilized, so are the closed-loop dynamics (5.14).

We assume for the remainder of the chapter that the gains $G(i)$ are picked so that $G(i) = \hat{F}(i)$ (so that $G(i)$ is nonzero only on the rows and columns indexed by $\uparrow i$), and such that $F(i)$ is picked so that $A(\uparrow i, \uparrow i) + B(\uparrow i, \uparrow i)F(i)$ is stable. Let us denote $\hat{\mathbf{F}} = \left\{\hat{F}(1), \ldots, \hat{F}(s)\right\}$.

5.4.5 Controller Realization

We now describe two explicit controller realizations. The natural controller realization arises from the closed-loop dynamics (5.14) along with the control law (5.10) to give:

$$X[t + 1] = AX[t] + AX_d[t] + B(U[t] + U_d[t]) \tag{5.17}$$

$$U[t] = \zeta(\hat{F} \circ \mu(X))[t]. \tag{5.18}$$

We remind the reader that $U_d = \text{diag}(U)Z$. As explained earlier, the plant states correspond to the diagonal entries of X whereas the controller states correspond to X_j^i for $j > i$.

While the above corresponds to a natural description of the controller, it is possible to specify an alternative realization. This is motivated from the following observtation. The control input U depends only on $\mu(X)$. Hence, rather than implementing controller states that track the state predictions X, it is natural to implement controller states that compute the corrections $\mu(X)$ directly.

Let us define $q(i) = \mu(X)_{\uparrow\uparrow i}^i$ (so that the vector $q(i)$ has only those components of $\mu(X)_j^i$ such that $i < j$), and $q_j(i) = \mu(X)_j^i$. Note also that $\mu(X)_j^j = x_j - \sum_{k<j} q_j(k)$ from (5.8). Let us define $A^{cl}(j) = A(\uparrow j, \uparrow j) + B(\uparrow j, \uparrow j)F(j)$. The closed-loop dynamics at subsystem j (corresponding to the jth column of (5.16) reduce to:

$$\begin{bmatrix} x_j - \sum_{k<j} q_j(k) \\ q(j) \end{bmatrix}[t+1] = \begin{bmatrix} A_{11}^{cl}(j) & A_{12}^{cl}(j) \\ A_{21}^{cl}(j) & A_{22}^{cl}(j) \end{bmatrix} \begin{bmatrix} x_j - \sum_{k<j} q_j(k) \\ q(j) \end{bmatrix}[t]. \tag{5.19}$$

Note that from (5.10) (keeping in mind that $\mu(X)_j^i = 0$ if $j \not\geq i$, and that $u_j = U_j^j$), the

155

control law assumes the form:

$$u_j = \sum_{k \leq j} \hat{F}^{(j)}(k)\mu(X)^k$$

$$= \sum_{k \leq j} F^{(j)}(k) \begin{bmatrix} x_k - \sum_{l < k} q_k(l) \\ q(k) \end{bmatrix} [t].$$

(Recall that $F^{(j)}(k)$ is the jth row of the matrix $F(k)$).

It may be verified from (5.19) and (5.10) that the explicit controller for subsystem $j \in P$ assumes the following form:

$$q(j)[t + 1] = A_{22}^{cl}(j)q(j)[t] + A_{21}^{cl}(j)\left(x_j - \sum_{k < j} q_j(k)\right)[t]$$

$$u_j[t] = \sum_{k \leq j} F^{(j)}(k) \begin{bmatrix} x_k - \sum_{l < k} q_k(l) \\ q(k) \end{bmatrix} [t]. \tag{5.20}$$

5.4.6 Structure of the Optimal Controller

Consider again the poset-causal system considered in (5.2). Consider the optimal control problem:

$$\underset{K}{\text{minimize}} \quad \|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\|^2$$

subject to K stabilizes P $\tag{5.21}$

$$K \in \mathcal{I}(\mathcal{P}).$$

The solution K^* is the $\mathcal{H}_2$-optimal controller that obeys the poset-causality information constraints described Section 5.2. The solution to this optimization problem was presented in Theorem 4.3 in Chapter 4. We now establish the relationship between the optimal controller K^* and the proposed controller architecture.

In Theorem 4.3, we obtain matrices $K(\uparrow i, \uparrow i)$ by solving a system of decoupled Riccati equations via $(K(\uparrow j, \uparrow j), Q(j), P(j)) = \text{Ric}(A(\uparrow j, \uparrow j), B(\uparrow j, \uparrow j), C(\uparrow j), D(\uparrow j))$.

156

Theorem 5.2. *The controller* (5.20) *with gains* $F(i) = K(\uparrow i, \uparrow i)$ *for all* $i \in P$ *is the optimal solution to the control problem* (5.21).

Proof. Recall that the optimal controller is given by (4.16). We will show that when we pick $F(i) = K(\uparrow i, \uparrow i)$, we recover this controller. By (4.15) $\mathrm{diag}(A^{cl}(j)) = \mathbf{A}$, and hence $\mathrm{diag}(A^{cl}_{22}(j)) = A_\Phi$ and $\mathrm{diag}(A^{cl}_{21}(j)) = B_\Phi$. Also note that:

$$\left[\sum_{k < j} q_j(k) \right]_{j \in P} = C_\Phi q. \tag{5.22}$$

Letting q be the vectorization of $q(j)$ for $j \in P$ via $q = [q(j)]_{j \in P}$, we may rewrite the dynamics in (5.20) as

$$q[t + 1] = A_\Phi q[t] + B_\Phi(x[t] - C_\Phi q[t]). \tag{5.23}$$

Further, note that the vectorization of the control law equation in (5.20) yields

$$u = \left[\sum_{k \leq j} K^{(j)}(\uparrow k, \uparrow k) \begin{bmatrix} x_k - \sum_{l < k} q_k(l) \\ q(k) \end{bmatrix} \right]_{j \in P} = C_Q \Pi_2 q + C_Q \Pi_1 (x - C_\Phi q).$$

As a result, the control law may be written as:

$$u = C_Q \Pi_2 q + C_Q \Pi_1 (x - C_\Phi q). \tag{5.24}$$

Combining (5.23), (5.24), we obtain that

$$u = \left[\begin{array}{c|c} A_\Phi - B_\Phi C_\Phi & B_\Phi \\ \hline C_Q(\Pi_2 - \Pi_1 C_\Phi) & C_Q \Pi_1 \end{array} \right] q,$$

which is precisely the same expression as (4.16). Since this corresponds to the optimal $\mathcal{H}_2$ controller, we have the required result. $\square$

157

Theorem 5.2 establishes that the controller architecture proposed in this chapter is also *optimal* in the sense of the $\mathcal{H}_2$ norm.

5.4.7 Φ and Γ revisited

In Section 4.4.2 in Chapter 4, we introduced a pair of transfer function matrices (Φ, Γ) (see (4.17)). In Section 4.4.3 we briefly mentioned an interpretation of Φ and Γ and provided an intuitive explanation for the structural form of the optimal controller derived in Theorem 4.4. We now re-examine these explanations more formally. We begin by interpreting Γ.

As a consequence of the separation theorem, we have that the local variables $\mu(X)$ satisfy the decoupled relationship (5.16). Defining $\mu(X)_i^j = q_i(j)$, we saw that the evolution of $\mu(X)$ may be rewritten as (5.19), where $F(j) = K(\uparrow j, \uparrow j)$ as determined by the solution to the Riccati equation. Also $A_{22}^{cl}(j) = A_\Phi(j)$, and $A_{21}^{cl}(j) = B_\Phi(j)$. Recalling that $\mu(X)_j^j = x_j - \sum_{k<j} q_j(k)$ and $\mu(X)_{\uparrow\uparrow j}^j = q(j)$, from (5.19) we have that for each $j \in P$

$$\mu(X)_{\uparrow\uparrow j}^j[t+1] = A_\Phi(j)\mu(X)_{\uparrow\uparrow j}^j[t] + B_\Phi(j)\left(x_j - \sum_{k<j}\mu(X)_j^k\right)[t]$$

$$\mu(X)_j^j[t] = \left(x_j - \sum_{k<j}\mu(X)_j^k\right)[t].$$

Rewriting this in transfer function form we obtain

$$\left[\mu(X)_j^j\right]_{j\in P} = \left[\begin{array}{c|c} A_\Phi - B_\Phi C_\Phi & B_\Phi \\ \hline -C_\Phi & I \end{array}\right]x$$

$$= \Gamma x.$$

Thus $\Gamma x = \left[\mu(X)_j^j\right]_{j\in P}$, the diagonal entries of $\mu(X)$. Hence the role of Γ is to compute the *diagonal entries* of $\mu(X)$.

Note that we showed in (5.19) that the dynamics of the local variables $\mu(X)^j$ are decou-

pled and satisfy

$$\mu(X)^j[t+1] = \left(A + B\hat{K}(\uparrow j, \uparrow j)\right)\mu(X)^j[t].$$

It is thus possible to compute $\mu(X)^j_{\uparrow\uparrow j}$ from $\mu(X)^j_j$ by noting that:

$$\mu(X)^j_{\uparrow\uparrow j} = (zI - A^{cl}_{22}(j))^{-1}A^{cl}_{21}(j)\mu(X)^j_j.$$

(This simply corresponds to computing $\mu(X)^j_{\uparrow\uparrow j}$ by *propagating* $\mu(X)^j_j$ using the its dynamical equation). Using the notation of Chapter 4 where $\Phi(j)$ represents the jth column of Φ, the above equation may be rewritten as

$$\mu(X)^j = \Phi(j)\mu(X)^j_j.$$

This clarifies the role of Φ; it is to compute the *off-diagonal* entries of $\mu(X)$ by propagating the diagonal entries. Note that this form of propagation is possible as a consequence of the crucial fact that the dynamics of the local variables $\mu(X)$ are decoupled.

In Theorem 4.4 in Chapter 4 we showed that the optimal controller was of the form:

$$u[t] = -\sum_{j \in P} \hat{K}(\uparrow j, \uparrow j)\Phi(j)(\Gamma x)_j[t].$$

This can be related to the controller architecture $U = \zeta(G \circ \mu(X))$. Note that the $u = \left[U^j_j\right]_{j \in P}$ (i.e. u corresponds to the diagonal entries of the local variable U). Extracting the diagonal entries of the control law (and using the fact that $\mu(X)$ is in the incidence algebra and $G(j)$ is non-zero only on the rows and columns corresponding to $\uparrow j$) we obtain that

$$u[t] = \sum_{j \in P} G(j)\mu(X)^j[t].$$

As pointed out earlier,

$$\Phi(j)(\Gamma x)_j = \mu(X)^j,$$

159

and $G(j) = -\hat{K}(\uparrow j, \uparrow j)$ so that the controller form reduces to

$$u[t] = -\sum_{j \in P} \hat{K}(\uparrow j, \uparrow j)\Phi(j)(\Gamma x)_j,$$

which is exactly the controller form stated in Theorem 4.4.

The role of the controller states in the optimal controller (4.16) is also now clear. The controller states q compute the differential improvements $\mu(X)$, since $q_i(j) = \mu(X)_i^j$.

5.5 A Block-Diagram Interpretation

It is possible to interpret the results of this chapter via a simple block-diagram approach. We remind the reader that X and U are the local state and input variables and that X_d and U_d are the downstream components. We define $X_f = X + X_d$ and $U_f = U + U_d$. The quantities $\text{vec}(X_f)$ and $\text{vec}(U_f)$ represent the standard vectorizations of X_f and U_f respectively. We use $\otimes$ to represent the standard Kronecker product of matrices [26]. In our approach, the signals X and U (recall that they are both in $\mathcal{I}(\mathcal{P})$) will also be vectorized. In this vectorization, we only consider the non-zero elements (i.e. elements in $\mathcal{I}(\mathcal{P})$) so that $\text{vec}(X)$ is a vector of length $\sum_i |\uparrow i|$, and $\text{vec}(U)$ is similarly defined. We also remind the reader that the plant is described by the transfer function:

$$G(z) = (zI - A)^{-1}B.$$

The elementary blocks that appear in our block-diagram representation are the following:

- The plant G, which maps the inputs u to the states x,

- The transfer functions which play the role of predicting the local state variables X^i from the states x_j and inputs u_j for $j \in \downarrow i$ via (5.14). We call all these transfer functions collectively the "simulator", because their role may be interpreted as that

160

of simulating upstream states,

- The map $\bar{\mu}$ which takes as input vec(X) and computes vec($\mu(X)$),

- The local gains $F(1), \ldots F(s)$,

- The map $\bar{\zeta}$ which takes as input vec($\mu(U)$) and computes vec(U).

In the closed-loop system all these transfer functions are interconnected as shown in Fig. 5-5.
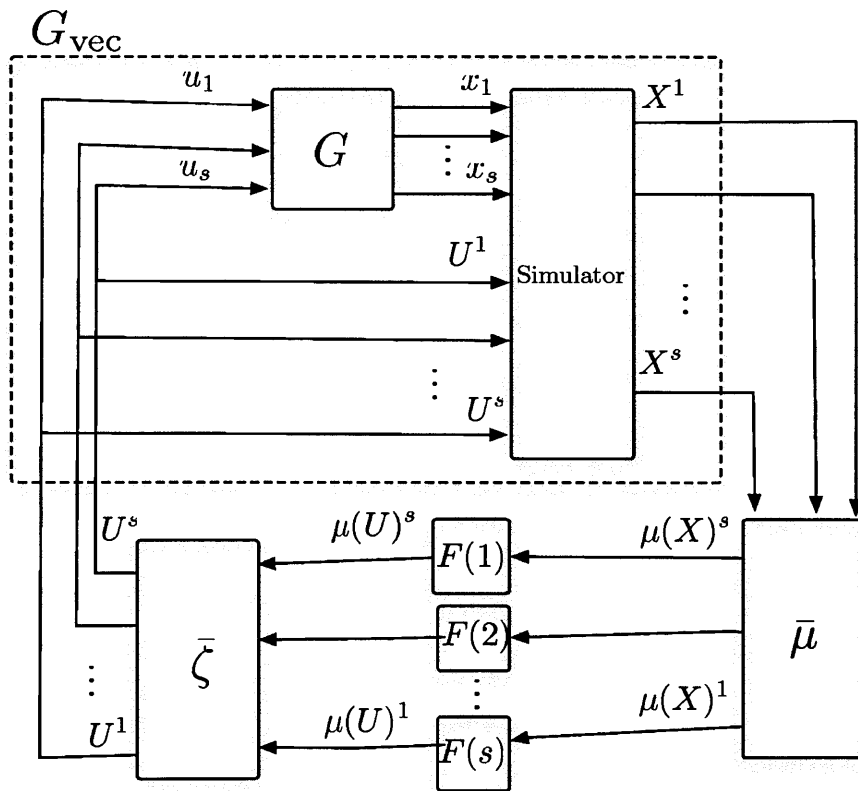

Figure 5-5: A block-diagram representation of the control architecture.

Note that the matrices $\bar{\zeta}$ and $\bar{\mu}$ are formally defined to be (the matrix representations) the linear maps that map vec(X) to vec($\zeta(X)$) and vec($\mu(X)$) respectively. Note that these are both completely defined via the maps ζ and μ. Their explicit matrix realizations are given

161

next. We remind the reader of some notation from Chapter 4. The vector e_i represents the ith standard unit vector, $E_{\uparrow i} = \left[e_j \right]_{j \in \uparrow i}$, and if $\{V_1, \ldots, V_s\}$ is a collection of matrices then diag$(V_k)_{k \in P}$ is a block diagonal matrix with the V_i along the diagonals. Let us define the projection matrix:

$$\Pi = \text{diag}(E_{\uparrow i})_{i \in P}.$$

Note that if $A \in \mathbb{R}^{s \times s}$, then Π acts on vec(A) by projecting onto those components that are in the incidence algebra. Using this notation, the matrices $\bar{\zeta}$ and $\bar{\mu}$ can be shown to have the explicit realization

$$\bar{\zeta} = \Pi^T(\zeta \otimes I)\Pi \qquad \bar{\mu} = \Pi^T(\mu \otimes I)\Pi. \tag{5.25}$$

Moreover, since ζ and μ are inverses of each other, the matrices $\bar{\zeta}$ and $\bar{\mu}$ are inverses of each other. An easy way to see this is that since ζ and μ are inverses of each other $\zeta \otimes I$ and $\mu \otimes I$ are inverses of each other. Moreover, both $\zeta \otimes I$ and $\mu \otimes I$ are block lower triangular. Since $\bar{\zeta}$ and $\bar{\mu}$ are just principal sub-matrices of $\zeta \otimes I$ and $\zeta \otimes I$ (corresponding to the rows and columns picked out by Π) these matrices are inverses of each other.

By standard algebraic manipulations it is possible to see that the variables X_f and U_f are related by the block diagonal map:

$$\text{vec}(X_f) = \begin{bmatrix} G & & \\ & \ddots & \\ & & G \end{bmatrix} \text{vec}(U_f).$$

While this representation is appealing, we point out that redundant copies of variables appear in X_f and U_f. Indeed, since $X_f = X + X_d = X + \text{diag}(X)\zeta^T$, we see that there are many redundant copies of the entries of diag(X) in X_f. To get rid of these redundant copies of the variables, we instead vectorize X and U directly. The corresponding map between

162

the two quantities is then given by:

$$\text{vec}(X) = G_{\text{vec}}\text{vec}(U).$$

Here G_{vec} is a block matrix where the (j, i) block is of size $|\uparrow j| \times |\uparrow i|$, and this block is given by $G(\uparrow j, i)$ and zero-padding remaining entries so that the block is of appropriate size. We point out that at this block level, this matrix is in the incidence algebra. We illustrate this with an example.

Example 5.10. *For the poset in Fig. 5-1(a), we have*

$$\text{vec}(X) = \begin{bmatrix} x_1 \\ x_2(1) \\ x_2 \end{bmatrix} \qquad \text{vec}(U) = \begin{bmatrix} u_1 \\ u_2(1) \\ u_2 \end{bmatrix}.$$

Furthermore, the map G is given by

$$G = \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix},$$

and the map G_{vec} is given by:

$$G_{vec} = \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & G_{22} & 0 \\ G_{21} & 0 & G_{22} \end{bmatrix}.$$

For this poset the matrices $\bar{\zeta}$ and $\bar{\mu}$ are given by:

$$\bar{\zeta} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & I \end{bmatrix} \qquad \bar{\mu} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -I & I \end{bmatrix}.$$

163

It is straightforward to verify that vec(X) $= G_{vec}vec(U)$*. The following important identity may be verified for this example:*

$$\bar{\mu}G_{vec}\bar{\zeta} = \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & G_{22} & 0 \\ 0 & 0 & G_{22} \end{bmatrix},$$

which is a block diagonal matrix.

As indicated in Fig. 5-5, the collective map from vec(U) to vec(X) (which collects the plant G and the simulation block into a single transfer function) is simply given by G_{vec}. Thus the block-diagram in Fig. 5-5 can be simplified to Fig. 5-6. The matrix G_{vec} satisfies
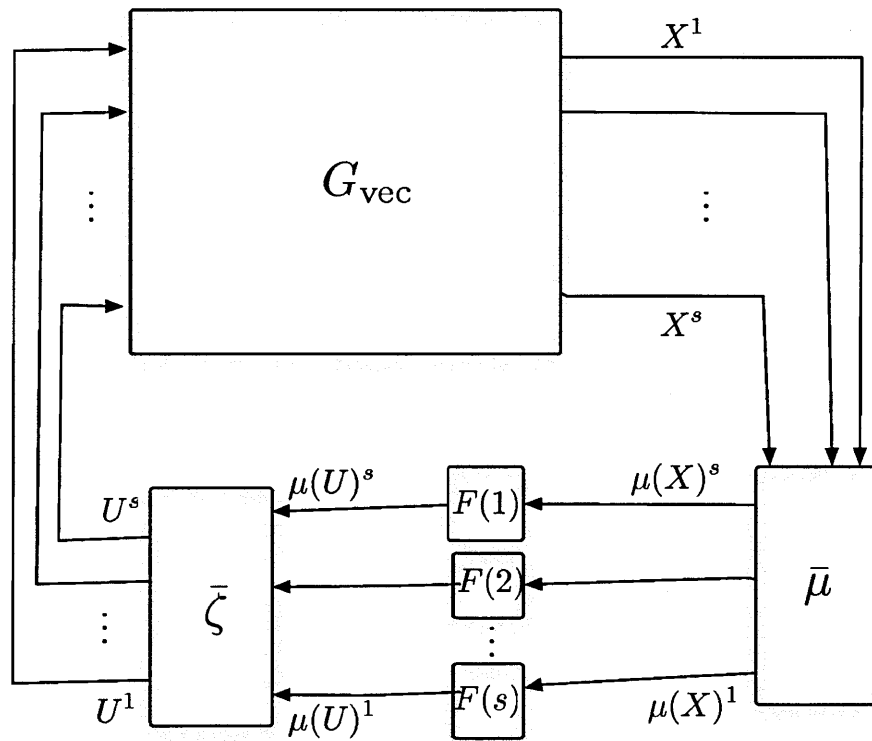

Figure 5-6: A simplified block-diagram representation of the control architecture.

the following important property.

Lemma 5.5. *The matrix G_{vec} can be block diagonalized via:*

$$\bar{\mu} G_{vec} \bar{\zeta} = \begin{bmatrix} G(\uparrow 1, \uparrow 1) & & \\ & \ddots & \\ & & G(\uparrow s, \uparrow s) \end{bmatrix}. \qquad (5.26)$$

Proof. Let

$$G_{\text{diag}} = \begin{bmatrix} G(\uparrow 1, \uparrow 1) & & \\ & \ddots & \\ & & G(\uparrow s, \uparrow s) \end{bmatrix}.$$

Since $\bar{\zeta}$ and $\bar{\mu}$ are inverses, using (5.25), it is sufficient to show that:

$$G_{\text{vec}} \Pi^T (\zeta \otimes I) \Pi = \Pi^T (\zeta \otimes I) \Pi G_{\text{diag}}.$$

The matrix on the left (as also the matrix on the right) is a block $|P| \times |P|$ matrix, where the (i, j) block may be seen to be of size $|\uparrow i| \times |\uparrow j|$. Some basic matrix manipulations reveal that in both the matrix on the left and on the right the (i, j) block is simply $G(\uparrow i, \uparrow j)$, thereby establishing the required result. □

In terms of this block-diagram approach, the role of $\bar{\mu}$ and $\bar{\zeta}$ become very transparent: it is simply to diagonalize the map G_{vec}. Once this diagonalization occurs, the controller simply applies a set of diagonal gains to stabilize the closed-loop. This also illustrates the separation principle at the block-diagram level. As mentioned in the preceding discussion, the architecture illustrated in block-diagram Fig 5-6 is also optimal, in that appropriate choice of the gains $F(i)$ yield optimal controllers.

Finally, we mention that the transfer function Γ is also easy to visualize, as shown in Fig. 5-7, it is simply the transfer from the states x to the vector $[\mu(X)_i^j]_{i \in P}$.
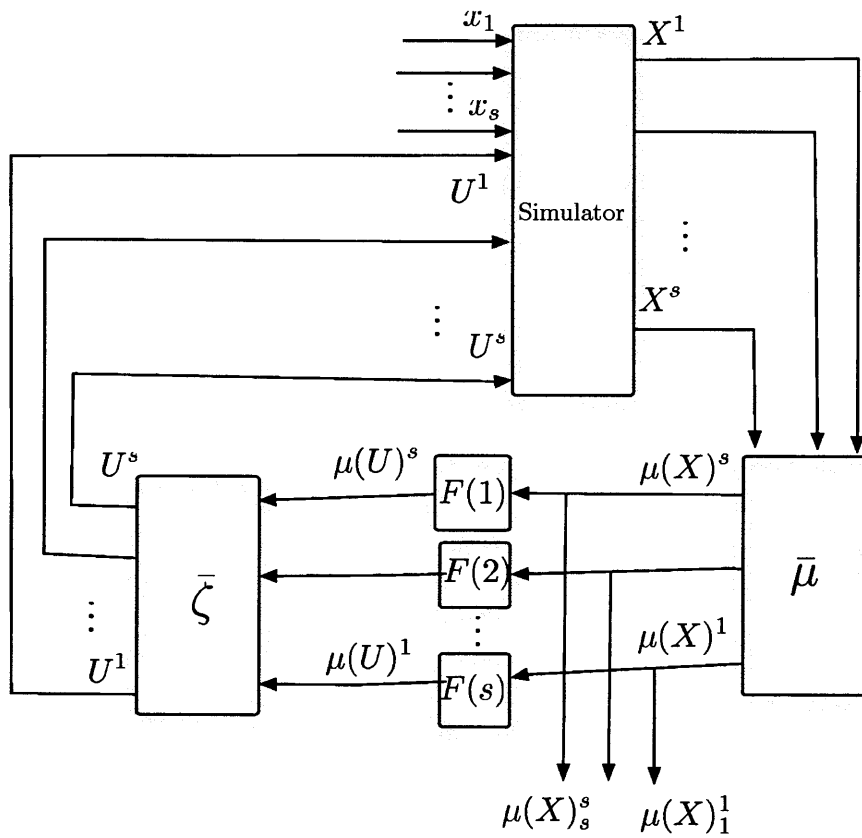
165

Figure 5-7: A block-diagram representation of Γ.

5.6 Conclusions

In this chapter we addressed an architectural question related to design of poset-causal controllers. We described an intuitive architecture that involved local state prediction at the different subsystems. Some natural ingredients involved in the control architecture included abstract notions of integration and differentiation along the poset. The notion of integration allowed us to fuse local state information at different subsystems. The notion of differentiation, intimately related to the Möbius inversion formula, described the local corrections in the state predictions. The proposed control law, which consisted of a combination of these concepts, had several appealing properties. We established an elegant separation principle for the control architecture and proved that the architecture was optimal in a formal sense.

Chapter 6

Conclusions

6.1 Posets, Decentralization, and Computation

In this thesis we studied a class of decentralized control problems. The class under consideration provided a natural way to model a generalized notion of causality or hierarchy among subsystems of a decentralized system. Using the key technical concept of a poset, we developed a new notion of poset-causality as a formal framework for the same. The class of poset-causal systems enabled us to model, in a natural way, unidirectional/acyclic information flow within systems.

We argued that this paradigm allowed us to unify, into one common theoretical framework, several classes of previously studied decentralized control problems. Examples include systems with nested information, time-delayed systems, as well as certain classes of spatially distributed systems. We showed that algebraic properties of posets (via the notion of an incidence algebra), along with the Youla parametrization, allowed us to reformulate seemingly non-convex optimal control problems to convex ones.

While convex formulations in the Youla domain are elegant from a theoretical point of view, from a computational standpoint there are severe limitations. This led us to examine more efficient state-space approaches. While state-space approaches for centralized control

are well-studied, fairly limited literature is available for the same for decentralized control.

This led us to study state-space approaches for the class of decentralized control problems at the heart of this thesis, namely poset-causal systems. Given a poset-causal system, we studied the problem of computing the $\mathcal{H}_2$-optimal poset-causal controller with state feedback. We first identified a key separability of the problem. Using this separability, we showed how to reduce the decentralized problem to a set of decoupled centralized problems, each of which could be solved using standard techniques. Exploiting these ideas, we gave explicit state-space formulae for the $\mathcal{H}_2$-optimal controller.

Having established efficiently computable state-space formulae for poset-causal systems, we then examined the problem of controller design from an architectural viewpoint. Exploiting posets' rich structure, we described a natural and intuitive controller architecture. Some of the essential ingredients for the same included local predictions of the global state at various subsystems (this was the role of the controller states), a notion of integration (this played the role of fusing downstream information at subsystems) and differentiation (this played the role of computing innovations in the state via Möbius inversion) on posets. The proposed controller architecture had two important properties: a separation principle and a certain optimality property.

6.2   Future Directions

As future research it would be interesting to extend the state-space results of Chapter 4 to more general settings. Firstly, we note that we assumed that subsystems in the system had access to downstream state information. It is important to address the case where the full downstream states are not available, rather only general outputs of the form $y_i = C_i x_i$ are available. Some partial progress on this problem has been made by Swigart and Lall for the so-called "Two-Player Case" with partial output feedback [53].

Another interesting direction would be to extend these state-space techniques to some

of the classes of problems mentioned in Chapter 3. These include more general poset-like structures described via Galois connections, systems with time-delays and spatially invariant systems.

Finally, we mention that the state-space approaches have only been studied in the context of the $\mathcal{H}_2$ norm in this thesis. It is also important to study the problem in the context of other norms that capture different qualitative behavior. For example the $\mathcal{H}_\infty$ norm is the "robust" counterpart, and is also amenable to state-space techniques in the centralized case [21, 16]. It would be interesting to extend these ideas for decentralized control problems.

Bibliography

[1] M. Aigner. *Combinatorial theory*. Springer-Verlag, 1979.

[2] S.M. Aji and R.J. McEliece. The generalized distributive law. *IEEE Transactions on Information Theory*, 46(2):325–343, 2000.

[3] M. Akkurt, G. P. Barker, and M. Wild. Structural matrix algebras and their lattices of invariant subspaces. *Linear Algebra and its Applications*, 394:25–38, 2005.

[4] B. Bamieh, F. Paganini, and M. A. Dahleh. Distributed control of spatially invariant systems. *IEEE Transactions on Automatic Control*, 47(7):1091–1107, 2002.

[5] B. Bamieh and P. Voulgaris. Optimal distributed control with distributed delayed measurements. In *Proceedings of the IFAC World Congress*, 2002.

[6] B. Bamieh and P. Voulgaris. A convex characterization of distributed control problems in spatially invariant systems with communication constraints. *Systems and Control Letters*, 54(6):575–583, 2005.

[7] S. P. Banks. *State-Space and Frequency-Domain Methods in the Control of Distributed Parameter Systems*. Peter Peregrinus, 1983.

[8] A. Bensoussan, G.P. Da Prato, M. C. Delfour, and S. K. Mitter. *Representation and Control of Infinite Dimensional Systems*, volume I, II. Birkhauser, 1992, 1993.

[9] V. Blondel and J. N. Tsitsiklis. NP-hardness of some linear control design problems. *SIAM Journal on Control and Optimization*, 35(6):2118–2127, 1997.

[10] V. Blondel and J. N. Tsitsiklis. A survey of computational complexity results in systems and control. *Automatica*, 36(9):1249–1274, 2000.

[11] S. P. Boyd and C. H. Barratt. *Linear Controller Design: Limits of Performance*. Prentice Hall, 1991.

[12] F. M. Callier and C. A. Desoer. *Linear System Theory*. Springer-Verlag, 1991.

[13] R. B. Carroll and D. A. Brask. A practical implementation for solutions to the algebraic matrix Riccati equation in a LQCM setting. *Journal of Economic Dynamics and Control*, 23(1):1–7, 1998.

[14] P. Cousot and R. Cousot. Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints. In *Conference Record of the Fourth Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, pages 238–252, Los Angeles, California, 1977. ACM Press, New York, NY.

[15] R. F. Curtain and A. J. Pritchard. Infinite-dimensional linear systems theory. *Lecture Notes in Control Information Sciences*, 8, 1978.

[16] J.C. Doyle, K. Glover, P.P. Khargonekar, and B.A. Francis. State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems. *Automatic Control, IEEE Transactions on*, 34(8):831–847, 1989.

[17] M. Erne, J. Koslowski, A. Melton, and G.E. Strecker. A primer on Galois connections. *York Academy of Science*, 1992.

[18] C. Fan, J. L. Speyer, and C. R. Jaenesch. Centralized and decentralized solutions of the linear-exponential-gaussian problem. *IEEE Transactions on Automatic Control*, 1994.

[19] W. Findeisen, F. N. Bailey, M. Brdys, K. Malinowski, P. Tatjewoki, and A. Wozniak. *Control and Coordination in Hierarchical Systems*. Wiley, 1980.

[20] B. A. Francis. *A course in $\mathcal{H}_\infty$ control theory*. Springer-Verlag New York, Inc., New York, NY, USA, 1986.

[21] P. Gahinet and P. Apkarian. A linear matrix inequality approach to H_∞ control. *International Journal of Robust and Nonlinear Control*, pages 421–448, 1994.

[22] A. Gattami and B. Bernhardsson. Minimax team decision problems. In *Proceedings of the American Control Conference*, 2007.

[23] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous Lattices and Domains, In Encyclopedia of Mathematics and its Applications*, volume 93. Cambridge University Press, 2003.

[24] A. N. Gündes and C. A. Desoer. *Algebraic Theory of Linear Feedback Systems with Full and Decentralized Compensators*. Lecture Notes in Control and Information Sciences, 142, Berlin; New York: Springer-Verlag, 1990.

[25] Y.-C. Ho and K.-C. Chu. Team decision theory and information structures in optimal control problems-part I. *IEEE Transactions on Automatic Control*, 17(1):15–22, 1972.

[26] R. Horn and C. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, 1991.

[27] F.R. Kschischang, B.J. Frey, and H.-A. Loeliger. Factor graphs and the sum-product algorithm. *IEEE Transactions on Information Theory*, 47(2):498–519, 2001.

[28] C. Langbort, R. S. Chandra, and R. D'Andrea. Distributed control design for systems interconnected over an arbitrary graph. *IEEE Transactions on Automatic Control*, 49(9):1502–1519, 2004.

[29] L. Lessard and S. Lall. Quadratic invariance is necessary and sufficient for convexity. In *Proceedings of the American Control Conference*, 2011.

[30] J. L. Lions. *Optimal Control of Systems Governed by Partial Differential Equations*. Springer-Verlag, 1971.

[31] M. D. Mesarovic, D. Macko, and Y. Takahara. *Theory of Hierarchical, Multilevel Systems*. Academic Press, 1970.

[32] S. K. Mitter and A. Sahai. Information and control: Witsenhausen revisited. *Lecture Notes in Control and Information Sciences*, 241:281–293, 1999.

[33] N. Motee and A. Jadbabaie. Optimal control of spatially distributed systems. *IEEE Transactions on Automatic Control*, 53(7):1616–1629, 2008.

[34] R. E. Mullans and D.L. Elliott. Linear systems on partially ordered time sets. In *Proceedings of the IEEE Conference on Decision and Control including the 12th Symposium on Adaptive Processes*, volume 12, pages 334–337, 1973.

[35] C. H. Papadimitriou and J. N. Tsitsiklis. On the complexity of designing distributed protocols. *Information and Control*, 53(3):211–218, 1982.

[36] C. H. Papadimitriou and J. N. Tsitsiklis. Intractable problems in control theory. *SIAM Journal on Control and Optimization*, 24:639–654, 1986.

[37] R. Radner. Team decision problems. *Annals of mathematical statistics*, 33:857–881, 1962.

[38] A. Rantzer. Linear quadratic team theory revisited. In *Proceedings of the American Control Conference*, 2006.

[39] G.-C. Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. *Probability theory and related fields*, 2(4):340–368, 1964.

[40] M. Rotkowitz. On information structures, convexity, and linear optimality. In *Proceedings of the 47th IEEE Conference on Decision and Control*, 2008.

[41] M. Rotkowitz and R. Cogill. Convex synthesis of distributed controllers for spatio-temporal systems with subadditive support functions. In *Forty-Sixth Annual Allerton Conference on Communication, Control, and Computing*, 2008.

[42] M. Rotkowitz, R. Cogill, and S. Lall. A simple condition for the convexity of optimal control over networks with delays. In *Proceedings of the 44th IEEE Conference on Decision and Control*, 2005.

[43] M. Rotkowitz and S. Lall. Decentralized control subject to communication and propagation delays. In *Proceedings of the 43th IEEE Conference on Decision and Control*, 2004.

[44] M. Rotkowitz and S. Lall. A characterization of convex problems in decentralized control. *IEEE Transactions on Automatic Control*, 51(2):274–286, 2006.

[45] N. Sandell, P. Varaiya, M. Athans, and M. Safonov. Survey of decentralized control methods for large scale systems. *IEEE Transactions on Automatic Control*, 23(2):108–128, 1978.

[46] C.W. Scherer. Structured finite-dimensional controller design by convex optimization. *Linear Algebra and its Applications*, 2002.

[47] P. Shah and P. A. Parrilo. A partial order approach to decentralized control. In *Proceedings of the 47th IEEE Conference on Decision and Control*, 2008.

[48] P. Shah and P. A. Parrilo. A partial order approach to decentralized control of spatially invariant systems. In *Forty-Sixth Annual Allerton Conference on Communication, Control, and Computing*, 2008.

[49] P. Shah and P. A. Parrilo. A poset framework to model decentralized control problems. In *Proceedings of the 48th IEEE Conference on Decision and Control*, 2009.

[50] E. Spiegel and C. J. O'Donnell. *Incidence Algebras*. Marcel Dekker, 1997.

[51] J. Swigart and S. Lall. An explicit state-space solution for a decentralized two-player optimal linear-quadratic regulator. In *Proceedings of the American Control Conference*, 2010.

[52] J. Swigart and S. Lall. Optimal synthesis and explicit state-space solution for a decentralized two-player linear-quadratic regulator. In *Proceedings of the 49th IEEE Conference on Decision and Control*, 2010.

[53] J. Swigart and S. Lall. Optimal controller synthesis for a decentralized two-player system with partial output feedback. In *Proceedings of the American Control Conference*, 2011.

[54] J. N. Tsitsiklis and M. Athans. On the complexity of decentralized decision making and detection problems. *IEEE Transactions on Automatic Control*, 30(5):440–446, 1985.

[55] D. D. Šiljak. *Decentralized Control of Complex Systems*. Academic Press, 1991.

[56] D. Del Vecchio and R. M. Murray. Discrete state estimators for a class of hybrid systems on a lattice. In *HSCC*, pages 311–325, 2004.

[57] M. Vidyasagar. *Control System Synthesis: A Factorization Approach*. MIT Press, 1985.

[58] P. Voulgaris. Control under structural constraints: an input-output approach. *Lecture Notes in Control and Information Sciences*, pages 287–305, 1999.

[59] P. Voulgaris. Control of nested systems. In *Proceedings of the American Control Conference*, 2000.

[60] P. Voulgaris. A convex characterization of classes of problems in control with specific interaction and communication structures. In *Proceedings of the American Control Conference*, 2001.

[61] S-H. Wang and E. Davison. On the stabilization of decentralized control systems. *IEEE Transactions on Automatic Control*, 18(5):473 – 478, 1973.

[62] H.S. Witsenhausen. A counterexample in stochastic optimum control. *SIAM J. Control*, 6(1):131–147, 1968.

[63] H.S. Witsenhausen. Separation of estimation and control for discrete time systems. *Proceedings of the IEEE*, 59(11):1557 – 1566, 1971.

[64] D. Zelazo and M. Mesbahi. Analysis and synthesis of networked dynamic systems with $\mathcal{H}_2$ performance. In *Proceedings of the American Control Conference*, 2009.

[65] Q.-C. Zhong. *Robust Control of Time-Delay Systems*. Springer, 2006.

[66] K. Zhou and J. C. Doyle. *Essentials of Robust Control*. Prentice Hall, 1998.