F.De Terán. The solution of the equation AX+BX*=0,
in *Linear and Multilinear Algebra* 61 (2013) 12, pp.1605-1628. Available
in http://dx.doi.org/10.1080/03081087.2012.750656

The solution of the equation $AX + BX^\star = 0$

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We give a complete solution of the matrix equation $AX + BX^\star = 0$, where $A, B \in \mathbb{C}^{m\times n}$ are two given matrices, $X \in \mathbb{C}^{n\times n}$ is an unknown matrix, and $\star$ denotes the transpose or the conjugate transpose. We provide a closed formula for the dimension of the solution space of the equation in terms of the Kronecker canonical form of the matrix pencil $A + \lambda B$, and we provide also an expression for the solution X in terms of this canonical form, together with two invertible matrices leading $A + \lambda B$ to the canonical form by strict equivalence.

Keywords: matrix equations, matrix pencils, Kronecker canonical form, transpose, conjugate transpose, Sylvester equation.

AMS Subject Classification: 15A21, 15A22, 15A24, 15B05.

1. Introduction

Matrix equations involving both the unknown X and its transpose X^T or its conjugate transpose X^* have appeared in the literature since the 1950's [11]. Particular cases of these equations are the *Sylvester-like equations*, whose interest has increased notably in the recent years, mostly because of their applications [1–5, 12, 16, 17]. In particular, the *generalized $\star$-Sylvester equation*, $AXB + CX^\star D = E$ (where $\star$ denotes both the transpose and the conjugate transpose), has been addressed in several recent references [10, 13, 15–17, 19], most of them devoted to provide numerical methods to find particular solutions for the general or special cases of the equation, or for systems of equations. The term "generalized" here stands for the fact that this equation is an extension of the so-called *$\star$-Sylvester equation* $AX + X^\star B = C$. This equation naturally arises when trying to reduce a block anti-triangular matrix to a block anti-diagonal one by using $\star$-congruence transformations, a reduction which is of interest, for instance, in palindromic generalized eigenvalue problems [1, 12].

Some of the main goals when dealing with (linear) equations are to obtain: (i) necessary and sufficient conditions for the existence of solutions; (ii) formulas for the dimension of the solution space; (iii) expressions for the solution; (iv) necessary and sufficient conditions for the existence of a unique solution; and (v) (efficient) algorithms to find the solution, when unique. All these problems have been addressed (and, essentially, solved) by different authors for the general $\star$-Sylvester equation or the homogeneous case $AX + X^\star B = 0$ of this equation [1, 2, 6, 7, 12, 14, 18].

The subject of the present paper is the matrix equation

$$AX + BX^\star = 0, \tag{1}$$

*Email: fteran@math.uc3m.es

ISSN: 0308-1087 print/ISSN 1563-5139 online

DOI: 10.1080/03081087.2012.750656
http://www.informaworld.com

where $A, B \in \mathbb{C}^{m\times n}$. This equation look like very much to the homogeneous $\star$-Sylvester equation, and it is also another particular case of the generalized $\star$-Sylvester equation. By contrast with the situation for the $\star$-Sylvester equation, none of the goals mentioned in the precedent paragraph have been addressed so far for equation (1). The present paper considers some of these questions and our contribution in this context is threefold. In the first place, we provide an explicit formula for the dimension of the solution space; in the second place, we give a complete description of the solution; and, in the third place, we give necessary and sufficient conditions for the existence of a unique solution.

Apart from their appearance, there are relevant similarities between equation (1) and the homogeneous $\star$-Sylvester equation $AX + X^\star B = 0$. This way, both the results and techniques contained in the present paper look like very much to the ones in [7], where the homogeneous $\star$-Sylvester equation is solved. Let us mention one interesting similarity between both equations, which is in the basis of the procedure followed to solve them. On the one hand, the solutions of $AX + X^\star B = 0$ are in one-to-one linear correspondence with the solutions of $\widetilde{A}X + X^\star \widetilde{B} = 0$, where $\widetilde{A} + \lambda \widetilde{B}^\star$ is a matrix pencil strictly equivalent to $A + \lambda B^\star$ [7, Theorem 1]. Hence, the solution of $AX + X^\star B = 0$ depends only on the Kronecker canonical form (KCF) of $A + \lambda B^\star$, and on two nonsingular matrices leading $A + \lambda B^\star$ to its KCF. Similarly, the solutions of equation (1) are in one-to-one correspondence with the solutions of $\widetilde{A}X + \widetilde{B}X^\star = 0$, where $\widetilde{A} + \lambda \widetilde{B}$ is a matrix pencil strictly equivalent to $A + \lambda B$. Moreover, this one-to-one correspondence is given by a congruence transformation (see Theorem 2.1). As a consequence, the solution of $AX + BX^\star = 0$ depends only on the KCF of $A + \lambda B$, and on two nonsingular matrices leading $A + \lambda B$ to its KCF. In particular, the dimension of the solution space of equation (1) depends only on the KCF of $A + \lambda B$.

There are, however, striking differences between equation (1) and $AX + X^\star B = 0$. For instance, in the second equation the unknown X may be rectangular, whereas in the first equation X must be square. Another difference arises when replacing $X^\star$ with X. In the second equation we get the Sylvester equation $AX + XB = 0$, whose solution depends on the Jordan canonical form of A and B [9, Ch. VIII, §1], whereas for the first equation we get $(A + B)X = 0$, which is a standard linear system, whose solution depends on the column space of $A + B$.

We want to mention also that equation (1) is equivalent to the system of equations $AX + BY = 0, Y = X^\star$, which can be seen as an extension of the system $AX = 0, X = X^\star$, the one considered in [8] (with $\star = T$). However, the solution has nothing to do with the one of this last system (actually, the dimension of the solution space of this last system depends only on the rank and the size of A).

The paper is organized as follows: In Section 2 we show how to reduce the problem of solving (1) by decoupling it into smaller equations and systems of matrix equations, taking advantage of the KCF of the pencil $A + \lambda B$. In Section 3 we display the dimension of the solution space of (1) in terms of this KCF. In Section 4 we prove the results from Section 3 by obtaining the dimension of all the equations and systems of equations obtained in the decoupling procedure explained in Section 2. In these proofs we have also obtained an expression for the solution of the corresponding equations, which gives in turn an expression for the solution of (1), provided that the nonsingular matrices leading $A + \lambda B$ to its KCF are known. Section 5 contains necessary and sufficient conditions for (1) to have a unique (trivial) solution, and also for $AX + BX^\star = C$ to have a solution for every right-hand side matrix C. In Section 6 we review the main contributions of the paper and present some related open problems.

1.1. *Notation*

Throughout the paper we will use the following notation: $M = M_1 \oplus M_2 \oplus \cdots \oplus M_d$ denotes a direct sum of blocks $M_1, \ldots, M_d$ or, in other words, a block-diagonal matrix or matrix pencil M, whose diagonal blocks are $M_1, \ldots, M_d$. Also, I_n will denote the identity matrix with size $n \times n$. Given a vector space V over $\mathbb{C}$, we will denote by $\dim V$ and $\dim_{\mathbb{R}} V$ the complex and the real dimension of V, respectively.

2. Reduction to KCF. Decoupling procedure

We show in this section that the solution of the equation (1) is in one-to-one linear correspondence with the solution of $\widetilde{A}X + \widetilde{B}X^\star = 0$, where $A + \lambda B$ and $\widetilde{A} + \lambda\widetilde{B}$ are *strictly equivalent* pencils, that is, $\widetilde{A} + \lambda\widetilde{B} = P(A + \lambda B)Q$, with P, Q nonsingular matrices. Using this fact, we can solve this last equation, with $\widetilde{A} + \lambda\widetilde{B}$ being the canonical form for strict equivalence of $A + \lambda B$ (namely, the KCF), and recover the original equation using the one-to-one correspondence. The idea to solve the last equation is to take advantage of the block-diagonal structure of the KCF. For this, we will use Lemma 2.3, which shows us how equation (1) is decoupled when the coefficient matrices are block-diagonal.

2.1. *Strict equivalence of pencils and the solution space*

We start with Theorem 2.1, which is key in the procedure to solve (1).

Theorem 2.1 : *Let $A + \lambda B$ and $\widetilde{A} + \lambda\widetilde{B}$ be two strictly equivalent matrix pencils, with $\widetilde{A} + \lambda\widetilde{B} = P(A + \lambda B)Q$, and P, Q nonsingular. Then Y is a solution of $\widetilde{A}Y + \widetilde{B}Y^\star = 0$ if and only if $X = QYQ^\star$ is a solution of $AX + BX^\star = 0$. As a consequence, the solution spaces of both equations are isomorphic via $Y \mapsto QYQ^\star = X$.*

Proof : Let Y be a solution of $\widetilde{A}Y + \widetilde{B}Y^\star = 0$, with $\widetilde{A}$ and $\widetilde{B}$ as in the statement. Then

$$PAQY + PBQY^\star = 0 \Leftrightarrow PA(QYQ^\star) + PB(QY^\star Q^\star) = 0 \Leftrightarrow AX + BX^\star = 0,$$

with $X = QYQ^\star$ as in the statement. Since Q is invertible, $Y \mapsto X$ is an isomorphism, and the result follows. $\square$

As a consequence of Theorem 2.1, the solution of $AX + BX^\star = 0$ can be recovered from the solution of $\widetilde{A}X + \widetilde{B}X^\star = 0$, where $\widetilde{A} + \lambda\widetilde{B} = P(A + \lambda B)Q$, once we know the nonsingular matrices P, Q. In particular, we may consider $\widetilde{A} + \lambda\widetilde{B}$ to be the KCF of $A + \lambda B$. Since the KCF plays a fundamental role in the following, we include this canonical form here, for the sake of completeness.

Theorem 2.2 : **(Kronecker canonical form)** *Each complex matrix pencil $A + \lambda B$, with $A, B \in \mathbb{C}^{m \times n}$, is strictly equivalent to a direct sum of blocks of the following types:*

(1) Right singular blocks:

$$L_\varepsilon = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \end{pmatrix}_{\varepsilon \times (\varepsilon+1)} .$$

(2) Left singular blocks: L_η^T, *where L_η is a right singular block.*

(3) Finite blocks: $J_k(\mu) + \lambda I_k$, *where* $J_k(\mu)$ *is a Jordan block of size* $k \times k$ *associated with* $\mu \in \mathbb{C}$,

$$J_k(\mu) := \begin{pmatrix} \mu & 1 & & & \\ & \mu & 1 & & \\ & & \ddots & \ddots & \\ & & & \mu & 1 \\ & & & & \mu \end{pmatrix}_{k \times k} .$$

(4) Infinite blocks: $N_u := I_u + \lambda J_u(0)$.

This pencil is uniquely determined, up to permutation of blocks, and is known as the Kronecker canonical form (KCF) *of* $A + \lambda B$.

We will denote the coefficient matrices of the right singular blocks by A_ε and B_ε, that is, $L_\varepsilon = A_\varepsilon + \lambda B_\varepsilon$, where

$$A_\varepsilon := \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{pmatrix}_{\varepsilon \times (\varepsilon+1)} \quad \text{and} \quad B_\varepsilon := \begin{pmatrix} 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}_{\varepsilon \times (\varepsilon+1)} .$$

We will say that $\mu \in \mathbb{C}$ is an *eigenvalue* of $A + \lambda B$ if there is some block $J_k(-\mu) + \lambda I_k$, with $k > 0$, in the KCF of $A + \lambda B$.

2.2. *Decoupling the equation*

In order to take advantage of the block-diagonal structure of the KCF of $A + \lambda B$, we state the following result, which shows how (1) can be decoupled when A and B are block-diagonal matrices conformally partitioned. We omit the proof, because it is straightforward.

Lemma 2.3: *Let* $\widetilde{A} = A_1 \oplus A_2 \oplus \cdots \oplus A_d$ *and* $\widetilde{B} = B_1 \oplus B_2 \oplus \cdots \oplus B_d$ *be two block-diagonal matrices in* $\mathbb{C}^{m \times n}$. *Let* $X = [X_{ij}]_{i,j=1}^d$ *be partitioned conformally with the partition of A and B. Then the equation* $\widetilde{A}X + \widetilde{B}X^\star = 0$ *is equivalent to the following system of matrix equations:*

(i) d **matrix equations:**

$$A_i X_{ii} + B_i X_{ii}^\star = 0, \qquad \textit{for } i = 1, \ldots, d, \tag{2}$$

together with

(ii) $\frac{d(d-1)}{2}$ **systems of** 2 **matrix equations with** 2 **unknowns:**

$$\begin{array}{l} A_i X_{ij} + B_i X_{ji}^\star = 0 \\ A_j X_{ji} + B_j X_{ij}^\star = 0 \end{array}, \qquad \textit{for } i, j = 1, \ldots, d, \;\; i < j \,. \tag{3}$$

Now, using theorems 2.1 and 2.2, and Lemma 2.3, we have the following procedure to solve equation (1):

`Procedure to solve` $AX + BX^\star = 0$.

Given $\widetilde{A} + \lambda \widetilde{B} = P(A + \lambda B)Q$, the KCF of $A + \lambda B$:

Step 1. Solve (2) and (3) for all blocks $A_i + \lambda B_i, A_j + \lambda B_j$ in the KCF of $A + \lambda B$ with $i < j$. This gives X_{ii}, X_{ij}, and X_{ji} for $i, j = 1, \ldots, d$ and $i < j$.

Step 2. Set $X = [X_{ij}]_{i,j=1}^{d}$, where X_{ij} are the solutions obtained in **Step 1**.
Step 3. Recover the solution of $AX + BX^\star$ by means of the linear transformation $X \mapsto QXQ^\star$, where X is the matrix in **Step 2**.

3. Dimension of the solution space

The solution space of (1), with $\star = T$, is a vector space over $\mathbb{C}$. It is, actually, a vector subspace of $\mathbb{C}^{n\times n}$. By contrast, when we consider $\star = *$ instead, this is not necessarily true. However, the solution space in this case is a vector space over the field of real numbers $\mathbb{R}$.

In the following, we are interested in the dimension of the solution space of (1), instead of the explicit expression for the solution. In theorems 3.1 and 3.2, we give a closed formula for, respectively, the complex dimension of $AX + BX^T = 0$ and the real dimension of $AX + BX^* = 0$. This formula comes from analyzing all the independent equations and systems of equations that arise when decoupling the equation (1) with $A + \lambda B$ in KCF. In the proof of the corresponding results, addressed in Section 4, we give an expression for the solution of all equations and systems of equations, so that the solution of the original equation (1) can be obtained following the procedure described at the end of Section 2.

Theorem 3.1 : (Breakdown of the dimension count for $AX + BX^T = 0$) *Let $A, B \in \mathbb{C}^{m\times n}$, and let the KCF of the pencil $A + \lambda B$ be*

$$\begin{aligned}\widetilde{A} + \lambda\widetilde{B} = {} & L_{\varepsilon_1} \oplus L_{\varepsilon_2} \oplus \cdots \oplus L_{\varepsilon_p} \\ & \oplus L_{\eta_1}^T \oplus L_{\eta_2}^T \oplus \cdots \oplus L_{\eta_q}^T \\ & \oplus N_{u_1} \oplus N_{u_2} \oplus \cdots \oplus N_{u_r} \\ & \oplus J_{k_1}(\mu_1) + \lambda I_{k_1} \oplus J_{k_2}(\mu_2) + \lambda I_{k_2} \oplus \cdots \oplus J_{k_s}(\mu_s) + \lambda I_{k_s}.\end{aligned}$$

Then the dimension of the solution space of the matrix equation

$$AX + BX^T = 0$$

depends only on $\widetilde{A} + \lambda\widetilde{B}$. It can be computed as the sum

$$d_{Total} = d_{right} + d_{fin} + d_{right,right} + d_{fin,fin} + d_{right,left} + d_{right,\infty} + d_{right,fin} + d_{\infty,fin}, \tag{4}$$

whose summands are given by:

1. *The dimension due to equation* (2) *corresponding to the right singular blocks:*

$$d_{right} = \sum_{i=1}^{p} (\varepsilon_i + 1).$$

2. *The dimension due to equation* (2) *corresponding to the finite blocks:*

$$d_{fin} = \sum_i \lfloor k_i/2 \rfloor + \sum_j \lceil k_j/2 \rceil,$$

where the first sum is taken over all blocks in $\widetilde{A} + \lambda\widetilde{B}$ of the form $J_{k_i}(1) + \lambda I_{k_i}$ and the second sum over all blocks of the form $J_{k_j}(-1) + \lambda I_{k_j}$.

3. *The dimension due to the systems of equations* (3) *involving a pair of right singular blocks:*

$$d_{right,right} = \sum_{\substack{i,j=1\\ i<j}}^{p} (\varepsilon_i + \varepsilon_j + 2).$$

4. *The dimension due to the systems of equations* (3) *involving a pair of finite blocks:*

$$d_{fin,fin} = \sum_{i,j} \min\{k_i, k_j\},$$

where the sum is taken over all pairs $J_{k_i}(\mu_i) + \lambda I_{k_i}, J_{k_j}(\mu_j) + \lambda I_{k_j}$ *of blocks in* $\widetilde{A} + \lambda\widetilde{B}$ *such that* $i < j$ *and* $\mu_i\mu_j = 1$.

5. *The dimension due to the systems of equations* (3) *involving a right singular block and a left singular block:*

$$d_{right,left} = \sum_{i,j} (\eta_j - \varepsilon_i - 1),$$

where the sum is taken over all pairs $L_{\varepsilon_i}, L_{\eta_j}^T$ *of blocks in* $\widetilde{A} + \lambda\widetilde{B}$ *such that* $\eta_j - \varepsilon_i > 1$.

6. *The dimension due to the systems of equations* (3) *involving a right singular block and an infinite block:*

$$d_{right,\infty} = p\sum_{i=1}^{r} u_i.$$

7. *The dimension due to the systems of equations* (3) *involving a right singular block and a finite block:*

$$d_{right,fin} = p\sum_{i=1}^{s} k_i.$$

8. *The dimension due to the systems of equations* (3) *involving an infinite block and a finite block:*

$$d_{\infty,fin} = \sum_{i,j} \min\{u_i, k_j\},$$

where the sum is taken over all pairs $N_{u_i}, J_{k_j}(\mu_j) + \lambda I_{k_j}$ *of blocks in* $\widetilde{A} + \lambda\widetilde{B}$ *with* $\mu_j = 0$.

Theorem 3.2 : (Breakdown of the dimension count for $AX + BX^* = 0$) *Let* $A, B \in \mathbb{C}^{m\times n}$ *be two complex matrices, and let the KCF of the pencil* $A + \lambda B$ *be*

$$\begin{aligned}\widetilde{A} + \lambda\widetilde{B} = {} & L_{\varepsilon_1} \oplus L_{\varepsilon_2} \oplus \cdots \oplus L_{\varepsilon_p}\\ & \oplus L_{\eta_1}^T \oplus L_{\eta_2}^T \oplus \cdots \oplus L_{\eta_q}^T\\ & \oplus N_{u_1} \oplus N_{u_2} \oplus \cdots \oplus N_{u_r}\\ & \oplus J_{k_1}(\mu_1) + \lambda I_{k_1} \oplus J_{k_2}(\mu_2) + \lambda I_{k_2} \oplus \cdots \oplus J_{k_s}(\mu_s) + \lambda I_{k_s}.\end{aligned}$$

Then the real dimension of the solution space of the matrix equation

$$AX + BX^* = 0$$

depends only on $\widetilde{A} + \lambda\widetilde{B}$. It can be computed as the sum

$$d^*_{Total} = d^*_{right} + d^*_{fin} + d^*_{right,right} + d^*_{fin,fin} + d^*_{right,left} + d^*_{right,\infty} + d^*_{right,fin} + d^*_{\infty,fin}, \tag{5}$$

whose summands are given by:

1. *The real dimension due to equation* (2) *corresponding to the right singular blocks:*

$$d^*_{right} = 2\sum_{i=1}^{p}(\varepsilon_i + 1).$$

2. *The real dimension due to equation* (2) *corresponding to the finite blocks:*

$$d^*_{fin} = \sum_i k_i,$$

   where the sum is taken over all blocks in $\widetilde{A} + \lambda\widetilde{B}$ of the form $J_{k_i}(\mu) + \lambda I_{k_i}$ with $|\mu| = 1$.
3. *The real dimension due to the systems of equations* (3) *involving a pair of right singular blocks:*

$$d^*_{right,right} = 2\sum_{\substack{i,j=1\\ i<j}}^{p}(\varepsilon_i + \varepsilon_j + 2).$$

4. *The real dimension due to the systems of equations* (3) *involving a pair of finite blocks:*

$$d^*_{fin,fin} = 2\sum_{i,j}\min\{k_i, k_j\},$$

   where the sum is taken over all pairs $J_{k_i}(\mu_i) + \lambda I_{k_i}$, $J_{k_j}(\mu_j) + \lambda I_{k_j}$ of blocks in $\widetilde{A} + \lambda\widetilde{B}$ such that $i < j$ and $\mu_i\overline{\mu}_j = 1$.
5. *The real dimension due to the systems of equations* (3) *involving a right singular block and a left singular block:*

$$d^*_{right,left} = 2\sum_{i,j}(\eta_j - \varepsilon_i - 1),$$

   where the sum is taken over all pairs L_{ε_i}, $L^T_{\eta_j}$ of blocks in $\widetilde{A} + \lambda\widetilde{B}$ such that $\eta_j - \varepsilon_i > 1$.
6. *The real dimension due to the systems of equations* (3) *involving a right singular block and an infinite block:*

$$d^*_{right,\infty} = 2p\sum_{i=1}^{r}u_i.$$

7. *The real dimension due to the systems of equations* (3) *involving a right singular block and a finite block:*

$$d^*_{right,fin} = 2p\sum_{i=1}^{s}k_i.$$

8. *The real dimension due to the systems of equations* (3) *involving an infinite block and a*

finite block:

$$d^{*}_{\infty,fin} = 2\sum_{i,j} \min\{u_i, k_j\},$$

where the sum is taken over all pairs $N_{u_i}, J_{k_j}(\mu_j) + \lambda I_{k_j}$ of blocks in $\widetilde{A} + \lambda\widetilde{B}$ with $\mu_j = 0$.

We want to notice that there are other interactions between canonical blocks in the KCF of $A + \lambda B$ that do not appear in the statement of theorems 3.1 and 3.2. As we will see in Section 4, the dimension of the solution space of the equations and systems of equations corresponding to these blocks is zero, so they do not contribute to the dimension of the solution space of (1). For this reason, we have omitted them in the statements.

It is worth to compare theorems 3.1 and 3.2 with theorems 3 and 4 in [7]. In that case, the dimension of the solution space for both $AX + X^{\star}B = 0$ depends on the KCF of $A + \lambda B^{\star}$, which may be different depending on whether $\star = T$ or $\star = *$. However, the dimension of the solution space of $AX + BX^{\star} = 0$ depends on the KCF of $A + \lambda B$, which is the same canonical form for both $\star = T$ and $\star = *$.

4. Proof of the dimension count. Expression for the solution

In this section, we prove theorems 3.1 and 3.2. For this, we obtain the solution of all the equations and systems of equations obtained when decoupling the equation $\widetilde{A}X + \widetilde{B}X^{\star} = 0$, with $\widetilde{A} + \lambda\widetilde{B}$ being the KCF of $A + \lambda B$. We first consider in Section 4.1 the case $\star = T$ and then, in Section 4.2, the case $\star = *$.

Along this section, we will make use several times of the following result, which is true for both the transpose and the conjugate transpose.

Lemma 4.1: *Let $X \in \mathbb{C}^{n\times n}$ be such that $X + AX^{\star}B = 0$, and $A, B \in \mathbb{C}^{n\times n}$ be such that A and $B^{\star}$ commute and at least one of A and B is nilpotent. Then $X = 0$.*

Proof: Since $X + AX^{\star}B = 0$, we have $X^{\star} = -B^{\star}XA^{\star}$, and replacing this in the original identity we get $X = AB^{\star}XA^{\star}B$. Now, since A and $B^{\star}$ commute and at least one of A and B is nilpotent, we have that $AB^{\star}$ is also nilpotent. Then the result is an immediate consequence of Lemma 4 in [7]. □

4.1. *The transpose case*

Following [7], we will denote the (vector) space of solutions of the equation (1), with $\star = T$, by $\mathcal{S}(A + \lambda B)$, that is

$$\mathcal{S}(A + \lambda B) := \left\{X \in \mathbb{C}^{n\times n} \,:\, AX + BX^T = 0\right\}.$$

Accordingly, we will denote the (vector) space of solutions of the system of equations (3), with $\star = T$, by $\mathcal{S}(A_i + \lambda B_i, A_j + \lambda B_j)$, that is,

$$\mathcal{S}(A_i+\lambda B_i, A_j+\lambda B_j) := \left\{(X_i, X_j) \in \mathbb{C}^{n\times 2n} \,:\, A_iX_i + B_iX_j^T = 0,\ A_jX_j + B_jX_i^T = 0\right\}.$$

For simplicity, we will use the notation X and Y for the unknowns, instead of X_i and X_j.

4.1.1. *Dimension of the solution space for single blocks*

We obtain in this section the dimension of the solution space of all equations (2) with $A_i + \lambda B_i$ being each of the four types of blocks in the KCF of $A + \lambda B$. In the proof of each of the results we also obtain an expression for the solution.

Lemma 4.2: (Right singular block) *The dimension of the solution space of*

$$A_\varepsilon X + B_\varepsilon X^T = 0 \tag{6}$$

is

$$\dim \mathcal{S}(L_\varepsilon) = \varepsilon + 1.$$

Proof: Set $X = [x_{ij}]_{i,j=1,\dots,\varepsilon+1}$. Then (6) is equivalent to

$$\begin{bmatrix} x_{21} & \dots & x_{2,\varepsilon+1} \\ \vdots & & \vdots \\ x_{\varepsilon+1,1} & \dots & x_{\varepsilon+1,\varepsilon+1} \end{bmatrix} + \begin{bmatrix} x_{11} & \dots & x_{\varepsilon+1,1} \\ \vdots & & \vdots \\ x_{1\varepsilon} & \dots & x_{\varepsilon+1,\varepsilon} \end{bmatrix} = 0. \tag{7}$$

This, in particular, implies $x_{ij} + x_{j,i-1} = 0$, for all $i = 2, \dots, \varepsilon + 1$, $j = 1, \dots, \varepsilon + 1$. Iterating this identity we get $x_{ij} = x_{i-1,j-1}$, for all $i, j = 2, \dots, \varepsilon + 1$, which means that X is a Toeplitz matrix, and $x_{j,1} = -x_{1,j-1}$, for all $j = 2, \dots, \varepsilon + 1$. Hence, X is determined by its first row entries $x_{11}, \dots, x_{1,\varepsilon+1}$. On the other hand, since (7) consists of $\varepsilon \cdot (\varepsilon + 1)$ equations and X has $(\varepsilon + 1)^2$ entries, the variables $x_{11}, \dots, x_{1,\varepsilon+1}$ must be free variables. Hence X is of the form:

$$X = \begin{bmatrix} a_1 & a_2 & \dots & a_{\varepsilon+1} \\ -a_1 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ -a_\varepsilon & \dots & -a_1 & a_1 \end{bmatrix},$$

with $a_1, \dots, a_{\varepsilon+1}$ arbitrary, and this is the general solution of (6). Then the result follows. □

Lemma 4.3: (Left singular block) *The dimension of the solution space of*

$$A_\eta^T X + B_\eta^T X^T = 0 \tag{8}$$

is

$$\dim \mathcal{S}(L_\eta^T) = 0.$$

Proof: Multiplying (8) on the left by A_η and using the identities

$$A_\eta A_\eta^T = I_\eta, \qquad A_\eta B_\eta^T = J_\eta(0), \tag{9}$$

we get $X + J_\eta(0)X^T = 0$. Now, Lemma 4.1 implies $X = 0$, and the result follows. □

Lemma 4.4: (Infinite block) *The dimension of the solution space of*

$$X + J_u(0)X^T = 0$$

is

$$\dim\, \mathcal{S}(N_u) = 0.$$

Proof: This is an immediate consequence of Lemma 4.1. □

The next result gives us the dimension of the solution space for single Jordan blocks. The result is Lemma 9 in [7], and we include it here for completeness, though we omit the proof.

Lemma 4.5: **(Finite block)** *The dimension of the solution space of*

$$J_k(\mu)X + X^T = 0$$

is

$$\dim\, \mathcal{S}(J_k(\mu) + \lambda I_k) = \begin{cases} 0\,, & \text{if } \mu \neq \pm 1, \\ \lfloor k/2 \rfloor\,, & \text{if } \mu = 1, \\ \lceil k/2 \rceil\,, & \text{if } \mu = -1. \end{cases}$$

4.1.2. Dimension of the solution space for pairs of blocks

In this section we obtain the dimension of the solution space of all systems of equations (3) with $A_i + \lambda B_i$ and $A_j + \lambda B_j$ being a pair of blocks in the KCF of $A + \lambda B$. In the proof of each of the results we get also an expression for the solution.

Lemma 4.6: **(Two right singular blocks)** *The dimension of the solution space of the system of matrix equations*

$$A_\varepsilon X + B_\varepsilon Y^T = 0 \tag{10}$$

$$A_\delta Y + B_\delta X^T = 0, \tag{11}$$

is

$$\dim\, \mathcal{S}(L_\varepsilon, L_\delta) = \varepsilon + \delta + 2.$$

Proof: Set $X = [x_{ij}]$, with $i = 1, \ldots, \varepsilon + 1; j = 1, \ldots, \delta + 1$ and $Y = [y_{ij}]$, with $i = 1, \ldots, \delta + 1; j = 1, \ldots, \varepsilon + 1$. Then (10) and (11) are equivalent to, respectively,

$$\begin{bmatrix} x_{21} & \cdots & x_{2,\delta+1} \\ \vdots & & \vdots \\ x_{\varepsilon+1,1} & \cdots & x_{\varepsilon+1,\delta+1} \end{bmatrix} + \begin{bmatrix} y_{11} & \cdots & y_{\delta+1,1} \\ \vdots & & \vdots \\ y_{1\varepsilon} & \cdots & y_{\delta+1,\varepsilon} \end{bmatrix} = 0 \tag{12}$$

and

$$\begin{bmatrix} y_{21} & \cdots & y_{2,\varepsilon+1} \\ \vdots & & \vdots \\ y_{\delta+1,1} & \cdots & y_{\delta+1,\varepsilon+1} \end{bmatrix} + \begin{bmatrix} x_{11} & \cdots & x_{\varepsilon+1,1} \\ \vdots & & \vdots \\ x_{1\delta} & \cdots & x_{\varepsilon+1,\delta} \end{bmatrix} = 0. \tag{13}$$

We see, from (12), that each entry y_{ij}, with $i = 1, \ldots, \delta + 1$ and $j = 1, \ldots, \varepsilon$, depends on an entry of X, and also, from (13), that each entry $y_{i,\varepsilon+1}$, with $i = 2, \ldots, \delta + 1$, depends on an entry of X. Hence, Y is completely determined by X, except for $y_{1,\varepsilon+1}$.

Now, (12) and (13) are equivalent to

$$x_{j+1,i} = -y_{ij}, \qquad \text{for } i = 1, \dots, \delta+1;\ j = 1, \dots, \varepsilon, \tag{14}$$

and

$$x_{ij} = -y_{j+1,i}, \qquad \text{for } i = 1, \dots, \varepsilon+1;\ j = 1, \dots, \delta. \tag{15}$$

Let $1 \le i \le \varepsilon$ and $1 \le j \le \delta$. Then, by (15), $x_{ij} = -y_{j+1,i}$ and, by (14), $-y_{j+1,i} = x_{i+1,j+1}$, so we conclude that $x_{ij} = x_{i+1,j+1}$. Hence, X is a Toeplitz matrix. Then X is of the form:

$$X = \begin{bmatrix} a_1 & a_2 & \dots & a_{\delta+1} \\ b_1 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ b_\varepsilon & \dots & b_1 & a_1 \end{bmatrix},$$

for arbitrary complex numbers $a_1, \dots, a_{\delta+1}$ and $b_1, \dots, b_\varepsilon$. Also, Y is of the form

$$Y = \begin{bmatrix} -b_1 & -b_2 & \dots & -b_\varepsilon & d \\ -a_1 & -b_1 & -b_2 & \dots & -b_\varepsilon \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -a_{\delta-1} & \dots & -a_1 & -b_1 & -b_2 \\ -a_\delta & \dots & -a_2 & -a_1 & -b_1 \end{bmatrix},$$

with $d \in \mathbb{C}$ arbitrary. Now, it is straightforward to check that every X and Y of the above form give a solution of the system (10)–(11), so this is the general solution of this system of equations. In particular, the dimension of the solution space is $\varepsilon + \delta + 2$, as claimed. □

Lemma 4.7: (Two left singular blocks) *The dimension of the solution space of the system of matrix equations*

$$A_\eta^T X + B_\eta^T Y^T = 0 \tag{16}$$

$$A_\gamma^T Y + B_\gamma^T X^T = 0, \tag{17}$$

is

$$\dim \mathcal{S}(L_\eta^T, L_\gamma^T) = 0.$$

Proof: Multiplying (16) on the left by A_η and (17) by A_γ, and using (9) we get

$$X + J_\eta(0) Y^T = 0 \tag{18}$$

and

$$Y + J_\gamma(0) X^T = 0. \tag{19}$$

Now, from (18) we have $X^T = -Y J_\eta(0)^T$ and, replacing this expression in (19) we obtain $Y = J_\gamma(0) Y J_\eta(0)^T$. From Lemma 4 in [7] we conclude $Y = 0$, hence $X = 0$, and the result follows. □

Lemma 4.8: (Two infinite blocks) *The dimension of the solution space of the system of matrix equations*

$$X + J_u(0)Y^T = 0 \tag{20}$$
$$Y + J_t(0)X^T = 0, \tag{21}$$

is

$$\dim \mathcal{S}(N_u, N_t) = 0.$$

Proof: Note that (20) and (21) are the same (with different sizes) as (18) and (19), respectively. Hence the result follows by using the same arguments as in the proof of Lemma 4.7. □

Next result is the same as Lemma 13 in [7], so we omit the proof.

Lemma 4.9: (Two finite blocks) *The dimension of the solution space of the system of matrix equations*

$$J_k(\mu)X + Y^T = 0$$
$$J_\ell(\nu)Y + X^T = 0,$$

is

$$\dim \mathcal{S}(J_k(\mu) + \lambda I_k, J_\ell(\nu) + \lambda I_\ell) = \begin{cases} \min\{k,\ell\}, & \textit{if } \mu\nu = 1, \\ 0, & \textit{otherwise.} \end{cases}$$

Lemma 4.10: (Right singular and left singular blocks) *The dimension of the solution space of the system of matrix equations*

$$A_\varepsilon X + B_\varepsilon Y^T = 0 \tag{22}$$
$$A_\eta^T Y + B_\eta^T X^T = 0, \tag{23}$$

is

$$\dim \mathcal{S}(L_\varepsilon, L_\eta^T) = \begin{cases} 0, & \textit{if } \eta - \varepsilon \leq 1, \\ \eta - \varepsilon - 1, & \textit{if } \eta - \varepsilon > 1. \end{cases}$$

Proof: Setting $X = [x_{ij}]$, for $i = 1, \ldots, \varepsilon + 1, j = 1, \ldots, \eta$, and $Y = [y_{ij}]$, for $i = 1, \ldots, \eta, j = 1, \ldots, \varepsilon + 1$, equations (22) and (23) are equivalent to

$$\begin{bmatrix} x_{21} & \ldots & x_{2\eta} \\ \vdots & & \vdots \\ x_{\varepsilon+1,1} & \ldots & x_{\varepsilon+1,\eta} \end{bmatrix} + \begin{bmatrix} y_{11} & \ldots & y_{\eta 1} \\ \vdots & & \vdots \\ y_{1\varepsilon} & \ldots & y_{\eta\varepsilon} \end{bmatrix} = 0$$

and

$$\begin{bmatrix} 0 & \ldots & 0 \\ y_{11} & \ldots & y_{1,\varepsilon+1} \\ \vdots & & \vdots \\ y_{\eta 1} & \ldots & y_{\eta,\varepsilon+1} \end{bmatrix} + \begin{bmatrix} x_{11} & \ldots & x_{\varepsilon+1,1} \\ \vdots & & \vdots \\ x_{1\eta} & \ldots & x_{\varepsilon+1,\eta} \\ 0 & \ldots & 0 \end{bmatrix} = 0,$$

which are, in turn, equivalent to

$$
\begin{array}{ll}
x_{i+1,j} = -y_{ji}, & \text{for } i = 1, \ldots, \varepsilon; j = 1, \ldots, \eta \\
x_{11} = \cdots = x_{\varepsilon+1,1} = 0, & \\
y_{\eta 1} = \cdots = y_{\eta,\varepsilon+1} = 0, & \\
x_{i,j+1} = -y_{ji}, & \text{for } i = 1, \ldots, \varepsilon+1; j = 1, \ldots, \eta - 1.
\end{array} \tag{24}
$$

From these equations we get

$$
\begin{array}{ll}
x_{i+1,j} = x_{i,j+1}, & \text{for } i = 1, \ldots, \varepsilon; j = 1, \ldots, \eta - 1 \\
x_{11} = \cdots = x_{\varepsilon+1,1} = 0 & \\
x_{2\eta} = -y_{\eta,1} = 0, \ldots, x_{\varepsilon+1,\eta} = -y_{\eta,\varepsilon} = 0. &
\end{array}
$$

Hence, X is of the form

$$
X = \begin{bmatrix} 0 \ldots \ 0 & c_1 & \ldots \ c_{\eta-\varepsilon-1} \\ \vdots \ \iddots \ \iddots & & \iddots \\ 0 \ c_1 \ \ldots \ c_{\eta-\varepsilon-1} & 0 \ldots & 0 \end{bmatrix}. \tag{25}
$$

Then, if $\eta - \varepsilon - 1 \leq 0$, we have $X = 0$, and also $Y = 0$. Otherwise, X depends on $\eta - \varepsilon - 1$ parameters and it is of the form (25). We note that, by (22) and (23), Y is completely determined by X.

It remains to show that (25) is the general solution for X in (22) and (23) when $\eta > \varepsilon + 1$. For this, take an X of the form (25), with $c_1, \ldots, c_{\eta-\varepsilon-1}$ arbitrary, and set Y defined by (24), that is, $y_{ji} = -x_{i+1,j}$, for $i = 1, \ldots, \varepsilon, j = 1, \ldots, \eta$, and $y_{1,\varepsilon+1} = -x_{\varepsilon+1,2}, \ldots, y_{\eta-1,\varepsilon+1} = -x_{\varepsilon+1,\eta}, y_{\eta,\varepsilon+1} = 0$, namely

$$
Y = \begin{bmatrix} 0 & \ldots \ 0 & -c_1 \\ \vdots & \iddots \ \iddots & \vdots \\ 0 & \iddots & -c_{\eta-\varepsilon-1} \\ -c_1 & \iddots & 0 \\ \vdots & \iddots \ \iddots & \vdots \\ -c_{\eta-\varepsilon-1} & 0 \ \ldots & 0 \\ 0 & 0 \ \ldots & 0 \end{bmatrix}. \tag{26}
$$

It is straightforward to check that (X, Y), with X as in (25) and Y as in (26), is a solution of (22) and (23). Then this is the general solution of the system and the result follows. □

Lemma 4.11: (Right singular and infinite blocks) *The dimension of the solution space of the system of matrix equations*

$$
A_\varepsilon X + B_\varepsilon Y^T = 0 \tag{27}
$$

$$
Y + J_u(0) X^T = 0, \tag{28}
$$

is

$$
\dim \mathcal{S}(L_\varepsilon, N_u) = u.
$$

Proof: By (28) we have $Y^T = -X J_u(0)^T$ and, replacing this expression in (27) we get $A_\varepsilon X = B_\varepsilon X J_u(0)^T$. By setting $X = [x_{ij}]$, for $i = 1, \ldots, \varepsilon + 1, j = 1, \ldots, u$, this last equation can be written in coordinates as

$$\begin{bmatrix} x_{21} & \dots & x_{2u} \\ \vdots & & \vdots \\ x_{\varepsilon+1,1} & \dots & x_{\varepsilon+1,u} \end{bmatrix} = \begin{bmatrix} x_{12} & \dots & x_{1u} & 0 \\ \vdots & & \vdots & \vdots \\ x_{\varepsilon 2} & \dots & x_{\varepsilon u} & 0 \end{bmatrix},$$

which is equivalent to the system

$$\begin{array}{ll} x_{2u} = \cdots = x_{\varepsilon+1,u} = 0, & \\ x_{ij} = x_{i-1,j+1}, & \text{for } i = 2, \dots, \varepsilon+1; j = 1, \dots, u-1. \end{array}$$

Hence, X is of the form

$$X = \begin{bmatrix} c_1 & \dots & c_u \\ \vdots & \iddots & \\ c_u & & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}, \qquad \text{if } \varepsilon+1 \geq u, \tag{29}$$

or

$$X = \begin{bmatrix} c_1 & \dots & c_{\varepsilon+1} & \dots & c_u \\ \vdots & \iddots & & \iddots & \\ c_{\varepsilon+1} & \dots & c_u & & 0 \end{bmatrix}, \qquad \text{if } \varepsilon+1 < u. \tag{30}$$

It is straightforward to check that if X is of the form (29) or (30) and $Y = -J_u(0)X^T$ then (X, Y) is a solution of (27) and (28), so (X, Y), with X of the form (29) or (30) and $Y = -J_u(0)X^T$, is the general solution of (27) and (28). As a a consequence, the result follows. □

Lemma 4.12: (Right singular and finite blocks) *The dimension of the solution space of the system of matrix equations*

$$A_\varepsilon X + B_\varepsilon Y^T = 0 \tag{31}$$

$$J_k(\mu)Y + X^T = 0, \tag{32}$$

is

$$\dim \mathcal{S}(L_\varepsilon, J_k(\mu) + \lambda I_k) = k.$$

Proof: By (32) we have $X = -Y^T J_k(\mu)^T$. Replacing this in (31) we conclude that the system (31)-(32) is equivalent to

$$X = -Y^T J_k(\mu)^T, \tag{33}$$

$$-J_k(\mu) Y A_\varepsilon^T + Y B_\varepsilon^T = 0. \tag{34}$$

On the other hand, with similar arguments, the system (50)–(51) in [7] is equivalent to

$$Y = -X^T A_\varepsilon^T, \tag{35}$$

$$-J_k(\mu) X^T A_\varepsilon^T + X^T B_\varepsilon^T = 0. \tag{36}$$

We want to point out, however, that Y in (35) is $k \times \varepsilon$, whereas in (33)–(34) it is $k \times (\varepsilon+1)$ (though $X \in \mathbb{C}^{(\varepsilon+1)\times k}$ in both cases). Since the only restriction on Y within (33)–(34) is given by (34), and similarly for X^T in (35) and (36), we conclude that Y is equal to X^T of Lemma 16 in [7]. Hence, $Y = [x_{ji}]$, with x_{ij} given by (55) in [7]. From this expression for Y we immediately get X using (33). In particular, the dimension of the solution space of (31)–(32) is as claimed in the statement. □

Lemma 4.13: **(Left singular and infinite blocks)** *The dimension of the solution space of the system of matrix equations*

$$A_\eta^T X + B_\eta^T Y^T = 0 \tag{37}$$

$$Y + J_u(0) X^T = 0, \tag{38}$$

is

$$\dim \mathcal{S}(L_\eta^T, N_u) = 0.$$

Proof: By (38) we have $Y^T = -X J_u(0)^T$. Replacing this expression for Y^T in (37), multiplying this equation on the left by A_η, and using (9) we get

$$X = J_\eta(0) X J_u(0)^T.$$

Since $J_\eta(0)$ and $J_u(0)^T$ are nilpotent, Lemma 4 in [7] implies $X = 0$, and this in turn implies $Y = 0$. □

Lemma 4.14: **(Left singular and finite blocks)** *The dimension of the solution space of the system of matrix equations*

$$A_\eta^T X + B_\eta^T Y^T = 0 \tag{39}$$

$$J_k(\mu) Y + X^T = 0, \tag{40}$$

is

$$\dim \mathcal{S}(L_\eta^T, J_k(\mu) + \lambda I_k) = 0.$$

Proof: By (40) we have $X = -Y^T J_k(\mu)^T$. Replacing this expression for X in (39), multiplying this equation on the left by B_η, and using the identities

$$B_\eta A_\eta^T = J_\eta(0)^T, \qquad B_\eta B_\eta^T = I_\eta,$$

we get

$$Y^T = J_\eta(0)^T Y^T J_k(\mu)^T.$$

Since $J_\eta(0)^T$ is nilpotent, Lemma 4 in [7] implies $Y^T = 0$, which in turn implies $Y = 0$ and $X = 0$. □

Lemma 4.15: (Infinite and finite blocks) *The dimension of the solution space of the system of matrix equations*

$$X + J_u(0)Y^T = 0 \quad (41)$$

$$J_k(\mu)Y + X^T = 0, \quad (42)$$

is

$$\dim \mathcal{S}(N_u, J_k(\mu) + \lambda I_k) = \begin{cases} \min\{u,k\}\,, & \text{if } \mu = 0, \\ 0\,, & \text{if } \mu \neq 0. \end{cases}$$

Proof: From (42), $X = -Y^T J_k(\mu)^T$ and, replacing this expression in (41) we get $J_u(0)Y^T - Y^T J_k(\mu)^T = 0$. This is a Sylvester equation, whose solution is [9, Ch. VIII, §1]:

- If $\mu \neq 0$, $Y^T = 0$, so $X = Y = 0$.
- If $\mu = 0$, then $Y^T = \Delta R^{-1}$, where Δ is an arbitrary *regular* upper triangular matrix (see [9, p. 218]) and R is an invertible matrix such that $R^{-1}J_k(0)^T R$ is in Jordan canonical form. Hence, $Y = R^{-T}\Delta^T$, and $X = -J_u(0)Y^T$.

The result now follows by counting the number of free variables in Y. □

4.2. *The conjugate transpose case*

In this subsection, and according to the notation used before, we will denote the vector space of solutions of the equation (1), with $\star = *$, by $\mathcal{S}^*(A + \lambda B)$, that is

$$\mathcal{S}^*(A + \lambda B) := \left\{ X \in \mathbb{C}^{n\times n} \,:\, AX + BX^* = 0 \right\}.$$

Accordingly, we will denote the (vector) space of solutions of the system of equations (3), with $\star = *$, by $\mathcal{S}^*(A_i + \lambda B_i, A_j + \lambda B_j)$, that is,

$$\mathcal{S}^*(A_i{+}\lambda B_i, A_j{+}\lambda B_j) := \left\{ (X_i, X_j) \in \mathbb{C}^{n\times 2n} \,:\, A_iX_i + B_iX_j^* = 0,\ A_jX_j + B_jX_i^* = 0 \right\}.$$

4.2.1. Dimension of the solution space for single blocks

This is the counterpart of Section 4.1.1 for equation (1) with $\star = *$. We solve all equations (2) with $\star = *$ obtained when replacing $A_i + \lambda B_i$ with a canonical block in the KCF of $A + \lambda B$.

Lemma 4.16: (Right singular block) *The real dimension of the solution space of*

$$A_\varepsilon X + B_\varepsilon X^* = 0 \quad (43)$$

is

$$\dim_{\mathbb{R}} \mathcal{S}^*(L_\varepsilon) = 2(\varepsilon + 1).$$

Proof: With similar arguments to the ones in the proof of Lemma 4.2, we can see that the solution, X, of (43) is also a Toeplitz matrix. Moreover, X is of the form:

$$X = \begin{bmatrix} a_1 & a_2 & \dots & a_{\varepsilon+1} \\ -\overline{a}_1 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ -\overline{a}_\varepsilon & \dots & -\overline{a}_1 & a_1 \end{bmatrix},$$

with $a_1, \dots, a_{\varepsilon+1}$ arbitrary, and this is the general solution of (43). Since X depends on $\varepsilon + 1$ complex variables, the result follows. □

Lemma 4.17: (Left singular block) *The real dimension of the solution space of*

$$A_\eta^T X + B_\eta^T X^* = 0$$

is

$$\dim_{\mathbb{R}} \mathcal{S}^*(L_\eta^T) = 0.$$

Proof: The proof is similar to the one of Lemma 4.3. Notice that (9) are still true if we replace T by $*$. □

Lemma 4.18: (Infinite block) *The real dimension of the solution space of*

$$X + J_u(0)X^* = 0$$

is

$$\dim_{\mathbb{R}} S^*(N_u) = 0.$$

Proof: This is an immediate consequence of Lemma 4.1. □

The following result is the same as Lemma 24 in [7], so we include here, for the sake of completeness, without proof.

Lemma 4.19: (Finite block) *The real dimension of the solution space of*

$$J_k(\mu)X + X^* = 0$$

is

$$\dim_{\mathbb{R}} \mathcal{S}^*(J_k(\mu) + \lambda I_k) = \begin{cases} k, & \textit{if } |\mu| = 1, \\ 0, & \textit{otherwise.} \end{cases}$$

4.2.2. Dimension of the solution space for pairs of blocks

This subsection is the counterpart of Subsection 4.1.2 for the system of equations (3) with $\star = *$, that is, we solve all systems of equations (3) with $A_i + \lambda B_i$ and $A_j + \lambda B_j$ being a pair of canonical blocks in the KCF of $A + \lambda B$.

Lemma 4.20: (Two right singular blocks) *The real dimension of the solution space of the system of matrix equations*

$$A_\varepsilon X + B_\varepsilon Y^* = 0 \tag{44}$$

$$A_\delta Y + B_\delta X^* = 0, \tag{45}$$

is

$$\dim_{\mathbb{R}} \mathcal{S}^*(L_\varepsilon, L_\delta) = 2(\varepsilon + \delta + 2).$$

Proof : Similar arguments to the ones in the proof of Lemma 4.6 give

$$X = \begin{bmatrix} a_1 & a_2 & \dots & a_{\delta+1} \\ b_1 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ b_\varepsilon & \dots & b_1 & a_1 \end{bmatrix}, \qquad Y = \begin{bmatrix} -\overline{b}_1 & -\overline{b}_2 & \dots & -\overline{b}_\varepsilon & d \\ -\overline{a}_1 & -\overline{b}_1 & -\overline{b}_2 & \dots & -\overline{b}_\varepsilon \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -\overline{a}_{\delta-1} & \dots & -\overline{a}_1 & -\overline{b}_1 & -\overline{b}_2 \\ -\overline{a}_\delta & \dots & -\overline{a}_2 & -\overline{a}_1 & -\overline{b}_1 \end{bmatrix},$$

and (X, Y), with X, Y as above, is the general solution of the system (44)–(45). In particular, $\dim_{\mathbb{R}} \mathcal{S}^*(L_\varepsilon, L_\delta) = 2(\varepsilon + \delta + 2)$, as claimed. □

The following two results are straightforward, using similar arguments to the ones in the proof of Lemma 4.7.

Lemma 4.21 : (Two left singular blocks) *The real dimension of the solution space of the system of matrix equations*

$$A_\eta^T X + B_\eta^T Y^* = 0$$
$$A_\gamma^T Y + B_\gamma^T X^* = 0,$$

is

$$\dim_{\mathbb{R}} S^*(L_\eta^T, L_\gamma^T) = 0.$$

Lemma 4.22 : (Two infinite blocks) *The real dimension of the solution space of the system of matrix equations*

$$X + J_u(0) Y^* = 0$$
$$Y + J_t(0) X^* = 0,$$

is

$$\dim_{\mathbb{R}} S^*(N_u, N_t) = 0$$

The result corresponding to a pair of Jordan blocks is exactly the same as Lemma 28 in [7], so we omit the proof.

Lemma 4.23 : (Two finite blocks) *The real dimension of the solution space of the system of matrix equations*

$$J_k(\mu) X + Y^* = 0$$
$$J_\ell(\nu) Y + X^* = 0,$$

is

$$\dim_{\mathbb{R}} S^*(J_k(\mu) + \lambda I_k, J_\ell(\nu) + \lambda I_\ell) = \begin{cases} 2\min\{k, \ell\}, & \text{if } \mu\overline{\nu} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4.24: (Right singular and left singular blocks) *The real dimension of the solution space of the system of matrix equations*

$$A_\varepsilon X + B_\varepsilon Y^* = 0 \tag{46}$$
$$A_\eta^T Y + B_\eta^T X^* = 0, \tag{47}$$

is

$$\dim_{\mathbb{R}} S^*(L_\varepsilon, L_\eta^T) = \begin{cases} 0, & \text{if } \eta - \varepsilon \leq 1, \\ 2(\eta - \varepsilon - 1), & \text{if } \eta - \varepsilon > 1. \end{cases}$$

Proof: With similar arguments to the ones in the proof of Lemma 4.10, we get

$$X = \begin{bmatrix} 0 \dots 0 & c_1 & \dots & c_{\eta-\varepsilon-1} \\ \vdots \; \cdot^{\cdot^{\cdot}} \; \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 \; c_1 \; \dots c_{\eta-\varepsilon-1} & 0 \dots & & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & \dots \; 0 & -\overline{c}_1 \\ \vdots & \cdot^{\cdot^{\cdot}} \; \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & \cdot^{\cdot^{\cdot}} & -\overline{c}_{\eta-\varepsilon-1} \\ -\overline{c}_1 & \cdot^{\cdot^{\cdot}} & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} \; \cdot^{\cdot^{\cdot}} & \vdots \\ -\overline{c}_{\eta-\varepsilon-1} & 0 \; \dots & 0 \\ 0 & 0 \; \dots & 0 \end{bmatrix}.$$

Again, a direct computation shows that (X, Y), with X, Y as above, is a solution of the system (46)-(47), so this is the general solution of this system. Hence, the result follows by counting the number of (real) free variables in X. □

Lemma 4.25: (Right singular and infinite blocks) *The real dimension of the solution space of the system of matrix equations*

$$A_\varepsilon X + B_\varepsilon Y^* = 0 \tag{48}$$
$$Y + J_u(0) X^* = 0, \tag{49}$$

is

$$\dim_{\mathbb{R}} S^*(L_\varepsilon, N_u) = 2u.$$

Proof: We, have, as in the proof of Lemma 4.11, that X is of the form (29) or (30), and $Y = -J_u(0)X^*$. Again, a direct computation shows that (X, Y), with X and Y of this form, is a solution of the system (48)–(49), so this is the general solution of the system. The result follows by counting the number of (real) free variables in X. □

Lemma 4.26: (Right singular and finite blocks) *The real dimension of the solution space of the system of matrix equations*

$$A_\varepsilon X + B_\varepsilon Y^* = 0 \tag{50}$$
$$J_k(\mu) Y + X^* = 0, \tag{51}$$

is

$$\dim_{\mathbb{R}} S^*(L_\varepsilon, J_k(\mu) + \lambda I_k) = 2k.$$

Proof: Reasoning as in the proof of Lemma 4.12, we conclude that the solution Y of the system (50)–(51) coincides with X^*, where X is the solution of the system (77)–(78) in [7]. Hence, $Y = [\overline{x}_{ji}]$, with x_{ij} given by (79) in [7]. From this, (51) gives $X = -Y^* J_k(\mu)^*$, so we have the general solution of the system (50)–(51). Now the result follows by counting the number of (real) free variables in Y. □

We omit the proof of the following two results, because they are similar to the ones of Lemma 4.13 and Lemma 4.14, respectively.

Lemma 4.27: (Left singular and infinite blocks) *The real dimension of the solution space of the system of matrix equations*

$$\begin{array}{l} A_\eta^T X + B_\eta^T Y^* = 0 \\ Y + J_u(0) X^* = 0, \end{array}$$

is

$$\dim_{\mathbb{R}} \mathcal{S}^*(L_\eta^T, N_u) = 0.$$

Lemma 4.28: (Left singular and finite blocks) *The real dimension of the solution space of the system of matrix equations*

$$\begin{array}{l} A_\eta^T X + B_\eta^T Y^* = 0 \\ J_k(\mu) Y + X^* = 0, \end{array}$$

is

$$\dim_{\mathbb{R}} \mathcal{S}^*(L_\eta^T, J_k(\mu) + \lambda I_k) = 0.$$

Lemma 4.29: (Infinite and finite blocks) *The real dimension of the solution space of the system of matrix equations*

$$\begin{array}{l} X + J_u(0) Y^* = 0 \\ J_k(\mu) Y + X^* = 0, \end{array}$$

is

$$\dim_{\mathbb{R}} \mathcal{S}^*(N_u, J_k(\mu) + \lambda I_k) = \begin{cases} 2\min\{u,k\}\,, & \text{if } \mu = 0, \\ 0\,, & \text{if } \mu \neq 0. \end{cases}$$

Proof: With similar arguments to the ones in the proof of Lemma 4.15, we get:

- If $\mu \neq 0$, then $X = Y = 0$.
- If $\mu = 0$, then $Y = R^{-1}\Delta$ and $X = -J_u(0)Y^*$, where Δ is an arbitrary *regular* upper triangular matrix (see [9, p. 218]) and R is an invertible matrix such that $R^{-1}J_k(0)^T R$ is in Jordan canonical form.

The result follows again by counting the number of (real) free variables in Y. □

5. Uniqueness of solution

In the following two results we give necessary and sufficient conditions for equation (1) to have a unique solution. We first consider in Theorem 5.1 the transpose case, and then in Theorem 5.2 the conjugate transpose case. The proof of the second result is similar to the proof of the first one, and we omit it.

Theorem 5.1 : *Let $A, B \in \mathbb{C}^{m\times n}$. Then the matrix equation $AX + BX^T = 0$ has only the trivial solution, $X = 0$, if and only if the following three conditions hold:*

(a) *The KCF of the matrix pencil $A + \lambda B$ has no right singular blocks.*
(b) *If $\mu \in (\mathbb{C} \setminus \{-1\}) \cup \{\infty\}$ is an eigenvalue of $A + \lambda B$ then $1/\mu$ is not an eigenvalue of $A + \lambda B$.*
(c) *The algebraic multiplicity of the eigenvalue $\mu = -1$ in $A + \lambda B$ is at most one.*

Notice, in particular, that it must be $m \geq n$, and that $\mu = 1$ can not be an eigenvalue of $A + \lambda B$.

Proof : The equation $AX + BX^T = 0$ has only the trivial solution if and only if the dimension of the solution space is zero. Looking at Theorem 3.1, this happens if and only if conditions (a–c) in the statement hold. □

Theorem 5.2 : *Let $A, B \in \mathbb{C}^{m\times n}$. Then the matrix equation $AX + BX^* = 0$ has only the trivial solution, if and only if the following two conditions hold:*

(a) *The KCF of the matrix pencil $A + \lambda B$ has no right singular blocks.*
(b) *If $\mu \in \mathbb{C} \cup \{\infty\}$ is an eigenvalue of $A + \lambda B$ then $1/\overline{\mu}$ is not an eigenvalue of $A + \lambda B$.*

Note that, in particular, it must be $m \geq n$, and that $A + \lambda B$ can not contain eigenvalues on the unit circle.

We want to point out that the equation $AX + BX^\star = 0$ may have a unique solution with $A+\lambda B$ being singular. Though, by part (a) in both theorems 5.1 and 5.2, the KCF of $A+\lambda B$ cannot contain right singular blocks, it may contain left singular blocks. Consider, for instance, the equation $AX + BX^\star = 0$, with

$$A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The equation has only the trivial solution $X = 0$, though the pencil $A + \lambda B$ is singular (it consists of just one left singular block L_1^T). However, the operator $X \mapsto AX + BX^\star$ is not invertible. Actually, this operator maps $\mathbb{C}$ to $\mathbb{C}^{2\times 1}$. In order for this operator to be invertible, we need the dimension of both the original and the final vector spaces to be the same. In general, this happens if and only if $m = n$. Then, part (a) in theorems 5.1 and 5.2 implies that $A + \lambda B$ is regular. This leads to the following two results.

Theorem 5.3 : *Let $A, B \in \mathbb{C}^{m\times n}$. Then the matrix equation $AX + BX^T = C$ has a unique solution, for every right-hand side matrix $C \in \mathbb{C}^{m\times n}$, if and only if the following three conditions hold:*

(a) *The matrix pencil $A + \lambda B$ is regular.*
(b) *If $\mu \in (\mathbb{C} \setminus \{-1\}) \cup \{\infty\}$ is an eigenvalue of $A + \lambda B$ then $1/\mu$ is not an eigenvalue of $A + \lambda B$.*
(c) *The algebraic multiplicity of the eigenvalue $\mu = -1$ in $A + \lambda B$ is at most one.*

Theorem 5.4 : *Let $A, B \in \mathbb{C}^{m\times n}$. Then the matrix equation $AX + BX^* = C$ has a unique solution, for every right-hand side matrix $C \in \mathbb{C}^{m\times n}$, if and only if the following two conditions hold:*

(a) *The matrix pencil $A + \lambda B$ is regular.*
(b) *If $\mu \in \mathbb{C} \cup \{\infty\}$ is an eigenvalue of $A + \lambda B$ then $1/\overline{\mu}$ is not an eigenvalue of $A + \lambda B$.*

6. Conclusions and open problems

We have presented a procedure to get an explicit solution of the equation $AX + BX^\star = 0$, with $A, B \in \mathbb{C}^{m\times n}$. We have given an explicit formula for the dimension of the solution space in terms of the Kronecker canonical form of the matrix pencil $A + \lambda B$, and also an explicit description of the solution in terms of the Kronecker canonical form and of two nonsingular matrices leading $A + \lambda B$ to this form. It remains as an open problem to obtain necessary and sufficient conditions for the existence of solutions for the non-homogeneous equation $AX + BX^\star = C$, with $C \in \mathbb{C}^{m\times n}$. Another related problem is to address the same questions for the solution of the more general equations $AXB + CX^\star D = 0$ and $AXB + CX^\star D = E$. This last equation has been considered within the last few years by several authors in different contexts, but providing a formula for the general solution of the equation remains as a challenging open problem.

Acknowledgments

The author wishes to thank an anonymous referee for a careful reading of the paper and several insightful comments. This work was partially supported by the Ministerio de Ciencia e Innovación of Spain through grant MTM-2009-09281.

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