Interrogating the Void: The Difficulty of Extracting Information From Many-Body Systems

by

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Abstract

In this thesis, I will explore some of the ways the information-theoretic properties of quantum many-body systems can be analyzed. I do this in two different settings. First, I will describe an approach to the "scrambling time problem," a conjecture of Susskind and Sekino that asserts that black holes can thermalize the information of objects that are dropped into them at the fastest rate consistent with unitarity. Specifically, I will analyze the dynamics of the Iizuka-Polchinksi model, a matrix model of a black hole whose response functions can be calculated exactly. Second, I will study the average information content of subsystems of a larger system. In particular, I will improve a result of Page giving the average entanglement entropy of such a subsystem in the ensemble of random, Haar-distributed states by refining it to a smaller, more physically relevant ensemble of states known as "matrix product states," which encode a notion of locality. In both these examples, fundamental obstacles arise that impede our analysis; I explain how these roadblocks are related to the difficulty of understanding the interactions between the exponentially large number the degrees of freedom such many-body systems contain.

Thesis Supervisor: John McGreevy
Title: Assistant Professor

Acknowledgments

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Contents

Chapter 1

Introduction

Characterizing the collective behavior of large numbers of interacting particles is an evergreen problem in physics. Even if one only considers systems where each particle only feels the influence of others close to it, there is tremendous richness in the macroscopic properties such systems can exhibit. Indeed, "studying many-body systems with short-range interactions" is a fairly accurate, albeit gross, characterization of the entire corpus of condensed-matter physics prior to the 1970's or so. This apparent myopia was not accidental - for decades, the problem of modeling many-body systems exhibiting long-range interactions was simply intractable. Nevertheless, many examples of such systems exist in nature, and recent advances have provided fruitful avenues for analyzing them. By exploring a few of these methods, this thesis will attempt to elucidate why these systems have proven so difficult to describe and analyze. The perspective this thesis will take is information-theoretic: by trying to calculate things like entanglement entropies and correlation functions, we will attempt to quantify "where" the information in the state of a strongly-interacting many-body system resides, how it can be extracted, and what it ultimately means. To this end, we will study two vastly different examples:

First, we will examine some aspects of black hole thermodynamics. Though a precise description of the internal quantum degrees of freedom of a black hole are unknown, it is suspected that the time evolution operator for a black hole can be accurately modeled as a random unitary operator. This is exciting, since recent dis-

coveries of Hayden, Preskill, Susskind, and Sekino show that such random unitary operators can thermalize information quickly. That is, they can quickly spread out the information contained in a pure quantum state over the system's internal degrees of freedom in such a way that any small subsystem is maximally entangled with the rest of the system. In fact, it is conjectured by Susskind and Sekino that certain quantum systems with black hole duals, known as "matrix models", can thermalize information at the fastest rate consistent with unitarity. By examining some approaches to tackling this conjecture, which the authors call the "scrambling time" problem, we will learn a great deal about how difficulties in analyzing many-body systems arise.

Second, we will consider the entanglement entropy of subsystems of larger systems in the matrix product state (MPS) formalism. It has long been known that most of the enormous Hilbert space of possible states of a many-body system is physically inaccessible in the sense that any "reasonable" Hamiltonian that could govern such a system's dynamics would take astronomical amounts of time to evolve any fiducial state of the system (e.g. the ground state) into those parts of the Hilbert space. The matrix product state formalism allows one to describe the states that are achievable in this sense in an elegant, computationally tractable way. Here, we will attempt to refine a result due to Page describing the average entanglement entropy of a subsystem of a bipartite system to the ensemble of matrix product states. In doing so, we will uncover more ways in which many-body systems can conceal information about their state.

Chapter 2

The Scrambling Time Problem

The scrambling time problem is to saturate a known bound due to Sekino and Susskind [13] regarding how quickly a quantum system can entangle the information content of objects that interact with it with its internal degrees of freedom. The reason why this problem is interesting is because we expect this extremal rate can be realized by black holes. This leads to a strange implication - if we assume (rightly) that black holes don't destroy information despite their appearances, we can deduce the precise quantum state of objects falling into the black hole almost immediately from the black hole's Hawking radiation provided we have enough information about the black hole itself. That is, if the conjecture is true, black holes aren't "black" at all! Furthermore, we expect that no other physical system in nature can entangle data so quickly. In this section, we will motivate these claims, starting from a general overview of black hole thermodynamics in the context of the black hole information paradox. With these tools, we will develop a precise statement of the scrambling time problem and explain why one particular class of black holes are good candidates for saturating Susskind's bound. We will then develop some novel identities which will allow us to compute various response functions in this model. Finally, we will comment on the results, vis a vis the theme of understanding how a strongly-interacting many-body system like a black hole can hide information about its state.

2.1 The Black Hole Information Paradox and Black Hole Complementarity

The assertion that it should be possible to learn anything at all about a black hole's microscopic state seems mild, but its justification is in fact one of the most important open questions in gravitational physics (and has been for decades). Indeed, in pure Einstein gravity, the no-hair theorem asserts that black holes are totally characterized by their mass M, their electric charge Q, and their angular momentum J. This is rather unsettling, as it implies that no matter how a black hole of a certain mass, charge, and angular momentum was formed, it is indistinguishable from any other black hole sharing the same values of those three parameters. Unfortunately, first attempts at solving this rather unsatisfying feature by semiclassical arguments not only fail to resolve this worry, but actually give rise to a contradiction, since they predict that black holes should evaporate, releasing their energy as thermal radiation. This was first established by Stephen Hawking in the 1970's, and is called Hawking radiation. The contradiction arises from this fact if we consider a black hole which is initally in some pure state. After it evaporates it will now be in some mixed state - namely the thermal ensemble - violating the unitarity of quantum mechanics. The information about the state of the black hole appears to have been lost. This is the famous black hole information paradox.

Many arguments exist to justify the phenomenon of Hawking radiation. The most rigorous demonstrations proceed by positing some scalar field theory in the black hole background, performing a mode expansion, and making a clever change of variables called a Bogoliubov transformation. Unfortunately, these arguments are rather formal and obscure the physics of the situation. We will instead opt for a less rigorous, but more illuminating proof that comes directly from analyzing the near-horizon behavior of a Schwartzschild black hole. The metric of such a black hole of mass M is given

by

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega_2^2, \tag{2.1}$$

where the spherical part of the metric is $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$, the Schwartschild factor is $f = 1 - \frac{R}{r}$, and R is the Schwartschild radius of the black hole. In units where $\hbar = G = c = 1$, the Schwartzschild radius R is $2M$. Near the horizon $r = R$, f tends to zero, so we may approximate the proper distance from the horizon $d\rho = \frac{dr}{\sqrt{f}} \approx \frac{dr}{\sqrt{f'(R)(r-R)}}$, from which we find $\rho = \frac{2\sqrt{r-R}}{f'(R)}$ and subsequently $f \approx f'(R)(r-R) = \kappa^2\rho^2$, where we have identified the surface gravity of the black hole $\kappa = \frac{f'(R)}{2}$. In these variables, the near-horizon Schwartzschild metric acquires the Rindler form:

$$ds^2 = -\rho^2\kappa^2 dt^2 + d\rho^2 + R^2 d\Omega_2^2 \tag{2.2}$$

Now, suppose we have some quantum field theory defined on this background. The notion of temperature in quantum field theory arises from analytic continuation of the partition function into imaginary time $\tau = it$, where the inverse temperature β is given by identifying τ with $\tau + \beta$. The horizon is regular, so we may analytically continue the metric in this way. We obtain

$$ds^2 = \rho^2\kappa^2 d\tau^2 + d\rho^2 + R^2 d\Omega_2^2 \tag{2.3}$$

$$= \rho^2 d\theta^2 + d\rho^2 + R^2 d\Omega_2^2, \text{ where } \theta = \kappa\tau \tag{2.4}$$

This metric manifestly describes the cartesian product of a cone in the $\theta - \rho$ coordinates and the two-sphere in the angular directions. We know that the horizon is regular, so to avoid any potential conical singularity at the horizon $\rho = 0$, we need to identify $\theta \sim \theta + 2\pi$. That is, τ is periodic in $2\pi/\kappa = \beta$, which gives the Hawking temperature $T = \kappa/2\pi$. This shows that if the notion of the temperature of a black hole can be constructed, the geometry of spacetime, not the details of any specific quantum field theory, constrain the temperature to be a certain value. Indeed, for

scalar field theory in this background, there is a unique vacuum state, called the Hawking-Hartle state, which is regular at the horizon. Though the construction of the Hawking-Hartle state is irrelevant for our purposes, it constructively shows that black holes can actually radiate. Furthermore, one can interpret the black hole information paradox vis a vis the Hawking-Hartle state by noting that the infalling modes of the state are causally disconnected from the Hawking radiation, and therefore the radiation cannot encode any information about those modes.

Resolving the black hole information paradox is obviously beyond the scope of this thesis, but many proposals resolving the information paradox have been made. One of the most widely accepted, known as the black hole complementary principle, is due to Susskind, Thorlacius, and Uglum [14]. In their proposal, when an object falls into a black hole, the information about its state is duplicated. First, in the frame of reference of the object, there is nothing special about the event horizon, and so its information content passes into the interior of the black hole. To an external observer, however, the object requires infinite proper time to reach the horizon. Instead of treating the horizon as being a purely geometric feature, Susskind, et al, instead propose that the horizon is covered by a "stretched membrane" of Planck-scale thickness, which is excited by the object. The information content of the object diffuses over the surface of the membrane, and is ultimately re-emitted as Hawking radiation, which is postulated to be non-thermal - it contains the information content of the object's state. In other words, the operator that governs the object's interaction with the stretched membrane is taken to be unitary and therefore consistent with quantum mechanics.

Hence, in this picture, the information content can be ascertained either from inside or outside the black hole. Of course, the quantum no-cloning theorem states that copying a quantum state itself violates unitarity, so it should not be possible to obtain both copies of the information: this sets a constraint on how quickly a black hole can release information into its surroundings. To see this, consider the following thought experiment. Suppose an observer falls into a black hole with some object, and we reconstruct the object's state from the Hawking radiation. Then, we jump

into the black hole and compare our state with the observer's. If it is possible for us to share our information before we run into the singularity, we have effectively cloned the state, violating unitarity. Thus, the time required to collect the Hawking radiation must be "sufficiently long" in some sense if unitarity is to be preserved. In the next section, we will determine how fast one can reconstruct information about a black hole's state from its radiation, assuming only the unitarity of the black hole's time evolution operator, as required by black hole complementarity. This will then yield Susskind's bound on how quickly a black hole can propagate information across its internal degrees of freedom.

2.2 Black Holes as Mirrors

The main goal of this section is to prove a result due to Hayden and Preskill [5][1] that black holes leak information out very quickly. We mean this in a purely an information-theoretic sense - namely, that each bit a black hole radiates has almost maximal information content. To this end, it will be useful for clarity of exposition to neglect practical matters, such as the time it takes for black holes to entangle information and the time it takes for black holes to produce Hawking radiation. These issues will be revisited in the next section. Furthermore, since we don't know quantum gravity, we can't actually model the black hole's dynamics exactly; rather, we're going to assume that its time evolution is well-approximated by a random unitary operator.

As a warmup, we'll begin by considering the following model of a "classical" black hole. Suppose k bits of unknown data falls into a black hole whose internal state consists of $n-k$ bits of information, and further suppose that we know these $n-k$ bits. After it falls into the hole, the state of the black hole grows, requiring n bits to describe. Then, the black hole instantaneously processes the new bit string; this process may be modeled as a permutation on the space of possible bit strings, which has cardinality 2^n. (We'd like this process to be a permutation rather than an

[1]This section is largely based on the discussion in this paper

arbitrary function per our assumption that black holes don't destroy information.) After it processes the string, it begins to radiate one bit at a time. Supposing we know which particular permutation the black hole uses, we want to answer: how many bits do we need to see before we know the k-bit string?

If we insist on perfect certitude, the answer is n because, after obtaining the entire bit string, we just take the preimage under the permutation. However, as is often the case in life and physics, things get a lot better if we relax our standards and only require that our probability of error be less than some $\epsilon = 2^{-c}$. Since we knew the state of the black hole before the data fell in, there are only 2^k possible strings we might observe. Thus, assuming the permutation the black hole applies is selected uniformly at random among the $(2^n)!$ possibilities, the probability that the first s bits we observe matches a string other than the correct one is bounded from above by $2^k 2^{-s}$, so to achieve our error bound, we simply need to read the first $s_{crit} = \lceil c + k \rceil$ bits. Thus, the black hole isn't really black at all. It reveals information about things that fall into it almost as fast as it possibly can.

The result actually generalizes to the quantum mechanical case, although a little technology from quantum statistical mechanics and analysis is required to see this. The precise details will not be given here, but the crucial observation that allows for this extension, due to Page [11], is the following: Suppose some quantum system described by some state space $\mathcal{H}$ is initially prepared in a randomly chosen pure state, and suppose the system admits a decomposition as $A \otimes B$, where A and B have dimension m and n. Then, the density matrix of the smaller of the two systems will, on average, have nearly maximum entropy - that is, it will appear to encode very little information. More to the point, the information content of the state of the system is contained almost entirely in correlations between the two subsystems, not the correlations among the degrees of freedom of each of them separately. We will explore the proof of this result and its consequences in much greater detail in the next chapter, but just knowing the statement will suffice for now.

In the quantum case, we're not going to ask that we be able to retrieve a complete classical description of the quantum state of whatever falls into the black hole. For our

purposes, it will be good enough to possess a subsystem that is perfectly correlated with the quantum state. If we possess such a system, we can do anything with the subsystem that we could have done with the quantum state, so this is a reasonable restriction. Under this framework, we can leverage Page's result for information retrieval as follows:

Suppose we have been observing the black hole so long that at least half the data that was originally contained in the black hole at the moment of its formation has been leaked to us. Call this system of previously-emitted radiation E. Then, by Page's result, the black hole's state B is nearly maximally entangled with the state of E. Now, suppose we see something (which we denote by M) fall into the black hole. It will be useful to suppose that M's state is maximally entangled with that of some auxiliary system N. After M falls into the hole, suppose the black hole applies a unitary transformation V that very strongly mixes the state of M with that of B. Then, the black hole starts radiating - denote the system describing the radiation after the application of V by R, and denote the state of the black hole after the transformation by B'.

Initially, B is maximally entangled with E, and M is maximally entangled with the joint system NE, while the state of the entire apparatus $BEMN$ is pure and will therefore remain so throughout our discussion. Then, V is applied, entangling B with M (and therefore N). Now, we collect the Hawking radiation. After enough bits have been emitted, Page's result applies; RE becomes almost maximally entangled with N, and since N was originally perfectly correlated with M, this means we've obtained the quantum information of M. The error we might incur is contained in the residual correlations that B' maintains with N. A more detailed analysis using the quantitative results of Page shows that this generalization is actually even better than the classical case - after receiving $c+k$ bits, we can actually bound the decoding failure rate by 2^{-2c}.

Hence, we have learned that black holes give up information as fast as the parameters of information theory will allow. Now the only thing protecting us from the possible unitarity violation described in the previous section are physical effects - the

time it takes for the black hole to apply an appropriate V transformation and the time required to obtain the Hawking radiation.

2.3 The Scrambling Time Bound

Let us first consider the time necessary to reconstruct the state of an object which has fallen into a black hole from its Hawking radiation. For the Schwartzschild black hole, the bound on the reconstruction time that this thought experiment implies can be computed easily, since the near-horizon geometry of the Schwartzschild black hole is flat, as we demonstrated in Section 2.1.1. Let ρ be the proper radial distance from the horizon, and let $\omega = t/2R$ be the Rindler time, where t is Schwartzchild time, and R is the black hole radius. Then, we can change to Kruskal-Szekeres coordinates. The lightcone coordinates near the horizon take the form $X^{\pm} = \pm\rho e^{\pm\omega}$, and therefore the curvature singularity occurs at $X^+X^- = R^2$.

Now consider some observer sitting outside the black hole who is observing the Hawking radiation emitted by the black hole in an effort to reconstruct the information of some object which has fallen into the black hole at past infinity ($X^- = 0$). Suppose it takes some Schwartzchild time t_* to do this, so that the observer obtains the information at $X^+ = R\exp(\omega_*)$, where $\omega_* = t_*/2R$. If the observer then jumps into the black hole, the fact that the curvature singularity is at $X^+X^- = R^2$ implies that he must hit the singularity at or before $X^- < R\exp(-\omega_*)$. Thus, the observer can only compare his state with the object's (and therefore violate the no-cloning theorem) if the object sends a message about its state betwee $X^- = 0$ and $X^- = R\exp(-\omega_*)$. By the uncertainty principle, this will take energy of order $R^{-1}\exp(\omega_*)$. The total mass of the black hole, which scales like R, has to be greater than the photon energy. Thus, $\exp(\omega_*) \lesssim R^2$, which implies $\omega_* \lesssim \log R$. Typically, this is a very weak bound, since the time required to retrieve half of the information in a black hole (necessary for the application of Page's result in the previous section) is of order R^2. But if we are like our powerful observers in the previous section - that is, we already know the state of the black hole - the situation is much less clear. In that case, the only thing

protecting us from unitarity violation is a quantity whose microscopic description is a matter of quantum gravity and therefore beyond our mortal ken - the time it takes for the black hole to mix up the qubit with its degrees of freedom (i.e. the time required to apply the V transformation of the previous section). We call this time the scrambling time, following [13].

A more precise description of the scrambling time may be given as follows: the initial joint state of the black hole and some object we throw into it is originally a pure state, and although it remains so, after some time it will evolve to a different state where the density matrix of every subsystem of sufficiently small size is essentially thermal, which means its entanglement entropy with the rest of the system is nearly maximal. According to Page, sufficiently small means that the number of degrees of freedom in its state space is a little less than half the total dimension of the state space of the system.

Although this definition is not rigorous and difficult to calculate even for simple systems (since the entanglement entropy is very hard to compute), Susskind and Sukino were still able to produce an example for which the bound is saturated: D0-brane black holes. The precise description of this system requires considerable machinery from string theory unrelated to our goals, so we will not give one. The key feature this theory exhibits, however, is that it admits a dual in terms of the language of matrix quantum mechanics, a formalism where calculations of key quantities like response functions are much more tractable than in a generic quantum field theory. By exploring some other matrix models of black holes, we can therefore hope to produce more evidence for the scrambling time problem.

2.4 The Iizuka-Polchinski Model: Definition and Computation of Response Functions

The matrix model we will focus our efforts on is due to Iizuka and Polchinski [7]. It consists of an N-dimensional matrix harmonic oscillator X_{ij} and a vector harmonic

oscillator ϕ_i (also with N components) which are coupled together by a trilinear interaction. Let the canonical conjugates to these fields be Π_{ij} and π_j. Then, the Hamiltonian is

$$H = \frac{1}{2}\,\mathrm{tr}(\Pi)^2 + \frac{m^2}{2}\,\mathrm{tr}(X^2) + \pi^\dagger(1 + \frac{gX}{M})\pi + M^2\phi^\dagger(1 + \frac{gX}{M})\phi \qquad (2.5)$$

Note that we have chosen units so that the mass is 1 for both fields, while the frequency of the oscillators are given by m and M. Just as for a regular harmonic oscillator, we can define ladder operators for the fields of this theory. Let $a_i = \frac{\pi_i^\dagger - iM\phi_i}{\sqrt{2M}}$ and $\bar{a}_i = \frac{\pi_i - iM\phi_i^\dagger}{\sqrt{2M}}$ be the lowering operators for the vector and its conjugate, respectively. Then, the Hamiltonian becomes

$$H = \frac{1}{2}\,\mathrm{tr}(\Pi)^2 + \frac{m^2}{2}\,\mathrm{tr}(X^2) + M(a^\dagger a + \bar{a}^\dagger\bar{a}) + g(a^\dagger X a + \bar{a}^\dagger X \bar{a}) \qquad (2.6)$$

The motivation for this model are twofold. First, it is a simplified version of the matrix model dual to the D0-brane black hole model that Susskind studied. By stripping away features like supersymmetry and additional fermionic fields, this model promises to be as simple as one could hope for while still capturing much of the essential features of the D0-brane black hole. Second, the interactions of this theory give it almost the same diagrammatic structure as a model pursued by Festuccia and Liu [3], which demonstrated that the low-energy behavior of black holes in anti de-Sitter space exhibited information loss. More precisely, correlation functions of that theory exhibit exponential decay at large timelike separation, a feature that we will show is preserved in this model. Since we're trying to study the black hole information paradox, this is a crucial feature.

Though the model is cast in the language of ordinary, non-relativistic quantum mechanics, its properties are most easily described in field-theoretic language. In particular, it admits a large-N limit, as the index structure of this theory is identical to the index structure of quantum chromodynamics (QCD). The matrix has the same structure as the gluon field, which is a $U(N)$ adjoint field, and the vector has the same structure as the quark field, which is a $U(N)$ fundamental field. Recall that

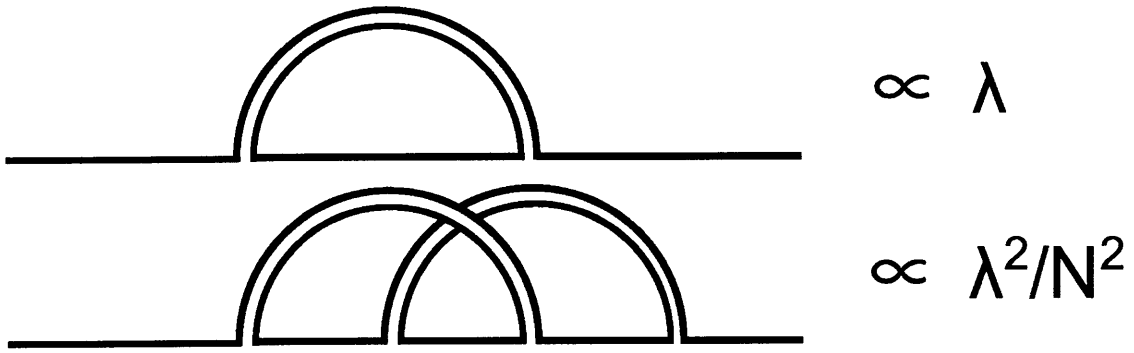

Figure 2-1: Illustration of power-counting in the large-N limit.

in the large-N limit, the size of the gauge group N is taken to infinity, while the coupling g is taken to zero in such a way that $g^2N \equiv \lambda$ is held constant. Then, if we represent the two indices of the adjoint by double-lines and the fundamental by single lines in Feynman diagrams, a perturbative expansion in powers of $1/N$ is equivalent to organizing all the Feynman diagrams by the genus of the lowest genus surface in which each Feynman diagram can be embedded [2]. To illustrate this point, a leading order term and a first order term for the fundamental two-point function is given in figure 2-1. Note that the crossings between the lines are carefully drawn to make evident how the indices of the fields are being contracted. The power-counting can be verified by noting that each vertex contributes a factor of g, and each closed loop contributes a factor of N, since any of value of the index can run the loop. As the example shows, the diagram which is non-planar is subleading in N. In general, it can be shown that as N becomes infinite, the planar diagrams dominate.

In the large-N limit, the Iizuka-Polchinski model turns out to have another property that makes it very attractive for our purposes: the coefficients of the large-N expansion are exactly computable because the Schwinger-Dyson equations that determine them collapse into a single algebraic relation. This is essential, since information loss is an intrinsically non-perturbative effect: the correlation functions of theories describing black holes decay exponentially at infinite N but not at finite N because of Poincare recurrences - at finite N the black hole has only finitely many states at a given energy. Hence, whatever the resolution of the information paradox or the

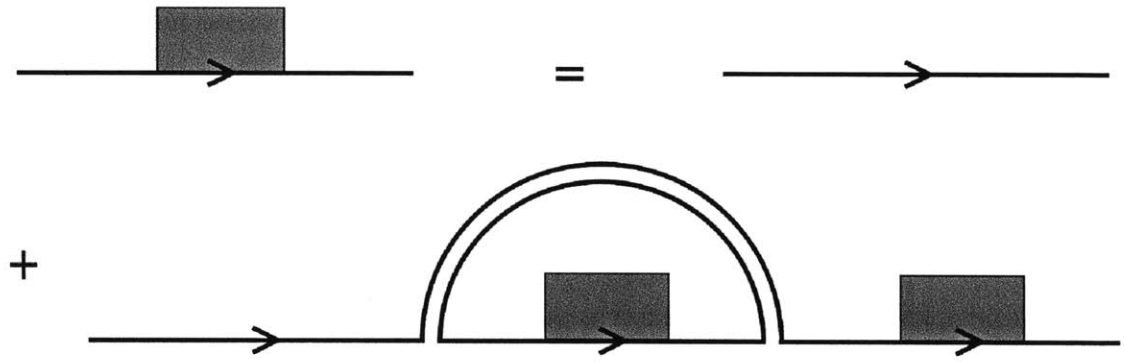

Figure 2-2: Feynman diagrams contributing to the planar correction to the two-point function in the Iizuka-Polchinski model. The lines with a shaded box represent the propagator with planar corrections. This figure is taken from [7].

scrambling time problem is, it will not be observed in any power expansion in N. Thus, a method for computing the correlation functions exactly is essential.

To illustrate this point, we describe the planar corrections to the fundamental two-point function at some temperature T. It is given by $\langle \mathcal{T} a(t) a^\dagger(t') \rangle_T \equiv \delta_{ij} G(T; t - t')$, and it was worked out in Iizuka and Polchinski's original paper. Consider the structure of an arbitrary planar diagram for the fundamental two-point function. The diagram with no vertices at all is the free propagator. If there are any vertices, then the adjoint that comes out of it must contract with a fundamental, since the Hamiltonian has no adjoint-adjoint self-interactions. By hypothesis, the processes that occur between the first vertex and the point where the adjoint contracts with a fundamental must be planar, and similarly for the processes that occur after the adjoint has contracted. These two processes cannot connect to each other, since that would violate planarity. The planar corrections to the fundamental two-point function therefore has only two diagrams contributing to it, as illustrated by Iizuka and Polchinski in figure 2-2.

From here, one can easily write down the equation that one obtains from these diagrams. At zero temperature in frequency space, the free fundamental propagator is $G_0 = \dfrac{i}{\omega + i\epsilon}$, and the free adjoint propagator is $K_0 = \dfrac{i}{\omega^2 - m^2 + i\epsilon}$. Then, the full

propagator with planar corrections satisfies

$$G(\omega) = G_0(\omega) - \lambda G_0(\omega)G(\omega)\int_{\infty}^{\infty}\frac{d\omega'}{2\pi}G(\omega')K_0(\omega-\omega') \tag{2.7}$$

Now, the time ordering ensures that the propagator is zero for $t < 0$ since the lowering operator annihilates the vacuum. Thus, the poles of G are in the upper half-plane in frequency-space. Furthermore, the coupling g has positive mass dimension, so the theory flows to the free theory at large energy, so there are also no poles at infinity. Thus, we can close the integral in the upper half-plane and evaluate the residue at $\omega' = \omega - m + i\epsilon$. The result is:

$$G(\omega) = \frac{i}{\omega}\left(1 - \frac{\lambda}{2m}G(\omega)G(\omega-m)\right) \tag{2.8}$$

This is an algebraic recursion which can be explicitly solved. Let $\nu = 2\lambda/m$. Then, the solution to 2.8 is

$$G(\omega) = \frac{2i}{\nu}\frac{J_{-\omega/m}(\nu/m)}{J_{-1-\omega/m}(\nu/m)} \tag{2.9}$$

where J_α is the Bessel function of the first kind. As one can verify explicitly, this indeed decays at large N. This holds even at finite temperature, although the discussion is more involved.

The fact that the Schwinger-Dyson equations collapse for the two-point function of the fundamental field appears to generalize to all the correlators that one might want to compute. The reasoning given for the two-point function, which computes the recursion by considering where the important "rainbows" are placed, works for higher point functions. One example are given in figure 2-3.

These algebraic features can be extended to functions we would actually like to compute: response functions. The general idea we would like to implement is as follows: we would like to perturb the system by applying some operator $\mathcal{O}$ at time t. Then, at some later time t', we would like to measure how the system responds to this perturbation. Given our rather powerful machinery for calculating the exact value of

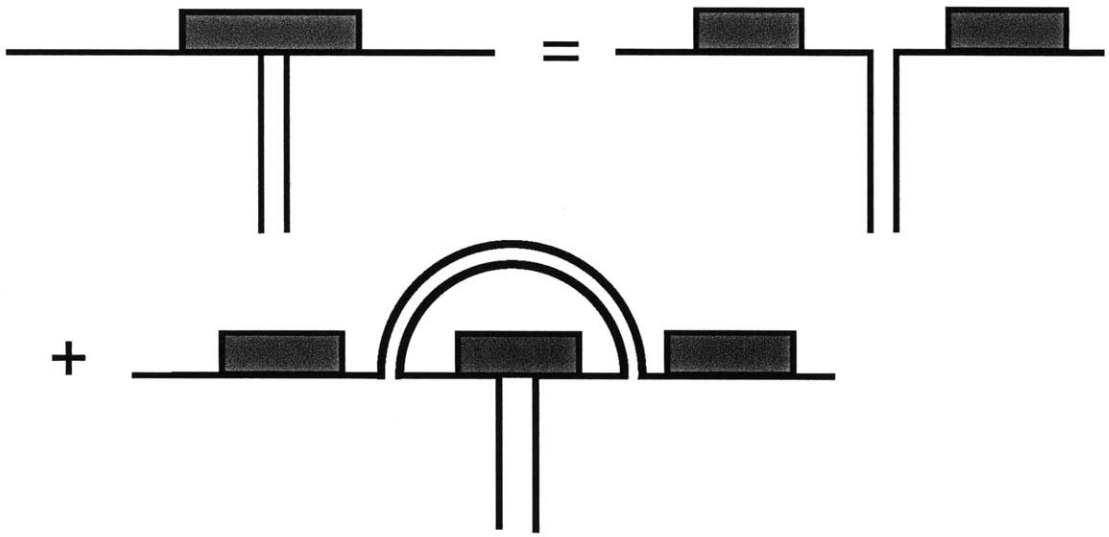

Figure 2-3: Recursive identity for computing a three-point function in the Iizuka-Polchinski model. Shaded boxes represent planar corrections to the corresponding propagator or vertex.

the correlation functions of this theory, it seems like we should be pretty close to our goal. All we need to determine is a suitable operator for measuring thermalization. Ideally, this would be something approximating the entanglement entropy, since that precisely measures how thoroughly mixed the degrees of freedom we perturb are with the rest of the system. The problem, however, is twofold:

First, the details of the black hole picture are so far removed from this theory that we don't know how to write down any approximation to the entanglement entropy. More precisely, it can be shown that the simplified theory we have been describing is only capable of encoding the physics of "very stringy" black holes whose size is of order the string scale. Hence, there is no smaller meaningful length scale one could define that would give rise to a natural notion of a local bipartition of the degrees of freedom in this system. To wit: the matrix multiplication one must carry out in the interaction term of the Iizuka-Polchinski Hamiltonian couples all the degrees of freedom of the fundamental to all the degrees of freedom of the matrix in essentially the same way. If we perturb some subset of the degrees of freedom in the matrix model, which of the other degrees of freedom are near to each other in the black hole that the matrix model is apparently describing? We need to be able to divide

our space into two physical parts in order to define an entanglement entropy, so answering this question is important if we wish to literally implement the definition of the entanglement entropy. Other ideas run into similar problems. In choosing a theory where many things can be computed, therefore, we have lost a lot of intuition. Indeed, though we might be able to construct a string theory dual to this system, the question of whether it even contains a black hole at all at finite temperature is not clear a priori.

One might hope, then, to compute a measure of thermalization from the top down: perhaps we might just try out some operators, and see what we get. This, too, runs into problems, since the raising and lowering operator structure of the theory is so simple. For example, suppose we perturb the system by one of the X degrees of freedom and measure the response using the number operator. That is, we could compute the retarded Green's function $\langle (a^\dagger a)(t > 0) X(0) \rangle$, which is the correlator given precisely by the diagrams drawn in figure 2-3. But, we know $X \propto A + A^\dagger$, where A and $A^\dagger$ are the raising and lowering operators of the theory. Thus, the vacuum will be transported to a state orthogonal to itself, and thus will be annihilated - the Green's function is identically zero. In fact, it is easy to see from this example that any simple combination of the fields and their associated creation and annihilation operators is essentially trivial. Thus, if it is at all possible to implement our plan, it will require a complicated choice for the perturbation we use and probably also the measure for thermalization.

It is worth mentioning that Iizuka and Polchinski themselves (along with an additional collaborator, Okuda) realized some of these pitfalls and wrote down a modification of the theory that involves an interaction that replaces the trilinear interaction with a charge-charge interaction [6]: $H \propto q_{li} Q_{il}$, where $q_{li} = -a_i^\dagger a_l$ and $Q_{il} = A_{ij}^\dagger A_{jl} - A_{ij} A_{jl}^\dagger$. This modification dramatically enriches the theory and allows one to make headway on the question of how a coherent space-time picture might emerge from the model. Their techniques involved many elegant ideas about the how the representation theory of $U(N) \times U(N)$ is connected to the combinatorics of Young tableaux and is beyond the scope of our discussion, but even granting these advances,

we nevertheless seem far from our goal.

The lesson we learn from all this is that we seem forced to trade away something for the simplicity of our model. In the original picture of a black hole, we knew, at least schematically, what the entanglement entropy meant and what the relevant quantities were, but we couldn't even write down an action for our system, much less compute anything. In our simplification of a dual description of the black hole, we can compute almost anything we like, but we have no idea what to compute because our simplifications have obscured the original physics to such a great extent. Hence, one reason many-body systems are good at hiding information is because they exhibit some kind of "conservation of confusion" principle: simpler descriptions of the physics necessarily obscure the relevant physical questions.

Chapter 3

The Entanglement Entropy of Matrix Product States

Though the scrambling time problem and its motivation are elegant and easy to understand, those ideas are not on solid theoretical footing. The key assumption we have been making the entire time is that a black hole's time evolution operator is described well by a random unitary operator. It is only under this assumption that Page's result for the average entanglement entropy of a subsystem can be applied. This is critical, since Page's result justifies why a black hole was mirror-like, and therefore, why it was even plausible for a possible unitarity violation to occur: the majority of randomly chosen states contain enough long-range correlations that any small part of the state is maximally entangled with the rest of the state, which allows one to quickly reconstruct the state of any object that is dropped into the black hole and subsequently thermalized.

If instead, the black hole's time evolution operator was restricted to some subspace of unitaries that correspond to local Hamiltonians, the black hole might not be mirror-like at all. To see this, suppose we model the black hole as being a system of n qubits whose time evolution can be approximated as a quantum circuit. At each time step (which can be no smaller than the Planck time), we apply some number of unitary transformations on pairs of neighboring qubits - at each time step, this is most this is $n/2$ local unitaries. Then, after some time t, we can apply at most $O(nt)$ local

unitaries. It is known that reconstructing a random unitary operator in $U(k)$ by local unitaries of this form requires order $O(k4^k)$ local unitaries [10]. However, for a system of n qubits, the dimension of the Hilbert space is 2^n. Hence, approximation of a randomly chosen unitary by a local quantum circuit requires time t doubly exponential in n, which is astronomically large for any system of nontrivial size. To illustrate, if we assume all constants are order unity, for a system of order 100 qubits, the required time is larger than the age of the universe. If we demand only that our circuit approximate the time evolution operator to some ϵ in the operator norm, this exponential growth persists.

Thus, if Page's result does not hold for a randomly chosen **local** unitary transformation, the thermalization step provides very strong protection against unitarity violation, rendering the scrambling time problem essentially moot. Hence, it is natural to see whether or not this is the case. To this end, we will proceed in three steps. First, we will review the details of Page's proof so that we know what quantities we need to calculate. Then, after restricting our attention to one-dimensional systems for simplicity, we will give a description of which states are "locally accessible" - these will turn out to be the so-called "matrix product states", or MPS. Finally, we will attempt to refine Page's result to this smaller ensemble of states. Here, we will meet yet more challenges arising from the many-body physics of the system.

3.1 The Average Entanglement Entropy of a Subsystem

In this section, we briefly review Page's result for the average entanglement entropy of a subsystem [11]. Consider some bipartite system which we partition into two parts, labeled A and B. Let this joint system's Hilbert space have dimension mn, where m and n are the dimensions of the Hilbert space describing A and B, respectively. Without loss of generality, take $m < n$. An arbitrary (possibly mixed) state of the system is therefore described some mn by mn density matrix ρ. If the state is pure,

then $\rho = |\psi\rangle\langle\psi|$, and $\rho^2 = \rho$. The reduced density matrices are defined by the partial traces of ρ: $\rho_A = \mathrm{tr}_B\rho$ and $\rho_B = \mathrm{tr}_A\rho$. If ρ describes a pure state, then ρ_A and ρ_B necessarily have the same eigenvalues. The entanglement entropy for a pure state is then defined to be $S = -\mathrm{tr}\rho_A \ln \rho_A = -\mathrm{tr}\rho_B \ln \rho_B$. By changing to a basis where the density matrices are diagonalized, this reduces to a sum over the eigenvalues of the reduced density matrices $S = -\sum \lambda_i \ln \lambda_i$.

We would like to calculate the expected entanglement entropy $\langle S\rangle$ in the ensemble of all pure states. We will represent an arbitrary state $|\psi\rangle$ as being the image of some fixed, arbitrary unit vector $|\psi_0\rangle$ under a randomly chosen matrix U. Then, the average $\langle S\rangle$ is naturally defined with respect to the *Haar measure*, which is the measure which is invariant under left-multiplication by unitary matrices. To clarify this property, we can write an explicit integral expression. If we write $S = f(U)$ for some function f of unitary matrices U, then $S = \int dU f(U)$. The Haar measure dU is then defined to be the measure which satisfies $S = \int dU f(U) = \int dU f(VU)$ for any unitary matrix V. It can be shown using techniques from measure theory that this suffices to completely determine the Haar measure for any compact topological group (not just $U(n)$), but we will ignore this point. Indeed, it can also be shown that in the case of $U(n)$, the Haar measure is just the standard volume form on the $2mn - 1$ sphere when we interpret the complex, mn-dimensional Hilbert space of the entire system as a real, $2mn$-dimensional Euclidean space after normalizing every nonzero vector.

By explicitly choosing a basis for the infinitesimal generators of $U(n)$, Lloyd and Pagels [9] showed that, for, the ensemble of random states distributed according to the Haar measure, the probability distribution for any particular set of eigenvalues $\{p_1, \ldots, p_m\}$ of the reduced density matrix ρ_A was given by

$$P(p_1, \ldots, p_m)dp_1 \ldots dp_m \propto \delta(1 - \sum p_i) \prod_{1 \le i < j \le m} (p_i - p_j)^2 \prod_{k=1}^{m} (p_k^{n-m} dp_k) \qquad (3.1)$$

Using this probability distribution, one can in principle compute $\langle S\rangle$ by evaluating

the integral

$$\langle S \rangle = -\int (\sum p_i \ln p_i) P(\{p_i\}) \prod dp_i \tag{3.2}$$

This was evaluated both numerically and analytically by Page - the result was that in the limit that $1 \ll m \ll n$,

$$\langle S \rangle = \ln m - \frac{m}{2n} + O(\frac{1}{mn}) \tag{3.3}$$

Hence, if m is much less than half the size of the entire system mn, the expected entanglement entropy is dominated by the leading term $\ln m$, which is precisely the maximal possible entropy of any quantum ensemble - i.e. a mixed state which has no quantum information at all. The interpretation is that, when A is a lot smaller than B, the reduced density matrix ρ_A usually cannot tell us anything about the randomly chosen pure state from which it was produced. All the information about this pure state is contained in correlations between A and B that are lost once we trace over the B degrees of freedom.

For our purposes, however, we would like to restrict the average to a smaller subset of states - namely, the ones that are "locally accessible" from some fiducial state of the system (say, the ground state). This means identifying an appropriate submanifold of $U(n)$ that specifies which states we're interested in, and then computing the induced measure from the Haar measure on this submanifold. Then, we could compute the distribution on the eigenvalues corresponding to this measure using some method similar to Lloyd and Pagels, and then integrate the entanglement entropy weighted by that distribution. We will begin this program by describing a good choice of submanifold: the "matrix product states".

3.2 Matrix Product State Formalism

3.2.1 Motivation: The AKLT state

The matrix product state formalism is a way of expressing quantum states in terms of traces of products of matrices. Though this seems extremely contrived and not at all useful upon first glance, it turns out that this general form is a natural way to describe the states that naturally arise from the time evolution of local Hamiltonians with an energy gap in one dimension. To illustrate how this works, we will study a particular model of a spin chain, known as the AKLT model [1], whose ground state admits a simple description in this form. In this model, we consider a circular chain of spin 1 particles. Number the sites by an index i and let S_i be the spin operator corresponding to that site. The AKLT Hamiltonian is:

$$H = \sum_i S_i \cdot S_{i+1} + (1/3)(S_i \cdot S_{i+1})^2 \tag{3.4}$$

The ground state is easy to figure out just from inspection of the Hamiltonian. Finding the ground state is tantamount to minimizing the energy due to the coupling between neighboring spins. Recall that a pair of spin 1/2 particles can be made to look like a spin 1 particle by putting them in an $S = 1$ (triplet) state. Thus, we can split up the degrees of freedom at each site into those of two spin 1/2 particles that are forced to be in a triplet state. Then, we can put each of the two spin 1/2 particles at each site in an $S = 0$ (singlet) configuration with a spin 1/2 particle at a neighboring site. This clearly minimizes the energies due to the couplings, as required.

The AKLT Hamiltonian is manifestly local - it only couples nearest neighbors. Furthermore, as we have just described, its ground state exhibits correlations only between neighboring sites - each virtual spin 1/2 particle knows only about how it must pair up with its partner and exactly one other neighboring spin. Hence, it provides a good setting for studying how states arising from local Hamiltonians might be represented efficiently. To this end, let's try to write the ground state explicitly. Following [12], the three basis states of the real spin 1 particles located at each site

can be written in terms of the virtual particles as follows:

$$|+\rangle = |\uparrow\uparrow\rangle \tag{3.5}$$

$$|0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \tag{3.6}$$

$$|-\rangle = |\downarrow\downarrow\rangle \tag{3.7}$$

The singlet state that neighboring auxiliary spins occupy is given by:

$$|S=0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{3.8}$$

Now, let's call the first virtual spin on each site a_i and the second b_i. We will take $\{|\uparrow\rangle, |\downarrow\rangle\}$ as a basis for each virtual spin. If we restrict our attention to the subspace consisting of the second virtual spin on site i and the first on site j, we can write the state of these spins as

$$\Sigma^{[i]} = \sum_{b_i, a_{i+1} \in \{\uparrow,\downarrow\}} \Sigma_{b_i a_{i+1}} |b_i\rangle |a_{i+1}\rangle \tag{3.9}$$

where

$$\Sigma = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{3.10}$$

Hence, the state consisting of an equal superposition of all the ways we can form singlet bonds on neighboring sites is given by

$$|\psi\rangle = \sum_{a,b} \Sigma_{b_1 a_2} \Sigma_{b_2 a_3} \dots \Sigma_{b_N a_1} |ab\rangle \tag{3.11}$$

Here, we have written $|ab\rangle = |a_1 b_1 \dots a_n b_n\rangle$, and the sums over a and b imply sums over each a_i and b_j separately. Now, we enforce the constraint that the two pairs of virtual spins on the same site must form a triplet. Introduce the projectors M^{σ}_{ab} which map the virtual spins to the real space - we must choose them in such a way

that only triplet combinations of spins are permitted. This can be easily achieved just by turning equations 3.5, 3.6, and 3.7 into matrix equations.

$$M^{\sigma}_{ab} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \tag{3.12}$$

Here, we have chosen $\{|+\rangle, |0\rangle, |-\rangle\}$ as a basis for the real spin 1 space. Thus, the projector P mapping the entire $|ab\rangle$ space to the $|\sigma\rangle$ space is given by

$$P = \sum_{\sigma,a,b} M^{\sigma_1}_{a_1 b_1} M^{\sigma_2}_{a_2 b_2} \dots M^{\sigma_N}_{a_N b_N} |\sigma\rangle\langle ab| \tag{3.13}$$

Thus, the final AKLT ground state is given by

$$P|\psi\rangle = \sum_{\sigma,a,b} M^{\sigma_1}_{a_1 b_1} \Sigma_{b_1 a_2} M^{\sigma_2}_{a_2 b_2} \Sigma_{b_2 a_3} \dots M^{\sigma_N}_{a_n b_n} \Sigma_{b_N a_1} |\sigma\rangle \tag{3.14}$$

$$= \mathrm{tr}(M^{\sigma_1} \Sigma M^{\sigma_2} \Sigma \dots M^{\sigma_N} \Sigma) |\sigma\rangle \tag{3.15}$$

As promised, the coefficients are entirely encapsulated as traces of products of matrices. The key lessons to glean from this example are twofold: First, products of matrices encode locality because the degrees of freedom located in any individual matrix only directly mix with neighboring matrices. If we choose the matrices to be very large, we have enough degrees of freedom to express long-range correlations, but otherwise, the structure of the state forces us to relate the degrees of freedom at one site with those at distant sites in an essentially trivial way. Second, we learn that the presence of an auxiliary Hilbert space allows us to bridge sites with each other naturally. This will be a generic feature of an abstract matrix product state.

Before moving on to consider the full MPS formalism, it is worth mentioning that the AKLT state actually has many interesting properties which we shall not mention here. For example, it was one of the earliest examples of a system that exhibited topological order: it has edge states protected by a discrete global symmetry [8]. In fact, with the tools we've discussed so far, we can actually see the existence of

the edge states by considering what happens if our chain of spins is not made into a circle - i.e. we let the last auxiliary spin remain unpaired at each end. Then, there is a degeneracy in the ground state given spanned by the four different spin eigenvectors these free spin-1/2 parts can occupy. Seeing the topological ordering is not so straightforward, so we will stop here, and move on to consider general matrix product states.

3.2.2 Systematics of the MPS Representation

Construction from the Singular Value Decomposition

The main tool used to build up general matrix product states (MPS) is the singular value decomposition (SVD): Given an arbitrary $n_a \times n_b$ matrix M, there exist matrices U, S, and V such that $M = USV^\dagger$ and such that the following three properties are satisfied:

1. U is an $n_a \times \min(n_a, n_b)$ matrix with orthonormal columns - i.e. $UU^\dagger = I$.
2. S is an $\min(n_a, n_b) \times \min(n_a, n_b)$ diagonal matrix. Its entries are called the *singular values* of M.
3. $V^\dagger$ is an $\min(n_a, n_b) \times n_b$ matrix with orthonormal columns - i.e. $V^\dagger V = I$.

One major reason the SVD is useful for our purposes is because it can be employed in a natural way to study the entanglement entropy of a quantum state. To see this, consider some arbitrary state in a bipartite system $|\psi\rangle = \psi_{ij}|i\rangle_A|j\rangle_B$. Here, the states $|i\rangle_A$ are an orthonormal basis for the A Hilbert space, and similarly for $|j\rangle_B$. Note that we are using the Einstein summation convention here, and will continue to do so for the rest of this thesis. Then, treating the coefficients ψ_{ij} as a matrix, we perform a singular value decomposition. Thus,

$$\begin{aligned}
\psi_{ij}|i\rangle_A|j\rangle_B &= U_{ia}S_aV^*_{ja}|i\rangle_A|j\rangle_B \\
&= S_a(U_{ia}|i\rangle_A)(V^*_{ja}|j\rangle_B) \\
&= S_a|a\rangle_A|a\rangle_B
\end{aligned}$$

In this form, we can compute $|\psi\rangle\langle\psi|$ and its partial traces over the A and B subspaces by inspection. The entanglement entropy is just $S = -\mathrm{tr}\rho_A \ln \rho_A = \mathrm{tr}\rho_B \ln \rho_B = \sum_a -S_a^2 \ln S_a^2$. Here, the sum is taken using $0 \log 0 = 0$ in the case that any of the singular values are 0. This is sensible, since $x \ln x$ approaches 0 as x tends to 0. Hence, one way the entanglement entropy of a state can be limited is by manually forcing some of the singular values to be zero. Then, the entanglement entropy for such a state must be strictly smaller than $\ln(\min(n_a, n_b))$. This, at least heuristically, realizes our desire to construct states that arise from a local Hamiltonian, since states that are "local" have fewer long-range correlations than "nonlocal" states - i.e. their entanglement entropy must be smaller.

With this motivation, we finally construct a general matrix product state, following the discussion of the review [12] closely. Suppose we have a one-dimensional chain of N sites, where at each site we have a particle whose state lies in a d-dimensional Hilbert space. On site i, label the basis states by an index $|\sigma_i\rangle$ ranging from 0 to $d-1$. Then, the joint space is spanned by the states $|\sigma_1\sigma_2 \ldots \sigma_N\rangle$ so that an arbitrary state $|\psi\rangle = c_{\sigma_1 \ldots \sigma_N} |\sigma_1 \ldots \sigma_N\rangle$. Now, we can reshape the $d \times d \times \cdots \times d$ array c into a matrix by taking the first index to be σ_1 and the second index to be the collection $\{\sigma_2, \ldots, \sigma_N\}$ (perhaps a less abstract way to think about it is by treating $\sigma_2 \ldots \sigma_N$ as a d-ary integer). Then, we can perform a singular value decomposition:

$$\begin{aligned} c_{\sigma_1 \ldots \sigma_N} &= c_{\sigma_1, \sigma_2 \ldots \sigma_N} \\ &= U_{\sigma_1, a_1} S_{a_1} (V^\dagger)_{a_1, \sigma_2 \ldots \sigma_n} \\ &= U_{\sigma_1, a_1} c_{a_1 \sigma_2 \ldots \sigma_N} \end{aligned}$$

In the last line, we have multiplied S and $V^\dagger$ together and reshaped the result back into an array. This procedure can be performed a second time, taking $a_1\sigma_2$ and $\sigma_3 \ldots \sigma_N$ as the two indices by which c is reshaped. The result is that $c_{\sigma_1 \ldots, \sigma_N} = A^{\sigma_1}_{a_1} A^{\sigma_2}_{a_1, a_2} c_{a_2 \sigma_3 \ldots \sigma_N}$. Here, we have suggestively reshaped the matrix $U_{a_1\sigma_2, a_2}$ into a collection of d matrices we label $A^{\sigma_2}_{a_1, a_2}$. By iterating this procedure until we run out

of σ indices, we obtain the result

$$c_{\sigma_1 \ldots \sigma_N} = A^{\sigma_1}_{a_1} A^{\sigma_2}_{a_1,a_2} \ldots A^{\sigma_{N-1}}_{a_{N_2},a_{N-1}} A^{\sigma_N}_{a_{N-1}} \tag{3.16}$$

$$= \mathrm{tr}\,(A^{\sigma_1} A^{\sigma_2} \ldots A^{\sigma_N}) \tag{3.17}$$

This is a trace of a product of matrices, as claimed, albeit a trivial one since A^{σ_1} and A^{σ_N} are all vectors - thus the product of all the A's is just a number at the end. To get an idea of how large these A-matrices might be, we can count the dimensions in the case that none of the singular values are 0. In that case, we can easily see that the dimensions of the A^{σ_i} are $(1 \times d), (d \times d^2), \ldots, (d^{N/2-1} \times d^{N/2}), (d^{N/2} \times d^{N/2-1}), \ldots, (d^2 \times d), (d \times 1)$. It is also worth noting that the fact that U has orthonormal columns implies a useful normalization property for the A matrices:

$$\begin{aligned} I = \delta_{a_l,a'_l} &= (U^\dagger)_{a_l,(a_{l-1}\sigma_l)} U_{(a_{l-1}\sigma_l),a'_l} \\ &= (A^{\sigma_l})^\dagger_{a_l,a_{l-1}} A^{\sigma_l}_{a_{l-1},a'_l} \\ &= A^{\sigma_l \dagger} A^{\sigma_l} \end{aligned}$$

Of course, we could have just as well performed the SVD's by reshaping the c array "from the right" - i.e. by taking $c_{\sigma_1 \ldots \sigma_N} = c_{\sigma_1 \ldots \sigma_{N-1},\sigma_N}$ and by performing the SVD's according to this reshaping. Then, we would have obtained

$$c_{\sigma_1 \ldots \sigma_N} = B^{\sigma_1}_{a_1} B^{\sigma_2}_{a_1,a_2} \ldots B^{\sigma_{N-1}}_{a_{N_2},a_{N-1}} B^{\sigma_N}_{a_{N-1}} \tag{3.18}$$

$$= \mathrm{tr}\,(B^{\sigma_1} B^{\sigma_2} \ldots B^{\sigma_N}) \tag{3.19}$$

Here, the B matrices are reshapings of the $V^\dagger$ terms that arise in the SVD's. As before, the trace is totally trivial since the product of all the B's is a number, and the B's satisfy a normalization property, this time given by $B^{\sigma_l} B^{\sigma_l \dagger} = I$. In general, although their dimensions will match, the B matrices obtained in this way will not be equal to the A matrices described before due to the different reshaping used.

This still does not exhaust all the possible ways to get a matrix product state, however, since we could have done some number of "left-reshapings" and then switched to "right-reshapings" in the middle. This will require us to keep the diagonal matrix containing the singular values at the site where we stopped left-reshaping. Call this site l. Then, we obtain the form:

$$c_{\sigma_1 \ldots \sigma_N} = \mathrm{tr}\,(A^{\sigma_1} A^{\sigma_2} \ldots A^{\sigma_l} S B^{\sigma_{l+1}} \ldots B^{\sigma_N}) \tag{3.20}$$

It is worth noting that in this form, the entanglement entropy can be read off easily, if we choose to partition the chain by putting sites 1 through l in one part, and sites $l+1$ to N in the other. Then, by taking the basis states for the first part to be $|a_l\rangle_A = (A^{\sigma_1} \ldots A^{\sigma_l})_{a_l} |\sigma_1 \ldots \sigma_l\rangle$ and similarly for the other part, we can write $|\psi\rangle S_{a_l} |a_l\rangle_A |a_l\rangle_B$ so that the entanglement entropy is $S = \sum_{a_l} -S_{a_l}^2 \ln S_{a_l}^2$.

The three possibilities just described for rewriting an arbitrary state as a matrix product state are known as left-canonical, right-canonical, and mixed-canonical matrix product states, respectively, due to the normalization properties on the A and B matrices. Although the lack of a natural decomposition of an arbitrary state into a matrix product state is somewhat inelegant, far more serious is the existence of gauge degrees of freedom. Write a generic matrix product state $|\psi\rangle = \mathrm{tr}\,(M^{\sigma_1} \ldots M^{\sigma_N}) |\sigma_1 \ldots \sigma_N\rangle)$. Then, for any set of invertible matrices X^i of the appropriate dimension, the transformation

$$M^{\sigma_i} \to X_i M^{\sigma_i} X_{i+1}^{-1} \tag{3.21}$$

produces the same matrix product state, if we identify $X_{n+1} = X_1$. This gauge freedom will prove important in our calculations later on.

Abstract Matrix Product States and Gauge Fixing

Though the previous discussion provides a good way to express any state as a matrix product state, our ultimate goal is to identify those states which are accessible from

the ground state of some local Hamiltonian. The connection between the singular value decomposition and the entanglement entropy described before provides a natural plan of attack. Suppose we restrict the size of the matrices M^σ in an MPS to have dimensions not larger than some $\chi \ll d^{N/2}$. This integer χ is known as the *bond dimension* of the MPS. If we think of the site index as specifying the location of the particle attached to that site, then any partition we will be interested in will take the form described for the mixed-canonical MPS - one part will be sites 1 through l for some l, and the other part will be sites $l+1$ to N. Then, the restriction on the dimensions of the MPS matrices will inherently limit how large the entanglement entropy can be since that limits how many nonzero singular values we can have.

We can estimate how restrictive this procedure will be by counting degrees of freedom. At most, the matrices will have χ^2 elements, and there are d of these matrices at each of the N sites. This yields a total of $Nd\chi^2$ degrees of freedom, which is very small compared with a total of d^N degrees of freedom for the entire Hilbert space.

Besides imposing this restriction on the bond dimensions, we may also want to impose some symmetries on the space of states we're considering. So far, we have considered MPS with *open boundary conditions* - that is, A^{σ_1} and A^{σ_N} are both vectors and similarly for the B matrices. We may instead consider more general states, where the first and last matrices are not necessarily vectors. Then, the trace we've been writing in all our expressions becomes meaningful, as the product of the A matrices are no longer just a scalar. These are called MPS with *periodic boundary conditions.* If we further assert that all the matrices at each site must be the same (which necessarily makes them square), we call the MPS obtained *translationally invariant* MPS with periodic boundary conditions.

Now, we discuss the issue of gauge fixing. We will now show there is a way to use the gauge freedom in equation 3.21 to greatly simplify the calculation of the reduced density matrices of the theory. Let the matrices S_i be the singular value matrices

obtained by bringing a state into MPS from by repeated SVD's. Then,

$$\begin{aligned}|\psi\rangle &= A^{\sigma_1}\dots A^{\sigma_N}|\sigma_1\dots\sigma_N\rangle \\ &= (A^{\sigma_1}S_1)(S_1^{-1}A^{\sigma_2}S_2)(S_2^{-1}A^{\sigma_3}S_3)\dots A^{\sigma_N}|\sigma_1\dots\sigma_N\rangle\end{aligned}$$

In the second line, we've used up the gauge freedom in conjugating by the S matrices. Now, define matrices Γ^{σ_i} by rescaling the rows of A by the elements of S_i as follows: $A^{\sigma_i} = S_{i-1}\Gamma^{\sigma_i}$. Inserting this definition, we obtain the following form, which we will call *the canonical gauge*:

$$|\psi\rangle = \Gamma^{\sigma_1}S_1\Gamma^{\sigma_2}S_2\Gamma^{\sigma_3}S_3\dots\Gamma^{\sigma_N}|\sigma_1\dots\sigma_N\rangle \tag{3.22}$$

Now, suppose we pick a particular site i and partition the system by choosing one part to consist of sites 1 through i, and the other part to consist of the rest of the sites. Then, let a_i index the entries of the diagonal matrix S_i. Define the states $|a_i\rangle_A$ and $|a_i\rangle_B$ as follows:

$$|a_i\rangle_A = (\Gamma^{\sigma_1}S_1\Gamma^{\sigma_2}\dots S_{i-1}\Gamma^{\sigma_i})_{a_i}|\sigma_1\dots\sigma_i\rangle \tag{3.23}$$

$$|a_i\rangle_B = (\Gamma^{\sigma_{i+1}}S_{i+1}\Gamma^{\sigma_{i+2}}\dots S_{N-1}\Gamma^{\sigma_N})_{a_i}|\sigma_{i+1}\dots\sigma_N\rangle \tag{3.24}$$

Then, we may write $|\psi\rangle$ as

$$|\psi\rangle = (S_i)_{a_i}|a_i\rangle_A|a_i\rangle_B \tag{3.25}$$

Thus, in the canonical gauge, the eigenvalues of *every* reduced density matrix may be read off immediately by squaring the appropriate S matrix corresponding to the desired bipartition of the system at a given site. Given our interest in computing entanglement entropies, this gauge will clearly be the most useful one for our purposes.

So far, we have described only the most basic properties of MPS. There exist many other features of MPS that will not be described that are nevertheless of considerable interest: for example, MPS can be used to efficiently compute correlation functions

and carry out density matrix renormalization group flow [12], a topic of great importance in condensed matter physics. For now, however, we will stop describing formal properties of MPS and return to the question of deriving analogues of Page's result for the MPS ensemble.

3.3 Probability Distributions Induced from the Haar Measure

Using the MPS formalism, we will now try to refine Page's result to the ensemble of matrix product states. Before starting, we can immediately perform two sanity checks to guide our intuition.

First, we can check immediately that the answer we will compute is not going to be the same as Page's result. This can be most easily seen in the canonical gauge, where the matrices Λ^i are the eigenvalues of the reduced density matrices. At some fixed bond dimension χ, the entanglement entropy is bounded above by $\log \chi$ for every i. If we fix χ to be much less than $d^{N/2}$, then in the ensemble of all Haar distributed states, the average entanglement entropy is of order $N \log d \gg \log \chi$. Thus, for small χ, the maximum possible entanglement entropy of an MPS is a lot smaller than the average entanglement entropy of a Haar-distributed random state. Hence, the average entanglement entropy in the MPS ensemble is not going to be the same as the average entanglement entropy in the ensemble of all possibls states.

Second, we can recover Page's result just by making no restriction on χ. This discussion will also help us set up some basic framework and notation. As mentioned in section 3.1, the Haar measure on the space of all possible states is just the standard volume form on the $2mn - 1$ sphere when we interpret the complex, mn-dimensional Hilbert space of the entire system as a real, $2mn$-dimensional Euclidean space. If we represent a general state as $c_{\sigma_1 \ldots \sigma_n} |\sigma_1 \ldots \sigma_n\rangle$, then the metric whose volume form is the Haar measure is just $g = \sum_\sigma |dc_{\sigma_1 \ldots \sigma_n}|^2$.

For simplicity of the discussion, let's specialize to the case where there are only

two sites (or alternatively, where we split the system into two equal pieces and treat the collection of indices at each site like a reshaped multi-index), so that one singular value decomposition $c = USV$ brings the state into the matrix product state form. Note that, for convenience of notation, we've renamed $V^\dagger$ to V in the SVD. We would like to compute the induced measure on the eigenvalues of this state, which are given by the elements of S^2.

To do this, we need to choose a convenient parameterization for the matrices U and V. One good one is given by noting that the infinitesimal generators of $U(N)$ is just given by the space of Hermitian matrices. Thus, let us write $U = e^{iH}$ and $V = e^{iK}$ so that

$$\mathrm{d}c = U i\mathrm{d}H S V + U\mathrm{d}S V + U S i\mathrm{d}K V \tag{3.26}$$

$$= U(i\mathrm{d}H S + \mathrm{d}S + iS\mathrm{d}K)V \tag{3.27}$$

Plugging into the metric and using the fact that S is real (and therefore $S\mathrm{d}S^\dagger = \mathrm{d}S S^\dagger$), we obtain

$$g = \mathrm{tr}[(i\mathrm{d}H S + \mathrm{d}S + iS\mathrm{d}K)(-iS^\dagger \mathrm{d}H + \mathrm{d}S^\dagger - i\mathrm{d}K S^\dagger)] \tag{3.28}$$

$$= \mathrm{tr}[(\mathrm{d}H S + S\mathrm{d}K)(S^\dagger \mathrm{d}H + \mathrm{d}K S^\dagger)] + \mathrm{tr}[\mathrm{d}S\mathrm{d}S^\dagger] \tag{3.29}$$

$$= S_a^2(\mathrm{d}H_{ab}\mathrm{d}H_{ba} + \mathrm{d}K_{ab}\mathrm{d}K_{ba}) + 2S_a\mathrm{d}H_{ab}S_b\mathrm{d}K_{ba} + \mathrm{d}S_a\mathrm{d}S_a \tag{3.30}$$

Note that the metric does not depend on H or K explicitly, and so it is invariant under translations of H and K. Thus, U and V are Haar-distributed, as we expect. We can now bring the metric into diagonal form by a clever change of variables. Let $\mathrm{d}H_{ab} = \frac{1}{\sqrt{2}}(\mathrm{d}A_{ab} + \mathrm{d}B_{ab})$ and $\mathrm{d}K_{ab} = \frac{1}{\sqrt{2}}(\mathrm{d}A_{ab} - \mathrm{d}B_{ab})$. Then, we obtain

$$g = \frac{1}{2}(S_a + S_b)^2 \mathrm{d}A_{ab}\mathrm{d}A_{ba} + \frac{1}{2}(S_a - S_b)^2 \mathrm{d}B_{ab}\mathrm{d}B_{ba} + \mathrm{d}S_a\mathrm{d}S_a \tag{3.31}$$

Since H and K were Hermitian, however, the real part of H must be symmetric, and the imaginary part of H must be antisymmetric. The same holds for K. Let $A^{(0)}$

and $A^{(1)}$ be the real and imaginary parts of A and similarly for B. Then, rewriting the metric in terms of only independent components, we obtain

$$g = \sum_a 2S_a^2 \left(\mathrm{d}A_{aa}^{(0)}\right)^2 + \sum_{a>b}(S_a + S_b)^2 \left(\left(\mathrm{d}A_{ab}^{(0)}\right)^2 + \left(\mathrm{d}A_{ab}^{(1)}\right)^2\right) + \sum_{a>b}(S_a - S_b)^2 \left(\left(\mathrm{d}B_{ab}^{(0)}\right)^2 + \left(\mathrm{d}B_{ab}^{(1)}\right)^2\right) + \sum_a \mathrm{d}S_a \mathrm{d}S_a \tag{3.32}$$

Thus, by inspection, we conclude that the Jacobian, which yields the volume form, is given by

$$\sqrt{\det g} \propto \left(\prod_a S_a^2\right)\left(\prod_{a>b}(S_a^2 - S_b^2)^2\right) \tag{3.33}$$

This is exactly the $n = m$ case of Page's original result given in equation 3.1. It should be noted that this derivation works even if the two pieces are not of equal dimension. In that case, either U or V is not square, but we can rectify this problem by regarding the non-square matrix as being a subset of the rows of some larger unitary. This creates additional directions that one needs to keep track of, and it can be shown that these give rise to the extra factor in equation 3.1 missing from equation 3.33. Furthermore, note that equation 3.32 is missing many of the components of the matrices. They are merely flat directions corresponding to the gauge freedom in the singular value decomposition described earlier.

Though this direct approach works well for our simplified discussion, it runs into a number of problems when we try to tackle the general problem. First, the computation is completely intractable when we consider the general case, where we have an arbitrary number of singular value decompositions to handle. Second, even in the case where have only two singular value decompositions (schematically, $c \sim U_1 U_2 S V$), it does not appear as though the measure induced on U_2 is the Haar measure - the explicit factor of U_2 does not just cancel out when we compute $|dc|^2$ because none of the variables here commute. Similar remarks hold for even larger numbers of singular value decompositions. Third, it is unclear how to implement the constraint that the

MPS be of some fixed bond dimension. As we saw in section 3.2.2, it amounts to restricting how many nonzero singular values may appear at each step of the singular value decomposition. Simply throwing away some of the degrees of freedom requires a complicated renormalization step at each SVD to preserve total probability. Though this is actually done in practice in the context of the density matrix renormalization group (DMRG), a numerical technique used to calculate the low-energy behavior of one-dimensional many body systems, it is unsuitable for our purposes. We argued that the MPS states constitute a more physical choice for the state space of a quantum many-body system than the usual Hilbert space since the Hilbert space is comprised mostly of states that can never be realized in any local system. It would not be worth micromanaging the complexities associated with these unphysical states given that we introduced MPS specifically to get rid of them.

Thus, it appears as though hoping that a Haar-distributed random state would induce simple Haar-distributed matrices in its singular value decompositions was rather wishful thinking. An alternative proposal due to Garnerone, de Oliveira, and Zanardi [4], motivated by efforts at generating efficient protocols for generation of matrix product states, is as follows. Suppose each site is described by a d-dimensional Hilbert space $\mathcal{H}_B$ and suppose we initialize our chain to be in the state $|0\rangle^{\otimes N} \in \mathcal{H}_B^{\otimes N}$. Introduce an ancilla in the state $|\phi_i\rangle \in \mathcal{H}_A = \mathbb{C}^\chi$. Then, on the kth site, pick a random, Haar-distributed unitary matrix U acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ and define the MPS matrices on site k by $A^{\sigma_i}_{\alpha,\beta}[k] = \langle \sigma_i, \alpha | U[k] | 0, \beta \rangle$, where the first index in each bra/ket indexes the ancilla Hilbert space, and the second indexes the physical Hilbert space. Since U is unitary, note that this procedure gives matrices A that satisfy the normalization condition $\sum_i A^i[k]^\dagger A^i[k] = I$. It can be assumed without loss of generality that the ancilla space decouples from the physical space in the last step into a state $|\phi_f\rangle$ so that the full state is given by $|\psi\rangle = \langle \phi_f | A^{\sigma_N} \dots A^{\sigma_1} | \phi_i \rangle | \sigma_N \dots \sigma_1 \rangle$. To clarify this procedure, a figure showing the steps schematically is given in figure 3-1.

There are two directions one might go from here. First, we could treat this definition as inducing a new measure on the space of matrix product states, and we could try to calculate the distribution on the eigenvalues of the reduced density matrix

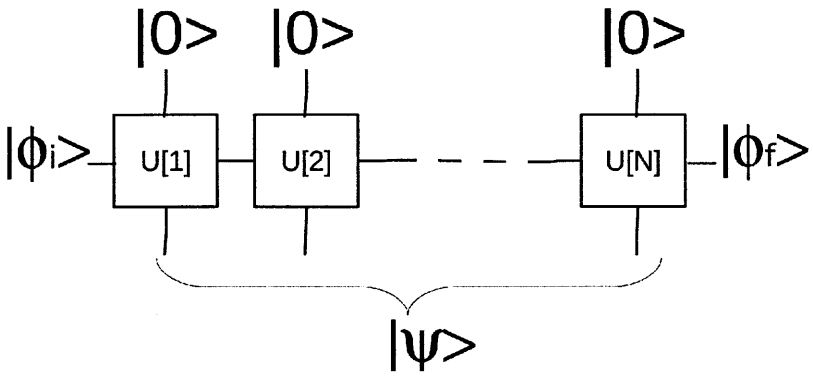

Figure 3-1: Diagram of Zanardi, et al.'s method for sequential generation of MPS from Haar-distributed unitary matrices using an ancilla. This diagram is taken from [4].

that we obtain in this way. Note that the answer we get may not necessarily be the one we computed in the direct calculation earlier in this section. There is, however, a different, interesting perspective on how to proceed using tools from analysis. Suppose we decide to work with homogeneous matrix product states - i.e. the states where the A matrices are the same at each site. Then, we may regard the computation of the entanglement entropy for some particular subsystem as being a function from the Haar-distributed unitaries over $\mathcal{H}_A \otimes \mathcal{H}_B$ to the real numbers. It is a result of measure theory that the unitary group exhibits a phenomenon called "concentration of measure," which states that for any such function f, the probability that $|f - \bar{f}| \geq \epsilon$ is bounded from above by $c_1 \exp(-c_2 \epsilon^2 d\chi/\eta^2)$. Here, $\bar{f}$ is the average value of f defined with respect to Haar-distributed unitaries, η is the Lipschitz constant for f[1], and c_1 and c_2 are universal constants that satisfy the inequality no matter what our function f is. Thus, instead of trying to compute the induced metric explicitly, perhaps it would be more fruitful to consider the analytic properties of the entanglement entropy, viewed as a function under this MPS construction.

Zanardi, et al. computed the Lipschitz constant for the case of f representing the expectation value of a local observable on the first $L \ll N$ sites. By a "local observable" we mean an observable which factors as the tensor product of Hermitian

[1]Recall that we call a function between two metric spaces (X, d_X) and (Y, d_Y) Lipschitz continuous if there exists $K > 0$ such that for all $x_1 \neq x_2 \in X$, we have $d_Y(f(x_1) - f(x_2)) \leq K d_X(x_1, x_2)$. The Lipschitz constant η for a Lipschitz continuous function f is the infimum of such K.

operators on each of the first L sites: $O = \left(\bigotimes_{k=1}^{L} O[k]\right)\left(\bigotimes_{k=L+1}^{N} I[k]\right)$. They found that the probability bound was $c_1 \exp(-c_2'\epsilon^2\chi/N^2)$, where $c_2' = c_2 k$, where k are some numerical constants. The key here is that, if we choose χ to be something polynomially large in N rather than exponential in N (as would be required for MPS to encompass the entire Hilbert space), the probability of deviation from the mean is still exponentially suppressed. If we can extend this result to cover the entanglement entropy - i.e. show that the entanglement entropy, regarded as a function from unitaries to the reals, is Lipschitz continuous, and calculate that the Lipschitz constant scales in a way that doesn't remove the exponential decay of the variance - we will have made significant progress in solving the problem of refining Page's result - we know that the mean has to be much, much smaller than Page's result, as argued in the beginning of this section. If we can, in addition, show the deviation is suppressed that dramatically, we will have fully proven a refinement of Page's result. This would definitively tell us that it is a mistake to apply Page's result to systems whose time evolution operator is not well-approximated by a random unitary matrix.

Heuristically, we anticipate that the entanglement entropy should have the nice analytic properties required for the concentration of measure result, as it basically consists of doing row operations to a matrix and then applying a smooth function to some of its elements. Further evidence comes from numerically computing the entanglement entropy using the canonical gauge by taking repeated singular value decompositions on the matrices obtained from Zanardi and collaborators' procedure. To see this procedure explicitly, write the coefficients as $A^{\sigma_1}A^{\sigma_2}\ldots$, suppressing auxiliary indices. An SVD on the first A, considered as a matrix whose rows are indexed by σ_1 and columns are indexed by the auxiliary index gives $U^{\sigma_1}SV^\dagger A^{\sigma_2}\ldots$. Now, using the entries of S, we can compute the entanglement entropy for the partition where the first site is separated from all the other sites. Then, define M_2^σ by $SV^\dagger A_2^\sigma$ and do another SVD, reshaping the matrix so that the columns are again indexed only by the auxiliary index. The S matrix obtained here yields the entanglement entropy if we separate the first two sites from the rest. Continuing in this way, we

can compute all the entanglement entropies.

The numerical result for $N = 40$ sites with physical Hilbert space dimension $d = 2$ and various choices of the bond dimension with open boundary conditions is shown in figure 3-2. Note that only half of the graph is given, since Zanardi's procedure explicitly breaks the symmetry between the left and right halves of the system, which is manifestly unphysical. To see why this occurs, write out the matrix product explicitly, keeping all auxiliary indices. Keeping in mind that the first and last matrices must be vectors in order to satisfy open boundary conditions, we obtain the structure

$$A_{1,a_1\sigma_1}A_{a_1,a_2\sigma_2}\cdots A_{a_{n-1},a_n\sigma_N}A_{a_n,1}$$

Zanardi's prescription requires us to take rows from unitary matrices - the normalization requirement for the A's is important in the proof of his results. Thus, we see that at the last site, where we must have a row vector, a_n is restricted to have only one value - i.e. the last matrix is just a number. Working backwards, we find that the matrices grow by powers of d at each subsequent site, until the kth site from the end, where d^k exceeds χ, and our constraint on the bond dimension forces us to take a subset of the rows of the random unitary at that site. Hence, a condition on the size of the matrices A has to be imposed by hand at the end of the chain in order to use Zanardi's procedure, breaking the symmetry between the left and right parts of the chain.

Note that our results for the entanglement entropy fall very short of saturating Page's bound. This is what we expected, since the entanglement entropy is necessarily limited by our choice of the bond dimension χ, as explained in section 3.2.2. A comparison for the $\chi = 50$ case is indicated in the figure. It is also notable that repeated trials produce very small deviations from the given results - for the $\chi = 50$ case, the variance in the value where the entanglement entropy levels off is only about 0.1 versus an average value of 3.5. Hence, the concentration of measure result appears to hold. Thus, we have demonstrated numerically a significantly stronger refinement

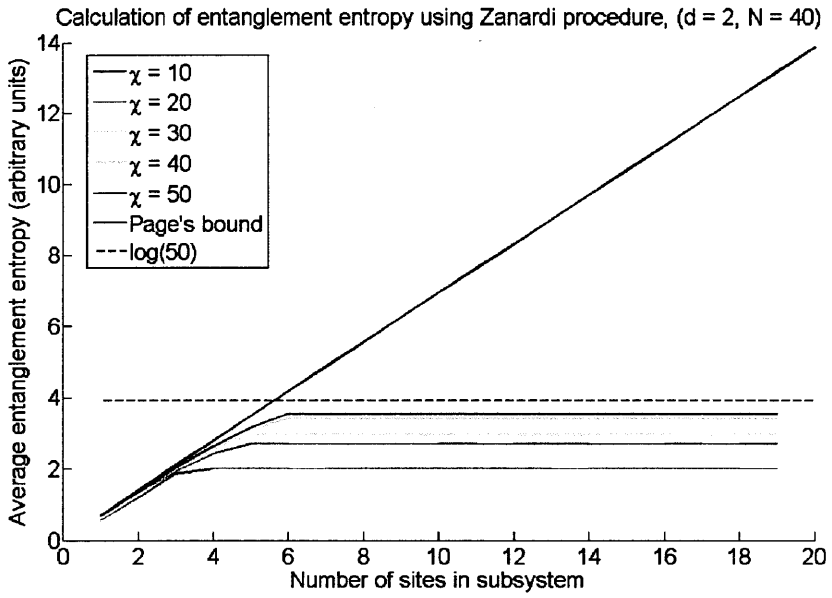

Figure 3-2: Numerical calculation of entanglement entropies in Zanardi, et al.'s procedure for a chain with $N = 40$ sites, physical Hilbert space dimension $d = 2$, and varying bond dimensions χ. Note that Page's bound is stricter than the bound arising from restricting the bond dimension if one of the subspaces has size less than $\log \chi$. Otherwise, the bound $\log \chi$ is stricter. The trapezoid shape formed by the convex hull of these two lines is therefore the absolute bound for any choice of χ.

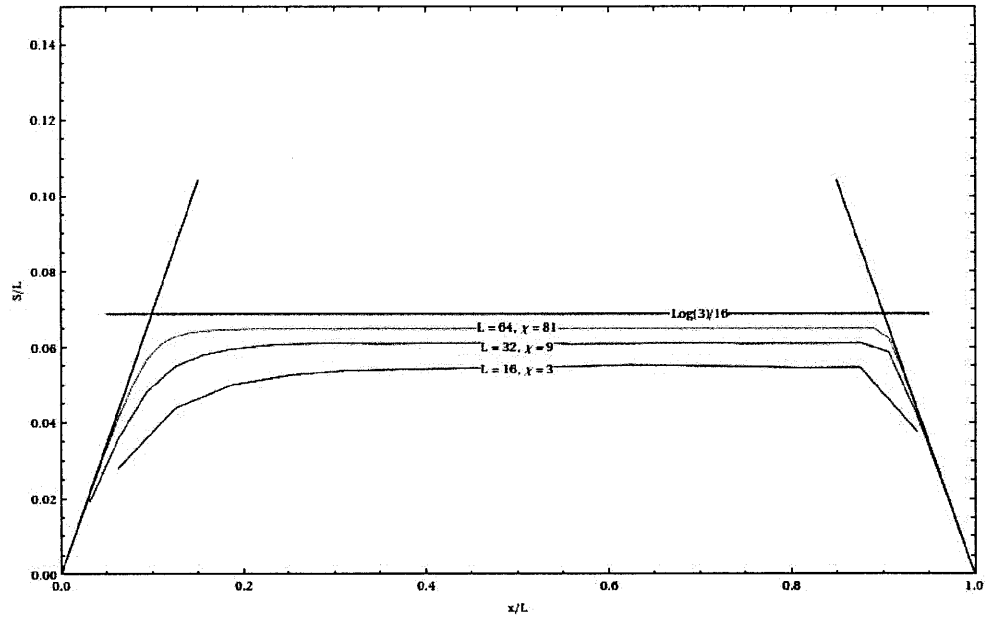

Figure 3-3: Numerical calculation of entanglement entropies per site using Zanardi, et al.'s procedure, for various choices for the total system size L. To allow comparison, we have chosen χ in each case so that the bound for the average entanglement entropy per site arising from our choices for χ is held constant at the value $\ln(3)/16$.

of Page's bound for the MPS ensemble.

Interestingly, though the result departs significantly from Page's prediction, it is very similar in spirit, since in the limit that the number of sites becomes very large, the average entanglement entropy gets closer and closer to saturating the bound imposed by the bond dimension. We can see this result by plotting the average entanglement entropy per number of sites for a variety of N with χ chosen so that $\log(\chi)/N$ is held constant. Figure 3-3 shows the results of this computation[2]. Even for the small number of sites we're probing, convergence towards the bound appears to be very fast. Thus, just like Page, we find that even in the restricted ensemble of MPS, the average state attains as much entanglement as possible.

Given the effort we've put into the problem so far, it is worthwhile to try to get some perspective on what we've accomplished. The entanglement entropy in the ensemble of matrix product states is evidently very, very small. For a macroscopic

[2]As an unrelated aside, it is worth noting that figure 3-3 also confirms the left-right symmetry breaking claimed earlier, as the approach towards the bound is faster on the right than on the left.

object with order Avogadro's number $\sim 10^{23}$ of sites, if we fix the bond dimension to be polynomial in the number of sites, we can bound the entanglement entropy by some constant multiple of $\ln 10^{23} \approx 54.7$. This is in stark contrast to the answer Page would predict, which is order 10^{23}. Trying to find the correlations in our ensemble of matrix product states is therefore much like trying to find a needle in a haystack. The apparently huge number of variables and couplings we have to manage in direct calculations are almost irrelevant and cannot contribute very much to the final answer. Our inability to calculate the answer in the obvious way therefore owes to the fact that a many-body system evolving under a local, gapped Hamiltonian has a lot of places to hide its information. In contrast, Page's result was almost immediate, since a typical, random unitary matrix does not obviously distinguish between sites, so we could learn a lot about the global properites of the system relatively easily. We did not have to answer any question about which sites and interactions were more important than others.

On the other hand, we have made progress in figuring out how the results of the previous chapter should be modified if we learn that black holes are not as nonlocal as we think: the answer is that those results will have to be seriously revised. The smallness of the entanglement entropy and the apparently sharp bounds on how big deviations from that mean value must be telling us that the thermalization time for a black hole would be so large that any possible unitarity violation is protected by an exponentially large time. Though our numerics strongly suggest that this is indeed the case, it would be desirable to have a more precise quantitative answer to this question, especially for systems that are not one-dimensional like the MPS ensemble. Research to that end is ongoing.

Chapter 4

Conclusions and Future Work

In this thesis, we approached the information theoretic properties of many-body systems from two perspectives. First, we attempted to understand black holes by examining a very specific, very simple model which we hoped would encode enough of the physics of the problem to be informative. On the other hand, our study of matrix product states started from the point of view that one should make no assumptions at all about the specific details of any particular system, instead opting for a statistical approach, where we tried to calculate what was typical of a certain class of Hamiltonians realized in nature. In both cases, however, we were unable to accurately handle the many degrees of freedom such system exhibited. In the scrambling-time problem, our efforts to reduce the problem to its simplest components made it hard for us to know what the physical questions of relevance were, a frustrating conclusion since computations in our model were so straightforward. In trying to refine Page's result to matrix product states, we found that our statistical approach rendered it hard to attack the problem directly since many-body systems have so many degrees of freedom that mix in a complicated way.

In both cases, however, we learned enough to recognize promising approaches for further research. The extension of the Iizuka-Polchinski model to include quadrilinear interactions opened up a new door for constructing a coherent space-time picture (and therefore, a concrete definition of a thermalization operator) using techniques from combinatorics and representation theory. In this new model, much of the algebraic

simplicity of the trilinear model is retained, but the overall picture is much richer. In the matrix product state formalism, the ensemble proposed by Zanardi, et al., may be amenable to both a direct computation of the induced measure on the space of matrix product states and a calculation of the distribution of the entanglement entropy by means of the concentration of measure. Interestingly, if the concentration of measure result derived for local observables holds for the entanglement entropy, we can conclude that the scrambling time problem isn't actually a problem at all if black holes evolve according to a local Hamiltonian. Our analysis of matrix product states provides evidence supporting this conjecture, although it would be desirable to expand our results to systems which are not limited to one dimension.

In physics, not knowing what the right question to ask is often the biggest barrier to progress. By learning more about why some questions don't lead to useful results, we have come some distance to finding avenues that do.

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