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The symmetry of mobility laws for viscous flow along arbitrarily patterned surfaces

Ken Kamrin[1,a)] and Howard A. Stone[2]

[1]Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

[2]Department of Mechanical and Aerospace Engineering, Princeton University, Princeton, New Jersey 08544, USA

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Generalizations of the no-slip boundary condition to allow for slip at a patterned fluid-solid boundary introduce a surface mobility tensor, which relates the shear traction vector tangent to the mean surface to an apparent surface velocity vector. For steady, low-Reynolds-number fluid motions over planar surfaces perturbed by arbitrary periodic height and Navier slip fluctuations, we prove that the resulting mobility tensor is always symmetric, which had previously been conjectured. We describe generalizations of the results to three other families of geometries, which typically have unsteady flow. © *2011 American Institute of Physics*. [doi:10.1063/1.3560320]

With recent advances in microfluidics, renewed interest has emerged in quantifying the effects of surface texture on fluid motion. When the scale of surface fluctuations is small compared to the size of macroscopic flow variations, it is advantageous to construct "effective boundary conditions,"[1–7] which mimic the far-field effects of a fluctuating boundary surface, but are applied instead on a smooth, mean surface. In three-dimensional flows, effective boundary conditions are generally tensorial in order to relate the fluid velocity to potentially misaligned applied stress tractions.[8–14] In this letter, we study Stokes flow along arbitrarily corrugated, nonuniformly hydrophobic, periodic surfaces, and present a proof of a previously posed conjecture regarding the symmetric nature of tensorial effective boundary conditions. Furthermore, we demonstrate symmetric properties in several related geometries involving surface fluctuations.

The arguments in this work are rooted in the reciprocal theorem of Lorentz,[15] which has been useful in other demonstrations of symmetric tensorial flow relations.[16,17] The theorem applies in Newtonian low-Reynolds-number flows where

$$\nabla \cdot \mathbf{T} = \mathbf{0}, \quad \nabla \cdot \mathbf{u} = 0, \quad \mathbf{T} = \eta(\nabla \mathbf{u} + \nabla \mathbf{u}^T) - p\mathbf{1}, \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, p is the pressure, $\mathbf{u}$ is the velocity field, and η is the viscosity.

Consider two flows that satisfy the Stokes equations (1) in some domain Ω; the flows differ due to different boundary conditions along $\partial\Omega$. Let the stress and velocity fields of the first flow be designated as $\mathbf{T}_1$ and $\mathbf{u}_1$, respectively, and those of the second be denoted by $\mathbf{T}_2$ and $\mathbf{u}_2$. The Lorentz reciprocal relation is

$$\int_{\partial\Omega} \mathbf{u}_1 \cdot \mathbf{t}_2 dS = \int_{\partial\Omega} \mathbf{u}_2 \cdot \mathbf{t}_1 dS, \tag{2}$$

where $\mathbf{t}=\mathbf{n}\cdot\mathbf{T}$ denotes the traction vector at a surface with outward unit normal $\mathbf{n}$.

[a)]Electronic mail: kkamrin@mit.edu.

We focus on a flow geometry appropriate to the study of effective boundary conditions.[14,18] Suppose an infinite rigid surface with arbitrary, periodic height fluctuations $H(x,y)$, i.e. a periodically perturbed plane, which is in contact with a tall layer of fluid. Let the period in the x and y directions be ℓ_x and ℓ_y, respectively [see Fig. 1(a)]. A Navier slip relation describes the interaction between the fluid and the rigid surface,

$$\mathbf{u}(x,y,H(x,y)) = b(x,y)(1 - \mathbf{nn}) \cdot \left.\frac{\partial \mathbf{u}}{\partial n}\right|_{(x,y,H(x,y))}. \tag{3}$$

The projection tensor $1-\mathbf{nn}$ enforces a purely tangential slip. Hence, the slip rate and shear rate along $H(x,y)$ are proportional. The proportionality scalar $b(x,y)$ is the Navier slip-length, which is 0 for no-slip surfaces and ∞ for perfect slip. We presume that $b(x,y)$ has the same periodicity as $H(x,y)$, but is otherwise arbitrary. By permitting b to depend on x and y, we allow the hydrophobicity of the surface to be spatially nonuniform.

Sufficiently far above the surface, at a height $z=z_H$, a horizontal shear traction $\boldsymbol{\tau}=(\tau_x,\tau_y,0)$ is applied. The traction induces a steady flow, which has some properties that can be determined at the outset. Of primary importance, the flow must approach a linear profile for sufficiently large values of z,

$$\mathbf{u}(x,y,z \gg D) = \boldsymbol{\tau} z/\eta + \mathbf{u}^s, \quad p(x,y,z \gg D) = P, \tag{4}$$

for some surface length scale D and constant P. The flow is akin to simple shear, but augmented by the possible addition of a constant "effective slip" velocity $\mathbf{u}^s=(u_x^s,u_y^s,0)$, which arises due to the interaction of the flow with the textured surface. It follows that $\mathbf{u}^s$ is uniquely defined up to the choice of the origin of the coordinate system, i.e. if one redefines $z=0$ to correspond to a new height, the same flow profile would correspond to a different slip velocity. To avert this issue, we define $z=0$ consistently at the peak height of the surface fluctuations.

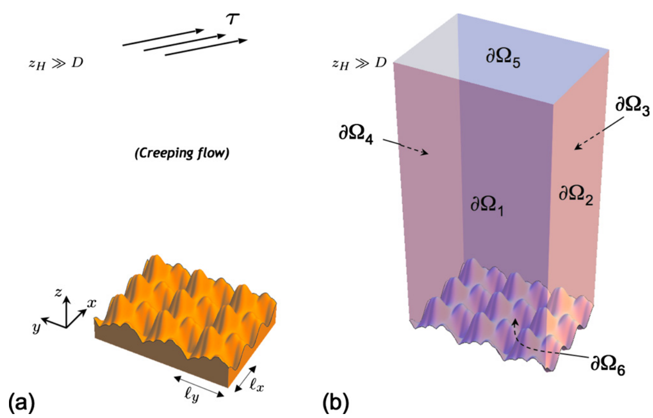

FIG. 1. (Color online) (a) Illustration of the setup and boundary conditions. The surface shape is periodic, but is otherwise arbitrary. The surface also has periodic Navier slip fluctuations. (b) The volume Ω is displayed with its six boundary faces labeled.

A basic linearity argument requires that $\mathbf{u}^s$ and $\boldsymbol{\tau}$ obey a relationship,

$$\begin{pmatrix} u_x^s \\ u_y^s \end{pmatrix} = \mathbf{M} \cdot \begin{pmatrix} \tau_x \\ \tau_y \end{pmatrix}, \quad (5)$$

for some 2×2 mobility tensor $\mathbf{M}$, which depends on the surface properties.

Is $\mathbf{M}$ always a symmetric tensor? This question is the motivating factor for the work herein. In Ref. 18, it was suggested without proof that $\mathbf{M}$ should be a symmetric matrix, using arguments based on molecular statistics and the commonly used Onsager–Casimir relations for linear thermodynamic response.[12,19,20] In Ref. 14, the method of domain perturbations was used to compute $\mathbf{M}$ up to second order, and the symmetry of $\mathbf{M}$ was verified assuming surface fluctuations with small curvature and slope. In particular cases such as parallel groove patterns, the symmetry of the mobility matrix is obvious from geometric considerations. However, when the surface shape and hydrophobicity are both arbitrary, the answer has not been given.

We now prove that $\mathbf{M}$ is always symmetric. Define the volume Ω as the prismatic shape depicted in Fig. 1(b). Its bottom surface is the rigid interface and its top surface a flat rectangle at height z_H. Other faces of Ω are vertical walls parallel to the x or y direction, spaced in a multiple of the corresponding periodicity.

Let $\mathbf{T}_1$ and $\mathbf{u}_1$ be the stress and velocity fields of the Stokes flow induced by shearing from above (at z_H) with stress $\boldsymbol{\tau}_1$, and let $\mathbf{T}_2$ and $\mathbf{u}_2$ correspond to the flow induced by shearing with an arbitrary traction $\boldsymbol{\tau}_2$. Applying Lorentz reciprocity to these two flows and dividing the boundary, as shown in Fig. 1(b), we have

$$\sum_{n=1}^{6} \int_{\partial\Omega_n} \mathbf{u}_1 \cdot \mathbf{t}_2 dS = \sum_{n=1}^{6} \int_{\partial\Omega_n} \mathbf{u}_2 \cdot \mathbf{t}_1 dS. \quad (6)$$

The periodicity of the flow requires that the integrals over parallel walls be of opposite sign but equal in magnitude,

$$\int_{\partial\Omega_1} \mathbf{u}_i \cdot \mathbf{t}_j dS = -\int_{\partial\Omega_3} \mathbf{u}_i \cdot \mathbf{t}_j dS, \qquad \int_{\partial\Omega_2} \mathbf{u}_i \cdot \mathbf{t}_j dS = -\int_{\partial\Omega_4} \mathbf{u}_i \cdot \mathbf{t}_j dS, \quad (7)$$

for $i,j \in \{1,2\}$ and $i \neq j$. Next, observe that at any point $(x,y,H(x,y))$ on $\partial\Omega_6$,

$$\mathbf{u}_1 \cdot \mathbf{t}_2 = \mathbf{u}_1 \cdot \left[\eta(1-\mathbf{nn}) \cdot \frac{\partial \mathbf{u}_2}{\partial n} - p_2 \mathbf{n} \right] = (\mathbf{u}_1 \cdot \mathbf{u}_2)\frac{\eta}{b}, \quad (8)$$

where we have used the slip condition and orthogonality of the fluid velocity and the rigid surface normal. Similarly, $\mathbf{u}_2 \cdot \mathbf{t}_1 = (\mathbf{u}_1 \cdot \mathbf{u}_2)(\eta/b)$, implying that

$$\mathbf{u}_1 \cdot \mathbf{t}_2 = \mathbf{u}_2 \cdot \mathbf{t}_1 \quad (9)$$

at any point on $\partial\Omega_6$. Likewise, we have

$$\int_{\partial\Omega_6} \mathbf{u}_1 \cdot \mathbf{t}_2 dS = \int_{\partial\Omega_6} \mathbf{u}_2 \cdot \mathbf{t}_1 dS. \quad (10)$$

Taking Eqs. (7) and (10) into account, Eq. (6) reduces to

$$\int_{\partial\Omega_5} \mathbf{u}_1 \cdot \mathbf{t}_2 dS = \int_{\partial\Omega_5} \mathbf{u}_2 \cdot \mathbf{t}_1 dS. \quad (11)$$

At $z=z_H$, Eq. (4) describes the flow, so the integral equality can be written as

$$\int_{\partial\Omega_5} (\boldsymbol{\tau}_1 z_H/\eta + \mathbf{u}_1^s) \cdot (\boldsymbol{\tau}_2 - P_2 \hat{\mathbf{z}}) dS = \int_{\partial\Omega_5} (\boldsymbol{\tau}_2 z_H/\eta + \mathbf{u}_2^s) \cdot (\boldsymbol{\tau}_1 - P_1 \hat{\mathbf{z}}) dS \quad (12)$$

for constant pressures P_1 and P_2 and unit vector $\hat{\mathbf{z}}$. Then, by orthogonality,

$$\int_{\partial\Omega_5} \mathbf{u}_1^s \cdot \boldsymbol{\tau}_2 dS = \int_{\partial\Omega_5} \mathbf{u}_2^s \cdot \boldsymbol{\tau}_1 dS. \quad (13)$$

Utilizing Eq. (5) and the fact that the flow and stress are uniform on $\partial\Omega_5$, the last equation directly implies that

$$\boldsymbol{\tau}_2 \cdot \mathbf{M} \cdot \boldsymbol{\tau}_1 = \boldsymbol{\tau}_1 \cdot \mathbf{M} \cdot \boldsymbol{\tau}_2 \quad (14)$$

for arbitrary choices of $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$.

Consequently, the tensor $\mathbf{M}$ is always symmetric. This result implies that regardless of the anisotropy of the surface texturing, there must always exist two orthogonal in-plane directions for which an applied shear stress aligns with the induced effective slip. Also, a symmetric $\mathbf{M}$ means only three scalar measurements are necessary to compute $\mathbf{M}$, which is advantageous for applications. Notice that the proof was carried out for a surface with both height and Navier slip fluctuations. Thus, it follows that $\mathbf{M}$ must be symmetric in the case of a no-slip surface with arbitrary periodic height fluctuations and the case of a flat surface with arbitrary periodic slip properties. We are not aware of a previous demonstration of these results.

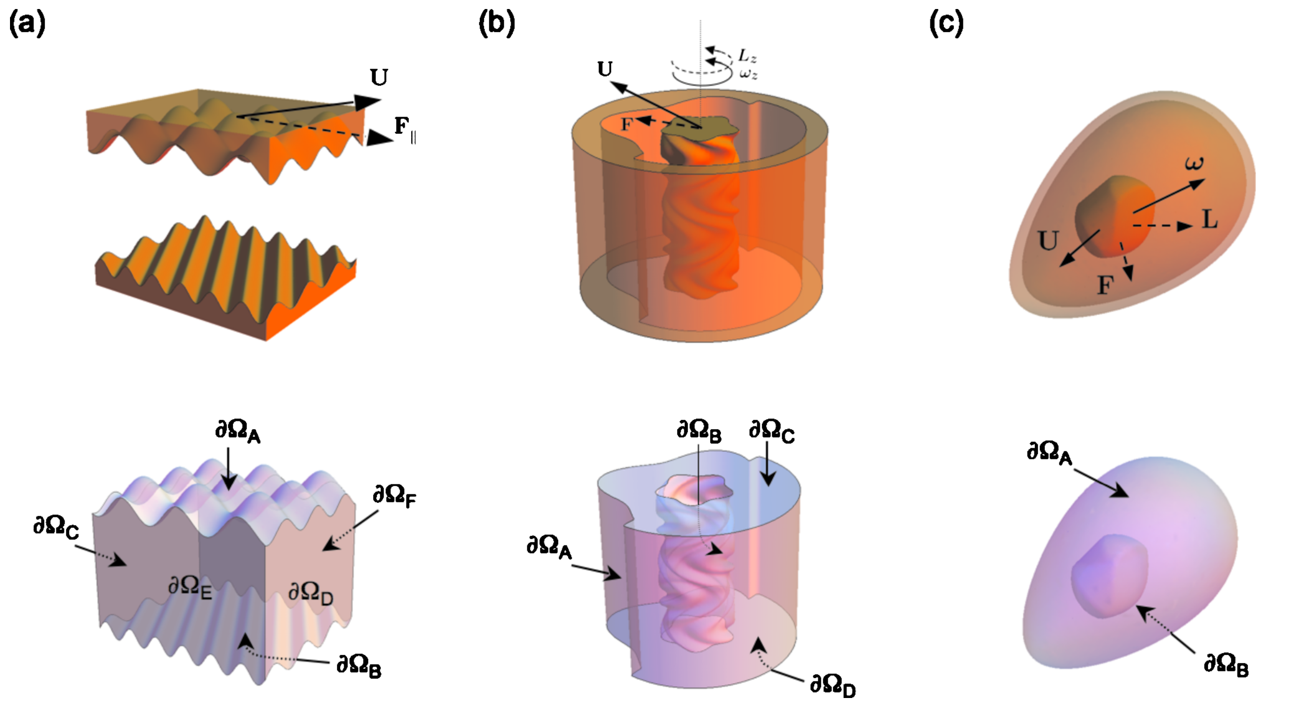

FIG. 2. (Color online) Representative geometries where, by reasoning analogous to Eqs. (6)–(14), the matrix relating the generalized force to velocity can be proven symmetric. In each case, all rigid surfaces are permitted an arbitrary Navier slip distribution of period equal to the period of the shape fluctuations. From (a) to (b) to (c), each successive case removes one periodicity requirement on the surface patterning.

By modifying this derivation, we next proceed to prove symmetry relations in a variety of related geometries, as presented in Fig. 2. First, as depicted in Fig. 2(a), consider a finite-height "sandwich" version of Fig. 1, with two surfaces of equal periodicity in the x and y directions, each having arbitrary shapes, $h_A(x,y)$ and $h_B(x,y)$, and Navier slip properties, $b_A(x,y)$ and $b_B(x,y)$, within a periodic cell. We suppose that the surfaces are separated by a layer of viscous fluid and that one surface is sheared with respect to the other. One important difference from the first example is that the current geometry does not generally support a steady flow. This caveat is not a problem as long as we specify that the quantities of interest are defined instantaneously.

Suppose for some particular relative positioning of the surfaces, shearing motion is induced by an instantaneous force applied to the periodic cell of the top surface. Let us refer to $\mathbf{F}_{\parallel}=(F_x,F_y)$ as the component of that force lying parallel to the surface, and let the resulting top-surface motion be $\mathbf{U}=(U_x,U_y)$. Note that the surface velocity cannot have a z-component by volume conservation and periodicity; however, the applied force can have a nonzero z-component that is ultimately unrelated to the flow. The in-plane force and velocity must be related linearly through a 2×2 mobility tensor $\mathbf{M}^{(a)}$ defined by

$$\mathbf{U}=\mathbf{M}^{(a)}\cdot\mathbf{F}_{\parallel}. \tag{15}$$

We shall now prove that $\mathbf{M}^{(a)}$ is symmetric.

Let $\mathbf{F}_{\parallel 1}$ and $\mathbf{F}_{\parallel 2}$ be two arbitrary forces, inducing flow fields $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively. We apply Lorentz reciprocity in the fluid region between the rigid surfaces and which is bounded laterally by a periodic cell, as pictured at the bottom of Fig. 2(a). Similar to the previous derivation, all integrals over surfaces $\partial\Omega_C$ through $\partial\Omega_F$ cancel due to flow periodicity, and integrals over $\partial\Omega_B$ cancel due to the reasoning in Eqs. (8) and (9). Hence, we find an integral relationship over the moving surface,

$$\int_{\partial\Omega_A}\mathbf{u}_1\cdot\mathbf{t}_2 dS=\int_{\partial\Omega_A}\mathbf{u}_2\cdot\mathbf{t}_1 dS. \tag{16}$$

For the moving surface, the Navier slip condition at any point on $\partial\Omega_A$ is written as

$$\mathbf{u}(x,y)-\mathbf{U}=b_A(x,y)(1-\mathbf{nn})\cdot\frac{\partial\mathbf{u}}{\partial n}. \tag{17}$$

Hence, we rewrite Eq. (16) as

$$\int_{\partial\Omega_A}\left[\mathbf{U}_1\cdot\mathbf{t}_2+\eta b_A\frac{\partial\mathbf{u}_2}{\partial n}\cdot(1-\mathbf{nn})\cdot\frac{\partial\mathbf{u}_1}{\partial n}\right]dS$$
$$=\int_{\partial\Omega_A}\left[\mathbf{U}_2\cdot\mathbf{t}_1+\eta b_A\frac{\partial\mathbf{u}_1}{\partial n}\cdot(1-\mathbf{nn})\cdot\frac{\partial\mathbf{u}_2}{\partial n}\right]dS.$$

Since the surface projection tensor $1-\mathbf{nn}$ is symmetric, the above reduces to

$$\int_{\partial\Omega_A}\mathbf{U}_1\cdot\mathbf{t}_2 dS=\int_{\partial\Omega_A}\mathbf{U}_2\cdot\mathbf{t}_1 dS. \tag{18}$$

Finally, since $\mathbf{U}_1$ and $\mathbf{U}_2$ are constant vectors and the integral of the traction on a surface gives the net force, it follows that

$$\mathbf{U}_1\cdot\mathbf{F}_{\parallel 2}=\mathbf{U}_2\cdot\mathbf{F}_{\parallel 1}. \tag{19}$$

Substituting $\mathbf{U}_{(1,2)}=\mathbf{M}^{(a)}\cdot\mathbf{F}_{\parallel(1,2)}$ in the above proves that $\mathbf{M}^{(a)}$ is symmetric.

Since $\mathbf{M}^{(a)}$ is symmetric for any instant and relative orientation of the surfaces, similar conclusions can be drawn for averaged quantities. For instance, for any given force $\mathbf{F}_{\parallel}$, let $\langle \mathbf{U} \rangle$ represent the ensemble-averaged velocity induced by $\mathbf{F}$, over all relative orientations of the surfaces with mean separation distance fixed. Since each realization possesses a symmetric mobility, we deduce that a symmetric tensor relates $\mathbf{F}_{\parallel}$ to $\langle \mathbf{U} \rangle$. Or, if we instead fix $\mathbf{U}$ and ensemble-average over force, the same argument switching $\mathbf{M}^{(a)}$ with $(\mathbf{M}^{(a)})^{-1}$ gives that $\mathbf{U}$ and $\langle \mathbf{F}_{\parallel} \rangle$ are also related by a symmetric tensor. Note that if the top face is flat and has uniform Navier slip, the induced flow profile is steady and all average quantities equate to the instantaneous.

By identifying the faces $\partial\Omega_E$ and $\partial\Omega_F$, we can wrap this flow environment into a new topological family representing flow between partially closed, z-periodic surfaces [see Fig. 2(b)] for the z-axis along the Ω_B axial center-of-mass. The derivation from the prior case carries over up to Eq. (18), with the modification that $\mathbf{U} \rightarrow (y,-x,0)\omega_z + \mathbf{U}$. Unlike the prior case, this geometry places no volume conservation constraints on $\mathbf{U}$ so $\mathbf{U}$ can be any three-dimensional vector. Defining the per-length force and z-torque on the inner body by $\mathbf{F}$ and L_z, we integrate Eq. (18) to show that the mobility relation between the forces and velocities must be symmetric, i.e.,

$$\begin{pmatrix} \mathbf{U} \\ \omega_z \end{pmatrix} = \mathbf{M}^{(b)} \cdot \begin{pmatrix} \mathbf{F} \\ L_z \end{pmatrix} \tag{20}$$

must be characterized by a symmetric 4×4 tensor $\mathbf{M}^{(b)}$.

Lastly, we can remove all external periodicity requirements on the surface patterning by considering flows between two arbitrarily shaped, arbitrarily hydrophobic fully closed surfaces [see Fig. 2(c)]. This geometry eliminates surfaces $\partial\Omega_C$ and $\partial\Omega_D$ from the prior case and can be obtained from Fig. 2(a) under a deformation that brings the edges of each bumpy surface to a single point. The inner body is free to rotate (ω) and translate ($\mathbf{U}$). Letting the corresponding force and torque vectors be $\mathbf{F}$ and $\mathbf{L}$, respectively, the mobility matrix $\mathbf{M}^{(c)}$ for this problem is 6×6 and is defined by

$$\begin{pmatrix} \mathbf{U} \\ \boldsymbol{\omega} \end{pmatrix} = \mathbf{M}^{(c)} \cdot \begin{pmatrix} \mathbf{F} \\ \mathbf{L} \end{pmatrix}. \tag{21}$$

Following analogous steps from before, Eq. (18) can be obtained for this geometry under the modification $\mathbf{U} \rightarrow \boldsymbol{\omega} \times \mathbf{r} + \mathbf{U}$ (origin at the center-of-mass of Ω_B), and the symmetry of $\mathbf{M}^{(c)}$ is found upon integrating the relation.

This final result reveals a connection to a classical finding of viscous flow theory—that the resistance tensor of an arbitrary particle moving in a viscous fluid is necessarily symmetric.[15] Note, however, that we have added a new ingredient, allowing all surfaces to have arbitrary Navier slip fluctuations. In fact, following similar lines, we can prove that other known symmetry relations for no-slip particles in viscous flow remain symmetric when the particle has nonuniform surface hydrophobicity, as is the case for the symmetry noted in Ref. 17 of the fourth-rank tensor $\mathbf{C}$ relating a particle's stresslet $\mathbf{S}$ to a far-field pure strain rate $\mathbf{E}$.

We also note that every result in this work could be rephrased into an elasticity problem, arrived at by replacing all velocities with displacements—the viscous stress law for the bulk becomes a small-strain elasticity law, the Lorentz reciprocal relation becomes the well-known Betti reciprocal relation for elasticity, and the Navier slip condition becomes an adhesion law $\boldsymbol{\delta} = \alpha(x,y)\gamma$ relating sliding displacement along the surface to the local shear strain. Such an adhesion law is essentially a traction-displacement relation, like those used in fracture mechanics to model cohesive-zone crack surfaces.[21,22] Our above analyses would then show that the corresponding elastic "mobility" tensors, which relate vectors of generalized force and total displacement, are necessarily symmetric.

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