Monomization of Power Ideals and Parking Functions

by

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Abstract

A zonotopal algebra is the quotient of a polynomial ring by an ideal generated by powers of linear forms which are derived from a zonotope, or dually it's hyperplane arrangement. In the case that the hyperplane arrangement is of Type A, we can rephrase the definition in terms of graphs. Using the symmetry of these ideals, we can find monomial ideals which preserve much of the structure of the zonotopal algebras while being computationally very efficient, in particular far faster than Gröbner basis techniques. We extend this monomization theory from the known case of the central zonotopal algebra to the other two main cases of the external and internal zonotopal algebras.

Thesis Supervisor: Alexander Postnikov
Title: Associate Professor

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Contents

Chapter 1

Introduction

1.1 Background on Power Ideals

Ideals generated by powers of linear forms have arisen in several areas recently. They appear in work on linear diophantine equations and discrete splines [6], the study of zonotopes [10][7], zonotopal Cox rings [18], ideals of fat points [9], and other areas. Of particular interest is the computation of the colengths of quotients by these ideals, as well as their Hilbert series. In many cases these computations have been connected to the computation of other statistics which are more germane to the problem in which they appear (eg.,[1],[18]).

In this work, we are concerned with a special class of power ideals associated with graphs. More specifically, in [10] Holtz and Ron identify three major classes of algebras that can be attached to a zonotope or, dually, to a hyperplane arrangement. They call these *zonotopal algebras*. Since the construction of these algebras depends only on the matroid theory of the hyperplane arrangement, these algebras can be redescribed in terms of graphs in the case that the hyperplane arrangement is of Type A. The simplest and most classical of these algebras, the so-called *central algebra*, has thus far received the most attention. Indeed, our main results will be generalizations of an idea of Postnikov and Shapiro [14] to the other major classes of zonotopal algebra, the *external* and *internal algebras*.

We recall here some of the background, history, and motivation for the study of these algebras. Throughout, we will be working in a finitely generated polynomial ring over $\mathbb{C}$. To be consistent with the literature we let $V = \mathbb{C}^n$ and make the following definition.

Definition 1. A *power ideal* is an ideal $I \in \mathbb{C}[V]$ that is generated by powers of linear forms such that I has finite colength. Equivalently, there is a collection of linear forms $h_1, \ldots, h_k$ which span V^*, and non-negative integers $r_1, \ldots, r_k$ such that $I = (h_1^{r_1}, \ldots, h_k^{r_k})$.

Now let X be any $n \times r$ matrix, thought of as a collection of r (column) vectors in V. Let $\mathcal{H}$ be the arrangement of hyperplanes dual to these vectors, i.e. $\mathcal{H} = \{H_i\}$ where $H_i = \{y \in V^* | \langle x_i, y \rangle = 0\}$. Now let ρ_{H_i} be a function on V^* which is 1 on H_i and 0 everywhere else, and let

$$\rho_{\mathcal{H}} = \sum_{i=1} (\rho_{H_i})^n$$

Then for any k we define the ideal in $\mathbb{C}[V]$

$$\mathcal{I}_{\mathcal{H},k} = \left\langle y^{\rho_{\mathcal{H}}(y)+k} | y \in V^* \right\rangle$$

Notice that we do allow vectors to appear with multiplicity in X, and likewise hyperplanes may appear with multiplicity in $\mathcal{H}$.

The importance of these ideals to approximation theory appears via their inverse systems, the definition and application of which we briefly describe now. Given our X, the *box spline* is defined as the distribution satisfying

$$\int_{\mathbb{R}^n} f(x)\, B(x|X) = \int_{[0,1]^r} f\left(t^T X\right) dt_1 \ldots dt_r$$

for any function $f \in \mathcal{C}(\mathbb{R}^r)$. It is known to be a piecewise polynomial function on the image of X (thought of as a map from $\mathbb{R}^r$ to $\mathbb{R}^n$). An important question in approximation theory, and particularly in computer graphics design, is to understand the space of functions generated by integer translates of $B(x|X)$, that is

$$\mathcal{S}(X) = \{B(x-\alpha|X) \,|\alpha \in \mathbb{Z}^n\}.$$

In particular, one would like to describe in some way the complexity of the functions in this space.

Now consider the *(Macaulay) inverse system* of our ideal $\mathcal{I}_{\mathcal{H},0}$. To define the inverse system, we first define a pairing on the polynomial ring. Given $f, g \in \mathbb{C}[x_1, \ldots, x_n]$, we let $\langle f, g\rangle$ be the result of interpreting each x_i in f as $\partial/\partial x_i$, applying the operator to g, and then evaluating the resulting polynomail at 0. The inverse system of an ideal I is then defined by

$$I^{\perp} := \{f \in \mathbb{C}[x_1, \ldots, x_n] \,|\, \langle f, g\rangle = 0 \text{ for all } g \in I\}.$$

The main theorem in this vein is due to Dahmen and Micchelli, although a similar result appeared simultaneously in a paper by Jia.

Theorem 2. *[5] [11] Any function in $\mathcal{S}(X)$ is a piecewise polynomial, and the polynomials appearing in it are all in $\mathcal{I}_{\mathcal{H},0}^{\perp}$. Furthermore the dimension of $\mathcal{I}_{\mathcal{H},0}^{\perp}$ is equal to the number of bases of the hyperplane arrangement $\mathcal{H}$.*

A fundamental and simple fact about inverse systems over a field of characteristic 0 is that it's dimension is equal to that of the quotient of the polynomial algebra modulo the ideal, and if the ideal is homogeneous as in our case, the same holds for the graded parts. So, if we are content to answer questions about the number of polynomials and their degrees, we can work with the commutative algebra, an object whose properties are often more familiar.

As for the enumerative theory of power ideals, the farthest reaching result to date is due to Ardila and Postnikov [1]. They establish a relationship amongst the ideals with respect to restriction and deletion, allowing them to relate the formula for its Hilbert Series to the Tutte polynomial of the hyperplane arrangement. One precise statement of their result is as follows.

Theorem 3. *[1] For a central hyperplane arrangement of r hyperplanes in V and for $k \geq 1$,*

$$\sum_{k\geq 1} \operatorname{Hilb}\left(\mathbb{C}\left[x_1,\ldots,x_n\right]/\mathcal{I}_{\mathcal{H},k},t\right) z^{k-1} = \frac{t^{r-n}}{(1-z)} T_{\mathcal{H}}\left(1+\frac{t}{1-tz},\frac{1}{t}\right)$$

where $T_{\mathcal{H}}$ is the Tutte polynomial of the hyperplane arragnement.

For the cases $k=-1,0,1$, the ideals $\mathcal{I}_{\mathcal{H},k}$ admit a slightly simpler description, and it is these cases that Holtz and Ron have dubbed the internal, central, and external zonotopal algebras, respectively. We now give this simpler description, and at the same time introduce the relationship to parking functions.

1.2 Parking functions

Let us recall some definitions and facts about parking functions and some of their generalizations. A *parking function* of length n is a sequence of non-negative integers $(a_1,\ldots,a_n)$ such that for $i\in[n]$

$$\#\left\{j|a_j<i\right\}\geq i.$$

Note that we allow $a_i=0$; the definition is often written for positive integers instead of non-negative, in which case the '$<$' above is replaced with '$\leq$'. The definition is equivalent to requiring that the increasing rearrangement $b_1\leq\ldots\leq b_n$ has the property $b_i<i$. The origin of the term parking function comes from the following interpretation. Suppose n cars arrive at a linear parking lot with parking spaces labelled between 0 and $n-1$. Each car, i, has a preferred parking space, a_i. The cars arrive in order and drive to their preferred spot. If it is already taken, then they drive until they reach the next empty spot and take that one. A preference function $a_\bullet$ is a parking function if (and only if) every car gets a parking spot without having to turn around.

Several generalizations have appeared in the literature. In [17], Stanley defines

k-parking functions similarly, with the following distinction. If $(b_1, \ldots, b_n)$ is the increasing rearrangement, then $b_i \leq k(i-1)$. A further generalization are the *$\vec{x}$-parking functions* of Yan [19]. Here we have a non-decreasing sequence $\vec{x} = (x_1, \ldots, x_n)$, and a sequence is an $\vec{x}$-parking function if the increasing rearrangement $b_1 \leq \ldots \leq b_n$ satisfies $b_i \leq x_i$. A k-parking function is an $\vec{x}$-parking function when $\vec{x} = (0, k, 2k, \ldots, nk)$.

A G-parking function is another broad generalization, appearing first in [14], and later in [3, 4]. Let G be a digraph on vertices labelled $0, 1, \ldots, n$. For this definition, and throughout the rest of this work, by graph we will mean a finite graph with no loops, but with multiple edges allowed. We will call 0 the root of G. For every non-empty subset $I \subset [n]$ and $i \in I$ define $d_I(i)$ to be the number of edges originating at i and terminating at a vertex *not* in I. A *G-parking function* is defined to be a function f assigning a non-negative integer to the vertices $1, \ldots, n$ such that for every non-empty subset I there is an $i \in I$ such that $f(i) < d_I(i)$.

Example 4. When $G = K_{n+1}$ the complete graph on n vertices, the G-parking functions are the same as the ordinary parking functions defined above.

Example 5. In fact we can recover Stanley's k-parking functions with this notion as well. Let G be the graph on $0, 1, \ldots, n$ with $a + 1$ edges between 0 and any other vertex and $b + 1$ edges between any two non-zero vertices. The corresponding G-parking functions will be $(a, b, 2b, \ldots, nb)$-parking functions, and in particular cover the k-parking case when $a = 0$ and $b = 1$.

We now give an algebraic reinterpretation of the G-parking functions. For any $I \subset [n]$, we define D_I to be the the total number of edges of G which originate at a vertex in I and terminate at a vertex outside of I. Explicitly,

$$D_I = \sum_{i \in I} d_I(i)$$

Now let k be any integer. If $D_I + k > 0$ for every I as above, we define a polynomial

p_I in the ring $\mathbb{C}[x_1, \ldots, x_n]$ given by

$$p_I = \left(\sum_{i \in I} x_i\right)^{D_I + k}$$

Note that this will always be the case when k is positive. We then define $\mathcal{I}_{G,k}$ to be the power ideal generated by all such p_I, and further define $\mathcal{A}_{G,k}$ to be the quotient $\mathbb{C}[x_1, \ldots, x_n] / \mathcal{I}_{G,k}$. if we interpret G as a representation of a type A hyperplane arrangement $\mathcal{H}$, these ideals are the same as the ideals $\mathcal{I}_{\mathcal{H},k}$ in the cases $k = -1, 0, 1$. Although it's easy to see $\mathcal{I}_{G,k} \subset \mathcal{I}_{\mathcal{H},k}$, showing identity is nontrivial, and the details can be found in [1].

Since $\mathcal{I}_{G,k}$ is a homogeneous ideal, $\mathcal{A}_{G,k}$ will have a basis of monomials. Given a monomial basis B of $\mathcal{A}_{G,k}$, the set of monomials $\mathcal{M} = \mathbb{C}[x_1, \ldots, x_n] \backslash B$ is an ideal, and B is the basis of *standard monomials* for $\mathbb{C}[x_1, \ldots, x_n] / \mathcal{M}$. We call any such $\mathcal{M}$ a *monomization* of the ideal $\mathcal{I}_{G,k}$. Our program is to find a monomization for the ideals $\mathcal{I}_{G,k}$ which is natural in some way and easy to compute. Such a theory would greatly simplify the study of the linear structure of the rings $\mathcal{A}_{G,k}$, and in particular the dimensionality.

In the case $k = 0$, the picture is especially beautiful. In [14], it is shown that the monomials $x^a = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ where $(a_1, \ldots, a_n)$ is a G-parking function give a monomial basis for $\mathcal{A}_{G,0}$. Translating this to the ideal $\mathcal{J}_{G,0}$ of monomials x^a where a is *not* a G-parking function, we find the following description. For every non-empty $I \subset [n]$, let

$$m_I = x_{i_1}^{d_I(i_1)} \cdot \ldots \cdot x_{i_r}^{d_I(i_r)}$$

Then, $\mathcal{J}_{G,0} = \langle m_I \rangle_{I \subset [n]}$ is a monomization of $\mathcal{I}_{G,k}$.

There are several features of this monomization we would like to emulate. Firstly, a set of generators for $\mathcal{J}_{G,0}$ is relatively easy to compute from G. Secondly, both the generators of $\mathcal{I}_{G,0}$ and $\mathcal{J}_{G,0}$ are indexed by the same set, and in particular are the same size. There is a third very useful feature of $\mathcal{J}_{G,0}$, which we would like to emulate, but won't be able to. Clearly the group $H = \mathrm{Aut}_*(G)$ of basepoint preserving automorphisms of G acts on $\mathbb{C}[x_1, \dots, x_n]$, and since it preserves the p_I, H also acts on $\mathcal{A}_{G,k}$. Since the definition of G-parking function is invariant under the action of H, we see that this basis is H-invariant. We can thus use the combinatorial structure of the G-parking functions to understand not just the linear structure of $\mathcal{A}_{G,0}$, but also its structure as an H-representation. For example, this shows that the multiplicity of the trivial representation of $\mathfrak{S}_n$ on $\mathcal{A}_{K_{n+1},0}$ is equal to $\frac{1}{n+1}\binom{2n}{n}$, the n^{th} Catalan number, from the corresponding well-known result for parking functions.

For $k \neq 0$, we cannot in general find an $\mathrm{Aut}_*(G)$ invariant basis of monomials, even when G is a complete graph.

Example 6. Let $G = K_3$, the triangle. Then $\mathcal{I}_{G,0} = \left(x^2, y^2, (x+y)^2\right)$. The parking functions are $\{(0,1), (1,0), (0,0)\}$, and indeed

$$\mathcal{A}_{G,0} = \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}y$$

Now for $k = 1$, $\mathcal{I}_{G,1} = \left(x^3, y^3, (x+y)^3\right)$. The monomials which are non-zero in $\mathcal{A}_{G,1}$ are $1, x, y, x^2, xy, y^2, x^2y$, and xy^2. As we can easily verify, however, the Hilbert series of $\mathcal{A}_{G,1}$ is $1 + 2t + 3t^2 + t^3$. In particular, any monomial basis must contain $1, x, y, x^2, xy$, and y^2, and must can contain exactly one of x^2y or xy^2. Thus there is no way to choose an $\mathfrak{S}_3$-invariant basis of monomials.

Nonetheless, our goal will be to achieve an analogous construction for $k = -1$ and $k = 1$ in the remainder of this thesis.

1.3 Outline

The rest of this thesis is organized as follows. In Chapter 2 we discuss the already known case of the central algebra. In this section, we introduce some of the tools that will be used to attack the external and internal algebras. Chapters 3 and 4 deal respectively with the external and internal algebras. While the proof outline of both cases is similar to that of the central algebra, the combinatorial details are quite different. Finally in Chapter 5, some remarks and conjectures are made regarding the work done in this paper as well as avenues for future investigation.

Chapter 2

Monotone Monomial Ideals and the Central Algebra

2.1 Monotone Monomial Ideals

The idea of monotone monomial ideals and their properties will be critical to securing tightness on many of the bounds in our results, and therefore proving monomization. They were introduced in [14] and used there to establish monomization of the central algebra. Following the publication of [1], much of the argument from that orginal paper can be simplified, and serves as the model for our development in the external and internal cases.

Therefore, let us recall some facts about monotone monomial ideals. Although the theory in [14] is developed over a general poset, we only need to use the case when the underlying poset is the boolean algebra. Since the definitions and properties are somewhat easier then, we restrict immediately to that case. Let $\{m_I\}$ be any collection of monomials in $\mathbb{C}[x_1, \ldots, x_n]$. Let $m_{J \setminus I}$ be the monomial formed from m_J by removing all x_i with $i \in I$, and let $\bar{I} = [n] \setminus I$. Then the collection $\{m_I\}$ is a monotone monomial family if

- $m_{I \setminus I} = 1$ and
- if $I \subset J$, then $m_{J \setminus \bar{I}}$ divides m_I.

The first condition simply says that m_I contains only the the variables x_i for $i \in I$. The second condition says that if i is in both I and J, then the degree of x_i in m_J is less than or equal to that degree in m_I.

Given an index set $I = \{i_1, \ldots, i_r\}$ and a monomial m_I in the variables $\{x_{i_1}, \ldots, x_{i_r}\}$, we say that a homogeneous polynomial p_I (also in these variables) is an *I-deformation* of m_I if the set of monomials not divisible by m_I gives a linear basis for the algebra $\mathbb{C}[x_{i_1}, \ldots, x_{i_r}]$. That is

$$\mathbb{C}[x_{i_1}, \ldots, x_{i_r}] = \langle R_{m_I} \rangle \oplus (p_I)$$

where R_{m_I} is the set of monomials not divisible by m_I.

In fact, as long as $\deg(m_I) = \deg(p_I)$, almost any polynomial is an I-deformation. In fact,

Proposition 7. *[14] If m_I is a monomial in $\mathbb{C}[x_1, \ldots, x_r]$, and $\{\alpha_i\}_{i \in I}$ is any collection of non-zero constants, then*

$$p_I = (\alpha_1 x_1 + \ldots + \alpha_r x_r)^{\deg(m_I)}$$

is an I-deformation of m_I.

Now take a monomial m_I for every non-empty subset $I \subset [n]$, and for each an I-deformation p_I. We say that the ideal $\mathcal{I} = (p_I)$ is a *deformation* of the ideal $\mathcal{J} = \{m_I\}$. Of course, almost by definition, for any fixed I, we have

$$\mathrm{Hilb}(\mathbb{C}[x_1, \ldots, x_n] / (m_I), t) = \mathrm{Hilb}(\mathbb{C}[x_1, \ldots, x_n] / (p_I), t).$$

Notice there are more variables in the ring than just those indexed by I. However, there is no guarantee that the Hilbert series of (the quotients by) $\mathcal{J}$ and $\mathcal{I}$ will be equal, and indeed they typically differ. However, if $\mathcal{J}$ is a monotone monomial ideal, we have the following result

Theorem 8. *[14] If $\mathcal{I}$ is a deformation of the monotone monomial ideal $\mathcal{J}$, then the set of monomials not in $\mathcal{J}$ linearly span the algebra $\mathbb{C}[x_1,\ldots,x_n]/\mathcal{I}$. In particular*

$$\mathrm{Hilb}\left(\mathbb{C}[x_1,\ldots,x_n]/\mathcal{I},t\right) \leq \mathrm{Hilb}\left(\mathbb{C}[x_1,\ldots,x_n]/\mathcal{J},t\right).$$

The other important theorem on monotone monomial ideals that we will need, and also factors into the proof of monomization for the central algebra is an inclusion exclusion formula for the dimension of the quotient algebra $\mathcal{B} := \mathbb{C}[x_1,\ldots,x_n,]/\mathcal{J}$ of a monotone monomial ideal.

Lemma 9. *[13][14] Let $\mathcal{J}$ be a monotone monomial ideal generated by $m_I = \prod_{i\in I} x_{i_1}^{\nu_I(i)}$. Then the dimension of $\mathcal{B}$ is equal to the alternating sum*

$$\sum_{I_1\subsetneq\ldots\subsetneq I_k} (-1)^k \prod_{i\in I_1}\left(\nu_{\{i\}}(i)-\nu_{I_1}(i)\right)\times\ldots\times\prod_{i\in I_k\setminus I_{k-1}}\left(\nu_{\{i\}}(i)-\nu_{I_k}(i)\right)\times\prod_{i\notin I_k}\nu_{\{i\}}(i) \tag{2.1.1}$$

where we include the empty chain of subsets where $k=0$.

2.2 The Central Algebra

The central zonotopal algebra for graphs is the quotient algebra of $\mathcal{I}_{G,0}$ from the introduction, that is, the case $k=0$. It's importance to spline theory was described there. The proof that G-parking functions give a monomization of this ideal was first proven by Postnikov and Shapiro, using the ideas just introduced. A crucial part of the proof lies in showing that (2.1.1) is equal to the number of spanning trees of the graph G. The idea is to show that term counts oriented subgraphs of G of a certain form, and to show that in the sum the only such subgraphs with non-zero weight are the spanning trees (with some canonical orientation), and that these have weight 1. This methodology is a theme for our proofs in the external and internal cases.

As the proof goes, this demonstrates only that the dimension of $\mathcal{B}_{G,0}$ is equal to the number of spanning tree. The rest of Postnikov and Shapiro's work involves introducing several other algebras, connecting $\mathcal{A}_{G,0}$ and $\mathcal{B}_{G,0}$. Thanks to the work of Ardila and Postnikov [1], this work is greatly simplified. Although the particular result is not a direct application of Theorem 3, it is very similar. Applied to the graphical case it says that

$$\operatorname{Hilb}\left(\mathcal{A}_{G,0}, t\right) = t^{|E(G)|-n} T_G\left(1, \frac{1}{t}\right)$$

where $|E(G)|$ is the number of edges in the graph. This says that dimension of the the r^{th} graded part is equal to the number of spanning trees with external activity $|E(G)|-n-r$. Because the dimension in particular is equal to the number of spanning trees, we get the equality

$$\dim(\mathcal{A}_{G,0}) = \dim(\mathcal{B}_{G,0}).$$

Combining this with Theorem 8 gives the main theorem for central zonotopal algebras.

Theorem 10. *The ideal $\mathcal{J}_{G,0}$ gives a monomization of $\mathcal{I}_{G,0}$. In particular*

$$\operatorname{Hilb}\left(\mathbb{C}\left[x_1, \ldots, x_n\right]/\mathcal{I}_{G,0}, t\right) = \operatorname{Hilb}\left(\mathbb{C}\left[x_1, \ldots, x_n\right]/\mathcal{J}_{G,0}, t\right).$$

Chapter 3

External Algebras

3.1 Background

We will now address the case of the external algebra, the quotient algebra of $\mathcal{I}_{G,1}$, which we denote $\mathcal{A}_{G,1}$. It was initially this algebra which led to Postnikov and Shapiro's study of the central case, despite the fact that the primary historical interest in zonotopal algebras was on the central algebra. There is, however, a remarkable connection between this algebra and geometry of the flag manifold. Precisely

Theorem 11. *[16] Let $\mathcal{A}_n$ be the (commutative) algebra generated by the curvature forms of the n canonical line bundles on the manifold of flags in $\mathbb{C}^{n+1}$, with wedge product as the product. Then this algebra is isomorphic to $\mathcal{A}_{K_{n+1},1}$.*

Since we will not be directly addressing this result, we do not go into a detailed description of this ring as it pertains to geometry. Roughly, though, it says that $\mathcal{A}_{K_{n+1},1}$ contains information about the intersection theory of the flag manifold which is finer than that of the (singular) cohomology ring. We hope, however, that this result motivates the initial interest in looking at the external algebra, aside from it's interest as a particularly well behaved subclass of power ideals.

3.2 Monomization of the External Algebra

Similar to the $k = 0$ case, we define a monomial m_I for any non-empty subset $I \subset [n]$. Namely, let

$$\nu_I(i) = \begin{cases} d_I(i) + 1 & i \in I \text{ is minimal} \\ d_I(i) & i \in I \text{ is not minimal} \\ 0 & i \notin I \end{cases}$$

and define

$$m_I = \prod_{i=1}^{n} x_i^{\nu_I(i)}$$

Notice that these monomials are as close to the center of the Newton polytope of p_I as possible. This remark is addressed further in Chapter 5. Let $\mathcal{J}_{G,1} = \langle m_I \rangle$. The main result of this section is that $\mathcal{J}_{G,1}$ is a monomization of $\mathcal{I}_{G,1}$.

Theorem 12. *The standard monomial basis of $\mathcal{J}_{G,1}$ gives a monomial basis for $\mathcal{I}_{G,1}$.*

We can use this to give a defintion of what we will call a $(G, 1)$-parking function. In the case of the complete graph, we believe this definition appeared first in [10].

Definition 13. For a graph G on the vertex set $\{0, 1, \ldots, n\}$, a $(G, 1)$-parking function is a function $f : [n] \to \mathbb{N}$ such that for any $I \subset [n]$,

$$f(i) < \begin{cases} \#\{\text{edges from } i \text{ out of } I\} + 1 & \text{if } i \text{ is minimal} \\ \#\{\text{edges from } i \text{ out of } I\} & \text{otherwise} \end{cases}$$

Example 14. Let's take the following examples. From the graph in Figure 3.2.1 we obtain the power ideal

$$\begin{aligned} \mathcal{I}_{G,1} = (& x_1^3, x_2^3, x_3^4, x_4^4, (x_1 + x_2)^5, (x_1 + x_3)^4, (x_1 + x_4)^6, (x_2 + x_3)^6, \\ & (x_2 + x_4)^4, (x_3 + x_4)^5, (x_1 + x_2 + x_3)^6, (x_1 + x_2 + x_4)^6, \\ & (x_1 + x_3 + x_4)^5, (x_2 + x_3 + x_4)^5, (x_1 + x_2 + x_3 + x_4)^5) \end{aligned}$$

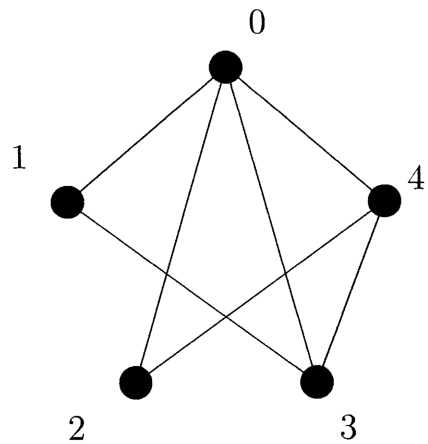


Figure 3.2.1: Graph for Example 14

and the monomization

$$\begin{aligned}\mathcal{J}_{G,1} = (&x_1^3, x_2^3, x_3^4, x_4^4, x_1^3x_2^2, x_1^2x_3^2, x_1^3x_4^3, x_2^3x_3^3, x_2^2x_4^2,\\ &x_3^3x_4^2, x_1^2x_2^2x_3^2, x_1^3x_2x_4^2, x_1^2x_3x_4^2, x_2^2x_3^2x_4, x_1^2x_2x_3x_4)\end{aligned}$$

From this, it is fairly easy to compute that the dimension of $\mathcal{B}_{G,1}$, and therefore $\mathcal{A}_{G,1}$, is equal to 82. Notice that many of the generators of $\mathcal{J}_{G,1}$ are redundant, and it easy to reduce. For example, clearly since we have x_1^3, we don't need $x_1^3x_2^3$, $x_1^3x_4^3$, or $x_1^3x_2x_4^2$. Continuing in this way we can reduce to a minimal set of 10 generators. A computation also gives that the Hilbert series of both $\mathcal{A}_{G,1}$ and $\mathcal{B}_{G,1}$ is $1 + 4t + 10t^2 + 18t^3 + 23t^4 + 18t^5 + 7t^6 + t^7$.

It is routine to check that the monomials defined above are a monotone monomial family, that is, $\mathcal{J}_{G,1}$ is a monotone monomial ideal. Indeed, the condition $m_{I \setminus I} = 1$, which simply states that m_I contains x_i only if $i \in I$, is satisfied by definition. To check the second condition, we examine the degree of x_i for $i \in I$ in m_I and m_J. Since $J \supset I$, the number of edges originating at vertex i and terminating outside J is less than or equal to those terminating outside I, ie. $d_J(i) \leq d_I(i)$. So there are two cases. If i is the smallest element of I, then either its degree goes from $d_I(i) + 1$ to $d_J(i) + 1$ (in the case that i is still the smallest element of J), or it goes from

$d_I(i)+1$ to $d_J(i)$. In both cases the degree drops or remains the same. If i is not the smallest element of I, then it can't be the smallest element of J, so the degree goes from $d_I(i)$ to $d_J(i)$. Therefore, in any case $\deg_{x_i}(m_I) \geq \deg_{x_i}(m_J)$.

Since each m_I is an I-deformation of p_I, the ideal $\mathcal{J}_{G,1}$ is a deformation of $\mathcal{I}_{G,1}$. Because of this, we can use Theorem 8 to conclude

$$\mathrm{Hilb}\left(\mathcal{A}_{G,1}, t\right) \leq \mathrm{Hilb}\left(\mathcal{B}_{G,1}, t\right).$$

The goal is demonstrate equality, for which it is enough to demonstrate the equality of the dimensions. By plugging $z=0$ into the formula from **??**, we get

$$\mathrm{Hilb}\left(\mathcal{A}_{G,1}, t\right) = t^{|E(G)|-n} T_G\left(1+t, \frac{1}{t}\right)$$

and in particular, the dimension is equal to the number of forests which are subgraphs of G.

To investigate the dimension of the of space $\mathcal{B}_{G,1}$, we first use (2.1.1) to find an expression for the dimension as an alternating sum. Let $\nu_I(i) = \deg_{x_i}(m_I)$. We recall here the formula

$$\sum_{I_1 \subsetneq \ldots \subsetneq I_k} (-1)^k \prod_{i \in I_1} \left(\nu(i) - \nu_{I_1}(i)\right) \times \ldots \times \prod_{i \in I_k \setminus I_{k-1}} \left(\nu(i) - \nu_{I_k}(i)\right) \times \prod_{i \notin I_k} \nu(i) \quad (3.2.1)$$

We give the following interpretation to the alternating sum. For a given chain $I_1 \subsetneq \ldots \subsetneq I_k$ the product counts the number of directed subgraphs H of G with the following properties

1. There is at most one edge originating at each $i \in [n]$, and there is no edge originating at 0.
2. If $i \in I_j$ for some j, then the edge originating at i must end in I_j as well.
3. If $i \in I_j$ is the minimal element of I_j, then i has an edge originating at it.

In other words we interpret the numbers in the product as follows. We take $\nu(i)$ to count "the edges incident on i, plus the possibility of no edge", and we take $\nu(i) - \nu_I(i)$

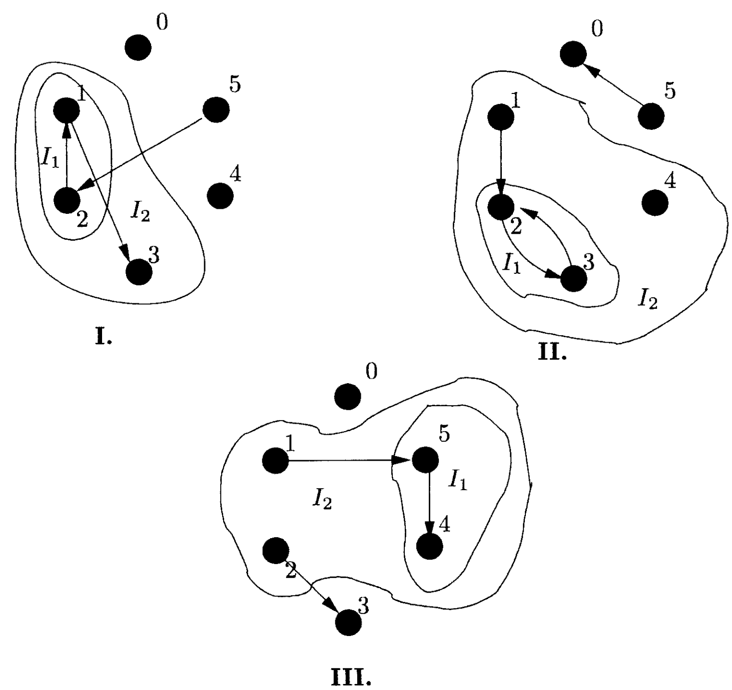


Figure 3.2.2: The digraphs in **I** and **II** are examples of allowable pairs for the displayed chain of subsets. **III** is a non-example because 2 connects to 3, and because 4 needs an edge originating at it.

to count "those edges and the possibility of no edge, less those edges which leave I, and also less the possibility of no edge if i is the minimal vertex of I". Clearly then, for a given chain of subsets, the unweighted product counts the number of subgraphs as described. We call a subgraph with the above properties together with the chain of subsets an *allowable pair*. See Figure 3.2.2 for some sample allowable pairs, and an example of a digraph which is not an allowable pair.

Let us note some properties of these subgraphs. Firstly, any subgraph of G satisfying the first condition appears in the sum with $k = 0$. Secondly, we can embed the set of forests canonically in this collection as follows. For any forest F of G, orient each edge of F so that each connected component has a unique sink at the minimal element of that component. Indeed, have the following lemma.

Lemma 15. *Every subforest F of G appears with this canonical orientation in the alternating sum exactly once with weight 1, namely with the empty chain of subsets.*

Proof. Suppose that the chain of subsets is non-empty. Then the minimal element of I_1 must have an edge originating at it. But then the forest is not canonically oriented. So if F appears with the canonical orientation, it must have the empty chain of subsets. Clearly, though, it does appear with the empty chain. $\square$

We claim that every other subgraph is cancelled out in the sum, so that the alternating sum is precisely equal to the number of subforests of G. To show this, we now construct an involution on the set of pairs $(H, I_1 \subsetneq \ldots \subsetneq I_k)$. The involution will only act on the chain of subsets, that is, it will leave H fixed. A pair will be fixed by the involution if and only if it corresponds to $(H, \emptyset)$ with H a canonically oriented forest. and it will take a chain of length k either to chain of length $k-1$ or length $k+1$. Since there are no other fixed points, this will show that any nonforest H is cancelled out in the alternating sum.

The involution will only use some subset of the vertices of H which we call *special.* We use the following algorithm to label the vertices of H special and non-special.

- Let v be the smallest unlabelled vertex.
- If v has an edge originating at it, label it and all remaining unlabelled vertices special and stop. Otherwise, label v non-special as well as any vertex such that the chain of edges originating from it terminates at v.
- Return to the first step.

Notice that 0 will always be chosen first (and labelled non-special), when none of the vertices are labelled. See Figure 3.2.3 for an example. We also have the following claim.

Proposition 16. *Each connected component is either composed entirely of non-special vertices or special vertices. Those labelled non-special are trees rooted at their minimal element.*

Proof. Suppose that i is non-special. Then i must lie on a directed path towards a terminal vertex, and in particular the path originating at i does not contain a circuit.

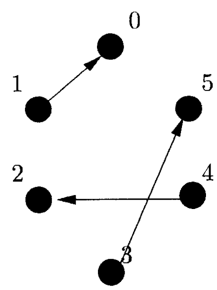


Figure 3.2.3: For this graph, vertices 3 and 5 are special, while the others are non-special.

This follows because the only way a vertex can be labelled non-special is in step 2 of the algorithm, and only as part of a path which terminates. Therefore, if i is non-special, then its connected component must be a tree.

If i is part of a tree T and non-special, we claim the tree is rooted, ie. it has a unique sink. More specifically, it is rooted at the end of the path originating from i. If this is the case, then every vertex of the tree was labelled non-special in the same step that i was. To see that it has a uniqe sink, let w be a sink in T. There is a unique undirected path $(w, v_1, v_2, \ldots, v_k, i)$ from w to i. The edge from w to v_1 must be oriented towards w, because w is a sink. Because each vertex can have at most one out-edge, this is the unique out-edge from v_1. Therefore the edge from v_2 to v_1 must be oriented towards v_1.

Continuing in this way, we conclude that the entire path is oriented from i to w. Therefore w is the vertex at the end of the path originating from i, and consequently is unique. The only thing left to see is that w is the minimal vertex of the tree. This is easy, though, since otherwise w would not have been chosen in step 1 of the algorithm, and this component would be labelled special. □

Note that the converse of the second part of the claim is false. It is perfectly feasible for a subtree of H to be oriented towards its minimal vertex and still be labelled special. We do however get the following corollary.

Corollary 17. *H is a canonically oriented forest if and only if the algorithm labels every vertex non-special.*

Proof. One direction is clear from the claim: If all the vertices are labelled non-special, then every connected component is a rooted tree oriented toward the minimal vertex, which is a canonically oriented forest. For the converse, suppose H is a canonically oriented forest, but some vertex is marked special. Then at some point in the algorithm, the least unlabelled vertex v has an out-edge. The path coming out of v must terminate at a vertex smaller than v since the forest is canonically oriented, but then this vertex must have been labelled non-special. This, in turn, would imply that v is labelled non-special, and we get a contradiction. $\square$

Corollary 18. *If $i \in I_j$, then i is special.*

Proof. Let w_j be the minimal vertex in I_j. Clearly w_j must be special, because if it were non-special then by Proposition 16 it would lie on a tree oriented towards its minimal vertex. However this isn't possible since there is an edge originating at w_j, and the entire path from w_j must lie within I_j by definition. However, if i is non-special, then the path originating at i must terminate at a non-special vertex v, and that vertex must lie in I_j. Because v must also lie in I_j it must be greater than w_j, but then w_j would have been chosen in step 1 of the above algorithm before v was, and v wouldn't have been marked non-special. This is a contradiction. $\square$

We now define the involution κ. Let $\mathcal{S}$ be the set of special vertices in H. Then

$$\kappa\left((H, I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq I_k)\right) = \begin{cases} (H, I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq I_k \subsetneq \mathcal{S}) & \text{if } \mathcal{S} \neq I_k \\ (H, I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq I_{k-1}) & \text{if } \mathcal{S} = I_k \end{cases}$$

If the chain of subsets is empty, and there are no special vertices, then κ does nothing. By Corollary 17 this means that H is a canonically oriented tree. Otherwise, the length of the chain of subsets is changed by κ. Therefore, the only fixed points are $(H, \emptyset)$ where H is a canonically oriented tree. Applying κ to formula 2.1.1 we get

Theorem 19. *The dimension of $\mathcal{B}_{G,1}$ is equal to the number of forests on G.*

This completes the proof of Theorem 12. As mentioned, we also get the following corollary

Corollary 20. $\mathrm{Hilb}\,(\mathcal{B}_{G,1}, t) = \mathrm{Hilb}\,(\mathcal{A}_{G,1}, t)$

From [1] we also have a combinatorial interpretation for the coeffients in $\mathrm{Hilb}\,(\mathcal{A}, t)$. If

$$\mathrm{Hilb}\,(\mathcal{A}, t) = \sum c_n t^n$$

then c_n is equal to the number of forests F on G with external activity equal to $|G| - |F| - n$. This generalizes the results of [15, 16]. It would be nice to find a bijection between the monomials of fixed degree and the forests of fixed external activity.

Chapter 4

Internal Algebras

We now examine the case $k = -1$, the internal algebra. We would like to proceed by analogy with the external algebra, choosing one monomial for each generator of $\mathcal{I}_{G,-1}$ as near to the center of the Newton poytope as possible, and making a lexicographically minimal choice to break ties. In other words, for a graph G, we let

$$\nu_I(i) = \begin{cases} d_I(i) - 1 & i \in I \text{ is minimal} \\ d_I(i) & i \in I \text{ is not minimal} \\ 0 & i \notin I \end{cases}$$

and define

$$m_I = \prod_{i=1}^{n} x_i^{\nu_I(i)}$$

This product will only be a monomial in $\mathbb{C}[x_1, \ldots, x_n]$ if $d_I(i) > 0$ when i is minimal in I. This will not be a case for a general graph, even in a simple case like the complete graph on $(0, 1, 2)$ with the edge $(0, 1)$ missing. In fact, for any vertex x let $N^0(x)$ be the neighborhood of x in $G \setminus \{0\}$. Now suppose there is a vertex x such that x is minimal in $N^0(x)$. If x is not connected to 0, then letting $I = N^0(x)$, we have $\nu_I(x) = -1$, and $m_I \notin k[x_1, \ldots, x_n]$. If, however, all vertices of G are connected to 0, then the product above will always be a monomial, and we define the ideal $\mathcal{J}_{G,-1}$ to be $\langle m_I \rangle$. Even in this case, though, we cannot achieve the same results as in the external case.

Example 21. For the graph G appearing in figure 4.0.1, the ideal $\mathcal{I}_{G,-1}$ has Hilbert Series $1 + 5t + 11t^2 + 15t^3 + 12t^4 + 3t^5$ while the ideal $\mathcal{J}_{G,-1}$ has Hilbert Series $1 + 5t + 11t^2 + 15t^3 + 12t^4 + 4t^5$.

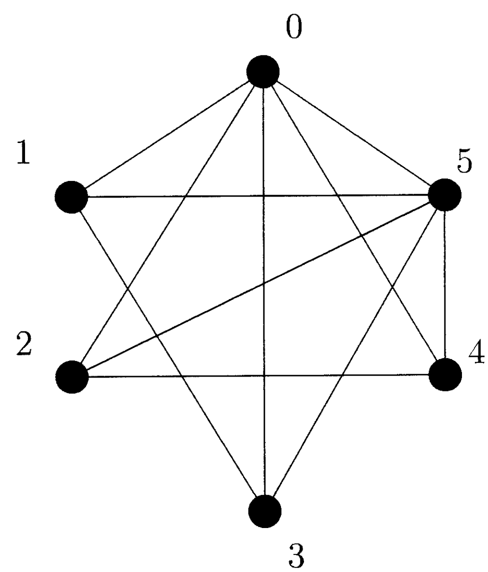


Figure 4.0.1:

Using the `nauty` [12] software package for exhaustive graph generation and `Singular` [8] for Hilbert Series computations, we have determined that this graph is the minimal counterexample. In other words, all graphs on a smaller number of vertices have $\text{Hilb}\left(\mathcal{I}_{G,-1}, t\right) = \text{Hilb}\left(\mathcal{J}_{G,-1}, t\right)$. Furthermore, all counterexamples thus discovered are missing a four-cycle, although this is certainly not a sufficient condition for inequality. It would be interesting and certainly illuminating to determine the conditions under which one can guarantee the equality of the two Hilbert Series.

Thus, we must impose some restriction on the graphs we examine. It turns out the condition that at least one edge exist between any two vertices is sufficient. Let $\mathcal{B}_{G,-1} = \mathbb{C}\left[x_1, \ldots, x_n\right] / \mathcal{J}_{G,-1}$. The main result of this section is

Theorem 22. *Let G be a graph with multiple edges on the vertex set $\{0, 1, \ldots, n\}$ with at least one edge between any two vertices. Then* $\text{Hilb}\left(\mathcal{B}_{G,-1}, t\right) = \text{Hilb}\left(\mathcal{A}_{G,-1}, t\right)$.

In particular, $\dim(\mathcal{B}_{G,-1}) = \dim(\mathcal{A}_{G,-1})$ *which is known to be the number of subtrees with no internal activity.*

Regarding parking functions, the corresponding notion is as follows

Definition 23. For a graph G with multiple edges on the vertex set $\{0, 1, \ldots, n\}$ such that there is at least one edge between any two vertices, a $(G, -1)$-parking function is a function $f : [n] \to \mathbb{N}$ such that for any $I \subset [n]$,

$$f(i) < \begin{cases} \#\{\text{edges from } i \text{ out of } I\} - 1 & \text{if } i \text{ is minimal} \\ \#\{\text{edges from } i \text{ out of } I\} & \text{otherwise} \end{cases}$$

The proof will be similar to the case of the external algebra. We first confirm that $\mathcal{J}_{G,-1}$ is a monotone monomial ideal, and therefefore that we can use the formula 2.1.1 to compute the dimension of $\mathcal{B}_{G,-1}$. Then we will find a combinatorial interpretation for this count in terms of subgraphs of G, and then show that this number agrees with the one for $\mathcal{A}_{G,-1}$ from Ardila-Postnikov.

Lemma 24. *$\mathcal{J}_{G,-1}$ is a monotone monomial ideal if and only if there is an edge between any two non-zero vertices in G.*

Proof. Again the first condition in the definition of monotone monomial family is obviously satisfied. For the second, suppose $i \in I \subsetneq J$. If i is not the minimal vertex of I, then it is also not the minimal vertex of J, so $\nu_I(i) = d_I(i) \geq d_J(i) = \nu_J(i)$. If i is minimal in I, but not in J, then $\nu_I(i) = d_I(i) - 1$, but because there is an edge from i to some vertex in $J \backslash I$, we have $d_J(i) \leq d_I(i) - 1$, and therefore, once again, $\nu_I(i) \geq \nu_J(i)$. Finally, if i is minimal in both I and J, we have $\nu_I(i) = d_I(i) - 1 \geq d_J(i) - 1 = \nu_J(i)$.

To see that the condition is necessary, assume that the edge (i, j) is not in G, with $i < j$. Then $\nu_{\{j\}}(j) = d_{\{j\}}(j) - 1$ while $\nu_{\{i,j\}}(j) = d_{\{j\}}(j)$. □

In this situation, we can use the formula 2.1.1 to determine the dimension of $\mathcal{B}_{G,-1}$. We now associate a combinatorial interpretation to the summands. Consider subgraphs H of G created according to the following method. For edges with multiplicity choose some fixed ordering. The ordering we will take on the edges will be the lexicographic ordering on the edges, with tie-breaks according to these fixed orderings. The method contains a loop, so some statements in parantheticals will only make sense on a complete reading:

1. Let $j = 1$.

2. Let x be the minimal (remaining) vertex in I_j (if I_j doesn't exist, skip to step 5).

3. Choose any edge at x which lies in I_j besides the minimal edge and connect it.

4. Now choose any other edge in I_j except the one we just came from and connect to it. If this vertex does not have an edge originating at it yet, repeat this step. Otherwise, choose the minimal remaining vertex in I_j, and repeat step 3. If there are no remaining vertices in I_j, increase j by one, and go to step 2.

5. After all the vertices in the subsets have edges originating at them, let x be the minimal vertex remaining, and connect it to some vertex except along the minimal edge to 0. Continue connecting until either 0 is reached, or a vertex which already has an edge is reached. At that time, repeat this step.

6. Stop when every vertex has an edge originating at it, or no edge may be chosen.

We first note that the only circumstance in which the method terminates before every vertex has an edge is if I_1 has only 1 or 2 vertices, and in the latter case, only if there is only one edge between these two vertices. For later reference, note that in this case the product in (2.1.1) is 0. If every vertex besides 0 has an edge originating at it in the end result, we say that the resulting graph H together with the chain of subsets is an *allowable pair*. See Figure 4.0.2 for an example on the complete graph.

Lemma 25. *For a fixed chain of subsets $I_1 \subset I_2 \subset \ldots \subset I_k$, the number of allowable*

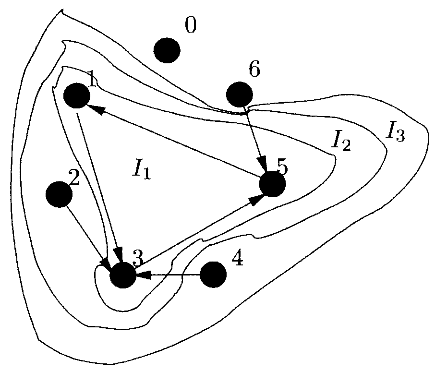


Figure 4.0.2: Allowable Pair

pairs is equal to

$$\prod_{i\in I_1}\left(\nu_{\{i\}}(i)-\nu_{I_1}(i)\right)\times\ldots\times\prod_{i\in I_k\backslash I_{k-1}}\left(\nu_{\{i\}}(i)-\nu_{I_k}(i)\right)\times\prod_{i\notin I_k}\nu_{\{i\}}(i)$$

Proof. As a first approximation to an interpretation of these numbers, we think of $\nu_{\{v\}}(v)$ as meaning "The number of edges out of v besides the minimal one to 0," and we think of $\nu_{\{v\}}(v)-\nu_{I_j}(v)$ as meaning "Remove from those edges all edges going out of I_j, and also remove one more if v is not minimal in I_j." This is only a first approximation for several reasons, not the least of which is that the second statement is vague.

For each vertex v, we first fix a list of edges incident to v. Then for each v we choose a natural number, depending on which subset(s) v lies in. If v is not in I_k, then the list is simply all edges at v, and we choose a number between 1 and $\nu_{\{v\}}(v)$. Otherwise, if $v\in I_j\backslash I_{j-1}$, the list is all edges at v in I_j, and we choose a number between 1 and $\nu_{I_j}(v)-\nu_{\{v\}}(v)$. We now use these numbers to construct a subgraph H_χ. Basically, we follow the steps in the method above, and at each step we (possibly) make a change to our list, and then choose the $\chi(v)^{\text{th}}$ edge on our list when the method gives us a choice. In order to ensure that H_χ is allowable, we do the following at each vertex v.

If v is the minimal vertex in I_j, then it won't have any edges coming into it yet. If it did, then that edge would have to come from within I_j and in fact tracing the whole chain back, each vertex would have to lie in I_j. In particular, the very last vertex in that chain would lie in I_j, but the only way this vertex could have been chosen is if it were the minimal remaining vertex in I_j, and this is impossible since v is that vertex. In this case, we don't change the list at all. Notice that for these vertices, the list is the correct length. In all other cases, some edge must be removed. If v is in $I_j \backslash I_{j-1}$ and we have arrived at v from some other vertex, cross that edge off the list, *then* choose the $\chi(v)^{\text{th}}$ edge on the list. If we arrived at v and v does not yet have an edge enterring it, remove the minimal edge in I_j at v from its list and then choose the $\chi(v)^{\text{th}}$ edge on the list. Finally, if v is not in I_k, we remove the minimal edge to 0 from it's list, unless it has an edge enterring it, in which case we remove that edge and choose the $\chi(v)^{th}$ edge on the list. In any of these cases, the list has length equal to our first approximation.

Clearly, because we have simply provided choices for the method above, we have created an allowable graph. As mentioned before the lemma, this will only fail if we remove all edges from the list, which can only happen in the cases described, in which case the number of allowable pairs and the product in the lemma are both 0. Furthermore, the number of choices for χ is clearly equal to that product, and each choice determines a unique graph H_χ. Finally, given any graph H determined via the method above, the choice made at each step determines such a χ in the obvious manner, so this number is in fact the correct count. □

Our goal, of course, is to show that this number is equal to the number of spanning trees with no internal activity. The first step towards this is to show that each such tree appears with total multiplicity 1.

Lemma 26. *If H is a tree constructed by the above method, then it is rooted at 0, the chain of subsets is empty, and the tree has no internal activity. Furthermore, any such tree can be constructed in this way.*

Proof. Since every non-zero vertex has an edge originating at it, if H is a tree, it clearly must be rooted at 0, as 0 is the only sink. If I_1 is not empty, then there are

$|I_1|$ vertices and as many edges between them, and therefore there must be a cycle. Therefore, if H is a tree, then I_1, and hence the chain, must be empty.

Now suppose that H has an internally active edge (i, j) with $i < j$. For this proof, it is easier to ignore the direction on the edge, and so even though we write it as an ordered pair, the ordering will be ignored for now. After removing the edge (i, j), let H_i be the component of the remaining graph containing i, and H_j be the component containing j. If $0 \in H_i$, then $(0, j)$ reconnects the graph. Since (i, j) is internally active, we must have $i = 0$, else $(0, j)$ is smaller. In the above method, the only way $j \to 0$ could have been chosen is if at that stage in the development of H, there was already an edge incident on j. The only way this can happen is if there is some vertex l which is less than j that was, at some time, chosen as the minimal remaining vertex, and the path chosen from it eventually led to j. In this case, however, $(0, l)$ would reconnect the graph, and that edge is less $(0, j)$ so that (i, j) would not be internally active, a contradiction. Therefore we could not have had $0 \in H_i$. So assume $0 \in H_j$. Since $i < j$, we can't have $j = 0$, but then $(i, 0)$ reconnects the graph and is less than (i, j), meaning that (i, j) is not internally active, contradiction. Therefore we also can't have $0 \in H_j$. In conclusion we cannot have an internally active edge, so H has no internal activity.

Now, given such a tree with no internal activity rooted at 0, it is clear how to construct it according to the method above. When given a choice of edge, we simply choose the edge in the tree. We must only check that this edge is allowed according to the method. But this is clear. The only edges which would not be allowed are edges that double-back, creating a cycle, or in a few instances, edges connecting to 0. But, if $(i, 0)$ is an edge in H, and at the current stage of the method there is no edge incident on i, then i must be the smallest vertex amongst those whose paths eventually lead through i. If some $j < i$ has its path to 0 through i, then it will be chosen before i in the method, and hence i would have an edge incident on it already. $\square$

The two lemmas above demonstrate that in the alternating sum expressing the dimension of $\mathcal{B}_{G,-1}$, every subtree of G with no internal activity appears exactly once

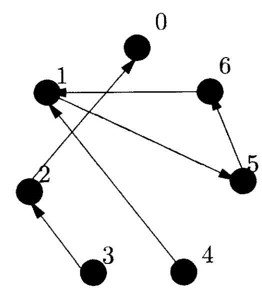


Figure 4.0.3: Vertices 1, 5, and 6 are special. All others are non-special

with (total) positive weight 1. We now must see that all other graphs H appear with weight 0. Towards this end we will define an involution on pairs $(H, I_1 \subset \ldots \subset I_k)$ as in the other cases. By analogy with the external case, we divide the vertices of H into two classes, which we call *special* and *non-special*, according to the following scheme. To begin, 0 is always a non-special vertex. Let $\mathrm{For}(v)$ be the set of vertices that we hit if we travel forward along the path originating at v, and let $\mathrm{Tail}(v)$ be the vertices w such that $v \in \mathrm{For}(w)$. Note that while $\mathrm{For}(v)$ may have only one branch, $\mathrm{Tail}(v)$ may actually have several branches. If $0 \in \mathrm{For}(v)$, then label v non-special. Now let x be the minimal vertex with $0 \notin \mathrm{For}(x)$. If any unlabelled vertex v points (directly) to x, *and* v is minimal in $\mathrm{Tail}(v)$, label it and any vertex in its tail non-special. Label all other vertices special. See Figure 4.0.3 for an example.

Let J be the set of all vertices eventually labelled special. Now define a function

$$\kappa (H, I_1 \subset \ldots \subset I_k) = \begin{cases} (H, I_1 \subset \ldots \subset I_k \subset J) & I_k \neq J \\ (H, I_1 \subset \ldots \subset I_{k-1}) & I_k = J \end{cases}$$

If it is well-defined, the operation is clearly a fixed-point free involution which changes the weight $(-1)^k$ in (2.1.1) from positive to negative or vice versa. Thus the only thing left to prove is that $\kappa (H, I_1 \subset \ldots \subset I_k)$ is an allowable pair, that is, that H is a subgraph which can be constructed with the method above using the new chain of subsets.

Lemma 27. *If $v \in I_j$ for some j, then v is special*

Proof. There's only two ways a vertex can be labelled non-special. If $v \in I_j$, then any vertex in $\text{For}(v)$ is also in I_j. Since 0 is not in any of the subsets, v cannot be labelled non-special in the first way. Therefore, suppose $v \in I_j$ is labelled non-special, but $0 \notin \text{For}(v)$. Let x be the minimal unlabelled vertex, as in the description of the labelling process. Then (v, x) is an edge of H, and v is minimal in $\text{Tail}(v)$. Since $v \in I_j$ and $x \in \text{For}(v)$, we have $x \in I_j$. Since the only vertices labelled so far have 0 in their forward path, none of the members of any I_j are labelled non-special yet. Therefore, x is the minimal element of I_j. So, in order for H to have been constructed according to the method, v must have an edge enterring it at the time it is selected by the method. This is impossible though, since v is the minimal element of $\text{Tail}(v)$, and therefore must have been chosen ahead of any of those vertices. This is a contradiction. □

This shows, in particular that $I_k \subset J$ always. If $I_k = J$, then consider the situation when we remove it from the chain of subsets. As we follow the method for constructing subgraphs using this chain, the only vertices whose rules have changed are those that were in $J \backslash I_{k-1}$. In fact, though, the choice of edges has become strictly more permissive, so H together with this chain is an allowable pair.

Contrary to this, in the case $I_k \neq J$, while these vertices are again the only ones for whom the rules regarding edge choosing have changed, the rules are more restrictive. However, by construction, if w non-special, so is $\text{Tail}(w)$, so we can not have (v, w) as an edge with $v \in J$ and $w \notin J$. The only problem that can arise, then, is if v connects to the minimal element of J, and this happens at a point in the method when v does not have an edge enterring it. This implies that v is minimal amongst $\text{Tail}(v) \cap J$. Suppose the minimal element w of $\text{Tail}(v)$ is not in J. Clearly $0 \notin \text{For}(w)$, so therefore for some element $u \in \text{For}(w)$ we have u connecting directly to x and u is minimal in $\text{Tail}(u)$. However, v is the element of $\text{For}(w)$ which connects to x, and so it must be minimal. This contradicts the minimality of w in $\text{Tail}(v)$. So v must be minimal in all of $\text{Tail}(v)$, but in this case it would have been marked non-special, and so not in J.

This completes the proof, and shows that $\kappa(H, I_1 \subset \ldots \subset I_k)$ is an allowable pair.

Since κ is a fixed-point free involution on set of allowable pairs when H is not a tree with no internal activity, while changing the sign of the weight in (2.1.1), we find that this together with Lemma 26 proves that the dimension of $\mathcal{B}_{G,-1}$ is equal to the number of spanning trees with no internal activity, completing the proof of Theorem 22. Again from the paper of Ardila-Postnikov, we also get an interpretation for the coefficients of the Hilbert Series, namely that it is the generating function for such trees with a given external activity.

Chapter 5

Observations, Conjectures, and Future Work

There are many issues raised throughout this thesis. Some purely conjectural, some obvious avenues for new research, and some suggestive of a deeper program and theory. We'll touch on a few of them here.

First and foremost is the unsatisfying requirement in Theorem 22 that the graph G contain an edge between any two vertices. Because the ideals $\mathcal{I}_{G,-1}$ can be defined, and have importance in the literature, for any graph G, a more general construction should be found to handle this case.

There is also the question of $\mathbb{C}[x_1, \ldots, x_n] / \mathcal{I}_{G,k}$ as a representation for $\mathrm{Aut}_*(G)$, especially for the symmetric group, when the graph is the complete graph. Although the monomization provides little in the way of help, trying to understand even the degree of the trivial representation would be an interesting program. Towards this end, we have computed the degree of the trivial representation in some small cases for the complete graph. The results so far are compiled in Table 5. The computations are very lengthy, but the code used is available by request.

Both (partial) sequences for the external and internal algebra appear in the Sloane database already. The partial sequence for the external case appears as the image of the Catalan numbers under the Riordan array $(1, x(1-x^2))$. Another way of saying

n	External	Central	Internal
2	2	1	0
3	4	2	1
4	10	5	2
5	27	14	7
6	78	42	22
7	?	132	73
8	?	429	?

Table 5.1: Degree of the trivial representation in the three zonotopal algebras for K_n.

this is that this sequence appears to have the generating function

$$\frac{2}{1+\sqrt{1-4x\left(1-x^2\right)}}.$$

The partial sequence for the internal case appears as the number of returns to the x-axis in all hill-free Dyck paths of semilength n. This gives the generating function

$$\frac{\left(1-\sqrt{1-4x}\right)^2}{\left(1+\sqrt{1-4x}+2x\right)^2}.$$

While both cases are just partial comparisons, because they are both related to the Catalan numbers, there may be some justification for conjecturing equality.

Another obvious area for future work is other zonotopes. A general theory of monomization for zonotopes is likely far off, and perhaps not even intelligible. However, it seems worthwhile to investigate monomization for hyperplane arrangements of types B,C, and D. Initial investigations suggest that the approach of this paper cannot be directly applied, but perhaps some minor modifications will work.

5.1 Computational Comments

Some words must be said regarding initial ideals and Gröbner bases. Given an ideal I in a polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, the initial ideal $\mathrm{in}(I)$ is defined with respect to a global ordering. A global ordering is a well-ordering on the monomials of the polynomial ring satisfying

- $x_i > 1$ for every variable x_i.
- If $m_1 > m_2$ and $m_3 \geq m_4$ then $m_1 m_3 > m_2 m_4$.

Given such an ordering and a polynomial f, we can define the leading monomial of f to be the monomial (with non-zero coefficient) in f which is greatest with respect to this ordering. The initial ideal, then, is the monomial ideal of those monomials which are leading monomials for elements of I.

Another way of saying this involves the Newton polytope. Given a polynomial

$$f = \sum_{\omega \in \mathbb{Z}^n} c_\omega x^\omega$$

the Newton polytope is the convex hull of the points ω for which $c_\omega \neq 0$. A monomial ordering is a linear form on the space of monomials (which distinguishes all the monomials), and the initial monomial is the monomial for which that form is maximal. It is, therefore, always a vertex of the Newton polytope. In contrast, the monomizations for all the algebras in this thesis are generated by taking the monomial for each generator of $\mathcal{I}_{G,k}$ which is closest to the *center* of the Newton polytope.

The initial ideal is extremely important as a computational, and even a theoretical, tool. The primary reason for this is that it is easy to define a flat deformation (over $\mathbb{P}^1$) from the ideal to its initial ideal. This deformation shows, for starters, that the Hilbert Series of the two ideals is the same. It also shows, somewhat more substantially, that the points corresponding to I and $\text{in}(I)$ lie on the same irreducible component of the Hilbert scheme.

It is not true in general that if $g_1, \ldots, g_m$ is a set of generators for I, then the leading monomials, $\text{in}(g_1), \ldots, \text{in}(g_m)$, of those generators generate the initial ideal. A Gröbner basis is a special collection of generators, $(f_1, \ldots, f_m) = I$, such that the initial ideal is generated by the leading terms, $(\text{in}(f_1), \ldots, \text{in}(f_m)) = \text{in}(I)$. The existence of a Gröbner basis is important, as is the development of methods for finding them. All algorithms for determining the Hilbert Series, or even the Hilbert function, first compute the initial ideal, and without something like a Gröbner basis, it's impossible to determine if one has chosen enough generators.

Unfortunately, algorithms, such as Buchberger's algorithm, for determining a Gröbner basis are exteremely slow, and indeed there is no known bound on the running time. In contrast to this, in the graphical case of the zonotopal algebras in this paper we have demonstrated that if one takes the center point (or closest such point) of the Newton polytope for each generator and generates this ideal, one obtains a monomial ideal with the same Hilbert series in constant time. Some examples of the amount of time taken using standard Gröbner basis techniques can be found in Table 5.2.

Note that running time is not the only limitation. Indeed, time was not the primary limitation in performing the comparison. Even on a computer with a memory cache of 24Gb, the computation runs out of memory even for graphs with just 8 vertices, essentially because the size of the Gröbner and its constituent parts gets so large. In contrast, the size of the monomization is, of course, fixed and equal to the number of generators of the original ideal. As an example, for the complete graph on 6 vertices, using the lexicographic ordering on monomials, the Gröbner basis found using `Singular` had 123 elements, while the monomization of course has $31 = 2^5 - 1$ generators. Perhaps, though, it is possible that a sort of centrally symmetric Gröbner theory could be developed for these ideals which would allow the case of the internal algebra to be solved by adding more generators, or modifying the generators used. Somehow the difficulty with Gröbner bases is that they ignore any symmetry in the ideal, looking only at the vertices (indeed *one* vertex) of the Newton polytope.

It should be added, even once the initial ideal or monomization has been found, the process of determining the Hilbert Series, although essentially combinatorial, is still fairly slow, and in fact is NP-complete [2]. This is a problem which we do not attempt to solve at this time.

The last connection with Gröbner theory regards the Hilbert scheme of points. Because the Hilbert scheme is a fine moduli space, the existence of flat deformation over $\mathbb{P}^1$ from I to $\mathrm{in}(I)$ implies that the two points on the Hilbert scheme are con-

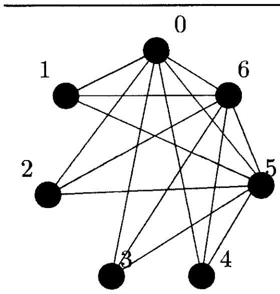
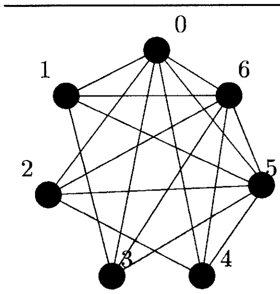
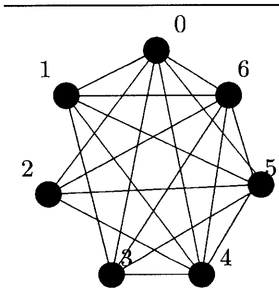
Graph	Time elapsed
	3 min. 5 sec.
	42 min. 6 sec.
	3 days 21 hrs. 32 min.

Table 5.2: Time elapsed in computing the Hilbert series of $\mathcal{A}_{G,1}$ using Gröbner basis techniques. Using monomization takes less than the timer in `Singular` can record.

nected by a smooth rational curve, and in particular lie on the same component. This strong geometric statement is something we would like to investigate regarding the monomizations. Because the connection between the geometry of the two fat points corresponding to $\mathcal{I}_{G,k}$ and $\mathcal{J}_{G,k}$ is anything but obvious, I would be inclined to conjecture that in general they do not lie on the same component. However, because $\mathcal{J}_{G,k}$ is a monomial ideal it has a tendency to lie at the intersection of several components, attempts to prove they lie on different components cannot be achieved solely by looking at tangent space computations, which is the primary computational tool in this regard. Therefore, greater knowledge of the component structure of the Hilbert scheme of points is necessary, and this is known to be an extremely difficult problem. Indeed, even in 3 variables, it is unknown which Hilbert schemes of points have more than one component.

The punctual Hilbert scheme is a subscheme of the Hilbert scheme of points corresponding to fat points at the origin. Deformations of these fat points, and therefore the geometry of the punctual Hilbert scheme, is even more subtle, and in fact little is known about them when the number of variables is greater than 2, but investigating the location of our ideals there would also be interesting.

Bibliography

[1] Federico Ardila and Alexander Postnikov. Combinatorics and geometry of power ideals. *Trans. Amer. Math. Soc.*, 2010.

[2] D. Bayer and M. Stillman. Computation of hilbert functions. *Journal of Symbolic Computing*, 14, 1992.

[3] Brian Benson, Deeparnab Chakrabarty, and Prasad Tetali. G-parking functions, acyclic orientations and spanning trees. *Discrete Mathematics*, 310(8):1340 – 1353, 2010.

[4] Denis Chebikin and Pavlo Pylyavskyy. A family of bijections between g-parking functions and spanning trees. *Journal of Combinatorial Theory, Series A*, 110(1):31 – 41, 2005.

[5] Wolfgang Dahmen and Charles A. Micchelli. On the local linear independence of translates of a box spline. *Studia Mathematica*, 82:243–262, 1985.

[6] Wolfgang Dahmen and Charles A. Micchelli. The number of solutions to linear diophantine equations and multivariate splines. *Transactions of the American Mathematical Society*, 308(2):509–532, 1988.

[7] C. De Concini and C. Procesi. The algebra of the box spline, 2006.

[8] G.-M.; Pfister G.; Schönemann H. Decker, W.; Greuel. SINGULAR 3-1-1 — A computer algebra system for polynomial computations. 2010. http://www.singular.uni-kl.de.

[9] J. Emsalem and A. Iarrobino. Inverse system of a symbolic power, i. *Journal of Algebra*, 174(3):1080 – 1090, 1995.

[10] Olga Holtz and Amos Ron. Zonotopal algebra, 2007.

[11] R.-Q. Jia. Local linear indepence of the translates of a box spline. *Constructive Approximation*, 1:175–182, 1985.

[12] Brendan McKay. Practical graph isomorphism. *Congressus Numerantium*, 30:45–87, 1981.

[13] H. Narushima. Principle of inclusion-exclusion on partially ordered sets. *Discrete Mathematics*, 42:243–250, 1982.

[14] Alexander Postnikov and Boris Shapiro. Trees, parking functions, syzygies, and deformations of monomial ideals. *Trans. Amer. Math. Soc.*, 356(8):3109–3142 (electronic), 2004.

[15] Alexander Postnikov, Boris Shapiro, and Mikhail Shapiro. Chern forms on flag manifolds and forests. In *Proceedings of the 10-th International Conference on Formal Power Series and Algebraic Combinatorics*. Fields Institute, Toronto, 1998.

[16] Alexander Postnikov, Boris Shapiro, and Mikhail Shapiro. Algebras of curvature forms on homogeneous manifolds. In *Differential topology, infinite-dimensional Lie algebras, and applications*, volume 194 of *Amer. Math. Soc. Transl. Ser. 2*, pages 227–235. Amer. Math. Soc., Providence, RI, 1999.

[17] Richard P. Stanley. Hyperplane arrangements, parking functions and tree inversions. In B. Sagan and R. Stanley, editors, *Mathematical Essays in Honor of Gian-Carlo Rota*, pages 259–375. Birkhäuser, Boston/Basel/Berlin, 1996.

[18] Bernd Sturmfels and zhiqiang xu. Sagbi bases of cox-nagata rings, 2008.

[19] Catherine H. Yan. Generalized parking functions, tree inversions, and multicolored graphs. *Advances in Applied Mathematics*, 27(2-3):641 – 670, 2001.